CHAPTER 1

Vectors and coordinate systems

Contents

We denote by $\mathbb{E}^2$ the Euclidean plane and by $\mathbb{E}^3$ the Euclidean space.

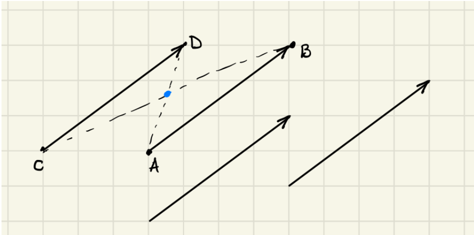
1.1 Geometric Vectors

Two points A and B in $\mathbb{E}^2$ or $\mathbb{E}^3$ can be assembled in an ordered pair (A,B). Such a pair contains the following geometric information:

- (distance) the distance from A to B,
- (direction) the direction from A to B,
- (location) the line segment $[A,B]$.

While the third bit of geometric information depends on the points A and B, the first two do not depend on these points. They can be abstracted with the following notion.

Definition. Two ordered pairs of points (A,B) and (C,D) are called *equipollent*, and we write $(A,B) \sim (C,D)$, if the line segments $[A,D]$ and $[B,C]$ have the same midpoints.



Proposition 1.1. For two ordered pairs of points (A,B) and (C,D) the following statements are equivalent:

1. $(A,B) \sim (C,D)$.
2. $ABDC$ is a parallelogram.
3. (A,B) and (C,D) have the same distance and direction.

Proposition 1.2. For any ordered pair of points (A,B) and any point O, there is a unique point X such that $(A,B) \sim (O,X)$.

Proposition 1.3. The equipollence relation is an equivalence relation.

Definition. The equivalence classes of the equipollence relation are called *vectors*. The vector containing the ordered pair (A,B) is denoted by $\overrightarrow{AB}$:

$$\overrightarrow{AB} = \{\text{ordered pairs } (X,Y) \text{ such that } (X,Y) \sim (A,B)\}$$

We say that $\overrightarrow{AB}$ *is represented by the pair* (A,B) or that (A,B) *is a representative of the vector* $\overrightarrow{AB}$. Notice that, by definition we have

- $\overrightarrow{AB} = \overrightarrow{CD}$ if and only if $(A,B) \sim (C,D)$, in particular all representatives of the vector $\overrightarrow{AB}$ define the *same* distance and the *same* direction, therefore
- if $\overrightarrow{AB} = \overrightarrow{CD}$ then $|AB| = |CD|$ and we define the *length* $|\overrightarrow{AB}|$ *of the vector* $\overrightarrow{AB}$ to be the distance $|AB|$ of the line segment $[A,B]$.

The set of all vectors for $\mathbb{E}^2$ and respectively for $\mathbb{E}^3$ are

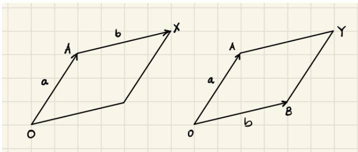
$$\mathbb{V}^2 = \{\overrightarrow{AB} : (A,B) \in \mathbb{E}^2 \times \mathbb{E}^2\} = \mathbb{E}^2 \times \mathbb{E}^2/\sim \quad \text{and} \quad \mathbb{V}^3 = \{\overrightarrow{AB} : (A,B) \in \mathbb{E}^3 \times \mathbb{E}^3\} = \mathbb{E}^3 \times \mathbb{E}^3/\sim .$$

Proposition 1.4. For any point O, the map ϕ_O defined by $\phi_O(A) = \overrightarrow{OA}$ is a bijection between points and vectors.

1.2 Vector space structure

Definition. Consider two vectors $\mathbf{a}$ and $\mathbf{b}$. If we fix a point O then, by Proposition 1.2, there is a unique point A such that $\mathbf{a} = \overrightarrow{OA}$ and for the point A there exists a unique point X such that $\mathbf{b} = \overrightarrow{AX}$. The *sum* of $\mathbf{a}$ and $\mathbf{b}$ is by definition the vector $\overrightarrow{OX}$ and we denote the sum by $\mathbf{a} + \mathbf{b}$.

Equivalently, for a fixed point O there are unique points A and B such that $\mathbf{a} = \overrightarrow{OA}$ and $\mathbf{b} = \overrightarrow{OB}$ and for the points O, A and B there is a unique point Y such that $OAYB$ is a parallelogram. It follows that $X = Y$ and therefore $\mathbf{a} + \mathbf{b} = \overrightarrow{OY} = \overrightarrow{OX}$.

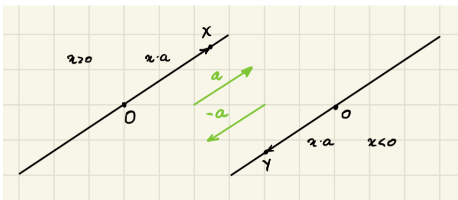


Proposition 1.5. The addition of vectors is well defined.

Proposition 1.6. The set of vectors $\mathbb{V}^2$ with addition is a commutative group. Similarly, $(\mathbb{V}^3, +)$ is a commutative group.

Definition. Consider a vector $\mathbf{a}$ and a scalar $x \in \mathbb{R}$. If we fix a point O then there is a unique point A such that $\mathbf{a} = \overrightarrow{OA}$. If $x > 0$ then there is a unique point X on the half-line $(OA$ such that $|OB| = x|OA|$. The *multiplication* of the vector $\mathbf{a}$ *by the scalar* x, denoted by $x \cdot \mathbf{a}$ (or simply by $x\mathbf{a}$), is by definition

$$x \cdot \mathbf{a} = \begin{cases} \overrightarrow{OX} & \text{for } \mathbf{a} \neq 0,\ x > 0 \text{ and } X \text{ as above,} \\ -(|x|\mathbf{a}) & \text{for } \mathbf{a} \neq 0,\ x < 0, \\ \overrightarrow{0} & \text{for } \mathbf{a} = 0 \text{ or } x = 0. \end{cases}$$



Proposition 1.7. The multiplication of scalars with vectors is well defined.

Proposition 1.8. For $\mathbf{a}, \mathbf{b} \in \mathbb{V}^3$ and $x, y \in \mathbb{R}$ we have

1. $(x + y) \cdot \mathbf{a} = x \cdot \mathbf{a} + y \cdot \mathbf{a}$
2. $x \cdot (\mathbf{a} + \mathbf{b}) = x \cdot \mathbf{a} + x \cdot \mathbf{b}$
3. $x \cdot (y \cdot \mathbf{a}) = (xy) \cdot \mathbf{a}$

4. $1 \cdot \mathbf{a} = \mathbf{a}$.

Theorem 1.9. The set of vectors $\mathbb{V}^2$ with vector addition and scalar multiplication is a vector space. Similarly, $(\mathbb{V}^3, +, \cdot)$ is a vector space.

Remark. The vector space structure consists in particular of two maps

$$+ : \mathbb{V}^2 \times \mathbb{V}^2 \to \mathbb{V}^2 \quad \text{and} \quad \cdot : \mathbb{R} \times \mathbb{V}^2 \to \mathbb{V}^2$$

(similarly for $\mathbb{V}^3$). With Proposition 1.4 we can define an 'addition' of vectors with points. For a vector $\mathbf{a}$ and a point O there is a unique point X such that $\mathbf{a} = \overrightarrow{OX}$, i.e. we have a map

$$+ : \mathbb{V}^2 \times \mathbb{E}^2 \to \mathbb{E}^2 \quad \text{given by} \quad \mathbf{a} + O = X.$$

This is an example of a group action. We say that the group $\mathbb{V}^2$ acts on the set of point $\mathbb{E}^2$ by *translations*.

Proposition 1.4 gives a way of identifying vectors with points. If O is a fixed point then we have a bijective map

$$\phi_O : \mathbb{E}^2 \to \mathbb{V}^2 \quad \text{given by} \quad \phi_O(A) = \overrightarrow{OA}.$$

(similarly for $\mathbb{V}^3$). With this map we can compare objects in $\mathbb{E}^2$ with objects in $\mathbb{V}^2$ and objects in $\mathbb{E}^3$ with objects in $\mathbb{V}^3$ as follows.

Theorem 1.10. Let S be a subset of $\mathbb{E}^2$ and let O be a point in S.

1. The set S is a line if and only if $\phi_O(S)$ is a 1-dimensional vector subspace of $\mathbb{V}^2$.
2. If S is a line then the vector subspace $\phi_O(S)$ is independent of the choice of O in S.
3. Two vectors $\overrightarrow{OA}, \overrightarrow{OB}$ are linearly dependent $\Leftrightarrow$ O, A, B are collinear.
4. $\dim \mathbb{V}^2 = 2$, i.e. $\mathbb{V}^2 \cong \mathbb{R}^2$.

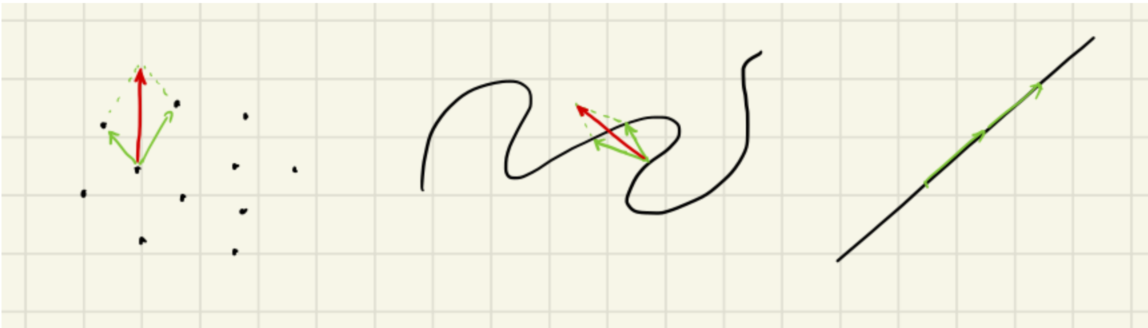
Similarly, let S be a subset of $\mathbb{E}^3$ and let O be a point in M.

5. The set S is a line if and only if $\phi_O(S)$ is a 1-dimensional vector subspace of $\mathbb{V}^3$.
6. If S is a line then the vector subspace $\phi_O(S)$ is independent of the choice of O in S.
7. Two vectors $\overrightarrow{OA}, \overrightarrow{OB}$ are linearly dependent $\Leftrightarrow$ O, A, B are collinear.
8. S is a plane if and only if $\phi_O(S)$ is a 2-dimensional vector subspace of $\mathbb{V}^3$.
9. If S is a plane then the vector subspace $\phi_O(S)$ is independent of the choice of O in S.
10. Three vectors $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$ are linearly dependent $\Leftrightarrow$ O, A, B, C are coplanar.
11. $\dim \mathbb{V}^3 = 3$, i.e. $\mathbb{V}^3 \cong \mathbb{R}^3$.

Remark. A different perspective on the above theorem is the following. Given a subset $S \subseteq \mathbb{E}^3$ we can consider the *set of vectors represented by pairs of points in* S, i.e. the set

$$V_S = \{\overrightarrow{AB} : A, B \in S\}.$$

The theorem says that S is a line if and only if V_S is a 1-dimensional vector subspace of $\mathbb{V}^3$ and S is a plane if and only if V_S is a 2-dimensional vector subspace of $\mathbb{V}^3$. A similar interpretation holds for $\mathbb{E}^2$ and $\mathbb{V}^2$.



1.3 Coordinate systems

Coordinates are a way of associating tuples of numbers to points in a space. Given a space S we want to have a subset $C \subseteq \mathbb{R}^n$ and a bijective map $C \to S$ which sets up a correspondence $(x_1, \dots, x_n) \leftrightarrow p$ between coordinates and points. In general there are many choices of such maps. The choice of such a map determines the control that you have over the object/space S. This in turn may depend on what you want to do with S. This semester we only consider Cartesian coordinate systems.

Fix two lines ℓ_1 and ℓ_2 in $\mathbb{E}^2$ which intersect in a (unique) point O. We can describe any point P in $\mathbb{E}^2$ as follows. By Theorem 1.10, $\phi_O(\ell_1)$ and $\phi_O(\ell_2)$ are 1-dimensional linearly independent vector subspaces of the two dimensional vector space $\mathbb{V}^2$. Thus, if we choose $\mathbf{i} \in \phi_O(\ell_1)$ and $\mathbf{j} \in \phi_O(\ell_2)$ two non-zero vectors, then $\{\mathbf{i}, \mathbf{j}\}$ is a basis of $\mathbb{V}^2$. Hence there are *unique* scalars x and y (in $\mathbb{R}$) such that

$$\phi_O(P) = x\mathbf{i} + y\mathbf{j}.$$

This gives a bijection between points P in $\mathbb{E}^2$ and pairs of real numbers (x, y) in $\mathbb{R}^2$. This bijection is the composition of two bijections:

1. the map ϕ_O giving the identification $\mathbb{E}^2 \cong \mathbb{V}^2$ and
2. the decomposition of vectors with respect to a basis giving the identification $\mathbb{V}^2 \cong \mathbb{R}^2$.

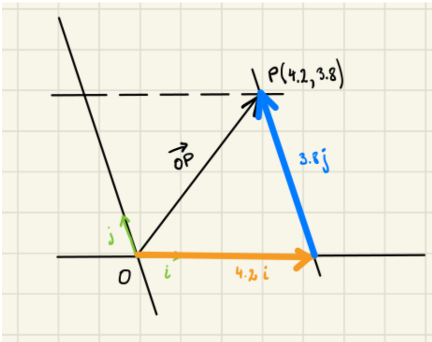
For this bijection we made two choices:

1. we chose the point O for the map ϕ_O and
2. we chose two vectors $\mathbf{i}$ and $\mathbf{j}$ which form a basis of $\mathbb{V}^2$.

Definition. A *coordinate system* for $\mathbb{E}^2$ is a triple $\mathcal{K} = (O, \mathbf{i}, \mathbf{j})$ where O is a point in $\mathbb{E}^2$ and $(\mathbf{i}, \mathbf{j})$ is a basis of $\mathbb{V}^2$. Given a coordinate system $\mathcal{K} = (O, \mathbf{i}, \mathbf{j})$, for any point $P \in \mathbb{E}^2$ there is a unique pair of scalars (x_P, y_P) such that

$$\overrightarrow{OP} = x_P \mathbf{i} + y_P \mathbf{j}.$$

The pair (x_P, y_P) is called *coordinates of P with respect to the coordinate system $\mathcal{K}$* and we write $P(x_P, y_P)$ when we want to indicate the coordinates. Most of the time, we use the letter x for the first coordinate and y for the second coordinate. We call ℓ_1 the x-axis and denote it by Ox. Similarly, ℓ_2 is the y-axis denoted by Oy. The point O is called the *origin* of the coordinate system $\mathcal{K}$.



Remark. When we fix a coordinate system $\mathcal{K} = (O, \mathbf{i}, \mathbf{j})$ with $\mathbf{i}$ and $\mathbf{j}$ of length 1 then we can interpret the coordinates of a point P as follows. For each such point P there is a unique parallelogram $OXPY$ with $X \in Ox$ and $Y \in Oy$ and the coordinates (x_P, y_P) of the point P with respect to $\mathcal{K}$ are the lengths of the sides of this parallelogram.

Similarly, if we fix three non-coplanar lines ℓ_1, ℓ_2 and ℓ_3 in $\mathbb{E}^3$ which intersect in a (unique) point O, we can describe any point P in $\mathbb{E}^3$ as follows. By Theorem 1.10, $\phi_O(\ell_1)$, $\phi_O(\ell_2)$ and $\phi_O(\ell_3)$ are 1-dimensional linearly independent vector subspaces of the two dimensional vector space $\mathbb{V}^3$. Thus, if we choose $\mathbf{i} \in \phi_O(\ell_1)$, $\mathbf{j} \in \phi_O(\ell_2)$ and $\mathbf{k} \in \phi_O(\ell_3)$ three non-zero vectors, then $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is a basis of $\mathbb{V}^3$. Hence there are *unique* scalars x, y and z (in $\mathbb{R}$) such that

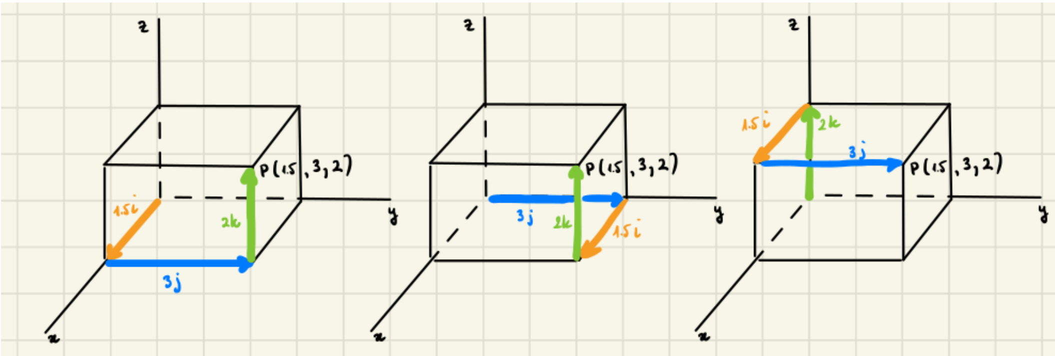
$$\phi_O(P) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

This gives a bijection between points P in $\mathbb{E}^3$ and triples of real numbers (x, y, z) in $\mathbb{R}^3$. As in the case of $\mathbb{E}^2$, this bijection is determined by the choice of O, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ and it is the composition of two bijections.

Definition. A *coordinate system* for $\mathbb{E}^3$ is a quadruple $\mathcal{K} = (O, \mathbf{i}, \mathbf{j}, \mathbf{k})$ where O is a point in $\mathbb{E}^3$ and $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ is a basis of $\mathbb{V}^3$. Given a coordinate system $\mathcal{K} = (O, \mathbf{i}, \mathbf{j}, \mathbf{k})$, for any point $P \in \mathbb{E}^2$ there is a unique triple of scalars (x_P, y_P, z_P) such that

$$\overrightarrow{OP} = x_P \mathbf{i} + y_P \mathbf{j} + z_p \mathbf{k}.$$

The triple (x_P, y_P, z_P) is called *coordinates of P with respect to the coordinate system $\mathcal{K}$* and we write $P(x_P, y_P, z_P)$ when we want to indicate the coordinates. Most of the time, we use the letter x for the first coordinate, y for the second coordinate and z for the third coordinate. We call ℓ_1 the x-axis and denote it by Ox. Similarly, ℓ_2 is the y-axis denoted by Oy and ℓ_3 is the z-axis denoted by Oz. The point O is called the *origin* of the coordinate system $\mathcal{K}$. The planes containing two coordinate axes are called *coordinate planes*. The plane containing Ox and Oy is denoted by Oxy and similarly for the other two.



Remark. When we fix a coordinate system $\mathcal{K} = (O, \mathbf{i}, \mathbf{j}, \mathbf{k})$ with $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ of length 1 then we can interpret the coordinates of a point P as follows. For each such point P there is a unique parallelepiped with vertices O, $X \in Ox$, $Y \in Oy$, $Z \in Oz$ such that P is opposite to O. Then the coordinates (x_P, y_P, z_P) of the point P with respect to $\mathcal{K}$ are the lengths of the sides of this parallelepiped.

Notation. We write points and vectors as column matrices when we work with their coordinates respectively with their components. For example, in dimension 2

$$[P]_{\mathcal{K}} = P(x_P, y_P) = \begin{bmatrix} x_P \\ y_P \end{bmatrix} \in \mathbb{E}^2 \quad \text{and} \quad [\mathbf{a}]_{\mathcal{K}} = a_x\mathbf{i} + a_y\mathbf{j} = \begin{bmatrix} a_x \\ a_y \end{bmatrix} \in \mathbb{V}^2.$$

With the subscript $\mathcal{K}$ we indicate the coordinate system with respect to which P has the indicated coordinates. For vectors, the subscript $\mathcal{K}$ indicates the basis $(\mathbf{i}, \mathbf{j})$ with respect to which the vector $\mathbf{a}$ has components a_x and a_y. Similarly, in dimension 3

$$[P]_{\mathcal{K}} = P(x_P, y_P, z_p) = \begin{bmatrix} x_P \\ y_P \\ z_P \end{bmatrix} \in \mathbb{E}^3 \quad \text{and} \quad [\mathbf{a}]_{\mathcal{K}} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \in \mathbb{V}^3.$$

1.4 Coordinates as projections

Let $\mathcal{K}$ be a coordinate system of $\mathbb{E}^2$. The correspondence between points and coordinates in the system $\mathcal{K}$

$$P \leftrightarrow \begin{bmatrix} x_P \\ y_P \end{bmatrix}_{\mathcal{K}}$$

automatically gives a map $\mathrm{Pr}_x : \mathbb{E}^2 \to Ox$ defined by $\mathrm{Pr}_x(P) = (x_P, 0)$ and a map $\mathrm{Pr}_y : \mathbb{E}^2 \to Oy$ defined by $\mathrm{Pr}_y(P) = (0, y_P)$. The map Pr_x is called *the projection on Ox along Oy* and Pr_y is called *the projection on Oy along Ox*.

If $\mathcal{B} = (\mathbf{i}, \mathbf{j})$ is a basis for $\mathbb{V}^2$ then for vectors, the correspondence

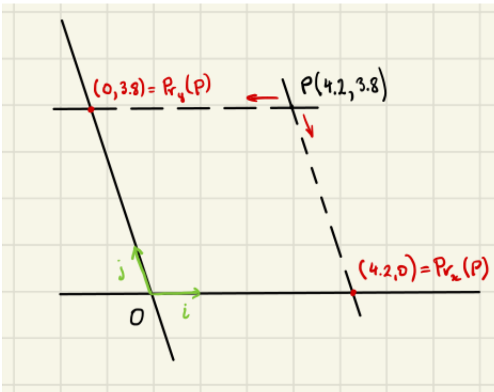
$$\mathbf{a} \leftrightarrow \begin{bmatrix} a_x \\ a_y \end{bmatrix}_{\mathcal{B}}$$

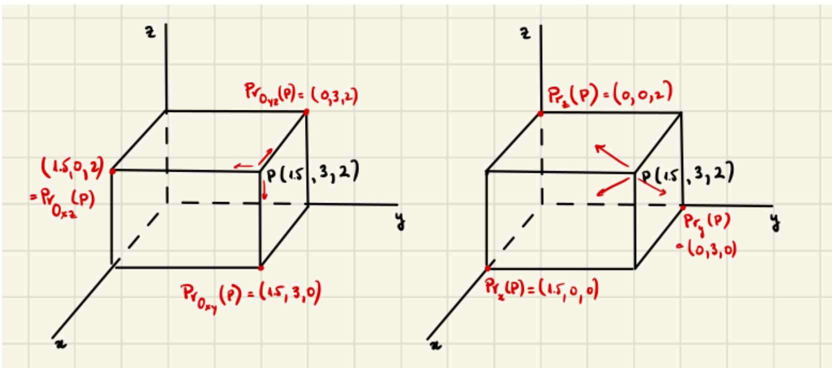
gives similar maps $\mathrm{pr}_x : \mathbb{V}^2 \to \mathbb{R}$ defined by $\mathrm{pr}_x(\mathbf{a}) = a_x$ and $\mathrm{pr}_y : \mathbb{V}^2 \to \mathbb{R}$ defined by $\mathrm{pr}_y(\mathbf{a}) = a_y$. These maps are related by

$$\overrightarrow{O\mathrm{Pr}_x(P)} = \mathrm{pr}_x(\overrightarrow{OP})\mathbf{i}, \quad \overrightarrow{O\mathrm{Pr}_y(P)} = \mathrm{pr}_y(\overrightarrow{OP})\mathbf{j} \quad \text{and by} \quad \overrightarrow{OP} = \mathrm{pr}_x(\overrightarrow{OP})\mathbf{i} + \mathrm{pr}_y(\overrightarrow{OP})\mathbf{j}.$$

It follows that

$$[P]_{\mathcal{K}} = P(x_P, y_P) = \begin{bmatrix} x_P \\ y_P \end{bmatrix} = \begin{bmatrix} \mathrm{pr}_x(\overrightarrow{OP}) \\ \mathrm{pr}_y(\overrightarrow{OP}) \end{bmatrix}.$$





Similarly, in dimension 3 we have maps defined by $\mathrm{Pr}_x(P) = (x_P, 0, 0)$, $\mathrm{Pr}_y(P) = (0, y_p, 0)$ and $\mathrm{Pr}_z(P) = (0, 0, z_P)$. The map Pr_x is called *the projection on Ox along Oyz* and similarly for the other two. These are projections on coordinate axes along coordinate planes. For vectors in $\mathbb{V}^3$ we have maps $\mathrm{pr}_x, \mathrm{pr}_y, \mathrm{pr}_z : \mathbb{V}^3 \to \mathbb{R}$ defined by $\mathrm{pr}_x(\mathbf{a}) = a_x$, $\mathrm{pr}_y(\mathbf{a}) = a_y$, $\mathrm{pr}_z(\mathbf{a}) = a_z$. They are related by

$$\overrightarrow{O\mathrm{Pr}_x(P)} = \mathrm{pr}_x(\overrightarrow{OP})\mathbf{i}, \quad \overrightarrow{O\mathrm{Pr}_y(P)} = \mathrm{pr}_y(\overrightarrow{OP})\mathbf{j} \quad \text{and} \quad \overrightarrow{O\mathrm{Pr}_z(P)} = \mathrm{pr}_z(\overrightarrow{OP})\mathbf{k}$$

and by

$$\overrightarrow{OP} = \mathrm{pr}_x(\overrightarrow{OP})\mathbf{i} + \mathrm{pr}_y(\overrightarrow{OP})\mathbf{j} + \mathrm{pr}_z(\overrightarrow{OP})\mathbf{k}.$$

It follows that

$$[P]_{\mathcal{K}} = P(x_P, y_p, z_P) = \begin{bmatrix} x_P \\ y_P \\ z_p \end{bmatrix} = \begin{bmatrix} \mathrm{pr}_x(\overrightarrow{OP}) \\ \mathrm{pr}_y(\overrightarrow{OP}) \\ \mathrm{pr}_z(\overrightarrow{OP}) \end{bmatrix}.$$

1.5 Changing coordinate systems

With a fixed coordinate system $\mathcal{K}$ of $\mathbb{E}^2$, the correspondence

$$P \leftrightarrow \begin{bmatrix} x_P \\ y_p \end{bmatrix}$$

identifies points with tuples of real numbers. A pair of numbers (x_P, y_P) has no geometric meaning in the absence of $\mathcal{K}$. Moreover, in a different coordinate system $\mathcal{K}'$ the point P will correspond to some other pair (x'_p, y'_P)

$$[P]_{\mathcal{K}} = \begin{bmatrix} x_P \\ y_p \end{bmatrix} \quad \text{and} \quad [P]_{\mathcal{K}'} = \begin{bmatrix} x'_P \\ y'_p \end{bmatrix}.$$

So when there is more than one coordinate system around, we need to understand how to translate the coordinates from one coordinate system to the other coordinate system. We will look at the details of how this is done later during the course.

CHAPTER 2

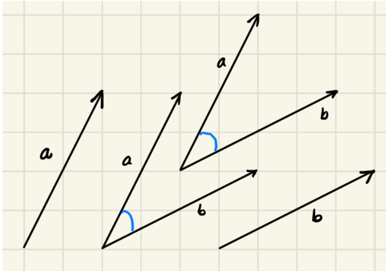
Scalar product and vector product

Contents

We denote by $\mathbb{E}^n$ the n-dimensional Euclidean space and by $\mathbb{V}^n$ the vector space associated to $\mathbb{E}^n$. For us $n = 1, 2, 3$.

2.1 Scalar product in $\mathbb{E}^n$

Definition. Let $\mathbf{a}, \mathbf{b} \in \mathbb{V}^n$ be two non-zero distinct vectors. Given a point $O \in \mathbb{E}^n$, there are unique points $A, B \in \mathbb{E}^n$ such that $\mathbf{a} = \overrightarrow{OA}$ and $\mathbf{b} = \overrightarrow{OB}$. The *angle* between $\mathbf{a}$ and $\mathbf{b}$ is the angle $\measuredangle AOB$ and we denote it by $\measuredangle(\mathbf{a}, \mathbf{b})$.



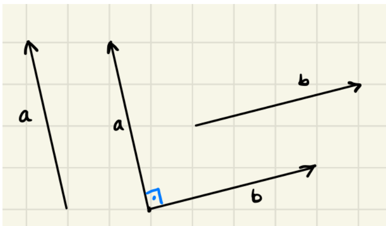
- The angle between two vectors is well defined: it does not depend on the choice of representatives of the vectors, it is a property of the two equipollence classes $\mathbf{a}$ and $\mathbf{b}$.

Definition. The *scalar product* (or, *dot product*) of two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{V}^n$ is the real number

$$\mathbf{a} \cdot \mathbf{b} = \begin{cases} 0, & \text{if one of the two vector is zero} \\ ||\mathbf{a}|| \cdot ||\mathbf{b}|| \cdot \cos\measuredangle(\mathbf{a}, \mathbf{b}) & \text{if both vectors are non-zero.} \end{cases}$$

- The scalar product is a map $\mathbb{V}^n \times \mathbb{V}^n \to \mathbb{R}$.
- Notice that $\mathbf{a}^2$ denotes $\mathbf{a} \cdot \mathbf{a}$ which by definition equals $||\mathbf{a}||^2$.
- For any vector $\mathbf{a} \in \mathbb{V}^n$ the *unit vector corresponding to* $\mathbf{a}$ is $\frac{1}{||\mathbf{a}||}\mathbf{a} = \frac{1}{\sqrt{\mathbf{a}\cdot\mathbf{a}}}\mathbf{a}$.
- When we divide a vector by its norm we say that we *normalize* the vector.
- A vector $\mathbf{a}$ is called a *unit vector* if $||\mathbf{a}|| = 1$.
- By normalizing a vector we obtain a unit vector with the same direction as the initial vector.

Definition. Let $\mathbf{a}, \mathbf{b} \in \mathbb{V}^n$ be two vectors. We say that $\mathbf{b}$ is *orthogonal to* $\mathbf{a}$ if $\mathbf{b} \cdot \mathbf{a} = 0$. When this is the case, we say that the two vectors are orthogonal to each other and we write $\mathbf{b} \perp \mathbf{a}$.



Definition. For $\mathbf{v} \in \mathbb{V}^n$, let $\mathbf{v}^\perp$ be the set of all vectors which are orthogonal to $\mathbf{v}$. One can show that if $\mathbf{v}$ is non-zero then $\mathbf{v}^\perp$ is an $(n-1)$-dimensional vector subspace of $\mathbb{V}^n$ and that $\mathbb{V}^n = \mathbf{v}^\perp \oplus \langle \mathbf{v} \rangle$. In particular, $\mathbb{V}^n$ has a basis $\mathbf{v}, \mathbf{v}_1, \dots, \mathbf{v}_{n-1}$ with $\mathbf{v}_1, \dots, \mathbf{v}_{n-1} \in \mathbf{v}^\perp$. Thus, any vector $\mathbf{w} \in \mathbb{V}^n$ has a unique decomposition

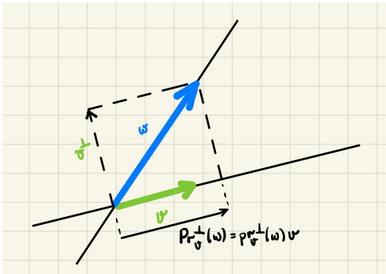
$$\mathbf{w} = w_0\mathbf{v} + w_1\mathbf{v}_1 + w_2\mathbf{v}_2 + \cdots + w_n\mathbf{v}_n.$$

The map $\mathrm{Pr}_{\mathbf{v}}^\perp : \mathbb{V}^n \to \langle \mathbf{v} \rangle$ defined by $\mathrm{Pr}_{\mathbf{v}}^\perp(\mathbf{w}) = w_0\mathbf{v}$ is the *orthogonal projection on* $\langle \mathbf{v} \rangle$. The map $\mathrm{pr}_{\mathbf{v}}^\perp : \mathbb{V}^n \to \mathbb{R}$ defined by $\mathrm{pr}_{\mathbf{v}}^\perp(\mathbf{w}) = w_0$ is the *orthogonal projection on the direction of* $\mathbf{v}$.

- The two maps defined above are very similar. They are linked by the following relation

$$\mathrm{Pr}_{\mathbf{v}}^{\perp}(\mathbf{w}) = \mathrm{pr}_{\mathbf{v}}^{\perp}(\mathbf{w})\mathbf{v}.$$

  So, the first one maps vectors to vectors, the second one maps vectors to scalars.



Proposition 2.1. Let $\mathbf{a}, \mathbf{b} \in \mathbb{V}^n$ be two non-zero vectors and let $\mathbf{a}_1 = \frac{1}{\|\mathbf{a}\|}$ and $\mathbf{b}_1 = \frac{1}{\|\mathbf{b}\|}\mathbf{b}$. Then

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \cdot \mathrm{pr}_{\mathbf{a}_1}^{\perp} \mathbf{b} = \|\mathbf{b}\| \cdot \mathrm{pr}_{\mathbf{b}_1}^{\perp} \mathbf{a}.$$

Proposition 2.2. For a non-zero vector $\mathbf{v} \in \mathbb{V}^n$, the maps $\mathrm{Pr}_{\mathbf{v}}^{\perp} : \mathbb{V}^n \to \langle \mathbf{v} \rangle$ and $\mathrm{pr}_{\mathbf{v}}^{\perp} : \mathbb{V}^n \to \mathbb{R}$ are linear.

Proposition 2.3. For any $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{V}^n$ and any $\lambda \in \mathbb{R}$ we have

1. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$,
2. $(\lambda\mathbf{a}) \cdot \mathbf{b} = \lambda(\mathbf{a} \cdot \mathbf{b})$,
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$,
4. $\mathbf{a} \cdot \mathbf{a} \geq 0$,
5. $\mathbf{a} \cdot \mathbf{a} = 0 \Leftrightarrow \mathbf{a} = 0$.

Definition. A basis $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ of $\mathbb{V}^n$ is called *orthogonal* if $\mathbf{e}_i \perp \mathbf{e}_j$ for $i \neq j$. A basis $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ is called *orthonormal* if it is orthogonal and all $\mathbf{e}_i$ are unit vectors.

A coordinate system $(O, \mathbf{e}_1, \ldots, \mathbf{e}_n)$ is called *orthogonal* or *orthonormal* if the basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is orthogonal or respectively orthonormal.

2.1.1 Scalar product in an orthonormal coordinate system

Proposition 2.4. Consider two vectors $\mathbf{a}(a_1, a_2, \ldots, a_n), \mathbf{b}(b_1, b_2, \ldots, b_n) \in \mathbb{V}^n$ with components relative to an *orthonormal basis*. Their scalar product is

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n. \tag{2.1}$$

- For an orthonormal basis $(\mathbf{i},\mathbf{j},\mathbf{k})$ of $\mathbb{V}^3$ and two vectors $\mathbf{a}(a_1,a_3,a_3),\mathbf{b}(b_1,b_2,b_3)\in\mathbb{V}^3$, we have

$$||\mathbf{a}|| = \sqrt{a_1^2+a_2^2+a_3^2} \tag{2.2}$$

$$\cos\measuredangle(\mathbf{a},\mathbf{b}) = \frac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}} \tag{2.3}$$

$$\mathbf{a}\perp\mathbf{b} \Leftrightarrow a_1b_1+a_2b_2+a_3b_3=0 \tag{2.4}$$

- In particular, for two points $A(A_1,A_2,A_3)$ and $B(B_1,B_2,B_3)$ we have

$$|AB| = \sqrt{(B_1-A_1)^2+(B_2-A_2)^2+(B_3-A_3)^2} = ||\overrightarrow{AB}||.$$

- For dimension 2, we have similar formulas with respect to an orthonormal basis of $\mathbb{V}^2$.
- In your Algebra 1 course you encountered the notion of a scalar product. You defined it as in (2.1). When the scalar product is defined like that, the basis with respect to which it is defined is orthonormal.

2.1.2 Gram-Schmidt process

It is possible to express different formulas as above for the norm of a vector or the cosine of an angle in a coordinate system which is *not* orthonormal. However, the simplest expressions for these formulas are obtained for orthonormal bases, which is why everybody uses them.

If we have a basis $\mathbf{e}_1,\mathbf{e}_2,\dots,\mathbf{e}_n$ of $\mathbb{V}^n$, how do we obtain an orthonormal basis? We can do this in two steps:

1. Construct an orthogonal basis $\mathbf{e}_1',\mathbf{e}_2',\dots,\mathbf{e}_n'$ as follows

$$\begin{aligned}
\mathbf{e}_1' &= \mathbf{e}_1\\
\mathbf{e}_2' &= \mathbf{e}_2 - \mathrm{Pr}^{\perp}_{\mathbf{e}_1'}(\mathbf{e}_2)\\
\mathbf{e}_3' &= \mathbf{e}_3 - \mathrm{Pr}^{\perp}_{\mathbf{e}_1'}(\mathbf{e}_3) - \mathrm{Pr}^{\perp}_{\mathbf{e}_2'}(\mathbf{e}_3)\\
&\vdots
\end{aligned}$$

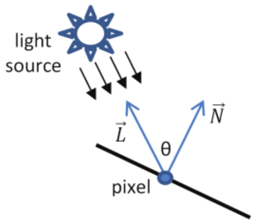
2. Normalize the vectors to obtain the basis

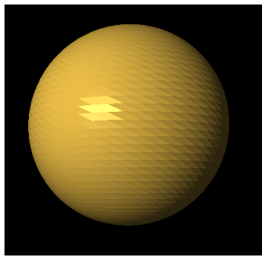
$$\left\{\frac{1}{||\mathbf{e}_1'||}\mathbf{e}_1',\dots,\frac{1}{||\mathbf{e}_n'||}\mathbf{e}_n'\right\}.$$

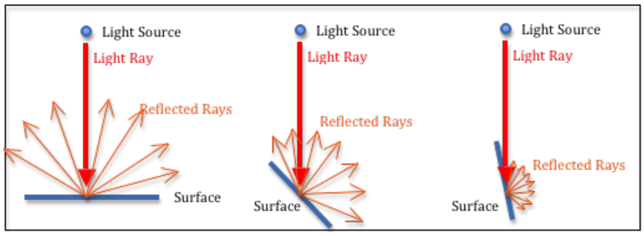
This process of obtaining an orthonormal basis from a given basis is the *Gram-Schmidt process*.

2.1.3 Applications

- [Lambert's cosine law] This is a law in optics, published by Johan Heinrich Lambert in 1760. It is used in computer graphics to model diffuse light. The law says that the luminous intensity observed from an ideal diffusely reflecting surface is directly proportional to the cosine of the angle θ between the direction of the incident light and the surface normal.





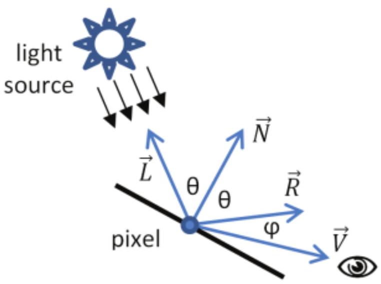


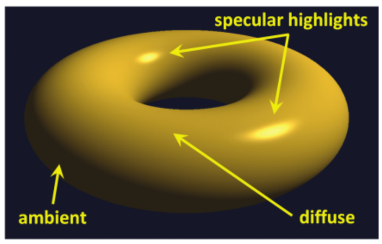
We will discuss normal vectors later during the course.

- [Specular light] The orthogonal reflection of a vector **b** in the vector **a** is

$$\mathrm{Ref}^{\perp}_{\mathbf{a}}(\mathbf{b}) = -\mathbf{b} + 2\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}$$

In computer graphics specular light is modeled per pixel/primitiv as follows. One calculates the vector $\overrightarrow{L}$ to the light source and the vector $\overrightarrow{V}$ to the camera. The normal $\overrightarrow{N}$ to the surface is given. Using the above formula one calculates the vector $\overrightarrow{R}$ which gives the direction in which the light is reflected. The intensity of the pixel is then proportional to $\cos(\varphi)^n$ where n is some constant which depends on the surface.





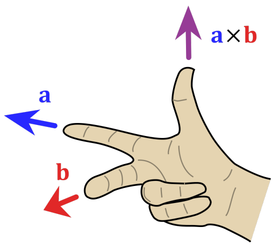
We will discuss normal vectors later during the course.

2.2 Vector product in $\mathbb{E}^3$

Definition. Let **u**, **v** be a basis of $\mathbb{V}^2$. If $O \in \mathbb{E}^2$ is any point, then there are unique points $X, Y \in \mathbb{E}^2$ such that $\mathbf{u} = \overrightarrow{OX}$ and $\mathbf{v} = \overrightarrow{OY}$. Rotate the plane such that **u** points downwards. If Y is in the right half-plane determined by the line OX, then we say that the basis $\mathbf{u}, \mathbf{v}$ is *right oriented*. If Y lies in the left half-plane we say that the basis $\mathbf{u}, \mathbf{v}$ is *left oriented*. A coordinate system $(O, \mathbf{u}, \mathbf{v})$ is left or right oriented if the basis $\mathbf{u}, \mathbf{v}$ is left respectively right oriented.

Let $\mathbf{u} = \overrightarrow{OX}, \mathbf{v} = \overrightarrow{OY}, \mathbf{w} = \overrightarrow{OZ}$ be a basis of $\mathbb{V}^3$. We say that the basis is *right oriented* if $\mathbf{u}$, $\mathbf{v}$ is a right oriented basis when observed from the point Z. We say that the basis is *left oriented* if $\mathbf{u}$, $\mathbf{v}$ is a left oriented basis when observed from the point Z. A coordinate system $(O, \mathbf{u}, \mathbf{v}, \mathbf{w})$ is left or right oriented if the basis $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is left respectively right oriented.

- There are many equivalent ways of deciding if a basis is left or right oriented. The Swiss liked the three-finger rule so much, they put it on their 200-franc banknotes:



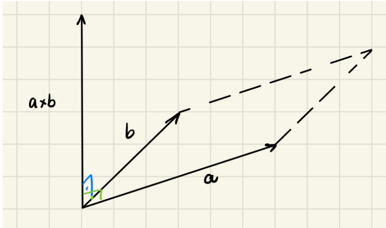
- A numerical way of checking this is by calculating a determinant (see Box product section).

Definition. Let $\mathbf{a}, \mathbf{b} \in \mathbb{V}^3$ be two vectors. The *vector product* (or *cross product*) of $\mathbf{a}$ and $\mathbf{b}$ is the vector denoted by $\mathbf{a} \times \mathbf{b}$ and defined by the following properties:

1. if $\mathbf{a}$ and $\mathbf{b}$ are collinear then $\mathbf{a} \times \mathbf{b} = 0$.
2. if $\mathbf{a}$ and $\mathbf{b}$ are not collinear, then
   (a) $||\mathbf{a} \times \mathbf{b}|| = ||\mathbf{a}|| \cdot ||\mathbf{b}|| \sin \measuredangle(\mathbf{a}, \mathbf{b})$,
   (b) $\mathbf{a} \times \mathbf{b} \perp \mathbf{a}$ and $\mathbf{a} \times \mathbf{b} \perp \mathbf{b}$,
   (c) $(\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b})$ is a right oriented basis of $\mathbb{V}^3$.

Notice that

- The vector product is a map $\mathbb{V}^3 \times \mathbb{V}^3 \to \mathbb{V}^3$, $(\mathbf{a}, \mathbf{b}) \mapsto \mathbf{a} \times \mathbf{b}$.
- The formula in (a) is valid also when $\mathbf{a} \parallel \mathbf{b}$.
- The angle $\measuredangle(\mathbf{a}, \mathbf{b})$ lies between 0 and π, so $\sin \measuredangle(\mathbf{a}, \mathbf{b}) \geq 0$.



- The norm $\|\mathbf{a}\times\mathbf{b}\|$ equals the area of the parallelogram spanned by the two vectors.
- We make the convention that the zero vector $0\in\mathbb{V}^3$ is collinear with any other vector (as in linear algebra where a set containing the zero vector is linearly dependent).
- Two vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if $\mathbf{a}\times\mathbf{b}=0$.

Proposition 2.5. For any $\mathbf{a},\mathbf{b}\in\mathbb{V}^3$ and any $\lambda\in\mathbb{R}$ we have

1. $\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}$.
2. $(\lambda\mathbf{a}\times\mathbf{b})=\mathbf{a}\times(\lambda\mathbf{b})=\lambda(\mathbf{a}\times\mathbf{b})$.
3. $\mathbf{a}\times(\mathbf{b}+\mathbf{c})=\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$ and $(\mathbf{a}+\mathbf{b})\times\mathbf{c}=\mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{c}$.

2.2.1 Vector product in a right oriented orthonormal coordinate system

Let $Oxyz=(O,\mathbf{i},\mathbf{j},\mathbf{k})$ be a right oriented orthonormal reference frame (coordinate system). The values of the vector product on the basis $\mathbf{i},\mathbf{j},\mathbf{k}$ of $\mathbb{V}^3$ are

$\times$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	0	$\mathbf{k}$	$-\mathbf{j}$
$\mathbf{j}$	$-\mathbf{k}$	0	$\mathbf{i}$
$\mathbf{k}$	$\mathbf{j}$	$-\mathbf{i}$	0

Given two vectors $\mathbf{a}(a_1,a_2,a_3)$ and $\mathbf{b}(b_1,b_2,b_3)$ in $\mathbb{V}^3$ we can calculate $\mathbf{a}\times\mathbf{b}$ in terms of the components of $\mathbf{a}$ and $\mathbf{b}$ relative to $Oxyz$:

$$\mathbf{a}\times\mathbf{b}=(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\times(b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k})=(a_2b_3-a_3b_2)\mathbf{i}+(-a_1b_3+a_3b_1)\mathbf{j}+(a_1b_2-a_2b_1)\mathbf{k}.$$

This can be arranged as a determinant:

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\a_1&a_2&a_3\\b_1&b_2&b_3\end{vmatrix}=\begin{vmatrix}a_2&a_3\\b_2&b_3\end{vmatrix}\mathbf{i}-\begin{vmatrix}a_1&a_3\\b_1&b_3\end{vmatrix}\mathbf{j}+\begin{vmatrix}a_1&a_2\\b_1&b_2\end{vmatrix}\mathbf{k}.$$

Therefore, the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$ has area

$$\mathrm{Area}_{\mathrm{Parall}}(\mathbf{a},\mathbf{b})=\|\mathbf{a}\times\mathbf{b}\|=\sqrt{\begin{vmatrix}a_2&a_3\\b_2&b_3\end{vmatrix}^2+\begin{vmatrix}a_1&a_3\\b_1&b_3\end{vmatrix}^2+\begin{vmatrix}a_1&a_2\\b_1&b_2\end{vmatrix}^2}$$

and the triangle ABC where $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AC} = \mathbf{b}$ has area

$$\text{Area}_{\triangle}(\mathbf{a},\mathbf{b}) = \frac{1}{2}\|\mathbf{a}\times\mathbf{b}\| = \frac{1}{2}\sqrt{\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}^2}.$$

If the three points lie in the Oxy plane, i.e. $A(x_A, y_A, 0)$, $B(x_B, y_B, 0)$, $C(x_C, y_C, 0)$ then

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_B - x_A & y_B - y_A & 0 \\ x_C - x_A & y_C - y_A & 0 \end{vmatrix} = \begin{vmatrix} x_B - x_A & y_B - y_A \\ x_C - x_A & y_C - y_A \end{vmatrix}\mathbf{k} = \begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix}\mathbf{k}.$$

Thus

$$\|\mathbf{a}\times\mathbf{b}\| = \pm\begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix} \quad \text{and} \quad \text{Area}_{\triangle}(\mathbf{a},\mathbf{b}) = \pm\frac{1}{2}\begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix}$$

where the sign '$\pm$' is chosen such that the value is positive. The value of the determinant could be negative. Without taking absolute value, we call the above determinant the *oriented area* of the triangle (or parallelogram).

This also gives a criterion for the collinearity of the points $A, B, C \in Oxy$: they are collinear if the above determinant is zero, i.e. if the area of the triangle ABC is zero.

2.2.2 Grassmann's vector product formula and Iacobi identity

The vector product is not associative. If $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ is a right oriented orthonormal basis then

$$(\mathbf{i}\times\mathbf{j})\times\mathbf{j} = \mathbf{k}\times\mathbf{j} = -\mathbf{i} \quad \text{and} \quad \mathbf{i}\times(\mathbf{j}\times\mathbf{j}) = 0.$$

Definition. In order to fix ideas, we call $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$ the *double vector product* of the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{V}^3$.

Theorem 2.6 (Grassmann's vector product formula)**.** For $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{V}^3$ we have

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}.$$

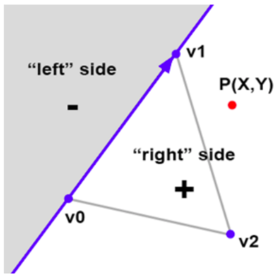
- The above theorem has computational value: instead of calculating two determinants we may calculate two scalar products.
- The property that makes up for the lack of associativity is the Jacobi identity.

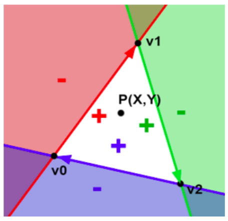
Proposition 2.7 (Jacobi identity)**.** For $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{V}^3$ we have

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} + (\mathbf{b}\times\mathbf{c})\times\mathbf{a} + (\mathbf{c}\times\mathbf{a})\times\mathbf{b} = 0.$$

2.2.3 Applications

- [Edge function] In computer graphics one often has to decide if a point lies inside a triangle/primitive or not.





In such cases one uses a so-called *edge function*. The oriented area of a triangle gives a way of constructing such an edge function.

Suppose you have a triangle with vertices $v_0(x_0, y_0)$, $v_1(x_1, y_1)$ und $v_2(x_2, y_2)$. A point $P(x_P, y_P)$ lies inside the triangle if and only if

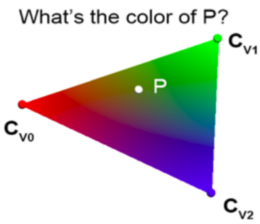
$$\begin{cases} A_{01}(x_P, y_P)A_{01}(x_2, y_2) > 0 \\ A_{12}(x_P, y_P)A_{12}(x_0, y_0) > 0 \\ A_{20}(x_P, y_P)A_{20}(x_1, y_1) > 0 \end{cases}$$

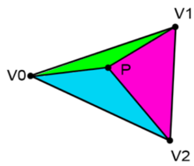
where $A_{01}(x_P, y_P)$= 2 times the oriented area of the triangle v_0v_1P, i.e.

$$A_{ij}(x,y) = \begin{vmatrix} x & y & 1 \\ x_i & y_i & 1 \\ x_j & y_j & 1 \end{vmatrix} = \begin{vmatrix} x - x_i & y - y_i & 0 \\ x_i & y_i & 1 \\ x_j - x_i & y_j - y_i & 0 \end{vmatrix} = \begin{vmatrix} x - x_i & y - y_i \\ x_j - x_i & y_j - y_i \end{vmatrix}.$$

Can you find a more efficient edge function?

- [Barycentric coordinates] In computer graphics one uses the vertices of a triangle/primitive to deduce attributes of the points inside the triangle/primitive. By attributes we mean color, normal vectors, texture coordinates, etc.





Definition. Consider a triangle ABC and a point P in the plane of the triangle. There exist unique scalars $\lambda_0, \lambda_1, \lambda_2$ such that $\lambda_0 + \lambda_1 + \lambda_2 = 1$ and

$$P = \lambda_0 A + \lambda_1 B + \lambda_2 C.$$

You read the above equality as follows: for any point O we have $\overrightarrow{OP} = \lambda_0 \overrightarrow{OA} + \lambda_1 \overrightarrow{OB} + \lambda_2 \overrightarrow{OC}$. These scalars are called barycentric coordinates of the point P.

If F = 2×area of the triangle $V_0V_1V_2$ (Fig.) then we have

$$P = \lambda_0 V_0 + \lambda_1 V_1 + \lambda_2 V_2$$

where

$$\lambda_0 = \frac{A_{12}(P)}{F}, \quad \lambda_1 = \frac{A_{20}(P)}{F}, \quad \lambda_2 = \frac{A_{01}(P)}{F} \quad \text{with } A_{ij} \text{ as before.}$$

Now, if you have attributes for the vertices of the triangle encoded as numerical values A_0, A_1, A_2, then one calculates the corresponding attribute for the point P like this:

$$A_P = \lambda_0 A_0 + \lambda_1 A_1 + \lambda_2 A_2.$$

2.3 Box product in $\mathbb{E}^3$

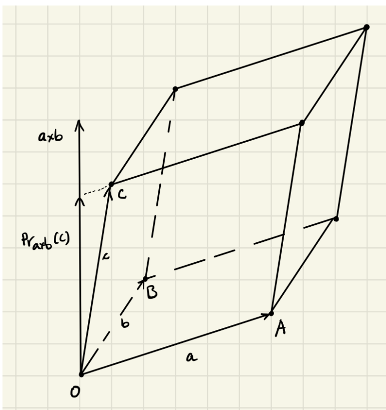
Definition. The *box product* of three vectors $\mathbf{a},\mathbf{b},\mathbf{c} \in \mathbb{V}^3$ is $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$ and we denote it by $[\mathbf{a},\mathbf{b},\mathbf{c}]$.

- Notice that the box product is a map

$$[\_,\_,\_] : \mathbb{V}\times\mathbb{V}\times\mathbb{V} \to \mathbb{R} \quad (\mathbf{a},\mathbf{b},\mathbf{c}) \mapsto [\mathbf{a},\mathbf{b},\mathbf{c}] = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}.$$

- The geometric interpretation of this product is the following.

Theorem 2.8. Let $\mathbf{a} = \overrightarrow{OA}, \mathbf{b} = \overrightarrow{OB}, \mathbf{c} = \overrightarrow{OC} \in \mathbb{V}^3$ be non-collinear vectors and let $\mathcal{P}$ be the parallelepiped spanned by the three vectors (i.e. $[OA]$, $[OB]$ and $[OC]$ are sides of the parallelepiped $\mathcal{P}$). Then $[\mathbf{a},\mathbf{b},\mathbf{c}]$ is the volume of $\mathcal{P}$ if $(\mathbf{a},\mathbf{b},\mathbf{c})$ is right oriented; it is minus the volume of $\mathcal{P}$ if $(\mathbf{a},\mathbf{b},\mathbf{c})$ is left oriented.



- It follows that the volume of the tetrahedron $OABC$ ist

$$\mathrm{Vol}_{OABC} = \left|\frac{1}{6}[\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}]\right|.$$

- The basis $(\mathbf{a},\mathbf{b},\mathbf{c})$ is right oriented if and only if $[\mathbf{a},\mathbf{b},\mathbf{c}] > 0$. It is left oriented if and only if $[\mathbf{a},\mathbf{b},\mathbf{c}] < 0$.
- In particular, an orthonormal basis $(\mathbf{a},\mathbf{b},\mathbf{c})$ is right oriented if and only if $[\mathbf{a},\mathbf{b},\mathbf{c}] = 1$. It is left oriented if and only if $[\mathbf{a},\mathbf{b},\mathbf{c}] = -1$.
- Three vectors $\mathbf{a},\mathbf{b},\mathbf{c}$ are coplanar if and only if $[\mathbf{a},\mathbf{b},\mathbf{c}] = 0$.

2.3.1 Box product in an orthonormal coordinate system

Let $(O,\mathbf{i},\mathbf{j},\mathbf{k})$ be a right oriented orthonormal reference frame (coordinates system) and consider three vectors

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}, \quad \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}, \quad \mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}.$$

We have

$$[\mathbf{a},\mathbf{b},\mathbf{c}] = (\mathbf{a}\times\mathbf{b})\mathbf{c} = (a_2b_3 - a_3b_2)c_1 + (-a_1b_3 + a_3b_1)c_2 + (a_1b_2 - a_2b_1)c_3.$$

or equivalently

$$[\mathbf{a},\mathbf{b},\mathbf{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

From the properties of the determinants we have

$$[\mathbf{a},\mathbf{b},\mathbf{c}] = [\mathbf{b},\mathbf{c},\mathbf{a}] = [\mathbf{c},\mathbf{a},\mathbf{b}] = -[\mathbf{b},\mathbf{a},\mathbf{c}] = -[\mathbf{a},\mathbf{c},\mathbf{b}] = -[\mathbf{c},\mathbf{b},\mathbf{a}].$$

The coplanarity condition for $\mathbf{a},\mathbf{b},\mathbf{c}$ is

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0.$$

The orientation of the basis is determined as follows

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \begin{cases} > 0 & \text{then } (\mathbf{a},\mathbf{b},\mathbf{c}) \text{ is right oriented} \\ = 0 & \text{then } (\mathbf{a},\mathbf{b},\mathbf{c}) \text{ is not a basis} \\ < 0 & \text{then } (\mathbf{a},\mathbf{b},\mathbf{c}) \text{ is left oriented} \end{cases}.$$

2.3.2 Applications

Look, this section is just saying that determinants of 3×3 matrices = volume in dimension 3. So this gives a geometric meaning to 3×3 determinants. Any applications of such determinants and any need to calculate volumes in dimension 3 is an application of the box product.

2.4 Further identities

Next to the Grassman identity and the Jacobi identity we have the following remarkable identities:

Theorem 2.9 (Lagrange identity)**.** For $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}\in\mathbb{V}^3$ we have

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})=(\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})-(\mathbf{b}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{d})=\begin{vmatrix}(\mathbf{a}\cdot\mathbf{c}) & (\mathbf{a}\cdot\mathbf{d})\\ (\mathbf{b}\cdot\mathbf{c}) & (\mathbf{b}\cdot\mathbf{d})\end{vmatrix}.$$

Theorem 2.10 (Triple vector product)**.** For $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}\in\mathbb{V}^3$ we have

$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})=\mathbf{b}\cdot[\mathbf{a},\mathbf{c},\mathbf{d}]-\mathbf{a}\cdot[\mathbf{b},\mathbf{c},\mathbf{d}]=\mathbf{c}\cdot[\mathbf{a},\mathbf{b},\mathbf{d}]-\mathbf{d}\cdot[\mathbf{a},\mathbf{b},\mathbf{c}].$$

- You can prove such identities by writing both the left- and the right-hand side in coordinates.

CHAPTER 3

Lines in dimension 2

Contents

Unless otherwise stated, any coordinate system that we fix will be a right oriented orthonormal coordinate system (reference frame). In this document $Oxy = (O, \mathbf{i}, \mathbf{j})$ is a coordinate system of $\mathbb{E}^2$. Recall that $\mathbb{V}^2$ is the vector space of vectors which can be represented with points in $\mathbb{E}^2$. Recall also that we have a bijective map $\phi_O : \mathbb{E}^2 \to \mathbb{V}^2$ given by $\phi_O(P) = \overrightarrow{OP}$.

3.1 Describing curves in $\mathbb{E}^2$

There are essentially three ways in which we can describe curves (or, more generally, geometric objects). For curves in $\mathbb{E}^2$ these are the following

1. Via parametrizations, where, for a given interval I, you use a map $\varphi : I \to \mathbb{E}^2$ to specify for each $t \in I$ a point in $\mathbb{E}^2$. For example, the circle of radius 1 centered at the origin is given by $\phi : [0, 2\pi) \to \mathbb{E}^2$ with $\phi(t) = (\cos(t), \sin(t))$.

2. Via global equations, where you describe the curve as the set of all points whose coordinates satisfy a given equation. For example, the circle of radius 1 centered at the origin is the set $\{P(x, y) \in \mathbb{E}^2 : x^2 + y^2 = 1\}$.

3. Via geometric properties, where you describe the curve as the set of all points satisfying a geometric property. For example, the circle of radius 1 centered at the point O is the set $\{P \in \mathbb{E}^2 : P \text{ is at distance 1 from } O\}$.

Notice that:

- The first two descriptions depend on the coordinate system while the third one does not.

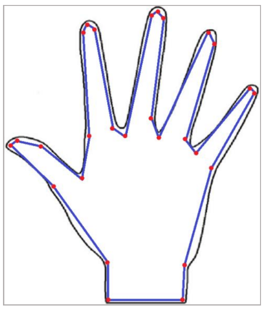
- The first two descriptions allow for concrete computational methods.

- In this course we focus on translating geometric properties into equations with respect to a coordinate system (a reference frame).

- One can think about the first approach as being local: when you vary the parameter t a little bit you move the point $\phi(t)$ a little bit, so this approach allows you to control the curve locally.

- One can think about the second approach as being global: indeed, you have one rule (one equation) for all points.

Computationally you will most often use an approximation of the curve that you are interested in. The simplest form of approximation is via line segments.



3.2 Lines in $\mathbb{E}^2$

3.2.1 Geometric description

Recall from Lecture 1 that we can describe lines as being sets of points S in $\mathbb{E}^2$ such that the set of vectors which can be represented by points in S form a 1-dimensional vector subspace of $\mathbb{V}^2$, i.e. for any $A \in S$

$$\phi_A(S) = \{\overrightarrow{AB} : B \in G\} \quad \text{is a 1-dimensional vector subspace of } \mathbb{V}^2.$$

3.2.2 Parametric equations - local description

If S is a line then for any two distinct points A, B in S the vector $\overrightarrow{AB}$ is called a *direction vector* of S. Since $\phi_A(S)$ is 1-dimensional, all direction vectors are linearly dependent and $\mathbf{v}$ is a direction vector for S if and only if it is linearly dependent on $\overrightarrow{AB}$. So, for any direction vector $\mathbf{v}$ of S there is a scalar $t \in \mathbb{R}$ such that

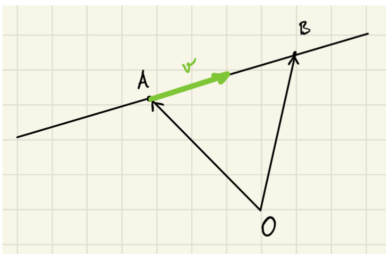
$$\overrightarrow{AB} = t\mathbf{v}.$$

Now, if you fix A and let B vary on the line then t varies in $\mathbb{R}$. Since ϕ_A is a bijection, the line S can be described as

$$S = \left\{B \in \mathbb{E}^2 : \overrightarrow{AB} = t\mathbf{v} \text{ for some } t \in \mathbb{R}\right\}.$$

In this description the point A is arbitrary but fixed. We sometimes refer to it as the *base point*. Now, the coordinates of a point B are the components of the vector $\overrightarrow{OB}$ and, rearranging the above equation we have

$$\overrightarrow{OB} = \overrightarrow{OA} + t\mathbf{v}. \tag{3.1}$$



So, we can describe the line S as being the set of points B in $\mathbb{E}^2$ which satisfy Equation (3.1) for some $t \in \mathbb{R}$. This is called the *vector equation* of the line S.

- The vector equation depends on the choice of the base point A.
- The vector equation depends on the choice of the direction vector $\mathbf{v}$.
- Hence, a line does not have a unique vector equation.
- The vector equation does not depend on the coordinate system. In the above description O can be any point in $\mathbb{E}^2$.

If we write Equation (3.1) in coordinates (relative to the coordinate system Oxy) then we obtain

$$\begin{cases} x = x_A + tv_x \\ y = y_A + tv_y \end{cases} \quad \text{or, in matrix form} \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_A \\ y_A \end{bmatrix} + t \begin{bmatrix} v_x \\ v_y \end{bmatrix} \tag{3.2}$$

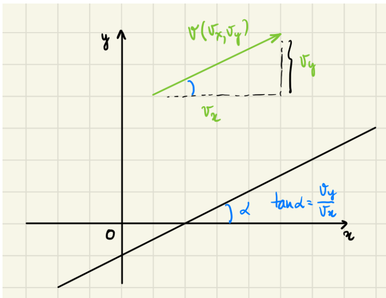
where $A = A(x_A, y_A)$, $\mathbf{v} = \mathbf{v}(v_x, v_y)$ and for different values t we obtain different points (x, y) on the line. The two equations in the system (3.2) are called *parametric equations* for the line S.

3.2.3 Cartesian equations - global description

It is possible to eliminate the parameter t in (3.2) by expressing it in both equations and setting the two expressions equal in order to obtain

$$\frac{x - x_A}{v_x} = \frac{y - y_A}{v_y}. \tag{3.3}$$

We refer to Equation (3.3) as *symmetric equation* of the line S. It could happen that v_x or v_y are zero. In that case, translate back to the parametric equations to understand what happens.



You can rearrange Equation (3.3) as

$$y = kx + m \quad \text{where} \quad k = \frac{v_y}{v_x} \quad \text{and} \quad m = -\frac{v_y}{v_x}x_A + y_A.$$

In this form we will call it *the equation where you can read-off the slope* k. Draw a picture to see that if $\alpha = \measuredangle(\mathbf{i}, \mathbf{v})$ then $k = \tan(\alpha)$. In fact, you can rearrange Equation (3.3) as

$$y - y_A = \tan(\alpha)(x - x_A).$$

If you like names, then you can call this *the equation of a line in* $\mathbb{E}^2$ *where you can read off the slope and a point on the line.*

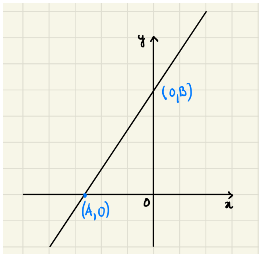
- We have just described a line with a linear equation (relative to the coordinate system Oxy).

Proposition 3.1. Every line in $\mathbb{E}^2$ can be described with a linear equation in two variables

$$ax + by + c = 0 \tag{3.4}$$

relative to a fixed coordinate system and any linear equation in two variables describes a line relative to a fixed coordinate system.

- Equation (3.4) is called a *Cartesian equation* of the line it describes.
- Notice that there are infinitely many Cartesian equations describing the same line, since you can multiply one equation by a non-zero constant.



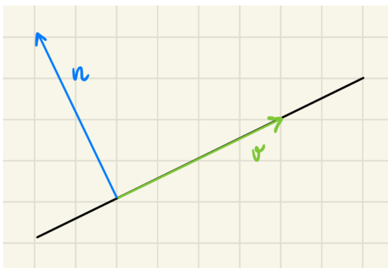
If a line is given with the Cartesian equation (3.4), you can rearrange it in the form

$$\frac{x}{A} + \frac{y}{B} = 1 \quad \text{where} \quad A = -\frac{c}{a} \quad \text{and} \quad B = -\frac{c}{b}.$$

In this form we have the *equation of the line where we can read off the intersection points with the coordinate axes* since the line intersects Ox in $(A,0)$ and it intersects Oy in $(0,B)$.

3.2.4 Normal vectors

Normal vectors play an important role both in abstract geometry and in applications. Here we consider normal vectors of lines (in $\mathbb{E}^2$!).



The idea is very simple: we discussed in Lecture 2 that if we fix a vector $\mathbf{n} \in \mathbb{V}^2$ then $\mathbf{n}^\perp$ is a vector subspace of $\mathbb{V}^2$ and $\mathbb{V}^2 = \langle \mathbf{n} \rangle \oplus \mathbf{n}^\perp$. The vector subspace $\langle \mathbf{n} \rangle$ is 1-dimensional (it is spanned by one vector) and $\dim(\mathbb{V}^2) = 2$. Therefore $\mathbf{n}^\perp$ must have dimension 1. So, returning to the description of a line as a set of points of the form

$$S = \left\{ B \in \mathbb{E}^2 : \overrightarrow{AB} = t\mathbf{v} \text{ for some } t \in \mathbb{R} \right\},$$

we may choose $\mathbf{n}$ such that a direction vector $\mathbf{v}$ for S is a generator of $\mathbf{n}^\perp$. In other words, we may choose $\mathbf{v} \in \mathbf{n}^\perp$ non-zero, i.e. we may choose $\mathbf{v}$ to be a non-zero vector orthogonal to $\mathbf{n}$. Then, we have

$$S = \left\{ B \in \mathbb{E}^2 : \mathbf{n} \perp \overrightarrow{AB} \right\} = \left\{ B \in \mathbb{E}^2 : \mathbf{n} \cdot \overrightarrow{AB} = 0 \right\},$$

where A is some point in S. In other words, the line S can be described as the set of point B such that

$$\mathbf{n} \cdot \overrightarrow{AB} = 0 \quad \text{or, equivalently} \quad \mathbf{n} \cdot (\overrightarrow{OB} - \overrightarrow{OA}) = 0$$

The vector $\mathbf{n}$ is perpendicular to the line S and any vector with this property is called a *normal vector* for the line S. If we look at this equation in coordinates, then $\overrightarrow{OA} = (x_A, y_A) = A \in S$ and with $\mathbf{n} = \mathbf{n}(n_x, n_y)$ we have

$$n_x(x - x_A) + n_y(y - y_A) = 0 \tag{3.5}$$

where (x, y) is an arbitrary point on the line S. This is a linear equation for S and since all other equations of S (in the coordinate system Oxy) are obtained form (3.5) by multiplying with a non-zero constant, we see that the coefficients of x and y in a Cartesian equation of S can be interpreted as the components of a normal vector.

3.2.5 Relative positions

- [Intersections] Assume we have two lines

$$\ell_1 : a_1x + b_1y + c_1 = 0 \quad \text{and} \quad \ell_2 : a_2x + b_2y + c_2 = 0$$

In order to determine if they intersect or not one has to discuss the system:

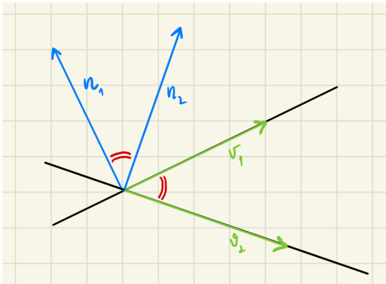
$$\begin{cases} \ell_1 : a_1x + b_1y + c_1 & = & 0 \\ \ell_2 : a_2x + b_2y + c_2 & = & 0 \end{cases} \tag{3.6}$$

Recall Lecture 12 of your Algebra course last semester. In the plane the situation is very simple:

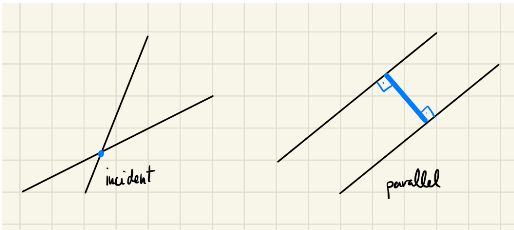
  - two lines either intersect, in a unique point, the coordinates of which will be a unique solution to (3.6); or
  - they don't intersect and (3.6) doesn't have solutions, in which case the lines are parallel; or
  - system (3.6) has infinitely many solutions in which cases $\ell_1 = \ell_2$.

Notice that (a_1, b_1) is a normal vector for ℓ_1 and (a_2, b_2) is a normal vector for ℓ_2, so the last two cases above occur if these two vector are proportional, which is equivalent to $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$, i.e. the matrix of coefficients for the system (3.6) has rank 1.

- [Angles] Two lines define two angles: if $\mathbf{v}_1$ is a direction vector for ℓ_1 and if $\mathbf{v}_2$ is a direction vector for ℓ_2 then the two angles described by ℓ_1 and ℓ_2 are $\measuredangle(\mathbf{v}_1,\mathbf{v}_2)$ and $\measuredangle(-\mathbf{v}_1,\mathbf{v}_2)$. They are supplementary angles so if you can measure one of them you know the other one. Recall from Lecture 2 that
$$\cos\measuredangle(\mathbf{v}_1,\mathbf{v}_2)=\frac{\mathbf{v}_1\cdot\mathbf{v}_2}{\|\mathbf{v}_1\|\cdot\|\mathbf{v}_2\|}. \tag{3.7}$$
Notice also that the two angles can also be described with normal vectors: if $\mathbf{n}_1$ and $\mathbf{n}_2$ are normal vectors for ℓ_1 and ℓ_2 respectively, then the two angles between ℓ_1 and ℓ_2 are $\measuredangle(\mathbf{n}_1,\mathbf{n}_2)$ and $\measuredangle(-\mathbf{n}_1,\mathbf{n}_2)$. So, if these vectors are known we may calculate
$$\cos\measuredangle(\mathbf{n}_1,\mathbf{n}_2)=\frac{\mathbf{n}_1\cdot\mathbf{n}_2}{\|\mathbf{n}_1\|\cdot\|\mathbf{n}_2\|}. \tag{3.8}$$
Clearly, if the two lines are parallel, then the two angles are 0° and 180°.



- [Distances]



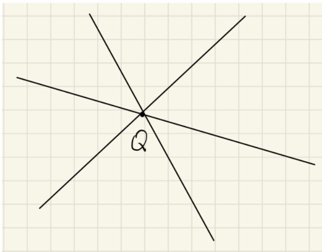
We first consider the distance from a point to a line

Proposition 3.2. Suppose you have a line $\ell : ax+by+c=0$ and a point $P(x_P,y_P)$ in $\mathbb{E}^2$. The distance from P to ℓ is
$$d(P,\ell)=\frac{|ax_P+by_P+c|}{\sqrt{a^2+b^2}}.$$

Now, considering the distances between two lines ℓ_1 and ℓ_2 we have the following cases:

- if ℓ_1 and ℓ_2 intersect then the distance between them is zero $d(\ell_1,\ell_2)=0$.
- if ℓ_1 and ℓ_2 do not intersect, i.e. if they are parallel, then
$$d(\ell_1,\ell_2)=d(P_1,\ell_2)=d(\ell_1,P_2)$$
for any point $P_1\in\ell_1$ and any point $P_2\in\ell_2$.

3.3 Bundle of lines



Definition. Let $Q \in \mathbb{E}^2$. The set $\mathcal{L}_Q$ of all lines in $\mathbb{E}^2$ passing through Q is called a *bundle of lines* and Q is called the *center* of the bundle $\mathcal{L}_Q$.

Proposition 3.3. If $\ell_1 : a_1x + b_1y + c_1 = 0$ and $\ell_2 : a_2x + b_2y + c_2 = 0$ are two distinct lines in the bundle $\mathcal{L}_Q$, then $\mathcal{L}_Q$ consists of lines having equations of the form

$$\ell_{\lambda,\mu} : \lambda(a_1x + b_1y + c_1) + \mu(a_2x + b_2y + c_2) = 0.$$

where $\lambda, \mu \in \mathbb{R}$ not both zero.

- In particular, if $Q = Q(x_0, y_0)$, $\ell_1 : x = x_0$ and $\ell_2 : y = y_0$ then

$$\mathcal{L}_Q = \Big\{\ell_{\lambda,\mu} : \lambda(x - x_0) + \mu(y - y_0) = 0 : \lambda, \mu \in \mathbb{R} \text{ not both zero}\Big\}.$$

- Bundles of lines are useful in praxis when a point Q is given as the intersection of two lines, but its coordinates are not known explicitly, and one wants to find the equation of a line passing through Q and satisfying some other conditions. For example, the condition that it contains some point P distinct from Q or that it is parallel to a given line.

- There is redundancy in the two parameters λ, μ, meaning that there are not two independent parameters here. If $\lambda \neq 0$ then one can divide the equation of $\ell_{\lambda,\mu}$ by λ to obtain

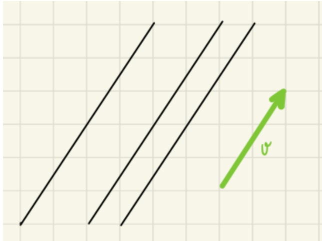
$$\ell_{1,t} : (a_1x + b_1y + c_1) + t(a_2x + b_2y + c_2) = 0.$$

  where $\frac{\mu}{\lambda} = t \in \mathbb{R}$. So $\ell_{1,\frac{\mu}{\lambda}}$ and $\ell_{\lambda,\mu}$ are in fact the same lines.

- A *reduced bundle* is the set of all lines $\mathcal{L}_Q$ passing through a common point Q from which we remove one line, i.e. it is $\mathcal{L}_Q \setminus \{\ell_2\}$ for some $\ell_2 \in \mathcal{L}_Q$. With the above notation and discussion, it is the set

$$\{\ell_{1,t} : (a_1x + b_1y + c_1) + t(a_2x + b_2y + c_2) = 0 : t \in \mathbb{R}\}$$

  The fact that we use one parameter instead of two, to describe almost all lines passing through Q, greatly simplifies calculations.

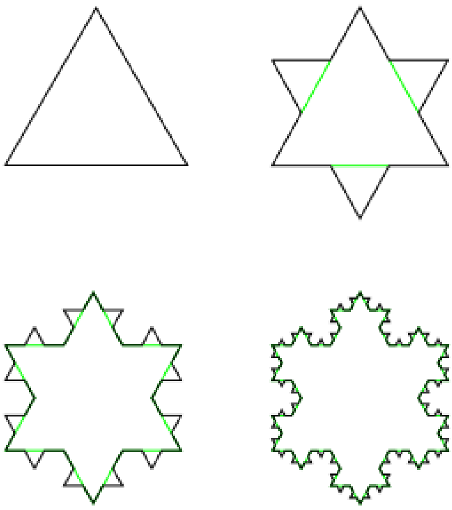


Definition. Let $\mathbf{v} \in \mathbb{V}^2$. The set $\mathcal{L}_{\mathbf{v}}$ of all lines in $\mathbb{E}^2$ with direction vector $\mathbf{v}$ is called an *improper bundle of lines*, and $\mathbf{v}$ is called a *direction vector* of the bundle $\mathcal{L}_{\mathbf{v}}$.

- The connection between bundles of lines and improper bundles of lines is best understood through projective geometry, where the improper bundle of lines is the set of all lines intersecting in the same point at infinity.

3.4 Strange curves in $\mathbb{E}^2$

There are 'curves' which can be described geometrically by a process, like the Koch curve (Koch snowflake):



It looks like a curve, but it has very little in common with the curves that you are familiar with:

- It has no tangent line, no derivatives.
- It has fractal dimension $\log_3(4) \sim 1.26186$ wherease as line segment has fractal dimension 1.

- It's better to call it a fractal than a curve.
- Notice that you describe it using line segments.

CHAPTER 4

Lines and planes in dimension 3

Contents

Unless otherwise stated, any coordinate system that we fix will be a right oriented orthonormal coordinate system (reference frame). In this document $Oxyz = (O, \mathbf{i}, \mathbf{j}, \mathbf{k})$ is a coordinate system of $\mathbb{E}^3$. Recall that $\mathbb{V}^3$ is the vector space of vectors which can be represented with points in $\mathbb{E}^3$. Recall also that we have a bijective map $\phi_O : \mathbb{E}^3 \to \mathbb{V}^3$ given by $\phi_O(P) = \overrightarrow{OP}$.

4.1 Planes in $\mathbb{E}^3$

4.1.1 Describing surfaces in $\mathbb{E}^3$

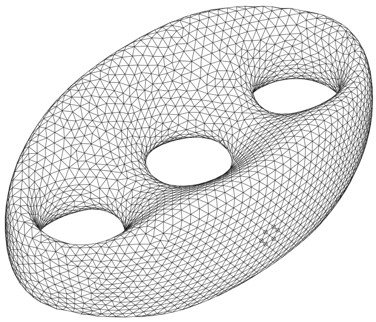
There are essentially three ways in which we can describe a surface in $\mathbb{E}^3$:

1. Via parametrizations, where, for two intervals $I_1, I_2 \subseteq \mathbb{R}$, you use a map $\varphi : I_1 \times I_2 \to \mathbb{E}^3$ to specify for each pair $(s,t) \in I_1 \times I_2$ a point in $\mathbb{E}^3$. For example, the sphere of radius 1 centered at the origin is given by $\phi : [0,2\pi) \times [0,\pi] \to \mathbb{E}^3$ with $\phi(s,t) = (\cos(s)\sin(t), \sin(s)\sin(t), \cos(t))$.

2. Via global equations, where you describe the surface as the set of all points whose coordinates satisfy a given equation. For example, the sphere of radius 1 centered at the origin is the set $\{P(x,y,z) \in \mathbb{E}^3 : x^2 + y^2 + z^2 = 1\}$.

3. Via geometric properties, where you describe the curve as the set of all points satisfying a geometric property. For example, the sphere of radius 1 centered at the point O is the set $\{P \in \mathbb{E}^3 : \text{ } P \text{ is at distance 1 from } O\}$.

Computationally you will most often use an approximation of the surface that you are interested in. The simplest form of approximation is via triangles.



4.1.2 Geometric description

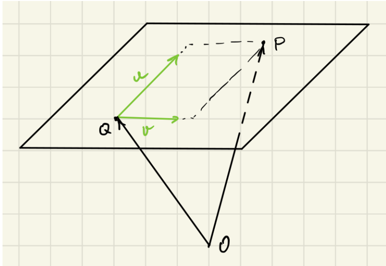
Recall from Lecture 1 that we can describe planes as sets of points S in $\mathbb{E}^3$ such that the set of vectors which can be represented by points in S form a 2-dimensional vector subspace of $\mathbb{V}^3$, i.e. for any $A \in S$

$$\phi_A(S) = \{\overrightarrow{AB} : B \in S\} \quad \text{is a 2-dimensional vector subspace of } \mathbb{V}^3.$$

4.1.3 Parametric equations - local description

Since $\phi_A(S)$ is 2-dimensional, any basis will contain two vectors. Let $\mathbf{v}$, $\mathbf{w}$ be a basis of the vector space $\phi_A(S)$. Thus, for any two points $A, B \in S$, the vector $\overrightarrow{AB}$ is a linear combination of the basis vectors, i.e. there exist unique scalars $s, t \in \mathbb{R}$ such that

$$\overrightarrow{AB} = s\mathbf{v} + t\mathbf{w}.$$



Now, if you fix A and let B vary in the plane S then s and t vary in $\mathbb{R}$. Since ϕ_A is a bijection, the plane S can be described as

$$S = \left\{ B \in \mathbb{E}^3 : \overrightarrow{AB} = s\mathbf{v} + t\mathbf{w} \text{ for some } s, t \in \mathbb{R} \right\}.$$

In this description the point $A \in S$ is arbitrary but fixed. We sometimes refer to it as the *base point*. Now, the coordinates of a point B are the components of the vector $\overrightarrow{OB}$ and, rearranging the above equation we have

$$\overrightarrow{OB} = \overrightarrow{OA} + s\mathbf{v} + t\mathbf{w}. \tag{4.1}$$

So, we can describe the plane S as the set of points B in $\mathbb{E}^3$ which satisfy Equation (4.1) for some $s, t \in \mathbb{R}$. This is called the *vector equation* of the plane S.

- The vector equation depends on the choice of the base point A.
- The vector equation depends on the choice of the vector $\mathbf{v}$ and $\mathbf{w}$. In analogy with the case of the line in $\mathbb{E}^2$ we may call such vectors direction vectors for the plane S.
- Hence a plane does not have a unique vector equation.
- The vector equation does not depend on the coordinate system. In the above description O can be any point in $\mathbb{E}^3$.

If we write Equation (4.1) in coordinates (relative to the coordinate system $Oxyz$) then we obtain

$$\begin{cases} x = x_A + sv_x + tw_x \\ y = y_A + sv_y + tw_y \\ z = z_A + sv_z + tw_z \end{cases} \quad \text{or, in matrix form} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_A \\ y_A \\ z_A \end{bmatrix} + s \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} + t \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix} \tag{4.2}$$

where $A = A(x_A, y_A, z_A)$, $\mathbf{v} = \mathbf{v}(v_x, v_y, v_z)$, $\mathbf{w} = \mathbf{v}(w_x, w_y, w_z)$ and for different values s and t we obtain different points (x, y, z) in the plane S. The three equations in the system (4.2) are called *parametric equations* for the plane S.

4.1.4 Cartesian equations - global description

As in the case of the line in $\mathbb{E}^2$, it is possible to eliminate the parameters s, t in (4.2) to obtain

$$\left(\frac{v_x}{w_x} - \frac{v_z}{w_z} \right) \left(\frac{x - x_A}{w_x} - \frac{y - y_A}{w_y} \right) = \left(\frac{v_x}{w_x} - \frac{v_y}{w_y} \right) \left(\frac{x - x_A}{w_x} - \frac{z - z_A}{w_z} \right). \tag{4.3}$$

We will not give this equation a name, because it is a bit much to keep in mind, and one has to make sense of what happens when the denominators are zero. The whole point here, is that *it is a linear equation* in x, y and z and that it can be obtained by eliminating the parameters in (4.2).

There is an easier way of describing S with a linear equation. For this you can interpret (4.2) as saying that the vector $\overrightarrow{AB}$ is linearly dependent on the vectors $\mathbf{v}$ and $\mathbf{w}$. With this in mind, the point $B(x,y,z)$ lies in the plane S if and only if

$$\begin{vmatrix} x-x_A & y-y_A & z-z_A \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = 0 \quad \Leftrightarrow \quad [\overrightarrow{AB},\mathbf{v},\mathbf{w}] = 0 \tag{4.4}$$

Recall that this was the coplanarity condition which we deduced when we discussed the box product. It says that the volume of the parallelepiped spanned by the vectors $\overrightarrow{AB},\mathbf{v},\mathbf{w}$ is zero. We may refer to this expression as the *zero-volume equation* of the plane S.

Proposition 4.1. Every plane in $\mathbb{E}^3$ can be described with a linear equation in three variables

$$ax + by + cz + d = 0 \tag{4.5}$$

relative to a fixed coordinate system and any linear equation in two variables describes a plane relative to a fixed coordinate system if the constants a,b,c are not all zero.

- Equation (4.5) is called the *Cartesian equation* of the plane it describes.
- Notice that there are infinitely many Cartesian equations describing the same plane, since you can multiply one equation by a non-zero constant.
- In a fixed coordinate system equations of a plane are the same up to multiplication by a non-zero scalar.

If a plane is given with the Cartesian equation (4.5), you can rearrange it in the form

$$\frac{x}{A}+\frac{y}{B}+\frac{z}{C}=1 \quad \text{where} \quad A=-\frac{d}{a} \quad ,B=-\frac{d}{b}. \quad \text{and} \quad C=-\frac{d}{c}.$$

In this form we have the *equation of the plane where we can read off the intersection points with the coordinate axes* since the line intersects Ox in $(A,0,0)$, it intersects Oy in $(0,B,0)$ and it intersects Oz in $(0,0,C)$.

4.1.5 Normal vectors

Normal vectors play an important role both in abstract geometry and in applications. Here we consider normal vectors of planes in $\mathbb{E}^3$.

The idea is again simple: we discussed in Lecture 2 that if we fix a vector $\mathbf{n} \in \mathbb{V}^3$ then $\mathbf{n}^\perp$ is a vector subspace of $\mathbb{V}^3$ and $\mathbb{V}^3 = \langle \mathbf{n} \rangle \oplus \mathbf{n}^\perp$. The vector subspace $\langle \mathbf{n} \rangle$ is 1-dimensional (it is spanned by one vector) and $\dim(\mathbb{V}^3) = 3$. Therefore $\mathbf{n}^\perp$ must have dimension 2. So, returning to the description of a plane as a set of points of the form

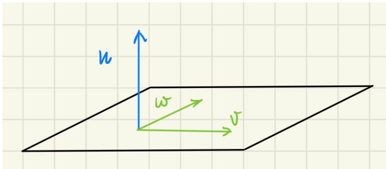
$$S = \left\{B \in \mathbb{E}^3 : \overrightarrow{AB} = s\mathbf{v} + t\mathbf{w} \text{ for some } s,t \in \mathbb{R}\right\},$$

we may choose $\mathbf{n}$ perpendicular to both $\mathbf{v}$ and $\mathbf{w}$. In other words, we may choose $\mathbf{n}$ such that $\mathbf{v}, \mathbf{w}$ are a basis of $\mathbf{n}^\perp$. Then, we have

$$S = \left\{B \in \mathbb{E}^3 : \mathbf{n} \perp \overrightarrow{AB}\right\} = \left\{B \in \mathbb{E}^3 : \mathbf{n} \cdot \overrightarrow{AB} = 0\right\},$$

where A is some point in S. In other words, the plane S can be described as the set of point $B \in \mathbb{E}^3$ such that

$$\mathbf{n} \cdot \overrightarrow{AB} = 0 \quad \text{or, equivalently} \quad \mathbf{n} \cdot (\overrightarrow{OB} - \overrightarrow{OA}) = 0.$$



The vector $\mathbf{n}$ is perpendicular to the plane S and any non-zero vector with this property is called a *normal vector* for the plane S. If we look at this equation in coordinates, then $\overrightarrow{OA} = (x_A, y_A, z_A) = A \in S$ and with $\mathbf{n} = \mathbf{n}(n_x, n_y, n_z)$ we have

$$n_x(x - x_A) + n_y(y - y_A) + n_z(z - z_A) = 0 \tag{4.6}$$

where (x, y, z) is an arbitrary point in the plane S. This is a linear equation for S and since all other equations of S (in the coordinate system Oxy) are obtained from (4.6) by multiplying with a non-zero constant, we see that the coefficients of x, y and z in a Cartesian equation of S can be interpreted as the components of a normal vector.

If a plane is given via parametric equations, with $\mathbf{v}$ and $\mathbf{w}$ as direction vectors. How can we obtain a normal vector? You can transform the parametric equations in a Cartesian equation and read off the coefficients of x, y, z or you recall that $\mathbf{v} \times \mathbf{w}$ is a vector which is perpendicular to both $\mathbf{v}$ and $\mathbf{w}$, thus, it is a normal vector for the plane.

4.1.6 Relative positions of two planes

- [Intersections] Assume that we have two planes

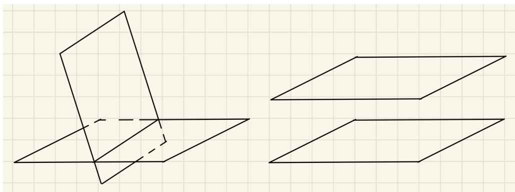
  $$\pi_1 : a_1x + b_1y + c_1z + d_1 = 0 \quad \text{and} \quad \pi_2 : a_2x + b_2y + c_2z + d_2 = 0.$$

  In order to determine if they intersect or not one has to discuss the system:

  $$\begin{cases} \pi_1 : a_1x + b_1y + c_1z + d_1 & = & 0 \\ \pi_2 : a_2x + b_2y + c_2z + d_2 & = & 0 \end{cases} \tag{4.7}$$

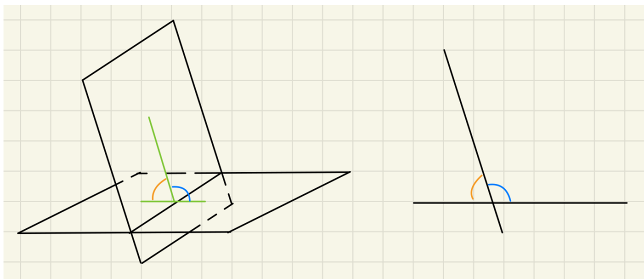
  Recall Lecture 12 of your Algebra course last semester. Let A be the matrix of the system and $\tilde{A}$ the extended matrix of the system.

- two planes either intersect in a line, the coordinates of the points on the line will be solutions to (4.7), this happens if the rank of A and the rank of $\tilde{A}$ equals 2; or
- they don't intersect and (4.7) doesn't have solutions, in which case the planes are parallel, this happens if the rank of A is strictly less than the rank of $\tilde{A}$; or
- the solution to system (4.7) depends on two parameters in which case $\pi_1 = \pi_2$, this happens if the rank of A and the rank of $\tilde{A}$ are equal to 1.



Notice that (a_1, b_1, c_1) is a normal vector for π_1 and (a_2, b_2, c_2) is a normal vector for π_2. These are the rows of the matrix A, so the last two cases above occur if these two vectors are proportional, since the rank of A equals 1.

- [Angles]



For planes we have the notion of *dihedral angle*. Two planes π_1 and π_2 define two dihedral angles which are supplementary.

These two angles can also be described with normal vectors: if $\mathbf{n}_1$ and $\mathbf{n}_2$ are normal vectors for π_1 and π_2 respectively, then the two angles between π_1 and π_2 are congruent to $\measuredangle(\mathbf{n}_1, \mathbf{n}_2)$ and $\measuredangle(-\mathbf{n}_1, \mathbf{n}_2)$. So, if these vectors are known we may calculate

$$\cos\measuredangle(\mathbf{n}_1, \mathbf{n}_2) = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\| \cdot \|\mathbf{n}_2\|}. \tag{4.8}$$

Clearly, if the two planes are parallel, then the two angles are 0° and 180°.

- [Distances] We first consider the distance from a point to a plane

Proposition 4.2. Suppose you have a plane $\pi : ax + by + cz + d = 0$ and a point $P(x_P, y_P, z_P)$ in $\mathbb{E}^3$. The distance from P to π is

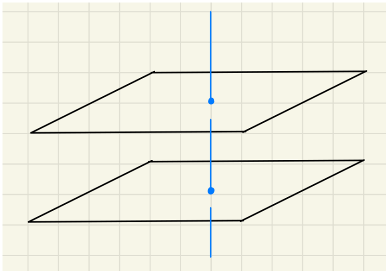
$$d(P, \pi) = \frac{|ax_P + by_P + cz_P + d|}{\sqrt{a^2 + b^2 + c^2}}. \tag{4.9}$$

Now, considering the distances between two planes π_1 and π_2 we have the following cases:

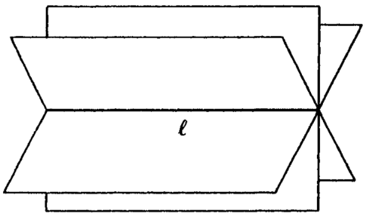
- if π_1 and π_2 intersect then the distance between them is zero $d(\pi_1, \pi_2) = 0$.
- if π_1 and π_2 do not intersect, i.e. if they are parallel, then

$$d(\pi_1, \pi_2) = d(P_1, \pi_2) = d(\pi_1, P_2)$$

for any point $P_1 \in \pi_1$ and any point $P_2 \in \pi_2$.



4.1.7 Bundles of planes



Definition. Let $\ell \subseteq \mathbb{E}^3$ be a line. The set Π_ℓ of all planes in $\mathbb{E}^3$ containing ℓ is called a *bundle of planes* and ℓ is called the *axis* of the bundle Π_ℓ.

Proposition 4.3. If $\pi_1 : a_1x + b_1y + c_1z + d_1 = 0$ and $\pi_2 : a_2x + b_2y + c_2z + d_2 = 0$ are two distinct planes in the bundle Π_ℓ, then Π_ℓ consists of planes having equations of the form

$$\pi_{\lambda,\mu} : \lambda(a_1x + b_1y + c_1z + d_1) + \mu(a_2x + b_2y + c_2z + d_2) = 0.$$

where $\lambda, \mu \in \mathbb{R}$ not both zero.

- Bundles of planes are useful in practice when a line ℓ is given as the intersection of two planes and one wants to find the equation of a plane containing ℓ and satisfying some other conditions. For example, the condition that it contains some point P which does not belong to ℓ or that it is parallel to a given line.

- There is redundancy in the two parameters λ, μ, meaning that there are not two independent parameters here. If $\lambda \neq 0$ then one can divide the equation of $\pi_{\lambda,\mu}$ by λ to obtain

$$\pi_{1,t} : (a_1x + b_1y + c_1z + d_1) + t(a_2x + b_2y + c_2z + d_2) = 0.$$

where $\frac{\mu}{\lambda} = t \in \mathbb{R}$. So $\pi_{1,\frac{\mu}{\lambda}}$ and $\pi_{\lambda,\mu}$ are in fact the same planes.

- A *reduced bundle* is the set of all planes Π_ℓ with axis ℓ from which *we remove one plane*, i.e. it is $\Pi_\ell \setminus \{\pi_2\}$ for some $\pi_2 \in \Pi_\ell$. With the above notation and discussion, it is the set

$$\{\pi_{1,t} : (a_1x + b_1y + c_1z + d_1) + t(a_2x + b_2y + c_2z + d_2) = 0 : t \in \mathbb{R}\}.$$

The fact that we use one parameter instead of two, to describe almost all planes containing ℓ, greatly simplifies calculations.

Definition. Let $\mathbb{W} \subseteq \mathbb{V}^3$ be a vector subspace of dimension 2. The set $\Pi_\mathbb{W}$ of all planes in $\mathbb{A}$ having associated vector subspace $\mathbb{W}$ is called an *improper bundle of planes*, and $\mathbb{W}$ is called the vector subspace associated to the bundle $\Pi_\mathbb{W}$.

- The connection between bundles of planes and improper bundles of planes is best understood through projective geometry, where we can think about the improper bundle of planes as the set of all planes intersecting in a line at infinity.

4.2 Lines in $\mathbb{E}^3$

4.2.1 Describing curves in $\mathbb{E}^3$

There are essentially three ways in which we can describe a curve in $\mathbb{E}^3$:

1. Via parametrizations, where, for an interval $I \subseteq \mathbb{R}$, you use a map $\varphi : I \to \mathbb{E}^3$ to specify for each $t \in I$ a point in $\mathbb{E}^3$. For example, *a* circle of radius 1 centered at the origin is given by $\phi : [0, 2\pi) \to \mathbb{E}^3$ with $\phi(s,t) = (\cos(t), \sin(t), 0)$.

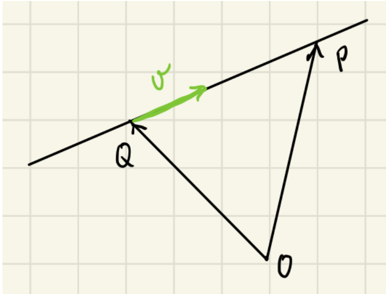
2. Via global equations, where you describe the curve as the set of all points whose coordinates satisfy *two* given equations. For example, the above circle can be described as the set $\{P(x,y,z) \in \mathbb{E}^3 : x^2 + y^2 = 1 \text{ and } z = 0\}$.

3. Via geometric properties, where you describe the curve as the set of all points satisfying some geometric properties. For example, the above circle can be described as the set of points $\{P \in \mathbb{E}^3 : P \text{ is at distance 1 from } O \text{ and } P \text{ lies in the plane } Oxy\}$.

4.2.2 Geometric description

Recall from Lecture 1 that we can describe lines in $\mathbb{E}^3$ as being sets of points S such that the set of vectors which can be represented by points in S form a 1-dimensional vector subspace of $\mathbb{V}^3$, i.e. for any $A \in S$

$$\phi_A(S) = \{\overrightarrow{AB} : B \in S\} \quad \text{is a 1-dimensional vector subspace of } \mathbb{V}^3.$$



4.2.3 Parametric equations - local description

If S is a line then for any two distinct points A, B in S the vector $\overrightarrow{AB}$ is called a *direction vector* of S. Since $\phi_A(S)$ is 1-dimensional, all direction vectors are linearly dependent and $\mathbf{v}$ is a direction vector for S if and only if it is linearly dependent on $\overrightarrow{AB}$. So, for any direction vector $\mathbf{v}$ of S there is a scalar $t \in \mathbb{R}$ such that

$$\overrightarrow{AB} = t\mathbf{v}.$$

Now, if you fix A and let B vary on the line then t varies in $\mathbb{R}$. Since ϕ_A is a bijection, the line S can be described as

$$S = \left\{ B \in \mathbb{E}^3 : \overrightarrow{AB} = t\mathbf{v} \text{ for some } t \in \mathbb{R} \right\}.$$

In this description the point $A \in S$ is arbitrary but fixed. We sometimes refer to it as the *base point*. Now, the coordinates of a point B are the components of the vector $\overrightarrow{OB}$ and, rearranging the above equation we have

$$\overrightarrow{OB} = \overrightarrow{OA} + t\mathbf{v}. \tag{4.10}$$

So, we can describe the line S as being the set of points B in $\mathbb{E}^3$ which satisfy Equation (4.10) for some $t \in \mathbb{R}$. This is called the *vector equation* of the line S.

- The vector equation depends on the choice of the base point A.
- The vector equation depends on the choice of the direction vector $\mathbf{v}$.
- Hence, a line does not have a unique vector equation.
- The vector equation does not depend on the coordinate system. In the above description O can be any point in $\mathbb{E}^3$.

If we write Equation (4.10) in coordinates (relative to the coordinate system $Oxyz$) then we obtain

$$\begin{cases} x = x_A + tv_x \\ y = y_A + tv_y \\ z = z_A + tv_z \end{cases} \quad \text{or, in matrix form} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_A \\ y_A \\ z_A \end{bmatrix} + t \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \tag{4.11}$$

where $A = A(x_A, y_A, z_A)$, $\mathbf{v} = \mathbf{v}(v_x, v_y, v_z)$ and for different values t we obtain different points (x, y, z) on the line. The three equations in the system (4.11) are called *parametric equations* for the line S.

4.2.4 Cartesian equations - global description

It is possible to eliminate the parameter t in (4.11) in order to obtain

$$\frac{x-x_A}{v_x}=\frac{y-y_A}{v_y}=\frac{z-z_A}{v_z}. \tag{4.12}$$

We refer to the Equations (4.12) as the *symmetric equations* of the line S. It could happen that v_x, v_y or v_z are zero. In that case, translate back to the parametric equations to understand what happens.

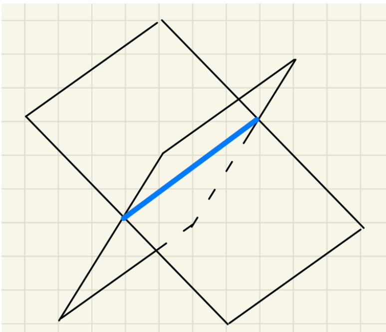
- For lines in $\mathbb{E}^3$ there is no notion of 'slope'.
- We have just described a line with *two* linear equations (relative to the coordinate system $Oxyz$).

Proposition 4.4. Every line in $\mathbb{E}^3$ can be described with two linear equations in three variables

$$\begin{cases} a_1x+b_1y+c_1z+d_1 &= 0\\ a_2x+b_2y+c_2z+d_2 &= 0 \end{cases} \tag{4.13}$$

relative to a fixed coordinate system and any compatible system of two linear equations of rank 2 in three variable describes a line relative to a fixed coordinate system.

- The Equations (4.5) are called *Cartesian equations* of the line it describes.
- The Equations (4.5) describe a line as an intersection of two planes.



4.2.5 Relative positions of two lines

- [Intersections] Assume we have two lines

$$\ell_1:\begin{cases} a_1x+b_1y+c_1z+d_1 &= 0\\ a_2x+b_2y+c_2z+d_2 &= 0 \end{cases} \quad\text{and}\quad \ell_2:\begin{cases} a_3x+b_3y+c_3z+d_3 &= 0\\ a_4x+b_4y+c_4z+d_4 &= 0 \end{cases}.$$

One way to determine if they intersect is to discuss the system:

$$\begin{cases} a_1x+b_1y+c_1z+d_1 &= 0\\ a_2x+b_2y+c_2z+d_2 &= 0\\ a_3x+b_3y+c_3z+d_3 &= 0\\ a_4x+b_4y+c_4z+d_4 &= 0 \end{cases}. \tag{4.14}$$

As you did in Lecture 12 of your Algebra course last semester. However, it is easier to discuss the relative positions of lines in $\mathbb{E}^3$ via their parametric equations:

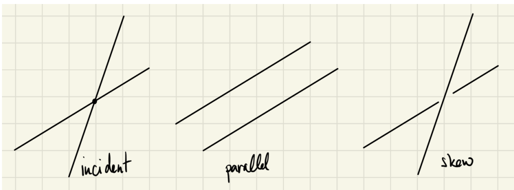
$$\ell_1 : \begin{cases} x = x_1 + tv_x \\ y = y_1 + tv_y \\ z = z_1 + tv_z \end{cases} \quad \text{und} \quad \ell_2 : \begin{cases} x = x_2 + tu_x \\ y = y_2 + tu_y \\ z = z_2 + tu_z \end{cases}$$

We have the following cases

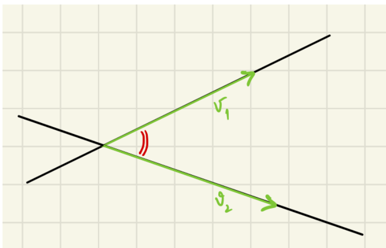
- If the direction vectors **v** and **u** are proportional then the two lines are parallel.
- If they are parallel and have a point in common then the two lines are equal.
- If they are not parallel then they are coplanar (they lie in the same plane) if

$$\begin{vmatrix} x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ v_x & v_y & v_z \\ u_x & u_y & u_z \end{vmatrix} = 0.$$

  in which case they intersect in exactly one point.
- If they are not parallel and they don't intersect then we say that the two lines ℓ_1 and ℓ_2 are *skew* relative to each other.



- [Angles]



Two lines define two angles: if $\mathbf{v}_1$ is a direction vector for ℓ_1 and if $\mathbf{v}_2$ is a direction vector for ℓ_2 then the two angles described by ℓ_1 and ℓ_2 are $\measuredangle(\mathbf{v}_1, \mathbf{v}_2)$ and $\measuredangle(-\mathbf{v}_1, \mathbf{v}_2)$. They are supplementary angles so if you can measure one of them you know the other one. Recall from Lecture 2 that

$$\cos \measuredangle(\mathbf{v}_1, \mathbf{v}_2) = \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{\|\mathbf{v}_1\| \cdot \|\mathbf{v}_2\|}. \tag{4.15}$$

Clearly, if the two lines are parallel, then the two angles are 0° and 180°.

- [Distances] We first consider the distance from a point to a lines

  Proposition 4.5. Suppose you have a line

  $$\ell_1 : \begin{cases} x = x_A + tv_x \\ y = y_A + tv_y \\ z = z_A + tv_z \end{cases} \quad \text{and a point} \quad P(x_P, y_P, z_P) \in \mathbb{E}^3.$$

  The distance from P to ℓ is

  $$d(P, \ell) = \frac{\|\overrightarrow{PA} \times \mathbf{v}\|}{\|\mathbf{v}\|}.$$

  Now, considering the distances between two lines ℓ_1 and ℓ_2. We have the following cases:

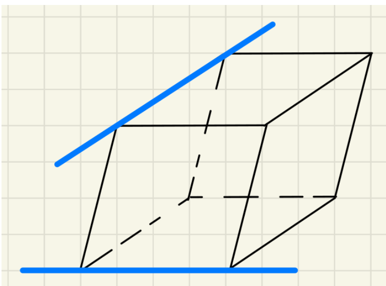
  - if ℓ_1 and ℓ_2 intersect then the distance between them is zero $d(\ell_1, \ell_2) = 0$.
  - if ℓ_1 and ℓ_2 do not intersect and they are parallel, then

    $$d(\ell_1, \ell_2) = d(P_1, \ell_2) = d(\ell_1, P_2)$$

    for any point $P_1 \in \ell_1$ and any point $P_2 \in \ell_2$.
  - if ℓ_1 and ℓ_2 are skew relative to each other, there is a unique plane π_1 containing ℓ_1 which is parallel to ℓ_2 and there is a unique plane π_2 which contains ℓ_2 and is parallel to ℓ_1, thus

    $$d(\ell_1, \ell_2) = d(\pi_1, P_2) = d(P_1, \pi_2)$$

    for any point $P_1 \in \ell_1$ and any point $P_2 \in \ell_2$.

    More concretely, if the two lines are

    $$\ell_1 : \begin{cases} x = x_1 + tv_x \\ y = y_1 + tv_y \\ z = z_1 + tv_z \end{cases} \quad \text{and} \quad \ell_2 : \begin{cases} x = x_2 + tu_x \\ y = y_2 + tu_y \\ z = z_2 + tu_z \end{cases}$$

    then

    $$\pi_1 : \begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ v_x & v_y & v_z \\ u_x & u_y & u_z \end{vmatrix} = 0$$

Hence, by (4.9),

$$d(\ell_1,\ell_2)=d(\pi_1,P_2)=\left|\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ v_x & v_y & v_z \\ u_x & u_y & u_z \end{vmatrix}}{\sqrt{a_x^2+a_y^2+a_z^2}}\right|.$$

where $\mathbf{a}$ is the normal vector of the plane π_1 corresponding to the above equation, i.e.

$$a_x=\begin{vmatrix} v_y & v_z \\ u_y & u_z \end{vmatrix},\quad a_y=\begin{vmatrix} v_z & v_x \\ u_z & u_x \end{vmatrix}\quad \text{and}\quad a_z=\begin{vmatrix} v_x & v_y \\ u_x & u_y \end{vmatrix}.$$

- [Common perpendicular line of two skew lines] With the notation in the previous paragraph consider the line

$$\ell:\begin{cases} \pi_1':\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ v_x & v_y & v_z \\ a_x & a_y & a_z \end{vmatrix}=0, \\ \pi_2':\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ u_x & u_y & u_z \\ a_x & a_y & a_z \end{vmatrix}=0. \end{cases}$$

viewed as the intersection of the two planes π_1' and π_2'. It is a calculation to check that

  - $\ell\perp\ell_1$ and $\ell\perp\ell_2$, and that
  - $\ell\cap\ell_1\neq\emptyset$ and $\ell\cap\ell_2\neq\emptyset$.

4.3 Relative positions of a line and a plane

- [Intersection] Consider the plane

$$\pi: ax+by+cz+d=0$$

and the line

$$\ell:\begin{cases} x=x_0+tv_x \\ y=y_0+tv_y \\ z=z_0+tv_z. \end{cases}$$

In order to see if they intersect, we check which points in ℓ satisfy the equation of π:

$$a(x_0+tv_x)+b(y_0+tv_y)+c(z_0+tv_z)+d=0 \quad\Leftrightarrow\quad (av_x+bv_y+cv_z)t+ax_0+by_0+cz_0+d=0. \tag{4.16}$$

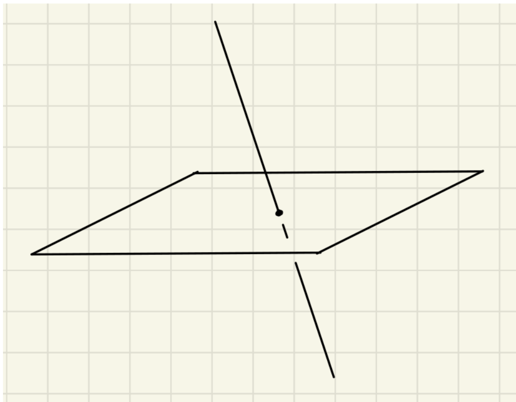
Notice that $\mathbf{n}(a,b,c)$ is a normal vector for π. The possibilities are

  - $\mathbf{n}\cdot\mathbf{v}=0$ in which case the line is parallel to the plane.
  - If $\ell\parallel\pi$ and Equation (4.16) is not satisfied, then the line lies outside the plane π.

  - If $\ell \parallel \pi$ and Equation (4.16) is satisfied, then any $t \in \mathbb{R}$ is a solution. In this case the line lies in the plane π.
  - If $\ell \nparallel \pi$ then Equation (4.16) has the unique solution

$$t_0 = -\frac{ax_0 + by_0 + cz_0 + d}{av_x + bv_y + cv_z}.$$

Hence, the point corresponding to the parammeter t_0 is the intersection point $\ell \cap \pi$.



- [Angle] Let $\mathbf{n}$ be a normal vector for the plane π and $\mathbf{v}$ be a direction vector for the line ℓ. If $\mathbf{n} \cdot \mathbf{v} < 0$ then the angle $\measuredangle(\mathbf{n}, \mathbf{v})$ is obtuse, so, replace $\mathbf{n}$ by $-\mathbf{n}$ or $\mathbf{v}$ by $-\mathbf{v}$ so that the angle $\measuredangle(\mathbf{n}, \mathbf{v})$ is acute and $\mathbf{n} \cdot \mathbf{v} > 0$.

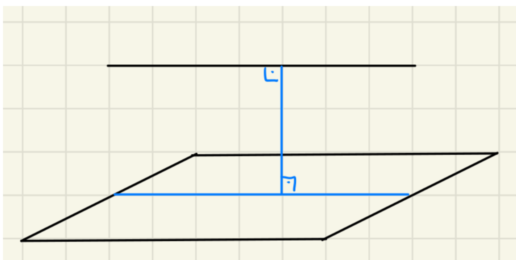
  The line ℓ and the plane π determine two angles: $90° \pm \measuredangle(\mathbf{n}, \mathbf{v})$ which can be calculated as usual with the scalar product if $\mathbf{n}$ and $\mathbf{v}$ are known.

- [Distance] For the distances between a plane π and a line ℓ we have the following cases:

  - if π and ℓ intersect, then the distance between them is zero $d(\ell, \pi) = 0$.
  - if π and ℓ do not intersect, i.e. if they are parallel, then

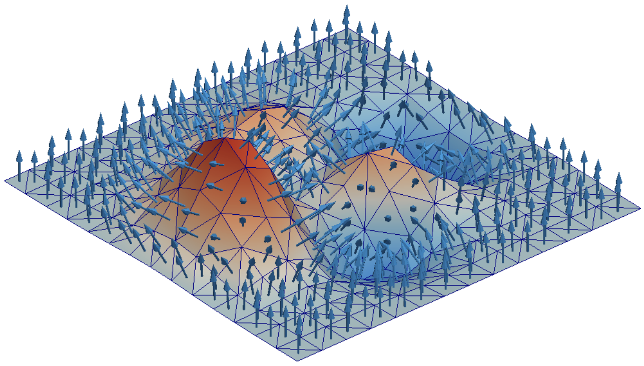
$$d(\ell, \pi) = d(P, \pi)$$

for any point $P \in \ell$.



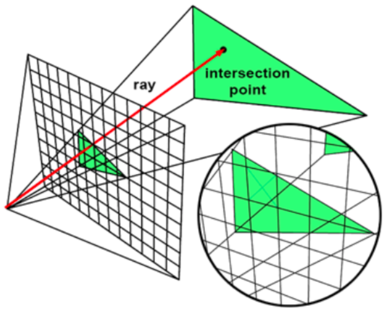
4.4 Applications

- [Triangulation] In computer graphics it is common to describe a surface by a triangulation, i.e. as a union of triangles:



- [Light] When modeling light on a surface one uses the normal vectors to the surface as described in Lecture 2. If the surface is given by a triangulation, the luminosity of each triangle can be calculated with the normal vector corresponding to that triangle, more precisely, with the normal vector to the plane determined by the corresponding triangle. Normal vectors are generally pre-computed, and are part of the 'model'.

- [Ray tracing]



Ray tracing algorithms consider rays passing through the camera and each pixel of the screen. These rays are intersected with the objects in the scene in order to decide what color to choose for the pixels. For this, one intersects the rays with the planes of the triangles. After finding the intersection points P one has to decide if P is inside or outside the triangle.