Lecture 14:
Solved problems on angular momentum

Exercise 1: Demonstrate that $L^2 = L_- L_+ + L_3^2 + L_3$

Solution

We know that:

a) $L^2 = L_1^2 + L_2^2 + L_3^2$

b) $L_- = L_1 - iL_2$

c) $L_+ = L_1 + iL_2$

d) $[L_+, L_-] = 2L_3$

Let us calculate $L_- L_+ = (L_1 - iL_2)(L_1 + iL_2) = L_1^2 + L_2^2 + i(L_1 L_2 - L_2 L_1)$. The term in brackets is the commutator $[L_1, L_2]$ which, neglecting $\hbar$ (we consider here to have $\hbar = 1$), is equal to $[L_1, L_2] = iL_3$.

We thus have $L_- L_+ = L_1^2 + L_2^2 - L_3$. If we add $L_3 + L_3^2$, we obtain: $L_- L_+ + L_3 + L_3^2 = L_1^2 + L_2^2 + L_3^2 = L^2$, which concludes our demonstration. ∎

Exercise 2: Demonstrate that, if J^2 has an eigenvalue $j(j+1)$, there exists $2j+1$ states that are eigenfunctions of J_3 with eigenvalues j, $j-1$, $j-2$, ..., $-j$.

Solution

First of all, we demonstrate that there exists an eigenfunction of J^2 with eigenvalue $j(j+1)$ which has the maximum eigenvalue for J_3, and that this is equal to j. In other words, there exists a state ψ such as $J^2\psi = j(j+1)\psi$ and, simultaneously, $J_3\psi = j\psi$. If the state ψ has the maximum eigenvalue of J_3, it immediately follows that $J_+\psi = 0$. So, remembering that $J^2 = J_- J_+ + J_3^2 + J_3$, we have:

$$J^2\psi = J_- J_+ \psi + J_3^2\psi + J_3\psi$$

The first term is equal to 0 (ψ is the state with maximum eigenvalue for J_3 implying that $J_+\psi = 0$), the second is equal to $j^2\psi$ and the third is equal to $j\psi$ giving: $J^2\psi = j(j+1)\psi$. On the other hand, there must be also a state with the *minimum* eigenvalue for J_3 meaning that I can only apply the operator J_- a certain amount of times (which I will denote as n). We will have:

$$J_3 J_-\psi = (j-1)J_-\psi, \quad J_3 J_-^2\psi = (j-2)J_-^2\psi, \quad J_3 J_-^3\psi = (j-1)J_-^3\psi, \quad ,...., \quad J_3 J_-^n\psi = 0$$

For this state of minimum eigenvalue of J_3 we have:

$$J^2 J_-^n \psi = (J_+ J_- + J_3^2 - J_3) J_-^n \psi = ((j-n)^2 - (j-n)) J_-^n \psi$$

However, also the state $J_-^n \psi$ is eigenfunction of J^2 with eigenvalue $j(j+1)$. We thus have the equality:

$$j(j+1) = (j-n)^2 - (j-n)$$

which implies $n = 2j$. However, n is a positive integer number (it represents the times I can apply J_- to the state with maximum eigenvalue of J_3) implying that there are $2j+1$ states all with the same eigenvalue for J^2.■.

Exercise 3: Calculate the spherical harmonics $Y_{\ell,m}$ associated with: a) ℓ=0, $m = 0$, a) ℓ=1, $m = 0$, a) ℓ=1, $m = 1$.

Solution

We know that a general spherical harmonic can be written as:

$$Y_\ell^m(\theta, \phi) = (-)^{(m+|m|)/2} i^\ell \sqrt{\frac{(2\ell+1)(\ell-|m|)!}{4\pi(\ell+|m|)!}} P_\ell^m(\cos\theta) e^{im\phi} \tag{1}$$

where the associated *Legendre* polynomials P_ℓ^m are defined as:

$$P_\ell^m(x) = (x^2-1)^{m/2} \frac{\mathrm{d}^m}{\mathrm{d}\, x^m} P_\ell(x) \tag{2}$$

and the *Legendre* polynomials P_ℓ are defined as:

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{\mathrm{d}^\ell}{\mathrm{d}\, x^\ell} (x^2-1)^\ell \tag{3}$$

We then start by calculating $P_0(x)$:

$$P_0(x) = 1 \tag{4}$$

and $P_1(x)$:

$$P_1(x) = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}\, x} (x^2-1) = x \tag{5}$$

Then, $P_0^0(x)$:

$$P_0^0(x) = 1 \tag{6}$$

and $P_1^0(x)$:

$$P_1^0(x) = x \tag{7}$$

and $P_1^1(x)$:

$$P_1^1(x) = (x^2-1)^{1/2}\frac{\mathrm{d}}{\mathrm{d}\,x}x = \sqrt{x^2-1} \tag{8}$$

We now have all the ingredients to calculate the various spherical harmonics (bear in mind that $x = \cos\theta$):

$$a) \quad Y_{0,0} = \frac{1}{\sqrt{4\pi}} \tag{9}$$

$$b) \quad Y_{1,0} = i\frac{3}{\sqrt{4\pi}}\cos\theta \tag{10}$$

$$c) \quad Y_{1,1} = -i\frac{3}{\sqrt{8\pi}}\sin\theta e^{i\phi} \tag{11}$$

Exercise 4: Consider the Hamiltonian:

$$\hat{H} = g\hat{L}\cdot\hat{s}$$

Here $\hat{L}$ is the angular momentum operator, $\hat{s}$ is the spin of the electron and g is a coupling constant.

a Making use of the commutation rule between $\hat{L}$ and $\hat{s}$, demonstrate that, the square of total angular momentum is given by:

$$\hat{J}^2 = \hat{L}^2 + \hat{s}^2 + 2\hat{L}\cdot\hat{s}$$

b If we have an electron with $\ell = 1$ and $s = 1/2$, what are the possible values of the eigenvalues of $\hat{J}_z$ and $\hat{J}^2$?

c Still considering $\ell = 1$ and $s = 1/2$, find the eigenvalues of the Hamiltonian (energy) for all the possible configurations of the total angular momentum.

Solution

$J^2 = (L+s)^2 = L^2 + s^2 + Ls + sL$. However $[L, s] = 0$ implying that $J^2 = L^2 + s^2 + 2Ls$. The Hamiltonian can thus be written as:

$$H = g(J^2 - L^2 - s^2)/2 \tag{12}$$

The possible values for the total angular momentum j are $j = \ell \pm s = 3/2$ or $1/2$. We thus have two possible energy eigenvalues:

$$\text{a)} \ \ j = 3/2 \ \ \ell = 1 \ \ s = 1/2 \ \ \rightarrow \ \ E = g/2 \tag{13}$$

or

$$\text{b)} \ \ j = 1/2 \ \ \ell = 1 \ \ s = 1/2 \ \ \rightarrow \ \ E = -g/2 \tag{14}$$