MFEM Electromagnetics Mini Applications

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1 Electromagnetics

The equations describing electromagnetic phenomena are know collectively as the "Maxwell Equations". They are usually given as:

$$\begin{aligned}
\nabla\times\vec{H} - \frac{\partial\vec{D}}{\partial t} &= \vec{J} &\quad (1)\\
\nabla\times\vec{E} + \frac{\partial\vec{B}}{\partial t} &= 0 &\quad (2)\\
\nabla\cdot\vec{D} &= \rho &\quad (3)\\
\nabla\cdot\vec{B} &= 0 &\quad (4)
\end{aligned}$$

Where equation (1) can be referred to as *Ampère's Law*, equation (2) is called *Faraday's Law*, equation (3) is *Gauss's Law*, and equation (4) doesn't generally have a name but is related to the nonexistence of magnetic monopoles. The various fields in these equations are:

Symbol	Name	SI Units
$\vec{H}$	Magnetic Field	Ampere/meter
$\vec{B}$	Magnetic Flux Density	Tesla
$\vec{E}$	Electric Field	Volts/meter
$\vec{D}$	Electric Displacement	Coulomb/meter^2
$\vec{J}$	Current Density	Ampere/meter^2
ρ	Charge Density	Coulomb/meter^3

In the literature these names do vary, particularly those for $\vec{H}$ and $\vec{B}$, but in this document we will try to adhere to the convention laid out above.

Generally we also need constitutive relations between $\vec{E}$ and $\vec{D}$ and/or between $\vec{H}$ and $\vec{B}$. These relations start with the definitions:

$$\begin{aligned}
\vec{D} &= \epsilon_0\vec{E} + \vec{P} &\quad (5)\\
\vec{B} &= \mu_0\vec{H} + \vec{M} &\quad (6)
\end{aligned}$$

Where $\vec{P}$ is the *polarization density*, and $\vec{M}$ is the *magnetization*. Also, ϵ_0 is the *permittivity of free space* and μ_0 is the *permeability of free space* which are both constants of nature. In many common materials the polarization density can be approximated as a scalar multiple of the electric field i.e. $\vec{P} = \epsilon_0 \chi \vec{E}$, where χ is called the *electric suscepctibility*. In such cases we usually use the relation $\vec{D} = \epsilon \vec{E}$ with $\epsilon = \epsilon_0(1 + \chi)$ and call ϵ the *permittivity* of the material.

The nature of magnetization is more complicated but we will take a very simplified view which is valid in many situations. Specifically, we will assume that either $\vec{M}$ is proportional to $\vec{H}$ yielding the relation $\vec{B} = \mu \vec{H}$ where $\mu = \mu_0(1 + \chi_M)$ and χ_M is the *magnetic susceptibility* or that $\vec{M}$ is a constant. The former case pertains to both diamagnetic and paramagnetic materials and the later to ferromagnetic materials.

Finally we should note that equations (1) and (3) can be combined to yield the equation of charge continuity

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0$$

which can be important in plasma physics and magnetohydrodynamics (MHD).

1.1 Static Fields

1.1.1 Electrostatics

Electrostatic problems come in a variety of subtypes but they all derive from Gauss's Law and Faraday's Law (equations (3) and (2)). When we assume no time variation, Faraday's Law becomes simply $\nabla \times \vec{E} = 0$. This suggests that the electric field can be expressed as the gradient of a scalar field which is traditionally taken to be $-\varphi$, i.e.

$$\vec{E} = -\nabla \varphi \tag{7}$$

where φ is called the *electric potential* and has units of Volts in the SI system. Inserting this definition into equation (3) gives:

$$-\nabla \cdot \epsilon \nabla \varphi = \rho \tag{8}$$

which is *Poisson's equation* for the electric potential. Where, clearly, we have assumed a linear constitutive relation between $\vec{D}$ and $\vec{E}$. If this relation happens to be nonlinear then Poisson's equation would need to be replaced with a more complicated nonlinear expression.

The solutions to equation (8) are non unique because they can be shifted by any additive constant. This means that we must apply a Dirichlet boundary condition at at least one point in the problem domain in order to obtain a solution. Typically this point will be on the boundry but it need not be so. Such a Dirichlet value is equivalent to fixing the voltage (aka potential) at one or more locations. Additionally, this equation admits a normal derivative boundary condition. This means setting $\hat{n} \cdot \vec{D}$ to a prescribed value on some portion of the boundary. This is equivalent to defining a surface charge density on that portion of the boundary.

1.1.2 Magnetostatics

Magnetostatic problems arise when we assume no time variation in Ampère's Law (equation (1)) which leads to:

$$\nabla\times\vec{H} = \vec{J}$$

In this case we'll assume a somewhat more general constitutive relation between $\vec{H}$ and $\vec{B}$:

$$\vec{B} = \mu\vec{H} + \vec{M}$$

Notice that we use μ rather than μ_0. This allows for paramagnetic and/or diamagnetic materials defined through μ as well as ferromagnetic materials represented by $\vec{M}$. This choice yields:

$$\nabla\times\mu^{-1}\vec{B} = \vec{J} + \nabla\times\mu^{-1}\vec{M}$$

Of course a nonlinear constitutive relation is possible and would lead to a rather complicated nonlinear equation for $\vec{B}$.

To reach an equation we can solve in the linear case we need to make use of equation (4) which implies that $\vec{B} = \nabla\times\vec{A}$ for some potential $\vec{A}$ which is called the *magnetic vector potential*. This gives the final form of the equation:

$$\nabla\times\mu^{-1}\nabla\times\vec{A} = \vec{J} + \nabla\times\mu^{-1}\vec{M} \tag{9}$$

Once again the potential is non unique so we must apply Dirichlet boundary conditions in order to arrive at a solution for $\vec{A}$.

more on appropriate boundary conditions...

more on magnetic scalar potential...

1.1.3 Statics Miniapp

Possible sources:

- Charge Density ρ
- Current Density $\vec{J}$
- Magnetization $\vec{M}$
- Surface Charge Density σ
- Surface Current Density $\vec{K}$

Additional Boundary Conditions:

- Voltage at at least one point
- Magnetic Vector Potential on some region (restrictions???)
- Magnetic Scalar Potential at at least one point (we may be able to choose this for the user)