Università della Svizzera italiana
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Optimization Methods **Spring 2024**

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Fourth assignment

Due date: Monday, 3 June 2024, 11:59 PM

Exercise 1 (20/100)

Consider the quadratic function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as:

$$f(x) = 7x^2 + 4xy + y^2 \tag{1}$$

where $\mathbf{x} = (x, y)^T$.

1. Write this function in canonical form, i.e. $f(x) = \frac{1}{2}x^T Ax - b^T x + c$, where A is a symmetric matrix.

   The function $f(x)$ can be written according to the canonical form definition previously described as follows:

$$f(x) = \frac{1}{2}\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 14 & 4 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

   It can be easily noticed that the matrix A is symmetric, the vector b is equal to $\begin{bmatrix} 0 & 0 \end{bmatrix}$ because there are not any 1° grade polynomials and the constant c is equal to 0. It is possible to prove that the canonical form is correct for the function $f(x)$ by computing the product of the matrices and the vectors as follows:

$$\begin{aligned} f(x) &= \frac{1}{2}\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 14 & 4 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + 0 \\ &= \frac{1}{2}(14x^2 + \underline{4xy} + \underline{4xy} + 2y^2) \\ &= 7x^2 + 4xy + y^2 \end{aligned}$$

2. Describe briefly how the Conjugate Gradient (CG) Method works and discuss whether it is suitable to minimize f from equation (1). Explain your reasoning in detail **(max. 30 lines)**.

   The Conjugate Gradient method is an optimization algorithm for solving linear systems of equations in the form $Ax = b$ where A is a symmetric positive definite matrix. The linear system can be interpreted as a minimization problem for a quadratic function with the matrix A s.p.d. which it can be described as :

$$\min_{x \in \mathbb{R}^n} f(x) = \frac{1}{2}x^T Ax - b^T x \tag{2}$$

The CG method is based on the idea of finding the minimum of the function $f(x)$ by generating a set of vectors that guarantee the conjugacy property with respect to the s.p.d matrix A in a lighter way, this property guarantees the method reaches the convergence in at most n steps. In this particular case, the iterative method is suitable to minimize the function $f(x)$ from equation (1) because previously I proved that the function could be written in the canonical form and it was possible to notice that the one correspond to the exact quadratic form expression as described in the minimization problem (2) in this case, thus,we know the matrix A is symmetric positive definite, which is the necessary conditions for the method to work properly. Note that the CG is used generally for problems in which we have a large sparse matrix due to efficiency in terms of memory usage and computational complexity at each iteration (that requires a matrix-vector multiplication) and its negative point is the fact that it is affected by rounding errors.In this case where the matrix A is very small, it would have been better to use the Gauss elimination or a factorization methods instead of the CG.

Exercise 2

Consider the following constrained minimization problem for $\mathbf{x} = (x, y, z)^T$

$$\begin{aligned} &\min f(x) := -3x^2 + y^2 + 2z^2 + 2(x + y + z) \\ &\text{s.t. } c(\mathbf{x}) = x^2 + y^2 + z^2 - 1 = 0 \end{aligned} \tag{3}$$

Write down the Lagrangian function and derive the KKT conditions for 3

To begin with, the Lagrangian function is a mathematical tool used to solve minimization or maximization optimization problems in which the objective function is subjected to a set of constraints that describe which is the feasible area of solutions. The Lagrangian function is defined as follows:

$$\mathcal{L}(x, \lambda) = f(x) - \sum_{i=1}^{n} \lambda_i c_i(x)$$

where $x \in \mathbb{R}^n$, $f(x)$ is the objective function, $c_i(x)$ are the constraints and λ_i are the Lagrange multipliers which are scalar values used as coefficients in the linear combination of the constraint gradients which allow to satisfy those conditions while we optimize the objective function. They are required to guarantee that the given solution is both a stationary point of the initial function and stationary point for the Lagrangian function.
As it is possible to notice, the optimization problem is subjected to only one equality constraint in this problem, thus, it can be possible to semplify the Lagrangian function (3) as follow:

$$\begin{aligned} \mathcal{L}(x, \lambda) &= f(x) - \lambda c(x) \qquad (4) \\ \nabla_x \mathcal{L}(x, \lambda) &= \nabla f(x) - \lambda \nabla c(x) \end{aligned}$$

As we proceed, it becomes important to also introduce the *Karush-Kuhn-Tucker conditions*: those requisites, also known as KKT conditions, are a set of first-order necessary conditions used to define if a solution is optimal for a constrained optimization problem. The KKT conditions are a generalization of the Lagrange multipliers method to equality and inequality constraints.
The theorem behind those necessary conditions, as introduced at the section ***"Theorem 12.1 (First-Order Necessary Conditions)"*** *page 321 of "Jorge Nocedal and Stephen J Wright. Numerical optimization. Springer, 1999"(1)* describes that:

Suppose that x^* is a local solution of (12.1), that the functions f and c_i in (12.1) are continuously differentiable, and that the LICQ holds at x^*. Then there is a Lagrange multiplier vector λ^*, *with components* λ_i^*, $i \in \mathcal{E} \cup \mathcal{I}$, *such that the following conditions are satisfied at* (x^*, λ^*)

$$\begin{aligned} \nabla_x \mathcal{L}(x^*, \lambda^*) &= 0, \\ c_i(x^*) &= 0, \quad \text{for all } i \in \mathcal{E}, \\ c_i(x^*) &\geq 0, \quad \text{for all } i \in \mathcal{I}, \\ \lambda_i^* &\geq 0, \quad \text{for all } i \in \mathcal{I}, \\ \lambda_i^* c_i(x^*) &= 0, \quad \text{for all } i \in \mathcal{E} \cup \mathcal{I}. \end{aligned}$$

where $\mathcal{E}$ and $\mathcal{I}$ are the sets of equality and inequality constraints, respectively, and the LICQ refers to the *Linear Independence Constraint Qualification property* that must be verified by the active constraint gradients at the point x^*. This attribute is important to guarantee that the point x^* is a local solution for the problem by verifying that the gradients of constraint are not linearly dependent between one another at that specific point; a deeper explanation of the LICQ property can be found in the section **"Definition 12.4 (LICQ)"** *page 320 of "Jorge Nocedal and Stephen J Wright. Numerical optimization. Springer, 1999"*(1).

In the case of the problem 3, the KKT conditions can be summarized in the following conditions :

$$\begin{aligned} \nabla_x \mathcal{L}(x^*, \lambda^*) &= 0, \\ c_0(x^*) &= 0, \quad \text{for all } i \in \mathcal{E}, \\ \lambda_0^* c_0(x^*) &= 0, \quad \text{for all } i \in \mathcal{E} \cup \mathcal{I} \end{aligned}$$

where $\mathcal{I} = \emptyset$, $\mathcal{E} = \{0\}$ in which $i = 0$ correspond to the unique equality constraint in the problem.

References

1. Jorge Nocedal and Stephen J Wright. Numerical optimization. Springer, 1999
2. "Notes on the optimization" available on iCorsi platform