Algebra-Seminar 8

25.11.2022

S5-[redacted] 3

$\langle a,b\rangle = \langle c,d,e\rangle$

$$\left.\begin{array}{l} a \in \langle c,d,e\rangle \\ b \in \langle c,d,e\rangle \end{array}\right\} \Rightarrow \underbrace{\langle a,b\rangle}_{\dim} \subseteq \underbrace{\langle c,d,e\rangle}_{\dim} \Rightarrow \langle a,b\rangle = \langle c,d,e\rangle$$

if both dimensions are equal

S5-8

$\mathbb{Z}_2^3$ $\quad K = \mathbb{Z}_2 = \{0,1\}$

$v_1 \quad v_2 \quad v_3$

$2^3-1 \quad 2^3-2 \quad 2^3-4$

$7 \cdot 6 \cdot 4$

S4-6 p prim

$\exists\ {}_{\mathbb{Z}_p}\mathbb{Z}$?

Suppose we can. Let $x \in \mathbb{Z}$, $x \neq 0$

$$\underbrace{\hat{0} \cdot x = \hat{p} \cdot x}_{\text{(the same mod } x)} = \underbrace{(\hat{1}+\hat{1}+\ldots+\hat{1})}_{p \text{ times}} \cdot x = \hat{1}\cdot x + \hat{1}\cdot x + \ldots + \hat{1}\cdot x = p \cdot x$$

impossible

S4-8 (i) $[-1,1] \leq {}_{\mathbb{R}}\mathbb{R}$?

$1 \in [-1,1]$

$2 \cdot 1 \notin [-1,1]$

NO

(iv) $C^0 = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ cont}\}$ (continuitatea nu influențează $+, \cdot$)

$f, g \in C^0$, $\forall \alpha, \beta \in \mathbb{R}$

$\forall x_0 \in \mathbb{R}$; $\lim_{x\to x_0}(\alpha f(x) + \beta f(x)) = \alpha \lim_{x\to x_0} f(x) +$

$+ \beta \lim_{x\to x_0}$

. $GL_n \to$ matr. inversabile

$SL_n \to$ matr. $\det = 1$

rang M. = câți vect. sunt liniar independenți

~~$f = f^+$~~

$$f(x) = \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2}$$

S5-(4) $\forall v \in \mathbb{R}^3 - S(v_S + v_{\mathcal{I}})$

(5) Solve the following linear systems by the Gauss and Gauss-Jordan methods:

(ii) $\begin{cases} 2x+5y+z=7 \\ x+2y-z=3 \\ x+y-4z=2 \end{cases}$ $\left(\begin{array}{ccc|c} 2 & 5 & 1 & 7 \\ 1 & 2 & -1 & 3 \\ 1 & 1 & -4 & 2 \end{array}\right) \overset{L_1 \leftrightarrow L_2}{\sim} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & 5 & 1 & 7 \\ 1 & 1 & 4 & 2 \end{array}\right)$

$\underset{L_3 \leftarrow L_3 - L_1}{\overset{L_2 \leftarrow L_2 - L_1}{\sim}} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & -1 & -3 & -1 \end{array}\right) \overset{L_3 \leftarrow L_3 + L_2}{\sim} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$

Gauss method: Revert to the system and solve it manually

$\begin{cases} x+2y-z=3 \\ y+3z=1 \end{cases} \Leftrightarrow \begin{cases} y = 1-3z \\ x = 3+z-2+6z \end{cases} \Leftrightarrow \begin{cases} x = 1+7\alpha \\ y = 1-3\alpha \\ z = \alpha \end{cases}$

(iii) $\begin{cases} x+y+z=3 \\ x-y+z=1 \\ 2x-y+2z=3 \\ x+z=4 \end{cases}$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 3 \\ 2 & -1 & 2 & 3 \\ 1 & 0 & 1 & 4 \end{array}\right) \underset{\substack{L_3 \leftarrow L_3-2L_1 \\ L_4 \leftarrow L_4-L_1}}{\overset{L_2 \leftarrow L_2-L_1}{\sim}} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & -2 & 0 & 2 \\ 0 & -3 & 0 & 1 \\ 0 & -1 & 0 & 3 \end{array}\right)$$

$$\overset{L_2 \leftrightarrow L_3}{\sim} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 1 & 0 & 3 \end{array}\right) \underset{L_4 \leftarrow L_4-L_2}{\overset{L_3 \leftarrow L_3-2L_2}{\sim}} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 1 & 0 & 3 \end{array}\right) \underset{L_4 \leftarrow L_4-L_2}{\overset{L_3 \leftarrow L_3-2L_2}{\sim}}$$

$$\begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \Rightarrow \text{system is incompatible}$$

(4) $\begin{cases} ax+y+z=1 \\ x+ay+z=a \\ x+y+az=a^2 \end{cases}, \ a \in \mathbb{R}$

$$\left(\begin{array}{ccc|c} a & 1 & 1 & 1 \\ 1 & a & 1 & a \\ 1 & 1 & a & a^2 \end{array}\right) \underset{L_3 \leftarrow L_3-L_2}{\overset{L_1 \leftarrow L_1-aL_2}{\sim}} \left(\begin{array}{ccc|c} 0 & 1-a^2 & 1-a & 1-a^2 \\ 1 & a & 1 & a \\ 0 & 1-a & a-1 & a^2-a \end{array}\right)$$

Gauss-Jordan Method

- Eliminate the zeros above the old pivots

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \overset{L_1 \leftarrow L_1 - 2L_2}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & -7 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x - 7z = 1 \\ y + 3z = 1 \end{cases} \Rightarrow \begin{cases} x = 1 + 7\alpha \\ y = 1 + 3\alpha \\ z = \alpha \end{cases}$$

8 (i) $\begin{cases} 2x + 2y + 3z = 3 \\ x - y = 1 \\ -x + 2y + z = 2 \end{cases}$

$$\left(\begin{array}{ccc|c} 2 & 2 & 3 & 3 \\ 1 & -1 & 0 & 1 \\ -1 & 2 & 1 & 2 \end{array}\right) \overset{L_1 \leftrightarrow L_2}{\sim} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 2 & 2 & 3 & 3 \\ -1 & 2 & 1 & 2 \end{array}\right) \overset{L_2 \leftarrow L_2 - 2L_1}{\sim} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 4 & 3 & 1 \\ 0 & 1 & 1 & 3 \end{array}\right)$$

$$\overset{L_2 \leftrightarrow L_3}{\sim} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 4 & 3 & 1 \end{array}\right) \overset{L_3 \leftarrow L_3 - 4L_2}{\sim} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -1 & -11 \end{array}\right)$$

Gauss :

$$\begin{cases} x - y = 1 \\ y + z = 3 \\ -z = -11 \end{cases} \Leftrightarrow \begin{cases} x = -7 \\ y = -8 \\ z = 11 \end{cases}$$

Algebra- Seminar 9

09.12.2022

(9.3) Compute the rank of the matrix by applying elementary operations. (Gauss) (No operations on columns!)

$$\begin{pmatrix} \beta & 1 & 3 & 4 \\ 1 & \alpha & 3 & 3 \\ 2 & 3\alpha & 4 & 7 \end{pmatrix}, \alpha, \beta \in \mathbb{R}$$

$$\begin{pmatrix} \beta & 1 & 3 & 4 \\ 1 & \alpha & 3 & 3 \\ 2 & \alpha & 4 & 7 \end{pmatrix} \underset{\cancel{L_3 \leftarrow L_3 -}}{\overset{L_1 \leftrightarrow L_2}{=\!=}} \begin{pmatrix} 1 & \alpha & 3 & 3 \\ \beta & 1 & 3 & 4 \\ 2 & \alpha & 4 & 7 \end{pmatrix} \underset{L_3 \leftarrow L_3 - 2L_1}{\overset{L_1,\ L_2 \leftarrow L_2 - \beta L_1}{=\!=}}$$

$$\cancel{\begin{pmatrix} 1 & \alpha & 3 & 3 \\ \beta-1 & 1-\alpha & 0 & 1 \\ 1 & 0 & 1 & 4 \end{pmatrix}} = \begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & 1-\beta\alpha & 3-3\beta & 4-3\beta \\ 1 & 0 & 1 & 4 \end{pmatrix} \underset{\cancel{L_3 \leftarrow L_3 - L_1}}{\overset{L_1,\ L_2}{=\!=}}$$

~~1 α 3 3~~

$$\begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & 1-\beta\alpha & 3-3\beta & 4-3\beta \\ 0 & -\alpha & -2 & 1 \end{pmatrix} \underset{L_3}{\overset{L_1,\ \cancel{L_2 \leftarrow L_2 + \beta L_1}}{=\!=}}$$

$$\overset{L_2 \leftrightarrow L_3}{=\!=} \begin{pmatrix} 1 & \alpha & 3 & 3 \\ 1 & 0 & 1 & 4 \\ 0 & 1-\beta\alpha & 3-3\beta & 4-3\beta \end{pmatrix} \underset{L_3 \leftarrow L_3 - \alpha L_2}{\overset{L_1,\ L_2}{=\!=}} \begin{pmatrix} 1 & \alpha & 3 & 3 \\ 1 & 0 & 1 & 4 \\ -\alpha & 1-\beta\alpha & 3-3\beta-\alpha & 4-3\beta-4\alpha \end{pmatrix}$$

$$\underset{L_1 \leftrightarrow L_2}{\sim}\begin{pmatrix} 1 & \alpha & 3 & 3 \\ \beta & 1 & 3 & 4 \\ 2 & 3\alpha & 4 & 7 \end{pmatrix} \underset{L_3 \leftarrow 2L_1}{\overset{L_2 \leftarrow L_2-\beta L_1}{\sim}} \begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & 1-\alpha\beta & 3-3\beta & 4-3\beta \\ 0 & \alpha & -2 & 1 \end{pmatrix} \overset{L_2 \leftarrow L_2 +}{\sim}$$

$$\begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & 1 & 3-5\beta & 4-2\beta \\ 0 & \alpha & -2 & 1 \end{pmatrix} \overset{L_3 \leftarrow L_3-\alpha L_2}{\sim} \begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & 1 & 3-5\beta & 4-2\beta \\ 0 & 0 & -3+5\beta\underset{-2}{} & 4\alpha-2\beta\alpha+1 \end{pmatrix}$$

$$\begin{cases} -3\alpha+5\overset{=\gamma}{\widetilde{\beta\alpha}}-2=0 \\ -4\alpha+2\beta\alpha+1=0 \end{cases} \Leftrightarrow \begin{cases} -3\alpha+5\gamma-2=0 \;\Big|\; \begin{matrix}-5\gamma \\ +2\end{matrix} \\ -4\alpha+2\gamma+1=0 \end{cases}$$

$$\begin{cases} \alpha = \dfrac{-5\gamma+2}{-3} \\ -4\left(\dfrac{-5\gamma+2}{-3}\right)+2\gamma+1=0 \Leftrightarrow 20\gamma-\overset{5}{\cancel{8}}-6\gamma-3=0 \end{cases}$$

$$14\gamma=11 \Rightarrow \boxed{\gamma=\frac{11}{14}}$$

$$\Rightarrow \alpha = \frac{25\cdot\frac{11}{14}+2}{-3} \Rightarrow \alpha = \frac{-55+28}{-3\cdot 14} \Rightarrow \boxed{\alpha = \frac{-27}{-3\cdot 14} = \frac{9}{14}} \Rightarrow \beta = \frac{11}{14}\cdot\frac{14}{9}$$

$$\boxed{\beta = \frac{11}{9}}$$

$$\Rightarrow \operatorname{rank} M = \begin{cases} 2, & \alpha=\frac{9}{14},\ \beta=\frac{11}{9} \\ 3, & \text{otherwise} \end{cases}$$

9.9) $S = \langle (1,0,4), (2,1,0), (1,1,-4) \rangle$

$T = \langle (-3,-2,4), (5,2,4), (-2,0,-8) \rangle$

Find a basis for each of $S, T, S+T$ and find $\dim S$, $\dim T$, $\dim(S+T)$, $\dim(S \cap T)$

$$\begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 0 \\ 1 & 1 & -4 \end{pmatrix} \overset{\substack{L_1 \leftarrow L_1 + L_3 \\ \cancel{L_3 \leftarrow L_3}}}{\sim} \begin{pmatrix} 2 & 1 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & -4 \end{pmatrix} \overset{L_1 \leftarrow L_2 - L_1}{\sim} \begin{pmatrix} 0 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & -4 \end{pmatrix}$$

$$\begin{pmatrix} -3 & -2 & 4 \\ 5 & 2 & 4 \\ -2 & 0 & -8 \end{pmatrix} \overset{L_1 \leftarrow L_1 + L_2}{\sim} \begin{pmatrix} 2 & 0 & 8 \\ 5 & 2 & 4 \\ -2 & 0 & -8 \end{pmatrix} \overset{L_1 \leftarrow L_1 + L_3}{\sim} \begin{pmatrix} 0 & 0 & 0 \\ 5 & 2 & 4 \\ -2 & 0 & 8 \end{pmatrix}$$

1) $\dim S = ?$

$$\begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 0 \\ 1 & 1 & -4 \end{pmatrix} \overset{\substack{L_2 \leftarrow L_2 - 2L_1 \\ L_3 \leftarrow L_3 - L_1}}{\sim} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 1 & -8 \end{pmatrix} \overset{L_3 \leftarrow L_3 - L_2}{\sim} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ rank $= 2 \Rightarrow \dim S = 2$

$B = ((1,0,4), (0,1,-8))$ – basis

2) $\dim T = ?$

$$\begin{pmatrix} -3 & -2 & 4 \\ 5 & 2 & 4 \\ -2 & 0 & -8 \end{pmatrix} \overset{L_1 \leftrightarrow L_3}{\sim} \begin{pmatrix} -2 & 0 & -8 \\ 5 & 2 & 4 \\ -3 & -2 & 4 \end{pmatrix} \overset{L_1 \leftarrow \frac{1}{2} L_1}{\sim} \begin{pmatrix} 1 & 0 & 4 \\ 5 & 2 & 4 \\ -3 & -2 & 4 \end{pmatrix}$$

$$\overset{\substack{L_2 \leftarrow L_2 - 5L_1 \\ L_3 \leftarrow L_3 + 3L_1}}{\sim} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 2 & -16 \\ 0 & -2 & 16 \end{pmatrix} \overset{L_3 \leftarrow L_3 + L_2}{\sim} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 2 & -16 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{rank} = 2 \Rightarrow$$

$\Rightarrow \dim T = 2$

matrix (using Gauss elimination)

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

pivot

$$\left(\begin{array}{ccc|ccc} \boxed{1} & 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & -2 & 0 & 1 & 0 \\ 2 & -2 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[L_2 \leftarrow L_2-2L_1]{L_3 \leftarrow L_3-2L_1} \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -3 & -6 & -2 & 1 & 0 \\ 0 & -6 & -3 & -2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{L_3 \leftarrow L_3-2L_2} \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -3 & -6 & -2 & 1 & 0 \\ 0 & 0 & 9 & 2 & -2 & 1 \end{array}\right) \xrightarrow{L_2 \leftarrow L_2 : (-3)} \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 9 & 2 & -2 & 1 \end{array}\right)$$

$$\cancel{L_3 \leftarrow L_3+6L_2} \quad \cancel{\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & \frac{2}{3} & -\frac{1}{3} & 0 \end{array}}$$

$$\xrightarrow{L_3 \leftarrow L_3 : 9} \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & \frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \end{array}\right)$$

$$\xrightarrow[L_1 \leftarrow L_1-2L_3]{L_2 \leftarrow L_2-2L_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{5}{9} & \frac{4}{9} & -\frac{2}{9} \\ 0 & 1 & 0 & \frac{2}{9} & \frac{1}{9} & -\frac{2}{9} \\ 0 & 0 & 1 & \frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \end{array}\right) \xrightarrow{L_1 \leftarrow L_1-2L_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{9} & \frac{2}{9} & \frac{2}{9} \\ 0 & 1 & 0 & \frac{2}{9} & \frac{1}{9} & -\frac{2}{9} \\ 0 & 0 & 1 & \frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \end{array}\right)$$

$$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix} \cdot \frac{1}{9} \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{= I_3} \checkmark$$

Seminar 10

Algebra

16.12.2022

Def: V, V' K-V.S.

$f: V \to V'$ linear map

$B = (v_1, v_2, \dots, v_n)$ - basis of V

$B' = (v_1', v_2', \dots, v_m')$ - basis of V'

$[f]_{B,B'} = \left([f(v_1)]_{B'} \; [f(v_2)]_{B'} \dots [f(v_n)]_{B'} \right)$

$\in M_{m,n}(K)$

- $\forall v \in V : [f(v)]_{B'} = [f]_{B,B'} \cdot [v]$

10.2 $f \in Hom_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^2)$ defined by $f(x,y,z) = (y, -x)$

$B = (v_1, v_2, v_3) = ((1,1,0), (0,1,1), (1,0,1))$

$B' = (v_1', v_2') = ((1,1), (1,-2))$

$E' = (e_1', e_2') = ((1,0), (0,1))$

Find $[f]_{BE'}$ and $[f]_{BB'}$.

$[f]_{BE'} = ([f(v_1)]_{E'} \; [f(v_2)]_{E'} \; [f(v_3)]_{E'})$

$v_1 = (1,1,0) \Rightarrow f(1,1,0) = (1,-1) = e_1' - e_2' \Rightarrow \alpha = 1, \beta = -1$

$v_2 = (0,1,1) \Rightarrow f(0,1,1) = (1,0) = e_1' \Rightarrow \alpha = 1, \beta = 0$

$v_3 = (1,0,1) \Rightarrow f(1,0,1) = (0,-1) = -e_2' \Rightarrow \alpha = 0, \beta = -1$

$\cancel{v_1' = (1,1) \Rightarrow} f(1,1) = (1,-1)$

$\cancel{v_2' = (1,-2) \Rightarrow} f(1,-2) = (-2,$

$[f]_{BE'} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$

$[f]_{BB'} = [f(v_1)]_{B'} \; [f(v_2)]_{B'} \; [f(v_3)]_{B'}$

$f(v_1) = (1,-1) = \alpha v_1' + \beta v_2' = \alpha(1,1) + \beta(1,-2) = (\alpha, \alpha) + (\beta, -2\beta) = (\alpha+\beta, \alpha - 2\beta) \Rightarrow$

$\Rightarrow \begin{cases} \alpha + \beta = 1 \\ \alpha - 2\beta = -1 \end{cases} \overset{\ominus}{\Rightarrow} 3\beta = 2 \Rightarrow \boxed{\beta = \frac{2}{3}} \Rightarrow \boxed{\alpha = \frac{1}{3}}$

$f(v_2) = (1,0) = \alpha v_1' + \beta v_2' = \alpha(1,1) + \beta(1,-2) = (\alpha+\beta, \alpha-2\beta) \Rightarrow$

$\Rightarrow \begin{cases} \alpha+\beta = 1 \\ \alpha-2\beta = 0 \end{cases} \overset{\ominus}{\Rightarrow} 3\beta = 1 \Rightarrow \boxed{\beta = \frac{1}{3}} \Rightarrow \boxed{\alpha = \frac{2}{3}}$

$f(v_3) = (0,-1) = \alpha v_1' + \beta v_2' = \alpha(1,1) + \beta(1,-2) = (\alpha+\beta, \alpha-2\beta) \Rightarrow$

$\Rightarrow \begin{cases} \alpha+\beta = 0 \\ \alpha-2\beta = -1 \end{cases} \overset{\ominus}{\Rightarrow} 3\beta = 1 \Rightarrow \boxed{\beta = \frac{1}{3}} \Rightarrow \boxed{\alpha = -\frac{1}{3}}$

$$[f]_{BB'} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

(4) $f \in \mathrm{End}_{\mathbb{R}}(\mathbb{R}^4)$

$$[f]_E = \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix}$$

(i) $v = (1,4,1,-1) \in \mathrm{Ker} f$, $v' = (2,-2,4,2) \in \mathrm{Im} f$

(ii) basis =?, ~~dim~~ $\dim \mathrm{Ker} f$, $\dim \mathrm{Im} f$

(iii) Define f

(i) $v \in V$: $[f(v)]_{B'} = [f]_{BB'}[v]_B$

$\mathrm{Ker} f = \{v \in \mathbb{R}^4 \mid f(v) = 0\} = \{v \in \mathbb{R}^4 \mid [f(v)] = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\}$

$= \{v \in \mathbb{R}^4 \mid [f]_E \cdot [v]_E = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\}$

$$\begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow v \in \mathrm{Ker} f$

$$\begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} -8 \\ 8 \\ -16 \\ -8 \end{pmatrix}$$

$\cdot\ v' \in \mathrm{Im} f \Leftrightarrow \exists\, v \in \mathbb{R}^4$ s.t. $f(u) = v' \Leftrightarrow \exists\, \overset{=(x,y,z,t)}{v} \in \mathbb{R}^4$ s.t. $[f]_E [v]_E = [v']_E$

$$\Leftrightarrow \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 4 \\ 2 \end{pmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x+y-3z+2t = 2 \\ -x+y+z+4t = -2 \\ 2x+y-5z+t = 4 \\ x+2y-4z+5t = 2 \end{cases}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 2 \\ -1 & 1 & 1 & 4 & -2 \\ 2 & 1 & -5 & 1 & 4 \\ 1 & 2 & -4 & 5 & 2 \end{array}\right) \underset{\substack{L_3 \leftarrow L_3 - 2L_1 \\ L_4 \leftarrow L_4 - L_1}}{\overset{L_2 \leftarrow L_2 + L_1}{\sim}} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 2 \\ 0 & 2 & -2 & 6 & 0 \\ 0 & -1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 3 & 0 \end{array}\right)$$

$L_1 \Leftrightarrow L_4$

~~$L_2 \leftarrow L_2 - 2L_1$~~

~~$L_3 \leftarrow L_3 + L_1$~~

~~$L_4 \leftarrow L_4 - L_1$~~

~~1 1 -3 2~~

$$\left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 2 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & -1 & 1 & -3 & 0 \\ 0 & 2 & -2 & 6 & 0 \end{array}\right) \underset{L_4 \leftarrow L_4 - 2L_2}{\overset{L_3 \leftarrow L_2 + L_3}{\sim}} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 2 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 2 \\ 0 & 1 & -1 & 3 & 0 \end{array}\right)$$

$$\begin{cases} x+y-3z+2t = 2 \\ y-z+3t = 0 \Rightarrow y = z-3t \end{cases} \Leftrightarrow \quad x+z-3t-3z+2t = 2$$

$x-2z-t = 2 \Rightarrow$ compatible

$\Rightarrow v' \in \mathrm{Im} f$

(ii) $\begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \Leftrightarrow$

$$\begin{cases} x+y-3z+2t=a \\ -x+y+z+4t=b \\ 2x+y-5z+t=c \\ x+2y-4z+5t=d \end{cases} \qquad \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & a \\ -1 & 1 & 1 & 4 & b \\ 2 & 1 & -5 & 1 & c \\ 1 & 2 & -4 & 5 & d \end{array}\right)$$

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix} \right\}$$

Algebra- Seminar 11

19.12.2022

Def.: V K-v.s.

$$\left.\begin{array}{l}\mathcal{B}=(v_1, v_2, \dots, v_n)\\ \mathcal{B}'=(v_1', v_2', \dots, v_n')\end{array}\right\} \text{basis of } V$$

$$T_{\mathcal{B},\mathcal{B}'} = [id_V]_{\mathcal{B}',\mathcal{B}} = \left([v_1']_{\mathcal{B}} \dots [v_n']_{\mathcal{B}}\right)$$

↳ base change matrix from $\mathcal{B}$ to $\mathcal{B}'$

$\forall v \in V$:

$$[v]_{\mathcal{B}} = [id]_{\mathcal{B}',\mathcal{B}} \cdot [v]_{\mathcal{B}'}$$

- $f, g : V \to V'$ linear maps

  $\alpha, \beta \in K$

$$[\alpha f + \beta g]_{\mathcal{B},\mathcal{B}'} = \alpha [f]_{\mathcal{B},\mathcal{B}'} + \beta [g]_{\mathcal{B},\mathcal{B}'}$$

- $f : V \to V'$, $g : V' \to V''$ linear maps

  $\mathcal{B}, \mathcal{B}', \mathcal{B}''$ bases of V, V', V''

  $\cancel{[g \circ f]_{\mathcal{B},\mathcal{B}''} = [g] \cdot [f]}$

$$[g \circ f]_{\mathcal{B},\mathcal{B}''} = [g]_{\mathcal{B}',\mathcal{B}''} [f]_{\mathcal{B},\mathcal{B}'}$$

- $f : V \to V'$ linear map

  $\mathcal{B}_1, \mathcal{B}_2$ - bases of V

  $\mathcal{B}_1', \mathcal{B}_2'$ - bases of V'

$$[f]_{\mathcal{B}_1,\mathcal{B}_1'} = [id]_{\mathcal{B}_2',\mathcal{B}_1'} \cdot [f]_{\mathcal{B}_2,\mathcal{B}_2'} \cdot [id]_{\mathcal{B}_1,\mathcal{B}_2}$$

$$= T_{\mathcal{B}_1',\mathcal{B}_2'} [f]_{\mathcal{B}_2,\mathcal{B}_2'}, T_{\mathcal{B}_2,\mathcal{B}_1}$$

$[f]_{B,B'} \neq [f]^{-1}_{B',B}$, however,

$$[id]_{B,B'} = [id]^{-1}_{B',B}$$

Exam

11.2 $B = (v_1, v_2) = ((1,2),(1,3))$
$B' = (v_1', v_2') = ((1,0),(2,1))$ } bases of $\mathbb{R}^2$

$f, g \in End_{\mathbb{R}}(\mathbb{R}^2)$

$$[f]_B = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$$

$$[g]_{B'} = \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix}$$

Determine the matrices $[2f]_B$, $[f+g]_B$, $[f \circ g]_{B'}$

$$[2f]_B = 2[f]_B = 2\begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ -2 & -2 \end{pmatrix}$$

$$[f+g]_B = [f]_B + [g]_B$$

$\cancel{[g]_{B',B}}$

$$[g]_B = [id]_{B'B} \cdot [g]_{B'} \cdot \underbrace{[id]_{BB'}}_{=[id]^{-1}_{B',B}}$$

$$[id]_{B'B} = ([v_1']_B, [v_2']_B)$$

$v_1' = (1,0) \Rightarrow [v_1]_B = \alpha v_1 + \beta v_2 = (1,0) \Rightarrow \alpha(1,2) + \beta(1,3) = (1,0) \Rightarrow$

$\Rightarrow (\alpha, 2\alpha) + (\beta, 3\beta) = (1,0) \Leftrightarrow (\alpha+\beta, 2\alpha+3\beta) = (1,0) \Rightarrow$

$\Rightarrow \begin{cases} \alpha+\beta = 1 \mid \cdot 2 \\ 2\alpha+3\beta = 0 \end{cases} \Leftrightarrow \begin{cases} 2\alpha+2\beta = 2 \\ 2\alpha+3\beta = 0 \end{cases} \overset{\ominus}{\Rightarrow} \boxed{\beta = -2} \Rightarrow \boxed{\alpha = 3}$

$v_2' = (2,1) \Rightarrow [v_2']_B = \alpha v_1 + \beta v_2 = (2,1) \Rightarrow \alpha(1,2) + \beta(1,3) = (2,1) \Rightarrow$

$\Rightarrow (\alpha, 2\alpha) + (\beta, 3\beta) = (2,1) \Rightarrow \begin{cases} \alpha+\beta = 2 \mid \cdot 2 \\ 2\alpha+3\beta = 1 \end{cases} \Leftrightarrow \begin{cases} 2\alpha+2\beta = 4 \\ 2\alpha+3\beta = 1 \end{cases} \overset{\ominus}{\Rightarrow} \boxed{\beta = -3} \Rightarrow \boxed{\alpha = 5}$

$$P_A(X) = \det(A - X I_n)$$

Step 2: eigenvalues = roots of P_A

$\lambda_1, \lambda_2, \dots, \lambda_k$

Step 3: For each eigenvalue λ_i solve the system given by:

$$A \cdot X = \lambda \cdot X, \; X \in \mathcal{M}_{n,m}(K)$$

$$(\Leftrightarrow A - \lambda I_n) \cdot X = O_n)$$

The solutions $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ are the eigenvectors corresponding to λ.

(5.11)

$$A = \begin{pmatrix} 3 & 1 & 0 \\ -4 & -1 & 0 \\ -4 & -8 & -2 \end{pmatrix}$$

$$P_A(X) = \det(A - X I_n)$$

$$A - X I_n = \begin{pmatrix} 3-X & 1 & 0 \\ -4 & -1-X & 0 \\ -4 & -8 & -2-X \end{pmatrix} \Rightarrow \det(A - X I_n) = (3-X)(1+X)(2+X) - 4(2+X)$$

$$= (3+2X-X^2)(2+X) - 8 - 4X = -X^3 + 7X + 6 - 8 - 4X = -X^3 + 3X - 2$$

$$= (-2-X)(X-1)^2 \quad X_1 = 1, \quad X_2 = -2$$

• The eigenvectors for $\lambda_1 = 1$ are

$$\left[\begin{pmatrix} 3 & 1 & 0 \\ -4 & -1 & 0 \\ -4 & -8 & -2 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] \underbrace{X}_{=O_3} \Leftrightarrow \begin{pmatrix} 2 & 1 & 0 \\ -4 & -2 & 0 \\ -4 & -3 & -3 \end{pmatrix} X = O_3 \; (\Rightarrow)$$

Ex

$$[id]_{B'B} = \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix} \Rightarrow \det [id]_{B'B} = 1$$

$$[id]_{B'B}^{-1} = \frac{1}{\det [id]_{B'B}} \cdot [id]_{B'B}^{*} = \begin{pmatrix} -3 & 2 \\ -5 & 3 \end{pmatrix}$$

$$[id]_{B'B}^{*} = \begin{pmatrix} -3 & 2 \\ -5 & 3 \end{pmatrix}$$

$$[g]_B = \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} -14 & -13 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} -3 & 2 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} 4 & -4 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} -3 & 2 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} 8 & -4 \\ -22 & 13 \end{pmatrix}$$

$$[f+g]_B = [f]_B + [g]_B = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} + \begin{pmatrix} 8 & -4 \\ -22 & 13 \end{pmatrix} = \begin{pmatrix} 9 & -2 \\ -23 & 12 \end{pmatrix}$$

$$[f \circ g]_{B'} = [f]_{BB'} [g]_{B'B}$$

$$[f]_{BB'} = [id]_{BB'} [f]_B$$

$$[f]_{BB'} = \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$[g]_{B'B} = [id]_{B'B} [g]_{B'}$$

$$[g]_{B'B} = \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} -14 & -13 \\ 5 & 4 \end{pmatrix} = \begin{pmatrix} 4 & -4 \\ -1 & 5 \end{pmatrix}$$

$$[f \circ g]_{B'} = [f]_{BB'} \cdot [g]_{B'B} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 4 & -4 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} 9 & -13 \\ -5 & 9 \end{pmatrix}$$

<u>Def</u>: $f \in \mathrm{End}_K(V)$

$\lambda \in K$ is an eigenvalue for f is $\exists v \in V \setminus \{0\}$ (called an generator for f corresponding to λ) s.t. $f(v) = \lambda v$.

If B, B' - basis for V then everything we said before can be said about the matrix $[f]_{B,B'}$

… of a matrix)

$$\left(\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ -4 & -2 & 0 & 0 \\ -4 & -8 & -3 & 0 \end{array}\right) \underset{L_3 \leftarrow L_3 + 2L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \left(\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -6 & -3 & 0 \end{array}\right)$$

$$\begin{cases} 2x + y = 0 \\ -6y - 3z = 0 \end{cases} \Leftrightarrow \begin{cases} x = \frac{y}{2} \\ z = -2y \end{cases}$$

$$\Rightarrow S(\lambda_1) = S(1) = \left\{\left(-\frac{y}{2}, y, -2y\right) \middle| \, y \in \mathbb{R} \right\} = \left\langle \left(-\frac{1}{2}, 1, -2\right) \right\rangle$$

ALGEBRĂ – Seminar 13

13.01.2023

(n,k) - code

message $\xrightarrow{\text{encode}}$ encoded vector

$m = (a_0, a_1, \dots, a_k) \in \mathbb{Z}_2^k$ $\qquad$ $v \in \mathbb{Z}_2^n$

$v = (c_0, c_1, \dots, c_{n-k-1}, a_0, \dots, a_{k-1})$

decode $\uparrow$ $\qquad$ channel

$v \xleftarrow[\text{correction}]{\text{error}}$ output vector v'

A code C is linear if its encoding function $\gamma : \mathbb{Z}_2^k \longrightarrow \mathbb{Z}_2^n$ is linear

if $\forall\, m_1, m_2 \in \mathbb{Z}_2^k : \gamma(m_1 + m_2) = \gamma m_1 + \gamma m_2$

(also $\forall\, \alpha \in \mathbb{Z}_2, \forall\, m \in \mathbb{Z}_2^k$

$\gamma(\alpha m) = \alpha \gamma(m)$

$G[\gamma]_{E,E'} = ([\gamma(e_1)]_{E'}\ [\gamma(e_2)]_{E'} \dots [\gamma(e_k)]_{E'})$

↳ generator matrix

To encode a message m with such a code it suffices to multiply it by G

i.e. $[\gamma(m)]_{E'} = G \cdot [m]_E$

$G = \left(\frac{M}{I_k}\right) \Rightarrow H = \left(I_{n-k} \mid M\right)$ parity check matrix

An (n,k) polynomial code generated by $p \in \mathbb{Z}_2[X]$

Step 1 : $m = (a_0, \dots, a_{k-1})$

$$\rightsquigarrow P_m = a_0 + a_1 X + \dots + a_{k-1} X^{k-1}$$

Step 2 : $Q_m = P_m \cdot X^{n-k}$

Step 3 : We divide Q_m by P

$$\underbrace{Q_m}_{\text{divident}} = \underbrace{P}_{\text{divisor}} \cdot \underbrace{q}_{\text{quotient}} + \underbrace{R_m}_{\text{remainder}}$$

Step 4 : The encoded polynomial is $T_m = Q_m - R_m = Q_m + R_m$

Step 5 : convert T_m to a vector :

$$T_m = b_0 + b_1 X + \dots + b_{n-1} X^{n-1}$$

The encoded vector is :

$$v = (b_0, b_1, \dots, b_{n-1})$$

We encode the message $m = 101$ using the $(6,3)$-code generated by

$P = 1 + X^2 + X^3 \in \mathbb{R}[X]$

Step 1 : $m = (1,0,1) \to P_m = 1 + X^2$

Step 2 : $Q_m = P_m \cdot X^{n-k} = (1+X^2) \cdot X^3 = X^5 + X^3$

Step 3 :

$$\begin{array}{l|l}
\cancel{X^5} + X^3 & X^3 + X^2 + 1 \\ \hline
\cancel{X^5} + X^4 + X^2 & X^2 + X \\ \hline
\cancel{X^4} + \cancel{X^3} + X^2 & \\
\cancel{X^4} + \cancel{X^3} + X & \\ \hline
X^2 + X &
\end{array}$$

$\to R_m = X^2 + X$

Step 4 : $T_m = X^2 + X + X^5 + X^3$

Step 5 : $v = (0,1,1,\underline{1,0,1})$

$\hookrightarrow$ the message is at the end (check)

(12.8) Determine the generator matrix and the parity check matrix for the (7,3) - code generated by $P = 1 + X^2 + X^3 + X^4 \in \mathbb{Z}_2^3[X]$

$$G = [\gamma]_{E,E'} = \left([\gamma(e_1)]_{E'}, [\gamma(e_2)]_{E'}\right)$$

$$P = 1 + x^2 + x^3 + x^4$$

S1
$m_1 = (1,0,0)$
$m_2 = (0,1,0)$
$m_3 = (0,0,1)$

$m_1 = (1,0,0) \longrightarrow P_{m_1} = 1$
$m_2 = (0,1,0) \rightarrow P_{m_2} = X$
$m_3 = (0,0,1) \rightarrow P_{m_3} = X^2$

S2: $Q_m = P_m \cdot X^{n-k} = 1 \cdot X^4 = X^4$

S3:
$$\begin{array}{r|l} X^4 & X^4 + X^3 + X^2 + 1 \\ \cline{2-2} X^4 + X^3 + X^2 + 1 & 1 \\ \cline{1-1} X^3 + X^2 + 1 & \end{array}$$

$\hookrightarrow r_{e_1} = 1 + X^2 + X^3$

~~$T_m = X^3 + X^2$~~ $T_{e_1} = Q_{e_1} + r_{e_1} = 1 + X^2 + X^3 + X^4$

$v_{e_1} = (1,0,1,1,1,0,0)$

$e_2 = (0,1,0) \rightarrow Pe_2 = X$

$Qe_2 = Pe_2 \cdot X^4 = X^5$

$$
\begin{array}{r|l}
X^5 & X^4+X^3+X^2+1 \\ \hline
X^5+X^4+X^3+X & X+1 \\ \cline{1-1}
X^4+X^3+X & \\
X^4+X^3+X^2+1 & \\ \cline{1-1}
X^2+X+1 \rightarrow Re_2 &
\end{array}
$$

$Te_2 = 1+X+X^2+X^5$

$ve_2 = (1,1,1,0,\underline{0,1,0})$

$$
\begin{array}{r|l}
X^6 & X^4+X^3+X^2+1 \\ \hline
X^6+X^5+X^4+X^2 & X^2+X \\ \cline{1-1}
X^5+X^4+X^2 & \\
X^5+X^4+X^3+X & \\ \cline{1-1}
X^3+X^2+X &
\end{array}
$$

$\hookrightarrow Re_3$

$Te_3 = X^6+X^3+X^2+X$

$ve_3 = (0,1,1,1,\underline{0,0,1})$

$$
G = \begin{array}{c}
\overbrace{}^{M} \\
\left(\begin{array}{ccc}
1 & 1 & 0 \\
0 & 1 & 1 \\
1 & 1 & 1 \\
1 & 0 & 1 \\ \hline
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right) \\
\underbrace{}_{I_3} \\
ve_1 \quad ve_2 \quad ve_3
\end{array}
\Rightarrow H = \left(\begin{array}{cccc|ccc}
1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 1
\end{array}\right)
$$

$\underbrace{}_{I_{7-3}} \quad \underbrace{}_{M}$

In $\mathbb{Z}_2^n$ we can define a matrix : $\forall v, v' \in \mathbb{Z}_2^n$ d

$d_H(v, v') = \#$ of all positions where v and v' disagree

Humming distance

$= w(v + v')$

$\# \text{ of } 1\text{'s}$

$$d(\mathcal{C}) = \min_{v, v' \in \mathcal{C}} d_H(v, v')$$

minimum Hummig distance

$d(\mathcal{C})$ = minimum number of columns in H that add up to a zero column.

<u>Theorem</u> : $\mathcal{C}$ linear code. We can detect at most $d(\mathcal{C}) - 1$ errors and we can correct at most $d(\mathcal{C}) - 1$ errors and we can correct at most $\lfloor \frac{d(\mathcal{C}) - 1}{2} \rfloor$ errors

12.5 Determine $d(\mathcal{C})$ if $G = \begin{pmatrix} P \\ I_n \end{pmatrix} \in \mathcal{M}_{9,4}(\mathbb{Z}_2)$

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

Discuss the error-detecting and error-correcting capabilities of this code and write down the parity check matrix.

$$H = \left(\begin{array}{ccccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\underbrace{\qquad\qquad}_{\substack{I_5 \\ (9-4)}} \qquad \underbrace{\qquad\qquad}_{P}$$

$$\left.\begin{array}{l} \text{no zero columns} \Rightarrow d(\mathcal{C}) \geq 1 \\ \text{no identical colums} \Rightarrow d(\mathcal{C}) > 2 \end{array}\right\} \Rightarrow d(\mathcal{C}) > 2$$

$c_2 + c_6 + c_9 = 0 \Rightarrow d(\mathcal{C}) = 3$

$\Rightarrow$ the code can detect at most 2 errors, we can correct at most 1 error

Algebra - Seminar 14

$\mathcal{C}$ linear code

H parity check matrix

$v \in \mathcal{C} \Leftrightarrow H[v]^{t} = 0$, $v \in \mathbb{Z}_2^{k n}$

(v is a codevector)

$H \cdot [v]$ = the syndrome associated to the vector we will call a

coset leader

the most likely errors for the vector in a coset

(13.2) Using the parity check matrix

$$H = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

and the syndrome and coset leader

Syndrome	000	001	010	011
Coset leader	00000	001000	01000	000010

Syndrome	100	101	110	111
Coset leader	100 000	000 110	000100	000001

Decode the words : 101110, 011000, 001011, 111111, 110011, 010101

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$e = 000010$

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$e = 000110$

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$e = 000110$

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$e = 010000$ → the only digit that gets changed

the corrected vector is 1 0 0 0 1 1

$m = 011$

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$e = 000010$

the corrected vector is: 010001

$m = 001$

(13.5) Construct a table of coset leaders and syndromes for the (7,4) code with parity check matrix

$$H = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}$$

Syndromes	000	001	010	011	100
Coset leader	000 000				

Syndromes	101	110	111
Coset leader			

The coset leader e of syndromes s is the "simplest" vector s.t.

$$H \cdot [e]_{E'} = [s]_{E'}$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = C_1 + C_3 + C_6 + C_7 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$A[v]_E = \sum_{\substack{i=1 \\ v=1}} c_i$$

↳ colums of A

$H \cdot [e_1]_{E'} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$ ~~coset leader = (1 0 0 0 0 0 0)~~

coset leader for $(1,0,0)$ is (1000000)

$$H \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \text{coset leader for } (0,1,0) \text{ is } (\cancel{0,1,0})$$

(13.8) Construct a table of coset leaders and syndromes for the $(7,3)$ code generated by $p = 1+x^2+x^3+x^4 \in \mathbb{Z}_2[x]$

$$\left(\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right)$$

0 0 0 0	0000000
0 0 0 1	0000100
0 0 1 0	0010000
0 1 0 0	0100000
0 1 0 1	0000110
0 1 1 0	01100000
0 1 1 1	0111000
1 0 0 0	1000000
1 0 0 1	0000011
1 0 1 0	0001100
1 0 1 1	0000100

~~1 0 1 1~~	
1 1 0 0	1100000
1 1 0 1	0000101
1 1 1 0	0000010
1 1 1 1	0100010

Decode 1101011, 001110

$$\left(\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right) \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

corrected ~~word~~ vector :

SEX

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BIROU

PULA