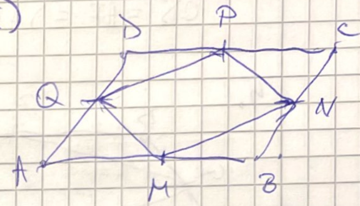
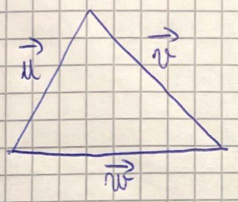
Seminar 1

1. $\underbrace{\overrightarrow{A_0A_1} + \overrightarrow{A_1A_2}}_{\overrightarrow{A_0A_2}} + \dots + \overrightarrow{A_nA_0} = 0$

$\overrightarrow{A_0A_3} \dots \overrightarrow{A_0A_0} = 0$ (point)



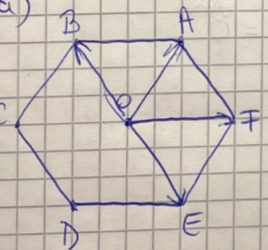
2. a) $u(7,3)$; $v(-2,-8)$; $w(-5,5)$



- calculate the length and check $l_1 + l_2 = l_3$ (sides)

c) - same as before : $7+3<11$ „False"

3. a)



$\overrightarrow{OF}$, $\overrightarrow{OE}$ repr. a basis (lin indep.)

$\overrightarrow{OA} = \overrightarrow{OB} + \overrightarrow{OF} = \overrightarrow{EO} + \overrightarrow{OF}$

$\overrightarrow{OB} = -\overrightarrow{OE} = \overrightarrow{EO}$

$\overrightarrow{OC} = -\overrightarrow{OF} = \overrightarrow{FO}$

$\overrightarrow{OD} = \overrightarrow{OE} + \overrightarrow{OC} = \overrightarrow{OE} - \overrightarrow{OF}$

b) $\vec{AB}+\vec{AC}+\vec{AD}+\vec{AE}+\vec{AF} = 3\vec{AD}$

$\vec{AB}+\vec{AC}+\vec{AE}+\vec{AF} = 2\vec{AD}$

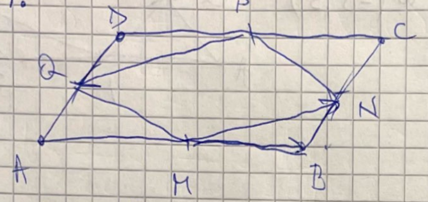
$\vec{AD} = \vec{AE}+\vec{ED} = \vec{AE}+\vec{AB}$

$\Rightarrow \vec{AC}+\vec{AF} = \vec{AD}$

but $\vec{AD} = \vec{AC}+\vec{CD}$

$\vec{CD} = \vec{AF} \Rightarrow \vec{AD} = \vec{AF}+\vec{AC}$

4.

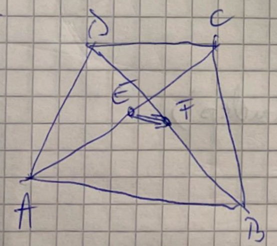


$\vec{MN} = \vec{MB}+\vec{BN} = \frac{1}{2}\vec{AB} + \frac{1}{2}\vec{BC}$

$\vec{PQ} = \vec{PD}+\vec{DQ} = \frac{1}{2}\vec{CD} + \frac{1}{2}\vec{DA}$

$\Rightarrow \vec{MN}+\vec{PQ} = \frac{1}{2}\big(\underbrace{\vec{AB}+\vec{BC}+\vec{CD}+\vec{DA}}_{\text{polyghon}}\big) = \vec{0}$

5.



$\vec{EF} = \frac{1}{2}(\vec{AB}+\vec{CD}) \overset{(1)}{=} \frac{1}{2}(\vec{AD}+\vec{CB})$

$\vec{AC} = A$

$\vec{AB} = \vec{AO}+\vec{OB}$, $\vec{CD} = \cancel{\vec{CO}+\vec{OD}}$

$\vec{EF} = \vec{EO}+\vec{OF}$

$(1) \Leftrightarrow \overrightarrow{EO} + \overrightarrow{OF} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{CD})$

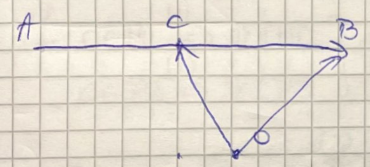
$\Leftrightarrow \overrightarrow{EO} + \overrightarrow{OF} = \frac{1}{2}(\overrightarrow{AO} + \overrightarrow{OB} + \overrightarrow{CO} + \overrightarrow{OD})$

$\Leftrightarrow -\frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OC}) + \frac{1}{2}(\overrightarrow{OB} + \overrightarrow{OD}) = \frac{1}{2}(\overrightarrow{AO} + \overrightarrow{CO}) + \frac{1}{2}(\overrightarrow{OB} + \overrightarrow{OD})$

$\Leftrightarrow \overrightarrow{EF} = \frac{1}{2}(-\overrightarrow{OA} - \overrightarrow{OC} + \overrightarrow{OB} + \overrightarrow{OD})$

7. $k = \left|\frac{CA}{CB}\right|$; $C \in [AB]$

$\forall O \Rightarrow \overrightarrow{OC} = \frac{1}{1+k}(\overrightarrow{OA} + k\overrightarrow{OB})$ $CA = kCB$



$\overrightarrow{AC} = k \cdot \overrightarrow{CB}$

$\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC} = k(-\overrightarrow{OC} + \overrightarrow{OB})$

$\overrightarrow{CB} = \overrightarrow{CO} + \overrightarrow{OB} = -\overrightarrow{OC} + \overrightarrow{OB}$

$\overrightarrow{OB} = \overrightarrow{OC} + \overrightarrow{CB}$

$\overrightarrow{OA} = \overrightarrow{OC} + \overrightarrow{CA} = \overrightarrow{OC} - k\overrightarrow{CB}$

$\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC} + \overrightarrow{CB} + \overrightarrow{OC} - k\overrightarrow{CB} =$

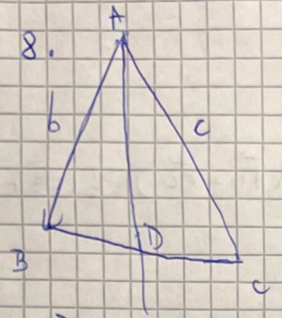
$= 2\overrightarrow{OC} + (1-k)(\overrightarrow{OB} - \overrightarrow{OC})$

$\overrightarrow{OA} + \overrightarrow{OB} = (k+1)\overrightarrow{OC} + (1-k)\overrightarrow{OB}$

$\overrightarrow{OA} + k\overrightarrow{OB} = (k+1)\overrightarrow{OC}$

b) $C\left(\frac{a_1 + k_1 b_1}{1+k}, \frac{a_2 + k b_2}{1+k}, \ldots, \frac{a_n + k b_n}{1+k}\right)$

8.



$\vec{AD} = ?$

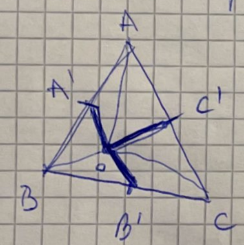
$\frac{BD}{DC} = \frac{BA}{CA}$ ... (solved)

$k = \frac{BD}{DC} = \frac{BA}{CA} = \frac{\|b\|}{\|c\|}$

$AD = \frac{1}{1+k}(\vec{AB} + k\vec{AC})$

9. $\vec{OA'} + \vec{OB'} + \vec{OC'} = \vec{OA} + \vec{OB} + \vec{OC}$

A', B', C' midpoints of the sides



$\vec{OA'} = \cancel{\vec{OA}}\ \frac{1}{2}(\vec{OA} + \vec{OB})$

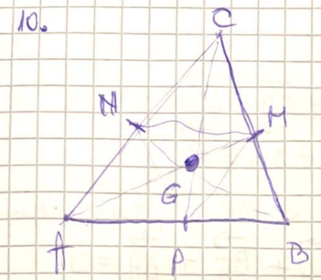
$\vec{OB'} = \frac{1}{2}(\vec{OB} + \vec{OC})$

$\vec{OC'} = \frac{1}{2}(\vec{OC} + \vec{OA})$

$OA' + OB' + OC' = \frac{1}{2}(2\vec{OA} + 2\vec{OB} + 2\vec{OC}) \Rightarrow$

$\Rightarrow OA' + OB' + OC' = OA + OB + OC$

10.



$\forall$ pt. O $\quad v = \vec{OA} + \vec{OB} + \vec{OC}$

(1) $\vec{v} = (\vec{OA} + \vec{OB}) + \vec{OC} = 2\vec{OM} + OC$

(2) $v = (\vec{OA} + \vec{OC}) + \vec{OB} = 2\vec{OP} + \vec{OB}$

(3) $v = (\vec{OB} + \vec{OC}) + \vec{OA} = 2\vec{ON} + \vec{OA}$

$$\text{if } \left.\begin{array}{l} O \in MC \overset{(1)}{\Rightarrow} v \parallel MC \\ O \in PB \overset{(2)}{\Rightarrow} v \parallel PB \end{array}\right\} \Rightarrow O = MC \cap PB$$

$\Rightarrow v = 0$

$\overset{(3)}{\Rightarrow} \vec{OA} = 2\vec{NO}$

in part. O, A, N collinear

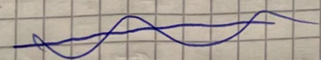
$\Rightarrow$ (Theorem of the median)

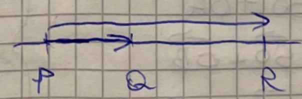
11. $P(3,-5)$; $Q(-1,2)$; $R(-5,9)$

- put them in a determinant ; if det $= 0 \Rightarrow$ P, Q, R colin.

PQ line, if $R \in PQ \Rightarrow$ colin.

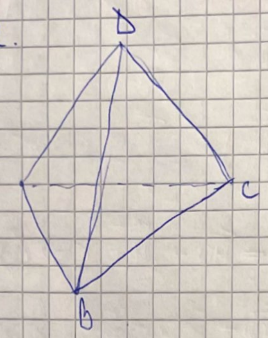
$$\begin{cases} 3a + b = -5 \\ -a + b = 2 \end{cases}$$





P, Q, R colin $\Leftrightarrow \vec{PQ}, \vec{PR}$ are lin. dep.

Seminar 2

12. 

$\vec{AB} + \vec{BC} + \vec{CD} = \vec{AD}$

$\vec{AD} + \vec{BC} + \vec{DB} = \vec{AC}$

(group is a v.s.; can move around

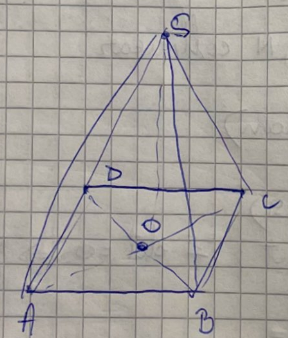
$\vec{AB} + \vec{CD} + \vec{BC} + \vec{DA} = 0$

13. $\vec{AD} + \vec{BC} = \vec{BD} + \vec{AC} \iff$

$\vec{AD} + \vec{BC} - \vec{BD} - \vec{AC} = 0$

$\underbrace{\vec{AD} + \vec{DB}} + \underbrace{\vec{BC} + \vec{CA}} = 0$

$\vec{AB} + \vec{BA} = 0 \iff \vec{AB} = \vec{AB}$ "True"

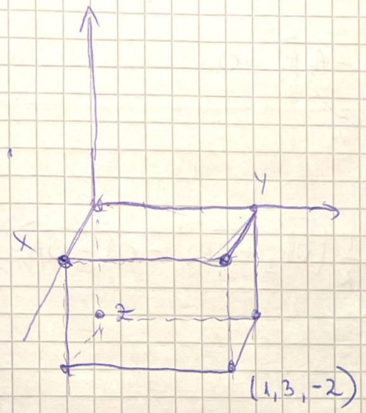
14. 

$$\left.\begin{aligned}
&\vec{SA} + \vec{SB} + \vec{SC} + \vec{SD} = 4\vec{SO} \\
&\vec{SA} = \vec{SO} + \vec{OA} \\
&\vec{SB} = \vec{SO} + \vec{OB} \\
&\vec{SC} = \vec{SO} + \vec{OC} \\
&\vec{SD} = \vec{SO} + \vec{OD}
\end{aligned}\right\} \Rightarrow$$

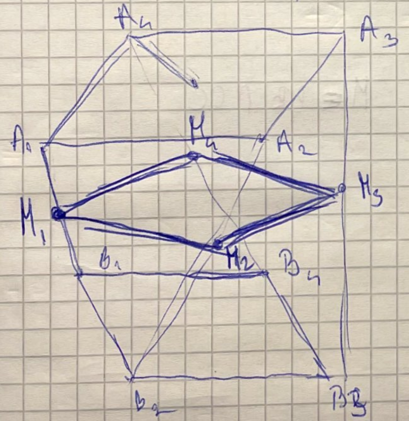
$$\Rightarrow 4\vec{SO} + \underbrace{\vec{OA} + \vec{OB} + \vec{OC} + \vec{OD}}_{=0 \text{ (they start in the origin)}} = 4\vec{SO} \quad \Big| \Rightarrow$$

$\Rightarrow \overrightarrow{4S0} = \overrightarrow{4S0}$

15. $x = 1;\ y = 3;\ z = -2$



15. E^3 (3^{rd} dim. plane)



$\overrightarrow{M_1M_2} + \overrightarrow{M_3M_4} \Leftrightarrow \overrightarrow{M_1M_2} = \overrightarrow{M_4M_3}$ (*)

- decompose relative to one point

$\overrightarrow{M_1M_2} = \frac{1}{2}\overrightarrow{B_1A_1} + \overrightarrow{A_1A_2} + \frac{1}{2}\overrightarrow{B_2A_2}$

$\overrightarrow{M_4M_3} = \frac{1}{2}\overrightarrow{B_4A_4} + \overrightarrow{A_4A_3} + \frac{1}{2}\overrightarrow{A_3B_3}$

$(*) \Leftrightarrow \overrightarrow{B_1A_1} + \overrightarrow{B_2A_2} = \overrightarrow{B_4A_4} + \overrightarrow{A_3B_3}$

$\overrightarrow{B_1A_2} = \overrightarrow{B_1B_1} + \overrightarrow{B_4A_4} + \overrightarrow{A_4A_1}$

$\overrightarrow{B_2A_2} = \overrightarrow{B_2B_3} + \overrightarrow{B_3A_3} + \overrightarrow{A_3A_2}$

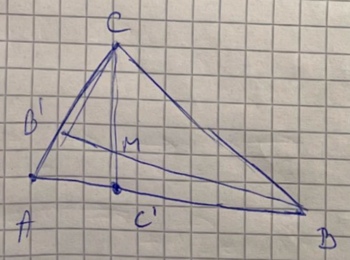
$(*) \Leftrightarrow \overrightarrow{B_1B_4} + \overrightarrow{A_4A_1} - \overrightarrow{B_2B_3} - \overrightarrow{A_3A_2} = 0$

$(*) \Leftrightarrow \overrightarrow{OM_2} - \overrightarrow{OM_1} = \overrightarrow{OM_3} - \overrightarrow{OM_4}$

$\frac{1}{\cancel{2}}(\overrightarrow{OA_2} + \overrightarrow{OB_2} - \overrightarrow{OA_1} - \overrightarrow{OB_1}) = \frac{1}{\cancel{2}}(\overrightarrow{OA_3} + \overrightarrow{OB_3} - \overrightarrow{OA_4} - \overrightarrow{OB_4})$

$\Leftrightarrow \underbrace{\overrightarrow{A_3A_4} + \overrightarrow{A_1A_4}}_{0} = \underbrace{\overrightarrow{B_2B_3} + \overrightarrow{B_4B_1}}_{0}$

Extra problems



$M \in BB' \cap CC'$

$\overrightarrow{AC'} = \lambda \overrightarrow{BC'}$ $\cdot\ O$

$\overrightarrow{AB'} = \mu \overrightarrow{CB'}$

$\overrightarrow{OM} = \dfrac{\overrightarrow{OA} - \lambda \overrightarrow{OB} - \mu \overrightarrow{OC}}{1 - \lambda - \mu}$

$M \in BB'$; $\overrightarrow{OM} = \overrightarrow{OB} + t\overrightarrow{BB'}$ (1) $\quad t, s \in \mathbb{R}$

$M \in CC'$; $\overrightarrow{OM} = \overrightarrow{OC} + s\overrightarrow{CC'}$ (2)

$(1) \Leftrightarrow \overrightarrow{BB'} = \overrightarrow{OB'} - \overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB'} - \overrightarrow{OB}$

$\overrightarrow{AB'} = \dfrac{\|\overrightarrow{AB'}\|}{\|\overrightarrow{AC}\|} \cdot \overrightarrow{AC} = \dfrac{\mu}{1-\mu}\overrightarrow{AC}$

$\vec{AC} = (\mu-1)\,CB'$

$(1-\mu)\vec{AB} = \mu\vec{CA} = -\mu\vec{AC}$

$BB' = \vec{OA} - \frac{\mu}{1-\mu}\vec{AC} - \vec{OB}$

$\Rightarrow \vec{OK} = (t-1)\vec{OB} - \frac{t\mu}{1-\mu}\underbrace{\vec{AC}}_{\vec{OC}-\vec{OA}} + t\vec{OA}$

$= (t-1)\vec{OB} - \frac{t\mu}{1-\mu}\vec{OC} + \left(\frac{t\mu}{1-\mu} + t\right)\vec{OA}$

$= (t-1)\vec{OB} - \frac{t\mu}{1-\mu}\vec{OC} + t\left(\frac{1}{1-\mu}\right)\vec{OA}$

$\vec{OH} = (s-1)\vec{OC} - \frac{s\lambda}{1-\lambda}\vec{OB} + s\left(\frac{1}{1-\lambda}\right)\vec{OA}$

$$\Rightarrow \begin{cases} t-1 = -\frac{s\lambda}{1-\lambda} \\ s-1 = -\frac{t\mu}{1-\mu} \\ \frac{t}{1-\mu} = \frac{s}{1-\lambda} \Rightarrow t = \frac{1-\mu}{1-\lambda}s \end{cases}$$

$s-1 = -\frac{\mu}{1-\lambda}s$

$\left(1+\frac{\mu}{1-\lambda}\right)s = 1$

$\Rightarrow \frac{1-\lambda+\mu}{1-\lambda}s = 1 \Rightarrow s = \frac{1-\lambda}{1-\lambda+\mu}$

$\rightsquigarrow \vec{OH} = \frac{-\mu}{1-\lambda+\mu}\vec{OC} - \frac{\lambda}{1-\lambda+\mu}\vec{OB} + \frac{1}{1-\lambda+\mu}\vec{OA}$

• ~~G centroid.~~ $\cancel{\vec{OG} = \frac{\vec{AB}+\vec{BC}+\vec{AC}}{3}}$ OG

• G centroid : $\overrightarrow{OG} = \dfrac{\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}}{3}$ (1) $\lambda = \mu = -1$

• I incenter : $\overrightarrow{OI} = \dfrac{a\overrightarrow{OA} + b\overrightarrow{OB} + c\overrightarrow{OC}}{a+b+c}$; $\lambda = -\dfrac{b}{a}$; $\mu = -\dfrac{c}{a}$

• H orthocenter : $\overrightarrow{OH} = \dfrac{\tan(\hat{A})\overrightarrow{OA} + \tan(\hat{B})\overrightarrow{OB} + \tan(\hat{C})\overrightarrow{OC}}{\tan\hat{A} + \tan\hat{B} + \tan\hat{C}}$

$\lambda = -\dfrac{\tan\hat{B}}{\tan\hat{A}}$; $\mu = -\dfrac{\tan\hat{C}}{\tan\hat{A}}$

• $\tan\hat{A} = \dfrac{|CC'|}{|AC'|}$

$\tan\hat{B} = \dfrac{|CC'|}{|BC'|}$ $\qquad \dfrac{|BC'|}{|AC'|} = \dfrac{\tan\hat{B}}{\tan\hat{A}}$

• ABC , A'B'C'

G , G' - centroids

$\overrightarrow{AA'} + \overrightarrow{BB'} + \overrightarrow{CC'} = 3\overrightarrow{GG'}$

$\cancel{\overrightarrow{AG} + \overrightarrow{GA'}} + \cancel{\overrightarrow{BG} + \overrightarrow{GB'}}$

$\cancel{= AA'}$

$\overrightarrow{AO} + \overrightarrow{OA'} + \overrightarrow{BO} + \overrightarrow{OB'} + \overrightarrow{CO} + \overrightarrow{OC'} = 3\overrightarrow{GO} + \cancel{3GO}\ 3\overrightarrow{OG'}$

(1) $\Rightarrow$ $3OG' = OA' + OB' + OC'$

$3GO = \cancel{A}AO + OB + CO$

$\Rightarrow$ "True"

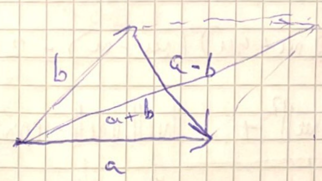
$\boxed{\text{Seminar 3}}$ $v \cdot v = v^2 = \|v\|^2$

1. m, n unit vec.

$\measuredangle(m, n) = 60^\circ$

$a = 2m + n$

$b = n - 2m$



• compute the norm

$\|a-b\|^2 = \cancel{\|2m+n-n+2m\| = \|m+3n\|}$ $(1,3)$ $(3,-1)$

$\cancel{\|a+b\|^2}$

$\cancel{= \|m+3n\|^2 = (m+3n)^2}$ $(m+3n) \cdot (m+3n) =$

$= (m+3n)^2 = \|m+3n\|^2 = m^2 + 6mn + 9n^2 =$

$= \|m\|^2 \cdot \cos 0 + 6\|m\| \cdot \|n\| \cdot \cos 60^\circ + 9\|n\|^2 \cdot \cos 0 =$

$= 1 + 6 \cdot \frac{1}{2} + 9 = 13 \Rightarrow \|a-b\|^2 = 13$

$\|a+b\|^2 = (3m-n)^2 = 9m^2 - 6mn + n^2 =$

$= 9\|m\|^2 - 6\|m\|\|n\| + \|n\|^2 = 9 - \overset{3}{\cancel{6}} + 1 = 7$

$\Rightarrow \|a+b\|^2 = 7$

2. m, n unit vectors

$\measuredangle(m, n) = 120^\circ$

$a = 2m + 4n$

$b = m - n$

$\measuredangle(a, b) = ?$

$\cancel{\|a-b\|}$

$\|a\|\|b\| \cdot \cos(\widehat{a, b})$

$\|a\| = \|2m + 4n\| = 2\|m\| + 4\|n\|$

$\begin{cases} 2\|m\| + 4\|n\| = 1 \\ \|m\| - \|n\| = 1 \quad / \cdot 2 \end{cases}$

$6\|n\| = -1 \Rightarrow$

$\Rightarrow \|n\| = -\frac{1}{6}$

$\cos\sphericalangle(a,b) = \dfrac{a\cdot b}{\|a\|\cdot\|b\|} = \dfrac{-3}{\sqrt{12\cdot 3}} = -\dfrac{3}{6} = -\dfrac{1}{2}$

$a\cdot b = (2m+4n)(m-n) = 2m^2 + 2mn - 4n^2 =$

$= 2\|m\|^2 + 2\|m\|\|n\|\cdot\cos(\widehat{m,n}) - 4\|n\|^2$

$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$

$1 \quad 1 \quad 1 \quad -\frac{1}{2} \quad 1$

$= \cancel{2+\sqrt{3}}\ 2 + 1 - 4 = -3$

$\cancel{\Rightarrow \cos\sphericalangle(a,b)}$

$\|a\| = (2m+4n)^2 = 4 - 16\cdot\frac{1}{2} + 16 = 12$

$\|b\| = (m-n)^2 = 1 + 1 + 1 = 3$

3. $a(2,1,0)$, $b(0,-2,1)$ w.r.t orthonormal basis

$a+b = 2\vec{i} - \vec{j} + \vec{k} = (2,-1,1)$

$\cancel{\|a+b\|^2 =}$

$a-b = 2\vec{i} + 3\vec{j} - \vec{k}$

$\cancel{\cos(\widehat{a,b})}$

$\cos(a+b, a-b) = \dfrac{(a-b)(a+b)}{\|a-b\|\,\|a+b\|} = 0$

$(2,1,1)\cdot(2,3,-1) =$

$(2\vec{i}+\vec{j}+\vec{k})\cdot(2\vec{i}+3-\vec{k}) = 2\cdot 2\cdot i^2 - 3j^2 - 1k^2 = 0$

4. (i, j, k) orthonormal

$q = 3i + j$ $(3, 1, 0)$; $p = i + 2j + \lambda k$ $(1, 2, \lambda)$, $\lambda \in \mathbb{R}$

$\lambda = ?$ a.î. $\cos(p, q) = \frac{5}{12}$

$\cos(p, q) = \frac{p \cdot q}{\|p\| \, \|q\|}$

$\cancel{p \cdot q =}$ $p \cdot q = \|p\| \, \|q\| \cdot \cos \dots$

$p \cdot q =$

$\cancel{\|p\| =}$

$\cancel{p \cdot q = \|p\| \|q\| =}$

$\|p\| \cancel{\cdot} \|q\| = (3i + j)(i + 2j \cancel{+ \lambda k}) =$

$= 3i^2 + 2j^2$

$\cos(\widehat{p q}) = \frac{5}{12} = \frac{p \cdot q}{\|p\| \cdot \|q\|}$

5. $(a_1 b_1 + \dots + a_n b_n)^2 \le (a_1^2 + \dots + a_n^2)(b_1^2 + \dots + b_n^2)$

6. $\triangle ABC$

$\vec{AB}^2 + \vec{AC}^2 - \vec{BC}^2 = 2 \vec{AB} \cdot \vec{AC}$

~~AB + AC +~~

$\vec{AB} + \vec{BC} + \vec{CA} = 0$ $\quad | \cdot (\,)^2$

$(\vec{AB} + \vec{BC} + \vec{CA})^2 =$

$\vec{AB}^2 - 2\vec{AB}\cdot\vec{AC} + \vec{AC}^2 = \vec{BC}^2$

$(\vec{AB} - \vec{AC})^2 = (\vec{BC})^2$

<u>Cosine rule</u>

$$\cos\hat{A} = \frac{b^2+c^2-a^2}{2bc}$$

~~7. ABCD tetrahedron~~

~~co~~

9. $v \perp a(4,-2,-3)$

$v \perp b(0,1,3)$ $\qquad \|v\| =$

$\|v\| = 26$; describes an acute angle with Ox $< 90^\circ$

$\Rightarrow \cos(\widehat{v,i}) > 0$

$\Rightarrow \dfrac{v \cdot i}{\|v\| \cdot \|i\|} =$

$= \dfrac{x}{\ldots} > 0 \Rightarrow x > 0$

$\cancel{4 \cdot v_1 + (-2v_2) + -3v_3}$

$\cancel{v_1 + v_2}$

$v(v_1, v_2, v_3)$

$$\begin{cases} 4x - 2y - 3z = 0 & (1) \\ y + 3z = 0 & (2) \\ x^2 + y^2 + z^2 = 26^2 \\ x > 0 \end{cases}$$

~~(1)+(2) => 4x - y = 0~~

$16 \cdot \frac{9}{144}$

$$\Leftrightarrow \begin{cases} x = -\frac{3}{4} z \\ y = -3z \\ \frac{9}{16} z^2 + \overset{16}{9z^2} + \overset{16}{z^2} = 676 \end{cases}$$

$$\cancel{\Leftrightarrow \frac{154}{16} z^2 = 676}$$

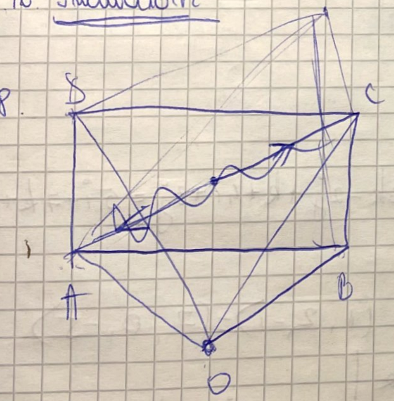
$$\cancel{z^2 = \frac{676 \cdot 16}{154}}$$

$$\frac{169}{16} z^2 = 676$$

- z must be negative because $-\frac{9}{4} \cdot z > 0$

16. <u>Induction</u>

8.



$\forall$ pt. O

$\vec{OA} = \vec{OB} + \vec{BA}$

$\vec{OC} = \vec{OD} + \vec{DC}$

$\vec{OB} = \vec{OD}$

$\vec{OA} \cdot \vec{OC} = (\vec{OB} + \vec{BA})(\vec{OD} + \vec{DC}) =$

$= \vec{OB}\vec{OD} + \vec{OB}\vec{DC} + \vec{BA} \cdot \vec{OD} + \underbrace{\vec{BA} \cdot \vec{DC}}_{0}$

$\boxed{\text{Seminar 4}}$

1. $a = i + 2j - 2k$

$b = 7i + 4j + 6k$

$a \times b$

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} i & j & k \\ 1 & 2 & -2 \\ 7 & 4 & 6 \end{vmatrix}$$

$$= 12i - 14j + 4k - 14k + 8i - 6j = 20i - 20j - 10k$$

2. $a(3, -1, -2)$

$b(1, 2, -1)$

$a \times b$

$(2a+b) \times b$

$(2a+b) \times (2a-b)$

$$a \times b = \begin{vmatrix} i & j & k \\ 3 & -1 & -2 \\ 1 & 2 & -1 \end{vmatrix} = i + 6k - 2j + k + 4i + 3j = 5i + 7k + j$$

$(2a+b) \cancel{\times b} = (6, -2, -4) + (1, 2, -1) = (7, 0, -5)$

$$\Rightarrow (2a+b) \times b = \begin{vmatrix} i & j & k \\ 7 & 0 & -5 \\ 1 & 2 & -1 \end{vmatrix} = 0 - 5j + 14k - 0 + 10i + 7j =$$

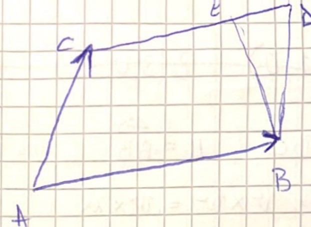
$$= 10i + 2j + 14k$$

$(\underbrace{2ab}_{=0} + b^2)$

$(2a - b) = (5, -4, -3)$

$$(2a+b)(2a-b) = \begin{vmatrix} i & j & k \\ 7 & 0 & -5 \\ 5 & -4 & -3 \end{vmatrix} = \sim$$

3. $\vec{AB}(6,0,1)$
$\vec{AC}(1.5,2,1)$



$d(AB, CD) = a$
$d(AC, BD) = b$

$BE \perp CD \Rightarrow m_{BE} \cdot m_{CD} = -1$

$\cancel{A_{\square} = \|a \times b\| \quad \|A \cdot E\| \, \|a \cdot CD\|}$

$$A_{\square} = \|a \times b\| = \begin{vmatrix} i & j & k \\ 6 & 0 & 1 \\ 1.5 & 2 & 1 \end{vmatrix} = 1.5j + 12k - 2i - 6j = $$
$$= -2i - 4.5j + 12k$$

$h_1 \cdot |AC| = \text{Area}_{\text{par. } ABCD} = \|\vec{AB} \times \vec{AC}\|$

$$h_1 = \frac{\|\vec{AB} \times \vec{AC}\|}{\|\vec{AC}\|} = \ldots$$

4. $a(2,3,-1)$; $b(1,-1,3)$ orthonormal basis

a) $\langle a, b\rangle^{\perp} = \{v \in V^3 : \underset{v \cdot a = 0}{v \perp a} \text{ and } \underset{v \cdot b = 0}{v \perp b}\} = \langle a \times b\rangle$

b) $p \in \langle a, b\rangle^{\perp}$ s.t. $p \cdot (2i - 3j + 4k) = 51$

$$a \times b = \begin{vmatrix} i & j & k \\ 2 & 3 & -1 \\ 1 & -1 & 3 \end{vmatrix} = 9i - j - 2k - 3k - 6j - i =$$
$$= 8i - 7j - 5k$$

$p = \alpha \cdot (a \times b) = (8\alpha i - 7\alpha j - 5\alpha k)$

$\Rightarrow (8\alpha i - 7\alpha j - 5\alpha k)(2i - 3j + 4k) = 16\alpha - 21\alpha - 20\alpha =$

$\Rightarrow -25\alpha = 51 \Rightarrow \alpha = -\frac{51}{25}$

~~5. $A(1,2,0)$; $B(3,0,-3)$, $C(5,2,6)$~~ orthonormal system

7. ΔABC ; $u = \vec{AB}$; $v = \vec{BC}$; $w = \vec{CA}$

$u \times v = v \times w = w \times u$

$u \times v = \|u\| \cdot \|v\| \cdot \sin(\hat{u,v})$

$v \times w = \|v\| \cdot \|w\| \cdot \sin(v, w)$

$u = (-v - w) \times v = \underbrace{-v \times v}_{0} - w \times v = v \times w$

$\|u \times v\| = \|v \times w\|$

$\|u\| \cdot \|v\| \cdot \sin(\hat{u,v}) = \|v\| \cdot \|w\| \cdot \sin(\hat{v,w})$

$\Rightarrow \frac{\sin(\hat{v,w})}{\|u\| \cdot \|v\|} = \frac{\sin(\hat{u,v})}{\|v\| \cdot \|w\|} = \ldots$

8. $a(2,-3,1)$; $b(-3,1,2)$; $c(1,2,3)$

$(a \times b) \times c$

9. $v \in \mathbb{V}^3$ fixed ; $v(v_1, v_2, v_3)$

$\phi : \mathbb{V}^3 \to \mathbb{V}^3 \qquad \phi(w) = v \times w$

$\times : \mathbb{V}^3 \times \mathbb{V}^3 \to \mathbb{V}^3$

det. the matrix of ϕ (w.r.t. a right oriented orthonormal basis.)

3x3 matrix

$$[\phi] = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \phi(i) & \phi(j) & \phi(k) \\ \downarrow & \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}$$

$$= \begin{vmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = i(v_2 w_3 - w_2 v_3)$$

I $w = i$ $\quad v \times i = (v_1 i + v_2 j + v_3 k) \times i$

II $w = j$ $\quad v \times j = v_2 j \times j + v_3 k \times i$

III $w = k$ $\quad = -v_2 k + v_3 j$

11. Jacobian:

$(a \times b) \times c + (b \times c) \times a + (c \times a) \times b = 0$

$\cancel{(a \cdot c)} \cdot b - \cancel{(b \cdot c)} \cdot a + \cancel{(a \cdot b)} \cdot c - \cancel{(a \cdot c)} \cdot b + \cancel{(c \cdot b)} \cdot a - \cancel{(a \cdot b)} \cdot c$

$= 0$

12. (i, j, k) ; (right-oriented basis)

$a = i + j$, $b = i - k$, $c = k$

(a, b, c) ? basis of V^3 $\iff \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{vmatrix} = -1 \neq 0$ (rank=3)

? left/right

det $= -1 < 0$ (negative) $\Rightarrow$ left-oriented

$\boxed{\text{Seminar 5}}$

1. Det param. eq. for l

a) $l \ni A(1,2)$ $\quad l \parallel a(3,-1)$

b) $l \ni$ origin , $l \parallel b(4,5)$

c) $l \ni M(1,7)$, $l \parallel O_y$

d) $l \ni M(2,4)$, $N(2,-5$

2. For l:

a) det Cartesian eq

b) det all direction vectors for l

1.a) $\begin{bmatrix} x \\ y \end{bmatrix} = \overset{A}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} + t \overset{a}{\begin{bmatrix} 3 \\ -1 \end{bmatrix}} = \begin{bmatrix} 1+3t \\ 2-t \end{bmatrix}$

b) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ $\quad \left.\begin{cases} x = 4t \\ y = 5t \end{cases}\right\} (2)$

c) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \left.\begin{cases} x = 1 \\ y = 7+t \end{cases}\right\} (3)$

d) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} + t \begin{bmatrix} 0 \\ -9 \end{bmatrix}$ $\quad \begin{cases} x = 2 \\ y = 4-9t \end{cases}$

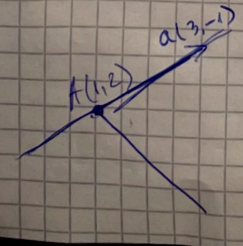
$l: \left.\begin{cases} x = 1+3t \\ y = 2-t \end{cases}\right\} (1)$

$\boxed{\text{Cartesian eq. = linear eq.}}$

$l = \{(1+3t, 2-t) : t \in \mathbb{R}\}$

$\hookrightarrow$ has direct vect. $\{t.a : t \in \mathbb{R} \setminus \{0\}\}$

$\{(3t, -t)$ -11- $\}$



(1) $\Rightarrow$ parametric eq.

(1) $\Rightarrow (t =) \dfrac{x-1}{3} = \dfrac{y-2}{1}$ gen. eq.: $\dfrac{x-x_A}{v_x} = \dfrac{y-y_A}{v_y}$

(2) $\Rightarrow (t =) \dfrac{x}{4} = \dfrac{y}{5} \Leftrightarrow \dfrac{x-0}{4} = \dfrac{y-0}{5}$

(x ~~var~~ fixed, y varies)

(3) $\Rightarrow \begin{cases} x = 1 \\ t = y-7 \end{cases}$

1. Det. for l

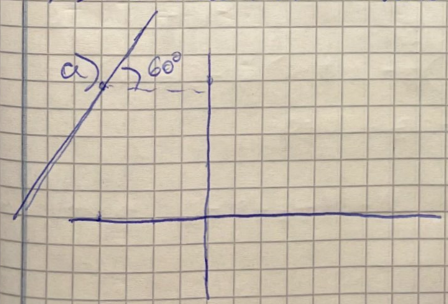
a) $l \ni A(-2,3)$, $60°$ with Ox

b) $l \ni B(1,7)$ $l \perp h(1,7)$

i) det. parametric eq.

ii) det. all normal vectors

a)

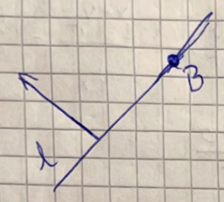


$y = mx + n$ (eq. of the line) ; m = slope

$m = tg\, 60° = \sqrt{3}$

$l \ni A(-2,3) \Rightarrow 3 = 2\sqrt{3} + n \Rightarrow n = 2\sqrt{3} + 3$

$n(n_x, n_y)$



$\boxed{l: n_x(x - x_B) + n_y(y - y_B) = 0}$ if vec. are orthogonal

b) $\Rightarrow l: 4(x-1) + 3(y-3) = 0$

5. l $\quad v(v_1, v_2) \Rightarrow n(-v_2, v_1)$ is a N.V. for

$v \cdot n = v_1(-v_2) + v_2 v_1 = 0 \Leftrightarrow v \perp n$

7. $M_1(1,2)$; $M_2(3,4)$, $M_3(5,-1)$ midpoints

det. param. eq and Cart. eq.

~~AB~~ $\frac{x_A + x_B}{2} = 1$

$\frac{y_A + y_B}{2} = 2$

AB: $\frac{x-1}{2} = \frac{y-2}{-5}$

$AB \parallel \overrightarrow{M_2M_3}(2,-5)$

8. $A(1,3)$, $B(-4,3)$; $C(2,8)$

a) lin. of the altitude from A

$\boxed{\text{Seminar 6}}$

1. Determine parametric eq. for the plane $\tilde{\pi}$:

a) $\tilde{\pi} \ni M(1,0,2)$ $\tilde{\pi} \parallel a_1(3,-1,1)$; $\tilde{\pi} \parallel a_2(0,3,1)$

b) $\tilde{\pi} \ni A(-2,1,1), B(0,2,3), C(1,0,-1)$

c) $\tilde{\pi} \ni A(1,2,1)$; $\tilde{\pi} \parallel i, j$

d) $\tilde{\pi} \ni M(1,7,1)$; $\tilde{\pi} \parallel O_{yz}$

e) $\tilde{\pi} \ni M_1(5,3,4), M_2(1,0,1)$, $\tilde{\pi} \parallel a(1,3,-3)$

f) $\tilde{\pi} \ni A(1,5,7)$, $\tilde{\pi} \supseteq Ox$

e) $\tilde{\pi}: P = M_1 + s\overrightarrow{M_1M_2} + ta$

a) $\tilde{\pi}: \begin{cases} x = x_m + a_{1x}s + a_{2x}t \\ y = y_m + a_{1y}s + a_{2y}t \\ z = z_m + a_{1z}s + a_{2z}t \end{cases}$

$\Leftrightarrow \tilde{\pi}: \begin{cases} x = 1 + 3s \\ y = -s + 3t \\ z = 2 + s + t \end{cases}$

b) $\overrightarrow{AB}(2,1,2)$ (v)

$\overrightarrow{AC}(3,-1,-2)$ (w)

$\tilde{\pi}: \begin{cases} x = -2 + 2s + 3t \\ y = 1 + s - t \\ z = 1 + 2s - 2t \end{cases}$

eq. for O_{yz} : $x = 0$

c) $\tilde{\pi}: \begin{cases} x = 1 + s \\ y = 2 + t \\ z = 1 \end{cases}$ $(s, t \in \mathbb{R})$

d) $v = j$

$w = k$

$\Rightarrow \tilde{\pi}: \begin{cases} x = 1 \\ y = 7 + s \\ z = 1 + t \end{cases}$ $(s, t \in \mathbb{R})$

f) $P = 0$

$v = \vec{OA}(1,5,7)$

$w = \vec{i}$

$\pi: \begin{cases} x = 0 + t \\ y = 5\theta \\ z = 7\theta \end{cases}$

2. Det. cart. eq. for:

$\pi: \begin{cases} x = 2 + 3u - 4v \\ y = 4 - v \\ z = 2 + 3\mu \end{cases} \quad v, \mu \in \mathbb{R}$

$P = Q + ua + vb$

$\vec{QP} = \mu a + vb$

$\begin{array}{c} \\ a \to \\ b \to \end{array} \begin{vmatrix} x-2 & y-4 & z-2 \\ 3 & 0 & 3 \\ -4 & -1 & 0 \end{vmatrix} = 0 \iff (x-2) \cdot \begin{vmatrix} 0 & 3 \\ -1 & 0 \end{vmatrix} -$

$-(y-4) \cdot \begin{vmatrix} 3 & 3 \\ -4 & 0 \end{vmatrix} + (z-2) \begin{vmatrix} 3 & 0 \\ -4 & -1 \end{vmatrix} = 0$

$\iff 3(x-2) - 12(y-4) - 3(z-2) = 0$

$\iff 3x - 12y - 3z - 00 = 0$

3. Det. param. eq. for π: $3x - 6y + z + 1 = 0$ ($\to$ cart. eq.)

$\tilde{\pi}: \begin{cases} x = x(\lambda) \\ y = y(\mu) \\ z = -3x + 6y - 1 \end{cases}, \; z = -3\lambda + 6\mu - 1$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}}_{\text{const.}} + \lambda \underbrace{\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}}_{\text{dep. on } \lambda} + \mu \underbrace{\begin{bmatrix} 0 \\ 1 \\ 6 \end{bmatrix}}_{\text{dep. on } \mu} ; \lambda, \mu \in \mathbb{R}$

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4. Det. an eq. for a plane $\tilde{\pi} \ni P(3,5,-7)$

$\tilde{\pi}$ intersects the coord.-axes in congruent segments

(drawing in Seminar 5)

$\tilde{\pi} \cap Ox = A$

$\tilde{\pi} \cap Oy = B$ $\qquad |OA| = |OB| = |OC|$

$\tilde{\pi} \cap Oz = C$

$\tilde{\pi} = \{p + sv + tw \mid t, s \in \mathbb{R}\}$

$v(v_1, v_2, v_3)$ $\qquad v(\alpha, 0, |\alpha|)$ $\qquad$ (1st case assumed

$v(v_1, 0, v_3)$ $\qquad w(\beta, |\beta|, 0)$ $\qquad$ we are in 1st cadran)

$v(1,0,1)$; $w(1,1,0)$

$$\begin{vmatrix} x-3 & y-5 & z+7 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 0$$

$\tilde{\pi} : \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

$\Rightarrow \tilde{\pi} \cap O_x = A(a,0,0) \quad \Rightarrow |OA| = |a|$

$\tilde{\pi} \cap O_y = B(0,b,0) \quad \Rightarrow |OB| = |b|$

$\tilde{\pi} \cap O_z = C(0,0,c) \quad \Rightarrow |OC| = |c|$

I $a = b = c$ $\qquad x + y + z = a \ni P$ $\quad a = 1$ $\qquad \tilde{\pi} : x + y + z - 1 = 0$

6. $\tilde{\pi}_1 : 2x + y - 2z + 6 = 0 \rightarrow n_1(2,1,-2)$ (normal vector)

$\tilde{\pi}_2 : 2x - 2y + z - 8 = 0 \rightarrow n_2(2,-2,1)$

$\tilde{\pi}_3 : x + 2y + 2z + 1 = 0 \rightarrow n_3(1,2,2)$

is a parallelipiped with faces in $\tilde{\pi}_1, \tilde{\pi}_2, \tilde{\pi}_3$ rectangular?

$\tilde{\pi}_1 \perp \tilde{\pi}_2 \Leftrightarrow n_1 \perp n_2 \Leftrightarrow n_1 \cdot n_2 = 0$

$4 - 2 - 2$

7. $A(1,0,-1)$

$B(0,2,3)$

$C(-2,1,1)$

$D(4,2,3)$

- check coplanarity.

• A, B, C, D coplanar $\Leftrightarrow [\vec{AB}, \vec{AC}, \vec{AD}] = 0$

(calc. the det.)

$$\begin{array}{c} \vec{AB} \rightarrow \\ \vec{AC} \rightarrow \\ \vec{AD} \rightarrow \end{array} \begin{vmatrix} x_B - x_A & y_B - y_A & z_B - z_A \\ - & \text{—} & \\ - & \text{— —} & \end{vmatrix} = 0$$

8. Det. cart. eq. for plane π where the orth. proj. of the origin on π is $A(1,-1,3)$

$n_x(x - x_p) + n_y(y - y_p) + n_z(z - z_p) = 0$

$n(n_x, n_y, n_z)$

$\vec{OA} \perp \pi \quad \vec{OA}(1,-1,3)$

$\pi: 1(x-1) - 1(y+1) + 3(z-3) = 0$

9. $\pi_1: x - 2y - 2z + 7 = 0$

$\pi_2: 2x - 4y - 4z + 17 = 0$

10, 11 $\rightarrow$ relative pos. of planes.

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