QUEEN'S UNIVERSITY BELFAST

PHY2004

Exam Time Table Code PHY2004

Any calculator, except one with pre-programmable memory, may be used in this examination.

Answer Books A, B and C

LEVEL 2

EXAMINATION CONTRIBUTING TO THE DEGREES OF BACHELOR OF SCIENCE (BSc) AND MASTER IN SCIENCE (MSci)

PHY2004
Electricity, Magnetism and Optics

Duration: 3 Hours

Wednesday, 2nd May 2018 2:30 PM - 5:30 PM

Examiners: Prof. P. Browning
Dr. P. van der Burgt
and the Internal Examiners

Answer ALL TEN questions in Section A
Answer TWO questions in Section B
Answer ONE question in Section C

Section A questions are worth 4 marks each

Section B and C questions are worth 20 marks each

Use a separate answer book for each Section.
Follow the instructions on the front of the answer book. Enter your Anonymous Code number ans Seat number, but NOT your name.

THE QUEEN'S UNIVERSITY OF BELFAST
DEPARTMENT OF PHYSICS AND ASTRONOMY

PHYSICAL CONSTANTS

Speed of light in a vacuum	$c = 3.00 \times 10^{8}\ \mathrm{ms^{-1}}$
Permeability of a vacuum	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{Hm^{-1}}$ $\approx 1.26 \times 10^{-6}\ \mathrm{Hm^{-1}}$
Permittivity of a vacuum	$\varepsilon_0 = 8.85 \times 10^{-12}\ \mathrm{Fm^{-1}}$
Elementary charge	$e = 1.60 \times 10^{-19}\ \mathrm{C}$
Electron charge	$= -1.60 \times 10^{-19}\ \mathrm{C}$
Planck Constant	$h = 6.63 \times 10^{-34}\ \mathrm{Js}$
Reduced Planck Constant	$\hbar = 1.05 \times 10^{-34}\ \mathrm{Js}$
Rydberg Constant for hydrogen	$R_\infty = 1.097 \times 10^{7}\ \mathrm{m^{-1}}$
Unified atomic mass unit	$1u = 1.66 \times 10^{-27}\ \mathrm{kg}$ $1u = 931\ \mathrm{MeV}$
1 electron volt (eV)	$= 1.60 \times 10^{-19}\ \mathrm{J}$
Mass of electron	$m_e = 9.11 \times 10^{-31}\ \mathrm{kg}$
Mass of proton	$m_p = 1.67 \times 10^{-27}\ \mathrm{kg}$
Mass of neutron	$m_n = 1.67 \times 10^{-27}\ \mathrm{kg}$
Molar gas constant	$R = 8.31\ \mathrm{JK^{-1}mol^{-1}}$
Boltzmann constant	$k = 1.38 \times 10^{-23}\ \mathrm{JK^{-1}}$
Avogadro constant	$N_A = 6.02 \times 10^{23}\ \mathrm{mol^{-1}}$
Gravitational constant	$G = 6.67 \times 10^{-11}\ \mathrm{Nm^{2}kg^{-2}}$
Acceleration of free fall on the Earth's surface	$g = 9.81\ \mathrm{ms^{-2}}$

MATHEMATICAL IDENTITIES

In answering the questions on this paper you may make use of any of the following:

Divergence theorem $\int_V \nabla \cdot \underline{E}\, dV = \oint_S \underline{E} \cdot d\underline{S}$

Stoke's Theorem $\int_S \nabla \times \underline{E} \cdot d\underline{S} = \oint \underline{E} \cdot d\underline{\ell}$

Identities

$$\nabla \times \nabla\left(Scalar\right) = \underline{0}$$

$$\nabla \cdot \nabla \times \left(Vector\right) = 0$$

$$\nabla \times \left(\nabla \times \underline{E}\right) = \nabla\left(\nabla \cdot \underline{E}\right) - \nabla^2 \underline{E}$$

$$\nabla\left(\psi\, \underline{E}\right) = \psi\, \nabla \cdot \underline{E} + \underline{E} \cdot \nabla \psi$$

$$\nabla \cdot \left(\underline{E} \times \underline{H}\right) = \underline{H} \cdot \left(\nabla \times \underline{E}\right) - \underline{E} \cdot \left(\nabla \times \underline{H}\right)$$

Material Equations $\underline{J} = \sigma \underline{E} \quad \underline{B} = \mu \underline{H} \quad \underline{D} = \varepsilon \underline{E}$

Poynting Vector $\underline{S} = \underline{E} \times \underline{H}$

Trigonometric identities

$$cos(\,A \pm B\,) = cos\,A\,cos\,B \mp sin\,A\,sin\,B$$

$$sin(\,A \pm B\,) = sin\,A\,cos\,B \pm sin\,B\,cos\,A$$

$$cos\,A + cos\,B = 2\,cos\left(\frac{A+B}{2}\right)cos\left(\frac{A-B}{2}\right)$$

$$sin\,A - sin\,B = 2\,cos\left(\frac{A+B}{2}\right)sin\left(\frac{A-B}{2}\right)$$

SECTION A

Use a section A answer book

Answer ALL 10 questions in this section

Full explanations of your answers are required to attain full marks

1. Discuss under what approximations the electrostatic regime is valid and show that these lead to the electric field being conservative.

[4]

2. Demonstrate, using Gauss' law, that the electric field at a distance *r* from a point-like charge *q* is given by: $\vec{E} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r}$ **[4]**

3. It is known that the divergence of the magnetic field is always equal to zero. Discuss how this property implies the absence of magnetic monopoles and show how a vector potential $\underline{A}$ can be obtained from it. **[4]**

4. Show that the magnetic field generated by an infinitely long solenoid with a density of loops *n* and carrying a current *I* is axial and has a magnitude $B = \mu_0 n I$ **[4]**

5. By considering the electric and magnetic energy densities in a volume V bounded by a surface S, the rate of change of the total electromagnetic energy W is described by the Poynting's Theorem

$$\int_S \underline{E} \times \underline{H} \cdot d\underline{S} + \int_V \underline{E} \cdot \underline{J}\, dV = -\frac{\partial W}{\partial t} \qquad \text{Eq.1}$$

Explain, in detail, the physical meaning of each term in Eq. 1. Which term is related to the Poynting vector? What are the SI units of this vector? **[4]**

6. Discuss how the presence of a temporally-varying magnetic field makes the electric field non-conservative. **[4]**

[EXAMINATION CONTINUED OVERLEAF]

7. Starting from the most general form of Maxwell's equations, derive the electromagnetic wave equation in absence of sources,

$$\nabla^2 \underline{E} = \mu\varepsilon \frac{\partial^2 \underline{E}}{\partial t^2} \qquad \text{Eq.2}$$

pointing out all the assumptions you make during your derivation. **[4]**

8. Describe the concept of impedance of vacuum and show how this can be derived from Maxwell's equations. **[4]**

9. Show how the relative permittivity of a material relates to its refractive index and discuss the physical meaning of the imaginary part of the relative permittivity. **[4]**

10. A beam of horizontally polarized light impinges onto two consecutive polarisers, the first with its transmission axis tilted by 30 degrees compared to the horizontal and the second with its transmission axis parallel to the horizontal. What is the percentage of light intensity escaping? Does this percentage change if the two polarisers are swapped? **[4]**

SECTION B

Use a Section B answer book

Answer TWO questions from this section

11. (a) A positive point charge Q_1 and a negative point charge Q_2 are separated by a distance h as shown in Fig. 1.

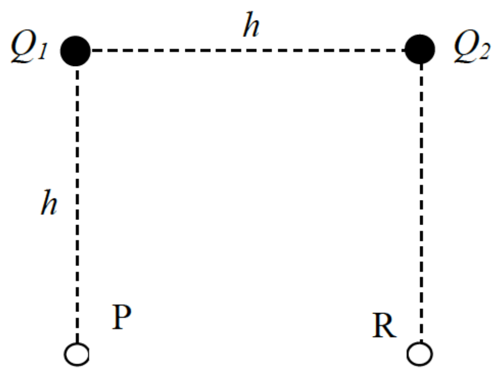

Fig. 1

(i) Calculate the magnitude of the electric field at point P and show its direction. **[4]**

(ii) Calculate the electrostatic potential at the point R. **[2]**

(b) Two long concentric conducting cylinders of radii a and b ($a<b$) carry equal and opposite charge per unit length λ (positive on the inner conductor). The space between the two conductors is filled with a dielectric of relative permittivity ε_r (Fig. 2).

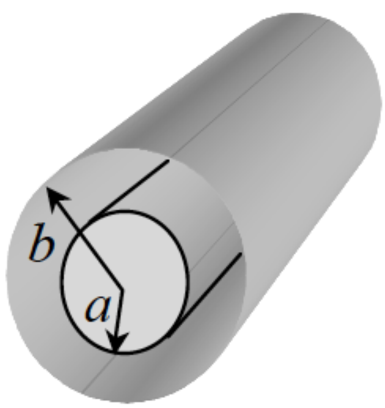

Fig.2

(i) Sketch the distribution of charges on the conductors and the dielectric. **[2]**

[QUESTION 11 CONTINUED OVERLEAF]

(ii) Calculate the magnitude of the electric field ***E***, polarisation ***P***, and electric displacement field ***D*** outside of the outer conductor ($r>b$). **[2]**

(iii) Calculate the magnitude of the electric field ***E***, polarisation ***P***, and electric displacement field ***D*** in the dielectric ($b>r>a$). **[3]**

(iv) Calculate the magnitude of the electric field ***E***, polarisation ***P***, and electric displacement field ***D*** inside the inner conductor ($r<a$). **[2]**

(v) Calculate the potential difference between the two conductors. **[3]**

(vi) Calculate the capacitance per unit length between the two conductors. **[2]**

[EXAMINATION CONTINUED OVERLEAF]

SECTION B

12 Consider a disc of conducting material of radius R, with a total charge Q deposited on it. The disc rotates around its axis with angular frequency ω, as in Fig. 3.

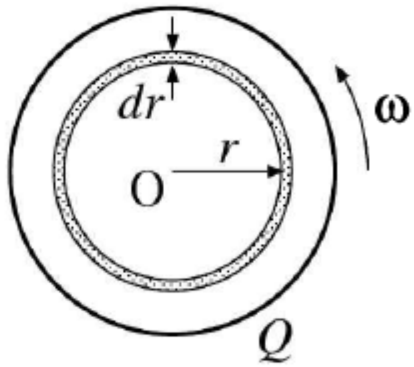

Fig. 3

(i) Calculate the areal charge density on the disc. **[1]**

(ii) Demonstrate that the infinitesimal area of a ring of width *dr* at a distance *r* from the centre is given by $dA=2\pi r\,dr$. **[2]**

(iii) Show that the charge contained in the thin ring is: $dq = Q\,2r\,dr/R^2$. **[2]**

(iv) Show that the infinitesimal current generated by each of these thin circular rings once in motion with an angular frequency ω is $dI = Q\omega r/(\pi R^2)\,dr$ **[4]**

(v) Using Biot-Savart law ($d\vec{B} = \frac{\mu_0}{4\pi}\frac{I d\vec{\ell}\times\vec{r}}{r^3}$), show that the amplitude of the magnetic field generated by a ring is, in its center, given by:

$$B = \frac{\mu_0 I}{2r} \quad \text{Eq. 3}$$

[5]

(vi) Show that the magnetic field generated by each of these thin rings at the centre of the disc is: $dB = \mu_0\omega Q/(2\pi R^2)dr$ **[2]**

(vii) Calculate the total magnetic field generated at the centre of the disc. What is the direction of the field? **[4]**

SECTION B

13 Fig. 4 shows a short section of a long cylindrical conductor of resistivity ρ and radius r carrying a steady current I

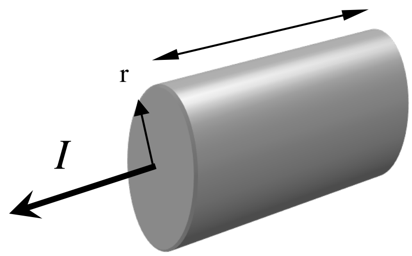

Fig. 4

(i) Show that the magnetic field B at the surface is given by $B = \mu_0 I/(2\pi r$ **[3]**

(ii) Redraw the diagram and show clearly the direction of the $\underline{E}$ and $\underline{H}$ vectors. **[2]**

(iii) Indicate the direction of the Poynting vector at the surface of the cylinder. **[1]**

(iv) Calculate the current density flowing through the material. **[1]**

(v) Show that the flow of electromagnetic power per unit area across the curved surface is given by

$$\frac{\rho I^2}{2\pi^2 r^3} \qquad \text{Eq. 4}$$

[3]

(vi) Using the result in Eq.4 show that the total power radiated by the cylinder is:

$$\frac{\rho I^2 \ell}{\pi r^2} \qquad \text{Eq. 5}$$

[3]

(vii) Show that this result is consistent with Ohm's law **[4]**

(viii) Assume now that the cylinder is immersed in a paramagnetic material with a relative permeability μ_r. Write the new expression for the magnetic fields $\vec{B}$, $\vec{H}$, and $\vec{M}$ **[3]**

SECTION B

14 **(a)** A plane electromagnetic wave of frequency ν = 10 THz is propagating in vacuum along the *z* direction. The wave is linearly polarised along x and the maximum electric field is E_0 = 50 V/m.

(i) Determine the wavelength and wavenumber of the wave. What part of the electromagnetic spectrum does this wave belong to? **[3]**

(ii) Write an expression for the magnitude of the electric and magnetic fields of the wave as a function of time *t* and of *z*. **[3]**

(iii) Show that the expressions derived in question (ii) are solutions of the wave equation for the electric and magnetic fields of an electromagnetic wave propagating in a vacuum with no external sources. **[4]**

(iv) Determine the energy density carried by the wave as a function of time. **[2]**

(v) Determine the average energy density carried by the wave. **[1]**

(b) **(i)** Show that for a plane boundary between two dielectrics the Fresnel amplitude reflection coefficient at normal incidence is given by

$$r = \frac{E_r}{E_i} = \frac{n_1 - n_2}{n_1 + n_2} \quad \text{Eq. 6}$$

where n_1 and n_2 are the refractive indices of the incident and transmitting media, respectively. E_r and E_i are the reflected and incident electric field amplitudes respectively. **[5]**

(ii) For what value of n_2 is the reflection minimised? For what value is it maximised? What do these cases physically correspond to? **[2]**

SECTION C

Use a Section C answer book

Answer <u>ONE</u> question from this section

15 **(a)** Assume two plane electromagnetic waves of frequency ω, wavenumber k with a relative phase difference δ. Both waves are propagating along the z direction and are linearly polarised, one along the x axis and the other along the y axis. The amplitude of each wave is E_{0x} and E_{0y}, respectively.

(i) Write the electric field of both waves as a function of position z and time t. **[2]**

(ii) Demonstrate that the combination of these two waves leads to the following equation governing the polarisation state:

$$\left(\frac{E_x}{E_{0x}}\right)^2 + \left(\frac{E_y}{E_{0y}}\right)^2 - \frac{2E_xE_y}{E_{0x}E_{0y}} cos\,\delta = sin^2\,\delta \qquad \text{Eq. 7}$$

[3]

(iii) What is the polarisation state of the resulting wave for a generic phase difference δ? **[2]**

(iv) For what values of δ and E_{0x}/E_{0y} is the resulting wave circularly polarised or linearly polarised? **[2]**

(b) The table below shows the real (n) and imaginary (n'') refractive indices of a number of materials for certain wavelengths.

Material	Wavelength (nm)	n'	n''
Al_2O_3 (sapphire)	500	1.61	0.00
Silicon	400	5.57	0.38
Aluminium	800	2.80	8.45

(i) Describe the physical meaning of the real and imaginary part of the refractive index. **[2]**

(ii) Which material in the table is transparent? Which one is the most strongly absorbing? **[1]**

[QUESTION 15 CONTINUED OVERLEAF]

(iii) Calculate the Brewster angle for all the materials in the table assuming propagation from air. What is the physical meaning of the Brewster angle? For what polarisation state does this phenomenon hold? **[4]**

(iv) Calculate the amplitude reflectance *r* and the intensity reflectance *R* for each material at normal incidence with air as the incident medium. **[4]**

You may assume: $r = (n_1 - n_2)/(n_1 + n_2)$, where n_1 and n_2 are the refractive indices of the incident and exit medium, respectively.

[EXAMINATION CONTINUED OVERLEAF]

SECTION C

16 (a) In the classical theory of optical dispersion, an electromagnetic wave (electric field: $E = E_0 e^{-i\omega t}$) is assumed to drive oscillations in the bound electrons (assume them to have mass m and charge e). Assume that these oscillations occur along the x direction. What are the main forces acting on them and how would you express them in terms of derivatives of x? **[4]**

(b) Why is the magnetic field of the electromagnetic wave neglected in this treatment? **[3]**

(c) Write down the full equation of motion for a bound electron. **[3]**

(d) The solution to the equation derived in (c) is written as:

$$x = \frac{(e/m)E_0 e^{-i\omega t}}{\left(\omega_0^2 - \omega^2\right) - i\gamma\omega} \quad \text{Eq. 8}$$

(i) Write the expression of the polarisation P as a function of x **[1]**

(ii) Show that the relative permittivity of the material can then be written as: **[3]**

$$\varepsilon_r = \hat{n}^2 = 1 + \frac{N(e^2/\varepsilon_0 m)}{\left(\omega_0^2 - \omega^2\right) - i\gamma\omega} = \varepsilon_r' + i\varepsilon_r'' \quad \text{Eq. 9}$$

(e) **(i)** Define the group velocity and phase velocity of an electromagnetic wave. **[2]**

(ii) Show that the group velocity v_g and phase velocity v_f of a monochromatic wave with wavelength λ propagating in a medium with refractive index n are linked by the following expression:

$$v_g = v_f + \frac{\lambda c}{n^2}\frac{\partial n}{\partial \lambda} \quad \text{Eq. 10}$$ **[4]**

END OF THE EXAMINATION PAPER