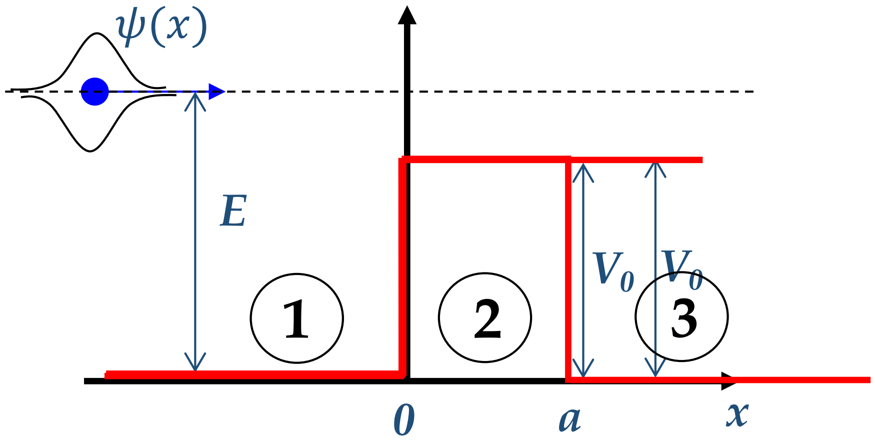
Part-2
Potential Barriers

Potential barrier : E > V_0



- **define potential :**

$$
\begin{aligned}
V(x) &= 0 && \text{if } x < 0 && \text{: Region 1}\\
&= V_0 < E && \text{if } 0 < x < a && \text{: Region 2}\\
&= 0 && \text{if } x > a && \text{: Region 3}
\end{aligned}
$$

Region 1 | **Region 2** | **Region 3**

- **write down T.I.S.E :**

$$\frac{\hbar^2}{2m}\frac{d^2\psi_1(x)}{dx^2} + E\,\psi_1(x) = 0$$

$$\frac{\hbar^2}{2m}\frac{d^2\psi_2(x)}{dx^2} + [E - V_0]\,\psi_2(x) = 0$$

$$\frac{\hbar^2}{2m}\frac{d^2\psi_3(x)}{dx^2} + E\,\psi_3(x) = 0$$

- **define general expression for the eigen function :**

moving along:

$$\psi_1(x) = \underbrace{A\exp(ik_1x)}_{+x} + \underbrace{B\exp(-ik_1x)}_{-x}$$

$$\psi_2(x) = \underbrace{C\exp(ik_2x)}_{+x} + \underbrace{D\exp(-ik_2x)}_{-x}$$

$$\psi_3(x) = \underbrace{F\exp(ik_3x)}_{+x} + \underbrace{\cancel{G\exp(-ik_3x)}}_{-x}$$

where, $k_1 = \sqrt{\frac{2mE}{\hbar^2}}$

A & B are amplitudes (constants)

where, $k_2 = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$

C & D are amplitudes (constants)

where, $k_3 = \sqrt{\frac{2mE}{\hbar^2}} = k_1$

F & G are amplitudes (constants) ; $G= 0$

- **apply boundary conditions to find A, B, C, D & F:**

  ✓ $\psi_1(x)|_{x=0} = \psi_2(x)|_{x=0} \quad \Rightarrow \quad A + B = C + D$

  ✓ $\left.\frac{d\psi_1(x)}{dx}\right|_{x=0} = \left.\frac{d\psi_2(x)}{dx}\right|_{x=0} \quad \Rightarrow \quad k_1(A - B) = k_2(C - D)$

  ✓ $\psi_2(x)|_{x=a} = \psi_3(x)|_{x=a} \quad \Rightarrow \quad C\exp(ik_2 a) + D\exp(-ik_2 a) = F\exp(ik_1 a)$ (note, $k_3 = k_1 = \sqrt{2mE}/\hbar$)

  ✓ $\left.\frac{d\psi_2(x)}{dx}\right|_{x=a} = \left.\frac{d\psi_3(x)}{dx}\right|_{x=a} \quad \Rightarrow \quad k_2\,[C\exp(ik_2 a) - D\exp(-ik_2 a)] = k_1 F\exp(ik_1 a)$

- **with some algebra** (rather messy ☹), **you can find**

$$\boxed{\mathrm{T} = \frac{J_T}{J_I} = \frac{F^2}{A^2} = \left[1 + \frac{\sin^2(k_2 a)}{4\frac{E}{V_0}\left(\frac{E}{V_0} - 1\right)}\right]^{-1}}$$

where, $k_2 = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$

- **now, let's try to make some sense of this…**

  Classically $T = 1$ and $R = 0$ for all $E/V_0 > 1$.
  But Quantum mechanically, this is not always the case, even if $E >> V_0$

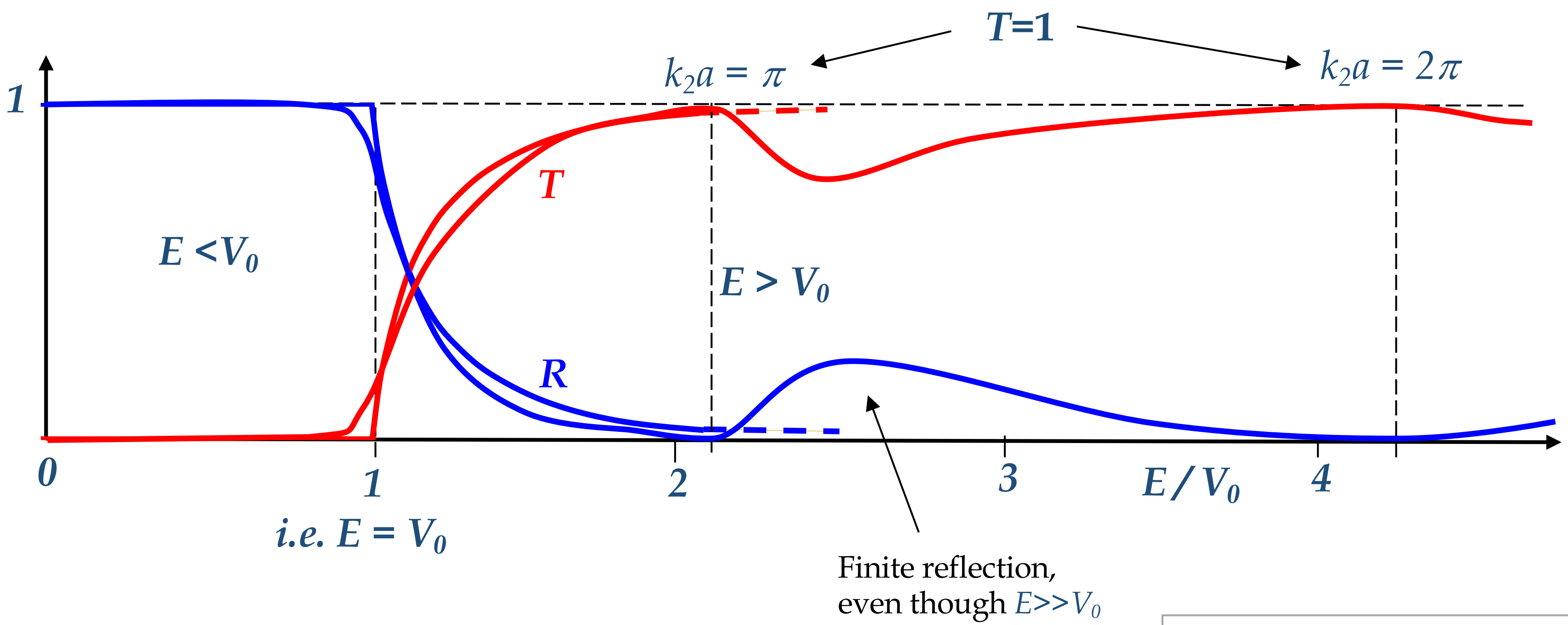
  However, there are some energies for which the barrier has no effect, and $T = 1$!

  This happens when (from the expression of T above) , $\sin(k_2 a) = 0 \quad \Rightarrow \quad k_2 a = m\pi$, where $m = 1,2, \ldots$

  i.e. when the de-Broglie wavelength in the region 2 (inside the barrier) $= \lambda_2 = \frac{2\pi}{k_2} = \frac{2a}{m}$

  i.e. when the width of the barrier, $a = m\,\frac{\lambda_2}{2}$, which leads to destructive interference between the reflections from the boundaries at x=0 and x=a

Plotting the reflection and transmission coefficient in this case,



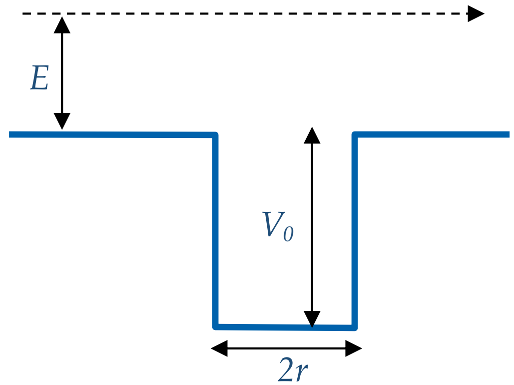
Ramsauer-Townsend Effect

When electrons scatter off atoms with tightly bound closed shells, the potential can be approximated by the potential well shown on right. The probability of the electron scattering reduces to zero when $k_2a = \pi$ (note in this case k_2 is different from the expression above).

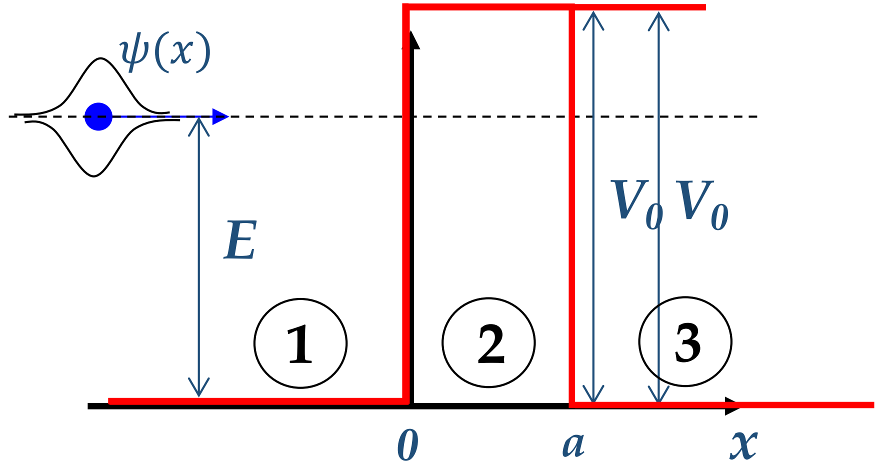
This effect was discovered by Ramsauer and Townsend in the early 1920s : they saw electrons of few eV energies pass through atoms as if there were not there.

This was probably the first experimental evidence for the wave nature of particles, although it was not interpreted as such at the time.

Potential seen by electron scattering off a noble gas atom



Potential barrier : $E < V_0$



- **define potential :**

$$V(x) = 0 \qquad if\ x < 0 \qquad \text{: Region 1}$$
$$= V_0 > E \qquad if\ 0 < x < a \qquad \text{: Region 2}$$
$$= 0 \qquad if\ x > a \qquad \text{: Region 3}$$

Region 1 | **Region 2** | **Region 3**

Note, $[E\text{-}V_0]$ is -ve

- **write down T.I.S.E :**

$$\frac{\hbar^2}{2m}\frac{d^2\psi_1(x)}{dx^2} + E\,\psi_1(x) = 0$$

$$\frac{\hbar^2}{2m}\frac{d^2\psi_2(x)}{dx^2} + [E - V_0]\,\psi_2(x) = 0$$

$$\frac{\hbar^2}{2m}\frac{d^2\psi_3(x)}{dx^2} + E\,\psi_3(x) = 0$$

- **define general expression for the eigen function :**

moving along:

$$\psi_1(x) = \underbrace{A\exp(ik_1x)}_{+x} + \underbrace{B\exp(-ik_1x)}_{-x}$$

$$\psi_2(x) = \underbrace{C\exp(ik_2x)}_{+x} + \underbrace{D\exp(-ik_2x)}_{-x}$$

$$\psi_3(x) = \underbrace{F\exp(ik_3x)}_{+x} + \underbrace{\cancel{G\exp(-ik_3x)}}_{-x}$$

where, $k_1 = \sqrt{\frac{2mE}{\hbar^2}}$

A & B are amplitudes (constants)

where, $k_2 = \sqrt{\frac{2m(E-V_0)}{\hbar^2}} = i\sqrt{\frac{2m(V_0-E)}{\hbar^2}} = ik_2'$

C & D are amplitudes (constants)

where, $k_3 = \sqrt{\frac{2mE}{\hbar^2}} = k_1$

F & G are amplitudes (constants) ; $G= 0$

Hence the general expressions of the eigen functions would be

$\psi_1(x) = A\exp(ik_1x) + B\exp(-ik_1x)$ $\psi_2(x) = C\exp(-k_2'x) + D\exp(k_2'x)$ $\psi_3(x) = F\exp(ik_1x)$

(note, $k_3 = k_1 = \sqrt{2mE}/\hbar$, $k_2 = ik_2'$)

- **apply boundary conditions to find** A**,** B**,** C**,** D **&** F**:**
  - ✓ $\psi_1(x)|_{x=0} = \psi_2(x)|_{x=0} \quad \Rightarrow \quad A + B = C + D$
  - ✓ $\left.\frac{d\psi_1(x)}{dx}\right|_{x=0} = \left.\frac{d\psi_2(x)}{dx}\right|_{x=0} \quad \Rightarrow \quad k_1(A - B) = ik_2'(C - D)$
  - ✓ $\psi_2(x)|_{x=a} = \psi_3(x)|_{x=a} \quad \Rightarrow \quad C\exp(-k_2'a) + D\exp(k_2'a) = F\exp(ik_1a)$
  - ✓ $\left.\frac{d\psi_2(x)}{dx}\right|_{x=a} = \left.\frac{d\psi_3(x)}{dx}\right|_{x=a} \quad \Rightarrow \quad k_2'\,[C\exp(-k_2'a) - D\exp(k_2'a)] = ik_1F\exp(ik_1a)$

- **with some algebra** (rather messy ☹), **you can find**

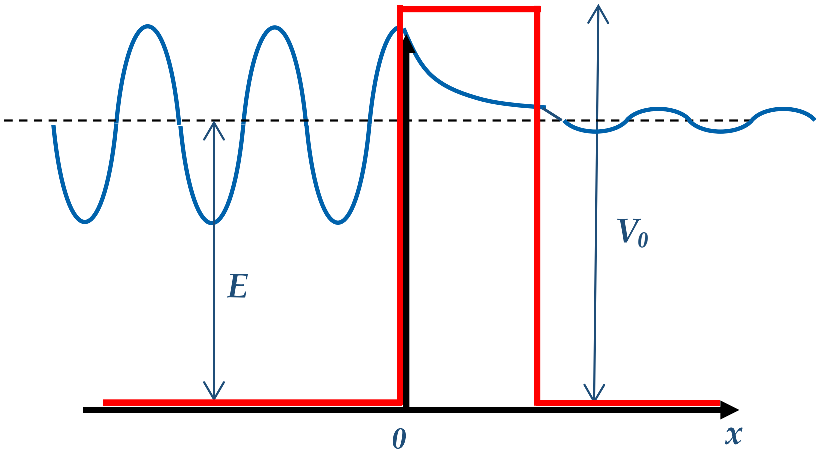
$$\mathrm{T} \approx 16\frac{E}{V_0}\left(1 - \frac{E}{V_0}\right)e^{-2k_2'a} \qquad \text{when} ,\ k_2'a \gg 1$$

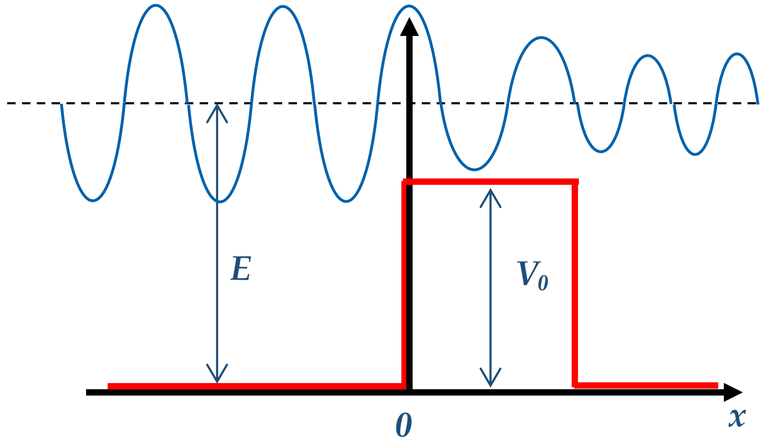
i.e. there is a finite probability of finding the particle at the other side of the barrier !!!

“ **QUANTUM TUNNELLING** ”

- The transmission coefficient is a function of E, V_0 , a and m.
- But most importantly, the expression is dominated by the exponential term $e^{-2k_2'a}$, where, $k_2' = \sqrt{2m(V_0 - E)}/\hbar$
  i.e. a small change in $(V_0 - E)$ and m can change the tunnelling probability significantly!

$E < V_0$ $E > V_0$





A fun problem!

What is the probability of you running through a wall? Let's assume mass of 50 kg and the wall of 5 cm thick representing an effective barrier height of 10^4 J

$$T \approx 16\frac{E}{V_0}\left(1-\frac{E}{V_0}\right)e^{-2k_2' a}$$

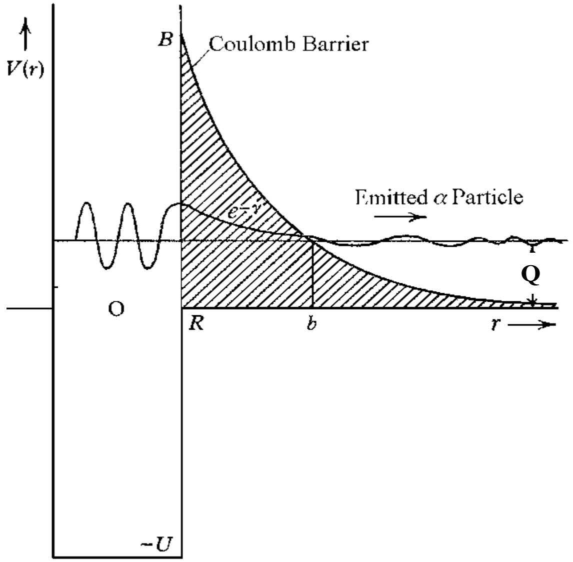
$$e^{-2k_2' a} = \exp\left(\frac{-2a\sqrt{2m(V_0-E)}}{\hbar}\right) \approx \exp\left(\frac{-2\times 0.05\sqrt{2\times 50\times 10^4}}{10^{-34}}\right) = \exp(-10^{32})$$

Not even a atom of my body will go through!

So, tunnelling occurs for very small particles, and thin, small barriers.

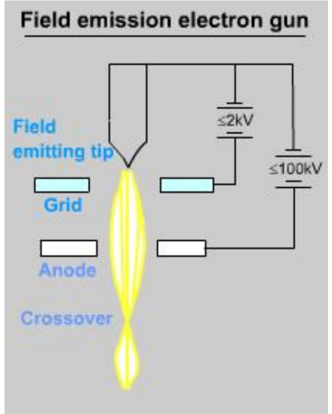
Some examples of Quantum tunnelling …

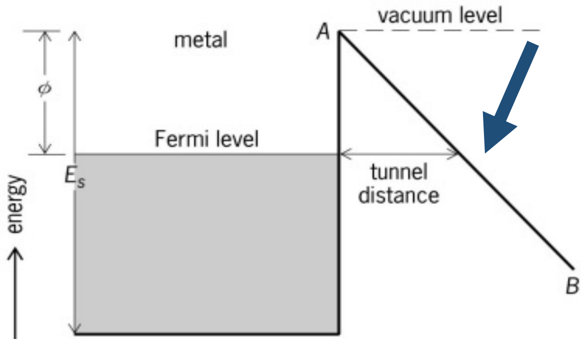
- **Alpha decay of a nucleus**
  - If 2 protons and 2 neutrons form an alpha particle, it experiences a potential with distance r as shown
  - Alphas can tunnel through the coulomb barrier
  - Probability (and thus half-life $t_{1/2}$) very dependent on energy of emitted alpha Q.



Nuclide	Half-life (years)	Decay Q-value (MeV)
${}^{241}_{95}Am$	432	5.64
${}^{238}_{92}U$	4.5×10^{9}	4.27
${}^{204}_{82}Pb$	1.4×10^{17}	1.97

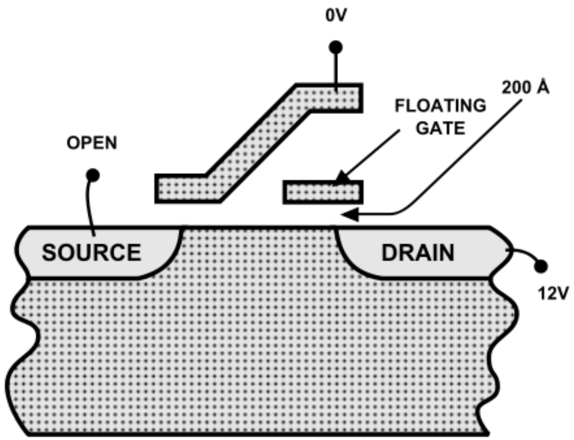
- **Field Emission**
  - In this type of electron gun, very strong (>10^9 V/m) is used to extract the electrons from a metal filament
  - Works by supressing the potential barrier by the applied voltage, so that electrons can tunnel through.
  - This gives much brighter source than guns operated via thermionic emission process, but needs a very good vacuum.



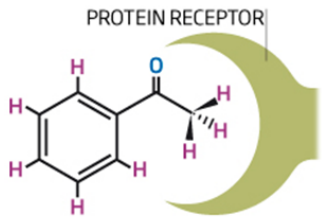


Some examples of Quantum tunnelling …

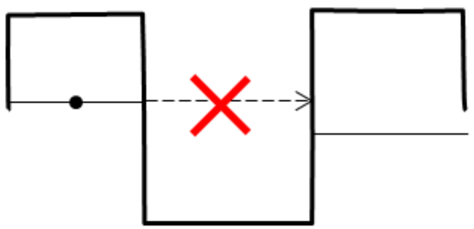
- **Flash Memory**
  - Based on "floating gate" transistor
  - If charged, the Floating Gate suppresses current from flowing from the Source to the Drain, i.e. a "0". If uncharged then stores a "1"
  - But how do you remove charge from the floating gate if it has no electrical connections?



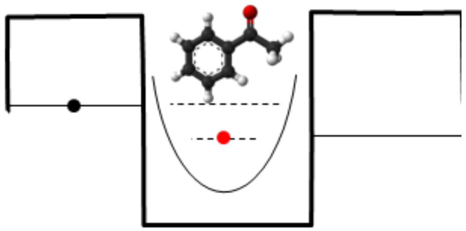
- **How does a nose generate the signals that the brain registers as smell ?**



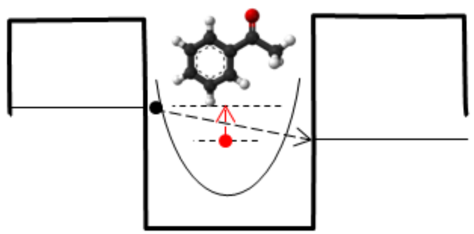
The fragrant molecules fits into the protein receptor like a key in a lock.



Electron has no state to tunnel across to



Molecule moves into receptor.
Molecule in ground vibrational level.

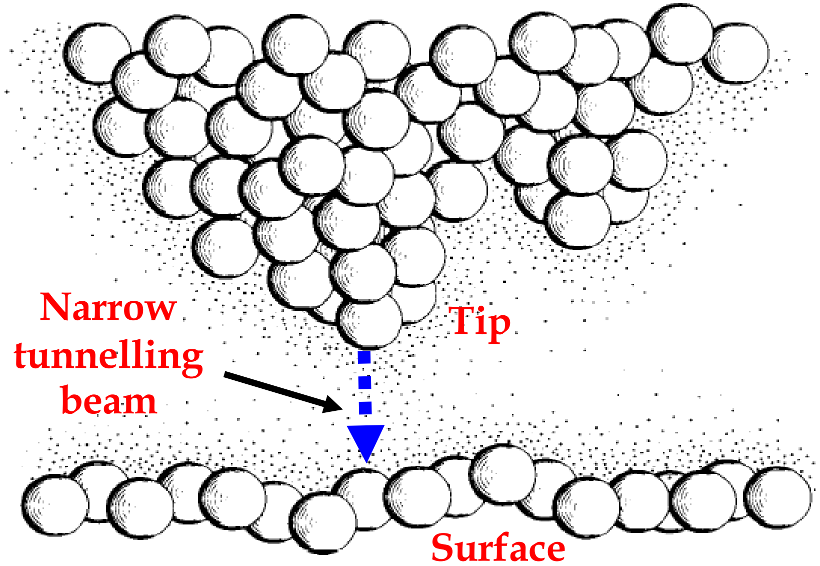


Electron excites a transition in the molecule and loses enough energy to tunnel to lower level.

http://www.newscientist.com/article/dn20130-fly-sniffs-molecules-quantum-vibrations.html

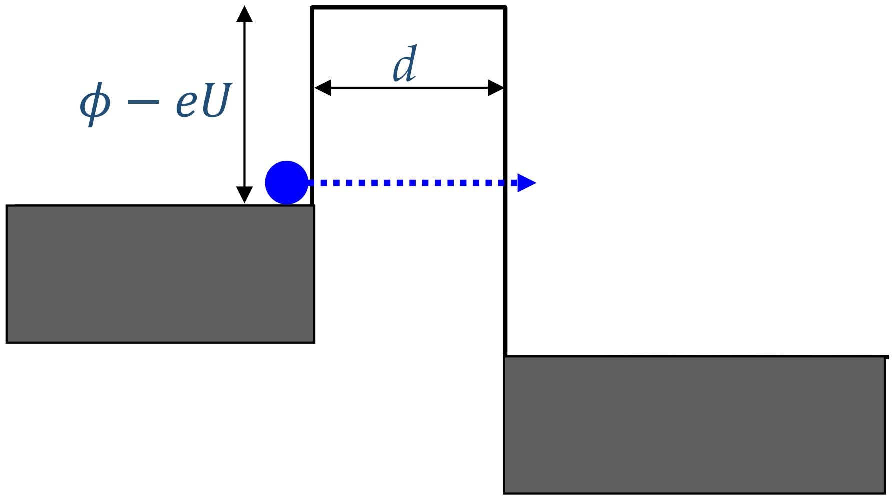
Some examples of Quantum tunnelling …

- **Scanning Tunnelling Microscope (STM)**
  - In a STM, a fine tip (only a few atoms wide at its point) is controlled with < 1 Å precision.
  - Electrons at the top of the conduction band in the tip require an energy equal to the work function to escape. If the tip is close enough to a surface there is a barrier and when a potential difference U is applied, electrons can tunnel into empty states in the surface.
  - Let us approximate this by a square barrier with height $\phi - eU$, where ϕ is the work function (≈ 5 eV for tungsten) and distance from the surface d in Å (10^{-10} m).



$$T \approx 16\frac{E}{V_0}\left(1-\frac{E}{V_0}\right)e^{-2k_2'a}$$

$$e^{-2k_2'a} = \exp\left(\frac{-2a\sqrt{2m(V_0-E)}}{\hbar}\right)$$
$$= \exp\left(\frac{-2d\sqrt{2m_e(\phi-eU)}}{\hbar}\right) \approx \exp(-1.03d\sqrt{5-U})$$

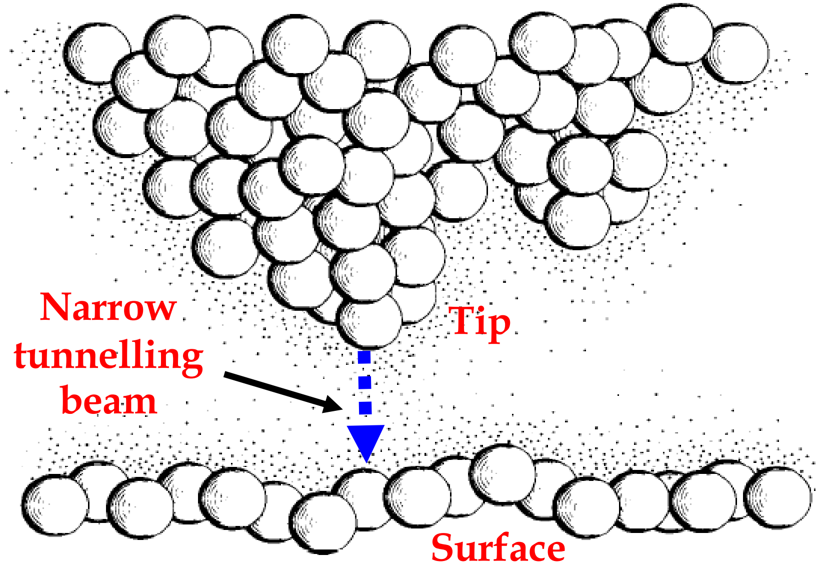


If $U = 0.1$ V,

d = 10 Å gives $T = 1.25\times10^{-10}$
d = 10.1 Å gives $T = 1.0\times10^{-10}$

- Due to the exponential dependence, the electron tunnelling current is very sensitive to the surface roughness giving very high resolution.

  (0.1Å ~ 1/10th of the width of one atom!)
- In scanning the surface, the tunnelling current is kept constant by moving the tip in and out.
- Although it is impossible to get a perfectly sharp tip, the lateral resolution is also very high. Again as tunnelling is sensitive to distance, electrons tunnel from only one atom!



This image is from the Nobel Physics prize winning lecture by Gerd Binnig who won in 1986 "for the design of the scanning tunneling microscope"

http://nobelprize.org/nobel_prizes/physics/laureates/1986/index.html
http://virtual.itg.uiuc.edu/training/AFM_tutorial/