Lecture 15:
Bose-Einstein and Fermi-Dirac statistics

We have seen that in Classical Mechanics the trajectory of a particle is uniquely defined and this has the intuitive consequence that different particles maintain their individuality even in the presence of other particles of the same nature. However, due to the uncertainty principle, the situation is intrinsically different in Quantum Mechanics whereby the trajectory of a particle is not a well defined concept. In this case, it will be impossible to follow the evolution of a single particle, whenever it is surrounded by other identical particles. We come then to another fundamental difference between Classical Mechanics and Quantum Mechanics; in the former, two identical particles can be distinguished only thanks to their history, whereas in the latter identical particles are fundamentally undistinguishable.

In order to see this more clearly, let us assume that $\psi(q_1, q_2)$ represents the total wavefunction of two identical particles. We assume here that q_i embeds all the properties of the i^{th} particle (position, momentum, spin...). If we swap the two particles, the total function cannot change, apart from a unitary phase:

$$\psi(q_1, q_2) \quad \rightarrow \quad e^{i\alpha}\psi(q_2, q_1) \tag{1}$$

α is not arbitrary; for instance, swapping the particles twice, must lead us to the original wavefunction:

$$e^{2i\alpha} = 1 \quad \rightarrow \quad e^{i\alpha} = \pm 1 \tag{2}$$

From this simple consideration, it follows that a wavefunction is either symmetric or anti-symmetric under an exchange of particles.

It is still to decide what sign to choose for the phase $e^{i\alpha}$. It is an *empirical* fact that particles with an integer spin (called *bosons*) have symmetric wavefunctions whereas particles with a half-integer spin (called *fermions*) have anti-symmetric wavefunctions. In other words, in a system with n bosons the wavefunction does not change sign if any two particles are swapped. On the other hand, the wavefunction changes sign whenever two fermions are swapped. These properties can be summarised by saying that fermions obey the Fermi-Dirac statisics, whereas bosons obey the Bose-Einstein statistics. Protons, electrons, neutrons, and nuclei with an odd number of nucleons are fermions; photons, kappa particles, pions, and nuclei with an even number of nucleons are bosons. It follows from the Fermi-Dirac statistics that the wavefunction of two fermions can be written as:

$$\Psi = \frac{1}{\sqrt{2}}[\psi(q_1, q_2) - \psi(q_2, q_1)] \tag{3}$$

From this formulation it is quite obvious that we cannot have two fermions with exactly the same properties ($q_1 = q_2$), since this configuration will directly lead to $\Psi = 0$. This is called the *Pauli's Exclusion Principle*. The main consequence of this principle is that two identical fermions may not occupy the same quantum state simultaneously.

A first striking consequence of these concepts is that they give a heuristic justification as to why we intuitively distinguish between waves and particles. In our classical understanding, it is difficult to associate fundamental particles such as the electrons with waves and, on the other hand, to associate electromagnetic waves to particles (photons). Why is that? From Quantum Mechanics, both treatments should be identical and interchangeable. However, we have seen that electrons are fermions, which implies that they cannot overpopulate the same quantum state. More precisely, there can be only one electron in a defined quantum state. Electrons thus cannot make 'agglomerates', justifying our intuitive representation of an electrons as an individual and well defined particle. On the other hand, photons are inclined to conglomerate into the same quantum state. It is thus common to find a high density of photons in a small region of space, justifying why we tend to picture the electromagnetic interaction as mediated by waves.

Another interesting phenomenon arises if we consider a bound state of two fermions. Let us assume that each of these fermions has an initial spin of $s_1 = s_2 = 1/2$. The total system composed of the two fermions will thus have a number of possible states given by $(2s_1 + 1)(2s_2 + 1) = 4$. Three of these states will have $s_{tot} = 1$ and one will have $s_{tot} = 0$. In other words the bound state of two fermions is a boson. We then come to the conclusion that, even if two independent particles with half-integer spin are fermions (and therefore cannot occupy exactly the same quantum state), an ensemble of pairs of particles with half-integer spin are bosons (and can therefore occupy the same quantum state).

As an example, this latter conclusion sits at the core of BSC (John Bardeen, Leon Neil Cooper, and John Robert Schrieffer) theory of superconductivity. This theory was proposed in 1957 and was worth the 1972 Nobel prize in physics. Superconductivity is a peculiar phenomenon in physics; at temperatures very close to the absolute zero, any material seems to dramatically drops its resistivity. In other words, electrons are almost free to move without any friction within the material. This phenomenon can be explained if we assume that electrons can create pair bound states. Two electrons couple with each other giving as a result a boson. These pairs of electrons then amass into the same quantum state creating a superfluid of electron pairs. The natural Coulombian repulsion is overcome by the tiny vibrations of the lattice at extremely low temperature. Therefore, even if single electrons are fermions and therefore avoid to occupy the same quantum state, a suitable external perturbation (such as the lattice vibrations in the material) allows for electrons to couple into pairs, which thus converge into the same quantum state (phenomenon usually called Bose-Einstein condensate) as if they were bosons.