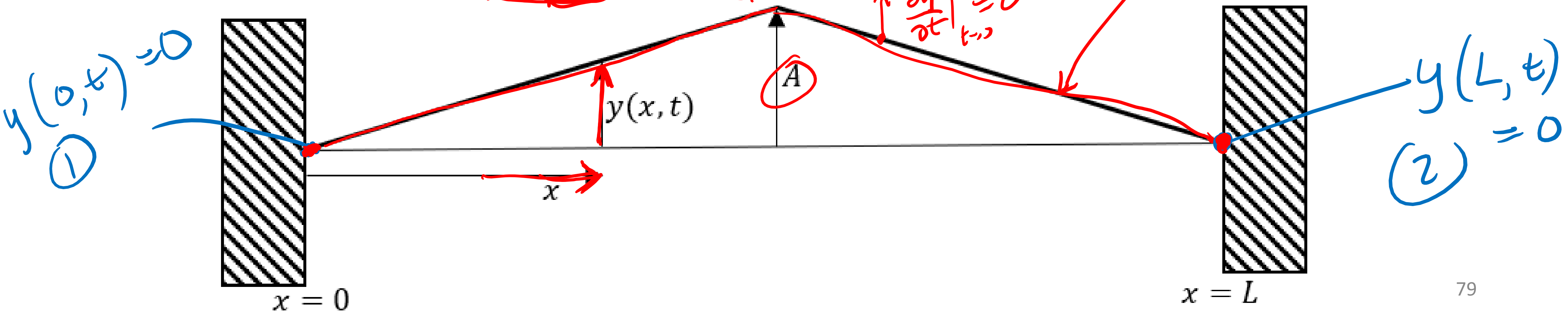
4.3 Separation of Variables

PDEs in which variables are separable

- Some linear PDEs can be solved by this process:
  1. Write solution as a **product of single variable functions**, e.g. $y(x,t) = f(x)g(t)$
  2. Substitute into PDE and **separate variables** on either side of the equation
  3. For equality to hold for all values, each side must equal a **separation constant**
  4. Solve the resulting ODEs
  5. Apply initial conditions and boundary condition find coefficients
  6. Obtain the most general possible (Fourier series solution)

Example: Waves on a wire $\boxed{\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}}$ $c^2 = T/\mu$

IC $t=0$ ③ $\left.\frac{\partial y}{\partial t}\right|_{t=0} = 0$ ④



PDEs in which variables are separable

$$\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}$$

1. $y(x,t) = \underbrace{f(x)g(t)}$
2. $\frac{1}{c^2}\frac{\partial^2}{\partial t^2}[f(x)g(t)] = \frac{\partial^2}{\partial x^2}[f(x)g(t)]$

$\frac{1}{c^2} f(x) \frac{d^2 g}{dt^2} = g(t) \frac{d^2 f}{dx^2}$ SEPARATE x, t DIVIDE BY fg

$\frac{1}{c^2}\frac{1}{g(t)}\frac{d^2 g(t)}{dt^2} = \frac{1}{f(x)}\frac{d^2 f(x)}{dx^2}$

3. $\underbrace{\frac{1}{c^2}\frac{d^2 g}{dt^2}\frac{1}{g(t)}}_{t} = \underbrace{\frac{d^2 f}{dx^2}\frac{1}{f(x)}}_{x} = \underbrace{-k^2}$ (separation constant)

c – WAVE VELOCITY

4. ODEs $\quad \underbrace{\frac{d^2 f}{dx^2} + k^2 f = 0} \quad \underbrace{\frac{d^2 g}{dt^2} + c^2 k^2 g = 0}$

$f(x) = C_1 \sin(kx) + C_2 \cos(kx)$ $\quad g(t) = C_3 \sin(ckt) + C_4 \cos(ckt)$

PDEs in which variables are separable

5. Apply Boundary Conditions $y(0,t) = y(L,t) = 0$

$f(x=0) = 0 = C_1 \sin(0) + C_2 \cos(0)$

$C_2 = 0$

$$f(L) = 0 = C_1 \underbrace{\sin(kL)}_{=0} + 0$$

$C_1 = 0$ (TRIVIAL)

$$\boxed{k = \frac{n\pi}{L}}$$

n - INTEGER

$f(x) = C_1 \sin\left(\frac{n\pi}{L} x\right)$

$\times$

$g(t) = C_3 \sin\left(c\frac{n\pi}{L} t\right) + C_4 \cos\left(c\frac{n\pi}{L} t\right)$

$y(x,t)$

$$y(x,t) = f(x)g(t) = \sin\left(\frac{n\pi}{L}x\right)\left[a_n \underbrace{\sin\left(\frac{n\pi}{L}ct\right)}_{=0} + b_n \underbrace{\cos\left(\frac{n\pi}{L}ct\right)}_{=1}\right]$$

$t=0$

$a_n = C_1 C_3$

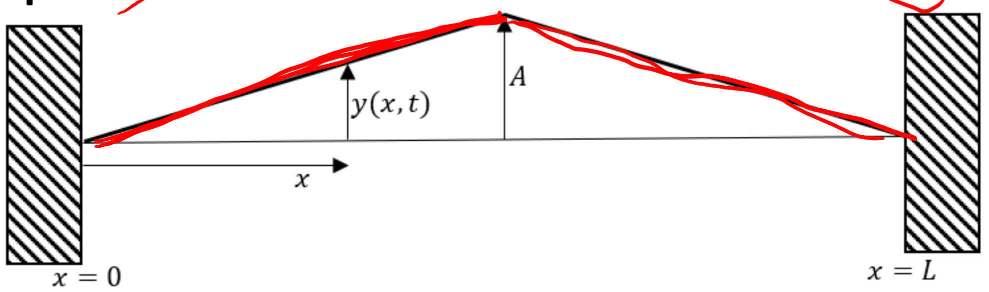
$b_n = C_1 C_4$

Most general solution – Fourier coefficients

6. Initial Conditions $y(x,0) = b_n \sin\left(\frac{n\pi}{L}x\right)$

But What if $y(x,0)$ is not sinusoidal? For example

$$y(x,0) = \frac{2Ax}{L} \text{ for } 0 < x < \frac{L}{2}$$
$$y(x,0) = 2A\left(1 - \frac{x}{L}\right) \text{ for } \frac{L}{2} < x < L$$

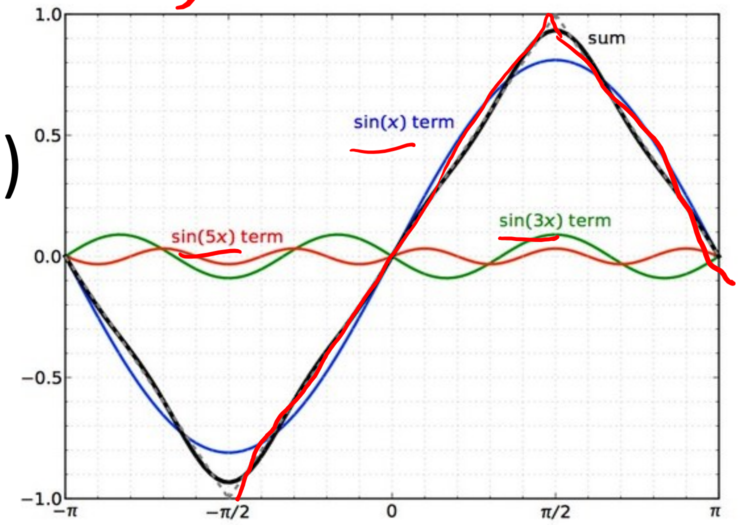


A superposition of sine/cosine wave solutions gives the most general solution.

$$y(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left(a_n \sin\left(\frac{n\pi}{L}ct\right) + b_n \cos\left(\frac{n\pi}{L}ct\right)\right)$$

This is a Fourier Series!

Coefficients found by Fourier methods(see Tom's notes)



$t = 0$ MULTIPLY BY $\sin\left(\frac{m\pi x}{L}\right)$ m - INTEGER

INTEGRATE IN x FROM $0 \to L$

Variable Separation – Heat Equation

$$\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2}$$

1. $T(x,t) = f(x)g(t)$
2. $f(x)\frac{dg}{dt} = D\, g(t)\frac{d^2 f}{dx^2}$ ÷ BY $f(x)\,g(t)$
3. $\frac{1}{g}\frac{dg}{dt} = \frac{D}{f}\frac{d^2f}{dx^2} = -k^2$
4.

$$\frac{dg}{dt} = -k^2 g \qquad \int\frac{dg}{g} = \int -k^2 dt$$

$$\frac{d^2f}{dx^2} + \frac{k^2}{D}f = 0$$

$$g(t) = A\exp(-k^2 t)$$

$$f(x) = B\sin\left(\frac{k}{\sqrt{D}}x\right) + C\cos\left(\frac{k}{\sqrt{D}}x\right)$$

$$T(x,t) = \exp(-k^2t)\,(B'\sin(\frac{kx}{\sqrt{D}}) + C'\cos(\frac{kx}{\sqrt{D}}))$$

$AB = B'$

$AC = C'$

1. Using the method of characteristics, find the solution to the following first order PDE

$$\frac{\partial u}{\partial t} + (1-x)\frac{\partial u}{\partial x} = -u$$

subject to the initial condition $u(x,0) = \exp(-x^2)$.

$$\frac{\partial u}{\partial t} + \frac{dx}{dt}\cdot\frac{\partial u}{\partial x} = \frac{du}{dt}$$

TOTAL DIFFERENTIAL $u(x,t)$

– x NOT CONSTANT

– x, t RELATED

$$\frac{du(x,t)}{dt} = \frac{\partial u}{\partial t}\cdot\underbrace{\frac{\partial t}{\partial t}}_{=1} + \frac{\partial u}{\partial x}\cdot\underbrace{\frac{\partial x}{\partial t}}_{\text{IF } x=f(t)\ \frac{dx}{dt}}$$

CHARACTERISTIC

CHARACTERISTIC TRAJECTORY

$$\frac{dx}{dt} = 1 - x$$

At $t = 0$ $x = x_0$

DETERMINES WHICH TRAJECTORY ON

$x_0 = \ldots\ldots$

$$\frac{du}{dt} = -u$$

INTEGRATION CONSTANT $C(x_0)$

$$\underbrace{u(x_0, t) = \ldots\ldots}_{\text{HOW } u \text{ EVOLVES ALONG A CHARACTERISTIC}}$$

SOLUTION $u(x,t)$ FOR ALL TRAJECTORIES

Consider a 2D region, $0 < x < 1$, $0 < y < 1$ with conducting walls, such that the electrostatic potential $\phi(x, y)$ at the boundaries is

$$\phi(0, y) = \phi(1, y) = 0 \quad \text{and} \quad \phi(x, 0) = \sin(4\pi x), \;\; \phi(x, 1) = \sin(8\pi x)$$

The spatial profile of the potential between the boundaries is determines by Laplace's eqaution

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

① $\phi(x, y) = X(x)\,Y(y)$

② SUBSTITUTE IN

③ SEPARATE x, y
SET EQUAL TO CONSTANT

$$\frac{d^2X}{dx^2} Y + X \frac{d^2Y}{dy^2} = 0 \qquad \div XY$$

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = -\lambda^2$$

$$\frac{d^2X}{dx^2} + \lambda^2 X \qquad\qquad \frac{d^2Y}{dy^2} - \lambda^2 Y = 0$$

$$e^{mx}$$

$$e^{i\lambda x} = \cos\lambda x + i\sin\lambda x$$

$$m^2 = -\lambda^2 \qquad m^2 = \lambda^2$$
$$= \pm i\lambda \qquad m = \pm\lambda$$

$$X = A\cos(\lambda x) + B\sin(\lambda x)$$

$$Y = C' e^{+\lambda y} + D' e^{-\lambda y}$$

OR

$$Y = C\,\underbrace{\tfrac{1}{2}\left(e^{+\lambda y} + e^{-\lambda y}\right)}_{\cosh(\lambda y)} + D\,\underbrace{\tfrac{1}{2}\left(e^{+\lambda y} - e^{-\lambda y}\right)}_{\sinh(\lambda y)}$$

Consider a 2D region, $0 < x < 1$, $0 < y < 1$ with conducting walls, such that the electrostatic potential $\phi(x, y)$ at the boundaries is

$$\phi(0, y) = \phi(1, y) = 0 \quad \text{and} \quad \phi(x, 0) = \sin(4\pi x), \;\; \phi(x, 1) = \sin(8\pi x)$$

The spatial profile of the potential between the boundaries is determines by Laplace's eqaution

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

which has separable solution

$$\phi(x, y) = \sum_{n=0}^{\infty} [A_n \cos \lambda_n x + B_n \sin \lambda_n x] [C_n \cosh \lambda_n y + D_n \sinh \lambda_n y]$$

where A_n, B_n, C_n, D_n and λ_n are arbitrary constants to be determined.

Using the first 3 boundary conditions above, prove:

(a) $A_n = 0$ for all n

(b) $\lambda_n = n\pi$ for $n = 0, 1, 2, \ldots$

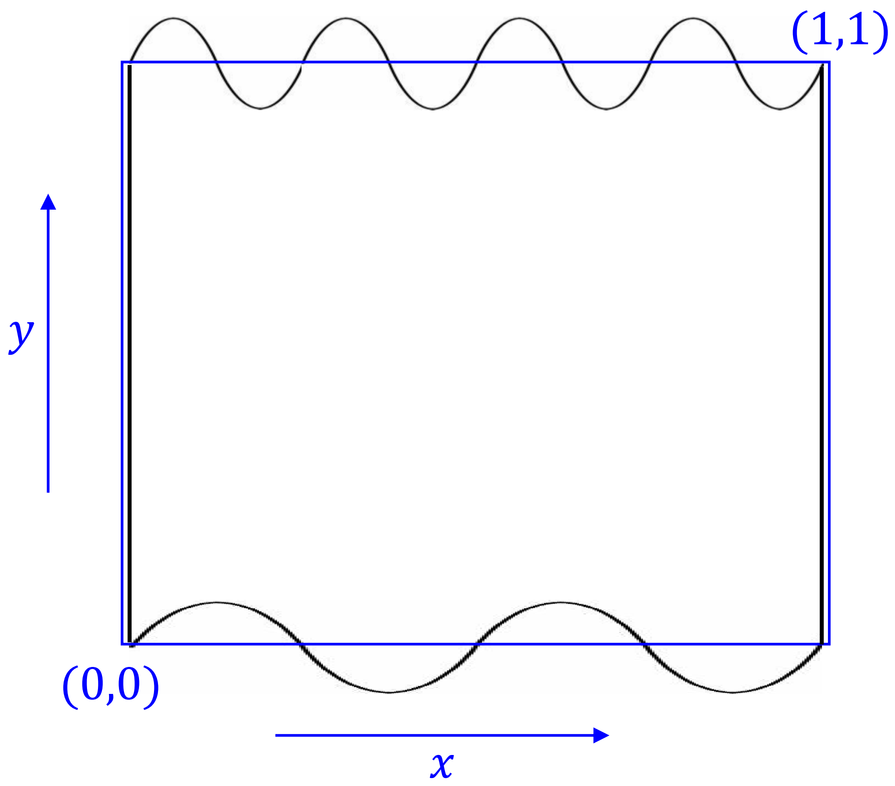
Use the last two relations determine an expression for the full solution.

You may use the follwoing relation without proof:

$$\int_0^1 \sin(n\pi x) \sin(m\pi x) dx = \begin{cases} 1/2 & m = n \\ 0 & m \neq n \end{cases}$$

Example of finding Fourier Coefficients

- Boundary Conditions
  - $\phi(0, y) = \phi(1, y) = 0$
  - $\phi(x, 0) = \sin(4\pi x)$
  - $\phi(x, 1) = \sin(8\pi x)$



Most general solution – Fourier coefficients

$x=0 \quad \phi(0,y)=0$

$$\phi(x,y)=\sum_{n=0}^{\infty}[A_n\cos(\lambda_n x)+B_n\underbrace{\sin(\lambda_n x)}_{=0}]\,[C_n\cosh(\lambda_n y)+D_n\sinh(\lambda_n y)]$$

$$\phi(0,y)=\sum_{n=0}^{\infty}A_n\,[C_n\cosh(\lambda_n y)+D_n\sinh(\lambda_n y)]=0$$

$A_n=0$ FOR ALL n

$x=1 \quad \phi(1,y)=0$

$$\phi(1,y)=\sum_{n=0}^{\infty}B_n\sin(\lambda_n)\,[C_n\cosh(\lambda_n y)+D_n\sinh(\lambda_n y)]=0$$

$\sin(\lambda_n)=0$

$\lambda_n=n\pi \quad n$ INTEGER

$$\phi(x,y)=\sum_{n=0}^{\infty}B_n\sin(n\pi x)\,[C_n\cosh(n\pi y)+D_n\sinh(n\pi y)]$$

$T(x,t)=\sum_{n=1}^{\infty}e^{-\lambda_n^2 Dt}(A_n\sin\lambda_n x+B_n\cos\lambda_n x)$

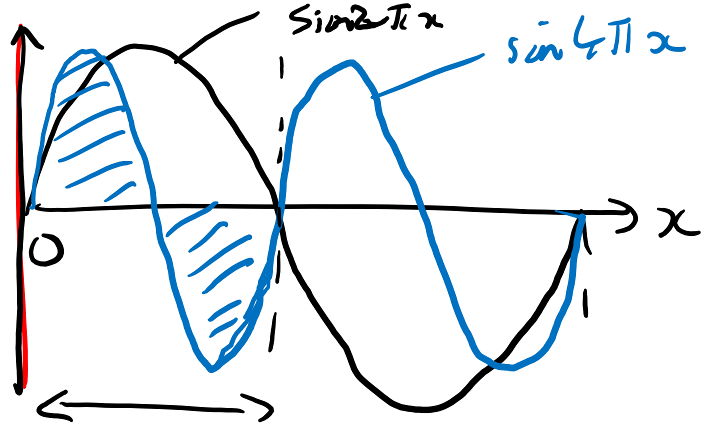
$x=0$

$0=\sum_{n=1}^{\infty}e^{-\lambda_n^2 Dt}(B_n)$

$x=1$

$0=\sum_{n=1}^{\infty}e^{-\lambda_n^2 Dt}(A_n\sin\lambda_n)$

Finding the Fourier coefficients



$$\phi(x,y) = \sum_{n=0}^{\infty} B_n \sin(n\pi x)\,[C_n \underbrace{\cosh(n\pi y)}_{=1,\ y=0} + D_n \underbrace{\sinh(n\pi y)}_{=0 \text{ when } y=0}]$$

Using boundary condition

$y=0$

$$\phi(x,0) = \sin(4\pi x) = \sum_{n=0}^{\infty} B_n C_n \sin(n\pi x)$$

Multiply by $\sin(m\pi x)$ and integrate w.r.t. x from 0 to 1

$$\int_0^1 \sin(m\pi x)\sin(4\pi x) = \sum_n^{\infty} B_n C_n \underbrace{\int_0^1 \sin(m\pi x)\sin(n\pi x)\,dx}_{\substack{=0 \quad m \neq n \\ =\frac{1}{2} \quad m=n}}$$

Let us take, for instance, $m = 2$

$$\underbrace{\int_0^1 \sin(2\pi x)\sin(4\pi x)\,dx}_{=0} = B_1 C_1 \underbrace{\int_0^1 \sin(2\pi x)\sin(\pi x)\,dx}_{=0} + B_2 C_2 \underbrace{\int_0^1 \sin(2\pi x)\sin(2\pi x)\,dx}_{=\frac{1}{2}} + B_3 C_3 \underbrace{\int_0^1 \sin(2\pi x)\sin(3\pi x)\,dx}_{=0} \ldots$$

$$0 = B_2 C_2 \left(\tfrac{1}{2}\right)$$

Therefore for all integers of m, $B_n C_n = 0$ except for $m = 4$ which gives

$$\int_0^1 \sin^2(4\pi x)\,dx = \frac{1}{2} = B_1 C_1 (0) + B_2 C_2 (0) + B_3 C_3 (0) + B_4 C_4 \left(\tfrac{1}{2}\right)$$

$$B_4 C_4 = 1$$

Finding the Fourier coefficients

Using boundary condition $\phi(x,1) = \sin(8\pi x)$

$$\phi(x,1) = \sin(8\pi x) = \underbrace{\sin(4\pi x)\cosh(4\pi)}_{n=4} + \sum_{n=0}^{\infty} B_n\, D_n \sin(n\pi x)\sinh(n\pi)$$

Multiply by $\sin(m\pi x)$ and integrate w.r.t. x from 0 to 1

$$\int_0^1 \sin(m\pi x)\sin(8\pi x)\,dx = \cosh(4\pi)\int_0^1 \sin(m\pi x)\sin(4\pi x)\,dx + \sum_{n=0}^{\infty} B_n D_n \sinh(n\pi)\int_0^1 \sin(m\pi x)\sin(n\pi x)\,dx$$

For any integer value of m except $(m = 4,8)$ yields

For $m = 4$

$$0 = \cosh(4\pi)\left(\tfrac{1}{2}\right) + B_4 D_4 \sinh(4\pi)\left(\tfrac{1}{2}\right)$$

$$B_4 D_4 = -\frac{\cosh(4\pi)}{\sinh(4\pi)}$$

For $m = 8$

$$\frac{1}{2} = \cos(4\pi)(0) + B_8 D_8 \sinh(8\pi)\left(\tfrac{1}{2}\right)$$

$B_n D_n = 0$ FOR ALL OTHER m

UNIQUE SOLUTION

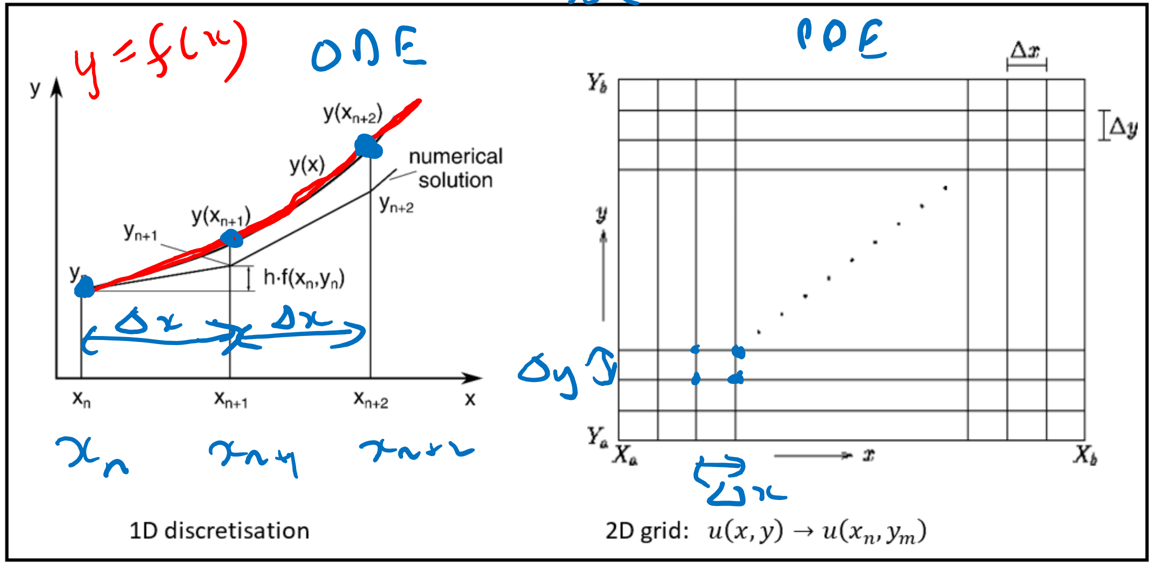
$$\phi(x,y) = \sin(4\pi x)\left[\cosh(4\pi y) - \frac{\cosh(4\pi)}{\sinh(4\pi)}\sinh(4\pi y)\right] + \frac{\sinh(8\pi y)}{\sinh(8\pi)}\sin(8\pi x)$$

5 Numerical Solutions

5.1 Finite Difference Methods
5.2 ODEs
5.3 PDEs

Discretization of Functions

$x_0 \leftrightarrow x_N$



1D discretisation

2D grid: $u(x,y) \rightarrow u(x_n, y_m)$

N — No. of point

$$N = \frac{x_N - x_0}{\Delta x}$$

$$M = \frac{y_M - y_0}{\Delta y}$$

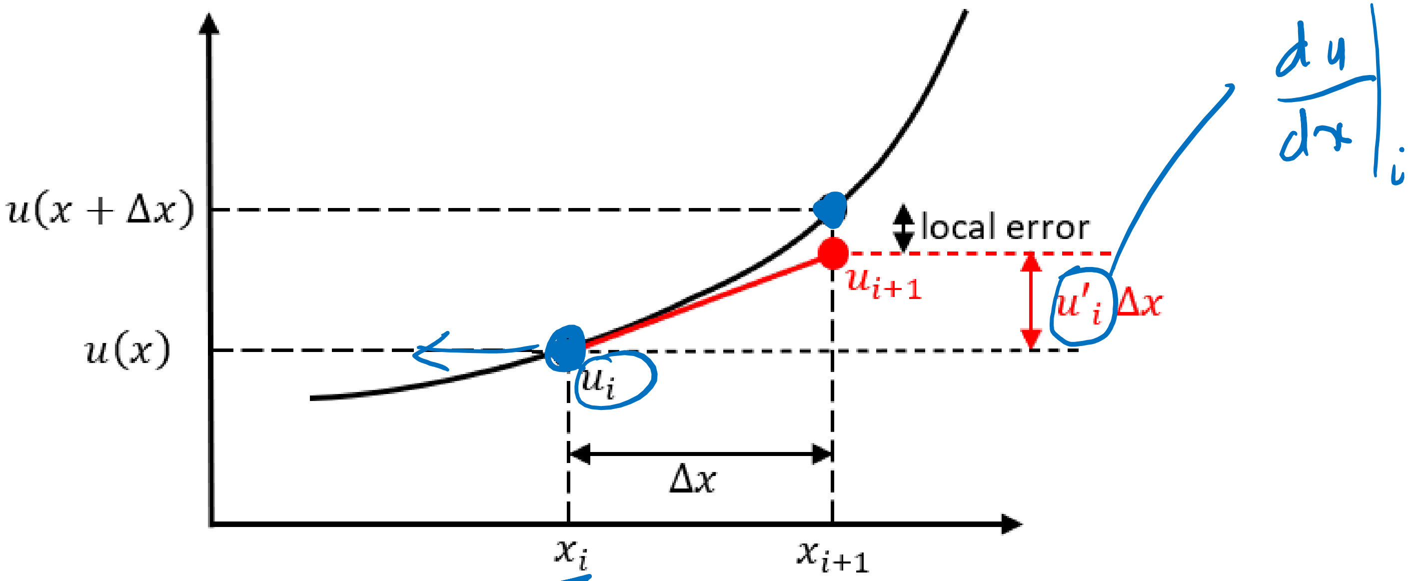
Function evaluated at discrete points – in 2D or more this is a grid of points

- $x = x_0, x_1, x_2, \dots x_n, \dots x_N,$
- $y = y_0, y_1, y_2, \dots y_m, \dots y_M,$

Value of a function at a point evaluated from the ***finite difference*** between surrounding points

ODEs – Euler's method (forward)

EXPLICIT



Function $u(x)$, x_{i+1} obtained by projecting from x_i using gradient at x_i

From Taylor expansion

CONTINUOUS $\quad u(x+\Delta x) = u(x) + u'(x)\Delta x + O(\Delta x^2)$

DISCRETE $\quad \boxed{u_{i+1} \approx u_i + u'_i \Delta x} \rightarrow$ EULER'S METHOD

FORWARD EULER (EXPLICIT)

$O(\Delta x)^2$ represents additional terms which depend on $(\Delta x)^2$ or higher powers – the order of the error in the approximation

Forward Euler Example

$u'(x) = -u(x)$ $\quad u_i' = \frac{du}{dx} = -u \quad x = 0,\ u = 1$

Easy analytical solution $u(x) = \exp(-x)$

$$u_{i+1} = u_i + u_i'\Delta x$$

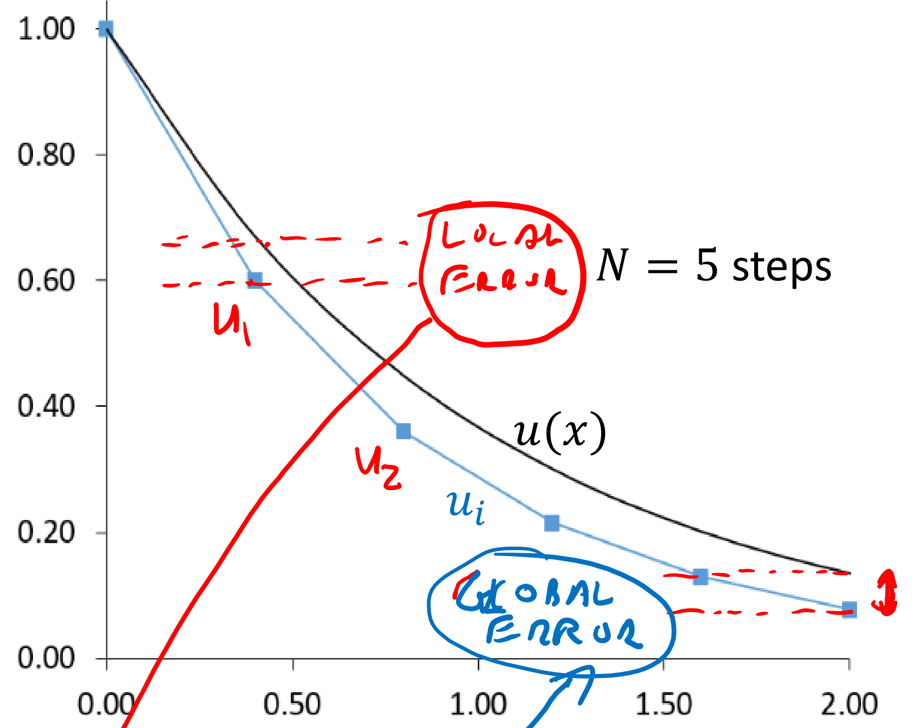
$$u_{i+1} = u_i - u_i\Delta x = u_i(1-\Delta x)$$

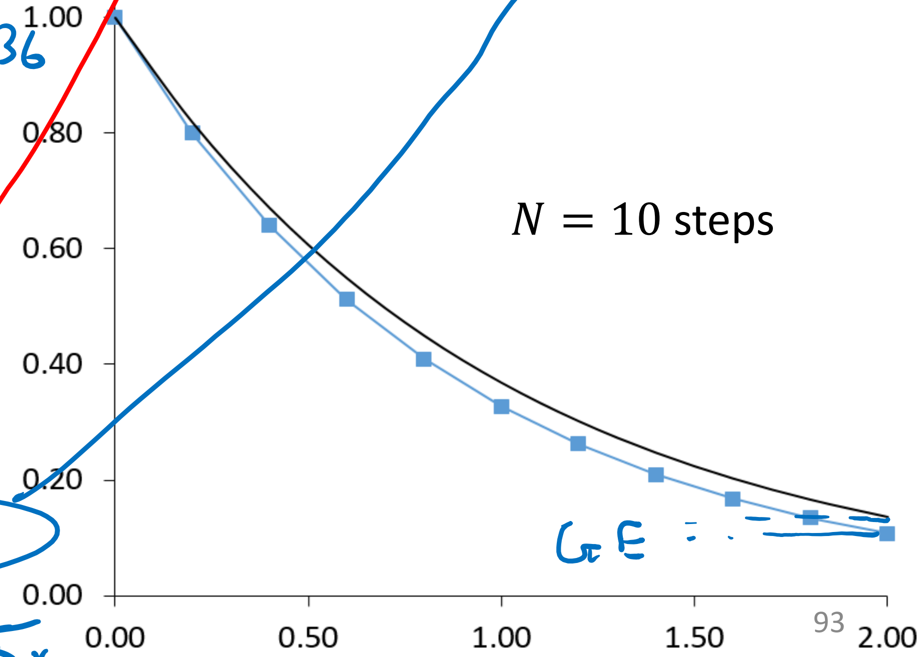
$$u_0 = 1$$

$$u_1 = 1(1-0.4) = 0.6$$

- What is the value of $u(x = 2)$? $\quad u_2 = u_1(1-\Delta x) = 0.6(1-0.4) = 0.36$
- For $N = 5, \Delta x = 0.4$ For $N = 10, \Delta x = 0.2$
- **Error = $\boldsymbol{u(x) - u_i}$**
- **Local Error LE** = Error from a single step $O(\Delta x)^2$
- **Global Error GE** = Error at the finishing point $O(\Delta x)$

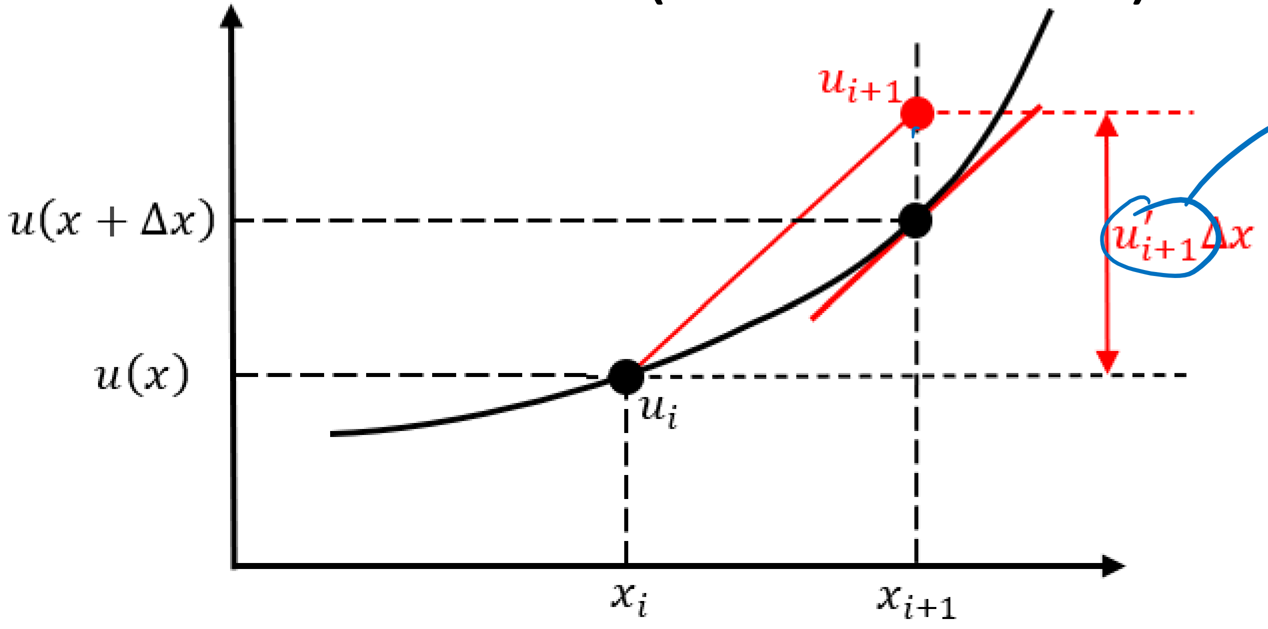
$$GE \approx LE \times N \approx O(\Delta x)^2 \cdot \frac{1}{\Delta x}$$





ODEs – Euler's method (backward)

IMPLICIT



Function $u(x)$, x_{i+1} obtained by projecting from x_i using gradient at x_{i+1}

From Taylor expansion

$$u_{i+1} \approx u_i + u'_{i+1}\Delta x$$

BACKWARD METHOD (IMPLICIT)

$$\frac{dy}{dx} = -y$$

$$u'_i = -u_i$$

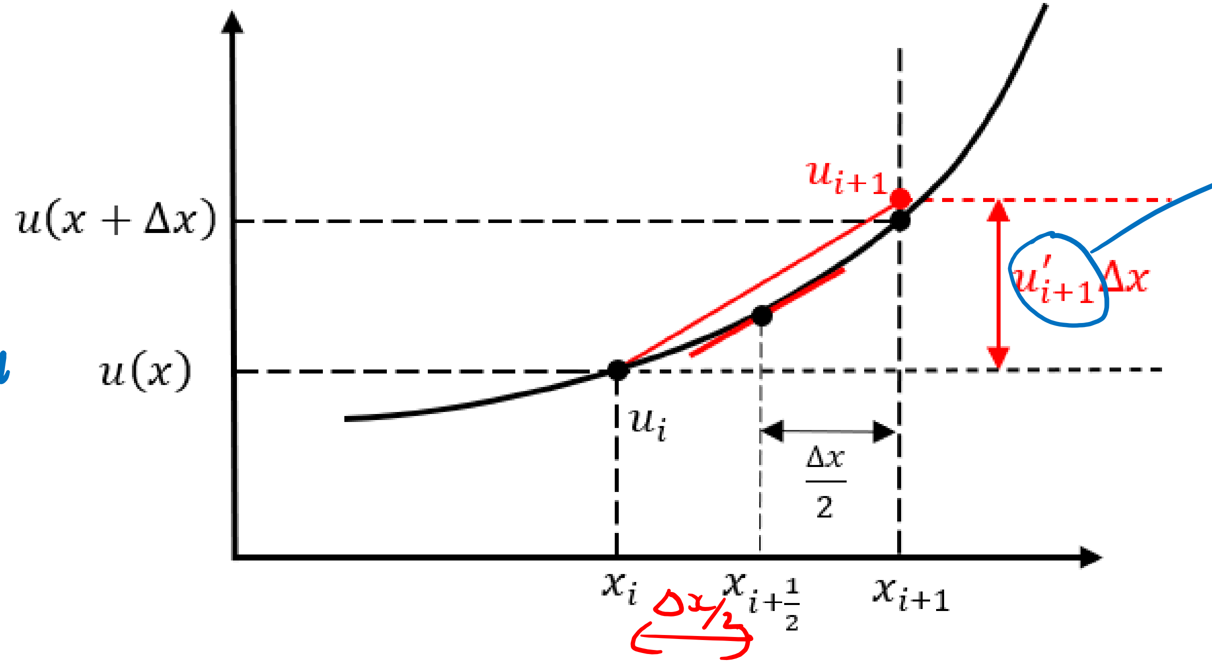
$N = 5 \quad \Delta x = 0.4$

$$u_1 = u_0 + u'_1 \Delta x$$

$$u_1 = u_0 - u_1 \Delta x$$

$$u_1 = \frac{u_0}{1+\Delta x} = \frac{1}{1+0.4}$$

ODEs – Midpoint method



$\frac{du}{dt} = u' = -u$

$u'_{i+\frac{1}{2}}$

GRADIENT AT MIDPOINT

Function $u(x)$, x_{i+1} obtained by projecting from x_i using gradient at $x_{i+\frac{1}{2}}$

$u_{i+1} = u_i + \underbrace{u'_{i+\frac{1}{2}}}\Delta x$

$u' = -u$

$u_{i+1} = u_i - u_{i+\frac{1}{2}}\Delta x$

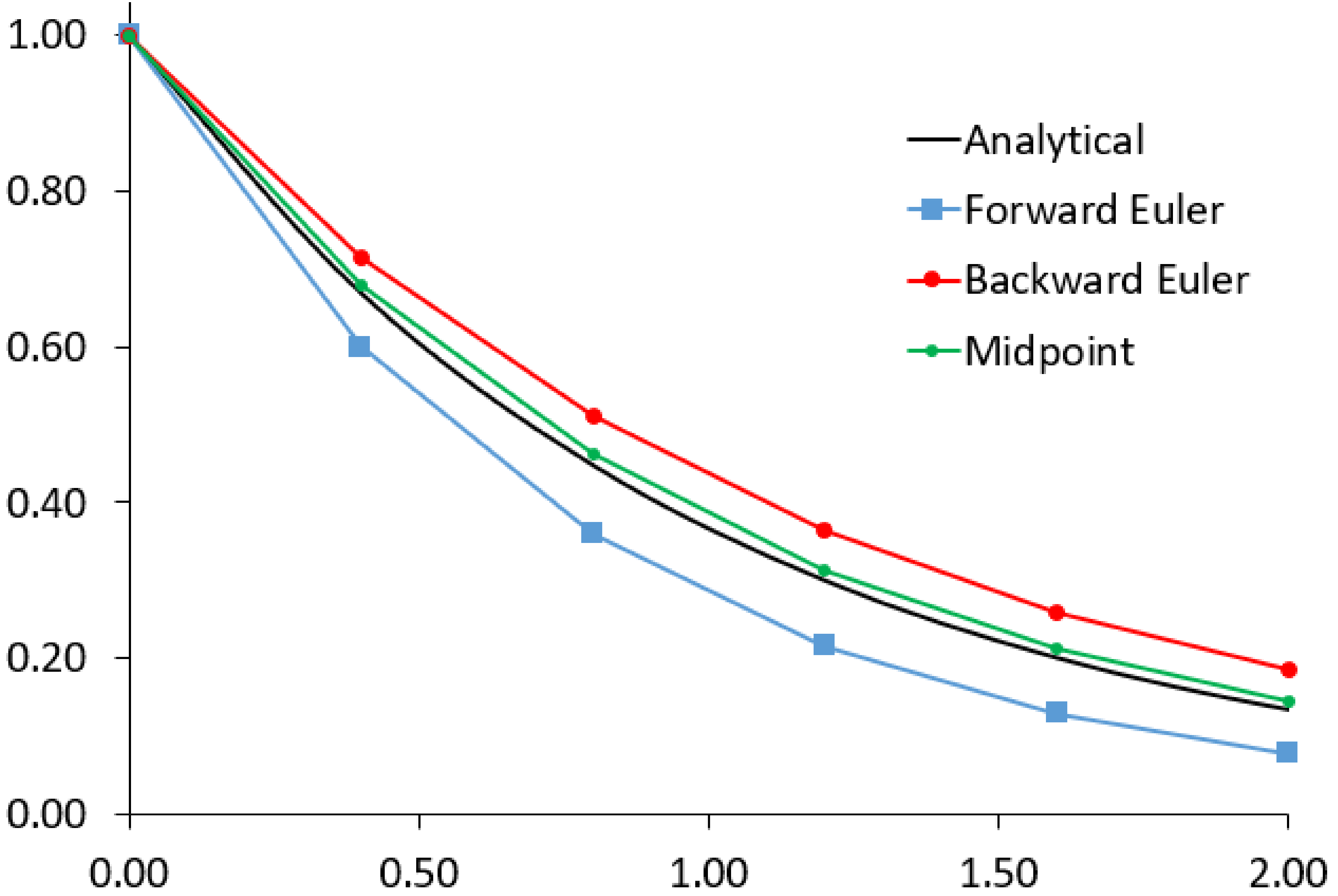
- But how do you find $u'_{i+1/2}$? Use forward Euler to find $u_{i+1/2}$.

$u_{i+\frac{1}{2}} = u_i + u'_i\frac{\Delta x}{2}$

$u_{i+1} = u_i - \left(u_i - u_i\frac{\Delta x}{2}\right)\Delta x$

$u_{i+1} = u_i\left(1 - \Delta x + \frac{(\Delta x)^2}{2}\right)$

ODEs – Forward Euler vs Backward Euler vs Midpoint



Runge- Kutta Methods

- Combines these different approximations

1. Forward Euler (increment $k_1 = u'\Delta x$)
2. Midpoint (k_2) – $u_{i+\frac{1}{2}}$ obtained using 1
3. Midpoint (k_3) – $u_{i+\frac{1}{2}}$ obtained using 2
4. Backward Euler (k_4) – u_{i+1} obtained using 3

- Overall step determined by weighted average of increments

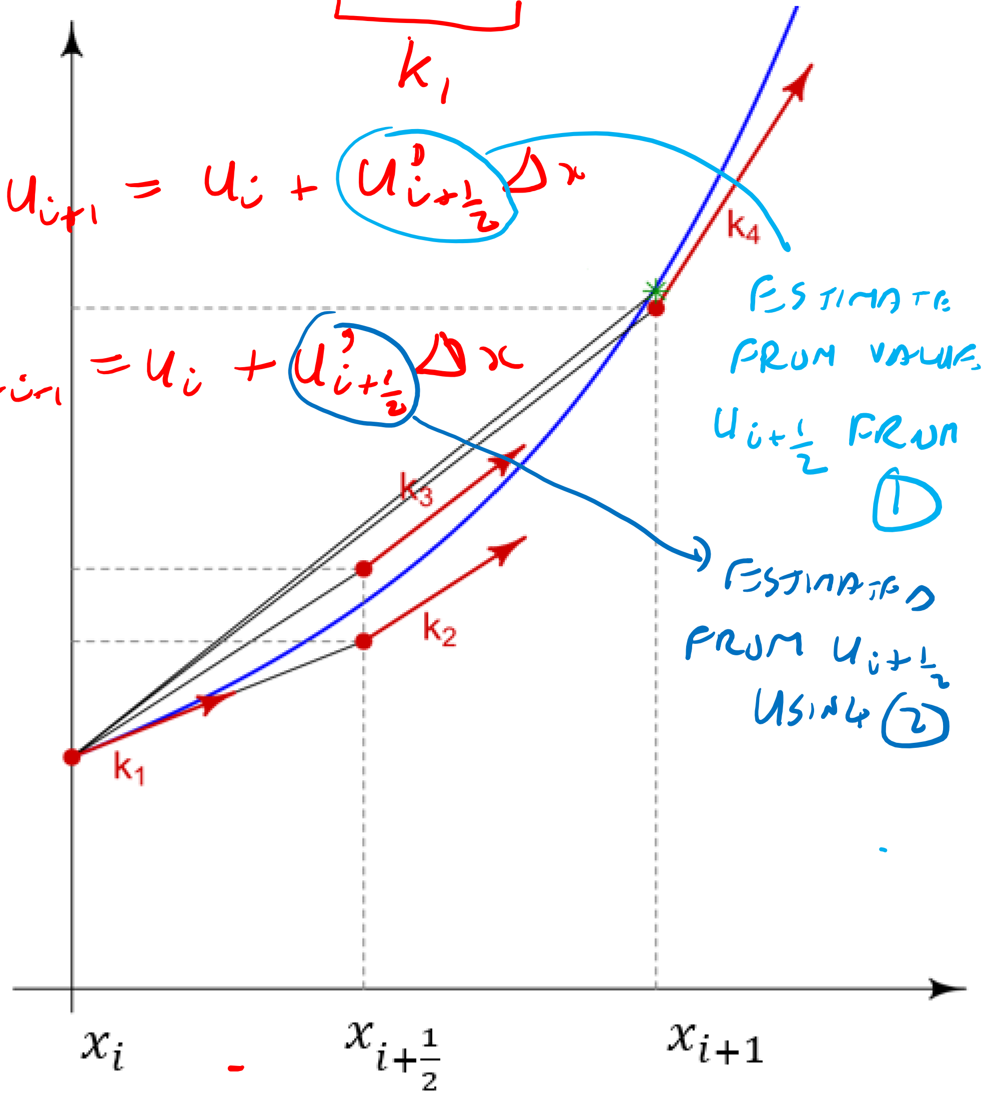
- $u_{i+1} \approx u_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

- Midpoint steps weighted double that of Euler methods

① $u_{i+1} = u_i + \underbrace{u_i' \Delta x}_{k_1}$

② $u_{i+1} = u_i + u'_{i+\frac{1}{2}} \Delta x$

③ $u_{i+1} = u_i + u'_{i+\frac{1}{2}} \Delta x$



Example – Aug 2020 paper

C.2 The SpaceX Falcon 9 rocket had a total mass of M_0= 550,000 kg just before launch. During the 1st stage, it expels burnt fuel at a speed of $u = 2770$ m/s and loses mass at a rate of $L = 2,500$ kg/s. At the end of the 1st stage the rockets mass reduces to $M = 150,000$ kg. The evolution of the velocity v and mass m of the rocket are governed by the following differential equation

$$\frac{dv}{dm} = \frac{g}{L} - \frac{u}{m}$$

(a) Solve this differential equation to show that the velocity of the rocket at the end of its 1st stage is given by

$$v_f = u \ln \frac{M_0}{M} - \frac{g}{L}(M_0 - M)$$

Calculate this velocity assuming g =9.81 ms^{-2} [6]

1st order ODE. Dependent variable v. Independent variable m. Solve analytically by separating variables and putting in the initial conditions – at launch $m = M_0$ and $v = u$ (see notes for solution)

(b) Using the Euler method, taking five steps with a step size of $\Delta m = -80,000$ kg, complete the table below and determine a numerical solution to the equation in part (a).

i	m (kg)	v_i (m s^{-1})	$\left(\frac{dv}{dm}\right)_i$ (m s^{-1} kg^{-1})
0	550,000	0	-0.00111
1	470,000		
2	390,000		
3	310,000		
4	230,000		
5	150,000		

$$V_{i+1} = V_i + \left(\frac{dv}{dm}\right)_i \Delta m$$

$$= V_i + \left(\frac{g}{L} - \frac{u}{m_i}\right)\Delta m$$

$$V_1 = V_0 + \underline{\left(\frac{g}{L} - \frac{u}{m_0}\right)}\Delta m$$

- $\dfrac{dv}{dm} = \dfrac{g}{L} - \dfrac{u}{m}$
- In discrete terms $\left(\dfrac{dv}{dm}\right)_i = \dfrac{g}{L} - \underline{\dfrac{u}{m_i}}$

i	m (kg)	v_i (m/s)	$\left(\frac{dv}{dm}\right)_i$
0	550,000	0 +	-0.00111 $\times \Delta m$
1	<u>470,000</u>	89	...
2	390,000	247	
3	310,000	501	
4	230,000	902	
5	150,000	1551	

(c) What is the global error of this numerical calculation? Roughly how many steps would be required for the Euler method to attain a solution which is within 25 m/s of the analytical solution.

- The analytical solution gives
- $v = -2770 \ln\left(\frac{550}{150}\right) + \frac{9.81}{2500}(150{,}000 - 550{,}000)$
- $v = 3599 - 1570 = 2029\ m/s$

$V_5 = 1551$

GLOBAL ERROR $= 478\ m/s \longrightarrow N = 5$

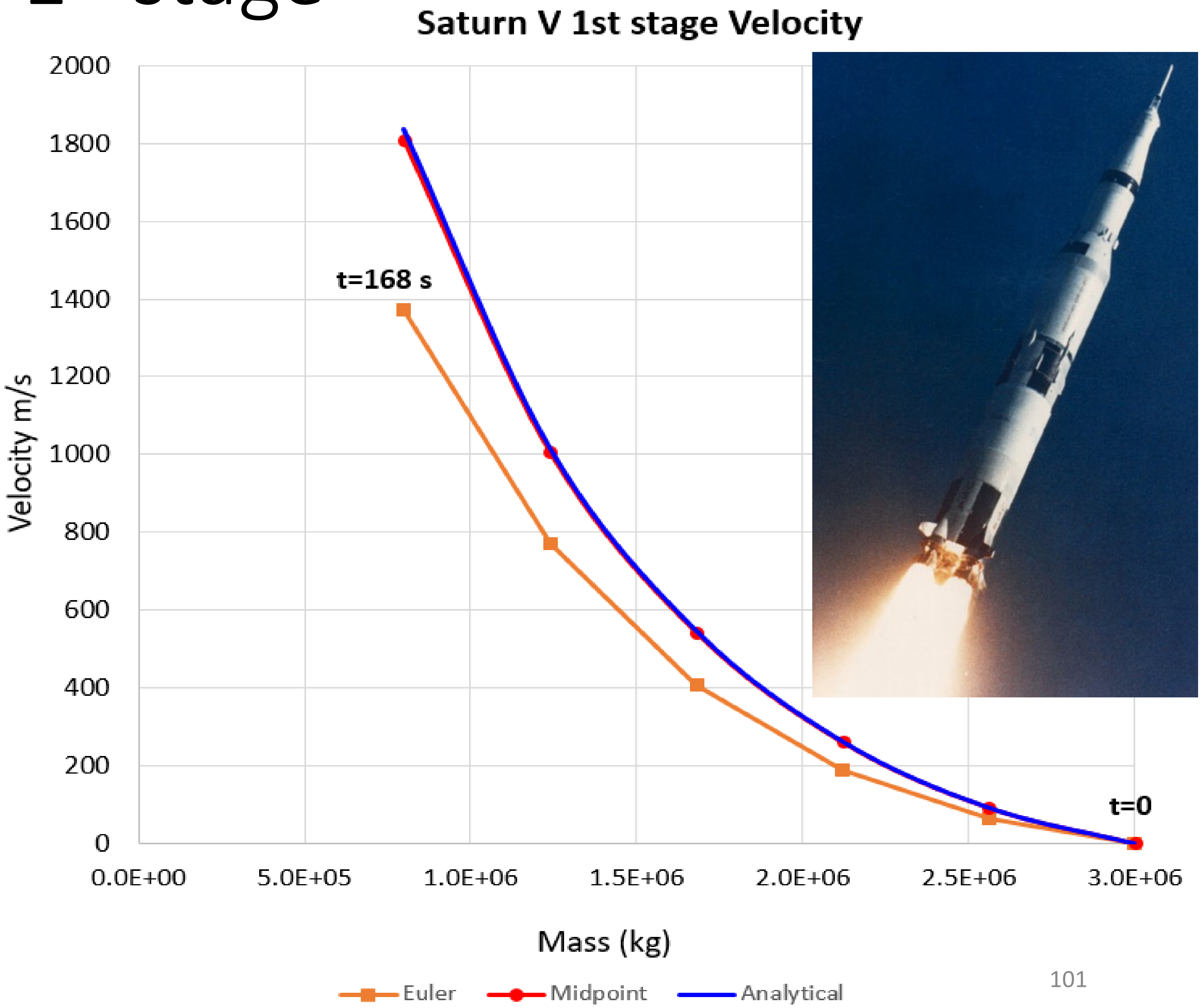
GE $25\ m/s \sim \approx N \approx 100$

Saturn V Rocket – 1st stage

The Saturn V rocket which took Apollo spacecraft to the Moon had a total mass of $M_0 = 3{,}000{,}000$ kg just before launch and during its 1st stage burned fuel at a rate of $L = 13{,}500$ kg/s so that gas was expelled out the engine nozzles at a speed of $u = 2600$ m/s. At the end of the 1st stage, the rocket's mass had reduced to $M = 800{,}000$ kg.

See this worked solution in the notes.

Also see car accelerating on a motorway. Example of non-linear equation.



Assignment 3

2. A straight aluminium bar with uniform cross section and length 1 m is heated uniformly to a temperature of 100°C. At $t = 0$ the ends at $x = 0,1$ are placed in good thermal contact with a heat reservoir at a temperature of 0°C. If the bar is insulated along its sides, then heat is only conducted along the bar (x direction) so that it is described by the 1D Heat Equation

$$\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2}$$

$$T(x,t) = \sum_{n=1}^{\infty} \exp\left(-\lambda_n^2 Dt\right)(A_n \underbrace{\sin(\lambda_n x)}_{=0} + \underbrace{B_n \cos(\lambda_n x)}_{=1})$$

[20]

(c) Using the boundary conditions as described at the start of the question, obtain expressions for B_n and λ_n

[20]

BOUNDARY CONDITIONS

$T(0,t) = 0$ $\quad$ $T(1,t) = 0$

$$0 = \sum_{n=1}^{\infty} B_n \, e^{-\lambda_n^2 Dt}$$

$$0 = B_1 e^{-\lambda_1^2 Dt} + B_2 e^{-\lambda_2^2 t} \ldots$$

$\underline{B_n = 0}$ for all n

$$0 = \sum_{n=1}^{\infty} e^{-\lambda_n^2 Dt} A_n \underbrace{\sin\lambda_n}$$

$\sin\lambda_n = 0$

$\lambda_n = n\pi$

n INTEGER

(d) Using the initial temperature distribution along the bar, determine values for the coefficients A_n for $n = 1 - 4$ and hence write down an approximate solution for $T(x,t)$ for this problem and determine the temperature at the centre of the bar 10 minutes later.

($D = 10^{-4}$ m^{2}s^{-1} for aluminium)

$\int_0^1 \sin(m\pi x)\sin(n\pi x) = \frac{1}{2}$ for $m = n$

$\int_0^1 \sin(m\pi x)\sin(n\pi x) = 0$ for $m \neq n$

$$T(x,t) = \sum_{n=1}^{\infty} e^{-(n\pi)^2 D t} A_n \sin(\lambda_n x)$$

INITIAL CONDITION $T(x,0) = 100$

$$100 = \sum_{n=1}^{\infty} A_n \sin(n\pi x)$$

$$100\int_0^1 \sin(m\pi x)\,dx = \sum_{n=1}^{\infty} A_n \underbrace{\int_0^1 \sin(m\pi x)\sin(n\pi x)\,dx}_{=0 \text{ UNLESS } n=m}$$

$$100\left[-\frac{1}{m\pi}\cos(m\pi x)\right]_0^1 = \frac{1}{2}A_m$$

$$100\left(-\frac{1}{m\pi}\cos(m\pi) + \frac{1}{m\pi}\cos(0)\right) = \frac{A_m}{2}$$

$$\frac{200}{m\pi}\left(1 - (-1)^m\right) = A_m \qquad m = 1,2,3,4$$

$m=1 \quad A_1 = \frac{400}{\pi}$

$m=2 \quad A_2 = 0$

$m=3 \quad A_3 = \frac{400}{3\pi}$

$m=4 \quad A_4 = 0$

$$T(0.5, 600) = 70^\circ C$$

Extra Question

$$\frac{\partial u}{\partial t} + (x+u)\frac{\partial u}{\partial x} = 0$$

subject to the initial condition $u(x,0) = x$

(i) Obtain an expression for u in terms of x_0, where $x = x_0$ when $t = 0$

(ii) Substitute your answer from (i) into an expression for $\frac{dx}{dt}$ and hence obtain u in terms of x, t

$$\frac{du}{dt} = 0$$

$$u(x_0, t) = C(x_0)$$

$$u(x,t) = x_0$$

$$\frac{dx}{dt} = x + u$$

$$= x + x_0$$

$$\int \frac{dx}{x+x_0} = \int dt$$

$$\ln(x+x_0) = t + D$$

$$t = 0 \quad x = x_0$$

$$D = \ln(2x_0)$$

$$\ln(x+x_0) - \ln(2x_0) = t$$

$$\ln\left(\frac{x+x_0}{2x_0}\right) = t$$

$$\frac{x+x_0}{2x_0} = e^t$$

$$x + x_0 = 2x_0 e^t$$

$$x = x_0(2e^t - 1)$$

$$u = x_0$$

$$u(x,t) = \frac{x}{2e^t - 1}$$

Numerical solutions to ODEs

FORWARD EULER (EXPLICIT)

$$u_{i+1} \approx u_i + \underbrace{u_i' \Delta x}$$

EXPLICITLY KNOWN
x_i, u_i

BACKWARD EULER (IMPLICIT)

$$u_{i+1} \approx u_i + \underbrace{u_{i+1}' \Delta x}$$

IMPLICITLY KNOWN
VALUE OF u_{i+1} NOT DIRECTLY
AVAILABLE

MIDPOINT METHOD

$$u_{i+1} \approx u_i + \underbrace{u_{i+\frac{1}{2}}' \Delta x}$$

FIND FROM SIMPLE FORWARD EULER

$$u_{i+\frac{1}{2}} = u_i + u_i' \frac{\Delta x}{2}$$

RUNGE-KUTTA

FORWARD EULER $K_1 = u_i' \Delta x$

MIDPOINT $K_2, K_3 = u_{i+\frac{1}{2}}' \Delta x$

BACKWARD EULER $K_4 = u_{i+1}' \Delta x$

$$u_{i+1} = u_i + \frac{1}{6}\left(K_1 + K_4 + 2(K_2 + K_3)\right)$$

Kahoot 7

2)

$$u_{i+1} = u_i + u_i' \Delta x + \underbrace{u_i'' \frac{(\Delta x)^2}{2!} + u_i''' \frac{(\Delta x)^3}{3!} \ldots\ldots}_{O((\Delta x)^2)}$$

$$LE \times N = GE$$

$$O(\Delta x^2) \times \frac{1}{\Delta x} = O(\Delta x)$$

4)

$$\Delta x = 0.5$$

$$u_0 = 1 \quad x_0 = 0 \qquad u_0' = u_0$$

$$u_1 = u_0 + u_0' \Delta x$$

$$= u_0 (1 + \Delta x) = 1.5$$

$$e^{0.5} = \underline{1.65}$$

5)

$$u_1 = u_0 + \overline{u_1' \Delta x} \; u_1$$

$$u_1 = \frac{u_0}{1 - \Delta x} = 2$$

Kahoot 7

$$u_1 = u_0 + u_{\frac{1}{2}} \Delta x$$

$$u_{\frac{1}{2}} = u_0 + u_0 \frac{\Delta x}{2}$$

$$u_1 = 1 + (1.25)(0.5)$$

$$= 1.625$$

$$= 1 + (1) \frac{0.5}{2}$$

$$= 1.25$$

Assignment 4

1. An energetic proton in the solar wind is moves in a region of space where the Earth's magnetic field has a magnitude of $B = 1.044 \times 10^{-5}$ T in the positive z direction. The proton has a velocity of $\dot{x} = 10^6$ m/s in the positive x direction ($\dot{y} = \dot{z} = 0$) as it passes through the origin at $t = 0$. The Lorentz force on the proton results in the following equations of motion

$$(1)\ m\frac{d\dot{x}}{dt} = eB\dot{y} \qquad (2)\ m\frac{d\dot{y}}{dt} = -eB\dot{x} \qquad (3)\ m\frac{d\dot{z}}{dt} = 0$$

While these are ODEs (the dependent variables x and y are each a function of t only) equations (1) and (2) are coupled since x, y are present in both.

(a)(i) By differentiating equation (1) and doing a substitution, solve these equations analytically using the initial conditions to obtain expressions for the components of the velocity $\dot{x}, \dot{y}, \dot{z}$ as a function of ω, t where $\omega = \frac{eB}{m}$. **[25]**

(ii) Using initial conditions again, obtain expressions for x, y, z as a function of t. Using Excel (or other software) plot out the motion of the proton in the x, y plane (make the scales on each axis the same length and plot points every 10^{-4} s until $t = 6 \times 10^{-3}$ s). **[20]**

$\underline{F} = q\underline{v} \times \underline{B}$

DEPENDENT VARIABLES

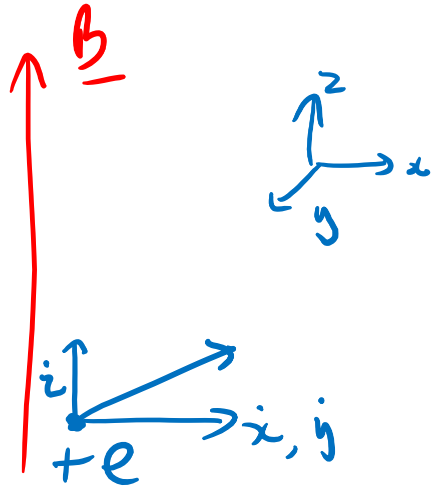
$\dot{x}, \dot{y}$ $\qquad \dot{x} = \frac{dx}{dt}$

$V_x \quad V_y$

COUPLED ODEs

$$m\frac{d^2\dot{x}}{dt^2} = eB\frac{d\dot{y}}{dt}$$

ORDER 2 ODE



(b)(i) Using the Euler method write down finite difference versions of equations (1) and (2) in terms of $\dot{x}_{i+1}$, $\dot{x}_i$, $\dot{y}_{i+1}, \dot{y}_i, \omega$ and Δt. Similarly write down expressions for x_{i+1} and y_{i+1} in terms of $x_i, y_i, \dot{x}_i, \dot{y}_i$ and Δt. **[15]**

(ii) Using Excel (or other) complete the following table using your finite difference equations with $\Delta t = 2\pi/10\omega$.

i	t_i	$\dot{x}_i$	$\dot{y}_i$	x_i	y_i
0	0	10^6	0	0	0
1	Δt	...	...	...	...
2	$2\Delta t$	...	...	...	...
3	$3\Delta t$	...	...	...	...
⋮	⋮	⋮	⋮	⋮	⋮
10	$10\Delta t$	...	...	...	...

[20]

$$\dot{x}_{i+1} = \dot{x}_i + \underbrace{\frac{d\dot{x}}{dt}\Big|_i}\Delta t$$

SIMPLE EULER

$$m\frac{d\dot{x}}{dt} = eB\dot{y}$$

$$m\frac{(\dot{x}_{i+1} - \dot{x}_i)}{\Delta t} = eB\dot{y}_i$$

TAYLOR SERIES

$$\dot{x}_{i+1} = \dot{x}_i + \frac{d\dot{x}}{dt}\Big|_i \Delta t \; \ldots$$

$$\boxed{\dot{x}_{i+1} = \dot{x}_i + \omega\dot{y}_i\Delta t}$$

$$\omega = \frac{eB}{m}$$

$$\dot{x}_1 = 10^6 + \omega(0)\Delta t$$
$$= 10^6$$

i	t_i	$\dot{x}_i$	$\dot{y}_i$	x_i	y_i
0	0	10^6	0	0	0
1	Δt	10^6		628	
2	$2\Delta t$				
3	$3\Delta t$				
⋮					

$$\dot{x}_i = \frac{dx_i}{dt} = \frac{x_{i+1} - x_i}{\Delta t}$$

$$\boxed{x_{i+1} = x_i + \dot{x}_i\Delta t}$$