Changing reference frames.

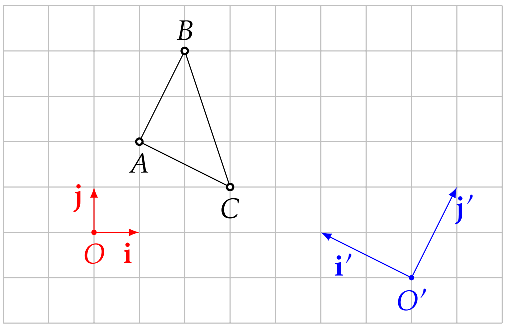
1. We consider two coordinate systems $\mathcal{K} = (O, \mathbf{i}, \mathbf{j})$ and $\mathcal{K}' = (O', \mathbf{i}', \mathbf{j}')$ where

$$[O']_{\mathcal{K}} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}, \quad [\mathbf{i}']_{\mathcal{K}} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \quad [\mathbf{j}']_{\mathcal{K}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

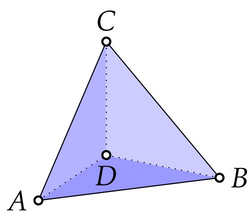
Determine the base change matrix from $\mathcal{K}$ to $\mathcal{K}'$ and the coordinates of the points

$$[A]_{\mathcal{K}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad [B]_{\mathcal{K}} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \quad [C]_{\mathcal{K}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

in the system $\mathcal{K}'$. Further, determine the base change matrix from $\mathcal{K}'$ to $\mathcal{K}$ and use it with the previously obtained coordinates to calculate $[A]_{\mathcal{K}}$, $[B]_{\mathcal{K}}$ and $[C]_{\mathcal{K}}$.



2. With the assumptions in the previous exercise, give parametric equations and Cartesian equations for the lines AB, AC, BC both in the coordinate system $\mathcal{K}$ and in the coordinate system $\mathcal{K}'$.



3. Consider the tetrahedron $ABCD$ and the coordinate systems

$$\mathcal{K}_A = (A, \overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}), \quad \mathcal{K}'_A = (A, \overrightarrow{AB}, \overrightarrow{AD}, \overrightarrow{AC}), \quad \mathcal{K}_B = (B, \overrightarrow{BA}, \overrightarrow{BC}, \overrightarrow{BD}).$$

Determine

a) the coordinates of the vertices of the tetrahedron in the three coordinate systems,

b) the base change matrix from $\mathcal{K}_A$ to $\mathcal{K}'_A$,

c) the base change matrix from $\mathcal{K}_B$ to $\mathcal{K}_A$.

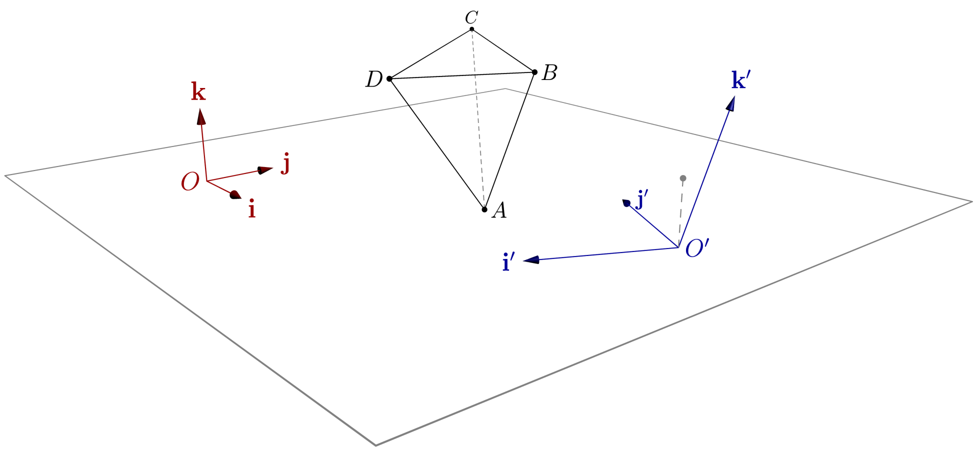
4. We consider the coordinate systems $\mathcal{K} = (O, \mathbf{i}, \mathbf{j}, \mathbf{k})$ and $\mathcal{K}' = (O', \mathbf{i}', \mathbf{j}', \mathbf{k}')$ where

$$[O']_{\mathcal{K}} = \begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix}, \quad [\mathbf{i}']_{\mathcal{K}} = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}, \quad [\mathbf{j}']_{\mathcal{K}} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \quad [\mathbf{k}']_{\mathcal{K}} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

Determine the base change matrix from $\mathcal{K}$ to $\mathcal{K}'$ and the coordinates of the points

$$[A]_{\mathcal{K}} = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}, \quad [B]_{\mathcal{K}} = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}, \quad [C]_{\mathcal{K}} = \begin{bmatrix} -3 \\ 7 \\ 1 \end{bmatrix}, \quad [D]_{\mathcal{K}} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}.$$

in the coordinate system $\mathcal{K}'$. Further, determine the base change matrix from $\mathcal{K}'$ to $\mathcal{K}$ and use it with the previously determined coordinates to calculate $[A]_{\mathcal{K}}$, $[B]_{\mathcal{K}}$, $[C]_{\mathcal{K}}$ and $[D]_{\mathcal{K}}$.



5. With the assumptions in the previous exercise, give parametric equations and Cartesian equations for the line AB and the plane ACD both in the coordinate system $\mathcal{K}$ and in the coordinate system $\mathcal{K}'$.

Projections and reflections on/in hyperplanes.

6. Consider $\mathbf{v}(2,1,1) \in \mathbb{V}^3$ and $Q(2,2,2) \in \mathbb{E}^3$.

a) Give the matrix form for the parallel projection on the plane $\pi : z = 0$ along the line $Q + \langle \mathbf{v} \rangle$.

b) Give the matrix form for the parallel reflection in the plane $\pi : z = 0$ along the line $Q + \langle \mathbf{v} \rangle$.

7. Determine the orthogonal projection of the point $A(2, 11, -5)$ on the plane $x + 4y - 3z + 7 = 0$ by determining the matrix form of the projection. (Compare your result with the previous seminar.)

8. Determine the orthogonal reflection of the point $P(6,-5,5)$ in the plane $2x-3y+z-4=0$ by determining the matrix form of the reflection. (Compare your result with the previous seminar.)

9. Determine the orthogonal projection of the line $\ell: 2x-y-1=0 \cap x+y-z+1=0$ on the plane $\pi: x+2y-z=0$ by determining the matrix form of the projection. (Compare your result with the previous seminar.)

10. Give Cartesian equations for the line passing through the point $M(1,0,7)$, parallel to the plane $\pi: 3x-y+2z-15=0$ and intersecting the line

$$\ell: \frac{x-1}{4} = \frac{y-3}{2} = \frac{z}{1}.$$

11. In $\mathbb{E}^3$, show that the orthogonal reflection $\mathrm{Ref}_{\pi}^{\perp}(x)$ in the plane $\pi: \langle n, x\rangle = p$ is given by

$$\mathrm{Ref}_{\pi}(x) = Ax + b$$

where $A = \left(I - 2\frac{nn^t}{||n||^2}\right)$ and $b = \frac{2p}{||n||^2}n$.

12. Give the matrix form for the orthogonal reflections in the planes

$$\pi_1 : 3x-4z=-1 \quad \text{and} \quad \pi_2 : 10x-2y+3z=4 \quad \text{respectively.}$$

13. Write down the vector forms and matrix forms for parallel projections and reflections in $\mathbb{E}^3$.

14. In $\mathbb{E}^2$, for the lines/hyperplanes

$$\pi: ax+by+c=0, \quad \ell: \frac{x-x_0}{v_1} = \frac{y-y_0}{v_2}$$

with $\pi \nparallel \ell$, deduce the matrix forms of $\mathrm{Pr}_{\pi,\ell}$ and $\mathrm{Ref}_{\pi,\ell}$.

15. Let H be a hyperplane and let $\mathbf{v}$ be a vector. Use the deduced compact matrix forms to show that

a) $\mathrm{Pr}_{H,\mathbf{v}} \circ \mathrm{Pr}_{H,\mathbf{v}} = \mathrm{Pr}_{H,\mathbf{v}}$ and

b) $\mathrm{Ref}_{H,\mathbf{v}} \circ \mathrm{Ref}_{H,\mathbf{v}} = \mathrm{Id}$.

Eigenvalues and eigenvectors.

1. Find the eigenvalues and eigenvectors of the following matrices in $\mathrm{Mat}_{2\times 2}(\mathbb{R})$:

$$\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},\quad \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix},\quad \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix},\quad \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}.$$

2. Consider the matrix

$$A = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$$

Show that A doesn't have eigenvectors when considered in $\mathrm{Mat}_{n\times n}(\mathbb{R})$. Show that A is diagonalizable when considered in $\mathrm{Mat}_{n\times n}(\mathbb{C})$ and find the eigenvectors of A.

3. Give the eigenvalues of $\mathrm{lin}(\mathrm{Pr}_{H,\mathbf{v}})$, $\mathrm{lin}(\mathrm{Ref}_{H,\mathbf{v}})$. What can you say about the eigenvectors?

4. Let $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map

$$\phi(x,y,z) = (x+y-z, y+z, 2x).$$

Find the matrix $\mathrm{M}_{\boldsymbol{b},\boldsymbol{b}}(\phi)$ where

$$\boldsymbol{b} = \{(1,1,0),(-1,0,1),(1,1,1)\}.$$

5. Calculate the eigenvalues and their algebraic and geometric multiplicities for the following matrices in $\mathrm{Mat}_{3\times 3}(\mathbb{R})$, and deduce whether or not they are diagonalizable:

$$\begin{bmatrix}-6 & 2 & -5\\ -4 & 4 & -2\\ 10 & -3 & 8\end{bmatrix},\quad \begin{bmatrix}1 & 0 & 0\\ 0 & -3 & -15\\ 0 & 2 & 8\end{bmatrix}.$$

6. Find the eigenvectors for each of the following symmetric matrices:

$$A = \begin{bmatrix}73 & 36\\ 36 & 52\end{bmatrix},\quad B = \begin{bmatrix}-94 & 180\\ 180 & 263\end{bmatrix}\quad \text{and}\quad C = \begin{bmatrix}128 & 240\\ 240 & 450\end{bmatrix}.$$

We will use these matrices to discuss examples of conic sections.

Rotations.

7. The vertices of a triangle are $A(1,1)$, $B(4,1)$ and $C(2,3)$. Determine the image of the triangle ABC under a rotation by $90°$ around C followed by an orthogonal reflection relative to the line AB.

8. Determine the sum-of-angles formulas for sine and cosine using rotation matrices.

9. Let T be the isometry obtained by applying a rotation of angle $-\frac{\pi}{3}$ around the origin after a transation with vector $(-2,5)$. Determine the inverse transformation, T^{-1}.

10. Determine the matrix form of a rotation with angle $45°$ having the same center of rotation as the rotation

$$f(\mathbf{x}) = \frac{1}{\sqrt{13}}\begin{bmatrix}2 & -3\\ 3 & 2\end{bmatrix}\mathbf{x} + \begin{bmatrix}1\\ -2\end{bmatrix}.$$

11. Determine the cosine of the angle of the rotation f given in the previous exercise and find the inverse rotation, f^{-1}.

12. Verify that the matrices

$$A = \frac{1}{3}\begin{bmatrix} -1 & 2 & -2 \\ -2 & -2 & -1 \\ -2 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \frac{1}{11}\begin{bmatrix} -9 & -2 & 6 \\ 6 & -6 & 7 \\ 2 & 9 & 6 \end{bmatrix}$$

belong to $SO(3)$. Moreover, determine the axis of rotation and the rotation angle.

13. Show that an isometry is bijective.

Circles

1. Find the equation of the circle:

a) of diameter $[A,B]$, with $A(1,2)$ and $B(-3,-1)$,

b) with center $I(2,-3)$ and radius $R=7$,

c) with center $I(-1,2)$ and passing through $A(2,6)$,

d) centered at the origin and tangent to $\ell: 3x-4y+20=0$,

e) passing through $A(3,1)$ and $B(-1,3)$ and having the center on the line $\ell: 3x-y-2=0$,

f) passing through $A(1,1)$, $B(1,-1)$ and $C(2,0)$,

g) tangent to both $\ell_1: 2x+y-5=0$ and $\ell_2: 2x+y+15=0$ if one tangency point is $M(3,-1)$.

2. For a circle $\mathcal{C}$ of radius R:

a) Use the parametrization $x \mapsto (x, \pm\sqrt{R^2-x^2})$ to deduce a parametrization of tangent lines to $\mathcal{C}$.

b) Use the parametrization $\theta \mapsto (R\cos(\theta), R\sin(\theta))$ to deduce a parametrization of tangent lines to $\mathcal{C}$.

c) Compare these to the equation of the tangent line $xx_0+yy_0=R^2$ where $(x_0,y_0)\in\mathcal{C}$.

Ellipses

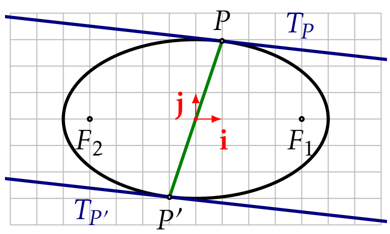
3. Determine the foci (focal points) of the Ellipse $9x^2+25y^2-225=0$

4. Determine the intersection of the line $\ell: x+2y-7=0$ and the ellipse $\mathcal{E}: x^2+3y^2-25=0$.

5. Determine the position of the line $\ell: 2x+y-10=0$ relative to the ellipse $\mathcal{E}: \frac{x^2}{9}+\frac{y^2}{4}-1=0$.

6. Determine an equation of a line which is orthogonal to $\ell: 2x-2y-13=0$ and tangent to the ellipse $\mathcal{E}: x^2+4y^2-20=0$.

7. A *diameter* of an ellipse is the line segment determined by the intersection points of the ellipse with a line passing through the center of the ellipse. Show that the tangent lines to an ellipse at the endpoints of a diameter are parallel.

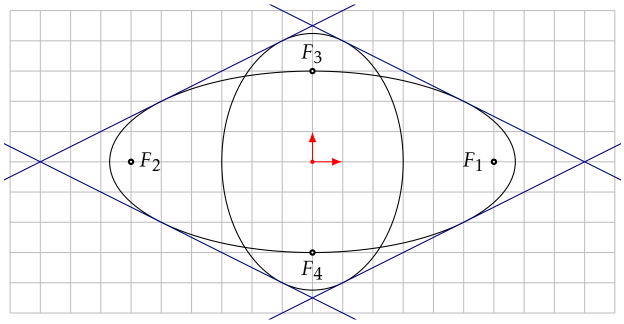


8. Consider the family of ellipses $\mathcal{E}_a : \frac{x^2}{a^2} + \frac{y^2}{16} = 1$. For what value $a \in \mathbb{R}$ is $\mathcal{E}_a$ tangent to the line $\ell : x - y + 5 = 0$?

9. Consider the family of lines $\ell_c : \sqrt{5}x - y + c = 0$. For what values $c \in \mathbb{R}$ is ℓ_c tangent to the ellipse $\mathcal{E} : x^2 + \frac{y^2}{4} = 1$?

10. Determine the common tangents to the ellipses

$$\frac{x^2}{45} + \frac{y^2}{9} = 1 \quad \text{and} \quad \frac{x^2}{9} + \frac{y^2}{18} = 1.$$



11. Consider the ellipse $\mathcal{E} : \frac{x^2}{4} + y^2 - 1 = 0$ with focal points F_1 and F_2. Determine the points M, situated on the ellipse, for which

a) the angle $\measuredangle F_1MF_2$ is right;

b) the angle $\measuredangle F_1MF_2$ is θ;

c) the angle $\measuredangle F_1MF_2$ is maximal.

12. Using a rotation of the coordinate system, find the equation of an ellipse centered at the origin, with focal points on the line $x = y$ and having the large diameter equal to 4 and the distance between the focal points equal to $2\sqrt{3}$.

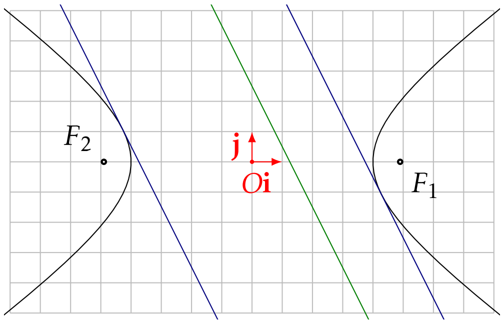
13. Consider the ellipse $\mathcal{E} : x^2 + 4y^2 = 25$. Find the chords on the ellipse which have the point $A(7/2, 7/4)$ as midpoint.

14. Consider the ellipse $\mathcal{E} : \frac{x^2}{25} + \frac{y^2}{9} = 1$. Determine the geometric locus of the midpoints of the chords on the ellipse which are parallel to the line $\ell : x + 2y = 1$.

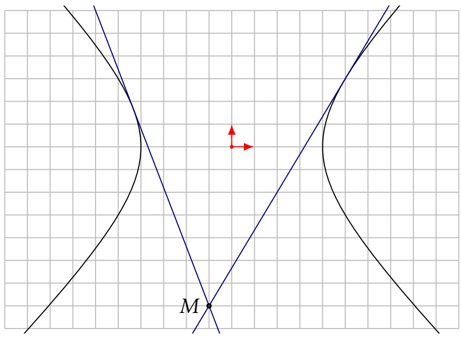
15. Using the gradient, prove the reflective properties of an ellipse.

1. Determine the intersection points between the line $\ell : 2x - y - 10 = 0$ and the hyperbola $\mathcal{H} : \frac{x^2}{20} - \frac{y^2}{5} - 1 = 0$.

2. Determine the tangents to the hyperbola $\mathcal{H} : \frac{x^2}{16} - \frac{y^2}{8} - 1 = 0$ which are parallel to the line $\ell : 4x + 2y - 5 = 0$.



3. Determine the tangents to the hyperbola $\mathcal{H} : x^2 - y^2 = 16$ which contain the point $M(-1, 7)$.



4. Determine the relations between the coordinates (x_P, y_P) of the point P such that P does not belong to any tangent line to the hyperbola

$$\frac{x^2}{4} - \frac{y^2}{9} = 1.$$

5. Find the area of the triangle determined by the asymptotes of the hyperbola $\mathcal{H} : \frac{x^2}{4} - \frac{y^2}{9} - 1 = 0$ and the line $\ell : 9x + 2y - 24 = 0$.

6. Find an equation for the tangent lines to:

a) the hyperbola $\mathcal{H} : \frac{x^2}{20} - \frac{y^2}{5} - 1 = 0$, orthogonal to the line $\ell : 4x + 3y - 7 = 0$;

b) the parabola $\mathcal{P} : y^2 - 8x = 0$, parallel to $\ell : 2x + 2y - 3 = 0$.

7. Find an equation for the tangent lines to:

a) the hyperbola $\mathcal{H}: \frac{x^2}{3} - \frac{y^2}{5} - 1 = 0$, passing through $P(1,-5)$;

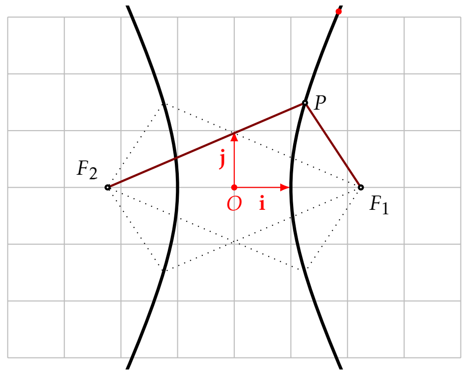
b) the paraola $\mathcal{P}: y^2 - 36x = 0$, passing through $P(2,9)$.

8. Consider the hyperbola $\mathcal{H}: x^2 - \frac{y^2}{4} - 1 = 0$ with focal points F_1 and F_2. Find the points M situated on the hyperbola such that

a) The angle $\measuredangle F_1MF_2$ is right;

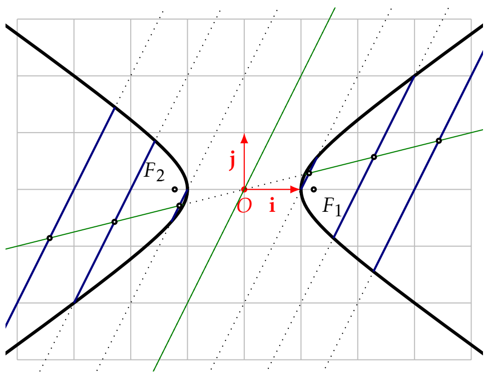
b) The angle $\measuredangle F_1MF_2$ is $60°$;

c) The angle $\measuredangle F_1MF_2$ is θ.



9. Consider the tangents to the parabola $\mathcal{P}: y^2 - 10x = 0$ passing through the point $P(-3,12)$. Calculate the distance from the point P to the chord of the parabola which is formed by the two contact points.

10. Consider the hyperbola $\mathcal{H}: x^2 - 2y^2 = 1$. Determine the geometric locus described by the mid-points of the chords of $\mathcal{H}$ which are parallel to the line $2x - y = 0$.



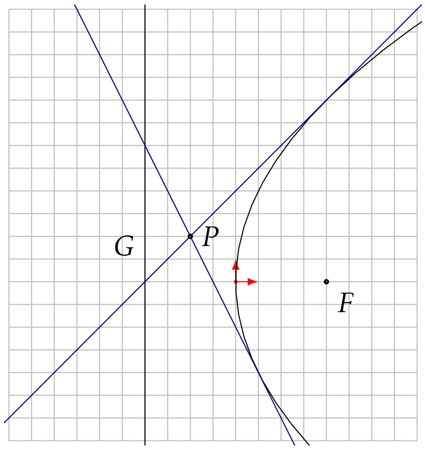
11. For which value k is the line $y = kx + 2$ tangent to the parabola $\mathcal{P} : y^2 = 4x$?

12. Consider the parabola $\mathcal{P} : y^2 = 16x$. Determine the tangents to $\mathcal{P}$ which are

a) parallel to the line $\ell : 3x - 2y + 30 = 0$;

b) perpendicular to the line $\ell : 4x + 2y + 7 = 0$.

13. Determine the tangents to the parabola $\mathcal{P} : y^2 = 16x$ which contain the point $P(-2, 2)$.



14. Using the gradient, prove the reflective properties of the hyperbola and of the parabola.

1. For each of the equations in Table 8.1 of Chapter 8 of the lecture notes, discuss the geometric locus of points satisfying them.

2. For each of the following matrices A, write down a quadratic equation with associated matrix A and find the matrix $M \in SO(2)$ which diagonalizes A.

a) $\begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix}$

b) $\begin{bmatrix} 5 & -13 \\ -15 & 5 \end{bmatrix}$

c) $\begin{bmatrix} 7 & -2 \\ -2 & 5/3 \end{bmatrix}$

3. Check the calculations in examples 8.3.2, 8.3.3 and 8.3.4 of Chapter 8 of the lecture notes.

4. For each of the following equations write down the associated matrix and bring the equation in canonical form.

a) $-x^2 + xy - y^2 = 0$,

b) $6xy + x - y = 0$.

5. In each of the following cases, decide the type of the quadratic curve based on the parameter $a \in \mathbb{R}$.

a) $x^2 - 4xy + y^2 = a$,

b) $x^2 + 4xy + y^2 = a$.

6. Consider the rotation R_{90° of $\mathbb{E}^2$ around the origin and the translation $T_{\mathbf{v}}$ of $\mathbb{E}^2$ with vector $\mathbf{v}(1,0)$.

a) Give the algebraic form of the isometries R_{90°, $T_{\mathbf{v}}$ and $T_{\mathbf{v}} \circ R_{90^\circ}$.

b) Determine the equations of the hyperbola $\mathcal{H} : \frac{x^2}{4} - \frac{y^2}{9} - 1 = 0$ and the parabola $\mathcal{P} : y^2 - 8x = 0$ after transforming them with R_{90° and with $T_{\mathbf{v}} \circ R_{90^\circ}$ respectively.

7. Find the canonical equation for each of the following cases

a) $5x^2 + 4xy + 8y^2 - 32x - 56y + 80 = 0$,

b) $8y^2 + 6xy - 12x - 26y + 11 = 0$,

c) $x^2 - 4xy + y^2 - 6x + 2y + 1 = 0$.

8. For each of the conics in the previous exercise, indicate the affine change of coordinates which brings the equation in canonical form.

9. Discuss the type of the curve

$$x^2 + \lambda xy + y^2 - 6x - 16 = 0$$

in terms of $\lambda \in \mathbb{R}$.

10. Using the classification of quadrics, decide what surfaces are described by the following equations.

a) $x^2 + 2y^2 + z^2 + xy + yz + zx = 1$,

b) $xy + yz + zx = 1$,

c) $x^2 + xy + yz + zx = 1$,

d) $xy + yz + zx = 0$.