Lecture 20:
Perturbation theory: time-dependent perturbations

Let us assume we have a non-degenerate system with an hamiltonian which consists of two parts:

$$H = H_0 + H'(t) \tag{1}$$

whereby the dynamics induced by H_0 is fully solved and H' is a small perturbation to the system. We assume now that H' explicitly depends on time. In this case the energy is obviously not conserved. The Schrödinger equation is then given by:

$$i\hbar\frac{\partial\Psi}{\partial t} = (H_0 + H'(t))\Psi \tag{2}$$

And we assume that $\Psi_k^{(0)}$ satisfies:

$$i\hbar\frac{\partial\Psi_k^{(0)}}{\partial t} = H_0\Psi_k^{(0)} \tag{3}$$

We write the general wavefunction Ψ, which satisfies Eq. 2 as a linear combination of $\Psi_k^{(0)}$:

$$\Psi = \sum_k a_k(t)\Psi_k^{(0)} \tag{4}$$

Substituting Eq. 4 in Eq. 2, we have:

$$i\hbar\sum_k \frac{\mathrm{d}\, a_k(t)}{\mathrm{d}\, t}\Psi_k^{(0)} = \sum_k a_k(t)H'(t)\Psi_k^{(0)} \tag{5}$$

Projecting over $\Psi_m^{(0)*}$, we have:

$$i\hbar\frac{\mathrm{d}\, a_m(t)}{\mathrm{d}\, t} = \sum_k a_k H'_{mk}(t) \tag{6}$$

whereby:

$$H'_{mk}(t) = \int \Psi_m^{(0)*}H'(t)\Psi_k^{(0)}\,\mathrm{d}\, x = V_{mk}e^{i\omega_{mk}t} \tag{7}$$

where we have defined:

$$\omega_{mk} = (E_m - E_k)/\hbar \tag{8}$$

and

$$V_{mk} = \langle\Psi_m|H'(t)|\Psi_k\rangle \tag{9}$$

Let us take $\Psi_n^{(0)}$ as an unperturbed solution of the hamiltonian H_0 so that $a_n^{(0)} = 1$ and $a_k^{(0)} = 0$ for $k \neq n$. We expand a_m to its first order: $a_m(t) \approx a_m^{(0)}(t) + a_m^{(1)}(t)$. The perturbation $a_m^{(1)}(t)$ is thus given by:

$$i\hbar \frac{\mathrm{d}\, a_m^{(1)}(t)}{\mathrm{d}\, t} = \sum_k H'_{mk}(t) a_k^{(0)} = H'_{mn}(t) \tag{10}$$

Integrating the equation we obtain:

$$a_m^{(1)}(t) = -\frac{i}{\hbar} \int V_{mn} e^{i\omega_{mk}t} dt \quad \text{with} \quad V_{mn} = \langle \Psi_m | H'(t) | \Psi_n \rangle \tag{11}$$

Let us now assume that the perturbation lasts for a finite period of time. In other words, the systems has a *transition* between the initial state in and the final state fin. The amplitude of transition $a_{in\to fin}$ is given by:

$$a_{in\to fin}^{(1)} = -\frac{i}{\hbar} \int_{-\infty}^{\infty} V_{in\to fin} e^{i\omega_{in\to fin}t} dt \tag{12}$$

The probability of finding the system in the state fin is thus given by:

$$Pin \to fin = |a_{in\to fin}^{(1)}|^2 = \frac{1}{\hbar^2} | \int_{-\infty}^{\infty} V_{in\to fin} e^{i\omega_{in\to fin}t} dt|^2 \tag{13}$$

It is instructive to see that if the pertubation occurs *adiabatically* (i.e., very slowly in time) the transition would not take place:

$$P_{in\to fin} \approx 0 \tag{14}$$

This happens when:

$$\left| \frac{\partial V_{in\to fin}}{\partial t} \right| \frac{1}{|V_{in\to fin}|} \ll \omega_{in\to fin} \tag{15}$$

which gives the definition of an adiabatic transformation.