



Fourth assignment

Due date: Monday, 3 June 2024, 11:59 PM

Exercise 1 (20/100)

Consider the quadratic function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined as:

$$f(x) = 7x^2 + 4xy + y^2 \quad (1)$$

where $\mathbf{x} = (x, y)^T$.

1. Write this function in canonical form, i.e. $f(x) = \frac{1}{2}x^T Ax - b^T x + c$, where A is a symmetric matrix.

The function $f(x)$ can be written according to the canonical form definition previously described as follows:

$$f(x) = \frac{1}{2} \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 14 & 4 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

It can be easily noticed that the matrix A is symmetric, the vector b is equal to $\begin{bmatrix} 0 & 0 \end{bmatrix}$ because there are not any 1° grade polynomials and the constant c is equal to 0. It is possible to prove that the canonical form is correct for the function $f(x)$ by computing the product of the matrices and the vectors as follows:

$$\begin{aligned} f(x) &= \frac{1}{2} \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 14 & 4 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + 0 \\ &= \frac{1}{2} (14x^2 + \underline{4xy} + \underline{4xy} + 2y^2) \\ &= 7x^2 + 4xy + y^2 \end{aligned}$$

2. Describe briefly how the Conjugate Gradient (CG) Method works and discuss whether it is suitable to minimize f from equation (1). Explain your reasoning in detail (**max. 30 lines**).

The Conjugate Gradient method is an optimization algorithm for solving linear systems of equations in the form $Ax = b$ where A is a symmetric positive definite matrix. The linear system can be interpreted as a minimization problem for a quadratic function with the matrix A s.p.d. which it can be described as :

$$\min_{x \in \mathbb{R}^n} f(x) = \frac{1}{2} x^T A x - b^T x \quad (2)$$

The CG method is based on the idea of finding the minimum of the function $f(x)$ by generating a set of vectors that guarantee the conjugacy property with respect to the s.p.d matrix A in a lighter way, this property guarantees the method reaches the convergence in at most n steps. In this particular case, the iterative method is suitable to minimize the function $f(x)$ from equation (1) because previously I proved that the function could be written in the canonical form and it was possible to notice that the one correspond to the exact quadratic form expression as described in the minimization problem (2) in this case, thus, we know the matrix A is symmetric positive definite, which is the necessary conditions for the method to work properly. Note that the CG is used generally for problems in which we have a large sparse matrix due to efficiency in terms of memory usage and computational complexity at each iteration (that requires a matrix-vector multiplication) and its negative point is the fact that it is affected by rounding errors. In this case where the matrix A is very small, it would have been better to use the Gauss elimination or a factorization methods instead of the CG.

Exercise 2

Consider the following constrained minimization problem for $\mathbf{x} = (x, y, z)^T$

$$\begin{aligned} \min f(x) &:= -3x^2 + y^2 + 2z^2 + 2(x + y + z) \\ \text{s.t. } c(\mathbf{x}) &= x^2 + y^2 + z^2 - 1 = 0 \end{aligned} \tag{3}$$

Write down the Lagrangian function and derive the KKT conditions for 3