Formule

- G – set
  operation on G } $(G, \cdot)$ – group if:

1. " $\cdot$ " is well defined (a stable part)
   $(\forall)\ x, y \in G$ s.t. $x \cdot y \in G$
2. " $\cdot$ " is associative
   $(\forall)\ x, y, z \in G$ s.t. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

} semigroup

3. " $\cdot$ " has neutral element
   $\exists!\ e \in G$ s.t. $x \cdot e = e \cdot x = x, (\forall)\ x \in G$

} monoid

4. inversibility
   $(\forall)\ x \in G, \exists\ x' \in G$ s.t. $x \cdot x' = x' \cdot x = e$

+5. commutative
   $(\forall)\ x, y \in G$ s.t. $x \cdot y = y \cdot x$ – for abelian group

Relations

- $\mathcal{R} = (A, B, R)$, A, B – sets
  - A – domain
    B – codomain
    R – graph ($\subseteq A \times B$)
- equivalence relation $\mathcal{R} = (A, A, R)$, so that:
  - reflexivity: $(\forall)\ x \in A$, $x \mathcal{R} x$ (in other words $(x, x) \in R$)
  - simetry: $(\forall)\ x, y \in A$, if $x \mathcal{R} y$ then $y \mathcal{R} x$
  - transitivity: $(\forall)\ x, y, z \in A$, if $x \mathcal{R} y$ and $y \mathcal{R} z$, then $x \mathcal{R} z$

* If $A = B$, the relation $\mathcal{R}$ is called homogeneous

Equivalence relations & partitions:

Let A – set

- $E(A)$ = the set of equivalences on A (r, s, t)
- $P(A)$ = the set of partitions of A

A homogeneous relation $r=(A,A,R)$ is called an equivalence relation if r has the prop. (r), (t), (s)

A/r = the quotient set $= \{ r\langle x\rangle / x \in A\}$
$r\langle x\rangle = \{y \in A / y\, r\, x\}$

• Subgroups:

- $(G,\cdot)$-group, $H \subseteq G$. H is a subgroup of G if:

i) $H \neq 0$

ii) $(\forall)\, x,y \in H:\ x\cdot y^{-1} \in H$

or

$(\forall)\, x,y \in H: \begin{cases} x\cdot y \in H \\ x^{-1} \in H \end{cases}$

the same!

Rings:

-$(A,+,\cdot)$ is a ring if:

i) $(A,+)$ - abelian group

ii) $(A,\cdot)$ - semigroup

iii) distributivity: $(\forall)\, x,y,z \in A:\ x\cdot(y+z) = xy + xz$

$(y+z)\cdot x = yx + zx$

! if $(A,\cdot)$ is a monoid, we have a ring with unity (unital ring)

! if $\cdot$ is commutative $\Rightarrow$ commutative ring

Def: If $(A,+,\cdot)$ is a ring, $B\subseteq A$. Then B is a subring of A is:

i) $B \neq 0$

ii) $\underset{\text{subgroup}}{(B,+) \leq (A,+)} \iff (\forall)\, x,y \in B$ s.t. $x-y \in B$

iii) $\underset{\text{sub semigroup}}{(B,\cdot) \leq (A,\cdot)} \iff (\forall)\, x,y \in B$ s.t. $x\cdot y \in B$

Def: $(G_1,*)$ and $(G_2,\#)$ - groups and let $f: G_1 \to G_2$. f is a group (homo)morphism if $(\forall)\, x,y \in G_1:\ f(x*y) = f(x) \# f(y)$

Def: $(A_1, \boxplus, \boxdot)$, $(A_2, \oplus, \odot)$ - rings. $f: A_1 \to A_2$ is ring (homo)morphism if:

$(\forall)\, x,y \in \boxed{A_1}:\ f(x \boxplus y) = f(x) \oplus f(y)$

$f(x \boxdot y) = f(x) \odot f(y)$

f - **ring isomorphism** if it is a **bijective ring homomorphism**

$\Rightarrow$ unital : $f(1_{A_1}) = 1_{A_2}$

Vector spaces :

Def : $(\underset{\text{"vectors"}}{V},+)$ - abelian group

$(K,+,\cdot)$ - field (corp comutativ)

"scalars"

$\cdot : K \times V$

V is **a vector space over** K (K-vector space, K^{v}) if :

- $(\forall)\ \alpha, \beta \in K, (\forall) v \in V : (\alpha+\beta)\cdot v = \alpha\cdot v + \beta \cdot v$
- $(\forall)\ \alpha \in K, (\forall) w \in V : \alpha\cdot(v+w) = \alpha\cdot v + \alpha\cdot w$
- $(\forall)\ \alpha, \beta \in K, (\forall) v \in V : (\alpha\cdot\beta)\cdot v = \alpha\cdot(\beta\cdot v)$
- $(\forall)\ v \in V : 1_K \cdot v = v$

Thm : If V is a K-v.s and $S \subseteq V$, then S is a **K-subspace** of V (denoted $S \leq_K V$) if:
or
$S \leq V$

i) $S \neq \emptyset$

ii) $(S,+) \leq (V,+) : (\forall) x, y \in S : x - y \in S$

iii) $(\forall)\ k \in K, (\forall) x \in S : kx \in S$

$\Updownarrow$!!

Thm : $S \leq V \iff \begin{cases} S \neq \emptyset \text{ (must contain } 0 \text{ } (0 \in S)) \\ (\forall)\ \alpha, \beta \in K, (\forall) v_1, v_2 \in S \text{ s.t. } \alpha v_1 + \beta v_2 \in S \end{cases}$

Def : V is K-v.s. , $X \leq V$, $\langle X \rangle = \bigcap\limits_{\substack{S \leq_K V \\ S \supseteq X}} S = \{ \sum\limits_{i=1}^{n} \alpha_i \cdot x_i \mid n \in \mathbb{N}, x_i \in X, \alpha_i \in K \}$ - **generated subspaces**

Def : V, W - K-v.s.

$f : V \to W$ is a **homomorphism** of K-vs (or a **K-linear map**) (denoted as $f \in$ **$\text{Hom}_K(V,W)$**) if:

i) $(\forall)\ v_1, v_2 \in V : f(v_1+v_2) = f(v_1) + f(v_2)$

ii) $(\forall)\ k \in K, (\forall) v \in V : f(k\cdot v) = k \cdot f(v)$

or (shorter) :

$\boxed{(\forall)\ v_1, v_2 \in V, (\forall)\ k_1, k_2 \in K : f(k_1 v_1 + k_2 v_2) = k_1 f(v_1) + k_2 \cdot f(v_2)}$

Def : V : K-v.s

$v_1, v_2, \dots v_n \in V$ are **linear independent** if $(\forall) \alpha_1, \dots \alpha_n \in K$ s.t. $\alpha_1 v_1 + \dots + \alpha_n v_n = 0$,
then $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$

Def: V: K-v.s

$v_1, v_2, \dots v_n \in V$ are **linear dependent** if $\exists\, \alpha_1, \dots \alpha_n \in K$ s.t. $\alpha_1 v_1 + \dots + \alpha_n v_n = 0$
(not all 0)

Def: V: K-v.s., $B = \{v_1, v_2, \dots v_n\}$

B - **basis for V** $\iff \begin{cases} \bullet\ \exists\ V = \langle v_1 \dots v_n \rangle \\ \bullet\ v_1, v_2, \dots v_n \text{ - linearly independent} \end{cases}$

or $\boxed{(\forall)\, v \in V, \exists!\, \alpha_1, \alpha_2 \dots \alpha_n \text{ such that: } v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n}$

If this is the case, we denote $[v]_B = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$

Def: V: K-v.s, $S, T \leq_K V$

$\left.\begin{array}{l} V = S+T \iff (\forall)\, v \in V, \exists\, s \in S, t \in T \text{ s.t. } v = s+t \\ V = S \oplus T \iff (\forall)\, v \in V, \exists!\, s \in S, t \in T \text{ s.t. } v = s+t \end{array}\right\}$ **sum of subspaces**

$V = S \oplus T \iff \begin{cases} V = S+T \\ S \cap T = 0 \end{cases}$

For every $S \leq_K V$, $\exists\, T \leq_K V$ s.t. $V = S \oplus T$

How do we find T?

Step 1: Find a basis $(v_1 \dots v_k)$ for S

Step 2: Complete this basis to a basis of V $(v_1, v_2, \dots v_k, v_{k+1}, \dots v_n)$

Step 3: $T = \langle v_{k+1}, \dots v_n \rangle$

Thm (1st dimension theorem):

$f: V \to V'$ - linear map

$\dim V = \underbrace{\dim(\ker f)}_{\text{def } f} + \underbrace{\dim(\operatorname{Im} f)}_{\text{rank}(f)}$

Thm (2nd - " -)

V: K-v.s, $S, T \leq_K V$

$\dim(S+T) = \dim(S) + \dim(T) - \dim(S \cap T)$

Kernel (ker f): $\ker f = \{v \in \mathbb{R}^{\boxed{3}} \mid f(v) = 0\}$

Image of f (Im f): $\operatorname{Im} f = \{f(\overbrace{x, y, z}) \mid x, y, z \in \mathbb{R}\}$

Other things:

* f - bijective $\Rightarrow f$ invertible $\Rightarrow \exists\, f^{-1}$
* $\det(i_n) = 1$, $(\forall)\, n \in \mathbb{N}^*$
* $A \cdot A^{-1} = A^{-1} \cdot A = i_n$, $A^{-1} = \frac{1}{\det A} \cdot A^*$

* sisteme:
  - incompatible – $\nexists$ solutions
  - compatible
    - det – unique solution (det $A \neq 0$, rank(A) = rank$(\bar{A})$ = nr de necunoscute)
    - nedet. – more than 1 solutions (det $A = 0$)