Any calculator, except one with pre-programmable memory, may be used in this examination.

LEVEL 2
Examination contributing to the Degrees of Bachelor of Science (BSc) and Master in Science (MSci)

PHY2006
Mathematical Physics

Duration: 3 hours plus additional 1 hour for upload of work

Friday 6th August 2021
09:30 AM – 1:30 PM

Examiners: Prof S Matthews, Dr F. Peters
and the internal examiners
Dr S Sim (s.sim@qub.ac.uk)

Answer ALL questions in Section A for 10 marks each.
Answer ONE question from Section B for 20 marks.
Answer ONE question from Section C for 20 marks.

If you have any problems or queries, contact the School Office at mpts@qub.ac.uk or 028 9097 1907, and the module lecturer J.Greenwood@qub.ac.uk

SECTION A

Answer ALL questions from Section A

A.1 Two functions $f(x)$ and $g(x)$ are given by

$$f(x) = 1 \qquad\qquad g(x) = x - \frac{1}{2}$$

$f(x)$ and $g(x)$ are orthogonal in the vector space where the inner product $< f(x)|g(x) >$ is defined by

$$< f(x)|g(x) >= \int_0^1 f(x)g(x)\, dx$$

Find the best fit function to $q(x) = \mathrm{e}^{-x}$ that can be constructed from these two orthogonal functions using this definition of the inner product. [10]

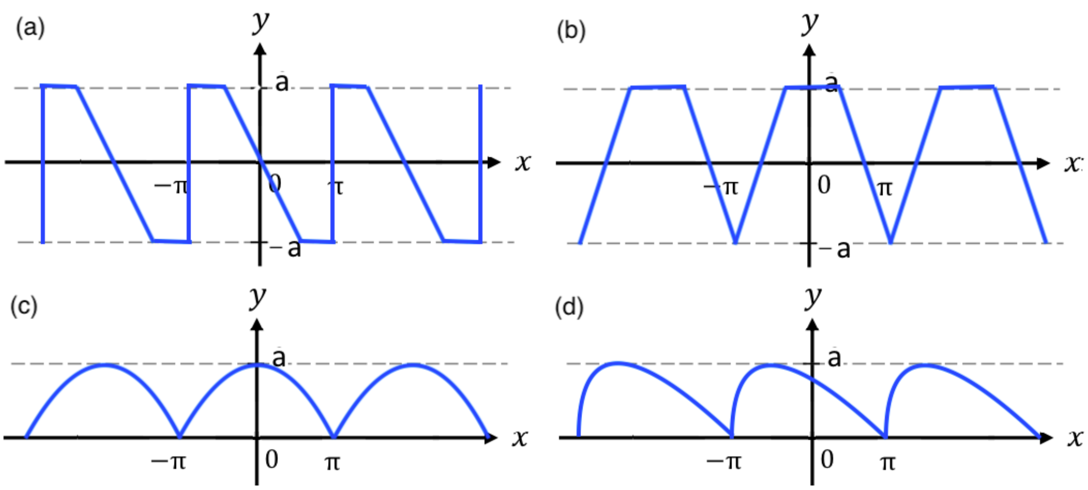

Figure 1: Four periodic functions for question **A.2**

A.2 Consider by inspection each of the four periodic functions shown in figure 1 and the Fourier series which could be used to represent them. For each of these four functions you should indicate for the equivalent Fourier series;

- If a_0 is zero or non-zero.
- If all the a_k values (for $k > 0$) are zero or if at least some of them will be non-zero.
- If all the b_k values are zero or if at least some of them will be non-zero. [10]

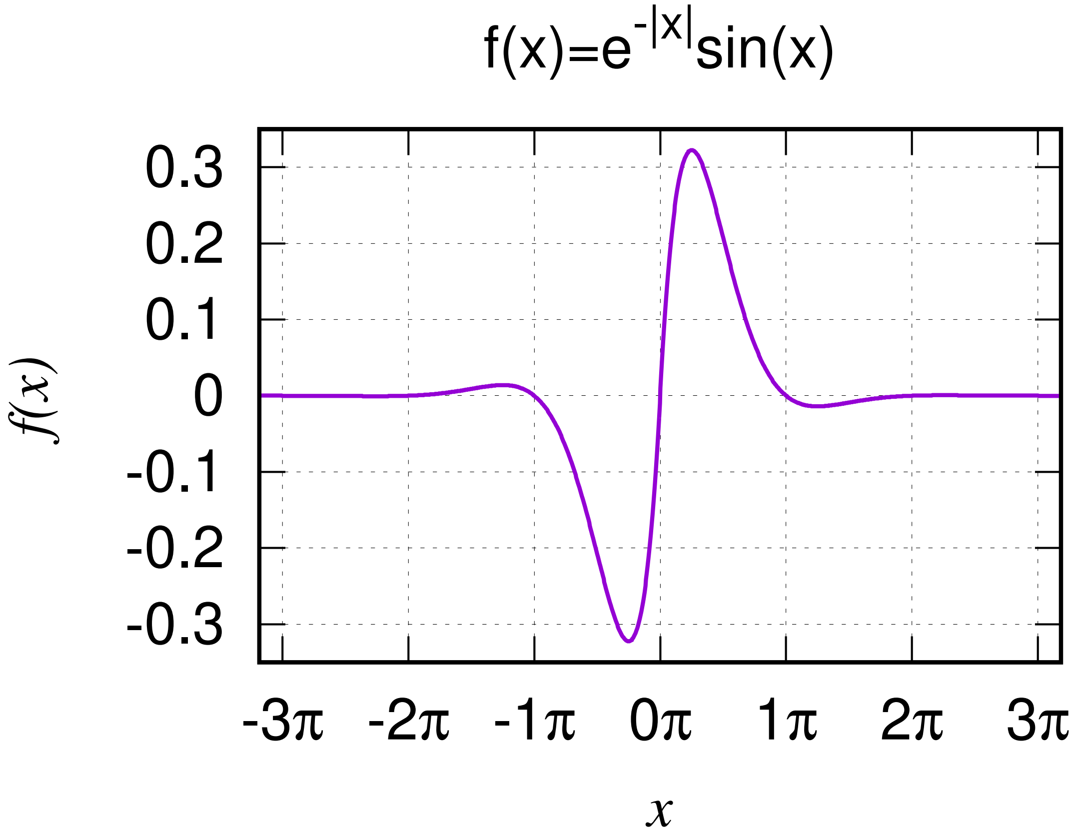

Figure 2: Function in question **A.3**

A.3 Calculate the Fourier transform of the function $f(x) = \mathrm{e}^{-|x|}\sin(x)$ shown in figure 2 which can be usefully defined by

$$f(x) = \mathrm{e}^{x}\sin(x) \qquad x < 0$$
$$f(x) = \mathrm{e}^{-x}\sin(x) \qquad x \geq 0$$

You should put the answer in the simplest terms possible.

Note that you will probably find it useful to use the substitution

$$\sin(x) = \frac{\mathrm{e}^{ix} - \mathrm{e}^{-ix}}{2i}$$

[10]

A.4 Show that

$$\psi(x,t) = A\exp\left(-\gamma x^2\right)\exp\left(\frac{-iEt}{\hbar}\right)$$

is a solution to the equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + \frac{1}{2}m\omega^2 x^2\psi$$

and hence obtain expressions for γ and E. [10]

A.5 Carefully explaining your answers, characterise the following partial differential equation in terms of the following:

- Order
- Linear : Non-linear
- Homogeneous : Inhomogeneous
- Elliptical : Parabolic : Hyperbolic : Mixed : Undefined

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$$

[10]

A.6 Using the method of characteristics, find the solution to the following first order partial differential equation

$$\frac{\partial u}{\partial t} + 2t\frac{\partial u}{\partial x} = -u$$

subject to the initial condition $u(x,0) = \sin x$ [10]

SECTION B

Answer ONE question from Section B

B.1 The function $f(x)$ is defined by

$$\begin{aligned}
f(x) &= \pi + x & -\pi \leq x < -\frac{\pi}{2} \\
f(x) &= 0 & -\frac{\pi}{2} \leq x < \frac{\pi}{2} \\
f(x) &= \pi - x & \frac{\pi}{2} \leq x < \pi \\
f(x) &= f(x + 2\pi)
\end{aligned}$$

(a) Carefully draw a sketch of the function $f(x)$. [5]

(b) Is $f(x)$ even, odd, or neither? [1]

(c) Indicate if any of the terms a_0, a_k and b_k of the Fourier series expansion of $f(x)$ are expected to be zero by inspection of the sketch of $f(x)$ and briefly explain your reasoning. [3]

(d) Determine the terms a_0, a_k and b_k of the Fourier series expansion of $f(x)$. *Note* that it is not necessary to explicitly calculate any terms that you have determined to be zero by inspection and your answer for non-zero terms may be an equation that depends on k, for example. [8]

(e) Determine individual expressions (equations/numerical values) for the terms a_0, a_1, a_2, a_3, a_4, b_1, b_2, b_3 and b_4. [3]

B.2 (a) A three dimensional subspace within the $\mathbb{R}^5$ vector space is defined by the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$;

$$\mathbf{a} = \begin{pmatrix} -1 \\ 2 \\ 1 \\ 2 \\ 3 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \\ 4 \end{pmatrix} \qquad \mathbf{c} = \begin{pmatrix} 4 \\ 2 \\ -2 \\ -4 \\ -3 \end{pmatrix}$$

i. Perform a calculation to demonstrate that the vectors $\mathbf{a}$ and $\mathbf{b}$ are not orthogonal. [1]

ii. Use Gram-Schmidt orthogonalisation to calculate from $\mathbf{b}$ a modified vector $\mathbf{b}'$, which is perpendicular to $\mathbf{a}$ [3]

iii. Determine $\mathbf{c}'$, a vector perpendicular to $\mathbf{a}$ and $\mathbf{b}'$, from $\mathbf{c}$ using Gram-Schmidt orthogonalisation [6]

(b) Consider carefully the inner product calculation method for the functions $f(x)$ and $g(x)$ and note the integral range of x from -1 to 1;

$$< f(x)|g(x) >= \int_{-1}^{1} (f(x))\,(g(x))\; dx$$

consider the functions $f(x)$, $g(x)$ and $h(x)$

$$f(x) = x \qquad g(x) = x^3 \qquad h(x) = x^5$$

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i. Use the Gram-Schmidt orthogonalisation procedure to generate a function $g'(x)$ from $g(x)$ which is orthogonal to $f(x)$ using the inner product defined above. [3]

ii. Now use Gram-Schmidt orthogonalisation to determine the function $h'(x)$ from $h(x)$, which is orthogonal to $f(x)$ and $g'(x)$ using the inner product as defined above. [6]

iii. predict without further calculation if the functions $f(x)$, $g'(x)$ and $h'(x)$ are orthogonal to the functions $j(x) = 1$ and $k(x) = x^2 - 1/3$. Explain your reasoning. [1]

SECTION C

Answer ONE question from Section C

C.1 A heated, solid stainless steel sphere of radius $R = 0.1$ m is dropped at time $t = 0$ into iced water acting as a heat reservoir at 0°C. The cooling of the sphere is governed by the Heat Equation in spherical coordinates where $T(r,t)$ is the temperature, r is the distance from the centre of the sphere and D is the heat diffusivity.

$$\frac{\partial T}{\partial t} = D\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)$$

(a) Use the separation of variables method with a solution of the form $T(r,t) = X(t)Y(r)$ to obtain the following ordinary differential equations where $-\lambda^2$ is the separation constant.

$$\frac{dX}{dt} + \lambda^2 DX = 0 \qquad\qquad \frac{d^2Y}{dr^2} + \frac{2}{r}\frac{dY}{dr} + \lambda^2 Y = 0$$

[5]

(b) By making the substitution $Y = u/r$ show the solution to these equations is

$$T(r,t) = \exp\left(-\lambda^2 Dt\right)\left(A\frac{\sin(\lambda r)}{r} + B\frac{\cos(\lambda r)}{r}\right)$$

[5]

(c) Given and the temperature at the centre of the sphere $T(0,t)$ is finite and the surface of the sphere is fixed at $T(R,t) = 0$, obtain allowed values for B and λ, and write down an expression for the most general solution as a Fourier series. [5]

(d) For most values of t only the first term in the Fourier series is significant. If the centre of the sphere is at a temperature of 60°C at a particular time, what is the temperature at a distance $r = 0.09$ m from the centre? [5]

$$\left[\lim_{x\to 0}\frac{\sin ax}{x} = a\right]$$

C.2 The velocity of a car v is governed by the ordinary differential equation

$$m\frac{dv}{dt} = F - Kv^2$$

F – forward thrust (force) of the engine

m = 2000 kg – mass of car

K = 1 kg m^{-1} – drag constant due to air resistance

(a) To solve this equation using the Euler method, show that the following finite difference equation should be used.

$$v_{i+1} = v_i + \frac{\Delta t}{m}\left(F - Kv_i^2\right)$$

[4]

(b) If car is travelling at a velocity of 30 m/s and the accelerator is pressed so that F= 5000 N. How long does it take the car to reach 55 m/s? Use time steps of Δt= 2 to complete the following table and determine the answer. [9]

i	t_i	v_i
0	0	30
1	2	34.1
2	4	⋮
3	6	⋮
4	8	⋮
⋮	⋮	⋮
⋮	⋮	⋮

(c) i. How much quicker would this take if there was no air resistance?

ii. What value of Δt would reduce the numerical error by about a factor of 5?

iii. Describe a numerical method which could give a more accurate answer for Δt = 2. [7]

END OF EXAMINATION