Lecture 4:
Solved problems on operators

4.1 Problem: *Find eigenvalues and eigenfunctions of the operator:* $A = x + \mathrm{d}/\mathrm{d}x$

A wavefunction ψ is an eigenfunction for the operator A, with eigenvalue λ if $A\psi = \lambda\psi$. We thus have:

$$x + \frac{\mathrm{d}\psi}{\mathrm{d}x} = \lambda\psi \rightarrow \frac{\mathrm{d}\psi}{\mathrm{d}x} = \psi(\lambda - x) \tag{1}$$

By separating variables we have:

$$\frac{\mathrm{d}\psi}{\psi} = \mathrm{d}x(\lambda - x) \rightarrow \int \frac{\mathrm{d}\psi}{\psi} = \int \mathrm{d}x(\lambda - x) \tag{2}$$

Solving the two integrals we have:

$$\log|\psi| = \lambda x - \frac{x^2}{2} \rightarrow \psi = e^{-x^2/2}e^{\lambda x} \tag{3}$$

Any function of the form above, with λ being any complex number (included zero), is eigenfunction of the operator A. In this case then, the eigenvalues are the whole ensemble of complex numbers $\mathbb{C}$. This is an example of a continuous set of eigenvalues (sometimes also called the *spectrum* of the operator A). ■

4.2 Problem: *Given two operators such as [L,M] = 1, calculate:* $[L, M^2]$

The commutator reads:

$$[L, M^2] \equiv LM^2 - M^2L \equiv LMM - MML \tag{4}$$

We must remember here that the order in which operators are written is important and cannot be changed at will, unless the commutation rules are used. In this case, LMM is in principle different from MML. However, we know that $[L, M] = LM - ML = 1$, $LM = 1 + ML$:

$$LMM - MML = (1 + ML)M - MML = M + MLM - MML \tag{5}$$

Now, we can factorise M (only because it is on the left side for all the members!) and we have:

$$M + MLM - MML = M(1 + LM - ML) \tag{6}$$

The term in parenthesis is 1 + the commutator $[L, M]$. We thus have:

$$[L, M^2] = 2M\blacksquare \tag{7}$$

4.3 Problem: *Find the commutation rule:* $[\vec{p}, \vec{A}(\vec{r})]$, *being* $\vec{A}(\vec{r})$ *the vector potential and* $\vec{p} = -i\hbar\nabla$ *the operator momentum*

The commutation rule of operators must be always intended as if applied to a generic wavefunction:

$$[-i\hbar\nabla, \vec{A}(\vec{r})]\psi = -i\hbar\nabla \cdot \left(\vec{A}(\vec{r})\psi\right) + i\hbar\vec{A}(\vec{r})\nabla\psi \tag{8}$$

This is equal to:

$$[-i\hbar\nabla, \vec{A}(\vec{r})]\psi = -i\hbar\left(\nabla \cdot \vec{A}(\vec{r})\right)\psi - i\hbar\vec{A}(\vec{r})\nabla\psi + i\hbar\vec{A}(\vec{r})\nabla\psi = -i\hbar\left(\nabla \cdot \vec{A}(\vec{r})\right)\psi \tag{9}$$

So, generally speaking the commutator is equal to:

$$[-i\hbar\nabla, \vec{A}(\vec{r})] = -i\hbar\nabla \cdot \vec{A}(\vec{r}) \tag{10}$$

We know that the vector potential is not univocally defined. If one adds the gradient of a scalar potential, the physical meaning of $\vec{A}(\vec{r})$ is unchanged. There are different possible choices of gauge, and we will show what is the result of this commutator if two of those are chosen: the Coulomb gauge and the Lorenz gauge.

Coulomb gauge:

$$\nabla \cdot \vec{A}(\vec{r}) = 0 \rightarrow [\vec{p}, \vec{A}(\vec{r})] = 0 \tag{11}$$

In the Coulomb gauge, momentum and vector potential commute.

Lorenz gauge:

$$\nabla \cdot \vec{A}(\vec{r}) + \frac{1}{c^2}\frac{\mathrm{d}\,V}{\mathrm{d}\,t} = 0 \rightarrow [\vec{p}, \vec{A}(\vec{r})] = -\frac{1}{c^2}\frac{\mathrm{d}\,V}{\mathrm{d}\,t} \tag{12}$$

In the Lorenz gauge, momentum and vector potential do not commute. $\blacksquare$

4.4 Problem: *Find the eigenvectors and eigenfunctions of the matrices* σ_1, σ_2*:*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{13}$$

We start with σ_1:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \det \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = 0 \tag{14}$$

The determinant is: $\lambda^2 - 1 = 0$ which implies $\lambda = \pm 1$. There are then only two possible eigenvalues. The related eigenvectors are:

$$\lambda = 1 \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to x_1 = x_2 \tag{15}$$

So any vector of the form:

$$\alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{16}$$

with $\alpha \in \mathbb{C}$ is an eigenvector of σ_1 with eigenvalue $\lambda = 1$.

Then, if $\lambda = -1$:

$$\lambda = -1 \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = - \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to x_1 = -x_2 \tag{17}$$

So, in this case any vector of the form:

$$\alpha \begin{pmatrix} +1 \\ -1 \end{pmatrix} \tag{18}$$

with $\alpha \in \mathbb{C}$ is an eigenvector of σ_1 with eigenvalue $\lambda = -1$.

We now proceed to consider σ_2:

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \det \begin{pmatrix} -\lambda & -i \\ i & -\lambda \end{pmatrix} = 0 \tag{19}$$

The determinant is $\lambda^2 + 1 = 0$ which implies $\lambda = \pm 1$. Thus, σ_1 and σ_2 have the same eigenvalues. In this case though, the related eigenvectors are:

$$\lambda = i \to \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = i \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to x_1 = ix_2 \tag{20}$$

This means that any vector of the form:

$$\alpha \begin{pmatrix} 1 \\ i \end{pmatrix} \tag{21}$$

with $\alpha \in \mathbb{C}$ is an eigenvector of σ_2 with eigenvalue $\lambda = 1$.

Analogously, we can show that any vector of the form:

$$\alpha \begin{pmatrix} 1 \\ -i \end{pmatrix} \tag{22}$$

with $\alpha \in \mathbb{C}$ is an eigenvector of σ_2 with eigenvalue $\lambda = -1$.■