PHY2004

Exam Time Table Code PHY2004

Any calculator, except one with pre-programmable memory, may be used in this examination.

Answer Books A, B and C

LEVEL 2

EXAMINATION CONTRIBUTING TO THE DEGREES OF BACHELOR OF SCIENCE (BSc) AND MASTER IN SCIENCE (MSci)

PHY2004
Electricity, Magnetism and Optics

Duration: 3 Hours

Friday, 17th August 2018 9:30 AM - 12:30 PM

Examiners: Prof. P. Browning
Dr. P. van der Burgt
and the Internal Examiners

Answer ALL TEN questions in Section A
Answer TWO questions in Section B
Answer ONE question in Section C

Section A questions are worth 4 marks each

Section B and C questions are worth 20 marks each

Use a separate answer book for each Section.
Follow the instructions on the front of the answer book. Enter your Anonymous Code number ans Seat number, but NOT your name.

THE QUEEN'S UNIVERSITY OF BELFAST
SCHOOL OF MATHEMATICS & PHYSICS

PHYSICAL CONSTANTS

Speed of light in a vacuum	$c = 3.00 \times 10^{8}\ \mathrm{ms^{-1}}$
Permeability of a vacuum	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{Hm^{-1}}$ $\approx 1.26 \times 10^{-6}\ \mathrm{Hm^{-1}}$
Permittivity of a vacuum	$\varepsilon_0 = 8.85 \times 10^{-12}\ \mathrm{Fm^{-1}}$
Elementary charge	$e = 1.60 \times 10^{-19}\ \mathrm{C}$
Electron charge	$= -1.60 \times 10^{-19}\ \mathrm{C}$
Planck Constant	$h = 6.63 \times 10^{-34}\ \mathrm{Js}$
Reduced Planck Constant	$\hbar = 1.05 \times 10^{-34}\ \mathrm{Js}$
Rydberg Constant for hydrogen	$R_\infty = 1.097 \times 10^{7}\ \mathrm{m^{-1}}$
Unified atomic mass unit	$1u = 1.66 \times 10^{-27}\ \mathrm{kg}$ $1u = 931\ \mathrm{MeV}$
1 electron volt (eV)	$= 1.60 \times 10^{-19}\ \mathrm{J}$
Mass of electron	$m_e = 9.11 \times 10^{-31}\ \mathrm{kg}$
Mass of proton	$m_p = 1.67 \times 10^{-27}\ \mathrm{kg}$
Mass of neutron	$m_n = 1.67 \times 10^{-27}\ \mathrm{kg}$
Molar gas constant	$R = 8.31\ \mathrm{JK^{-1}mol^{-1}}$
Boltzmann constant	$k = 1.38 \times 10^{-23}\ \mathrm{JK^{-1}}$
Avogadro constant	$N_A = 6.02 \times 10^{23}\ \mathrm{mol^{-1}}$
Gravitational constant	$G = 6.67 \times 10^{-11}\ \mathrm{Nm^{2}kg^{-2}}$
Acceleration of free fall on the Earth's surface	$g = 9.81\ \mathrm{ms^{-2}}$

continued …

MATHEMATICAL IDENTITIES

In answering the questions on this paper you may make use of any of the following:

Divergence theorem

$$\int_V \nabla \cdot \underline{E}\, dV = \oint_S \underline{E} \cdot d\underline{S}$$

Stoke's Theorem

$$\int_S \nabla \times \underline{E} \cdot d\underline{S} = \oint \underline{E} \cdot d\underline{\ell}$$

Identities

$$\nabla \times \nabla\left(Scalar\right) = \underline{0}$$

$$\nabla \cdot \nabla \times \left(Vector\right) = 0$$

$$\nabla \times \left(\nabla \times \underline{E}\right) = \nabla\left(\nabla \cdot \underline{E}\right) - \nabla^2 \underline{E}$$

$$\nabla\left(y\, \underline{E}\right) = y\, \nabla \cdot \underline{E} + \underline{E} \cdot \nabla y$$

$$\nabla \cdot \left(\underline{E} \times \underline{H}\right) = \underline{H} \cdot \left(\nabla \times \underline{E}\right) - \underline{E} \cdot \left(\nabla \times \underline{H}\right)$$

Material Equations $\quad \underline{J} = \sigma \underline{E} \quad \underline{B} = \mu \underline{H} \quad \underline{D} = \varepsilon \underline{E}$

Poynting Vector $\quad \underline{S} = \underline{E} \times \underline{H}$

Trigonometric identities

$$cos(A \pm B) = cos\, A\, cos\, B \mp sin\, A\, sin\, B$$
$$sin(A \pm B) = sin\, A\, cos\, B \pm sin\, B\, cos\, A$$

$$cos\, A + cos\, B = 2\, cos\left(\frac{A+B}{2}\right) cos\left(\frac{A-B}{2}\right)$$
$$sin\, A - sin\, B = 2\, cos\left(\frac{A+B}{2}\right) sin\left(\frac{A-B}{2}\right)$$

continued …

SECTION A

Use a section A answer book

Answer ALL 10 questions in this section

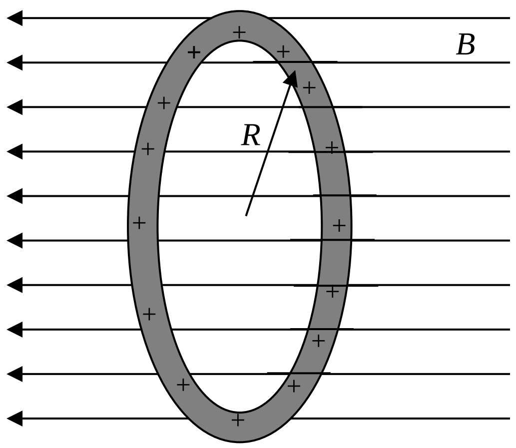
Full explanations of your answers are required to attain full marks

1 Discuss under what approximation the electrostatic regime is valid and show that this leads to the electric field being conservative. **[4]**

2 What is the microscopic origin of the polarization induced in a dielectric subject to an external electric field? **[4]**

3 Explain why the vector potential is not uniquely defined and provide an example of a typical Gauge choice. **[4]**

4 The diagram below shows a thin insulating ring holding a charge per unit length λ on its circumference. The ring sits in a uniform magnetic field, which is orthogonal to the plane of the ring. The magnetic field reduces linearly with time t, according to $B = B_o(1 - t/T)$, over a period T.



Re-draw the diagram and indicate the direction of the electric field component, at the ring circumference, induced by the magnetic field. Show that this field is given by

$$E = \frac{B_0 R}{2T}$$ **[4]**

continued …

SECTION A

5 Discuss how the presence of a temporally-varying magnetic field makes the electric field non-conservative. **[4]**

6 By considering the electric and magnetic energy densities in a volume V, bounded by a surface S, the rate of change of the total electromagnetic energy W is described by the Poynting's Theorem

$$\int_S \underline{E} \times \underline{H} \cdot d\underline{S} + \int_V \underline{E} \cdot \underline{J} dV = \frac{\partial W}{\partial t}$$

Explain, in detail, the physical significance of the terms in the expression above, defining clearly what you understand by the Poynting vector. What are the SI units of this vector? **[4]**

7 Assuming Snell's law show that the Brewster angle between two materials of refractive index n_1 and n_2 is given by

$$tan\theta_B = \frac{n_2}{n_1}$$

For what polarization state is the Brewster angle valid? Motivate your answer **[4]**

8 What is meant by dichroism and birefringence? **[4]**

9 The specific rotary power of a sugar solution is 3.0 deg/cm. If a 10 cm glass cell filled with the solution is placed between otherwise crossed polarisers, calculate the fraction of light intensity that will be transmitted through the system? (*Ignore any reflection losses at the polarisers or cell windows*). **[4]**

10 A beam of vertically polarized light impinges onto two consecutive polarisers, the first with its transmission axis tilted by 20 degrees compared to the horizontal and the second with its transmission axis parallel to the vertical. What is the percentage of light intensity escaping? Does this percentage change if the two polarisers are swapped? **[4]**

continued …

SECTION B

Use a Section B answer book

Answer <u>TWO</u> questions from this section

11 **(a)** Write down the differential form of Gauss's law in free space. How should this law be modified in a dielectric medium? **[2]**

(b) The following figure (Fig. 2) show charges placed near or inside a cubic Gaussian surface. For each case state the total flux of the electric field passing through the surface. **[3]**

(i)

(ii)

(iii)

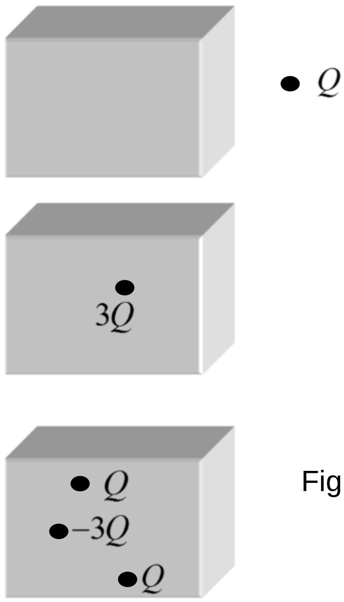


Fig.2

(c) Two positive point charges Q_1 and Q_2 are separated by a distance h as shown in Fig. 3.

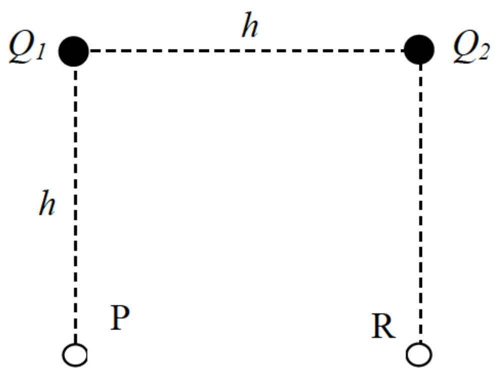


Fig. 3

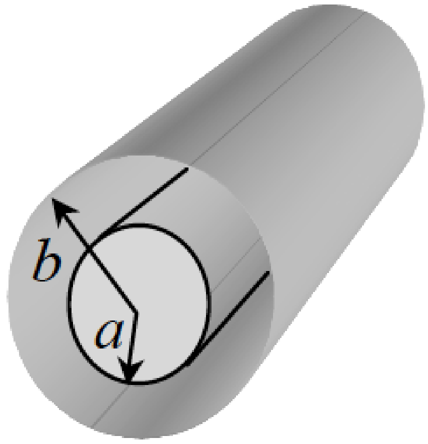
[QUESTION 11 CONTINUED OVERLEAF]

(i) Calculate the magnitude of the electric field generated at point P and qualitatively show its direction. **[3]**

(ii) Calculate the potential generated at the point R. **[1]**

(d) Two long concentric conducting cylinders of radii a and b ($a<b$) carry equal and opposite charge per unit length λ (positive on the inner conductor). The space between the two conductors is filled with a dielectric of relative permittivity ε_r (Fig.4).

Fig. 4



(i) Sketch the distribution of charges on the conductors and the dielectric. **[2]**

(ii) Sketch the direction of the resulting electric field. **[1]**

(iii) Calculate the amplitude of the electric field $\boldsymbol{E}$, polarisation $\boldsymbol{P}$, and electric displacement field $\boldsymbol{D}$ outside of the outer conductor ($r > b$). **[1]**

(iv) Calculate the amplitude of the electric field $\boldsymbol{E}$, polarisation $\boldsymbol{P}$, and electric displacement field $\boldsymbol{D}$ in the dielectric ($b > r > a$). **[3]**

(v) Calculate the amplitude of the electric field $\boldsymbol{E}$, polarisation $\boldsymbol{P}$, and electric displacement field $\boldsymbol{D}$ inside the inner conductor ($r < a$). **[1]**

(vi) Calculate the potential difference between the two conductors. **[2]**

(vii) Calculate the capacitance per unit length of the whole system. **[1]**

continued …

SECTION B

12 **(a)** The time-independent form of Ampère's circuital law, is

$$\oint \underline{B} \cdot \underline{dl} = \mu_0 \int \underline{J} \cdot \underline{dS} = \mu_0 I$$

Use this expression to derive the magnetic field associated with an infinitely long and straight wire carrying a constant current. **[4]**

(b) Inside a magnetised medium the effect of the magnetisation $\underline{M}$ may be regarded as equivalent to a volume current density $\underline{J}_V \left(= \nabla \wedge \underline{M}\right)$. Show that this leads to the introduction of the magnetic intensity vector $\underline{H}$, where

$$\underline{B} = \mu_0(\underline{H} + \underline{M}) = \mu_0 \mu_r \underline{H},$$

shows further that this leads to

$$\oint \underline{H} \cdot \underline{dl} = \int_S \underline{J} \cdot \underline{dS} = I$$ **[4]**

(c) By considering the electric and magnetic fields in the vicinity of a charging (*time varying*) parallel-plate capacitor, discuss why the above expression is incomplete. **[2]**

(d) Using the charge conservation equation

$$\nabla \cdot \underline{J} + \frac{\partial \rho}{\partial t} = 0,$$

show that the complete, time-varying, differential form of Ampère's cicuital law is

$\nabla \cdot \underline{H} = \underline{J} + \frac{\partial \underline{D}}{\partial T}$ (you may assume $\nabla \cdot \underline{D} = \rho$) **[4]**

(e) Figure 2 shows part of a long, conducting cylinder of relative permeability μ_R, carrying a steady, current I uniformly distributed over its cross-section.

[QUESTION 12 CONTINUED OVERLEAF]

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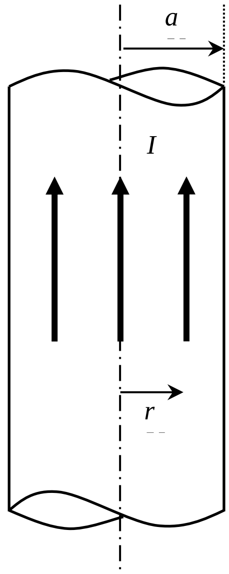
SECTION B



[QUESTION 12 CONTINUED]

(i) Reproduce the diagram on the left and, indicate, *unambiguously*, the direction of the static vector fields $\underline{B}, \underline{H}$ and $\underline{M}$, for $r < a$.

(ii) Use Ampère's circuital law to show that $H = Ir/2\pi a^2$.

(iii) Hence, determine an expression for B and show that

$$M = \frac{Ir}{2\pi a^2}(\mu_r - 1)$$ **[6]**

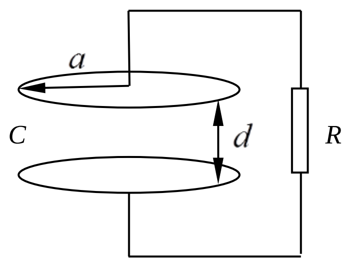
13 **(a)** Write down the definition of the Poynting vector and explain its meaning. **[3]**

(b) A plane-wave, travelling along the z direction, is described by the expressions

$$E_x = E_{0x} e^{ikz} e^{-i\omega t} \quad \text{and} \quad H_y = H_{0y} e^{ikz} e^{-i\omega t}$$

Show that the time average value of the Poynting vector is $\langle \underline{S} \rangle = \frac{1}{2} Re(\underline{E} \wedge \underline{H}^*)$ where $\underline{H}^*$ is the complex conjugate of $\underline{H}$. **[5]**

(c) A parallel plate capacitor (as in figure below) of capacitance C is initially charged to a potential ψ_0 and then completely discharged through a resistance R.



[QUESTION 13 CONTINUED OVERLEAF]

continued …

(i) Write down the time-dependent expression of the electric field $\boldsymbol{E}$ during discharge. **[3]**

(ii) Write down the time-dependent expression of the magnetic field $\boldsymbol{H}$ during discharge. **[3]**

(iii) Calculate the resulting pointing vector and indicate its direction. **[2]**

(iv) By integrating the energy flow over time, show that the total energy flowing from the capacitor during discharge is $W_C = \frac{1}{2}C\psi_0{}^2$. **[4]**

continued …

SECTION B

14 **(a)** A plane electromagnetic wave of frequency $\nu = 1$ THz is propagating in vacuum along the x direction. The wave is linearly polarised along y and the maximum electric field is $E_0 = 100$ V/m.

(i) Determine the wavelength and wavenumber of the wave. What part of the electromagnetic spectrum does this wave belong to? **[3]**

(ii) Write an expression for the amplitude of the electric and magnetic fields of the wave as a function of time t and of x. **[3]**

(iii) Show that the expressions derived in question **(ii)** are solutions of the wave equation for the electric and magnetic fields of an electromagnetic wave propagating in a vacuum with no external sources. **[4]**

(iv) Determine the energy density carried by the wave as a function of time. **[2]**

(v) Determine the average energy density carried by the wave. **[1]**

(b) **(i)** Show that for a plane boundary between two dielectrics the Fresnel amplitude reflection coefficient at normal incidence is given by

$$r = \frac{E_r}{E_i} = \frac{n_1 - n_2}{n_1 + n_2}$$

where n_1 and n_2 are the refractive indices of the incident and transmitting media, respectively. E_r and E_i are the reflected and incident electric field amplitudes respectively. **[5]**

(ii) For what value of n_2 is the reflection minimised? For what value is it maximised? What do these cases physically correspond to? **[2]**

continued …

SECTION C

Use a Section C answer book

Answer <u>ONE</u> question from this section

15 **(a)** Derive the electromagnetic wave equation in vacuum from Maxwell's equations:

$$\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \qquad \text{Eq. 1}$$

explaining in detail all the assumptions you make during your derivation. **[4]**

(b) Show that an equation analogous to Eq. 1 would hold for the magnetic field of the wave. **[4]**

(c) Using Eq. 1 show that, at optical frequencies, the refractive index of a dielectric is given by

$$n = \sqrt{\varepsilon_r} = c/v,$$

where c and v are the phase velocities in vacuum and in the dielectric, respectively. **[4]**

(d) Show that, for a plane boundary between two dielectrics, the Fresnel amplitude reflection coefficient at normal incidence in the visible region is given by

$$r = \frac{E_r}{E_i} = \frac{n_1 - n_2}{n_1 + n_2}$$

where n_1 and n_2 are the refractive indices of the incident and transmitting media and E_r and E_i are the reflected and incident electric field amplitudes.
You may assume that the relationship between the E and H field amplitudes in a plane wave is $E = ZH$, where $Z = \sqrt{\mu / \varepsilon}$. **[5]**

(e) Given that the refractive index of glass is about 1.4, what percentage of visible optical energy is reflected from a domestic window? If this is less than 100% where does the rest go? **[3]**

SECTION C

16 **(a)** In the classical theory of optical dispersion, a time-varying electric field $E = E_0 e^{-i\omega t}$ is assumed to drive bound electrons of mass m and charge e. Their equation of motion may be described by

$$m\frac{d^2x}{dt^2} + m\gamma\frac{dx}{dt} + m\omega_0^2 x = eE_0 e^{-i\omega t}$$

Explain the physical meaning of the four terms in this expression. **[4]**

(b) The solution to this equation is written as

$$x = \frac{(e/m)E_0 e^{-i\omega t}}{(\omega_0^2 - \omega^2) - i\gamma\omega}.$$

(i) If there are N such electrons per unit volume in a material show that its relative permittivity ε_r as a function of the angular frequency ω of the driving field is

$$\varepsilon_r = \hat{n}^2 = 1 + \frac{N(e^2/\varepsilon_0 m)}{(\omega_0^2 - \omega^2) - i\gamma\omega} = \varepsilon_r' + i\varepsilon_r''$$ **[2]**

(ii) What is $\hat{n}$ in the above expression? **[2]**

(iii) What is the significance of the ω_0 term? **[2]**

(iv) In relation to the optical properties of a material, what is the significance of the imaginary part (ε_r'') of the permittivity? **[2]**

(v) Explain how resonance can be reached in this configuration, and discuss its physical implications. **[2]**

[QUESTION 16 CONTINUED OVERLEAF]

continued …

[QUESTION 16 CONTINUED]

(c) **(i)** Sketch the variation of the real and imaginary parts of $\hat{n}$ as a function of wavelength in the vicinity of the resonance of a single electron and show in what region the material is transparent **[3]**

(ii) On your diagram indicate one region of normal dispersion and a region of anomalous dispersion. **[3]**

END OF EXAMINATION