DYNAMIC S.

Seminar 1

- 2 Tests - seminar 3 (1 hour) - 10% nota finală
  - seminar 6

- Diferential equation - equation where the variable is a function
- Aplications - Physics
  - Biology
  - Chemistry

Problems:

1.1.1. $\varphi: \mathbb{R} \to \mathbb{R}$, $\varphi(t) = 2e^{3t}$, $(\forall) t \in \mathbb{R}$ is a solution of the Initial Value Problem (IVP) $\begin{cases} x' = 3x & (1) \text{ - ecuation} \\ x(0) = 2 & (2) \text{ - initial condition} \end{cases}$

Represent the corresponding integral curve and describe it's long term behavior.

- integral curve = graphical representation of a solution of some differential equations.
- long term behavior = the behavior of some function (if it is periodic, oscillatory, bounded...)

$x = \varphi(t)$

$\varphi(0) = 2 \cdot e^{3 \cdot 0} = 2$ ✓

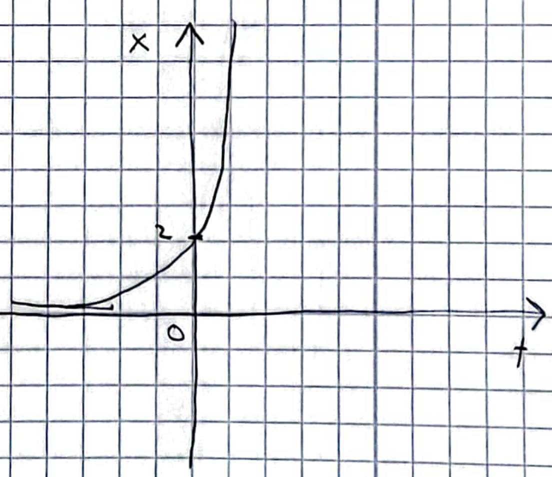
$\varphi(t)' = 2 \cdot e^{3t} \cdot 3 = 3 \cdot 2e^{3t} \overset{(1)}{\Rightarrow} 3\varphi(t)$ ✓

$\Rightarrow \varphi(t)$ is the solution for the IVP

$\lim\limits_{t \to -\infty} \varphi(t) = \lim\limits_{t \to -\infty} 2e^{3t} = 0$

$\lim\limits_{t \to \infty} \varphi(t) = \infty$

$\varphi(0) = 2 \Rightarrow (0,2) \in G_\varphi$



$\varphi \neq$ oscilatory $\Leftrightarrow$ the graphical rep of φ with Ot is not 0

$G_\varphi \cap Ot = \emptyset$

$e \neq$ periodic

$e(t)$ - periodic $\Leftrightarrow e(t) = e(t+T)$

$e' > 0 \Rightarrow e$ is increasing

It is only bounded on the left side, $e(a) = 0$

1.1.3. Show that the function $e(t) = e^{-2t} \cos t$, $(\forall) t \in \mathbb{R}$.

IVP: $\begin{cases} x'' + 4x' + 5x = 0 & (1) \\ x(0) = 1 & (2) \\ x'(0) = -2 & (3) \end{cases}$

$e'(t) = \cancel{e^{-2t} \cdot (-2t)} = -2e^{-2t} \cos t - \sin(t) e^{-2t} \Rightarrow (2)$

$e''(t) = 3e^{-2t} \cos t + 4e^{-2t} \sin t$

$e(0) = e^{-2 \cdot 0} = 1 \Rightarrow (2)$ is True

$e'(0) = -2e^{-2 \cdot 0} = -2 \Rightarrow (3)$ is True

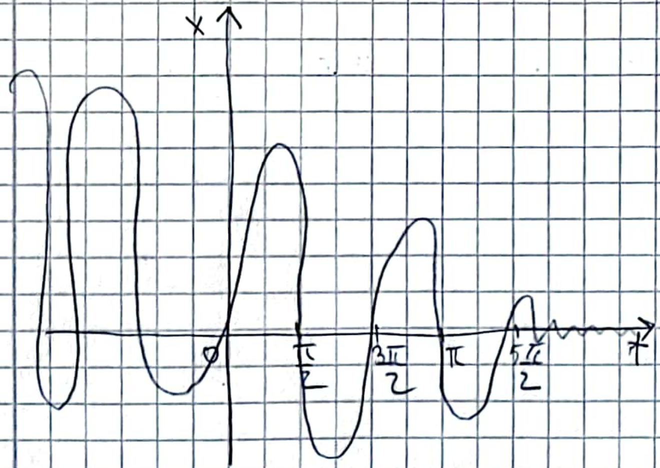
$\cancel{-4e^{-2t} + 8e^{-2t} + 5e^{-2t}} =$

$\cancel{3e^{-2t}} \cos t + \cancel{4e^{-2t} \sin t} + 4(-2e^{-2t} \cos t - \cancel{\sin(t) e^{-2t}}) +$
$+ \cancel{5e^{-2t}} \cos t = 0 \Rightarrow (1)$ is True

$\Rightarrow e(t)$ solution for IVP

$\lim\limits_{t \to \pm\infty} e(t)$ - can't be computet because of the cos.

$G_f \cap Ot \Rightarrow e(t) = 0 \Rightarrow \cos t = 0 \Rightarrow t = \frac{\pi}{2} + 2k\pi$



- The amplitude is decreasing
- Oscilates
- not bounded

1.1.7 Pay attention to the particular form of the function !!!!

Decide if:
$\varphi: \mathbb{R} \to \mathbb{R}$, $\varphi(t) = \cos t$, $(\forall)\, t \in \mathbb{R}$, is a solution for:

Homework
- $x' + x = 0$ $\qquad \varphi' = -\sin t \qquad \varphi''' = \sin t$
- $x'' - x = 0$ $\qquad \varphi'' = -\cos t \qquad \varphi^{IV} = \cos t$
- $x''' + x' = 0$
- $x^{IV} + x'' = 0 \Rightarrow \cos t - \cos t = 0$ T $\Rightarrow$ it is solution

1.1.8. Find all constant solutions of:

a) $x' = x - x^3$ $\qquad$ d) $x' = x^2 + x + 1$

b) $x' = \sin x$ $\qquad$ e) $x' = x + 4x^3$

Homework (c) $x' = \dfrac{x+1}{2x^2+5}$ $\qquad$ f) $x' = -1 + x + 4x^3$

Homework

a) $x' = x - x^3$ $\qquad \varphi(t) = c$

Because the function is a function $\Rightarrow \varphi' = 0$

$0 = x - x^3$

$0 = c - c^3$

$c^3 = c$

$c_1 = 0 \qquad c_2 = 1 \qquad c_3 = -1$

b) $\varphi(t) = c \Rightarrow \varphi'(t) = 0.$

$0 = \sin c \Rightarrow c = 0 + k\pi \Rightarrow c = k\pi,\ k \in \mathbb{Z}$

$\Rightarrow c \in \{ k\pi \mid k \in \mathbb{Z} \}$

11.9. $x_1, x_2, x_3 : \mathbb{R} \to \mathbb{R}$

(i) $x_1(t) = 1$ $x_2(t) = t$ $x_3(t) = t^2$

Prove that them are liniary independent

Liniary indep.: $\sum_{m=1}^{3} x_m c_m = 0 \Rightarrow c_1 = c_2 = c_3$

(ii) $a, b, c = ?$ real s.t. $x(t) = at^2 + bt + c$ is a sol. for:

- $x' - 5x = 2t^2 + 3$

Homework

- $x^{IV} = 0$
- $x''' = 0$

(i) x_1, x_2, x_3 are liniary ind. $\Leftrightarrow [c_1 x_1 + c_2 x_2 + c_3 x_3 = 0 \Rightarrow c_1 = c_2 = c_3]$

$c_1 \cdot 1 + c_2 \cdot t + c_3 \cdot t^2 = 0$, $(\forall)\, t \in \mathbb{R}$

We give 3 values for t, because that holds for $(\forall)\, t$

$$t = 0 \Rightarrow \begin{cases} c_1 + 0 + 0 = 0 & \Rightarrow c_1 = 0 \\ c_1 + c_2 + c_3 = 0 & \Rightarrow c_2 + c_3 = 0 \\ c_1 - c_2 + c_3 = 0. & \Rightarrow c_2 = c_3 \end{cases} \Rightarrow c_1 = c_2 = c_3 = 0$$

($t_2 = 1 \Rightarrow$ second equation, $t_3 = -1 \Rightarrow$ third equation)

$\Rightarrow x_1, x_2, x_3$ are liniary independent

(ii) $x'(t) = 2at + b$

$x' - 5x = 2t^2 + 3 \Rightarrow 2at + b - \underline{5at^2} + 5bt - 5c = 2at^2 + b$

$$\begin{cases} -5a = 2 & \Rightarrow a = -\frac{2}{5} \\ 2a - 5b = 0 & \Rightarrow b = \frac{79}{25} \\ b - 5c = 3 & \Rightarrow c = -\frac{75}{125} \end{cases}$$

SD

SEMINAR 2

Theoretical Part:

- $x^{n} + a_1 x^{(n-1)} + \dots + a_{n-1} x' + a_n = 0$

  differential eq.

- the characteristic eq: $r^n + a_1 r^{n-1} + \dots + a_{n-1} r + a_n = 0$

  $r_1 \dots r_n$ - roots.

we need to work only with this kind of x

$\rightarrow r_1 \in \mathbb{R} \Rightarrow r_1 = e^{r_1 t}$ (simple)

$\rightarrow r_1 = r_2 = \dots \in \mathbb{R}$ (multiple) $x_1 = e^{r_1 t}, x_2 = te^{r_2 t}, x_3 = t^2 e^{r_3 t} \dots$

$\rightarrow r_{1,2} = \alpha \pm i\beta \Rightarrow \begin{cases} r_1 = e^{\alpha t} \cos \beta t \\ r_2 = e^{\alpha t} \sin \beta t \end{cases}$

Exercises:

1.4.2. Find the linear homog. diff. eq. with const coeff. of the min. order that has sol:

(a) e^{-3t} and e^{5t} $\Rightarrow$ $x_1 = e^{-3t}$ $\rightarrow$ simple real root $\Rightarrow r_1 = -3$
$x_2 = e^{5t}$ $\rightarrow$ simple real root $\Rightarrow r_2 = 5$

$\Rightarrow$ the characteristic eq. $(r - r_1)(r - r_2) = 0 \Leftrightarrow (r+3)(r-5) = 0$

$\Rightarrow r^2 - 2r - 15 = 0$!

$\Rightarrow$ the diff. eq.: $x'' - 2x' - 15x = 0$

if we don't write x then the diff eq. is not homogeneous

- If we are asked to find "min. order" we can't add other x_3, x_4 as solutions.

$\rightarrow$ the general sol: $x = c_1 x_1 + c_2 x_2$

$x = c_1 e^{-3t} + c_2 e^{5t}, \quad c_1, c_2 \in \mathbb{R}$

ⓑ $5e^{-3t}$ and $-3e^{5t}$ $\Rightarrow$ $x_1 = e^{-3t}$
$x_2 = e^{5t}$!

= Se rezolvă la fel =

ⓒ $5e^{-3t}$ $3e^{5t}$ - constantele trebuie ignorate:

$$\begin{cases} x_1 = e^{-3t} \\ x_2 = e^{5t} \end{cases}$$

ⓓ $5te^{-3t}$ and $-3e^{5t}$

$$\Rightarrow \begin{cases} \cancel{x} = e^{-3t} \\ x_2 = te^{-3t} \\ x_3 = e^{5t} \end{cases}$$

~~$r_{1,3}$ = simple root~~

~~r_2 = multiple root~~

$\Rightarrow$ $r_{1,2}$ = real double root $= -3$

r_3 = real simple root $= 5$

$\Rightarrow$ $(r+3)^2 (r-5) = 0$

$(r^2+6r+9)(r-5) = 0$

$r^3 + r^2 - 21r - 45 = 0$

$\Rightarrow$ diff. eq. : $x''' + x'' - 21x' - 45 = 0$

$\Rightarrow$ general solution : $x = c_1 e^{-3t} + c_2 te^{-3t} + c_3 e^{5t}$, $c_1 c_2 c_3 \in \mathbb{R}$

ⓔ $(+5-3t)e^{-3t} = 5e^{-3t} - 3te^{-3t}$

$x_1 = e^{-3t} \Rightarrow r_1 = -3$

(That is all because $x_3 = e^{-3t} = x_1$)

$x_2 = te^{-3t} \Rightarrow r_2 = -3$

$\Downarrow$

double root.

$\Rightarrow (r+3)^2 = r^2 + 6r + 9$.

$\Rightarrow$ diff eq. : $x'' + 6x' + 9x = 0$

$\Rightarrow$ gen. sol. : $c_1 e^{-3t} + c_2 te^{-3t} = 0$, $c_1, c_2 \in \mathbb{R}$

BRUNNEN

(f) $\sin 3t = e^{0 \cdot t} \sin 3t \Rightarrow \begin{cases} x_1 = e^{0 \cdot t} \sin 3t \\ x_2 = e^{0 \cdot t} \cos 3t \end{cases} \rightarrow \alpha = 0, \beta = 3$

$r_{1,2} = \alpha \pm i\beta \Rightarrow r_1 = +3i$
$r_2 = 3i$

$\Rightarrow (r - r_1) \cdot (r - r_2) = (r - 3i)(r + 3i) = 0$
$= r^2 + 9 = 0$

$\Rightarrow$ diff eq. : $x'' + 9x = 0$.

$\Rightarrow$ gen. eq. : $c_1 \cdot \sin 3t + c_2 \cdot \cos 3t = 0$, $c_1 c_2 \in \mathbb{R}$.

(o) $(t-1)^2 = t^2 - 2t + 1 \Rightarrow \begin{cases} x_1 = e^{0t} \\ x_2 = e^{0t} \cdot t \\ x_3 = e^{0t} \cdot t^2 \end{cases}$

$\Rightarrow r_{1,2,3} = 0$

$\Rightarrow$ ch. eq. : $r^3 = 0$

$\Rightarrow$ diff. eq. : $x''' = 0$

$\Rightarrow$ gen. eq. : $c_1 + c_2 t + c_3 t^2$, $c_1, c_2, c_3 \in \mathbb{R}$.

1.4.4. Find the solution for each of the following IVP, $m \in \mathbb{R}$

$\begin{cases} x'' + \pi^2 x = 0 \\ x(0) = 0 \\ x'(0) = m \end{cases}$, $x = x(t)$

Non homogeneous eq : $x'' + \pi x = t$ ← we got only the variable

IVP - initial valid problem

Step. 1. $\Rightarrow$ char. eq: $r^2 + \pi^2 = 0$

$r^2 = -\pi^2$

$r_{1,2} = \pm\sqrt{-\pi} \Rightarrow \begin{cases} r_1 = -i\pi \\ r_2 = i\pi \end{cases} \Rightarrow$ We have complex root $\Rightarrow$ 3rd case

$\Downarrow$

$\alpha = 0$, $\beta = \pi$

Step 2: $x_1 = e^{0t} \cos \pi t = \cos \pi t$
$x_1' = e^{0t} \sin \pi t = \sin \pi t$

Step 3: $\Rightarrow$ gen. sol: of diff. eq: $x = c_1 x_1 + c_2 x_2 = c_1 \cos \pi t + c_2 \sin \pi t$ where $c_1, c_2 \in \mathbb{R}$

Step 4. (apply the conditions) $x(0) = c_1 \cos \pi \cdot 0 + c_2 \sin(\pi \cdot 0) = 0$

$\Rightarrow c_1 = 0$

$$x' = -c_1 \sin \pi t \cdot \pi + c_2 \pi \cos \pi t$$
$$= -c_1 \pi \sin \pi t + c_2 \cdot \pi \cos \pi t$$
$$x'(0) = -c_1 \pi \sin \pi \cdot 0 + c_2 \cdot \pi \cos \pi \cdot 0 = \eta$$
$$\Rightarrow c_2 = \frac{\eta}{\pi}, \quad \eta \in \mathbb{R}$$

Step 5 (Replace the const) : $\Rightarrow$ gen sol of IVP : $x(t) = \frac{\eta}{\pi} \sin \pi t$

1.4.5. BVP : $\begin{cases} x'' + x = 0 \\ x(0) = 0 \\ x(\pi) = 0 \end{cases}$

BVP - doesn't have always an unique solution.

Step 1. $\Rightarrow$ ch. eq. : $r^2 + 1 = 0$

$r^2 = -1 \Rightarrow \begin{cases} r_1 = -i \\ r_2 = i \end{cases} \Rightarrow 3^{rd}$ form.

$\Rightarrow \alpha = 0, \quad \beta = \pm 1$

Step 3: $\Rightarrow \begin{cases} x_1 = e^{0t} \cos t \\ x_2 = e^{0t} \sin t \end{cases}$

Step 4: (gen sol + apply conditions) $\Rightarrow$ gen sol. $x = c_1 \cdot \cos t + c_2 \sin t$

- $x(0) = 0 \Rightarrow c_1 \cdot \cos 0 + c_2 \sin 0 = 0 \Rightarrow c_1 = 0$
- $x(\pi) = 0 \Rightarrow c_1 \cdot \cos \pi + c_2 \sin \pi = 0 \Rightarrow c_1 = 0$

Step 5. (Replace constants) $\Rightarrow$ gen sol of BVP : $x = c_2 \sin t, \quad c_2 \in \mathbb{R}$.

1.1.15

- $x' + x = 0 \quad | \cdot e^t$

$x' \cdot e^t + x \cdot e^t = 0$

We can write it like: $x' \cdot e^t + x \cdot (e^t)' = 0$

$\Rightarrow (x \cdot e^t)' = 0 \quad | \int dt$

$x \cdot e^t = C \Rightarrow x = C \cdot e^{-t}$

- $x' + x = 1 + t \quad | \cdot e^t$

$x' \cdot e^t + x \cdot (e^t)' = (1+t)\, e^t \quad | \int dt$

$x \cdot e^t = \int (1+t) e^t \, dt$

$g' = e^t \Rightarrow g = e^t$

$f = 1+t \Rightarrow f' = 1$

$\Rightarrow (1+t)\, e^t - \int e^t \, dt = (1+t)e^t - e^t = t \cdot e^t + C$

$\Rightarrow x = t + C \cdot e^{-t}$

- $t x' + 2x = 1 \quad | \cdot t$

$t^2 x' + 2t\, x = t$

$t^2 x' + (t^2)' x = t$

$(x \cdot t^2)' = t \quad | \int dt$

$x \cdot t^2 = \int t \, dt$

$x \cdot t^2 = \frac{t^2}{2} + C \Rightarrow x = \frac{1}{2} + \frac{C}{t^2}$

1.4.6 , $\lambda \in \mathbb{R}$, $\lambda = ?$, $(\exists)$ nonnull 2π - per sol.

$$x'' + \lambda x = 0$$

$\Rightarrow$ ch eq. $r^2 + \lambda = 0 \Rightarrow r^2 = -\lambda$

I. $\lambda < 0 \Rightarrow r^2 > 0 \Rightarrow r_1, r_2 \in \mathbb{R} \Rightarrow x(t) = c_1 \cdot e^{r_1 t} + c_2 e^{r_2 t}$, $r_{1,2} = \pm\sqrt{-\lambda}$

This is never periodic because the exponentials are never periodic.

II $r^2 = 0 \Rightarrow x(t) = c_1 e^{0t} + c_2 \cdot t e^{0t} = c_1 + c_2 t$, $c_1, c_2 \in \mathbb{R}$

Not periodic because polynomial func. are not periodic.

III. $-\lambda < 0 \Rightarrow r_{1,2} = \pm i\sqrt{\lambda} \Rightarrow \alpha = 0, \beta = \sqrt{\lambda}$

$\Rightarrow x(t) = c_1 e^{0t} \cos\sqrt{\lambda}t + c_2 e^{0t} \sin t\sqrt{\lambda}$

$x(t) = c_1 \cos t\sqrt{\lambda} + c_2 \sin t\sqrt{\lambda}$, $c_1, c_2 \in \mathbb{R}$

Because sin and cos are 2 periodic functions

$\Rightarrow x(t)$ can be periodic

The main period $= \sqrt{\lambda}t = 2\pi \Rightarrow t = \frac{2\pi}{\sqrt{\lambda}}$

The general period $= n\pi$ but $n\pi = 2\pi$

$\Rightarrow \frac{2\pi}{\sqrt{\lambda}} = 2\pi$

$\Rightarrow \lambda = n^2 \in \mathbb{N}^*$

1.4.7 $\mu \in \mathbb{R}$, $\omega > 0$ s.t. $x'' + \mu x' + \omega^2 x = 0$ - se face la fel

SEMINAR 3

1.2.5.

a) $x_{P(a)} = ae^t$, $x' - 2x = e^t$

b) $x_{P(b)} = be^{-t}$, $x' - 2x = e^{-t}$

c) $x_p = ?$, $x' - 2x = 5e^t - 3e^{-t}$ (superpoz. princip.)

d) $x = \quad = ?$, $x' - 2x = 5e^t - 3e^{-t}$

Superpoz principle: c) $x_p = 5x_{P(a)} - 3x_{P(b)}$

a) $(a \cdot e^t)' - 2ae^t = e^t$

$ae^t - 2ae^t = e^t$

$-ae^t = e^t \Rightarrow a = -1$

$\Rightarrow x_{P(a)} = -e^t$

b) $x_{P(b)} = be^{-t}$

$(be^{-t})' - 2(be^{-t}) = e^{-t}$

$-be^{-t} - 2be^{-t} = e^{-t}$

$-3b = 1$

$b = -\frac{1}{3} \Rightarrow x_{P(b)} = -\frac{1}{3}e^{-t}$

c) $x_p = x_{p_1} + x_{p_2} = 5x_{P(a)} - 3x_{P(b)} =$

$= 5 \cdot (-e^t) - 3(-\frac{1}{3}e^{-t})$

$= -5e^t + e^{-t}$

d) $x = x_h + x_p$ — o avem de la c

?

$x_h = x' - 2x = 0$

$\Rightarrow$ Ec ch: $r - 2 = 0 \Rightarrow r = 2$

$\Rightarrow x_h = ce^{2t} \Rightarrow x = ce^{2t} - 5e^t + e^{-t}$, $c \in \mathbb{R}$

SEMINAR 4

a) $u_2 = ?$

$$A u_2 = 2 u_2 \Leftrightarrow \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 2 \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \begin{pmatrix} a + 3b = 2a \\ a - b = 2b \end{pmatrix} \Leftrightarrow$$

$$u_2 = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{cases} a + 3b = 0 \\ a - b = 0 \end{cases} \Rightarrow u_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

A is a diagonalizable matrix over $\mathbb{R}$

$$\begin{cases} \text{Conclusion: } \lambda_1, \lambda_2 = \mp 2 \in \mathbb{R} \text{ (real eigenvalues)} \\ W(u_1, u_2) = \begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} = 4 \neq 0 \Rightarrow \text{linearly independent} \end{cases}$$

b) $X = c_1 e^{\lambda_1 t} u_1 + c_2 e^{\lambda_2 t} u_2$, $c_1, c_2 \in \mathbb{R}$ – general solution.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = X = c_1 e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c e^{2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 e^{-2t} + 3 c_2 e^{2t} \\ -c_1 e^{-2t} + c_2 e^{2t} \end{pmatrix} \quad (1)$$

$X' = AX$ $\qquad A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}$

solution of

c) $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

(1) written by components $\begin{cases} x_1 = c_1 e^{-2t} + 3c_2 e^{2t} \\ x_2 = -c_1 e^{-2t} + c_2 e^{2t} \end{cases}$

2) using the reduction method find the gen solution:

$$\begin{cases} x_1' = x_1 + 3x_2 \\ x_2' = x_1 - x_2 \end{cases} \qquad \begin{aligned} x_1 &= x_1(t) \\ x_2 &= x_2(t) \end{aligned}$$

Step 1: We choose one of the eq. and we derivate with respect to t.

$$x_1' = x_1 + 3x_2 \quad |()'_t$$

$$x_1'' = x_1' + 3x_2'$$

Step 2 → Replace x_1' and x_2' from the main system:

$$x_1'' = (x_1 + 3x_2) + 3(x_1 - x_2)$$

$$= 4x_1$$

Remark: Sometimes it is needed another step: if the terms of x_2 is not canceled out and we apply another substitution.

We take x_2 from the first eq.

After we have to obtain a diff eq. of sec. order with const. coef.

$$x_1'' - 4x_1 = 0$$

Step 3: the chr. eq: $r^2 - 4 = 0$

$r_1 = -2 \quad r_2 = 2$ eigen values.

chr eq: $\det(A - \lambda I_n) = 0$

$$x_1 = c_1 e^{-2t} + c_2 e^{2t}$$

$$\Rightarrow x_2 = \frac{1}{3}(x_1' - x_1)$$

$$= \frac{1}{3}\left(-2c_1 e^{-2t} + 2c_2 e^{2t} - c_1 e^{-2t} - c_2 e^{2t}\right)$$

$$= \frac{1}{3}\left(-3c_1 e^{-2t} + c_2 e^{2t}\right)$$

$$= -c_1 e^{-2t} + \frac{1}{3} c_2 e^{2t}$$

$$\Rightarrow \begin{cases} x_1 = c_1 e^{-2t} + c_2 e^{2t} \\ x_2 = -c_1 e^{-2t} + \frac{1}{3} c_2 e^{2t} \end{cases}$$

e) $e^{tA} = ?$ by using the general solution from above $X' = AX$ on that:

$E(t) = e^{tA}$ satisfies the IVP: $\begin{cases} E'(t) = AE(t) \\ E(0) = I_2 \end{cases}$

Solution:

$$e^{tA} = \begin{pmatrix} X_1 & X_2 \end{pmatrix}$$

first col. second col.

• X_1 = sol of IVP $\begin{cases} x' = Ax \\ x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{cases}$

• X_2 = sol of IVP $\begin{cases} x' = Ax \\ X(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{cases}$

In order to solve IVPs from here we will use the general sol found at d) and then impose the corresponding initial condition.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{-2t} + c_2 e^{2t} \\ -c_1 e^{-2t} + \frac{1}{3} c_2 e^{2t} \end{pmatrix}$$

$$\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} c_1 e^0 + c_2 e^0 \\ -c_1 e^0 + \frac{1}{3} c_2 e^0 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 \\ -c_1 + \frac{1}{3} c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{cases} c_1 + c_2 = 1 \\ -c_1 + \frac{1}{3} c_2 = 0 \end{cases} \quad +$$

$$\frac{4}{3} c_2 = 1 \quad | \cdot 3$$

$$4 c_2 = 3 \Rightarrow c_2 = \frac{3}{4} \Rightarrow c_1 = \frac{1}{4}$$

$$X_1 = \begin{pmatrix} \frac{1}{4} e^{-2t} + \frac{3}{4} e^{2t} \\ -\frac{1}{4} e^{-2t} + \frac{1}{4} e^{2t} \end{pmatrix}$$

$X_2 = ?$

$$\begin{cases} c_1 + c_2 = 0 \\ -c_1 + \frac{1}{3} c_2 = 1 \end{cases} \quad +$$

$$c_2 = \frac{3}{4} \Rightarrow c_1 = -\frac{3}{4}$$

$$\Rightarrow X_2 = \begin{pmatrix} -\frac{3}{4} e^{-2t} + \frac{3}{4} e^{2t} \\ \frac{3}{4} e^{-2t} + \frac{1}{4} e^{2t} \end{pmatrix}$$

$$e^{tA} = (X_1 \quad X_2) = \begin{pmatrix} \frac{1}{4} e^{-2t} + \frac{3}{4} e^{2t} & -\frac{3}{4} e^{-2t} + \frac{3}{4} e^{2t} \\ -\frac{1}{4} e^{-2t} + \frac{1}{4} e^{2t} & \frac{3}{4} e^{-2t} + \frac{1}{4} e^{2t} \end{pmatrix}$$

f) Find e^{tA} in another method.

e^{tA} - using the eigen values and eigen vectors.

A = diagonalizable matrix in $\mathbb{R}$. $\sim D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

where λ_1, λ_2 are the eigenvalues of A

$e^{tA} = P \cdot D \cdot P^{-1}$, $P = (u_1, u_2) = \begin{pmatrix} 1 & 3 \\ -1 & 1 \end{pmatrix}$

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24 - From Seminar 4 FIG

For each $k>0$ we consider a diff. eq.

$x' = -k(x-21)$

a) Find the flow.

b) An experiment revealed the following fact:

a cup of tea of initial temp of $49°C$ has the temp. after 10 minutes of $37°C$

Find another initial temp. s.t after 20 min we will have again $37°C$

a) flow of an eq. a set of a solutions of an IVP of the form: $\begin{cases} x' = -k(x-21) \\ x(0) = \eta \end{cases}$

1) Find sol → Impose cond →

$x' = -k(x-21) = -kx + 21k$

$x' + kx = 21k$

Step 1: Find sol for: $x' + kx = 0$

Step 2: Part sol. x_p

Step 3: Gen sol.

Step 1: $x' + kx = 0$

$x' = -kx$

$\frac{dx}{dt} = -kx$

$\frac{dx}{x} = -k\,dt$

$\int \frac{dx}{x} = -k \int dt$

$\ln|x| = -kt + \ln|c|$

$\ln|x| = \ln e^{-kt} + \ln|c|$

$\ln|x| = \ln(|c| e^{-kt})$

$x_h = ce^{-kt}, \ c \in \mathbb{R}$

Step 2: $x_p = c(t) \cdot e^{-\ln t}$ need to be a sol to the initial eq.

$$[c(t)e^{-\ln t}]' + \ln [c(t)e^{-\ln t}] = 21 \ln .$$

$$c'(t)e^{-\ln t} - \cancel{\ln c(t)e^{-\ln t}} + \cancel{\ln c(t)e^{\ln t}} = 21 \ln$$

The terms with $c(t)$ <u>need</u> to cancel out

$$c'(t) = e^{\ln t}\, 21 \ln \quad \Big| \int$$

$$c(t) = 21 \cancel{\ln} \cdot \frac{1}{\cancel{\ln}} e^{\ln t} = 21 e^{\ln t}$$

$$\Rightarrow x_p = 21 e^{\ln t} \cdot e^{-\ln t} = 21$$

Step 3: $x = ce^{-\ln t} + 21$

$\to$ initial cond $x(0) = \eta$

$$x(0)\; ce^{0} + 21 = \eta \Rightarrow \eta = c + 21 \to c = \eta - 21$$

$$\Rightarrow x = (\eta - 21)e^{-\ln t} + 21$$

Denote $\ell(t, \eta) = (\eta - 21)e^{-\ln t} + 21$

moment in time   initial temp

a) $\to$ in. temp: $x(0) = 49°C \xrightarrow{10'} 37°C$

$t=0$ (initial time)

$$\ell(10, 49°) = (49 - 21)e^{-\ln \cdot 10} + 21 = 37 = \ln = -\frac{1}{10}\ln\left(\frac{4}{7}\right)$$

$\to$ in. temp $= ? \xrightarrow{20'} 37°C$

$$\ell(20, \tilde{\eta}) = (\tilde{\eta} - 21) \cdot e^{-\ln \cdot 20} + 21 = 37°$$

$$(\tilde{\eta} - 21)e^{2\ln \frac{4}{7}} + 21 = 37 \to (\tilde{\eta} - 21)e^{\ln \frac{16}{49}} = 16$$

$$(\tilde{\eta} - 21) \cdot e^{\ln \frac{16}{49}} = 16 \qquad \tilde{\eta} = \frac{16 + 21 e^{\ln \frac{16}{49}}}{e^{\ln \frac{16}{49}}}$$

$$\tilde{\eta} = 70°$$

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