PHY2006
Partial Differential Equations

Lecture Slides

1. Introduction/Revision

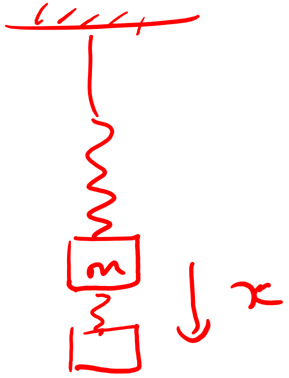
1.1. Ordinary Differential Equations (ODEs)

Types of Variables

- Dependent variable – function/variable being differentiated
  - $\boldsymbol{u}(x), \frac{d\boldsymbol{u}}{dx}, \frac{d^2\boldsymbol{u}}{dx^2}$

MASS ON SPRING

$x(t)$



NEWTON'S 2ND LAW

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt}$$

$$\frac{du}{dx} \qquad u(x)$$

- Independent variables – variable which function depends on
  - $f(\boldsymbol{t}), \psi(\boldsymbol{x})$

TIME t, SPACE x, y, z

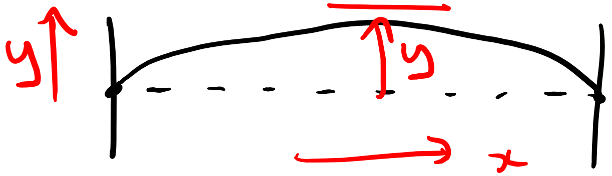
- Partial Differentials - dependent variable is function of more than 1 independent variable
  - $\boxed{u(x,y)}, \Psi(x,y,z,t)$
  - Partial derivative denoted by ∂ – differentiated w.r.t. one independent variable

PARTIAL DIFFERENTIAL

$\left.\frac{\partial u}{\partial x}\right|$ WHILE y CONSTANT

WAVES ON STRING

$y(x$ $\quad y(x,t)$



$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

Classification of Differential Equations

- Order - maximum number of times a function has been differentiated
  - $\frac{du}{dx} - 1$, $\frac{d^2u}{dx^2} - 2$, $\frac{\partial^3 u}{\partial x^3} - 3$, $\frac{\partial^2 u}{\partial x \partial y} - 2$

$$\frac{d^3y}{dt^3} + \omega_0^2 \frac{dy}{dt} = 0$$

ORDER 3

- Linearity - dependent variable and its differentials must only appear with a power of 1
  - Linear 2nd order ODE $A(x)\frac{d^2u}{dx^2} + B(x)\frac{du}{dx} + C(x)u = f(x)$

$$\frac{d^2y}{dx^2} + a\sqrt{y} = 0$$

$$\frac{d}{dx}(y^2) + y = 0$$

$$2y\frac{dy}{dx} + \cancel{y} = 0$$

- Homogeneity - $f(x) = 0$

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = cxy$$

ORDER 2

LINEAR

HOMOGENEOUS

Linearity - Superposition

- Solutions to linear differential equations obey the principle of superposition
  - For a linear ODE $A(x)\frac{d^2u}{dx^2} + B(x)\frac{du}{dx} + C(x)u = f(x)$

SOLUTIONS u_1, u_2

ALSO SOLUTION $u = au_1 + bu_2$ a, b CONSTANT

- For wave solutions of DEs this results in interference in many physical phenomena – water waves, sound, light, quantum mechanics
  - E.g. $\psi = \psi_1 + \psi_2$
  - $|\psi|^2 = |\psi_1 + \psi_2|^2 = (\psi_1 + \psi_2)(\psi_1^* + \psi_2^*)$ $= |\psi_1|^2 + |\psi_2|^2 + \underbrace{\psi_1^*\psi_2 + \psi_2^*\psi_1}_{\text{INTERFERENCE}}$

Solutions of 1st order ODEs

- 1st order separation of variables $\frac{du}{dx} = f(x)g(u)$
- $\int \frac{du}{g(u)} = \int f(x)dx$
- $\frac{du}{dx} = 2u(x+1)$

$$\int \frac{du}{u} = 2\int (x+1)\,dx$$

$$\ln u = 2\left(\frac{1}{2}x^2 + x + c\right)$$

$$u = \exp\left(x^2 + x + 2c\right)$$

$$= \underbrace{e^{2c}}_{A} e^{(x^2+x)}$$

Solutions of 1st order ODEs

- 1st order integrating factor $\dfrac{du}{dx} + f(x)u = g(x)$
- Integrating factor $\exp\left(\int f(x)dx\right)$
- $x\dfrac{dy}{dx} + (x-1)y = x^2 \quad \rightarrow \quad \dfrac{dy}{dx} + \left(1 - \dfrac{1}{x}\right)y = x$

$-\ln x = \frac{1}{x}$

I.F. $\exp\left(\int\left(1-\frac{1}{x}\right)dx\right) = \exp(x - \ln x) = e^x e^{-\ln x} = \frac{e^x}{x}$

$$\underbrace{\frac{e^x}{x}\frac{dy}{dx} + \frac{e^x}{x}\left(1-\frac{1}{x}\right)y}_{} = e^x$$

$$\frac{d}{dx}\left[\frac{e^x}{x}y\right] = e^x \xrightarrow{\text{INTEGRATE}} \frac{e^x}{x}y = e^x + C$$

$$y = x\left(1 + Ce^{-x}\right)$$

$$\frac{dy}{dx} = 2xy$$

$$\int \frac{dy}{y} = \int 2x\,dx$$

$y = \phi$

$x = 0$

$$\ln y = x^2 + c$$

$$\exp(\ln y) = e^{x^2 + c}$$

$$y = e^{x^2 + c}$$

$$1 = e^c \times 1$$

$$y = e^{x^2}$$

$\frac{du}{dx}$ ← DEPENDENT ← INDEPENDENT

ORDER – NO. TIMES DIFF. u

LINEARITY – $u, \frac{du}{dx}$ POWER 1

$u\frac{du}{dx}$

HOMOGENEITY

LHS – TERMS IN u

RHS – ZERO

SUPERPOSITION

1ST ORDER

– SEPARATION OF VARIABLES

$$\frac{du}{dx} = \frac{x}{u}$$

$$\int u\,du = \int x\,dx$$

$$\frac{1}{2}u^2 = \frac{1}{2}x^2 + c$$

– INTEGRATING FACTOR

$$\frac{dy}{dx} = e^{-x^2} - 2yx$$

ORDER 1

LINEAR

INHOMOGENEOUS

Solutions of 1st order ODEs - Example

- Find the solution to the differential equation given that when $x = 0, y = 2$

$$\frac{dy}{dx} = e^{-x^2} - 2xy$$

- $\frac{dy}{dx} + 2xy = e^{-x^2}$
- Integrating factor $\exp \int 2x\, dx = e^{x^2}$

I.F.

$$e^{x^2}\frac{dy}{dx} + 2xye^{x^2} = 1$$

$$\frac{d}{dx}\left[e^{x^2}y\right] = 1$$

$$e^{x^2}y = x + C$$

$$C = 2$$

Solutions of 2nd order ODEs – constant coefficients

a, b, c constants

- For equations of the type $a\frac{d^2u}{dx^2} + b\frac{du}{dx} + cu = 0$
- Trial solution Ae^{mx} generates a quadratic equation – characteristic equation

$a A m^2 e^{mx} + b A m e^{mx} + c A e^{mx} = 0$

$am^2 + bm + c = 0$

- General Solution $u = C_1 e^{m_1 x} + C_2 e^{m_2 x}$ depends on if
  - m_1, m_2 are real
  - $m_1 = m_2$ $\rightarrow$ $u = (C_1 x + B)e^{m_1 x}$
  - m_1, m_2 are complex
- Example $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 5x = 0$

$m^2 + 4m + 5 = 0$

$m = \frac{-4 \pm \sqrt{16 - 20}}{2} = -2 \pm i$

$u = C_1 e^{-2t} e^{+it} + C_2 e^{-2t} e^{-it}$

$u = e^{-2t}\left(\cos t \underbrace{(C_1 + C_2)}_{C_3} + \sin t \underbrace{(C_1 - C_2)i}_{C_4}\right)$

$e^{it} = \cos t + i \sin t$

EULER'S FORMULA

Solutions of 2nd order inhomogeneous ODEs

- For equations of the type $a\frac{d^2u}{dx^2} + b\frac{du}{dx} + cu = f(x)$
  - Find the complementary function u_{CF}, set $f(x) = 0$ and solve as before
  - Find particular integral u_{PI}
  - General solution $u = u_{CF} + u_{PI}$
- Example $\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 3x = t + 1$
  - Find CF:

$m^2 - 4m + 3 = 0$

$(m-3)(m-1) = 0 \quad m = 1, 3$

  - Try PI: $x = At + B$

$\frac{dx}{dt} = A \quad \frac{d^2x}{dt^2} = 0$

$-4A + 3(At + B) = t + 1 \quad 3A = 1 \quad A = \frac{1}{3} \quad B = \frac{7}{9}$

  - General Solution:

$x = C_1 e^t + C_2 e^{3t} + \frac{1}{3}t + \frac{7}{9}$

Solutions of 1st order ODEs – Radioactive Decay Example

In a radioactive decay chain X $\Rightarrow$ Y $\Rightarrow$ Z with decay constants λ_1 and λ_2 ($\lambda = \frac{\ln 2}{t_{1/2}}$).

a. Write down equations for the rate of change of the number of each of the nuclei

b. When is the is the number of Y nuclei maximum?

$N_X = N_{0X}$ X

λ_1

$N_Y = 0$ Y

λ_2

$N_Z = 0$ Z

$t = 0$

(a) The rate of change of a population is to the number of nuclei times the decay constant, giving the following coupled differential equations

$$\frac{dN_X}{dt} = -\lambda_1 N_X \qquad N_X = N_{0X} e^{-\lambda_1 t}$$

$$\frac{dN_Y}{dt} = +\lambda_1 N_X - \lambda_2 N_Y \qquad \frac{dN_Y}{dt} + \lambda_2 N_Y = \lambda_1 N_{0X} e^{-\lambda_1 t}$$

$$\text{I.F.} \quad \exp\left(\int \lambda_2 \, dt\right) = e^{\lambda_2 t}$$

$$\frac{d}{dt}\left[e^{\lambda_2 t} N_Y\right] = \lambda_1 N_{0X} \exp(-\lambda_1 t + \lambda_2 t)$$

Solutions of 1st order ODEs – Radioactive Decay Example

$$N_Y e^{\lambda_2 t} = \lambda_1 N_{0X} \int e^{(\lambda_2-\lambda_1)t}\, dt$$

$$N_Y e^{\lambda_2 t} = \lambda_1 N_{0X} \frac{e^{(\lambda_2-\lambda_1)t}}{\lambda_2 - \lambda_1} + C$$

$$N_Y = N_{0X} \frac{\lambda_1}{\lambda_2-\lambda_1} e^{-\lambda_1 t} + C e^{-\lambda_2 t} \qquad \underline{t = 0, N_Y = 0}$$

$$C = -N_{0X} \frac{\lambda_1}{\lambda_2-\lambda_1}$$

$$N_Y = N_{0x} \frac{\lambda_1}{\lambda_2-\lambda_1}\left(\underline{e}^{-\lambda_1 t} - \underline{e}^{-\lambda_2 t}\right)$$

Solutions of 1st order ODEs – Radioactive Decay Example

^{99}Mo is used as a radioactive "generator" to produce ^{99m}Tc which is the main radio isotope used for gamma imaging in medicine. ^{99}Mo has a half life of 65.9 hours and ^{99m}Tc has a half life of 6.01 hours. When needed, all the ^{99m}Tc is extracted from the generator, but how long should you ideally wait to repeat this process?

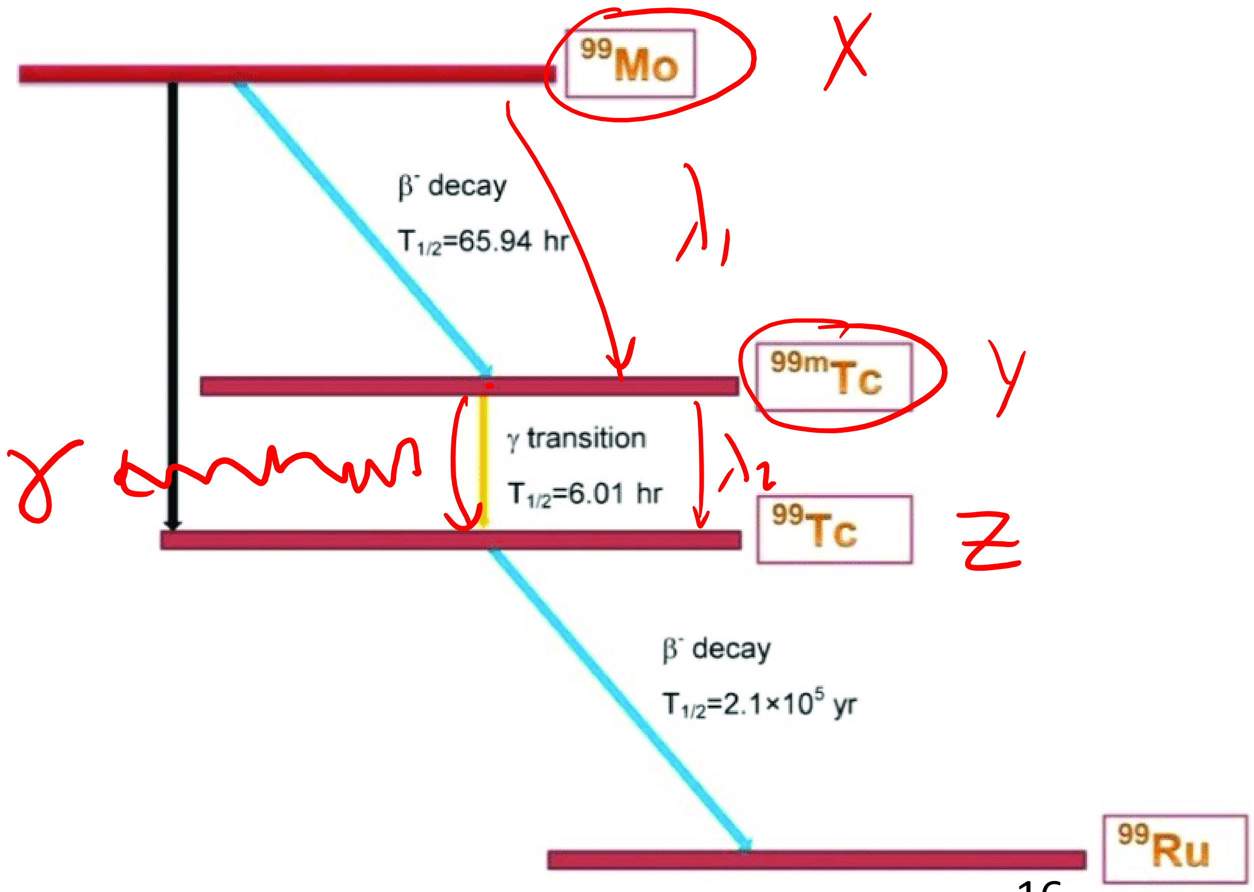
t_1

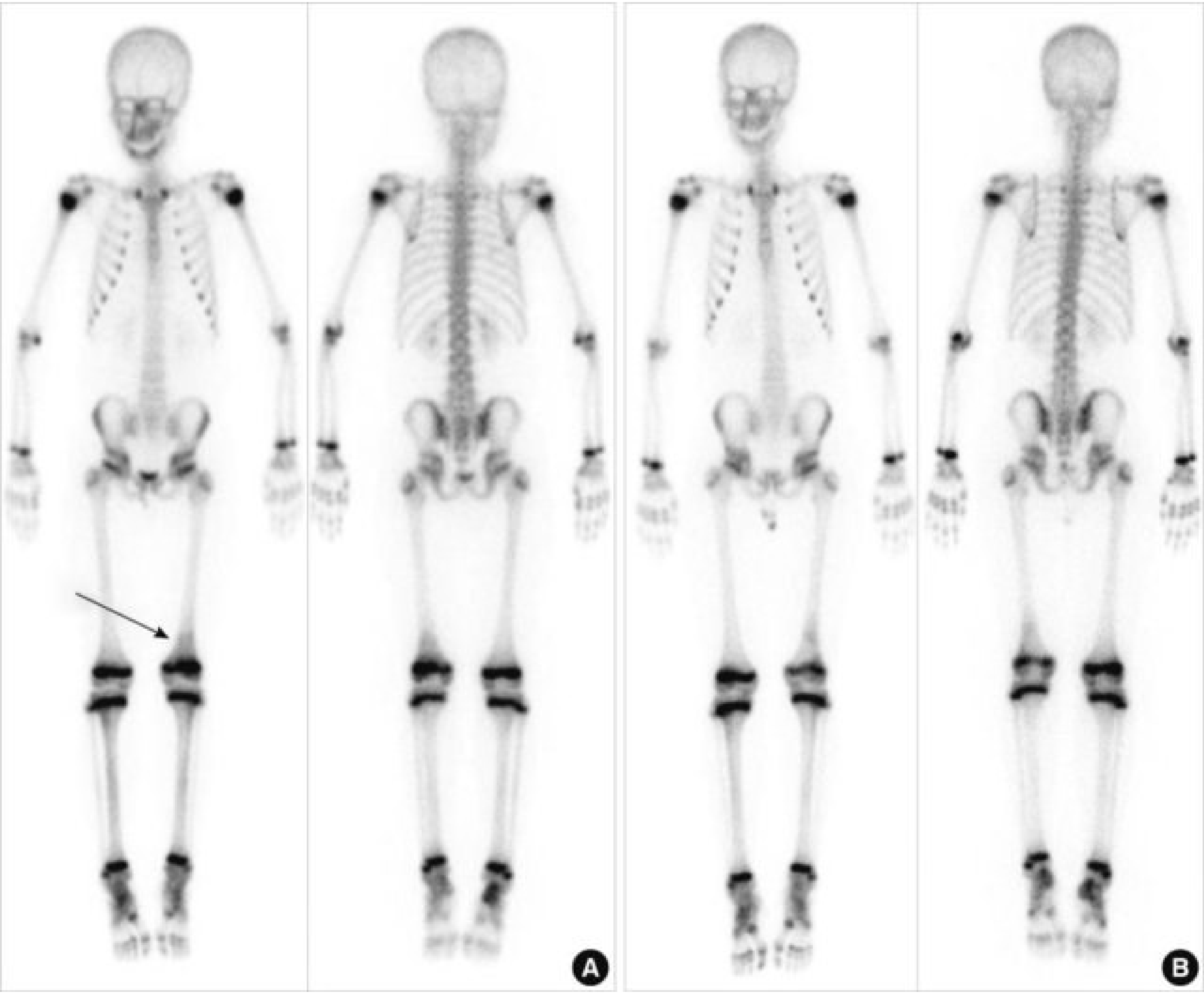
t_2

$$N_Y = N_{0X}\frac{\lambda_1}{\lambda_2-\lambda_1}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right)$$

$\lambda_1 = 0.0105$ hr^{-1}

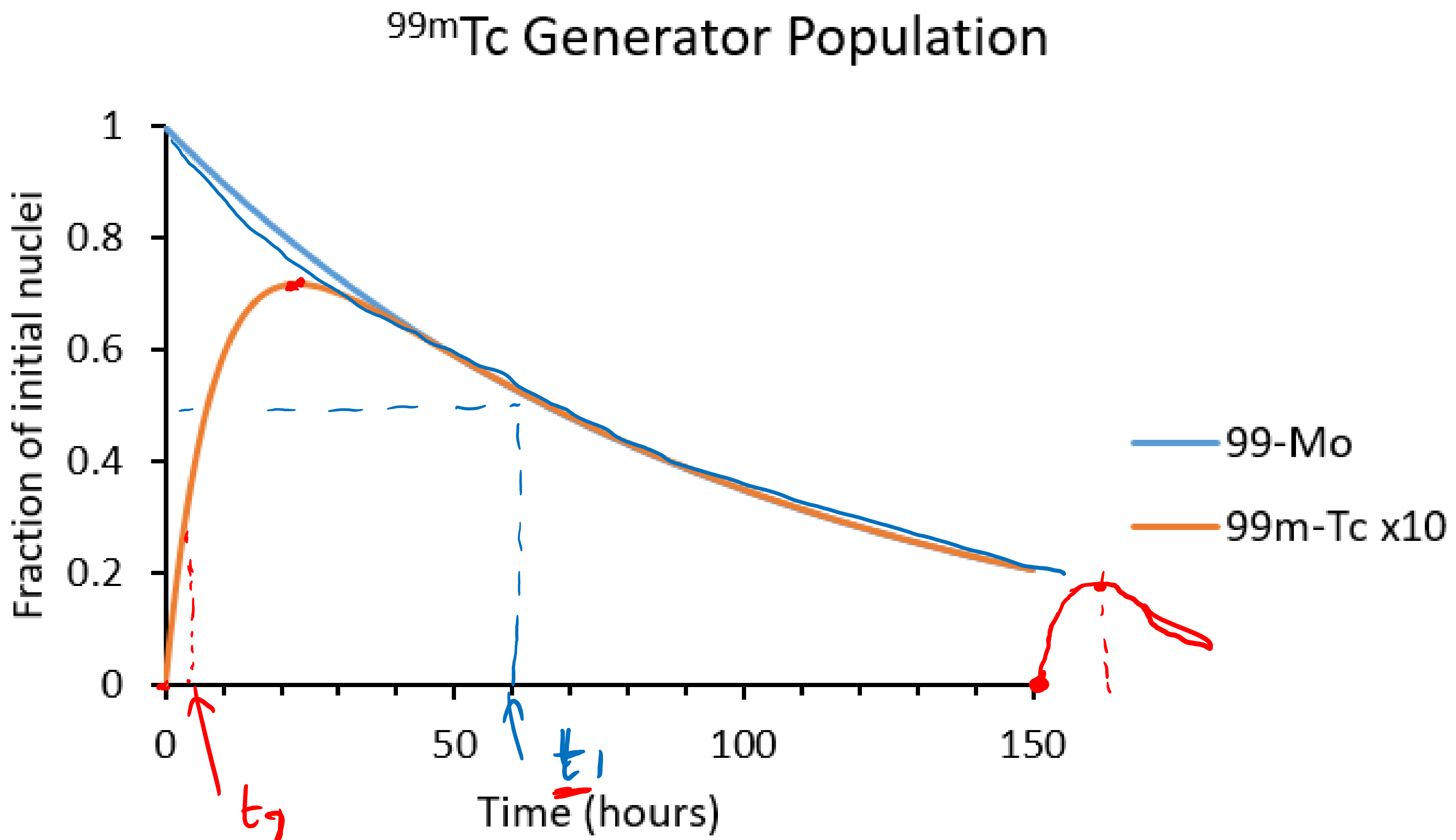
$\lambda_2 = 0.116$ hr^{-1}




A
B

Solutions of 1st order ODEs – Radioactive Decay Example

$$N_Y = N_{0X}\frac{\lambda_1}{\lambda_2-\lambda_1}\left(e^{-\lambda_1 t}-e^{-\lambda_2 t}\right)$$

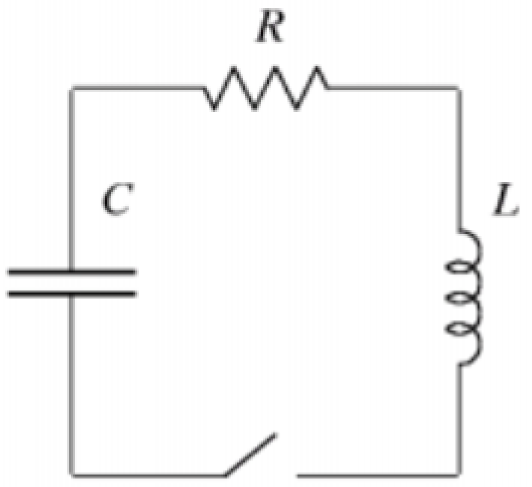


$$t_{max} = \frac{1}{\lambda_2-\lambda_1}\ln\frac{\lambda_2}{\lambda_1} = 22.8 \text{ hours}$$

Solutions of 2nd order ODEs - Example

a. An electrical circuit contains a resistor R, capacitor C and inductor L. By considering the potential difference across each component when the switch is closed, obtain a 2nd order differential equation in terms of the charge Q stored in the circuit

- Applying Kirchhov's 2nd law the net potential difference in a complete circuit is zero



- $L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0$
- Find characteristic equation $Q = Ae^{mt}$

Solutions of 2nd order ODEs - Example

b. If R = 2 Ω, L = 0.01 H, C = 10^{-6} F, the capacitor is initially uncharged with a current of 1 A flowing, obtain an expression for the current flowing in the circuit at a time t later

$$\omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = \sqrt{10^8 - 10^4} \approx 10^4 \qquad \frac{R}{2L} = 100$$

$$Q = \exp\left(-\frac{R}{2L}t\right)(A\sin\omega t + \cancel{B\cos\omega t})$$

$$t = 0, Q = 0, \ I = \frac{dQ}{dt} = \cancel{0.1}\ 1 \qquad B = 0$$

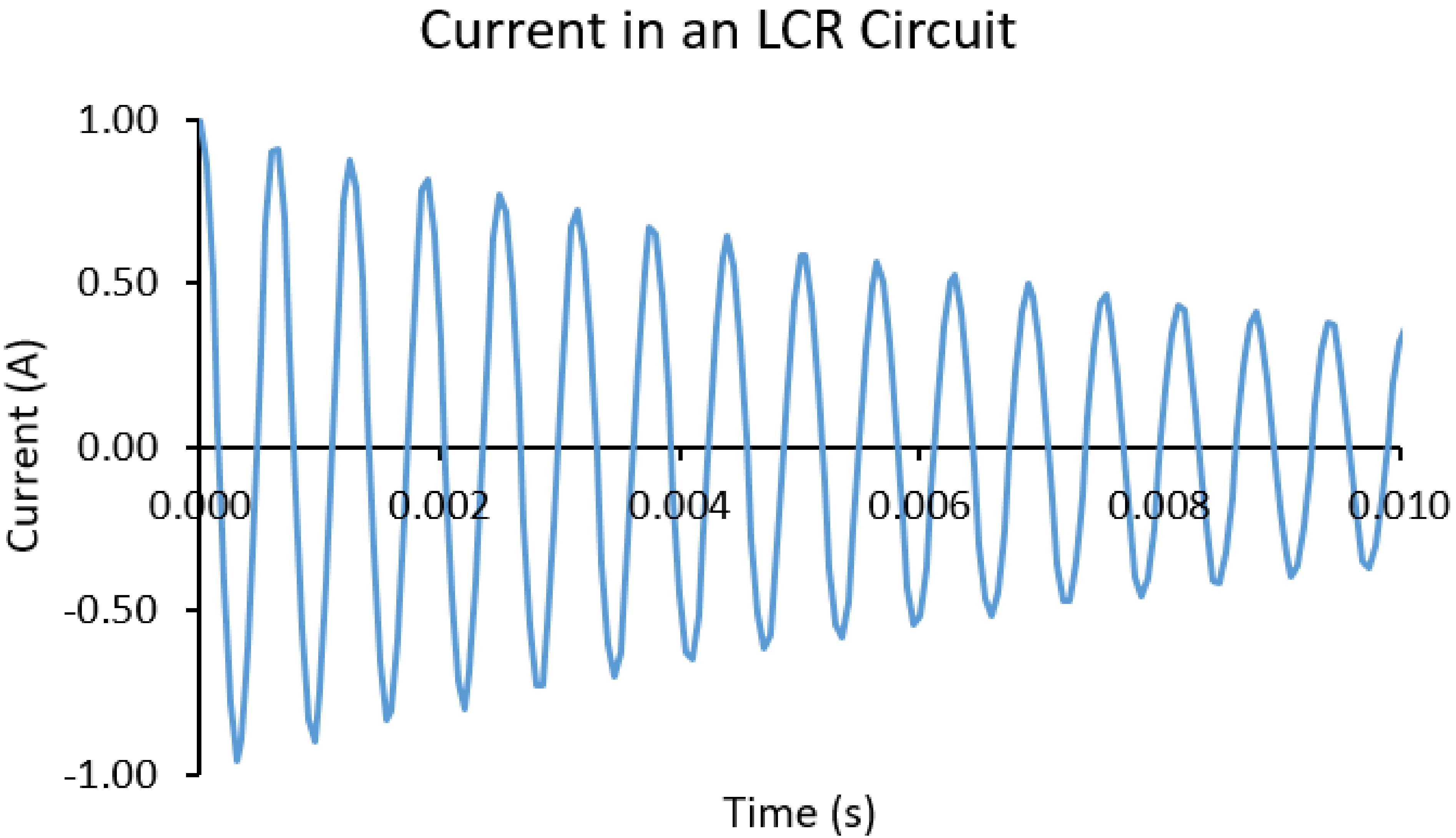
$$I = \frac{dQ}{dt} = -\frac{R}{2L}\exp\left(-\frac{R}{2L}t\right)A\sin\omega t + \exp\left(-\frac{R}{2L}t\right)A\omega\cos\omega t$$

$$I = 1 \quad t = 0 \qquad A\omega = 1 \quad A = 10^{-4}$$

$$I = \exp(-100t)\left(\cos(10^4 t) - 0.01\sin(10^4 t)\right)$$

Solutions of 2nd order ODEs - Example

$$I = \exp(-100t)\left(\cos(10^4 t) - 0.01\sin(10^4 t)\right)$$



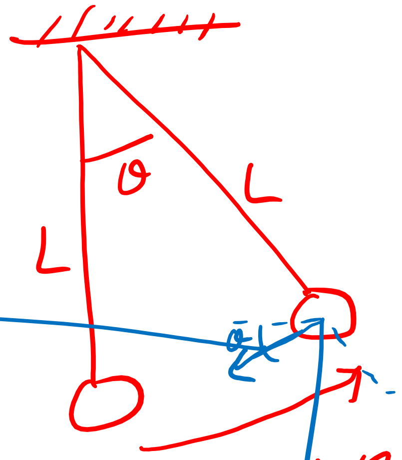
$m^2 + m - 6 = 0$

$(m+3)(m-2) = 0$

$m = 2, -3$

$Ae^{2t} + Be^{-3t}$

$\boxed{\text{KAHOOT 1a}}$



mg sin θ

$s = L\theta$

$\cancel{m}\frac{d^2s}{dt^2} = -\cancel{m}g\sin\theta$

mg

Other Solutions to ODEs

$a_{n+1} = \frac{a_n}{n+1}$

$a_1 = 1$

$a_2 = \frac{a_1}{1+1} = \frac{1}{2}$

$a_3 = \frac{a_2}{2+1} = \frac{a_2}{3} = \frac{1}{6}$

Power Series Solutions

Linearizing a non-linear ODE

$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$

$\sin\theta \approx \theta$

Power Series Solutions to 2nd order ODEs

- Linear, homogeneous ODEs have the form $A(x)\frac{d^2u}{dx^2} + B(x)\frac{du}{dx} + C(x)u = 0$
- Some can be solved by a power series solution

$$u(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

- For example $\boxed{\frac{d^2u}{dx^2} + u = 0}$
  - $\frac{du}{dx} = a_1 + 2a_2 x + 3a_3 x^2 \ldots = \sum_{n=1}^{\infty} n a_n x^{n-1}$
  - $\frac{d^2u}{dx^2} = 2a_2 + 6a_3 x + 12a_4 x^2 \ldots = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$ SHIFT INDEX
- Now let $i = n - 2$, then
  - $\frac{d^2u}{dx^2} = \sum_{i=0}^{\infty}(i+2)(i+1)a_{i+2}x^i$
- $\frac{d^2u}{dx^2} + u = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} a_n x^n = 0$

Power Series Solutions to 2nd order ODEs

$$\sum_{n=0}^{\infty} \underline{(n+2)(n+1)a_{n+2}}x^n + \sum_{n=0}^{\infty} \underline{a_n}x^n = 0$$

- Equate each power of x

$(n+2)(n+1)a_{n+2} + a_n = 0$

$a_{n+2} = -\frac{a_n}{(n+2)(n+1)}$

$a_0 \rightarrow a_2 \rightarrow a_4 \rightarrow \ldots\ldots$

$a_2 = -\frac{a_0}{(2)(1)} = -\frac{a_0}{2}$

$a_4 = -\frac{a_2}{(4)(3)} = \frac{a_0}{(4)(3)(2)(1)}$

- This is a ***recurrence relationship***
- Solutions
  - $u(x) = \underline{a_0}\left(1 - \frac{x^2}{\overset{2!}{(2)(1)}} + \frac{x^4}{\overset{4!}{(4)(3)(2)(1)}} - \cdots\right)$ $= a_0 \cos x$
  - $u(x) = a_1\left(x - \frac{x^3}{(3)(2)(1)} + \frac{x^5}{(5)(4)(3)(2)(1)} - \cdots\right)$ $= a_0 \sin x$
- Giving general solution

$u(x) = A\cos x + B\sin x$