Task 3.1

Theoretical part.

Minimal Spanning Tree.

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G_n contains n verticies, $n>2$.

cond 1: $|i-j| \leq 2$ & $\left\lfloor \frac{i-1}{4} \right\rfloor = \left\lfloor \frac{j-1}{4} \right\rfloor$

cond2: $|i-j| \leq 4$ & $(i+j) \bmod 4 = 0$

$weight(j,i) = w(j,i) = i+j+(|i-j|-1)^2$

I. First, let us consider v_j vertex with index j as newly added. Under such an assumption $j>i$, then:

cond 1: $j-i \leq 2$ & $\left\lfloor \frac{i-1}{4} \right\rfloor = \left\lfloor \frac{j-1}{4} \right\rfloor$

cond 2: $j-i \leq 4$ & $(i+j) \bmod 4 = 0 \Rightarrow$

$\Rightarrow$ cond 2: $j-i \leq 4$ & $((i \bmod 4 + j \bmod 4 = 0) \vee (i \bmod 4 + j \bmod 4 = 4))$

move 1.) For some given j there're 2 i's satisfying cond 1:

$i=j-1$ and $i=j-2$

let us consider a table below, where $n \in \mathbb{Z}^+$ and $g(x) = \left\lfloor \frac{x-1}{4} \right\rfloor$:

j	$g(j)$	$g(i), i=j-1$	$g(i), i=j-2$
$4\cdot n+3$	n	n	n
$4\cdot n+2$	n	n	$n-1$
$4\cdot n+1$	n	$n-1$	$n-1$
$4\cdot n$	$n-1$	$n-1$	$n-1$
$4\cdot(n-1)+3$	$n-1$	$n-1$	$n-1$
$4\cdot(n-1)+2$	$n-1$	$n-1$	$n-2$
$4\cdot(n-1)+1$	$n-1$	$n-2$	$n-2$
$4\cdot(n-1)$	$n-2$	$n-2$	$n-2$

From the table on the left handside we can see that initial condition cond 1 holds for:

$i=j-1$ & $j \in [n\cdot 4+2; n\cdot 4+4], n \in \mathbb{Z}$

and

$i=j-2$ & $j \in [n\cdot 4+3; n\cdot 4+4], n \in \mathbb{Z}$.

move 2.)

For now let us consider cond 2 for $j > i$ as j is a new node in the graph here:

$$j - i \leq 4 \;\&\; ((i \bmod 4 + j \bmod 4 = 0) \vee (i \bmod 4 + j \bmod 4 = 4))$$

To begin with, there're 4 possible values for i as the range of 4 is given:

$i = j-1$
$i = j-2$,
$i = j-3$
$i = j-4$

and generally there're 4 possible values for j:

$j = n \cdot 4 + 0$
$j = n \cdot 4 + 1$
$j = n \cdot 4 + 2$, $n \in \mathbb{Z}^+$
$j = n \cdot 4 + 3$

Let us assemble all the combinations of those in a table, where ~~m(x,y)~~ $m(x, y) = x \bmod 4 + y \bmod 4$:

j	i	~~m x,y)~~ $m(i,j)$	satisfiability
$n \cdot 4+0$	$j-1$	3	0
$n \cdot 4+1$	$j-1$	1	0
$n \cdot 4+2$	$j-1$	3	0
$n \cdot 4+3$	$j-1$	5	0
$n \cdot 4+0$	$j-2$	2	0
$n \cdot 4+1$	$j-2$	4	1
$n \cdot 4+2$	$j-2$	2	0
$n \cdot 4+3$	$j-2$	4	1
$n \cdot 4+0$	$j-3$	1	0
$n \cdot 4+1$	$j-3$	3	0
$n \cdot 4+2$	$j-3$	5	0
$n \cdot 4+3$	$j-3$	3	0
$n \cdot 4+0$	$j-4$	0	1
$n \cdot 4+1$	$j-4$	1	0
$n \cdot 4+2$	$j-4$	2	0
$n \cdot 4+3$	$j-4$	3	0

From the table we get that there're 3 combinations of j and i such that cond 2 is satisfied:

$j = n \cdot 4 + 1 \;\&\; i = j - 2$
$j = n \cdot 4 + 3 \;\&\; i = j - 2$
$j = n \cdot 4 \;\&\; i = j - 4$

node 3.)

Now to count the weights in the MST of G_n for some n, let's find the i (i's) s.t. $w(n,i)$ is minimal.

$w(n,i) = n+i+((n-i)-1)^2$ for $n>i$

for $k = \cancel{k}\, n-l$

$\cancel{w(n,k)}\, w(n, n-l) = n+n-l+(n-n+l-1)^2 = 2n-l+l^2-2l+1 = 2n+l^2-3l+1$

~~#~~ Since l^2-3l grows from $l=3$, l must be as small as possible if it has to be 3 or greater.

For $l=1$:

$w(n,k) = w(n,n-1) = n+n-1+(n-n+1-1)^2 = 2n-1$

For $l=2$:

$w(n,k) = w(n,n-2) = n+n-2+(n-n+2-1)^2 = 2n-2+1 = 2n-1$

$\Rightarrow$ For $l=1$ and $l=2$ $w(n,n-l)$ is the same, means it is the least, since for other l $w(n,n-l)$ is greater than $2n-1$.

From node 1 and node 2 we know that any natural ~~number~~ node n with any $n \bmod 4$ can be connected to $n-1$ or $n-2$ with fixed and minimal cost of $2n-1$ per edge.

Suppose, we know cost of MST for G_{n-1} ($MST(G_{n-1})$), then

$MST(G_n) = MST(G_{n-1}) + 2n-1$

$MST(G_2) = 3$

$MST(G_3) = 3+2\cdot 3-1 = 8$

$MST(G_4) = MST(G_3)+2\cdot 4-1 = 3+2\cdot 3-1+2\cdot 4-1 = 15$

$MST(G_n) = [redacted]\ 3+2\cdot\left(\frac{n+3}{2}\right)(n-2)-1\cdot(n-2) = 3+(n+2)(n-2) =$

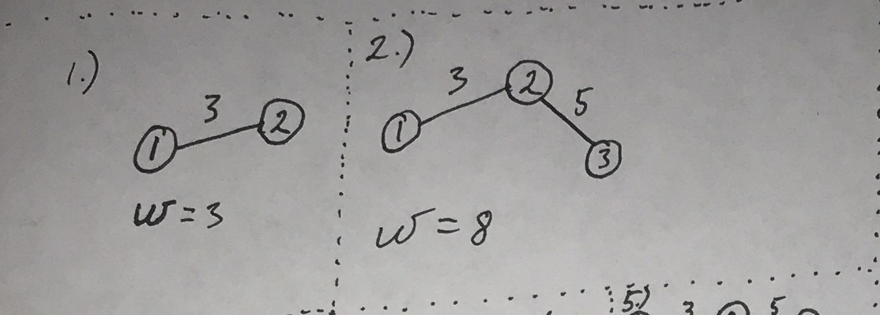
$= n^2-4+3 = n^2-1$

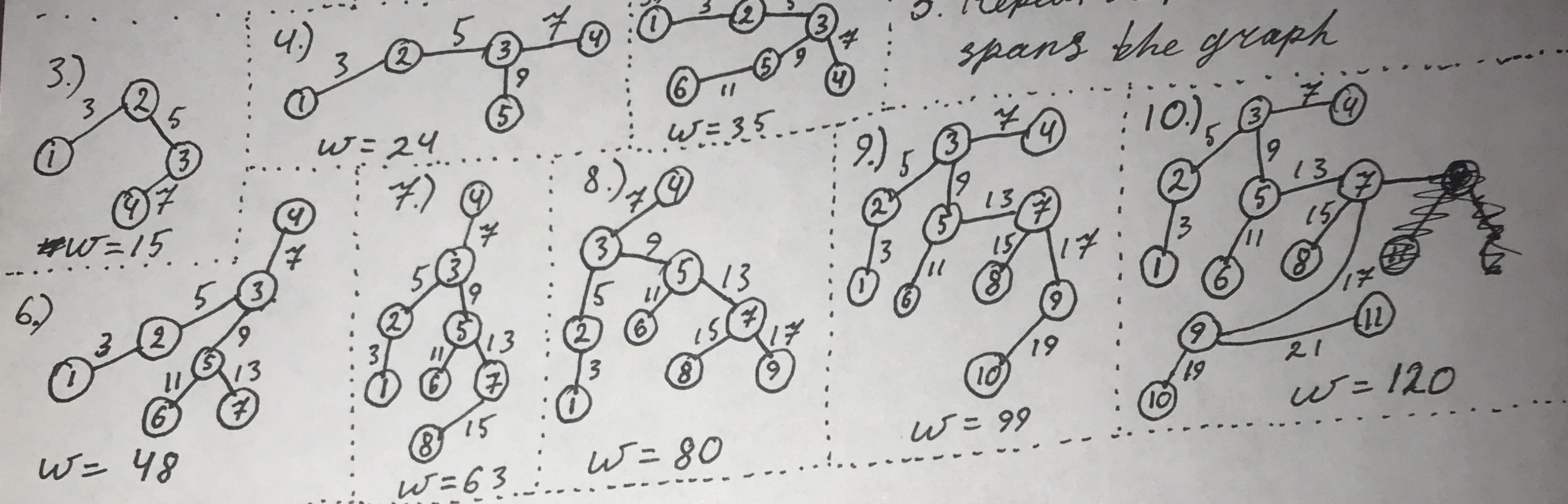
Answer: $\underline{n^2-1}$.

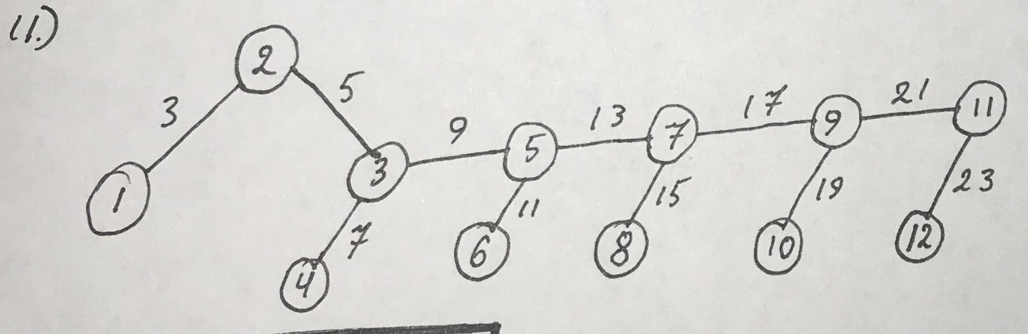
II Verification.

Using Prim's Algorithm for building MST for G_{12}:

1. Choose an arbitrary (any) vertex in the graph.
2. Add a minimum-weight tree and the vertex it leads to, that connects a chosen vertex with the one not yet in the tree.
3. Repeat step 2 untill the tree spans the graph







$\boxed{w = 143}$

From 3.1.1 (I) we got weight ~~equal~~ of the MST of G_n equal $w = n^2 - 1$.

Applying the formula above to the G_{12} we get:

$w = n^2 - 1 = 12^2 - 1 = 144 - 1 = \underline{143}$

Answer verified.

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Task 3.2

Shortest path.

1.) I suggest to ~~to~~ store the information about all the paths from every vertex to any by means of the ~~array of~~ array list of lengths and the 2-dimentional array list of trails: L and M.

```java
ArrayList<ArrayList<Vertex>> M;
ArrayList<Integer> L;
```

2.) All the changes are supposed to be done in the very end of the methods addVertex and addEdge, alike those suggested in Task 2.1, for points 1. and 2. respectively:

```java
for (int i = 0; i < vertices.size(); i++) {
    vertex.M.add(new ArrayList<>());
    vertex.L.add(Integer.MAX_VALUE);
}
vertex.L.set(vertex.idx, 0);
```

```java
for (Vertex v : vertices) {
    if (v.L.get(to.idx) > v.L.get(from.idx) + weight) {
        v.M.set(to.idx, v.M.get(from.idx));
        v.M.get(to.idx).add(to);
    }
}
```

3.) for adding a vertex:

while all the inside operations take constant time (lines 2, 3), they are located in for loop that ~~is~~ ~~ta~~ is linear of V, last line takes constant time as well, and hence overall time complexity is $T(V) = O(V)$.

for adding an edge:

if condition and its insides take const, although they are put into for loop that is linear of V, hence overall time complexity is $T(V) = O(V)$