1. Let $A_0,\dots,A_n$ be the vertices of a polygon. Determine $\overrightarrow{A_0A_1} + \overrightarrow{A_1A_2} + \cdots + \overrightarrow{A_{n-1}A_n} + \overrightarrow{A_nA_0}$.

2. In each of the following cases, decide if the indicated vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ can be represented with the vertices of a triangle:

a) $\mathbf{u}(7,3)$, $\mathbf{v}(-2,-8)$, $\mathbf{w}(-5,5)$.

b) $\mathbf{u}(7,3)$, $\mathbf{v}(2,8)$, $\mathbf{w}(-5,5)$.

c) $\|\mathbf{u}\| = 7$, $\|\mathbf{v}\| = 3$, $\|\mathbf{w}\| = 11$.

d) $\mathbf{u}(1,0,1)$, $\mathbf{v}(0,1,0)$, $\mathbf{w}(2,2,2)$.

3. Let $ABCDEF$ be a regular hexagon centered at O.

a) Express the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$, $\overrightarrow{OD}$ in terms of $\overrightarrow{OE}$ and $\overrightarrow{OF}$.

b) Show that $\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD} + \overrightarrow{AE} + \overrightarrow{AF} = 3\overrightarrow{AD}$.

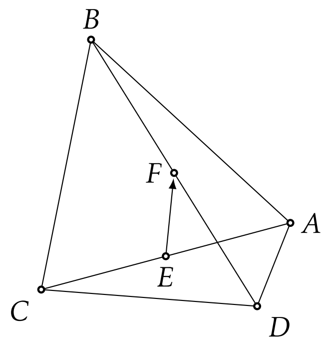
4. Let $ABCD$ be a quadrilateral. Let M, N, P, Q be the midpoints of $[AB]$, $[BC]$, $[CD]$ and $[DA]$ respectively. Show that

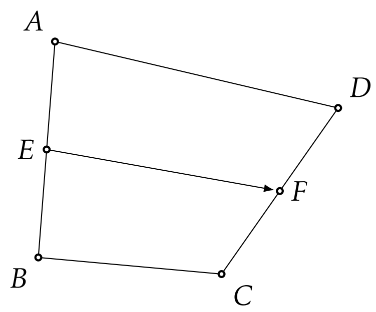
$$\overrightarrow{MN} + \overrightarrow{PQ} = 0.$$

Deduce that the midpoints of the sides of an arbitrary quadrilateral form a parallelogram.

5. Let $ABCD$ be a quadrilateral. Let E be the midpoint of $[AC]$ and let F be the midpoint of $[BD]$. Show that

$$\overrightarrow{EF} = \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{CD}\right) = \frac{1}{2}\left(\overrightarrow{AD} + \overrightarrow{CB}\right).$$





6. Let $ABCD$ be a quadrilateral. Let E be the midpoint of $[AB]$ and let F be the midpoint of $[CD]$. Show that

$$\overrightarrow{EF} = \frac{1}{2}\left(\overrightarrow{AD} + \overrightarrow{BC}\right).$$

Deduce that the length of the midsegment in a trapezoid is the arithmetic mean of the lengths of the bases.

7. Let $k = \frac{|CA|}{|CB|}$ be the ratio in which the point $C \in [AB]$ divides the segment $[AB]$. Show that for any point O we have

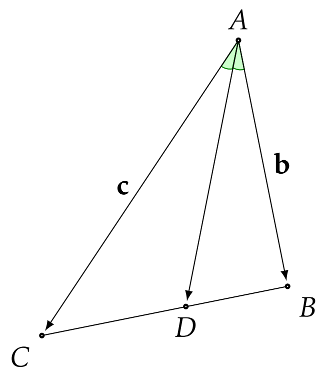
$$\overrightarrow{OC} = \frac{1}{1+k}\left(\overrightarrow{OA} + k\,\overrightarrow{OB}\right).$$

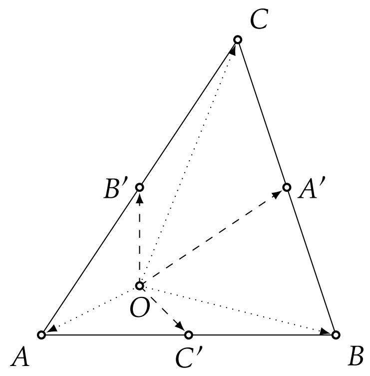
Deduce that C has coordinates

$$\left(\frac{a_1+kb_1}{1+k}, \frac{a_2+kb_2}{1+k}, \dots, \frac{a_n+kb_n}{1+k}\right)$$

where $A = A(a_1, \dots, a_n)$ and $B = B(b_1, \dots, b_n)$.

8. Let ABC be a triangle and let $D \in [BC]$ be such that AD is an angle bisector. Express $\overrightarrow{AD}$ in terms of $\mathbf{b} = \overrightarrow{AB}$ und $\mathbf{c} = \overrightarrow{AC}$.

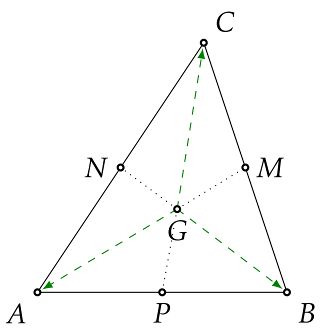




9. Let A', B' and C' be midpoints of the sides of a triangle ABC. Show that for any point O we have

$$\overrightarrow{OA'} + \overrightarrow{OB'} + \overrightarrow{OC'} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}.$$

10. Show that the medians in a triangle intersect in one point and deduce the ratio in which the common intersection point divides the medians.



11. In each of the following cases, decide if the given points are collinear:

a) $P(3,-5)$, $Q(-1,2)$, $R(-5,9)$.

b) $A(11,2)$, $B(1,-3)$, $C(31,13)$.

c) $P(1,0,-1)$, $Q(0,-1,2)$, $R(-1,-2,5)$.

d) $A(-1,-1,-4)$, $B(1,1,0)$, $C(2,2,2)$.

12. Let $ABCD$ be a tetrahedron. Determine the sums

a) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$,

b) $\overrightarrow{AD} + \overrightarrow{BC} + \overrightarrow{DB}$,

c) $\overrightarrow{AB} + \overrightarrow{CD} + \overrightarrow{BC} + \overrightarrow{DA}$.

13. Let $ABCD$ be a tetrahedron. Show that

$$\overrightarrow{AD} + \overrightarrow{BC} = \overrightarrow{BD} + \overrightarrow{AC}.$$

14. Let $SABCD$ be a pyramid with apex S and base the parallelogram $ABCD$. Show that

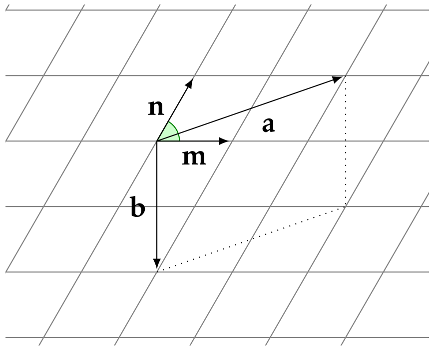
$$\overrightarrow{SA} + \overrightarrow{SB} + \overrightarrow{SC} + \overrightarrow{SD} = 4\overrightarrow{SO}$$

where O is the center of the parallelogram.

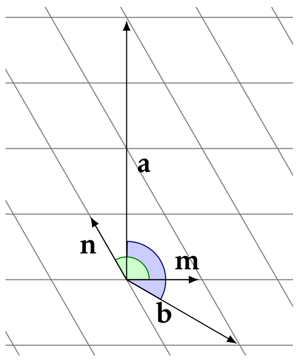
15. Give the coordinates of the vertices of the parallelepiped whose faces lie in the coordinate planes and in the planes $x = 1$, $y = 3$ and $z = -2$.

16. In $\mathbb{E}^3$ consider the parallelograms $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$. Show that the midpoints of the segments $[A_1B_1]$, $[A_2B_2]$, $[A_3B_3]$ and $[A_4B_4]$ are the vertices of a parallelogram.

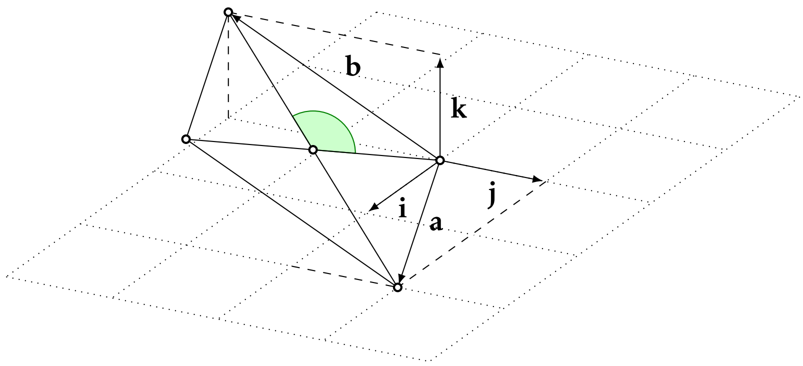
1. Let $\mathbf{m}$ and $\mathbf{n}$ be two unit vectors such that $\measuredangle(\mathbf{m},\mathbf{n}) = 60°$. Determine the length of the diagonals in the parallelogram spanned by the vectors $\mathbf{a} = 2\mathbf{m}+\mathbf{n}$ and $\mathbf{b} = \mathbf{m}-2\mathbf{n}$.



2. Let $\mathbf{m}$ and $\mathbf{n}$ be two unit vectors such that $\measuredangle(\mathbf{m},\mathbf{n}) = 120°$. Determine the angle between the vectors $\mathbf{a} = 2\mathbf{m}+4\mathbf{n}$ and $\mathbf{b} = \mathbf{m}-\mathbf{n}$.



3. You are given two vectors $\mathbf{a}(2,1,0)$ and $\mathbf{b}(0,-2,1)$ with respect to an orthonormal basis. Determine the angles between the diagonals of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$.



4. Let $(\mathbf{i},\mathbf{j},\mathbf{k})$ be an orthonormal basis. Consider the vectors $\mathbf{q} = 3\mathbf{i}+\mathbf{j}$ and $\mathbf{p} = \mathbf{i}+2\mathbf{j}+\lambda\mathbf{k}$ with $\lambda \in \mathbb{R}$. Determine λ such that the cosine of the angle $\measuredangle(\mathbf{p},\mathbf{q})$ is $5/12$.

5. Using the scalar product, prove the Cauchy-Bunyakovsky-Schwarz inequality, i.e. show that for any $a_1,\dots,a_n,b_1,\dots,b_n \in \mathbb{R}$ we have

$$(a_1b_1+\dots+a_nb_n)^2 \le (a_1^2+\dots+a_n^2)(b_1^2+\dots+b_n^2).$$

6. Let ABC be a triangle. Show that

$$\overrightarrow{AB}^2+\overrightarrow{AC}^2-\overrightarrow{BC}^2 = 2\overrightarrow{AB}\cdot\overrightarrow{AC}$$

and deduce the law of cosines in a triangle.

7. Let $ABCD$ be a tetrahedron. Show that

$$\cos(\measuredangle(\overrightarrow{AB},\overrightarrow{CD})) = \frac{AD^2+BC^2-AC^2-BD^2}{2\cdot AB\cdot CD}.$$

This is a $3D$-version of the law of cosine.

8. Let $ABCD$ be a rectangle. Show that for any point O

$$\overrightarrow{OA}\cdot\overrightarrow{OC} = \overrightarrow{OB}\cdot\overrightarrow{OD} \quad \text{and} \quad \overrightarrow{OA}^2+\overrightarrow{OC}^2 = \overrightarrow{OB}^2+\overrightarrow{OD}^2.$$

9. Consider the vector $\mathbf{v}$ which is perpendicular on $\mathbf{a}(4,-2,-3)$ and on $\mathbf{b}(0,1,3)$. If $\mathbf{v}$ describes an acute angle with Ox and $\|\mathbf{v}\| = 26$ determine the components of $\mathbf{v}$.

10. Show that the Gram-Schmidt orthogonalization process yields an orthonormal basis.

11. In an orthonormal basis, consider the vectors $\mathbf{v}_1(0,1,0)$, $\mathbf{v}_2(2,1,0)$ and $\mathbf{v}_3(-1,0,1)$. Use the Gram-Schmidt process to find an orthonormal basis containing $\mathbf{v}_1$.

12. In $\mathbb{E}^2$, show that the orthogonal reflection of a vector $\mathbf{b}$ parallel to $\mathbf{a}$ is

$$\mathrm{Ref}_{\mathbf{a}}^{\|}(\mathbf{b}) = \mathbf{b} - 2\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a} = \mathbf{b} - 2\,\mathrm{Pr}_{\mathbf{a}}^{\perp}(\mathbf{b}).$$

Show that the orthogonal reflection of a vector $\mathbf{b}$ in the vector $\mathbf{a}$ is

$$\mathrm{Ref}_{\mathbf{a}}^{\perp}(\mathbf{b}) = -\mathbf{b} + 2\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a} = -\mathbf{b} + 2\,\mathrm{Pr}_{\mathbf{a}}^{\perp}(\mathbf{b}) = -\mathrm{Ref}_{\mathbf{a}}^{\|}(\mathbf{b}).$$

13. Let $\mathbf{v} \in \mathbb{V}^n$ be a vector. Show that

a) The set $\mathbf{v}^{\perp}$ is a vector subspace of $\mathbb{V}^n$.

b) There is a basis $\mathbf{v},\mathbf{v}_2,\mathbf{v}_3,\dots,\mathbf{v}_n$ of $\mathbb{V}^n$ with $\mathbf{v}_2,\dots,\mathbf{v}_{n-1}$ a basis of $\mathbf{v}^{\perp}$.

14. Fix $\mathbf{v} \in \mathbb{V}^3$ and let $\phi : \mathbb{V}^3 \to \mathbb{R}$ be the map $\phi(\mathbf{w}) = \mathbf{v}\cdot\mathbf{w}$. Is the map linear? Explain why. Give the matrix of ϕ relative to an orthonormal basis. What changes if we define ϕ by $\phi(\mathbf{w}) = \mathbf{w}\cdot\mathbf{v}$?

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1. Let $(\mathbf{i},\mathbf{j},\mathbf{k})$ be a right oriented orthonormal basis of $\mathbb{V}^3$. Consider the vectors $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = 7\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$. Determine $\mathbf{a} \times \mathbf{b}$ in terms of the given basis vectors.

2. With respect to a right oriented orthonormal basis of $\mathbb{V}^3$ consider the vectors $\mathbf{a}(3,-1,-2)$ and $\mathbf{b}(1,2,-1)$. Calculate

$$\mathbf{a} \times \mathbf{b}, \quad (2\mathbf{a} + \mathbf{b}) \times \mathbf{b}, \quad (2\mathbf{a} + \mathbf{b}) \times (2\mathbf{a} - \mathbf{b}).$$

3. Determine the distances between opposite sides of a parallelogram spanned by the vectors $\overrightarrow{AB}(6,0,1)$ and $\overrightarrow{AC}(1.5,2,1)$ if the coordinates of the vectors are given with respect to a right oriented orthonormal basis.

4. Consider the vectors $\mathbf{a}(2,3,-1)$ and $\mathbf{b}(1,-1,3)$ with respect to an orthonormal basis.

a) Determine the vector subspace $\langle \mathbf{a}, \mathbf{b} \rangle^{\perp}$.

b) Determine the vector $\mathbf{p}$ which is orthogonal to $\mathbf{a}$ and $\mathbf{b}$ and for which $\mathbf{p} \cdot (2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) = 51$.

5. Consider the points $A(1,2,0)$, $B(3,0,-3)$ and $C(5,2,6)$ with respect to an orthonormal coordinate system.

a) Determine the area of the triangle ABC.

b) Determine the distance from C to AB.

6. Let $ABCD$ be a quadrilateral in $\mathbb{E}^3$ and let E, F be the midpoints of $[AB]$ and $[CD]$ respectively. Denote by K, L, M and N the midpoints of the segments $[AF]$, $[CE]$, $[BF]$ and $[DE]$ respectively. Prove that $KLMN$ is a parallelogram.

7. Let ABC be a triangle and let $\mathbf{u} = \overrightarrow{AB}$, $\mathbf{v} = \overrightarrow{BC}$, $\mathbf{w} = \overrightarrow{CA}$. Show that

$$\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{w} = \mathbf{w} \times \mathbf{u}.$$

and deduce the law of sines in a triangle.

8. With respect to a right oriented orthonormal coordinate system consider the vectors $\mathbf{a}(2,-3,1)$, $\mathbf{b}(-3,1,2)$ and $\mathbf{c}(1,2,3)$. Calculate $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ and $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$.

9. Fix $\mathbf{v} \in \mathbb{V}^3$ and let $\psi : \mathbb{V}^3 \to \mathbb{V}^3$ be the map $\phi(\mathbf{w}) = \mathbf{v} \times \mathbf{w}$. Is the map linear? Explain why. Give the matrix of ϕ relative to a right oriented orthonormal basis. What changes if we define ϕ by $\phi(\mathbf{w}) = \mathbf{w} \times \mathbf{v}$?

10. Let $(\mathbf{i},\mathbf{j},\mathbf{k})$ be a right oriented orthonormal basis. Determine the matrices of the linear maps $\phi, \psi : \mathbb{V}^3 \to \mathbb{V}^3$ defined by $\phi(\mathbf{v}) = \mathbf{w} \times \mathbf{v}$ and $\psi(\mathbf{v}) = \mathbf{v} \times \mathbf{u}$ where

a) $\mathbf{w} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k}$,

b) $\mathbf{w} = \mathbf{i} + \mathbf{k}$,

c) $\mathbf{u} = 2\mathbf{i} - \mathbf{j}$,

d) $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

11. Prove the following identities:

a) the Jacobi identity,

b) the Lagrange identity,

c) the formula for the cross product of two cross products.

12. Let $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ be a right oriented orthonormal basis. Consider the vectors $\mathbf{a} = \mathbf{i} + \mathbf{j}$, $\mathbf{b} = \mathbf{i} - \mathbf{k}$ and $\mathbf{c} = \mathbf{k}$. Determine if

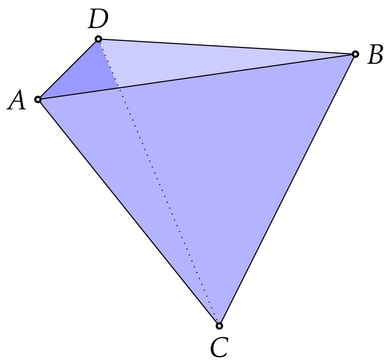
a) $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is a basis of $\mathbb{V}^3$,

b) if it is a basis, decide if it is left or right oriented.

13. The points $A(1,2,-1)$, $B(0,1,5)$, $C(-1,2,1)$ and $D(2,1,3)$ are given with respect to an orthonormal coordinate system. Are the four points coplanar?

14. Let $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ be an orthonormal basis and consider the vectors $\mathbf{u} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $\mathbf{w} = \mathbf{k}$. Determine the matrix of the linear map $\phi : \mathbb{V}^3 \to \mathbb{R}$ defined by $\phi(\mathbf{v}) = [\mathbf{v}, \mathbf{u}, \mathbf{w}]$.

15. Determine the volume of the tetrahedron with vertices $A(2,-1,1)$, $B(5,5,4)$, $C(3,2,-1)$ and $D(4,1,3)$ given with respect to an orthonormal system.



16. The volume of a tetrahedron $ABCD$ is 5. With respect to an orthonormal system $Oxyz$ the vertices are $A(2,1,-1)$, $B(3,0,1)$, $C(2,-1,3)$ and $D \in Oy$. Determine the coordinates of D.

17. With respect to an orthonormal system consider the vectors $\mathbf{a}(8,4,1)$, $\mathbf{b}(2,2,1)$ and $\mathbf{c}(1,1,1)$. Determine a vector $\mathbf{d}$ satisfying the following properties

a) the angles of $\mathbf{d}$ with $\mathbf{a}$ and with $\mathbf{b}$ are congruent,

b) $\mathbf{d}$ is orthogonal to $\mathbf{c}$,

c) $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ and $(\mathbf{a}, \mathbf{b}, \mathbf{d})$ have the same orientation.

All objects considered here are in the plane $\mathbb{E}^2$.

1. Determine parametric equations for the line ℓ in the following cases:

a) ℓ contains the point $A(1,2)$ and is parallel to the vector $\mathbf{a}(3,-1)$,

b) ℓ contains the origin and is parallel to $\mathbf{b}(4,5)$,

c) ℓ contains the point $M(1,7)$ and is parallel to Oy,

d) ℓ contains the points $M(2,4)$ and $N(2,-5)$.

2. For the lines ℓ in the previous exercise

a) give a Cartesian equation for ℓ,

b) describe all direction vectors for ℓ.

3. Determine a Cartesian equations for the line ℓ in the following cases:

a) ℓ has slope -5 and contains the point $A(1,-2)$,

b) ℓ has slope 1 and is at distance 2 from the origin,

c) ℓ contains the point $A(-2,3)$ and has an angle of $60°$ with the Ox-axis,

d) ℓ contains the point $B(1,7)$ and is orthogonal to $\mathbf{n}(4,3)$.

4. For the lines ℓ in the previous exercise

a) give parametric equations for ℓ,

b) describe all normal vectors for ℓ.

5. Consider a line ℓ. Show that

c) if $\mathbf{v}(v_1,v_2)$ is a direction vector for ℓ then $\mathbf{n}(v_2,-v_1)$ is a normal vector for ℓ,

d) if $\mathbf{n}(n_1,n_2)$ is a normal vector for ℓ then $\mathbf{v}(n_2,-n_1)$ is a direction vector for ℓ.

6. Consider the points $A(1,2)$, $B(-2,3)$ and $C(4,7)$. Determine the medians of the triangle ABC.

7. Let $M_1(1,2)$, $M_2(3,4)$ and $M_3(5,-1)$ be the midpoints of the sides of a triangle. Determine Cartesian equations and parametric equations for the lines containing the sides of the triangle.

8. Let $A(1,3)$, $B(-4,3)$ and $C(2,9)$ be the vertices of a triangle. Determine

a) the length of the altitude from A,

b) the line containing the altitude from A.

9. Determine the circumcenter of the triangle with vertices $A(1,2)$, $B(3,-2)$, $C(5,6)$.

10. Determine the angle between the lines $\ell_1 : y = 2x + 1$ and $\ell_2 : y = -x + 2$.

11. Let $A(1,-2)$, $B(5,4)$ and $C(-2,0)$ be the vertices of a triangle. Determine the equations of the angle bisectors for the angle $\measuredangle A$.

12. Let A' be the orthogonal reflection of $A(10,10)$ in the line $\ell : 3x + 4y - 20 = 0$. Determine the coordinates of A'.

13. Determine Cartesian equations for the lines passing through $A(-2,5)$ which intersect the coordinate axes in congruent segments.

14. Determine Cartesian equations for the lines situated at distance 4 from the line $12x - 5y - 15 = 0$.

15. Determine the values k for which the distance from the point $(2,3)$ to the line $8x + 15y + k = 0$ equals 5.

16. Consider the points $A(3,-1)$, $B(9,1)$ and $C(-5,5)$. For each pair of these three points, determine the line which is equidistant from them.

17. The point $A(3,-2)$ is the vertex of a square and $M(1,1)$ is the intersection point of its diagonals. Determine Cartesian equations for the sides of the square.

18. Determine a point on the line $5x - 4y - 4 = 0$ which is equidistant to the points $A(1,0)$ and $B(-2,1)$.

19. The point $A(2,0)$ is the vertex of an equilateral triangle. The side opposite to A lies on the line $x + y - 1 = 0$. Determine Cartesian equations for the lines containing the other two sides.

1. Determine parametric equations for the plane π in the following cases:

a) π contains the point $M(1,0,2)$ and is parallel to the vectors $\mathbf{a}_1(3,-1,1)$ and $\mathbf{a}_2(0,3,1)$,

b) π contains the points $A(-2,1,1)$, $B(0,2,3)$ and $C(1,0,-1)$,

c) π contains the point $A(1,2,1)$ and is parallel to $\mathbf{i}$ and $\mathbf{j}$,

d) π contains the point $M(1,7,1)$ and is parallel coordinate plane Oyz,

e) π contains the points $M_1(5,3,4)$ and $M_2(1,0,1)$, and is parallel to the vector $\mathbf{a}(1,3,-3)$,

f) π contains the point $A(1,5,7)$ and the coordinate axis Ox.

2. Determine Cartesian equations for the plane π in the following cases:

a) $\pi : x = 2+3u-4v,\ y = 4-v,\ z = 2+3u$;

b) $\pi : x = u+v,\ y = u-v,\ z = 5+6u-4v$.

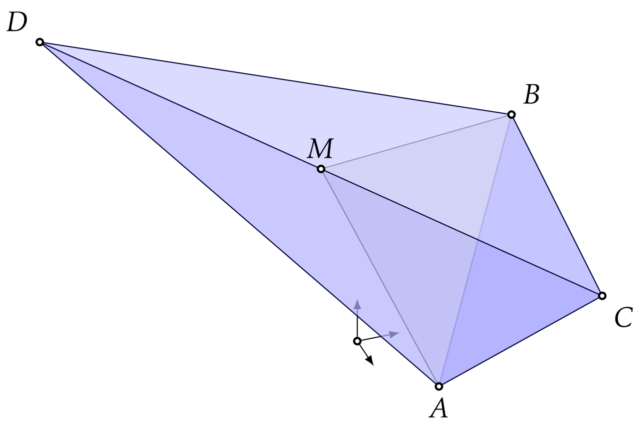
3. Determine parametric equations for the plane π in the following cases:

a) $3x-6y+z=0$;

b) $2x-y-z-3=0$;

4. Determine an equation for each plane passing through $P(3,5,-7)$ and intersecting the coordinate axes in congruent segments.

5. Let $A(2,1,0)$, $B(1,3,5)$, $C(6,3,4)$, $D(0,-7,8)$ be vertices of a tetrahedron. Determine a Cartesian equation of the plane containing $[AB]$ and the midpoint of $[CD]$.



6. Show that a parallelepiped with faces in the planes $2x+y-2z+6=0$, $2x-2y+z-8=0$ and $x+2y+2z+1=0$ is rectangular.

7. Show that the points $A(1,0,-1)$, $B(0,2,3)$, $C(-2,1,1)$ and $D(4,2,3)$ are coplanar.

8. Determine a Cartesian equation of the plane π if $A(1,-1,3)$ is the orthogonal projection of the origin on π.

9. Determine the distance between the planes $x-2y-2z+7=0$ and $2x-4y-4z+17=0$.

10. Determine the relative positions of the planes in the following cases

a) $\pi_1 : x+2y+3z-1=0$, $\pi_2 : x+2y-3z-1=0$.

b) $\pi_1 : x+2y+3z-1=0$, $\pi_2 : 2x+y+3z-2=0$, $\pi_3 : x+2y+3z+2=0$.

11. Show that the planes

$$\pi_1 : 3x+y+z-1=0, \quad \pi_2 : 2x+y+3z+2=0, \quad \pi_3 : -x+2y+z+4=0$$

have a point in common.

12. Show that the pairwise intersection of the planes

$$\pi_1 : 3x+y+z-5=0, \quad \pi_2 : 2x+y+3z+2=0, \quad \pi_3 : 5x+2y+4z+1=0$$

are parallel lines.

13. Determine parametric equations for the line ℓ in the following cases:

a) ℓ contains the point $M_0(2,0,3)$ and is parallel to the vector $\mathbf{a}(3,-2,-2)$,

b) ℓ contains the point $A(1,2,3)$ and is parallel to the Oz-axis,

c) ℓ contains the points $M_1(1,2,3)$ and $M_2(4,4,4)$.

14. Give Cartesian equations for the lines ℓ in the previous exercise.

15. Determine parametric equations for the line contained in the planes $x+y+2z-3=0$ and $x-y+z-1=0$.

16. Consider the lines $\ell_1 : x=1+t, y=1+2t, z=3+t, t\in\mathbb{R}$ and $\ell_2 : x=3+s, y=2s, z=-2+s, s\in\mathbb{R}$. Show that ℓ_1 and ℓ_2 are parallel and find the equation of the plane determined by the two lines.

1. Determine parametric equations of the line passing through $P(5,0,-2)$ and parallel to the planes $\pi_1 : x-4y+2z=0$ and $2x+3y-z+1=0$.

2. Determine an equation of the plane containing $P(2,0,3)$ and the line $\ell : x=-1+t, y=t, z=-4+2t, t\in\mathbb{R}$.

3. For the points $A(2,1,-1)$ and $B(-3,0,2)$, determine

a) an equation of the bundle of planes passing through A and B,

b) the plane π from the bundle, which is orthogonal to Oxy,

c) the plane ρ from the bundle, which is orthogonal to π.

4. Determine the relative positions of the lines $x=-3t, y=2+3t, z=1, t\in\mathbb{R}$ and $x=1+5s, y=1+13s, z=1+10s, s\in\mathbb{R}$.

5. Let $A(1,2,-7)$, $B(2,2,-7)$ and $C(3,4,-5)$ be vertices of a triangle. Determine the equation of the internal angle bisector of $\measuredangle A$.

6. Determine the angles between the plane $\pi_1 : x-\sqrt{2}y+z-1=0$ and the plane $\pi_2 : x+\sqrt{2}y-z+3=0$.

7. Determine the parameter m for which the line $x=-1+3t, y=2+mt, z=-3-2t$ doesn't intersect the plane $x+3y+3z-2=0$.

8. Determine the values a and d for which the line $\frac{x-2}{3}=\frac{y+1}{2}=\frac{z-3}{-2}$ is contained in the plane $ax+y-2z+d=0$.

9. Determine the values a and c for which the line $3x-2y+z+3=0 \cap 4x-3y+4z+1=0$ is perpendicular to the plane $ax+8y+cz+2=0$.

10. Determine the orthogonal projection of the point $A(2,11,-5)$ on the plane $x+4y-3z+7=0$.

11. Determine the orthogonal reflection of the point $P(6,-5,5)$ in the plane $2x-3y+z-4=0$.

12. Consider the point $A(1,3,5)$ and the line $\ell : 2x+y+z-1=0 \cap 3x+y+2z-3=0$.

a) Determine the orthogonal projection of A on ℓ.

b) Determine the orthogonal reflection of A in ℓ.

13. Determine the planes which pass through $P(0,2,0)$ and $Q(-1,0,0)$ and which form an angle of $60°$ with the z-axis.

14. Determine the orthogonal projection of the line $\ell : 2x-y-1=0 \cap x+y-z+1=0$ on the plane $\pi : x+2y-z=0$.

15. Determine the coordinates of a point A on the line $\ell : \frac{x-1}{2}=\frac{y}{3}=\frac{z+1}{1}$ which is at distance $\sqrt{3}$ from the plane $x+y+z+3=0$.

16. The vertices of a tetrahedron are $A(-1,-3,1)$, $B(5,3,8)$, $C(-1,-3,5)$ and $D(2,1,-4)$. Determine the height of the tetrahedron relative to the face ABC.