Lecture 1:
The Lagrangian and Hamiltonian formalism

The classical equation of motion of a point-like particle is:

$$\frac{\mathrm{d}\,\vec{p}}{\mathrm{d}\,t} = \vec{F}, \tag{1}$$

where $\vec{p} = m\dot{\vec{r}}$ is the particle momentum (from now on, a dot indicates a first derivative in respect to time). If the force $\vec{F}$ is conservative (i.e., it admits a scalar potential, $\vec{F} = -\nabla V$), we can demonstrate that there exists a scalar quantity E that is conserved. This quantity is defined as:

$$E = T + V \qquad \text{with} \qquad T = \frac{\vec{p}^2}{2m}. \tag{2}$$

If the potential has spherical symmetry ($V(\vec{r}) = V(r)$), then also the angular momentum ($\vec{L} = \vec{r} \times \vec{p}$) is conserved.

Newton mechanics can be formally treated by considering a single function of the generalised coordinates (r_i), their first derivatives ($\dot{r}_i$), and time (t) called *Lagrangian function of the system*. This is defined as:

$$L(r_i, \dot{r}_i, t) = T - V \tag{3}$$

For any given system, once the Lagrangian of the system is determined, its whole temporal evolution is known. The equation of motion of a system is given by the *Euler-Lagrange equation*:

$$\frac{\partial L}{\partial r_i} - \frac{\mathrm{d}}{\mathrm{d}\,t}\frac{\partial L}{\partial \dot{r}_i} = 0 \tag{4}$$

1.a Example:

Let us see how the Euler-Lagrange equation allows to derive the Lorentz force for a charged particle in an electromagnetic field. The potential associated to the field is (calling $\vec{v} = \dot{\vec{r}}$):

$$V = e\phi(\vec{r}) - e\vec{A}\cdot\vec{v} \tag{5}$$

Thus, the Lagrangian is:

$$L = \frac{\vec{p}^2}{2m} - e\phi(\vec{r}) + e\vec{A}\cdot\vec{v} \tag{6}$$

The first term of the Euler-Lagrange equation gives:

$$\frac{\partial L}{\partial r_i} = -e\nabla\phi + e\nabla(\vec{A}\cdot\vec{v}). \tag{7}$$

The second term gives instead:

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\frac{\partial L}{\partial v_i} = \frac{\mathrm{d}}{\mathrm{d}\,t}\left(m\vec{v} + e\vec{A}\right) = m\vec{a} + e\frac{\mathrm{d}\,\vec{A}}{\mathrm{d}\,t} = \vec{F} + e\frac{\mathrm{d}\,\vec{A}}{\mathrm{d}\,t}, \tag{8}$$

where Newton's second law has been used. Combining the two terms, we obtain:

$$-e\nabla\phi + e\nabla(\vec{A}\cdot\vec{v}) - \vec{F} - e\frac{\mathrm{d}\,\vec{A}}{\mathrm{d}\,t} = 0. \tag{9}$$

If we consider the vectorial identity:

$$\nabla\left(\vec{a}\cdot\vec{b}\right) = (\vec{a}\cdot\nabla)\,\vec{b} + \left(\vec{b}\cdot\nabla\right)\vec{a} + \vec{a}\times\left(\nabla\times\vec{b}\right) + \vec{b}\times(\nabla\times\vec{a}), \tag{10}$$

we put $\vec{a} = \vec{A}$ and $\vec{b} = \vec{v}$, and we take into account that $\nabla\times\vec{v} \equiv 0$, we obtain:

$$-e\nabla\phi + e\vec{v}\times\left(\nabla\times\vec{A}\right) - \vec{F} - e\frac{\mathrm{d}\,\vec{A}}{\mathrm{d}\,t} = 0. \tag{11}$$

If we remember that the electric field is defined as $\vec{E} = -\nabla\phi - \frac{\partial\vec{A}}{\partial t}$ and the magnetic field is defined as $\vec{B} = \nabla\times\vec{A}$, we obtain:

$$\vec{F} = e\left(\vec{E} + \vec{v}\times\vec{B}\right), \tag{12}$$

which is the Lorentz equation. ■

Despite the powerfulness of this mathematical tool, it still suffers from a non-trivial limitation; the variables upon which it is dependent, are not independent from each other: $\delta\dot{r}_i = \mathrm{d}\,\delta r_i/\,\mathrm{d}\,t$. This itch in the theory can be overcome if we adopt a transformation of the Lagrangian function:

$$H(r_i, p_i) = \Sigma_i p_i \dot{r}_i - L \qquad \text{with} \qquad p_i \equiv \frac{\partial L}{\partial \dot{r}_i} \tag{13}$$

The function H is called *Hamiltonian function of the system* and it is now a function of the *independent* canonical variables (r_i, p_i). In the simple case of $L = \frac{\vec{p}^2}{2m} - V$, the Hamiltonian represents the total energy of the system: $H = \frac{\vec{p}^2}{2m} + V$. The equations of motion are now all differential equations of the first order in t and read:

$$\dot{r}_i = \frac{\partial H}{\partial p_i}; \qquad \dot{p}_i = -\frac{\partial H}{\partial r_i}. \tag{14}$$

These are called the Hamiltonian equations or canonical equations. For any given physical system, the knowledge of the Hamiltonian (or, in most cases, of the energy) of the system is formally sufficient to know its whole temporal evolution.