RECAP WEEK 1

TYPES

ORDER $\rightarrow$ $\frac{dy}{dx}$ - 1 $\quad \frac{d^2y}{dx^2}$ - 2 $\quad \frac{\partial^2 y}{\partial x \partial y}$ ORDER

LINEARITY $\rightarrow$ $y, \frac{dy}{dx} \dots \quad \cancel{y^2}, \cancel{y^{1/2}}, \cancel{y\frac{dy}{dx}}$

HOMOGENEITY $\rightarrow$

ALL TERMS IN y = ALL OTHER TERMS

= 0 (HOMOGENEOUS)

SOLNS 1ST ORDER

$$\frac{dy}{dx} = f(x)\,g(y)$$

HOMOGENEOUS

SEPARATION OF VARIABLES

$$\int \frac{dy}{g(y)} = \int f(x)\,dx$$

$$\frac{dy}{dx} + \underbrace{f(x)}\, y = g(x)$$

INHOMOGENEOUS

INTEGRATING FACTOR I.F.

$$I.F. = \exp\left(\int f(x)\,dx\right)$$

$$\int \frac{d}{dx}\left[I.F. \times y \right] dx = \int I.F. \times g(x)\, dx$$

$$I.F. \times y \quad = \quad ''$$

SOLUTIONS TO 2ND ORDER, LINEAR CONSTANT COEFFICIENTS

HOMOGENEOUS

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \qquad a, b, c \text{ CONSTANTS}$$

CHARACTERISTIC EQUATION $\quad y = Ae^{mx}$

(QUADRATIC) $\qquad$ SOLN m_1, m_2

GENERAL SOLUTION $\quad y = Be^{m_1 x} + Ce^{m_2 x}$

INHOMOGENEOUS

GENERAL SOLUTION = COMPLEMENTARY FUNCTION + PARTICULAR INTEGRAL

ORDER 2 - NON-CONSTANT COEFFICIENT

POWER SERIES $\quad y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 \text{---}$

$$x \frac{d^2y}{dx^2} + y = 0$$

$$\sum_{n=2}^{\infty} a_n n(n-1) x^{n-1} = 0$$

$n \longrightarrow n+1$

$$\frac{dy}{dx} = \sum_{n=1}^{\infty} a_n n x^{n-1}$$

$$\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} a_{n+2} (n+2)(n+1) x^n$$

$n \longrightarrow n+2$

Power Series Solution Example

- Consider a more challenging example $(x^2+1)\frac{d^2u}{dx^2} - 4x\frac{du}{dx} + 6u = 0$
- Substituting in the power series solution for each derivative

$$(x^2+1)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 4x\sum_{n=1}^{\infty} n a_n x^{n-1} + 6\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 4\sum_{n=1}^{\infty} n a_n x^n + 6\sum_{n=0}^{\infty} a_n x^n = 0$$

- we shift the index of the 2nd term by 2 ($n \rightarrow n$+2)

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

- Equate the powers of x
  - $n = 0$ x^0 $\quad 2a_2 + 6a_0 = 0 \longrightarrow a_2 = -3a_0$
  - $n = 1$ x^1 $\quad 6a_3 - 4a_1 + 6a_1 = 0 \longrightarrow a_3 = -\frac{a_1}{3}$

Power Series Solution Example

$$\sum_{n=2}^{\infty}\underbrace{n(n-1)a_n}x^n+\sum_{n=0}^{\infty}\underbrace{(n+2)(n+1)a_{n+2}}x^n-\sum_{n=1}^{\infty}\underbrace{4na_n}x^n+\sum_{n=0}^{\infty}\underbrace{6a_n}x^n=0$$

- For $n \geq 2$

$$a_{n+2}(n+2)(n+1)+a_n(\underbrace{n^2-5n+6}_{(n-2)(n-3)})=0$$

$$\underbrace{a_{n+2}=-a_n\frac{(n-2)(n-3)}{(n+1)(n+2)}}$$

RECURRENCE RELATIONSHIP

- When $n=2,3$ $a_4=0$ and $a_5=0$ series is truncated (all higher terms also = 0)

$\to a_6=0 \to a_8=0$ $\quad \to a_7=0 \to a_9=0$

- Solutions

EVEN $\quad u(x)=a_0-3a_0x^2$

ODD $\quad u(x)=a_1x-\frac{a_1}{3}x^3$

- General solution $u(x)=a_0(1-3x^2)+a_1\left(x-\frac{1}{3}x^3\right)$

Linearising ODEs

– PERTURBATION TECHNIQUE

- Possible if dependent variable has a small disturbance from equilibirum $u = \bar{u} + \Delta u$, $\Delta u \ll \bar{u}$

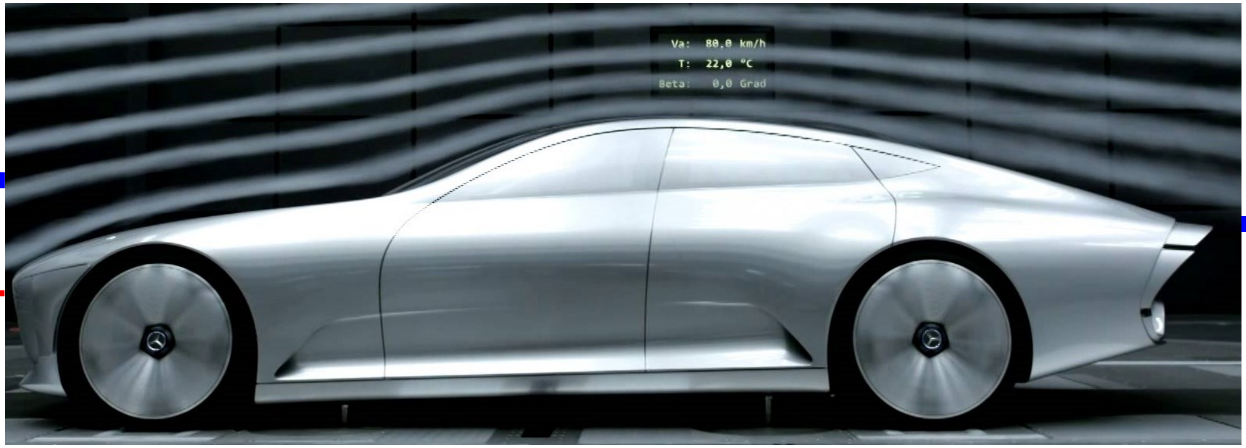
AVERAGE (CONSTANT)

NEW DEPENDENT VARIABLE

- Example – wind resistance of a car on motorway

Engine Thrust $= pF_{max}$

Velocity v



Wind Resistance $= -\frac{1}{2}\rho A C v^2$

ρ - air density, C - drag coefficient, A – cross sectional area, p – accelerator pedal (0-1)

- Newton's 2nd law
  - $m\frac{dv}{dt} = p(t)F_{max} - \frac{1}{2}\rho A C v^2$

$v = \bar{v} + \Delta v$ IF $\bar{v} \gg \Delta v$

$\bar{v}^2 + 2\bar{v}\Delta v + (\Delta v)^2 \rightarrow 0$

$$m\frac{d(\Delta v)}{dt} = p F_{max} - \frac{1}{2}\rho A C(\bar{v} + 2\bar{v}\Delta v)$$

Accelerating car

$$\frac{d(\Delta v)}{dt} + \frac{\rho A C \bar{v}}{m}\Delta v = p(t)\frac{F_{max}}{m} - \frac{1}{2m}\rho A C \bar{v}^2$$

- Consider the situation where a car of mass 1000 kg and maximum thrust $F_{max} =$ 3000 N is travelling at $\bar{v} = 30$ m/s (67 mph) when the accelerator pedal depressed by 30%, i.e. $p = 0.3$. In this equilibrium situation $\Delta v = 0$
  - $p(t)F_{max} = \frac{1}{2}\rho A C \bar{v}^2$

$$\rho A C = \frac{2pF_{MAX}}{\bar{v}^2} = 2\,kg\,m^{-1}$$

$$\frac{d(\Delta v)}{dt} + \frac{2\bar{v}}{m}\Delta v = p\frac{F_{MAX}}{m} - \frac{\bar{v}^2}{m}$$

- If at $t = 0$, the driver doubles the force applied to overtake another car so that $p(t) = 0.6$, how long does it take for the car to reach 33 m/s (74 mph)?
  - $D = p\frac{F_{max}}{m} - \frac{\bar{v}^2}{m} = 0.9\ \mathrm{m^2 s^{-2} kg^{-1}}$
  - $\frac{d(\Delta v)}{dt} + \frac{2\bar{v}}{m}\Delta v = D$

$$I.F.\quad \exp\left(\int \frac{2\bar{v}}{m}dt\right) = e^{\frac{2\bar{v}}{m}t}$$

$$\frac{d}{dt}\left[e^{\frac{2\bar{v}}{m}t}\Delta v\right] = De^{2\bar{v}/m\, t}$$

Accelerating car

- General Solution $\Delta v = A \exp\left(-\frac{2\bar{v}}{m}t\right) + \frac{Dm}{2\bar{v}}$
- Apply initial conditions $t = 0,\ \Delta v = 0$

$$A = -\frac{Dm}{2\bar{v}}$$

$$\Delta v = \underbrace{\frac{Dm}{2\bar{v}}}_{15\,m/s}\left(1 - e^{-\frac{2\bar{v}}{m}t}\right)$$

- $\Delta v = \underline{+3\ m/s}$

$$3 = 15\left(1 - e^{-0.06t}\right)$$

$$e^{-0.06t} = 0.8$$

$$t = 3.72\,s$$