Seminar 1

1.1

	$+$	$-$	$/$	$*$
$\mathbb{N}$	Yes	No	No	Yes
$\mathbb{Z}$	Yes	Yes	No	Yes
$\mathbb{Q}$	Yes	Yes	No*	Yes
$\mathbb{R}$	Yes	Yes	No*	Yes
$\mathbb{C}$	Yes	Yes	No*	Yes

* / nu e definită pe tot $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ (pt că nu se poate împărți la 0)

1.4 $x*y = x+y+xy$. Prove that:

i) $(\mathbb{R}, *)$ is a commutative monoid

1. "$*$" well defined

$$\left.\begin{array}{l}\left.\begin{array}{l} x\in\mathbb{R} \\ y\in\mathbb{R}\end{array}\right| \Rightarrow x\cdot y\in\mathbb{R} \\ \left.\begin{array}{l} x\in\mathbb{R} \\ y\in\mathbb{R}\end{array}\right| \Rightarrow x+y\in\mathbb{R}\end{array}\right| \Rightarrow x*y\in\mathbb{R} \Leftrightarrow \text{"}*\text{" - well defined}$$

2. associativity

$(\forall)\, x,y,z\in\mathbb{R}$ s.t. $(x*y)*z = x*(y*z)$

$$\left.\begin{array}{l}(x*y)*z = (x+y+xy)*z = x+y+xy+z+xz+yz+xyz \\ \qquad = x+y+z+xy+xz+yz+xyz \\ x*(y*z) = x*(y+z+yz) = x+y+z+yz+xy+xz+xyz \\ \qquad = x+y+z+xy+xz+yz+xyz\end{array}\right| \Rightarrow \text{"}*\text{" is associative}$$

3. neutral elem

$\exists!\, e\in\mathbb{R}$ st $x*e = e*x = x$, $(\forall)\, x\in\mathbb{R}$

$$\left.\begin{array}{l} x*e = \cancel{x}+e+xe = \cancel{x} \Rightarrow e(x+1)=0 \Rightarrow e=0\in\mathbb{R} \\ \qquad \Rightarrow x=-1,\ (\forall)\, x\in\mathbb{R},\ \text{false!}\end{array}\right| \Rightarrow e=0$$

4. commutativity

.....

from (1),(2),(3),(4) $\Rightarrow (\mathbb{R}, *)$ - commutative monoid

ii) $[-1,+\infty)$ is a stable subset of $(\mathbb{R}, *)$

$(\forall)\, x,y\in[-1,+\infty) \Rightarrow x*y\in[-1,+\infty)$

$x*y = x+y+xy = x(y+1)+y+1-1 = (x+1)(y+1)-1$

$x \in [-1,+\infty) \Rightarrow x \geq -1 \,|+1 \Rightarrow x+1 \geq 0$

$y \in [-1,+\infty) \Rightarrow y \geq -1 \,|+1 \Rightarrow y+1 \geq 0$.

$(x+1)(y+1) \geq 0 \,|-1 \Rightarrow (x+1)(y+1)-1 \geq -1 \Rightarrow x*y \geq -1 \Rightarrow x*y \in [-1,+\infty) \Rightarrow$ concl.

1.5 $(\mathbb{N},*)$, $x*y = g.c.d\,(x,y)$. Prove that:

i) $(\mathbb{N},*)$ – commutative monoid

1. well defined

Let $x = n\cdot k$
$y = m\cdot k$
$x,y \in \mathbb{N}$
$\Big| \Rightarrow m,n,k \in \mathbb{N}$

$x*y = \gcd(n\cdot k, m\cdot k) = k \in \mathbb{N} \Rightarrow x*y \in \mathbb{N} \Rightarrow$ "$*$" well defined

2. associativity

Let $x = n\cdot k$
$y = m\cdot k$, $m,n,l,k \in \mathbb{Z}$
$z = l\cdot k$

"$*$" – associative $\Leftrightarrow (\forall)\, x,y,z \in \mathbb{N}$ s.t. $(x*y)*z = x*(y*z)$

$(x*y)*z = [\gcd(x,y)]*z = k*z = \gcd(k,z) = k$
$x*(y*z) = x*[\gcd(y,z)] = x*k = \gcd(x,k) = k$
$\Big| \Rightarrow$ "$*$" is associative

3. neutral element

$\exists!\, e \in \mathbb{N}$ s.t. $x*e = e*x = x, (\forall)\, x \in \mathbb{N}$

$x*e = x \Leftrightarrow \gcd(x,e) = x$
we know that $0 \,|\, n, (\forall)\, n \in \mathbb{N}$ and $\gcd(0,n) = n, (\forall)\, n \in \mathbb{N}$
$\Big| \Rightarrow e = 0 \in \mathbb{N}$

4. commutativity

....

from (1),(2),(3),(4) $\Rightarrow (\mathbb{N},*)$ – commutative monoid

ii) $D_n = \{x \in \mathbb{N} \,/\, x|n\}$ – stable subset of $(\mathbb{N},*)$ + $(D_n,*)$ – commutative monoid

$(\forall)\, x,y \in D_n \Rightarrow x*y \in D_n$

$x \in D_n \Leftrightarrow x|n$

$y \in D_n \Leftrightarrow y|n$

Let $x*y = d \Rightarrow \gcd(x,y) = d \Rightarrow \begin{cases} d|x \\ d|y \end{cases}$

We know that $d|x$, $x|n$ $\Big| \Rightarrow d|n$, $\gcd(x,y) = d$ $\Big| \Rightarrow \gcd(x,y) | n \Rightarrow \gcd(x,y) \in D_n \Rightarrow x*y \in D_n \Rightarrow$ concl

$(D_n,*)$ – comm. monoid:

1. well defined

– same ca mai sus cu stable subset

2. associativity

$(\forall)\, x,y,z \in D_n$ s.t. $(x*y)*z = x*(y*z)$

$D_n \subseteq \mathbb{N}$
$(\mathbb{N},*)$ – associative
$\Big| \Rightarrow (D_n,*)$ – associative

3. neutral element

$x|n \Rightarrow$ we can take $x = n*k$, $(\forall)\, k \in \mathbb{N}^*$

$\gcd(n\cdot k, e) = n\cdot k \Rightarrow e = n \in D_n$

iii) Fill in the table operation "$*$" on D_6

$D_6 = \{ x \in \mathbb{N} \mid x \mid 6 \} \Rightarrow x = \{1, 2, 3, 6\}$

$*$	1	2	3	6
1	1	1	1	1
2	1	2	1	2
3	1	1	3	3
6	1	2	3	6

1.6 all the finite stable subsets of $(\mathbb{Z}, \cdot)$

$\rightarrow \{0\}, \{1\}, \{0, 1\}, \{-1, 0, 1\}, \{-1, 1\}$

1.7 $(G, \cdot)$ - group. Show that:

i) G is abelian $\Leftrightarrow (\forall) x, y \in G, (xy)^2 = x^2 \cdot y^2$

G is abelian group $\Rightarrow$ "$\cdot$" is commutative $\Rightarrow x\cdot y = y \cdot x, (\forall)\ x, y \in G$

$x^2 \cdot y^2 = (x \cdot x) \cdot (y \cdot y) = x \cdot y \cdot x \cdot y = (x \cdot y) \cdot (x \cdot y) = (x \cdot y)^2$

$(xy)^2 = (x \cdot y) \cdot (x \cdot y) = x^2 \cdot y^2$

$(xy \cdot x) \cdot y = (xxy) \cdot y \mid \cdot y^{-1} \Rightarrow xyx = xxy \Rightarrow x(yx) = x(xy)$ $\Rightarrow x = x, T$

we know that $xy = yx$

ii) If $x^2 = 1, (\forall) x \in G$, then G - abelian

$(\forall)\ x, y \in G, (xy)^2 = 1$

but also $x^2 = y^2 = 1$

$\Rightarrow (xy)^2 = x^2 y^2 \overset{i)}{\Longrightarrow} G$ - abelian

1.8 "$\cdot$" - operation on A, $X, Y \subseteq A$, "$*$" op. on $\mathcal{P}(A)$

$X * Y = \{ x \cdot y \mid x \in X, y \in Y \}$

Prove that:

i) If $(A, \cdot)$ is a monoid, then $(\mathcal{P}(A), *)$ - monoid

1. "$*$" well defined on $\mathcal{P}(A)$