RELATIVISTIC ELECTRODYNAMICS

The fact that Maxwell's equations were not consistent with a Galilean transformation between inertial frames provided a major impetus for advancing relativity theory. Einstein's postulates set out to address this. By the first postulate all Physical laws, including EM Theory, should be identical in all inertial frames, this being facilitated by the second postulate which allowed for the constancy of the speed of light.

Hence, Maxwell's equations must be *covariant* with respect to the Lorentz Transformation. To show that this is indeed true we must be able to express the equations in the form of 4-vector equations.

To do this we must first rewrite Maxwell's equations in terms of the vector and scalar potentials, rather than in the usual form in terms of electric and magnetic field parameters.

From there we need to formulate 4-vector expressions, that will permit us to bring the equations together in covariant form.

[Note – to avoid confusion between 3-D and 4-D vectors we will use script characters such as $\boldsymbol{\mathcal{E}}$ $\boldsymbol{\mathcal{B}}$ $\boldsymbol{\mathcal{J}}$ to denote 3-D vectors and Roman characters to denote 4-D vectors]

<u>Maxwell's Equations (ME)</u>

$$\nabla . \boldsymbol{\mathcal{E}} = \frac{\rho}{\epsilon_0} \qquad \text{----- (4.1)}$$

$$\nabla \times \boldsymbol{\mathcal{E}} = -\frac{\partial \boldsymbol{\mathcal{B}}}{\partial t} \qquad \text{----- (4.2)}$$

$$\nabla . \boldsymbol{\mathcal{B}} = 0 \qquad \text{----- (4.3)}$$

$$\nabla \times \boldsymbol{\mathcal{B}} = \mu_0 \boldsymbol{\mathcal{J}} + \frac{1}{c^2}\frac{\partial \boldsymbol{\mathcal{E}}}{\partial t} \qquad \text{----- (4.4)}$$

where:

$$\mu_0 \epsilon_0 = \frac{1}{c^2}$$

and:

$$\boldsymbol{\mathcal{J}} = \rho \boldsymbol{u}$$

$\boldsymbol{\mathcal{J}}$ - current density, ρ - charge density, $\boldsymbol{u}$ - charge velocity

These are ME expressed in terms of laboratory observables, the electric field strength and magnetic flux density, and represent physical effects such as Gauss' law, etc.

Taking the divergence of eqn (4.4):

$$\nabla . \nabla \times \mathcal{B} = \mu_0 \nabla . \mathcal{J} + \frac{1}{c^2} \nabla . \frac{\partial \mathcal{E}}{\partial t}$$

Leading to:

$$\nabla . \mathcal{J} + \frac{\partial \rho}{\partial t} = 0 \qquad \text{----- (4.5)}$$

which is known as the equation of continuity.

We introduce the concept of a vector potential, denoted $\mathcal{A}$, and a scalar potential, denoted ϕ

These can be defined *via*

$$\mathcal{B} = \nabla \times \mathcal{A} \qquad \text{----- (4.6)}$$

and

$$\mathcal{E} = -\nabla\phi - \frac{\partial \mathcal{A}}{\partial t} \qquad \text{----- (4.7)}$$

Combining these equations gives us an expression relating the vector and the scalar potentials, known as the Lorentz condition:

$$\nabla . \mathcal{A} = -\frac{1}{c^2} \frac{\partial \phi}{\partial t} \qquad \text{----- (4.8)}$$

Substituting from (4.6) into (4.4):

$$\nabla \times \nabla \times \mathcal{A} \equiv \nabla \nabla . \mathcal{A} - \nabla^2 \mathcal{A} = \mu_0 \mathcal{J} + \frac{1}{c^2} \frac{\partial \mathcal{E}}{\partial t}$$

Substituting from (4.8) and (4.7):

$$-\frac{1}{c^2} \frac{\partial}{\partial t} \nabla\phi - \nabla^2 \mathcal{A} = \mu_0 \mathcal{J} - \frac{1}{c^2} \frac{\partial}{\partial t} \nabla\phi - \frac{1}{c^2} \frac{\partial^2 \mathcal{A}}{\partial t^2}$$

Therefore:

$$\boxed{\nabla^2 \mathcal{A} - \frac{1}{c^2} \frac{\partial^2 \mathcal{A}}{\partial t^2} = -\mu_0 \mathcal{J}} \qquad \text{----- (4.9)}$$

Taking div of eqn (4.7):

$$\nabla . \mathcal{E} = -\nabla . \nabla\phi - \nabla . \frac{\partial \mathcal{A}}{\partial t}$$

Substituting from (4.1):

$$\frac{\rho}{\epsilon_0} = -\nabla^2 \phi - \frac{\partial}{\partial t} \nabla . \mathcal{A}$$

Substituting from (4.8):

$$\boxed{\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho}{\epsilon_0}}$$ ----- (4.10)

Note that equations (4.9) and (4.10) have the form of a wave equation.

These equations, along with equation (4.8), are completely equivalent to ME (4.1) – (4.4).

<u>The 4-D Del Operator</u>

Examining eqns (4.9) and (4.10) we observe that on the LHS they have the form of a 4-D equation, with the variable being operated on with respect to the three spatial parameters and by a time parameter.

Let us define a 4-D del operator:

$$\boxdot = \left(\frac{\partial}{\partial X_1}, \frac{\partial}{\partial X_2}, \frac{\partial}{\partial X_3}, \frac{\partial}{\partial X_4}\right)$$

where

$$\frac{\partial}{\partial X_1} = \frac{\partial}{\partial X}, \qquad \frac{\partial}{\partial X_2} = \frac{\partial}{\partial Y}, \qquad \frac{\partial}{\partial X_3} = \frac{\partial}{\partial Z}, \qquad \frac{\partial}{\partial X_4} = \frac{\partial}{ic\partial t} = -\frac{i}{c}\frac{\partial}{\partial t}$$

Hence 4-D Grad of a scalar ϕ:

$$\boxdot\,\phi = \left(\nabla\phi, -\frac{i}{c}\frac{\partial\phi}{\partial t}\right)$$

4-D Div of a 4-vector $\boldsymbol{A} = (\boldsymbol{\mathcal{A}}, A_4)$:

$$\boxdot.\boldsymbol{A} = \nabla.\boldsymbol{\mathcal{A}} - \frac{i}{c}\frac{\partial A_4}{\partial t}$$

4-D Laplacian (<u>**d'Alembertian**</u>):

$$\boxdot^2 = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

of a scalar ϕ:

$$\boxdot^2\,\phi = \nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}$$

of a 4-vector $\boldsymbol{A}$:

$$\boxdot^2\,\boldsymbol{A} = (\boxdot^2\,\boldsymbol{\mathcal{A}}, \boxdot^2\,\boldsymbol{A_4})$$

We can rewrite eqns (4.9) and (4.10) in terms of the d'Alembertian:

$$\boxdot^2 \boldsymbol{\mathcal{A}} = -\mu_0 \boldsymbol{\mathcal{J}}$$
----- (4.11)

$$\boxdot^2 \phi = -\frac{\rho}{\epsilon_0}$$
----- (4.12)

However, note that eqns (4.11) and (4.12) are still 3-dimensional; $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{J}}$ are 3-D vectors.

Let us define a 4-vector for current density

4-CURRENT DENSITY

$$\boldsymbol{J} = \rho_0 \boldsymbol{U}$$

where ρ_0 is proper charge density (the charge density in its rest frame) and $\boldsymbol{U}$ is the 4-vector for velocity.

[Note $\rho = \gamma(u)\rho_0$, similar to $m = \gamma(u)m_0$]

$$\boldsymbol{J} = (J_1, J_2, J_3, J_4) = (J_X, J_Y, J_Z, ic\rho)$$

This is consistent with the continuity equation (4.5)

$$\boxdot . \boldsymbol{J} = \boldsymbol{\nabla} . \boldsymbol{\mathcal{J}} + \frac{\partial \rho}{\partial t} = 0$$
----- (4.13)

Let us define a 4-vector for Potential:

4-POTENTIAL

$$\boldsymbol{A} = (A_1, A_2, A_3, A_4) = \left(A_X, A_Y, A_Z, \frac{i}{c}\phi\right)$$

combining the vector potential $\boldsymbol{\mathcal{A}} = (A_X, A_Y, A_Z)$ and the scalar potential ϕ as the space-like and time-like components.

Then we can rewrite eqn (4.8) as:

$$\boxdot . \boldsymbol{A} = 0$$
----- (4.14)

and eqns (4.11) and (4.12) as:

$$\boxdot^2 \boldsymbol{A} = -\mu_0 \boldsymbol{J}$$
----- (4.15)

Equations (4.14) and (4.15) are 4-vector expressions of ME (4.1) - (4.4), thus Maxwell's Equations are covariant as required.

ELECTROMAGNETIC FIELD TENSOR

We have seen that Maxwell's Equations, expressed in terms of a 4-vector potential $\boldsymbol{A}$, are covariant.

We can always transform the 4-potential $\boldsymbol{A}$ between inertial frames, as we know that its transformation properties satisfy:

$$A_\mu{}' = a_{\mu\nu} A_\nu$$

However, in order to discover how the measurable parameters $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{B}}$ transform under a Lorentz Transformation, it is useful to define a 2nd rank tensor, with components given in terms of $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{B}}$, to describe the electromagnetic field.

$$\Gamma_{\mu\nu} = \frac{\partial A_\nu}{\partial X_\mu} - \frac{\partial A_\mu}{\partial X_\nu} \qquad \text{----- (4.16)}$$

[*Note: This is the formal definition of a covariant tensor*]

A 2nd rank tensor $\Gamma_{\mu\nu}$ is a set of quantities $(\Gamma_{11} \dots \dots .. \Gamma_{44})$ whose transformation properties are the same as those of the displacement vector:

$$\Gamma'_{\mu\nu} = a_{\mu\alpha}\, a_{\nu\beta}\, \Gamma_{\alpha\beta}$$

[recall that the repeated indices on the RHS mean that we need to sum over α and β from 1 – 4].

$$\Gamma_{\mu\nu} = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} \\ \Gamma_{21} & \Gamma_{22} & \Gamma_{23} & \Gamma_{24} \\ \Gamma_{31} & \Gamma_{32} & \Gamma_{33} & \Gamma_{34} \\ \Gamma_{41} & \Gamma_{42} & \Gamma_{43} & \Gamma_{44} \end{pmatrix}$$

From eqn (4.16):

$$\Gamma_{11} = \frac{\partial A_X}{\partial X} - \frac{\partial A_X}{\partial X} = 0$$

and similarly:

$$\Gamma_{22} = \Gamma_{33} = \Gamma_{44} = 0$$

Hence the leading diagonal of $\Gamma_{\mu\nu}$ is composed of 0 elements.

To discover the other ($\Gamma_{11} \dots \dots \Gamma_{33}$) space-like components of $\Gamma_{\mu\nu}$ we recall the definition of $\boldsymbol{\mathcal{A}}$ in eqn (4.6):

$$\boldsymbol{\mathcal{B}} = \boldsymbol{\nabla} \times \boldsymbol{\mathcal{A}}$$

Now

$$\boldsymbol{\nabla} \times \boldsymbol{\mathcal{A}} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial X} & \dfrac{\partial}{\partial Y} & \dfrac{\partial}{\partial Z} \\ A_x & A_Y & A_Z \end{vmatrix}$$

Hence, for example:

$$(\boldsymbol{\nabla} \times \boldsymbol{\mathcal{A}})_X = \frac{\partial A_Z}{\partial Y} - \frac{\partial A_Y}{\partial Z} = \mathcal{B}_X$$

In this way we can show that:

$$\Gamma_{12} = \frac{\partial A_Y}{\partial X} - \frac{\partial A_X}{\partial Y} = -\Gamma_{21} = \mathcal{B}_Z$$

$$\Gamma_{31} = \frac{\partial A_X}{\partial Z} - \frac{\partial A_Z}{\partial X} = -\Gamma_{13} = \mathcal{B}_Y$$

$$\Gamma_{23} = \frac{\partial A_Z}{\partial Y} - \frac{\partial A_Y}{\partial Z} = -\Gamma_{32} = \mathcal{B}_X$$

Thus we know all the space-like 3-D components.

To discover the other time-like components of $\Gamma_{\mu\nu}$ we recall the definition of ϕ in eqn (4.7):

$$\boldsymbol{\mathcal{E}} = -\boldsymbol{\nabla}\phi - \frac{\partial \boldsymbol{\mathcal{A}}}{\partial t}$$

Hence, for example:

$$\mathcal{E}_X = -\frac{\partial \phi}{\partial X} - \frac{\partial \mathcal{A}_X}{\partial t}$$

In this way we can show that:

$$\Gamma_{41} = \frac{\partial A_X}{\partial X_4} - \frac{\partial A_4}{\partial X}$$

$$\Gamma_{41} = -\frac{i}{c}\frac{\partial A_X}{\partial t} - \frac{i}{c}\frac{\partial \phi}{\partial X}$$

$$\Gamma_{41} = -\frac{i}{c}\left(\frac{\partial A_X}{\partial t} + \frac{\partial \phi}{\partial X}\right) = \frac{i}{c}\,\mathcal{E}_X = -\Gamma_{14}$$

Similarly:

$$\Gamma_{42} = \frac{i}{c}\,\mathcal{E}_Y = -\Gamma_{24}$$

$$\Gamma_{43} = \frac{i}{c}\,\mathcal{E}_Z = -\Gamma_{34}$$

Thus we can write the antisymmetric tensor $\Gamma_{\mu\nu}$ in terms of $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{B}}$:

$$\Gamma_{\mu\nu} = \begin{pmatrix} 0 & \mathcal{B}_Z & -\mathcal{B}_Y & -\frac{i}{c}\,\mathcal{E}_X \\ -\mathcal{B}_Z & 0 & \mathcal{B}_X & -\frac{i}{c}\,\mathcal{E}_Y \\ \mathcal{B}_Y & -\mathcal{B}_X & 0 & -\frac{i}{c}\,\mathcal{E}_Z \\ \frac{i}{c}\,\mathcal{E}_X & \frac{i}{c}\,\mathcal{E}_Y & \frac{i}{c}\,\mathcal{E}_Z & 0 \end{pmatrix}$$

Maxwell's equations may also be expressed in terms of the covariant tensor $\Gamma_{\mu\nu}$:

$$\frac{\partial \Gamma_{\mu\nu}}{\partial X_\lambda} + \frac{\partial \Gamma_{\nu\lambda}}{\partial X_\mu} + \frac{\partial \Gamma_{\lambda\mu}}{\partial X_\nu} = 0$$

----- (4.17)

$$\frac{\partial \Gamma_{\mu\nu}}{\partial X_\nu} = \mu_0\, J_\mu$$

----- (4.18)

Equations (4.17) and (4.18) are equivalent to eqns (4.14) and (4.15), again demonstrating a form of Maxwell's equations in covariant form.

TRANSFORMATION EQUATIONS FOR EM FIELD

By definition of 2nd rank tensor, the EM Field Tensor transforms according to:

$$\Gamma'_{\mu\nu} = a_{\mu\alpha}\, a_{\nu\beta}\, \Gamma_{\alpha\beta}$$

Recall:

$$a_{\mu\nu} = \begin{pmatrix} \gamma & 0 & 0 & i\frac{\gamma v}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\frac{\gamma v}{c} & 0 & 0 & \gamma \end{pmatrix}$$

Consider Γ'_{14} : $(\mu = 1, \nu = 4)$

Einstein summation rule dictates summing over α and β. This will give us the 16 terms we would expect from a double matrix multiplication, for each new matrix element.

$$\begin{aligned} \Gamma'_{14} = {} & a_{11}a_{41}\Gamma_{11} + a_{12}a_{41}\Gamma_{21} + a_{13}a_{41}\Gamma_{31} + a_{14}a_{41}\Gamma_{41} \\ & + a_{11}a_{42}\Gamma_{12} + a_{12}a_{42}\Gamma_{22} + a_{13}a_{42}\Gamma_{32} + a_{14}a_{42}\Gamma_{42} \\ & + a_{11}a_{43}\Gamma_{13} + a_{12}a_{43}\Gamma_{23} + a_{13}a_{43}\Gamma_{33} + a_{14}a_{43}\Gamma_{43} \\ & + a_{11}a_{44}\Gamma_{14} + a_{12}a_{44}\Gamma_{24} + a_{13}a_{44}\Gamma_{34} + a_{14}a_{44}\Gamma_{44} \end{aligned}$$

The first line expanded $\alpha = 1 - 4$ for $\beta = 1$

The second line expanded $\alpha = 1 - 4$ for $\beta = 2$

The third line expanded $\alpha = 1 - 4$ for $\beta = 3$

The fourth line expanded $\alpha = 1 - 4$ for $\beta = 4$

The 16 terms simplify readily because most of the off-diagonal matrix elements in $a_{\mu\nu}$ and the leading diagonal elements in $\Gamma_{\mu\nu}$ are 0.

$$a_{12} = a_{13} = a_{21} = a_{23} = a_{24} = a_{31} = a_{32} = a_{34} = a_{42} = a_{43} = 0$$

$$\Gamma_{11} = \Gamma_{22} = \Gamma_{33} = \Gamma_{44} = 0$$

Hence:

$$\Gamma'_{14} = a_{11}a_{44}\Gamma_{14} + a_{14}a_{41}\Gamma_{41}$$

But $\Gamma_{\mu\nu}$ is antisymmetric. Hence:

$$\Gamma_{41} = -\Gamma_{14}$$

Therefore:

$$\Gamma'_{14} = (a_{11}a_{44} - a_{14}a_{41})\,\Gamma_{14}$$

Substituting from the LT matrix $a_{\mu\nu}$

$$\Gamma'_{14} = (\gamma^2 - \frac{v^2}{c^2}\,\gamma^2)\,\Gamma_{14}$$

$$= \gamma^2 \left(1 - \frac{v^2}{c^2}\right)\Gamma_{14} = \Gamma_{14}$$

Hence:

$$\Gamma'_{14} = -i\frac{\mathcal{E}'_X}{c} = -i\frac{\mathcal{E}_X}{c}$$

Therefore:

$$\mathcal{E}'_X = \mathcal{E}_X$$

Consider Γ'_{42} : $(\mu = 4, \nu = 2)$

By a similar expansion and equating 14 of the terms to zero it can be shown that:

$$\Gamma'_{42} = -i\,\frac{\gamma v}{c}\,\mathcal{B}_z + i\,\frac{\gamma}{c}\,\mathcal{E}_Y$$

Therefore:

$$i\frac{\mathcal{E}'_Y}{c} = i\,\frac{\gamma}{c}\,(\mathcal{E}_Y - v\,\mathcal{B}_z)$$

$$\mathcal{E}'_Y = \gamma\,(\mathcal{E}_Y - v\,\mathcal{B}_z)$$

The complete set of transformations are:

$$\mathcal{E}'_X = \mathcal{E}_X$$

$$\mathcal{E}'_Y = \gamma\,(\mathcal{E}_Y - v\,\mathcal{B}_z)$$

$$\mathcal{E}'_Z = \gamma\,(\mathcal{E}_Z + v\,\mathcal{B}_Y)$$

$$\mathcal{B}'_X = \mathcal{B}_X$$

$$\mathcal{B}'_Y = \gamma\,(\mathcal{B}_Y + \frac{v}{c^2}\,\mathcal{E}_z)$$

$$\mathcal{B}'_Z = \gamma\,(\mathcal{B}_Z - \frac{v}{c^2}\,\mathcal{E}_Y)$$
