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PHY2004

Full Maxwell's equations – macroscopic form

Macroscopic form = equations in matter/simplified for accommodating bound charge

Integral form

Relates fields in a region of space to fields on a boundary – useful for systems with symmetry

Gauss' law:

$$\oint_S \vec{D} \cdot \mathrm{d}\vec{S} = \int_V \rho_f \mathrm{d}V$$

$$\oint_S \vec{B} \cdot \mathrm{d}\vec{S} = 0$$

Faraday's law:

$$\oint_C \vec{E} \cdot \mathrm{d}\vec{l} = -\frac{\mathrm{d}}{\mathrm{d}t} \int \vec{B} \cdot \mathrm{d}\vec{S}$$

Ampere's law:

$$\oint_C \vec{H} \cdot \mathrm{d}\vec{l} = \int \vec{J}_f \cdot \mathrm{d}\vec{S} + \frac{\mathrm{d}}{\mathrm{d}t} \int \vec{D} \cdot \mathrm{d}\vec{S}$$

Differential form

Calculating the local field at a point

free charge density

$$\nabla \cdot \vec{D} = \rho_f$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

free current

$$\nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$$

Lecture 12:
Electromagnetic waves

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Mathematical formalism

Starting with Faraday's law:

$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} = -\mu \frac{\partial \underline{H}}{\partial t}$$

Take the curl of both sides (remembering that curl is commutative):

$$\nabla \times \nabla \times \vec{E} = -\mu \frac{\partial}{\partial t}(\nabla \times \vec{H})$$

On the right hand side, we have the curl of the magnetic field that from the Maxwell-Ampere law, we know to be equal to:

$$\nabla \times \nabla \times \vec{E} = -\mu \frac{\partial}{\partial t}\left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right)$$

We can express this entirely in terms of the electric field by substituting in for the current and the displacement field.

Conductivity of material relating current to E-field: $\vec{J} = \sigma \vec{E}$

Displacement field: $\vec{D} = \epsilon \vec{E}$

$$\nabla \times \nabla \times \vec{E} = -\mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

Mathematical formalism

We can express this equation in a more elegant manner by using a vector identity:

$$\boxed{\nabla \times \nabla \times \vec{E} = -\mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2}}$$

Vector identity

$$\nabla \times \nabla \times \vec{E} = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E}$$

$$\boxed{\nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -\mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2}}$$

In absence of external charges, charge density $\rho=0$, and therefore:

Gauss's law: $\nabla \cdot \vec{E} = 0$

Mathematical formalism

$$\nabla(\cancel{\nabla \cdot \vec{E}}) - \nabla^2 \vec{E} = -\mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

$\nabla \cdot \vec{E} = 0$

This leads to we then obtain:

$$\boxed{\nabla^2 \underline{\underline{E}} = \mu\sigma \frac{\partial \underline{\underline{E}}}{\partial t} + \mu\varepsilon \frac{\partial^2 \underline{\underline{E}}}{\partial t^2}}$$

Similar arguments can be applied (try!) to the magnetic field, giving an analogous expression:

$$\boxed{\nabla^2 \underline{\underline{H}} = \mu\sigma \frac{\partial \underline{\underline{H}}}{\partial t} + \mu\varepsilon \frac{\partial^2 \underline{\underline{H}}}{\partial t^2}}$$

Which equation do we start with to find B or H?

Mathematical formalism

If we have a perfect dielectric (conductivity $\sigma = 0$) then the equation reduces to:

$$\boxed{\nabla^2 \underline{E} = \mu\varepsilon \frac{\partial^2 \underline{E}}{\partial t^2}}$$

This equation is formally identical to the **general wave equation**:

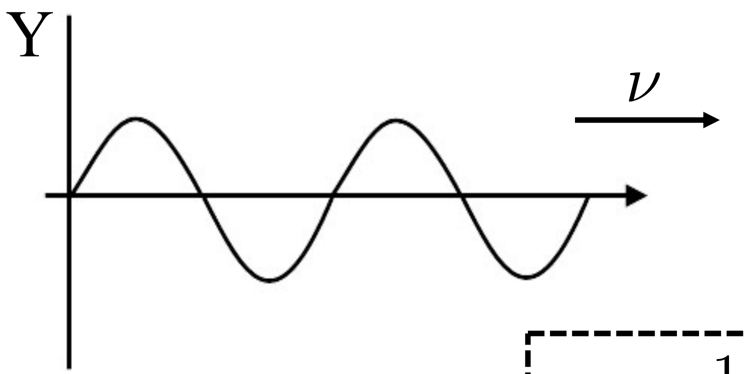
Vector Laplacian operator:

$$\nabla^2 Y = \frac{1}{\upsilon^2} \frac{\partial^2 Y}{\partial t^2}$$

Second derivative of Y with respect to t

$$\nabla^2 \vec{Y} = \nabla^2 Y_x \hat{x} + \nabla^2 Y_y \hat{y} + \nabla^2 Y_z \hat{z}$$

where Y is the disturbance propagating at a velocity ν (phase velocity of the wave).



By comparison, we can see that the speed of the wave described by our expression is:

$$\boxed{\nu = \frac{1}{\sqrt{\mu\epsilon}}}$$

Speed of light

We have shown that electric (and magnetic) fields can propagate in the form of waves and that their velocity depends on the permeability and permittivity of the medium.

For propagation in vacuum we find: $$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 2.998 \times 10^8 \ \text{m/s}$$

This is an important result for several reasons:

- This calculated value from Maxwell (1864), matched pretty well with measurements from Roemer (1676) and Bradley (1728) for the **speed of light**
- This ties together electricity, magnetism and light into a single theory.
- c is constant and as fundamental constants are unchanged in inertial reference frames this was all Einstein needed to formulate **Special Relativity**.

Looking at the phase velocity of the wave in other medium, we find it is slower than the speed in a vacuum (**for most materials**) and is dependent on the material properties (and frequency of light):

$$c' = \frac{1}{\sqrt{\mu \varepsilon}} < c$$

Plane waves

Looking at the wave equation you can see that it is a linear, second order, homogeneous partial differential equation:

$$\nabla^2 \underline{E} = \mu\varepsilon \frac{\partial^2 \underline{E}}{\partial t^2}$$

This type of equation can have many solutions and these solutions can be linearly combined to create other solutions.

Partial Differential Equations:

-Linear: derivatives are of first power of E with no cross terms.

-Second order: highest derivative present is second derivatives.

-Homogeneous: All terms include E or its derivatives (no sources).

The *second* simplest solution is represented by the **plane wave** that reads:

Maximum amplitude of the wave

Complex form

$$\vec{E}(z,t) = E_0 \cos(kz - \omega t)\hat{\mathbf{x}} = E_0 Re[e^{i(kz-\omega t)}]\hat{\mathbf{x}}$$

Phase of wave

Complex form of wave equation solution

Maximum amplitude of the wave — Complex form

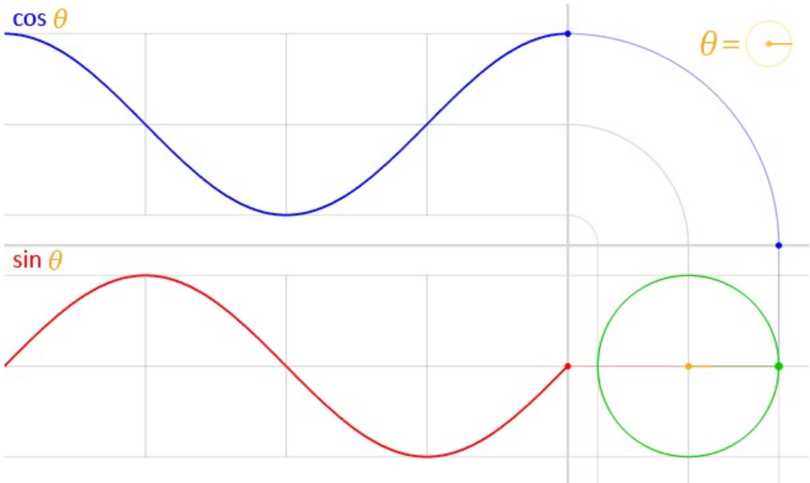
$$\vec{E}(z,t) = E_0 \cos(kz - \omega t)\hat{\mathbf{x}} = E_0 Re[e^{i(kz-\omega t)}]\hat{\mathbf{x}}$$

We will use Euler's theorem to switch between these two forms so here's a little refresher.

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

A complex number can be represented in polar or cartesian form:

$$z = re^{i\theta} = x + iy \qquad \text{Where: } \mathrm{Re}[z] = x = \frac{z + z*}{2} \text{ and } \mathrm{Im}[z] = y = \frac{z - z*}{2}$$



From Wikimedia Commons – L. Vieira

Using this form can make differentiation much easier/cleaner:

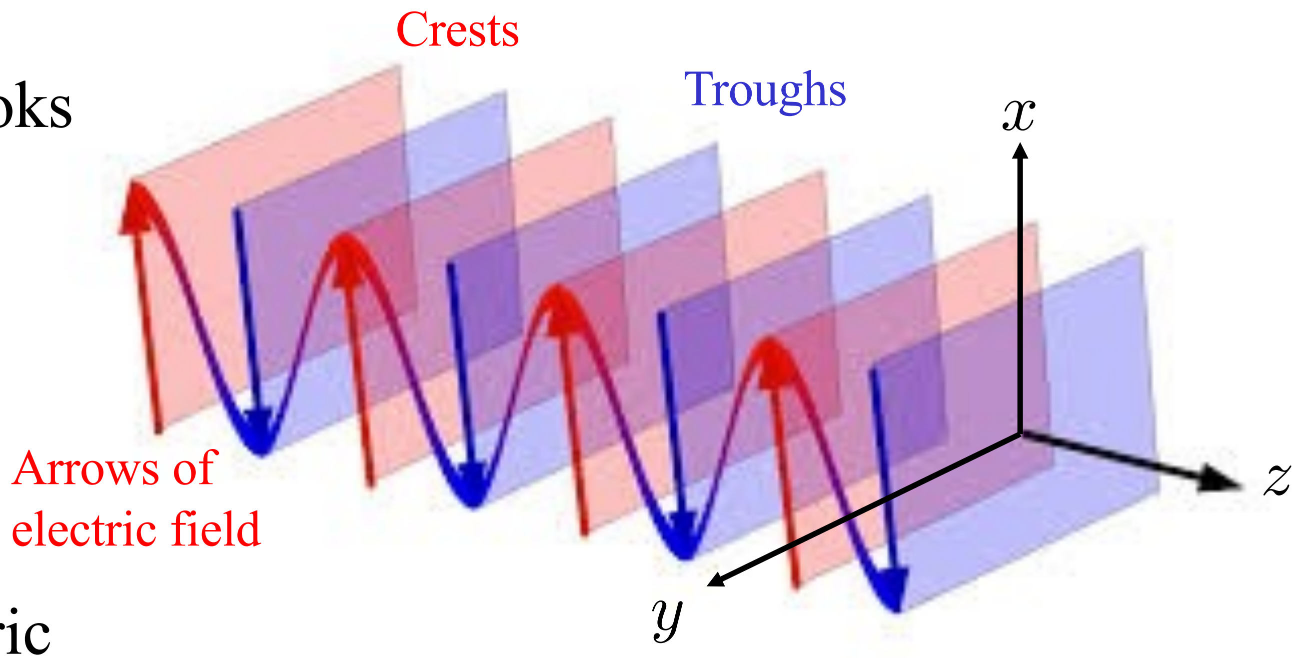
$$\frac{\mathrm{d}}{\mathrm{d}t}e^{ax} = ae^{ax}$$

Plane waves

What does this look like if we visualize it:

$$\vec{E}(z,t) = E_0 \cos(kz - \omega t + \phi)\hat{\mathbf{x}} = E_0 Re[e^{i(kz-\omega t+\phi)}]\hat{\mathbf{x}}$$

You can see that it looks like a sine or cosine wave with amplitude varying along z and continue forever.



Notice that our Electric field is pointed along the x axis and flips direction every half wavelength.

Finally, along the y axis, the wave continues forever, it forms a plane with all the crests lined up.

Plane waves

What does this look like if we visualiz

$$\vec{E}(z,t) = E_0 \cos(kz \quad t+\phi)]\hat{\mathbf{x}}$$

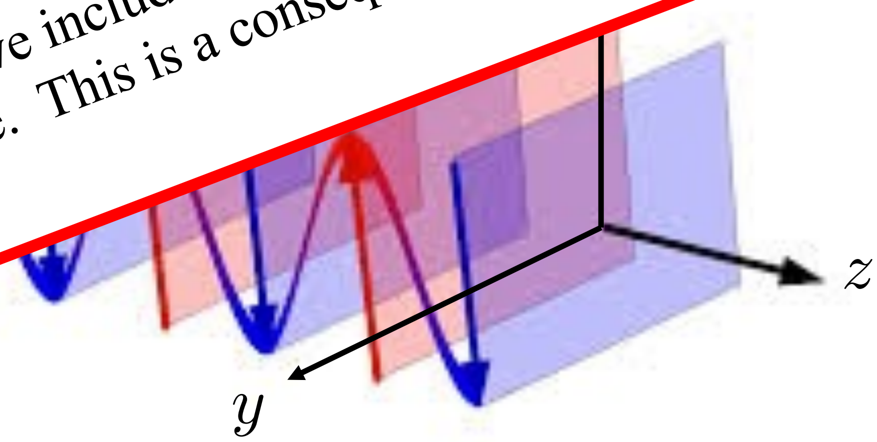
WARNING: It's important to note that the plane wave is unphysical! It has infinite extension in time (along the z axis) and in space (along the y-axis) so it would contain an infinite amount of energy!

BUT: Any solution of the wave equation can be represented as a sum of plane waves. The more plane waves we include, the shorter we can make our "super-positioned" final wave. This is a consequence of the Fourier theorem.

You can

li

c



Noti tric field ed along the x axis and flips direction every half wavelength.

Finally, along the y axis, the wave continues forever, it forms a plane with all the crests lined up.

Electromagnetic waves

Taking our simple unphysical solution, the wave equation only features the electric field (or magnetic field) where the amplitude of our electric field is varying.

$$\vec{E}(z,t) = E_0 \cos(kz - \omega t)\hat{\mathbf{x}}$$

But we used Maxwell's equations relating the electric and magnetic fields to derive this and we know that if we have a changing electric field or a changing magnetic field we'll generate the other.

Try this by substituting the expression for *E* above into Faraday's law.

You'll retrieve a similar expression for *B* or *H*.

BUT the vector direction of the magnetic field will be perpendicular to *E* and to the direction of travel of the wave (along z) – these are **TRANSVERSE WAVES**.

Use your right-hand rule to check that they will have the relative orientation shown on the right.

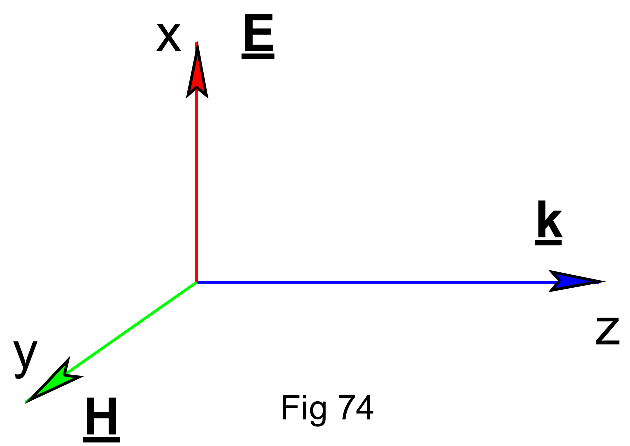
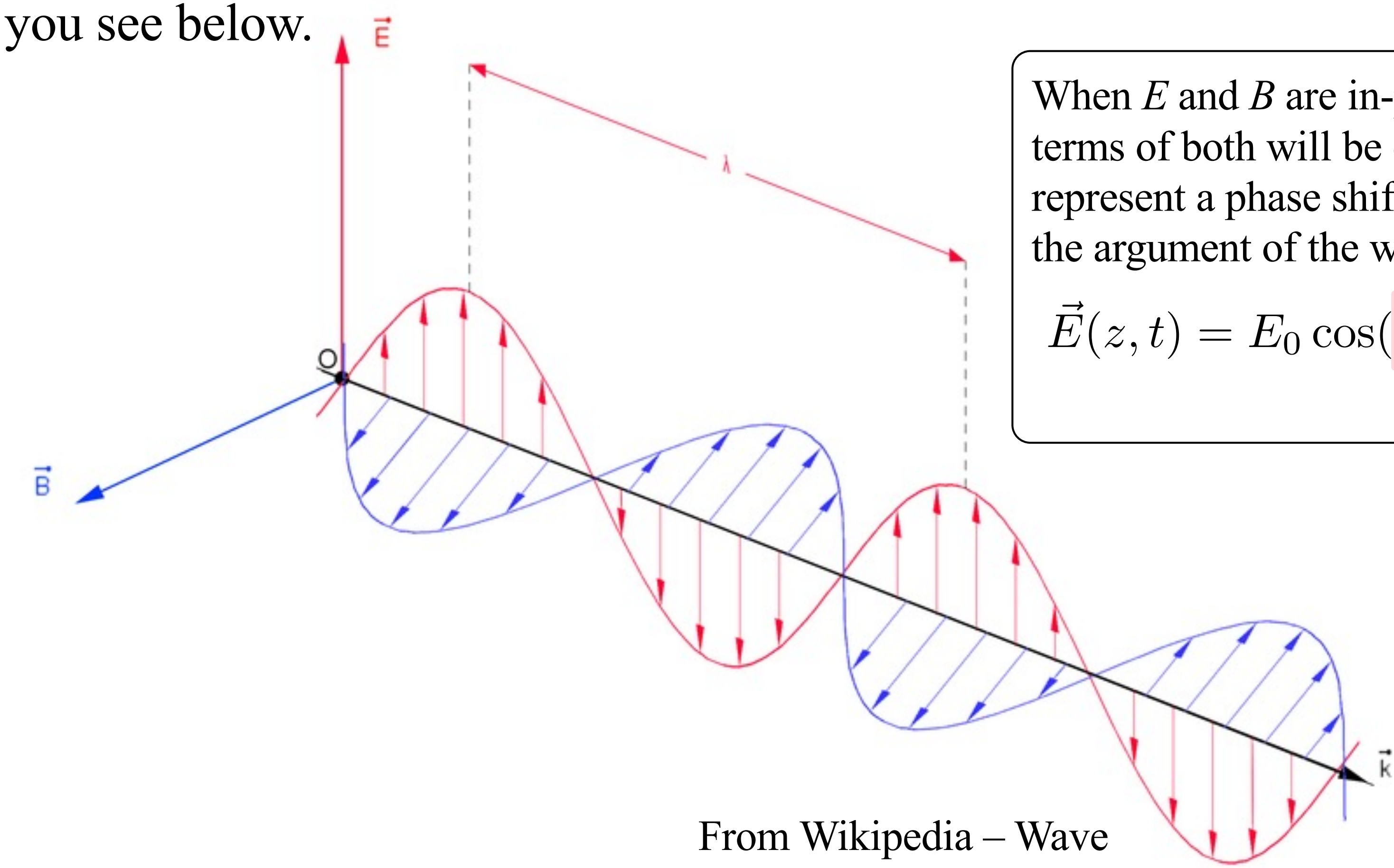


Fig 74

Electromagnetic waves

With both the electric and magnetic fields oscillating **in-phase** we have the structure that you see below.



When E and B are in-phase, the phase terms of both will be equal. We represent a phase shift by adding φ to the argument of the wave..

$$\vec{E}(z,t) = E_0 \cos(kz - \omega t + \phi)\hat{\mathbf{x}}$$

Phase of wave

From Wikipedia – Wave

This is a particular characteristic of a **linearly polarized plane wave**. You'll be learning more about polarization in the last third of the course.

Wave vector and frequency

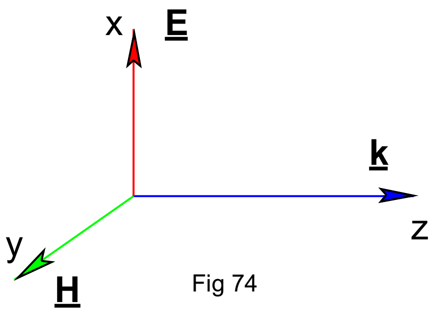


Fig 74

Looking at this sketch of axes again you'll notice that the axis along which the wave is propagating is also assigned a vector ***k***, the wavevector, and so really we should treat it as a vector.

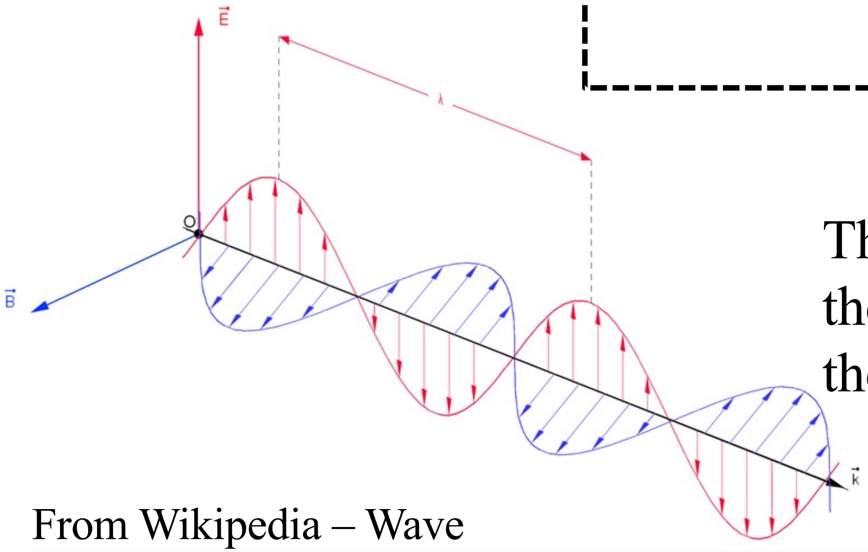
$$\vec{E}(z,t) = E_0 Re[e^{i(\vec{k}\cdot\hat{z}-\omega t)}]\hat{\mathbf{x}}$$

We know that we always have: $\nabla\cdot\vec{B} = 0$
And in the absence of charges: $\nabla\cdot\vec{E} = 0$

$$\nabla\cdot\vec{E} = \left(\cancel{\frac{\partial\vec{E}}{\partial x}} + \cancel{\frac{\partial\vec{E}}{\partial y}} + \frac{\partial\vec{E}}{\partial z}\right) = i\vec{k}\cdot\vec{E} = 0$$

perpendicular vectors

Show the same is true for the magnetic field.



The **magnitude of the wavevector** is often called the wavenumber and is inversely proportional to the **wavelength**, λ, of the wave.

$$|\vec{k}| = \frac{2\pi}{\lambda}$$

Units of inverse length

From Wikipedia – Wave

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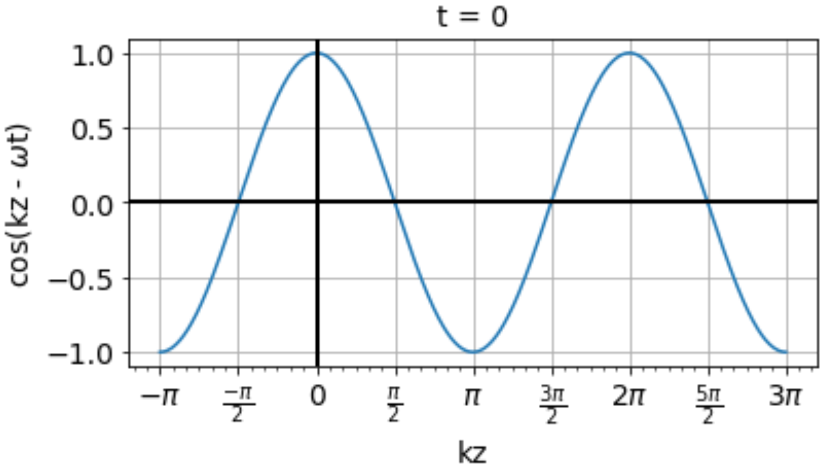
Phase velocity (again)

Both the wavenumber and the angular frequency of the wave appear inside the phase term in our mathematical expression for the wave.

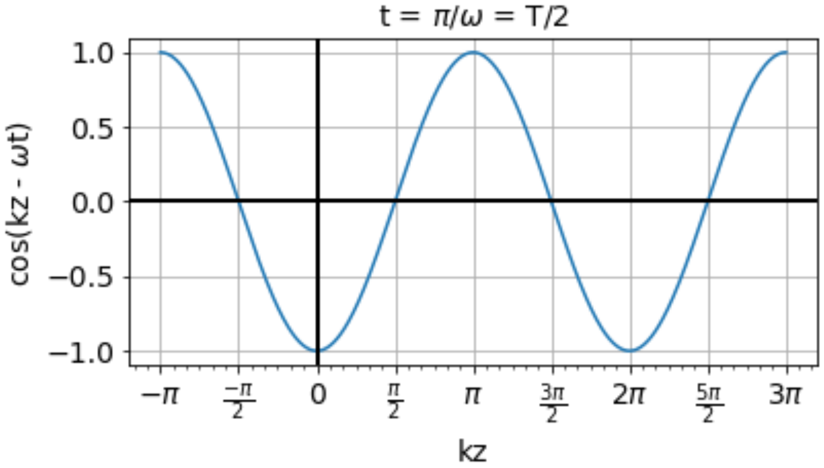
Phase of wave - dimensionless

$$\vec{E}(z,t) = E_0 \cos(\boxed{kz - \omega t})\hat{\mathbf{x}}$$

This argument represents where we are in our wave – we are travelling with the wave.



After time T/2 $\longrightarrow$



The velocity with which a fixed phase point on our wave propagates is called the phase velocity and it is equal to ω/k - this is also called **'the speed of light'.**

Refractive index

We know that the speed of light in vacuum is given by: $c = \dfrac{1}{\sqrt{\mu_0 \varepsilon_0}} = 2.998 \times 10^8$ m/s

whilst in a medium it is given by: $c' = \dfrac{1}{\sqrt{\mu \varepsilon}}$

The ratio of the speeds is $\boxed{\dfrac{c}{c'} = \sqrt{\varepsilon_r \mu_r} = n}$ which is called the **refractive index** of the material.

Note that since, in the optical frequency region (λ : 400-700 nm), $\mu_r \sim 1$, this is usually simplified to :

$$\boxed{n \sim \sqrt{\varepsilon_r}}$$

But importantly, remember the phase velocity in a medium is dependent on the frequency of the light, because the permittivity of a material is not constant – we'll do more on this in a couple of lectures time.

The refractive index is a great tool to determine how light behaves when it propagates between media. You'll look at this more in the optics section.