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Optimization Methods **Spring 2024**

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Fourth assignment **Due date:** Monday, 3 June 2024, 11:59 PM

Exercise 1 (20/100)

Consider the quadratic function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as:

$$f(x) = 7x^2 + 4xy + y^2 \tag{1}$$

where $\mathbf{x} = (x, y)^T$.

1. Write this function in canonical form, i.e. $f(x) = \frac{1}{2}x^T Ax - b^T x + c$, where A is a symmetric matrix.

   The function $f(x)$ can be written according to the canonical form definition previously described as follows:

   $$f(x) = \frac{1}{2}\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 14 & 4 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

   It can be easily noticed that the matrix A is symmetric, the vector b is equal to $\begin{bmatrix} 0 & 0 \end{bmatrix}$ because there are not any 1° grade polynomials and the constant c is equal to 0. It is possible to prove that the canonical form is correct for the function $f(x)$ by computing the product of the matrices and the vectors as follows:

   $$\begin{aligned} f(x) &= \frac{1}{2}\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 14 & 4 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + 0 \\ &= \frac{1}{2}(14x^2 + \underline{4xy} + \underline{4xy} + 2y^2) \\ &= 7x^2 + 4xy + y^2 \end{aligned}$$

2. Describe briefly how the Conjugate Gradient (CG) Method works and discuss whether it is suitable to minimize f from equation (1). Explain your reasoning in detail **(max. 30 lines)**.

   The Conjugate Gradient method is an optimization algorithm for solving linear systems of equations in the form $Ax = b$ where A is a symmetric positive definite matrix. The linear system can be interpreted as a minimization problem for a quadratic function with the matrix A s.p.d. which it can be described as :

   $$\min_{x \in \mathbb{R}^n} f(x) = \frac{1}{2}x^T Ax - b^T x$$

The CG method is based on the idea of finding the minimum of the function $f(x)$ by generating in a lighter way a set of vectors that guarantee the conjugacy property with respect to the s.p.d matrix A. The CG method is suitable to minimize the function $f(x)$ from equation (1) because the matrix A is symmetric and positive definite, which are the necessary conditions for the method to work properly.