1.2. Multivariable Calculus

Partial/total derivatives, Notation

Partial Derviatives

- Denoted by ∂
  - For function $u(x, y)$, possible partial derivatives

$$\left.\frac{\partial u}{\partial x}\right|_y \equiv \frac{\partial u}{\partial x}$$

$\leftarrow$ y constant

Differentiate u w.r.t. x, keep all other independent variables constant

$$\frac{\partial^2 u}{\partial x^2}, \quad \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

COMMUTATIVE

- Example $u(x, y) = \frac{x}{y} + x^2 y$

$$\frac{\partial u}{\partial x} = \frac{1}{y} + 2xy \qquad \frac{\partial^2 u}{\partial x^2} = 0 + 2y \qquad \frac{\partial^2 u}{\partial x \partial y} = -\frac{1}{y^2} + 2x$$

$$\frac{\partial u}{\partial y} = -\frac{x}{y^2} + x^2 \qquad \frac{\partial^2 u}{\partial y^2} = \frac{2x}{y^3}$$

Partial Derviatives

- Product rule
  - $u(x, y) = A(x, y)B(x, y)$

$$\frac{\partial u}{\partial x} = \frac{\partial A}{\partial x} B + A \frac{\partial B}{\partial x}$$

- Chain rule
  - $u(x, y) = A\big(B(x, y)\big)$

$$\frac{\partial u}{\partial x} = \frac{\partial A}{\partial B} \cdot \frac{\partial B}{\partial x}$$

- Example $\boxed{u = x\sin(xy)}$, $A = x, B = \sin C, C = xy$
  - $\dfrac{\partial u}{\partial x} = \underbrace{\dfrac{\partial A}{\partial x}} B + A \underbrace{\dfrac{\partial B}{\partial C} \cdot \dfrac{\partial C}{\partial x}}$

$$\frac{\partial B}{\partial x}$$

$$u = A \cdot B$$
$$= A \cdot \sin C$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x) \cdot \sin C + x \frac{\partial}{\partial C}(\sin C) \cdot \frac{\partial}{\partial x}(xy)$$
$$= \sin(xy) + xy\cos(xy)$$

Total Derivatives

- For multi-variable functions, e.g. $u(x, y)$
  - $\frac{\partial}{\partial x}$ differentiation with respect to (w.r.t.) x while y constant
  - $\frac{d}{dx}$ differentiation w.r.t. x but y is non-constant
  - Total derivative $\frac{du}{dx}$ can only be obtained if there are constraints on x and y.

TOTAL PARTIAL

$$\boxed{\frac{du}{dx} \neq \frac{\partial u}{\partial x}}$$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} \cdot \underbrace{\frac{\partial x}{\partial x}}_{=1} + \frac{\partial u}{\partial y} \cdot \underbrace{\frac{\partial y}{\partial x}}_{\frac{dy}{dx}}$$

$y = f(x)$

TOTAL DIFFERENTIAL

- Example $u(x, y) = \frac{x}{y} + x^2 y$ constraint $x = \cos y$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}$$

$$\frac{\partial u}{\partial x} = \frac{1}{y} + 2xy \qquad \frac{\partial u}{\partial y} = -\frac{x}{y^2} + x^2 \qquad 1 = -\sin y \frac{dy}{dx}$$

$$\frac{du}{dx} = \frac{1}{y} + 2xy + \left(-\frac{x^2}{y^2} + x^2\right)\left(-\frac{1}{\sin y}\right)$$

Shorthand Notation used in various texts

Ordinary Derivatives

$$\frac{du(x)}{dx} \equiv \cancel{f}u'(x) \qquad \frac{d^2u(x)}{dx^2} \equiv u''(x)$$

$$\frac{d^{(n)}u(x)}{dx} \equiv u^{(n)}(x)$$

Time derivatives

$$\frac{dx}{dt} = \dot{x} \qquad \frac{d^2x}{dt^2} = \ddot{x} \qquad \frac{d^3x}{dt^3} = \dddot{x}$$

Partial Derivatives

$$\frac{\partial u}{\partial x} \equiv \underline{u'_x} \equiv \underline{\partial_x u}$$

$$\frac{\partial^2 u}{\partial x^2} \equiv u''_x \equiv \partial_{xx} u \equiv \partial^2_x u \qquad \frac{\partial^2 u}{\partial x \partial y} = u''_{xy} \equiv \partial_{xy} u$$

Derivative evaluated at a constant value of an independent variable

$$\left.\frac{df}{dx}\right|_{x=a} = f'(a)$$

1.3. Differential Operators

Grad, Div, Curl Definitions and Meaning

Gradient - Grad

- Gradient of a line $\boxed{u(x) \text{ is } \frac{du}{dx}}$
- What is the gradient at a point on a surface $u(x,y)$?

Magnitude and direction of greatest rate of change

$\underline{\hat{i}}, \underline{\hat{j}}, \underline{\hat{k}}$ UNIT VECTORS

DIRECTION x, y, z

$\underline{i}$

∇ NABLA (DEL OPERATOR)

$$\nabla = \frac{\partial}{\partial x}\underline{\hat{i}} + \frac{\partial}{\partial y}\underline{\hat{j}} + \frac{\partial}{\partial z}\underline{\hat{k}}$$

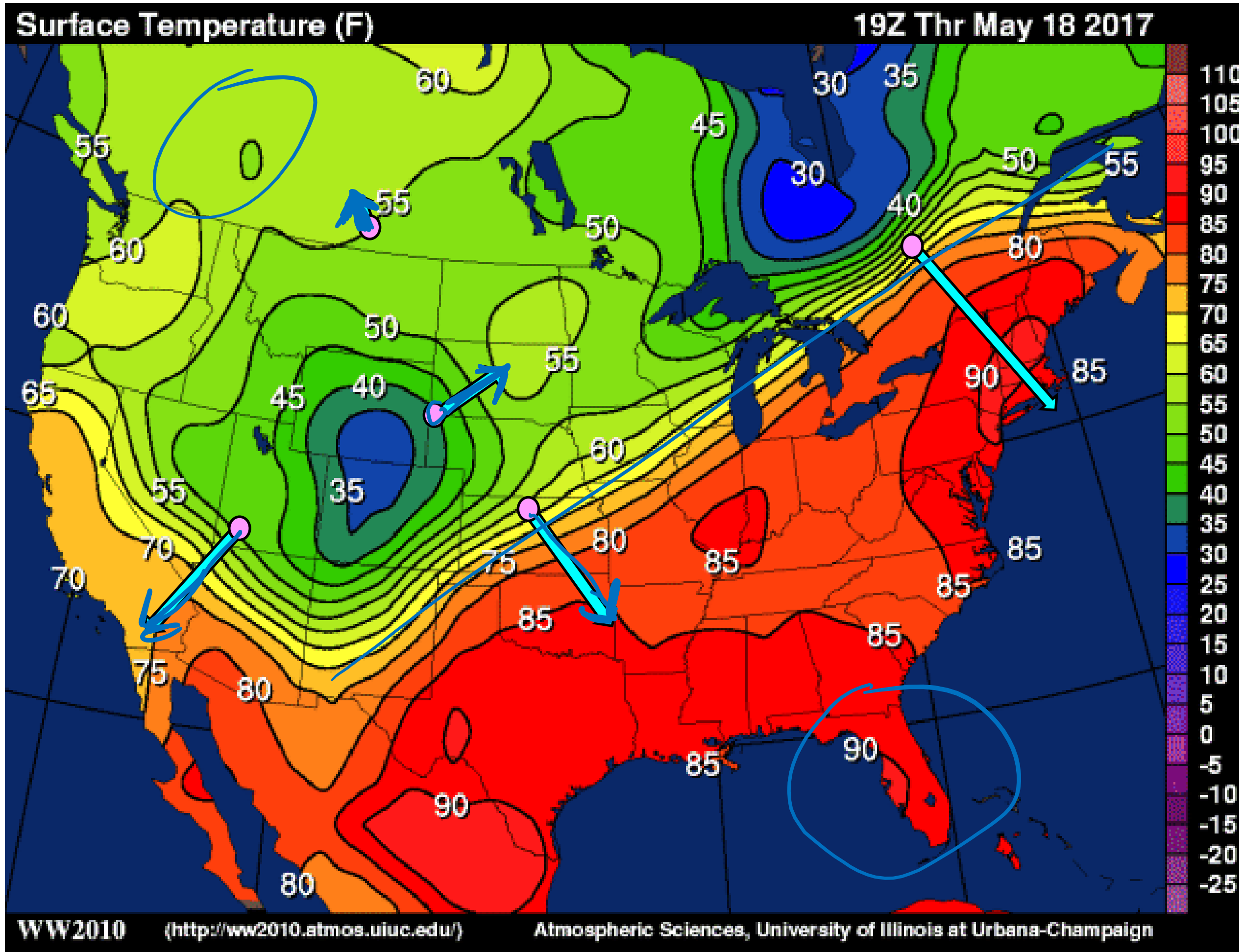
- Example: Temperature $T(x,y,z)$ Gradient

$$\text{GRAD}(T) = \nabla T = \frac{\partial T}{\partial x}\underline{\hat{i}} + \frac{\partial T}{\partial y}\underline{\hat{j}} + \frac{\partial T}{\partial z}\underline{\hat{k}}$$

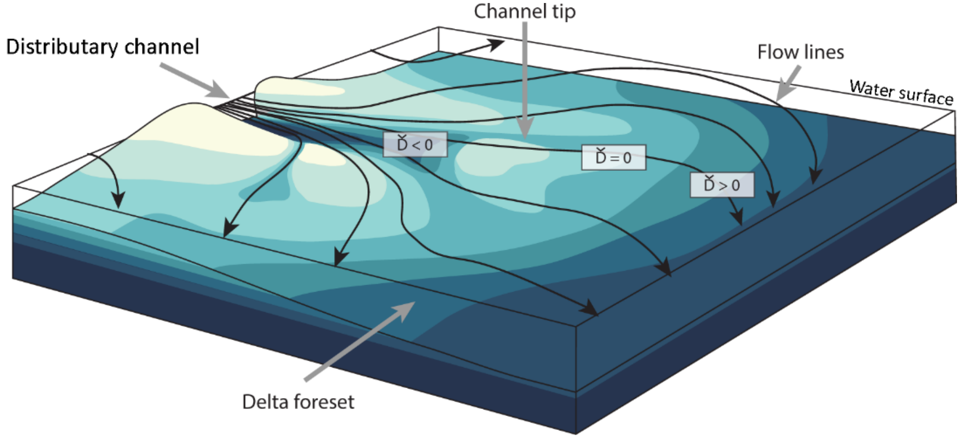
$$T(x,y,z) = A + Bz - Cx$$

$$\underset{\text{SCALAR}}{\nabla T} = \underbrace{B\underline{\hat{k}} - C\underline{\hat{i}}}_{\text{VECTOR}}$$

- Grad acts on a scalar to give a vector



Divergence - Div



- Scalar product of Del operator with a vector field

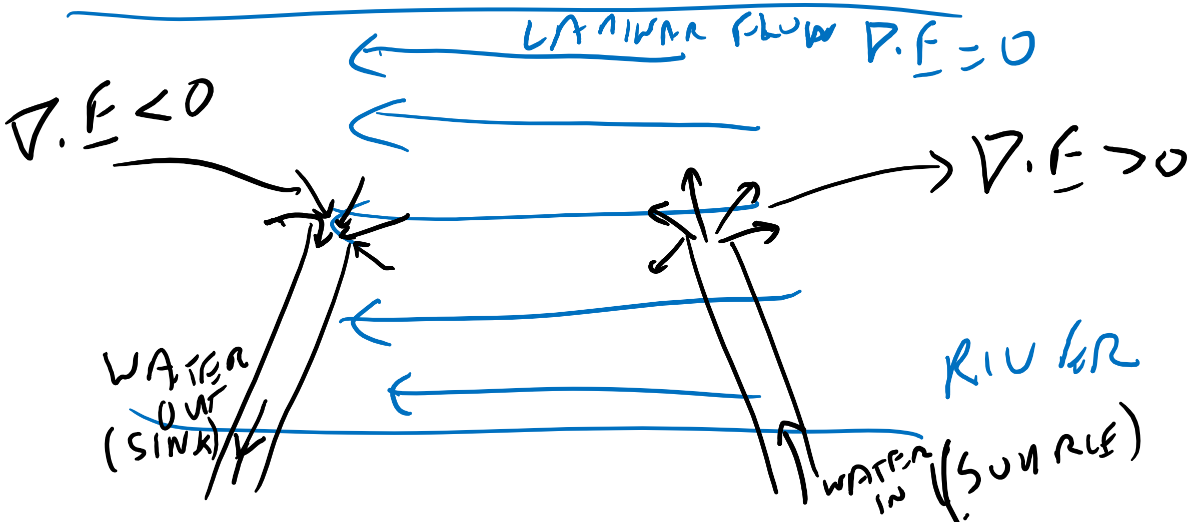
VECTOR SCALAR

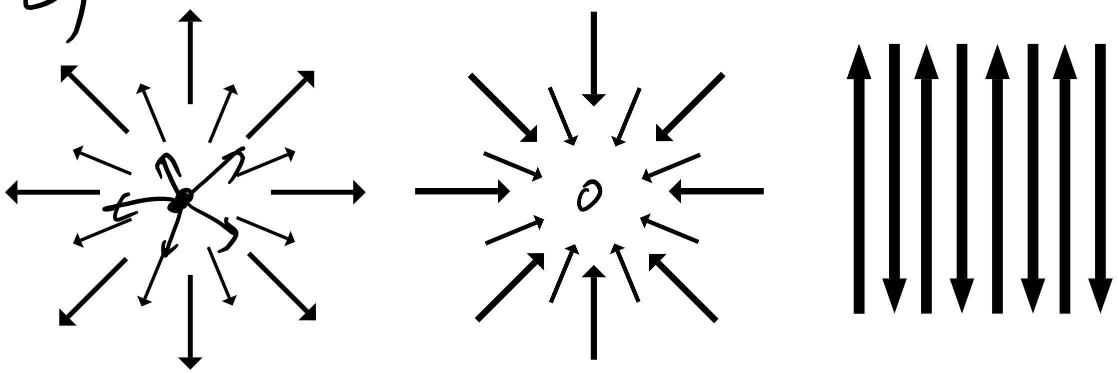
- $$\underline{\nabla . \boldsymbol{F}} = \underline{\text{Div}} \, \boldsymbol{F} = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} . \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

- ***F*** can represent flux of a quantity, e.g. directional flow of a fluid per unit area

$\nabla \cdot \underline{F}$ REPRESENTS EXPANSION (>0)

COMPRESSION (<0)





$\partial/\partial x(\mathbf{V}_x) > 0$	$\partial/\partial x(\mathbf{V}_x) < 0$	$\partial/\partial x(\mathbf{V}_x) = 0$
$\partial/\partial y(\mathbf{V}_y) > 0$	$\partial/\partial y(\mathbf{V}_y) < 0$	$\partial/\partial y(\mathbf{V}_y) = 0$
$\underline{\nabla\cdot(\mathbf{V}) > 0}$	$\underline{\nabla\cdot(\mathbf{V}) < 0}$	$\underline{\nabla\cdot(\mathbf{V}) = 0}$
SOURCE	SINK	

Curl

- Cross Product of Del operator with a vector field
- Curl $\boldsymbol{F} = \boldsymbol{\nabla} \times \boldsymbol{F} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \widehat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{\boldsymbol{i}} - \left(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z}\right)\hat{\boldsymbol{j}} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\widehat{\boldsymbol{k}}$

VECTOR

VECTOR

- $\boldsymbol{\nabla} \times \boldsymbol{F}$ represents the degree of circulation of $\boldsymbol{F}$ about a point in space
    - CURL PERPENDICULAR TO CIRCULATION DIRECTION

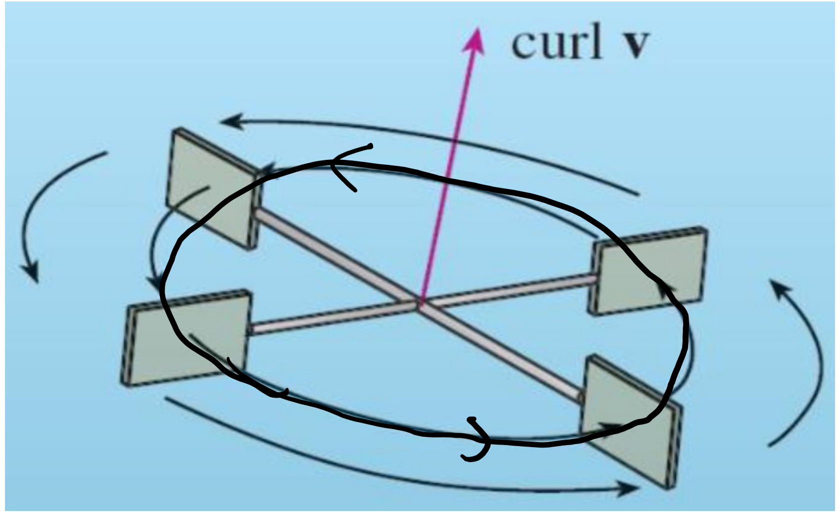
RIGHT HAND — FINGERS $\underline{F}$

— THUMB $\nabla \times \underline{F}$

$\boxed{\nabla \times \underline{F} = 0}$ FIELD IS CONSERVATIVE

WORK DONE IS INDEPENDENT OF PATH

E.G. GRAVITY



Laplacian ∇^2

- $\text{Div.}(\text{Grad }\Phi) = \boldsymbol{\nabla}.(\boldsymbol{\nabla}\Phi) = \boxed{\nabla^2}\Phi$
- Can act on scalar or vector quantity
- Wave equation in 1D
- $\underbrace{\frac{\partial^2\Phi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\Phi}{\partial t^2}}$
- Wave equation in 3D
- $\nabla^2\Phi = \frac{1}{v^2}\frac{\partial^2\Phi}{\partial t^2}$

$$\nabla^2 = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} + \frac{\partial}{\partial z^2}$$

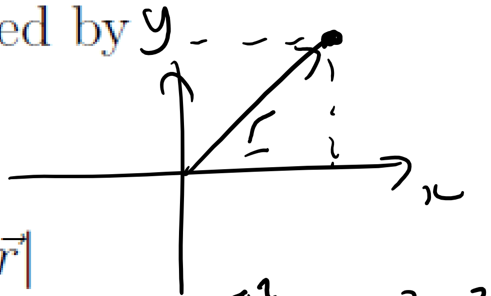
SCALAR OPERATOR

SCALAR $\longrightarrow$ SCALAR

VECTOR $\longrightarrow$ VECTOR

Example – PHY2006 exam 2019-20

A.4 Consider the velocity distribution, $\vec{v}$, for points in the (x, y)-plane defined by



$$\boxed{\vec{v}} = \vec{\Omega} \times \vec{r}$$

where $\quad \vec{\Omega} = (0, 0, \cos(\pi r)) \qquad \vec{r} = (x, y, 0) \qquad$ and $\qquad r = |\vec{r}|$

$|r|^2 = x^2 + y^2$

(a) Calculate the velocity $\vec{v}$ at all points in the (x, y)-plane in terms of x, y and r.

Note that $\vec{v} = \vec{\Omega} \times \vec{r}$

$$\underline{v} = \underline{\Omega} \times \underline{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & \cos(\pi r) \\ x & y & 0 \end{vmatrix} = -y\cos(\pi r)\,\hat{i} - \left(-x\cos(\pi r)\right)\hat{j}$$

$$= -y\cos(\pi r)\,\hat{i} + x\cos(\pi r)\,\hat{j}$$

Example – PHY2006 exam 2019-20

(b) Draw a schematic diagram to show the velocity vector field; indicate the direction and magnitude of the velocity at different points in the (x, y)-plane with arrows. (*Hint:* Consider how the velocity depends on r and consider the cases $r = 1$ and $r = 2$.) **[2]**

$$\underline{v} = -y\cos(\pi r)\,\underline{i} + x\cos(\pi r)\,\underline{j}$$

$r = 1 \quad \cos(\pi r) = -1$

$r = 2 \quad \cos(2\pi) = +1$

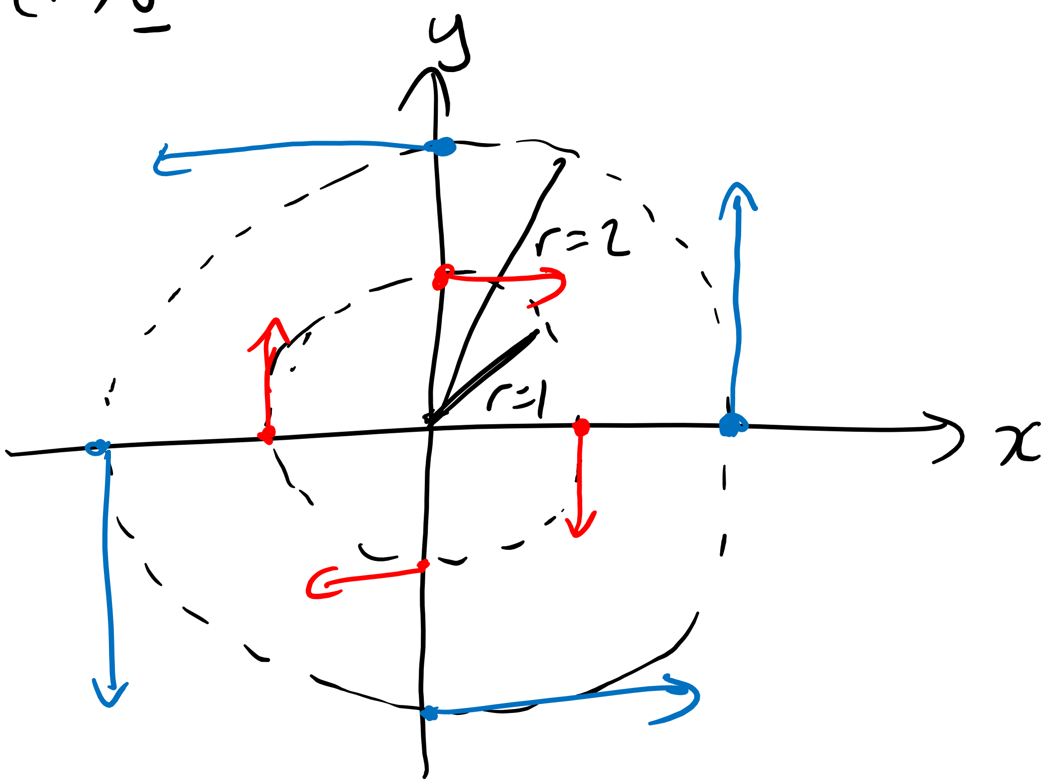
$$\underline{v}(r=1) = y\,\hat{\underline{i}} - x\,\hat{\underline{j}}$$

$1 \qquad 0$

$$\underline{v}(r=2) = -y\,\hat{\underline{i}} + x\,\hat{\underline{j}}$$

$0 \qquad 2$

$2 \qquad 0$



Example – PHY2006 exam 2019-20

(c) Calculate the divergence of the velocity; div $\vec{v} = \nabla \cdot \vec{v}$

(d) Calculate the rotational of the velocity; curl $\vec{v} = \nabla \times \vec{v}$

$$r^2 = x^2 + y^2$$

$$2r\frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

(c)

$$\nabla \cdot \underline{v} = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \cdot \begin{pmatrix} v_x = -y\cos(\pi r) \\ v_y = x\cos(\pi r) \\ v_z = 0 \end{pmatrix} = -y\left(-\pi\sin(\pi r)\frac{\partial r}{\partial x}\right) + x\left(-\pi\sin(\pi r)\frac{\partial r}{\partial y}\right)$$

$$\nabla \cdot \underline{v} = \pi\frac{xy}{r}\sin\pi r - \pi\frac{xy}{r}\sin\pi r = 0$$

(d)

$$\nabla \times \underline{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y\cos(\pi r) & x\cos(\pi r) & 0 \end{vmatrix}$$

$$\nabla \times \underline{v} = \left(2\cos(\pi r) - \pi\sin(\pi r)\right)\hat{k}$$

1.4. Taylor’s Theorem

Power Series Expansion

Taylor Expansion

- $u(x)$ is a continuous single–valued function of x with continuous derivatives
- Near $x = a$ the value of the function can be approximated by

$$u(a + \Delta x) = u(a) + \Delta x\, u'(a) + \frac{(\Delta x)^2}{2!} u''(a) + \frac{(\Delta x)^3}{3!} u'''(a) \ldots\ldots = \sum_{n=0}^{\infty} \frac{(\Delta x)^n}{n!} u^{(n)}(a)$$

$a = 0$ EXPANDING AROUND $x = 0$ MACLAURIN SERIES

- Example expand $\boxed{u(x) = \ln x}$ around $x = 1$ $(a = 1)$

$u' = \frac{1}{x}$ $\quad u'' = -\frac{1}{x^2}$ $\quad u''' = \frac{2}{x^3}$

$$u(1 + \Delta x) = 0 + \Delta x - \frac{(\Delta x)^2}{2} + \frac{(\Delta x)^2}{3} - \ldots\ldots$$

$u = \underset{=0}{\sin x}$ AROUND $x = 0$ (MACLAURIN)

$u' = \underset{=1}{\cos x}$ $\quad u'' = \underset{=0}{-\sin x}$ $\quad u''' = \underset{=-1}{-\cos x}$

$$u(\Delta x) = \Delta x - \frac{(\Delta x)^3}{3!} + \frac{(\Delta x)^5}{5!} \ldots\ldots$$

Definition of Derivatives

- Taylor's theorem can be used to define derivative
- 1st order $u(a+\Delta x) \approx u(a) + \Delta x\, u'(a)$ + ...

$$u'(a) = \left.\frac{du}{dx}\right|_a = \lim_{\Delta x \to 0} \frac{u(a+\Delta x) - u(a)}{\Delta x}$$

- 2nd order $\boxed{\left.\frac{d^2u}{dx^2}\right|_a = \lim_{\Delta x\to 0} \frac{\left.\frac{du}{dx}\right|_{a+\Delta x} - \left.\frac{du}{dx}\right|_a}{\Delta x}}$ but using Taylor we can also express
  - $u(a+\Delta x) \approx u(a) + \Delta x\, u'(a) + \frac{\Delta x^2}{2!} u''(a)$
  - $u(a-\Delta x) \approx u(a) - \Delta x\, u'(a) + \frac{\Delta x^2}{2!} u''(a)$

$$u(a+\Delta x) + u(a-\Delta x) \approx 2u(a) + (\Delta x)^2 u''(a)$$

$$u''(a) = \left.\frac{d^2u}{dx^2}\right|_a = \lim_{\Delta x\to 0} \frac{u(a+\Delta x) + u(a-\Delta x) - 2u(a)}{(\Delta x)^2}$$

- Approximate Taylor expression for derivatives key for numerical solutions of DEs

TAYLOR SERIES

$$u(a+\Delta x) = \underset{0\text{TH}}{u(a)\frac{(\Delta x)^0}{0!}} + \underset{1\text{ST}}{u'(a)\frac{(\Delta x)^1}{1!}} + \underset{2\text{ND}}{u''(a)\frac{(\Delta x)^2}{2!}} + u'''(a)\frac{(\Delta x)^3}{3!} \cdots$$

$$= u(a) + \Delta x\, u'(a) + \frac{(\Delta x)^2}{2!}u''(a) \cdots$$

$$u(x) = \frac{1}{1-x}$$ AT $x=0$ ($a=0$ MACLAURIN)

$$v = 1 - x \qquad \frac{dv}{dx} = -1$$

$$u(x) = v^{-1}$$

$$u'(x) = -v^{-2}\frac{dv}{dx} = v^{-2}$$

$$u''(x) = -2v^{-3}\frac{dv}{dx} = 2v^{-3}$$

$$u'''(x) = -6v^{-4}\frac{dv}{dx} = 6v^{-4}$$

	$x=0,\ v=1$	
$u'(0)$	1	$1!$
$u''(0)$	2	$2!$
$u'''(0)$	6	$3!$

$$u(0+x) = \overset{1}{\cancel{u(0)}} + x\,\cancel{u'(0)} + \frac{x^2}{\cancel{2!}}\cancel{u''(0)} + \frac{x^3}{\cancel{3!}}\cancel{u'''(0)} \cdots$$