Any calculator, except one with pre-programmable memory, may be used in this examination.

LEVEL 2
Examination contributing to the Degrees of Bachelor of Science (BSc) and Master in Science (MSci)

PHY2001
Quantum and Statistical Physics

Duration: 3 hours plus additional 1 hour for upload of work

Friday 13th of August 2021
09:30 AM – 1:30 PM

Examiners: Prof S Matthews, Prof F. Peters
and the internal examiners
Dr S Sim (s.sim@qub.ac.uk)

Answer ALL questions in Section A for 4 marks each.
Answer TWO questions from Section B for 20 marks each.
Answer ONE question from Section C for 20 marks.

If you have any problems or queries, contact the School Office at mpts@qub.ac.uk or 028 9097 1907, and the module lecturer s.sim@qub.ac.uk

THE QUEEN'S UNIVERSITY OF BELFAST
DEPARTMENT OF PHYSICS AND ASTRONOMY

PHYSICAL CONSTANTS

Speed of light in a vacuum	$c = 3.0 \times 10^{8}\ \mathrm{ms}^{-1}$
Permeability of a vacuum	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{Hm}^{-1}$
	$\approx 1.26 \times 10^{-6}\ \mathrm{Hm}^{-1}$
Permittivity of a vacuum	$\varepsilon_0 = 8.85 \times 10^{-12}\ \mathrm{Fm}^{-1}$
Elementary charge	$e = 1.6 \times 10^{-19}\ \mathrm{C}$
Electron charge	$= -1.6 \times 10^{-19}\ \mathrm{C}$
Planck Constant	$h = 6.63 \times 10^{-34}\ \mathrm{Js}$
Reduced Planck Constant	$\hbar = 1.05 \times 10^{-34}\ \mathrm{Js}$
Rydberg Constant for hydrogen	$R_\infty = 1.097 \times 10^{7}\ \mathrm{m}^{-1}$
Unified atomic mass unit	$1u = 931\ MeV = 1.66 \times 10^{-27}\ \mathrm{kg}$
1 electron volt (eV)	$= 1.6 \times 10^{-19}\ \mathrm{J}$
Mass of electron	$m_e = 9.11 \times 10^{-31}\ \mathrm{kg}$
Mass of proton	$m_p = 1.67 \times 10^{-27}\ \mathrm{kg}$
Mass of neutron	$m_n = 1.67 \times 10^{-27}\ \mathrm{kg}$
Molar gas constant	$R = 8.31\ \mathrm{JK}^{-1}\mathrm{mol}^{-1}$
Boltzmann constant	$k = 1.38 \times 10^{-23}\ \mathrm{JK}^{-1}$
Avogadro constant	$N_A = 6.02 \times 10^{23}\ \mathrm{mol}^{-1}$
Gravitational constant	$G = 6.67 \times 10^{-11}\ \mathrm{Nm}^{2}\mathrm{kg}^{-2}$
Acceleration of free fall on the Earth's surface	$g = 9.81\ \mathrm{ms}^{-2}$

CONTINUED...

SECTION A
Answer ALL 10 questions from this section
Full explanations of your answers are required to attain full marks

1. The total power irradiated by a blackbody, per unit surface area, is given by Stefan's Law;

$$P = \sigma T^4$$

where $\sigma = 5.67 \times 10^{-8}$ W m^{-2} K^{-4}.

Assuming that the surface temperature of the star Sirius is 9940 K and its diameter is 117,000 km, estimate the rate at which the rest mass is lost due to radiation.

[4]

2. A gamma photon is emitted from an excited state of a radioactive nucleus. If the lifetime of the excited state is about 1 ps, what is the minimum uncertainty in momentum of the gamma photon?

[4]

3. A particle is described by the following eigenfunction:
$\psi = A(1+2ix)$ $\quad 0 \le x \le 2$
$\psi = 0$ $\quad$ elsewhere
What is the probability of finding the particle in the range $0.75 \le x \le 1.25$?

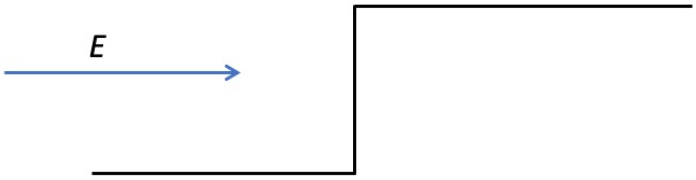
[4]

4. An electron is trapped in a finite 1-D square potential well that is deep enough to allow at least four bound states. Sketch the eigenfunction and corresponding probability density for the n=4 state.

[4]

CONTINUED...

5. The diagram below shows a particle of total energy E approaching a barrier with a potential energy of 20eV. Which of the following particles penetrates deeper into the barrier? Justify your answer.



(a) 5 eV electron

(b) 10 eV electron

(c) 5 eV proton

(d) 10 eV proton

[4]

6. A scanning tunnelling microscope (STM) is used to scan a conducting surface which is biased by -0.75 V with respect to the tip of the STM. Estimate the distance the tip has to be moved away from the surface to lower the tunnelling current by 25%. The work function of the tip is 4.1 eV.

[4]

7. Explain the physical significance of the magnetic quantum number m_l for an electron in a hydrogen atom.

[4]

8. Consider a system of four distinguishable particles that have access to a set of single-particle states with energies 0, ε_0, $2\varepsilon_0$, $3\varepsilon_0$, $4\varepsilon_0$ etc. If the total energy of the system is $U = 3\varepsilon_0$, what is the most probable distribution of particles in the single-particle states? Show all your working and explain your answer.

[4]

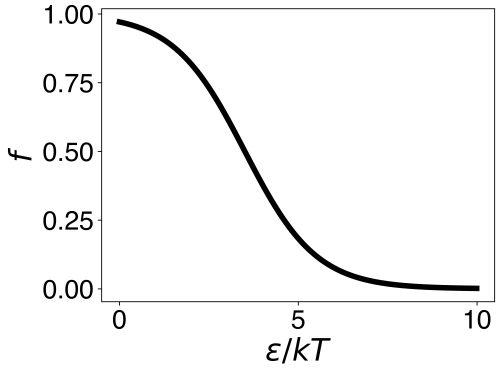
9. Discuss what is meant by the statement that particles are either *distinguishable* or *indistinguishable* in statistical mechanics. Your answer should clearly define these terms and explain why the "distinguishability" of particles is important when studying statistical mechanics.

[4]

CONTINUED...

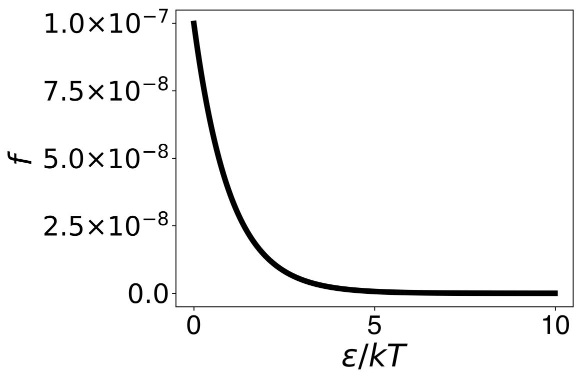
10. The figures below show the distribution functions (f) as a function of ε/kT for two different physical systems. In both cases, T denotes the temperature of the system and ε is a particle energy. For each case, explain whether the system is a dilute gas, or is governed by Fermi-Dirac or Bose-Einstein statistics.

(a)



[2]

(b)



[2]

SECTION B

Answer <u>TWO</u> questions from this section

11. **(a)** The one-dimensional time-independent Schrödinger equation (TISE) for a non-relativistic particle is;

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}+[V(x)-E]\psi(x)=0$$

(i) For a particle of mass m trapped in an infinite 1-D square potential well in the region $-a/2 \leq x \leq a/2$, obtain the odd and even parity eigenfunctions of the TISE and an expression for the allowed energies of this system.

[7]

(ii) For the eigenfunction corresponding to n=3 determine the value for the normalisation constant. Hint:$\int cos^2(kx)dx = x/2 + sin(2kx)/4k + C$.

[3]

(iii) Determine $< x >$ for the eigenstate in part (ii).

[2]

(iv) Show for this system that the fractional difference in energies between adjacent eigenvalues is given by

$$\frac{\Delta E_n}{E_n} = \frac{2n+1}{n^2}$$

[2]

(v) Comment on how the expression in (iv) is related to Bohr's correspondence principle.

[1]

(b) The potential energy function for a *quantum bouncing ball* of mass *m* is:

$$V(h) = mgh \quad \text{for } h > 0$$
$$V(h) = \infty \quad \text{for } h \leq 0$$

Plot $V(h)$ as a function of h and sketch the wave function associated with the n=3 state.

[5]

CONTINUED...

12. Consider $\psi = Cx\exp\left(-\gamma x^2\right)$ representing an eigenfunction of a simple harmonic oscillator with $V(x) = \frac{1}{2}m\omega^2x^2$.

(a) By substituting this solution into the time-independent Schrödinger equation, obtain an expression for γ and hence the energy of this state.

[8]

(b) Obtain an expression for the normalization constant A for this state.
Hint: $\int_0^\infty x^2\exp\left(-kx^2\right) = \sqrt{\pi}/4k^{3/2}$

[3]

(c) Sketch the potential energy function and the eigenfunction for this state.

[3]

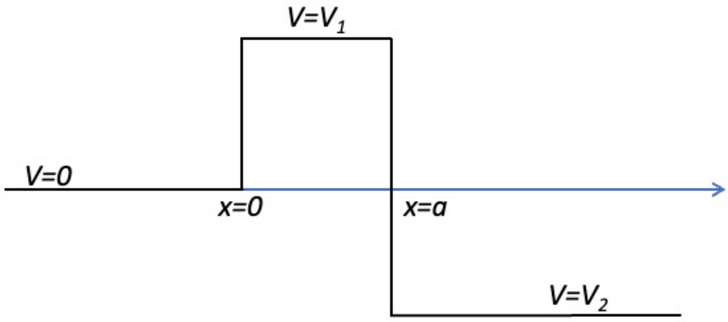
(d) Does this state correspond to the ground, first excited, second excited, or third excited state? Explain why.

[2]

(e) Draw the probability density function for this state. Briefly describe how this differs to what is expected for a classical particle in a simple harmonic potential.

[4]

13. Let us assume a stream of non-relativistic electrons of kinetic energy E travelling along the *x*-axis experiences an abrupt change in potential from 0 to V_1 at $x = 0$, where $E > V_1$. At $x = a$, the electrons experience another step change in potential from V_1 to V_2, as shown in the sketch below.



(a) Starting from the "time independent" Schrödinger equation for the particle wavefunctions in different regions (i.e. $x < 0$, $0 < x < a$ and $x > a$), write down the general solutions for the allowed eigenfunctions.

[5]

(b) Draw a suitably labelled diagram to appropriately represent the eigenfunctions in all regions.

[3]

(c) Using appropriate boundary conditions to ensure the eigenfunctions are well behaved, show that there will be 100% transmission of electrons to the region $x > a$ when $V_2 = 0$ and that the wavelength of the eigenfunction in the region $0 < x < a$ is $2a$.

[12]

14. **(a)** The equation below is the radial Schrödinger wave equation for the hydrogen atom. Without going into full mathematical detail, explain how this equation is obtained from the full three-dimensional Schrödinger wave equation.

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \frac{2\mu}{\hbar^2}\left(E + \frac{e^2}{4\pi\varepsilon_0 r} - \frac{l(l+1)\hbar^2}{2\mu r^2}\right)R = 0$$

[7]

(b) Explain what each of the terms inside the brackets represent and how the quantum number l is related to the orbital angular momentum of the electron.

[5]

(c) The radial wave function for an electron in the first excited state is given by

$$R(r) = Ar\, e^{-r/b},$$

where A and b are constants. Calculate the most probable and the average distances of the electron from the proton. [Note: $\int_0^\infty x^n \exp(-Kx)\, dx = n!/K^{n+1}$]

[8]

SECTION C

Answer ONE question from this section

15. **(a)** **(i)** Consider a system of weakly interacting particles in equilibrium at temperature T. Carefully discuss the circumstances under which it is valid to assume that the number of particles occupying a state of energy ε is directly proportional to $\exp(-\varepsilon/kT)$.

[4]

(ii) For an ideal gas, explain why the number of particles travelling with speed between v and $v+\mathrm{d}v$ is directly proportional to $v^2\exp(-mv^2/2kT)\mathrm{d}v$ where m is the particle mass.

[4]

(b) Approximate analysis of the first few energy levels of oxygen atoms yield energies and degeneracies as listed in the following table:

	Energy (eV)	degeneracy
Level-1	0.00	9
Level-2	1.97	5
Level-3	4.19	1
Level-4	9.15	5
Level-5	9.52	3

(i) For an equilibrium system of oxygen atoms, calculate the temperature at which the number of atoms occupying "Level-4" will be equal to the number in "Level-3".

[4]

(ii) Estimate the fraction of the oxygen atoms that would occupy "Level-5" at the temperature you calculated in part (b)(i).

[4]

(iii) More accurate analysis shows that the 9 states constituting "Level-1" are not perfectly degenerate, but actually span a range of energies approximately 0.0 – 0.3 eV. Explain whether accounting for this fact would significantly alter your answer to part (b)(ii), and comment on whether the alteration would increase or decrease the fraction compared to your calculation in (b)(ii).

[4]

CONTINUED...

16. For a three-dimensional system of non-relativistic particles, the density of states (expressed in terms of energy ε) can be written

$$g(\varepsilon)\mathrm{d}\varepsilon = G\frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\varepsilon^{1/2}\mathrm{d}\varepsilon$$

where V is the volume of the system, m is the particle mass and G is an appropriate degeneracy factor.

(a) A box of volume 10 m^3 contains 2.7 kg of neon atoms at a temperature of 87 K.

(i) For this system, estimate the number of single-particle states that have energy below 0.0075 eV.

[4]

(ii) Using your answer to (a)(i), explain whether this system of neon atoms can be described as a *dilute gas*.

[You may assume that neon atoms have mass of 20 a.m.u. and spin of zero.]

[3]

(b) A typical white dwarf star contains approximately 3.6×10^{56} electrons and has a radius of around $6,000$ km.

(i) Assuming that such a star can be approximated as a sphere of uniform density, estimate its Fermi energy and Fermi temperature.

[7]

(ii) Calculate the rest mass energy ($m_e c^2$) for the electron, and comment on the implications of comparing this value with the answer you obtained in part (b)(i).

[3]

(iii) Massive white dwarf stars may have mean densities that exceed the typical case by more than a factor of 100. Discuss how the approach used for estimating the Fermi energy would need to be altered for such a case. You do not need to present detailed calculations, but your answer should clearly explain the principles of the physics involved.

[3]

END OF EXAMINATION