PHY2006 - LINEAR ALGEBRA AND FOURIER ANALYSIS

TOM FIELD

Abstract. This synopsis covers linear algebra and Fourier analysis in PHY2006.

Contents

Date: October 31, 2021.

1. Linear Algebra - Introduction

(return to contents)

Linear algebra is related to linear operations, such as the sets of linear equations;

$$\begin{aligned} a_{11}u_1 + a_{12}u_2 &= v_1 \\ a_{21}u_1 + a_{22}u_2 &= v_2 \end{aligned} \tag{1.1}$$

which we know as simultaneous equations, as we saw in PHY1002. Note that the a_{ij} terms are constants and that variables u_i and v_i never appear in non-linear terms such as $\sin(u_i)$, v_i^2, e^{u_i} and $\sqrt{v_i}$.

Now we can represent the simulataneous equations in (1.1) with the matrix equation below.

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{1.2}$$

Equation (1.2) can be represented as

$$\mathbf{A}\mathbf{u} = \mathbf{v} \tag{1.3}$$

where

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Thus $\mathbf{A}$ represents a matrix whilst $\mathbf{u}$ and $\mathbf{v}$ represent vectors.

In this part of the course we will be considering vectors in general.

1.1. **Refresher questions.** Calculate the following;

(a) $\begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 7 \end{pmatrix} =$

(b) $\begin{pmatrix} 1 & 2 & 1 \\ 2 & -1 & 0 \\ 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} =$

(c) $\begin{pmatrix} 1 & 0 & 1 & 3 & 4 \\ 2 & 0 & 2 & 1 & 7 \\ 1 & 1 & 1 & 7 & 1 \\ 0 & 1 & 2 & 9 & 6 \\ 7 & -1 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} -1 \\ 7 \\ 2 \\ 3 \\ -2 \end{pmatrix} =$

(d) $\begin{pmatrix} 1 & 0 & -1 \\ 7 & 1 & 1 \\ 0 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 2 & 0 & 1 \\ 0 & 7 & 0 \end{pmatrix} =$

1.2. **Answers.**

(a) $\begin{pmatrix} 11 \\ 10 \end{pmatrix}$ (b) $\begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix}$ (c) $\begin{pmatrix} 2 \\ -9 \\ 27 \\ 26 \\ -20 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & -7 & 1 \\ 9 & 7 & 8 \\ 4 & 21 & 2 \end{pmatrix}$

2. Position vectors on two-dimensional planes

(return to contents)

Here we will consider many important fundamental properties of vectors and 'vector spaces' for the special case of vectors on a flat two dimensional plane. It is convenient to do this because we can look at diagrams on a page of the vectors. It is really important to get a good grasp of this material for vectors in two dimensions because later on we will consider more complicated vectors. It will help to be familiar with the essential concepts before we turn to more complicated vectors.

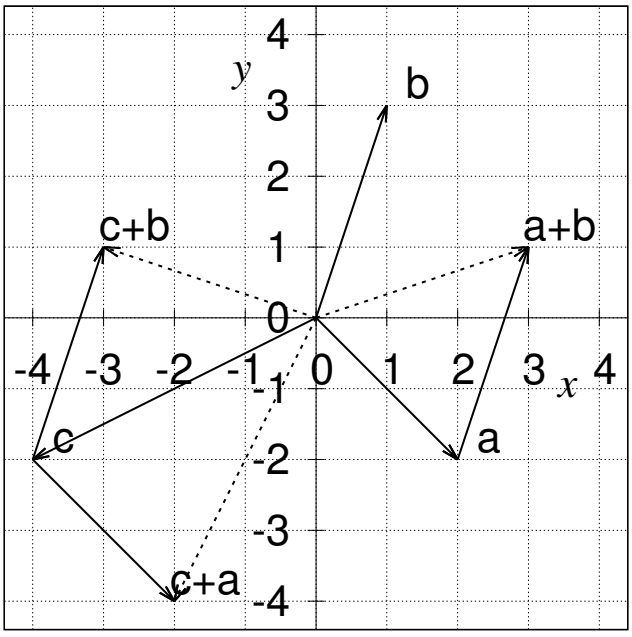

FIGURE 1. Three vectors (**a**, **b** and **c**) in a two dimensional flat plane are shown along with the combinations **a**+**b**, **a**+**c** and **b**+**c**. The combination vectors are shown as dotted arrows and **a**, **b** and **c** are shown in black

Consider Figure 1, which shows three vectors **a**, **b** and **c** along with the vectors combinations $\mathbf{a}+\mathbf{b}$, $\mathbf{a}+\mathbf{c}$ and $\mathbf{b}+\mathbf{c}$. We can represent these vectors **a**, **b** and **c** in terms of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ unit vectors or as column vectors.

$$\mathbf{a} = 2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} = \begin{pmatrix} 2 \\ -2 \end{pmatrix} \quad \mathbf{b} = \hat{\mathbf{i}} + 3\hat{\mathbf{j}} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \mathbf{c} = -4\hat{\mathbf{i}} - 2\hat{\mathbf{j}} = \begin{pmatrix} -4 \\ -2 \end{pmatrix} \tag{2.1}$$

Here column vector format is used for position vectors

2.1. **Vector space closure: Vector addition, subtraction and multiplication by a scalar.** Figure 1 shows several vectors in a flat two dimensional plane. This plane contains an infinite number of vectors which can be thought of as arrows from the origin to the infinite number of points in the plane. The infinite set of vectors pointing to points in the plane is called a 'vector space'. An important rule for vector spaces is that when any two vectors in the field are added together or one is subtracted from another the resultant vector is a member of the vector space. Similarly, if a vector is multiplied by a scalar the resultant vector must be a member of the vector space. The property of vector spaces is called '**closure**'.

To illustrate closure Figure 1 shows how vector addition gives resultant vectors that also lie in the same two dimensional plane. Closure is also illustrated below for addition of two column vectors from equation 2.10 above;

$$\mathbf{a}+\mathbf{b}=\begin{pmatrix}2\\-2\end{pmatrix}+\begin{pmatrix}1\\3\end{pmatrix}=\begin{pmatrix}3\\1\end{pmatrix}$$

Subtraction shows closure as well, for example;

$$\mathbf{a}-\mathbf{b}=\begin{pmatrix}2\\-2\end{pmatrix}-\begin{pmatrix}1\\3\end{pmatrix}=\begin{pmatrix}1\\-5\end{pmatrix}$$

and multiplication of vectors by a scalar also gives closure, for example;

$$0.5\mathbf{a}=0.5\begin{pmatrix}2\\-2\end{pmatrix}=\begin{pmatrix}1\\-1\end{pmatrix}\qquad 2.5\mathbf{b}=2.5\begin{pmatrix}1\\3\end{pmatrix}=\begin{pmatrix}2.5\\7.5\end{pmatrix}$$

Note that the result of addition, subtraction and multiplication by a scalar are all vectors in the same flat two dimensional plane, which points to closure.

In formal mathematics, of course, it is necessary to prove closure rather than look at a few examples. The proof is not complicated, but is not given here.

2.2. **Inner products, dot products, and norm for measuring vectors.** The inner product is a way of combining two vectors to get a scalar result which is useful for measuring length and comparing two vectors. The inner product between two general vectors $\mathbf{u}$ and $\mathbf{v}$ is written as $\langle\mathbf{u},\mathbf{v}\rangle$.

Properties of the inner product are described in more detail below, but here, for vectors on a flat two dimensional plane, the dot product is the inner product. For example, the inner product of the vectors $\mathbf{a}$ and $\mathbf{b}$ from above is calculated with

$$\langle\mathbf{a},\mathbf{b}\rangle=\begin{pmatrix}2\\-2\end{pmatrix}\cdot\begin{pmatrix}1\\3\end{pmatrix}=2-6=-4$$

The 'norm' of a general vector $\mathbf{u}$ is written as $||\mathbf{u}||$ and calculated with

$$||\mathbf{u}||=\sqrt{\langle\mathbf{u},\mathbf{u}\rangle}$$

and so for example for the vector **a** above the norm is calculated with

$$||\mathbf{a}|| = \sqrt{\langle \mathbf{a}, \mathbf{a} \rangle} = \sqrt{\begin{pmatrix} 2 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \end{pmatrix}} = \sqrt{4+4} = 2\sqrt{2} \approx 2.83$$

Can you see what the norm is equivalent to for the vector **a**?

Let us condsider a general two dimensional vector $\mathbf{r} = \binom{x}{y}$. Now we can calculate the norm of **r** with

$$||\mathbf{r}|| = \sqrt{\langle \mathbf{r}, \mathbf{r} \rangle} = \sqrt{\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}} = \sqrt{x^2 + y^2}$$

Thus, the norm is equivalent to the length of two dimensional geometrical vectors. The norm is also equal to the length of three dimensional geometrical vectors. In general, it can be helpful to think about the norm as length.

Note that if we consider the norm $||\mathbf{a} - \mathbf{b}||$ we measure the distance between vectors **a** and **b**. This is an important point.

2.3. **orthogonality; vectors which are perpendicular are orthogonal.** An important concept working with vectors is orthogonality. In general if two vectors **u** and **v** are orthogonal then their inner product is zero, *i.e.* $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

We know for geometrical vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ the angle between the two vectors, θ, can be calculated with

$$\mathbf{r}_1 \cdot \mathbf{r}_2 = |\mathbf{r}_1||\mathbf{r}_2| cos\theta$$

where $|\mathbf{r}_i|$ is the length of vector $\mathbf{r}_i$, which we now know we can also write as $||\mathbf{r}_i||$.

When two geometrical vectors are perpendicular the angle between the vectors is 90° or $\pi/2$ radians and, of course, $\cos\theta = 0$. Thus, if two vectors are perpendicular the dot product between them will be zero and their inner product will be zero and so they are orthogonal.

2.4. **Making vectors orthogonal: Gramm-Schmidt orthogonalization.** The vectors **a** and **b** shown in figure 1 are not orthogonal. They are not perpendicular. To make them perpendicular we can add or subtract a scalar multiple of the vector **a** to the vector **b** to make a new vector, which we will call $\mathbf{b}'$. The process of making $\mathbf{b}'$ from **b** and **a** is shown in figure 2.

Thus, given that $\mathbf{b}'$ is **b** with a scaled component of **a** added or subtracted we can write down the following equation for $\mathbf{b}'$.

$$\mathbf{b}' = \mathbf{b} + k\mathbf{a} \tag{2.2}$$

Now if **a** and $\mathbf{b}'$ are perpendicular then $\langle \mathbf{a}, \mathbf{b}' \rangle = 0$ and hence,

$$0 = \langle \mathbf{a}, \mathbf{b}' \rangle = \langle \mathbf{a}, \mathbf{b} + k\mathbf{a} \rangle \tag{2.3}$$

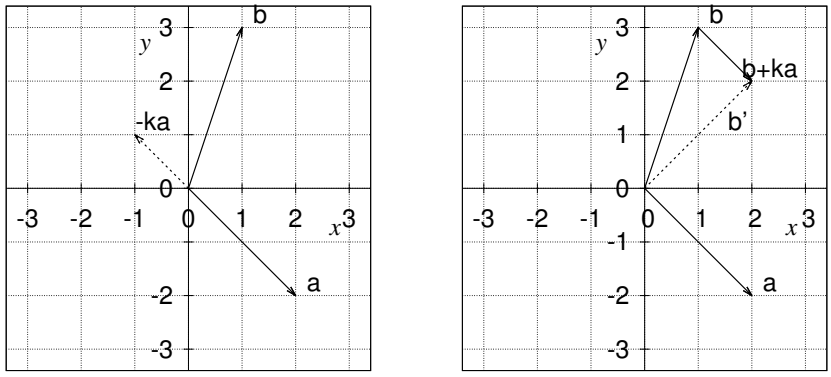

FIGURE 2. Orthogonalization is shown. (a) vectors $\mathbf{a}$ and $\mathbf{b}$ are shown with $-k\mathbf{a} = \frac{\langle \mathbf{a},\mathbf{b}\rangle}{\langle \mathbf{a},\mathbf{a}\rangle}\mathbf{a}$. (b) the vector $\mathbf{b}' = \mathbf{b} + k\mathbf{a}$ is shown. $\mathbf{b}'$ is vector which is orthogonal, or perpendicular, to $\mathbf{a}$. To make $\mathbf{b}'$ a scalar multiple $k\mathbf{a}$ is added to $\mathbf{b}$ as described in the text. This is the Gramm Schmidt orthogonalization process.

Now it turns out that with inner products we can expand the inner product on the right hand side above because of the following rules for inner products:

$$\langle \mathbf{u}, \mathbf{v} + \mathbf{w}\rangle = \langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{u}, \mathbf{w}\rangle \qquad \langle \mathbf{u}, \lambda\mathbf{v}\rangle = \lambda\langle \mathbf{u}, \mathbf{v}\rangle \tag{2.4}$$

where $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors and λ is a scalar.

Using these rules we can determine the value of the scalar k which makes $\mathbf{b}'$ perpendicular to $\mathbf{b}$

$$0 = \langle \mathbf{a}, \mathbf{b}'\rangle = \langle \mathbf{a}, \mathbf{b} + k\mathbf{a}\rangle = \langle \mathbf{a}, \mathbf{b}\rangle + \langle \mathbf{a}, k\mathbf{a}\rangle \tag{2.5}$$

$$0 = \langle \mathbf{a}, \mathbf{b}\rangle + k\langle \mathbf{a}, \mathbf{a}\rangle \tag{2.6}$$

$$-k\langle \mathbf{a}, \mathbf{a}\rangle = \langle \mathbf{a}, \mathbf{b}\rangle \tag{2.7}$$

$$\Rightarrow \quad k = -\frac{\langle \mathbf{a}, \mathbf{b}\rangle}{\langle \mathbf{a}, \mathbf{a}\rangle} \tag{2.8}$$

Thus we can use equation 2.8 to calculate the value of k to insert into equation 2.2 to calculate $\mathbf{b}'$, the modified vector based on $\mathbf{b}$ which is perpendicular, or orthogonal, to $\mathbf{a}$. We can summarize the process with the equation;

$$\mathbf{b}' = \mathbf{b} + k\mathbf{a} \quad \Rightarrow \qquad \mathbf{b}' = \mathbf{b} - \frac{\langle \mathbf{a}, \mathbf{b}\rangle}{\langle \mathbf{a}, \mathbf{a}\rangle}\mathbf{a} \tag{2.9}$$

The process of making vectors orthogonal to each other is called Gramm Schmidt orthogonalization. This process is illustrated in figure 2.

2.4.1. *Example question.* Use the Gramm Schmidt orthogonalization procedure to calculate $\mathbf{b}'$ from $\mathbf{b}$ and $\mathbf{a}$. Note that these vectors are equal to;

$$\mathbf{a} = \begin{pmatrix} 2 \\ -2 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \qquad \mathbf{b}' = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

2.5. **Basis sets of vectors.** A basis set of vectors is a set of vectors that can be used to generate every different vector in a vector space in a single unique way. Thus, every vector in a vector space can be generated by addition of each basis set vector multiplied by a scalar. For example, in figure 1 the vectors $\mathbf{a}$ and $\mathbf{b}$ form a basis set which can be used to generate all the vectors in the two dimensional flat plane shown. For example the vector $\mathbf{c}$ in figure 1 is equal to $-\frac{5}{4}\mathbf{a} - \frac{3}{2}\mathbf{b}$;

(2.10)
$$-\frac{5}{4}\mathbf{a} - \frac{3}{2}\mathbf{b} = -\frac{5}{4}\begin{pmatrix} 2 \\ -2 \end{pmatrix} - \frac{3}{2}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -2.5 \\ 2.5 \end{pmatrix} + \begin{pmatrix} -1.5 \\ -4.5 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \end{pmatrix} = \mathbf{c}$$

Note that all basis sets for the flat two dimensional plane we are considering must have two vectors. For example, the set of vectors $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ are not a basis set because there are two many vectors and there would be more than one way of making each vector.

2.6. **Orthogonal basis sets.** It is not so easy to determine how to represent vectors as a sum of basis set vectors unless the vectors in the basis set are orthogonal. With an orthogonal basis set it is easier to determine how to make up a vector from basis set vectors.

For example, consider a basis set $\{\mathbf{u}, \mathbf{v}\}$ where $\mathbf{u}$ and $\mathbf{v}$ are orthogonal. a general vector $\mathbf{r}$ will be a sum of scalar multiples of the basis set vectors;

$$\mathbf{r} = \alpha\mathbf{u} + \beta\mathbf{v} \tag{2.11}$$

now if we take the inner product of $\mathbf{r}$ with $\mathbf{u}$;

$$\langle \mathbf{u}, \mathbf{r} \rangle = \langle \mathbf{u}, \alpha\mathbf{u} + \beta\mathbf{v} \rangle \tag{2.12}$$

$$= \alpha\langle \mathbf{u}, \mathbf{u} \rangle + \beta\langle \mathbf{u}, \mathbf{v} \rangle \tag{2.13}$$

Now given $\mathbf{u}$ and $\mathbf{v}$ are orthogonal we know that $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ so equation 2.13 simplifies to

$$\langle \mathbf{u}, \mathbf{r} \rangle = \alpha\langle \mathbf{u}, \mathbf{u} \rangle \tag{2.14}$$

$$\Rightarrow \qquad \alpha = \frac{\langle \mathbf{u}, \mathbf{r} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \tag{2.15}$$

A similar calculation can be made to find β and the result is that we can represent r with the expression;

$$\mathbf{r} = \frac{\langle \mathbf{u}, \mathbf{r} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle}\mathbf{u} + \frac{\langle \mathbf{v}, \mathbf{r} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}\mathbf{v} \tag{2.16}$$

This procedure may be described as projecting a vector onto a basis set.

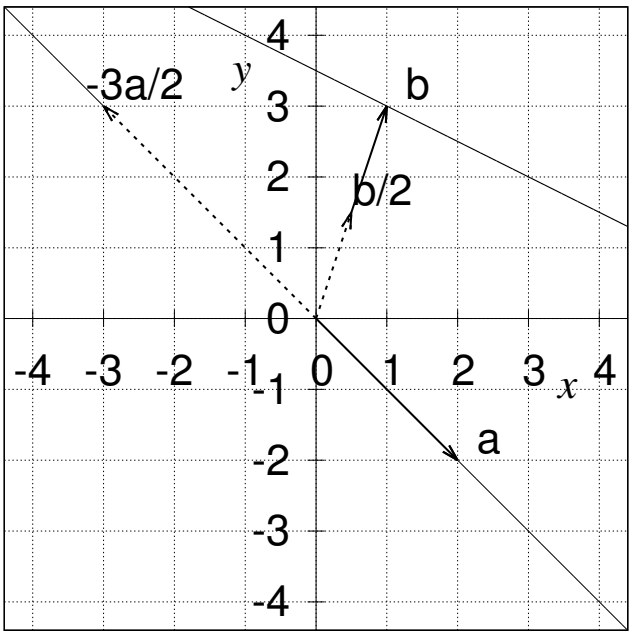

FIGURE 3. Vectors **a** and **b** point from the origin to points on two lines. Examples of vectors obtained by multiplying **a** and **b** by scalars are shown as dashed vectors; $-\frac{3}{2}\mathbf{a}$ and $\frac{1}{2}\mathbf{b}$. Note that $-\frac{3}{2}\mathbf{a}$ lies on the line which passes through the origin, but $\frac{1}{2}\mathbf{b}$ does not lie on the same line as **b**. Thus, vectors from the origin to the line through the origin form a closed vector subspace and the set of vectors to the line which does not pass through the origin is not closed and does not form a vector subspace.

2.7. **Vector subspaces.** If we consider any line which passes through the origin and lies in the two dimensional plane of the vector space then it forms a 'vector subspace' of the two dimensional vector space. If we take any vectors which lie along the line and add them, subtract them or multiply them by a scalar we will always get another vector that lies along the same line in the plane. Thus, there is closure for vectors that lie along a line which passes through the origin.

Consider a line that does not pass through the origin. if we multiply one of the vectors from the origin to a point in this line by a scalar will we get another vector which points from the origin to a point along the same line? If we do not then we do not have closure and the set of vectors which point from the origin to the points in the line is not a vector subspace.

Figure 3 shows the effect of multiplication by a scalar on a vector to a point on a line through the origin and a different vector to a point on a line that does not pass through the origin. It is clear that the set of vectors to points on the line through the origin form a vector subspace, but the set of vectors to the line which does not pass through the origin do not form a vecctor subspace because the set is not closed, *viz*. closure is not obeyed.

2.8. **Closest points.** Let us consider a flat two dimensional plane that has a vector subspace, which is a line in the plane. Let us consider the subspace is a line, which everypoint on the line is a multiple of the vector **a** shown in figure 3. Now, if we want to find the point on a line that is closest to a point on the plane there is a method to find it. Let us consider the point defined by the vector **b**. Now figure 2(a) will be helpful here to visualize what is happening. If we project the vector to the point in the plane onto a basis set for the line, which is the vector subspace then we will get the closest point on the line to the point on the plane. Now in the case of a line as the subspace, its basis set is simply a single vector in the direction of the line. So for example if we want to find the closest point to **b** on the line defined by the vector **a** we need to project **b** onto the vector **a**. The result, the closest point to **b** is normally written as $\hat{\mathbf{b}}$, which can be confusing as $\hat{\mathbf{b}}$ can also mean the unit vector in the direction of **b**

$$\hat{\mathbf{b}} = \frac{\langle \mathbf{a}, \mathbf{b} \rangle}{\langle \mathbf{a}, \mathbf{a} \rangle} \mathbf{a} \tag{2.17}$$

$$= \frac{\begin{pmatrix} 2 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix}}{\begin{pmatrix} 2 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \end{pmatrix}} \begin{pmatrix} 2 \\ -2 \end{pmatrix} \tag{2.18}$$

$$\hat{\mathbf{b}} = \frac{-4}{8} \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \frac{-1}{2} \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{2.19}$$

Now, looking at Figure 2(a) the vector labelled $-k\mathbf{a}$ is the closest point to **b** on the line defined by **a** and the position of the closest point agrees with the calculation above of $\hat{\mathbf{b}}$.

3. Vector Spaces

(return to contents)

We are familiar with position vectors which can be used to define positions relative to some reference point, which is generally called the 'Origin'. Here we are going to look more generally at different types of 'Vector Spaces'.

A vector space is a set of vectors, which follow the following axioms for vector addition and multiplication by a scalar.

3.1. **Vector space axioms .**

(1) **Closure under vector addition:** The sum of two vectors in a vector space gives another vector in the same vector space. So if vectors $\mathbf{u}$, $\mathbf{v}$ are elements of a vector space V then the sum of the vectors $\mathbf{u}+\mathbf{v}$ is also an element of V.
(2) **Commutativity under addition:** $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$. In words the order of addition is not important.
(3) **Associativity under addition:** $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
(4) **Additive identity** There exists a zero vector $\mathbf{0}$ for which $\mathbf{u}+\mathbf{0}=\mathbf{u}$.
(5) **Additive inverse** For every vector $\mathbf{u}$ there exsits an inverse vector under addition $-\mathbf{u}$ such that for which $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
(6) **Closure under scalar multiplication:** The multiplication of a vector in a vector space by a scalar gives another vector in the same vector space. So if $\mathbf{u}$ is an element of V then $k\mathbf{u}$ is also an element of V.
(7) **Distributivity of vector sums:** $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$.
(8) **Distributivity of scalar sums:** $(k+m)\mathbf{u}=k\mathbf{u}+m\mathbf{u}$.
(9) **Compatability of scalar and field multiplication:** $k(m\mathbf{u})=(km)\mathbf{u}$. For interest the name of this property is derived from scalar multiplication $m\mathbf{u}$ and field multiplication km. The scalars k and m are members of a 'field', such as the field of real numbers.
(10) **Identity for scalar multiplication:** There exists a unit scalar 1 such $1\mathbf{u}=\mathbf{u}$.

If the scalars, k and m, are real numbers then it is a 'Real Vector Space', but if the scalars are complex numbers then it is a 'Complex Vector Space'. More formally we might say that in a real vector space the scalars are members of the field of real numbers whereas for complex vector spaces the scalars are members of the field of complex numbers. In mathematics, a 'Field' is set where multiplication, addition, subtraction and division are defined and they behave in the same was as equivalent operations behave on rational and real numbers.

3.2. **Vector spaces: notation.** Where a vector space is a set of column vectors with n real numbers the vector space is defined as a vector space in $\mathbb{R}^n$, which is pronounced "r-n". The $\mathbb{R}$ represents real numbers and the n represents the number of dimensions. Thus for position vectors in two dimensions, such as $\mathbf{u}$ and $\mathbf{v}$ above are vectors in $\mathbb{R}^2$ or "r-two".

If a vector space is a set of column vectors with n complex numbers the vector space is defined as a vector space in $\mathbb{C}^n$,

3.3. **Example of a vector space: position vectors in 2D.** Let us consider the vector space of position vectors in two dimensions, which we have seen is a vector space in $\mathbb{R}^2$. As above, two general vectors, $\mathbf{u}$ and $\mathbf{v}$, are given by

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad\qquad \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Where u_1, u_2, v_1 and v_2 are, of course real numbers.

We will not consider all the axioms above in Section 3.1, but we will consider closure under addition and scalar multiplication:

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix} \qquad k\mathbf{u} = k \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} ku_1 \\ ku_2 \end{pmatrix}$$

It should be evident that the addition of two general vectors $\mathbf{u}$ and $\mathbf{v}$ gives a vector $\mathbf{u} + \mathbf{v}$, which is also a member of this vector space. Therefore, the vector space of two dimensional position vectors is closed under addition. Similarly the multiplication of a general vector $\mathbf{u}$ by a scalar k gives $k\mathbf{u}$ which is also a member of this vector space. Thus, this vector space is also closed under scalar multiplication.

Note, that if we had multiplied by a complex number, z, to get $z\mathbf{u}$ then the elements zu_1 and zu_2 would not necessarily be real numbers and thus two dimensional position vectors are not closed under multiplication by complex numbers.

3.4. **Example of a vector space: matrices.** A set of matricies, $\mathbf{M}_{kl}$, with k rows and l columns

$$\mathbf{M} = \begin{pmatrix} m_{11} & m_{12} & \dots & m_{1l} \\ m_{21} & m_{22} & \dots & m_{2l} \\ \dots & \dots & \dots & \dots \\ m_{k1} & m_{k2} & \dots & m_{kl} \end{pmatrix} \tag{3.1}$$

with real elements m_{ij} is a real vector space. If the matrix elements m_{ij} are complex then the set of matrices is a complex vector space.

For example, consider two by two matrices $\mathbf{M}_{22}$ and two general matrices $\mathbf{a}$ and $\mathbf{b}$

$$\mathbf{a} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \qquad\qquad \mathbf{b} = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$$

$$\mathbf{a} + \mathbf{b} = \begin{pmatrix} a_1 + b_1 & a_2 + b_2 \\ a_3 + b_3 & a_4 + b_4 \end{pmatrix} \qquad k\mathbf{a} = \begin{pmatrix} ka_1 & ka_2 \\ ka_3 & ka_4 \end{pmatrix}$$

Thus, it should be clear that the axioms for closure under addition and scalar multiplication are satisfied for the set of 2×2 matrices.

3.5. **Example of a vector space: polynomials.** It is not, perhaps, obvious at first glance that polynomials may be considered as vectors, but let us consider the set of polynomials with order up to n, P_n and a general polynomial from this set $a(x)$;

$$a(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$$

If the coefficients a_i are real numbers then P_n, the set of polynomials with degree up to n, can be considered as a real vector space.

For example, let us consider P_2, the set of polynomials of degree up to two, and two general polynomials in this set $a(x) = \mathbf{a}$ and $b(x) = \mathbf{b}$;

$$\mathbf{a} = a(x) = a_0 + a_1x + a_2x^2 \qquad \mathbf{b} = b(x) = b_0 + b_1x + b_2x^2$$

Now if we consider vector addition and multiplication by a scalar

$$\mathbf{a} + \mathbf{b} = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 \quad k\mathbf{a} = ka_0 + ka_1x + ka_2x^2$$

It should be evident that $\mathbf{a} + \mathbf{b}$ and $k\mathbf{a}$ are both members of the set of polynotmials of degree up to 2 as all the sums $(a_i + b_i)$ and products ka_i are real numbers. Thus, this set of polynomials is closed under vector addition and scalar multiplication. The same argument can be used for any polynomial vector space of any degree n.

Let us consider vector space axioms 4 and 5 above; additive identity and additive inverse. The zero vector, $\mathbf{0}$, is given by

$$\mathbf{0} = 0 + 0x + 0x^2$$

Clearly

$$\mathbf{a} + \mathbf{0} = (a_0 + 0) + (a_1 + 0)x + (a_2 + 0)x^2 = a_0 + a_1x + a_2x^2 = \mathbf{a}$$

so axiom 4, additive identity is satisfied with $\mathbf{0}$.

Given the same general order two polynomial vector, $\mathbf{a}$, the additive inverse is $-\mathbf{a}$ given by

$$-\mathbf{a} = -a(x) = -a_0 + -a_1x + -a_2x^2$$

so

$$\mathbf{a} + (-\mathbf{a}) = (a_0 + -a_0) + (a_1 + -a_1)x + (a_2 + -a_2)x^2 = 0 + 0x + 0x^2 = \mathbf{0}$$

which satisfies axiom 5, additive inverse.

The other axioms for vectors spaces will not be considered explicitly here, but it can be demonstrated that for any set, P_n, of polynomials with order up to n forms a vector space.

3.6. **Example of a vector space: real-valued functions .** The set of functions $f(x)$ which give real values for in the range $[a, b]$ for a vector space. The range $[a, b]$ means that $a \leq x \leq b$. This set of functions can be labelled $F[a, b]$.

In general if two functions $f(x)$ and $g(x)$ are members of $F[a, b]$ then $f(x) + g(x)$ will also be a member of the set. For example,

$$f(x) = \sin(x) \qquad g(x) = x^3 - x^2 \qquad f(x) + g(x) = \sin(x) + x^3 - x^2$$

3.7. **Vector subspaces of vector spaces.** A subset of the vector space is a vector subspace provided that it includes the additive identity, or zero vector $\mathbf{0}$, and is closed under addition and scalar multiplication.

For example, if we take three dimensional space $\mathbb{R}^3$ then the xy two dimesional plane where $z = 0$ is a two dimensional vector subspace in $\mathbb{R}^2$. Similarly, a line through the origin is a vector subspace in $\mathbb{R}^1$. An example of such a line through the origin is the line defined by the equation $x = y = z$.

3.8. **Linear transformations between vector spaces.** A function $f(x)$ takes a number x and generates a new number. A linear transformation transforms a vector into a new vector. In general we might write

$$T : V \to W$$

where T is a linear transformation that maps vectors from a vector space V, the domain, to a new vector space W, the co-domain.

The rules, or axioms, for linear transformations are

(1) A vector $\mathbf{u}$ in V is transformed to a unique vector $T(\mathbf{u})$ in W.
(2) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$.
(3) $T(k\mathbf{u}) = kT(\mathbf{u})$.

For example, a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ might have the effect

$$T(\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}) = \begin{pmatrix} 2a_1 \\ -a_1 + a_3 \\ a_2 - a_3 \end{pmatrix}$$

and here T could be represented by the matrix

$$T = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \qquad \text{as} \begin{pmatrix} 2 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 2a_1 \\ -a_1 + a_3 \\ a_2 - a_3 \end{pmatrix}$$

Linear transformations do not always have to be between vector spaces with the same number of dimensions. So, for example, consider a transformation that takes all three dimensional vectors and squashes them on to a two dimensional plane ($T : \mathbb{R}^3 \to \mathbb{R}^2$);

$$T(\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}) = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Here T could be represented by the matrix

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \text{as} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

3.9. **Isomorphisms between vector spaces .** An invertible linear transformation is said to be an isomorphism. Invertible here means reversible. If the transformation is reversible then there is a transformation T^{-1} for which

$$T^{-1}(T(\mathbf{u})) = \mathbf{u} \qquad \text{and} \quad T(T^{-1}(\mathbf{v})) = \mathbf{v}$$

For example, the transformation below from a maximum 2nd order polynomial to $\mathbb{R}^3$ is an isomorphism;

$$a_0 + a_1 x + a_2 x^2 \underset{T^{-1}}{\overset{T}{\longrightarrow}} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

Similarly, 2×2 matrices and $\mathbb{R}^4$ vectors are isomorphic;

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \underset{T^{-1}}{\overset{T}{\longrightarrow}} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$$

Finally, note that, of course, isomorphisms must be between vector spaces or the same dimension so that a transfromation from $T : \mathbb{R}^3 \to \mathbb{R}^2$ is not invertible. The example above for $T : \mathbb{R}^3 \to \mathbb{R}^3$ is invertible;

$$T = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \qquad T^{-1} = \begin{pmatrix} 0.5 & 0 & 0 \\ 0.5 & 1 & 1 \\ 0.5 & 1 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \underset{T^{-1}}{\overset{T}{\longrightarrow}} \begin{pmatrix} 2a_1 \\ -a_1 + a_3 \\ a_2 - a_3 \end{pmatrix}$$

3.10. **The linear superposition principle.** A consequence of these axioms for linear transformations in Section 3.8 above is that

$$T(k\mathbf{u} + m\mathbf{v}) = kT(\mathbf{u}) + mT(\mathbf{v}).$$

and more generally

$$T(k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + ... + k_n\mathbf{u}_n) = k_1T(\mathbf{u}_1) + k_2T(\mathbf{u}_2) + ... + k_nT(\mathbf{u}_n)$$

which is the principle of linear superposition.

The principle of linear superposition is very important in physics. For example if two or more waves interfere with each other we add up the contribution of each wave to calculate the net interference effect, which may be constructive inteference, destructive intereference, or something in between. We add up the contribution of each wave because of the linear superposition principle.

The principle of linear superposition is very useful in many areas of physics, but it can break down. For example, at high light intensities non-linear effects can be exploited to double and triple the frequency of light. Such work with high intesity light is often described as 'non-linear optics'.

4. Inner Products and Inner Product Spaces

(return to contents)

The dot product for column vectors is familiar to us;

$$\mathbf{u}\cdot\mathbf{v}=\begin{pmatrix}u_1\\u_2\\\vdots\\u_n\end{pmatrix}\cdot\begin{pmatrix}v_1\\v_2\\\vdots\\v_n\end{pmatrix}=u_1v_1+u_2v_2+\ldots+u_nv_n$$

In the dot product takes two column vectors with the same dimension and gives a scalar result. The inner product for vector spaces is similar, but generalized so that it can be applied to different types of vector spaces, such as functions and matrices. For two vectors $\mathbf{u}$ and $\mathbf{v}$ in a vector space V the inner product is written as $\langle\mathbf{u},\mathbf{v}\rangle$.

If a vector space has an inner product which satisfies the inner product axioms then it is named an 'Inner Product Space'

4.1. **Inner product axioms.** For all vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ and scalars k in a vector space V

(1) $\langle\mathbf{u},\mathbf{v}\rangle$ gives a unique scalar value.
(2) $\langle\mathbf{u},\mathbf{v}\rangle=\langle\mathbf{v},\mathbf{u}\rangle^*$ where * means complex conjugate.
Thus, if $\langle\mathbf{u},\mathbf{v}\rangle$ is real $\langle\mathbf{u},\mathbf{v}\rangle=\langle\mathbf{v},\mathbf{u}\rangle$.
(3) $\langle\mathbf{u}+\mathbf{v},\mathbf{w}\rangle=\langle\mathbf{u},\mathbf{w}\rangle+\langle\mathbf{v},\mathbf{w}\rangle$.
(4) $\langle\mathbf{k}u,\mathbf{v}\rangle=k\langle\mathbf{u},\mathbf{v}\rangle$
(5) $\langle\mathbf{u},\mathbf{u}\rangle\geq 0$
(6) $\langle\mathbf{u},\mathbf{u}\rangle=0\iff\mathbf{u}=\mathbf{0}$

Note that axioms 5 and 6 mean that the inner product of a vector with itself must be positive except the zero vector has an inner product with itself of zero.

The dot product is the inner product for column vectors and satisfies all the axioms above.

4.2. **Norm and length.** The norm of a vector, $||\mathbf{u}||$, can be thought of as its length. The norm is defined by

$$||\mathbf{u}||=\sqrt{\langle\mathbf{u},\mathbf{u}\rangle}$$

Thus for a vector $\mathbf{u}$ in $\mathbb{R}^n$ the norm, or length, of $\mathbf{u}$ is given by

$$||\mathbf{u}||=\sqrt{u_1^2+u_2^2+\ldots+u_n^2}$$

which is Pythagoras' theorem generalized to $n-$dimensions.

4.3. **Axioms of the norm.**

(1) $||\mathbf{u}||\geq 0$
(2) $||\mathbf{u}||=0\iff\mathbf{u}=\mathbf{0}$
(3) $||k\mathbf{u}||=k||\mathbf{u}||$
(4) $||\mathbf{u}+\mathbf{v}||\leq||\mathbf{u}||+||\mathbf{v}||$

Axioms 1,2 and 3 are direct consequences of axioms for the inner product. Axiom 4 is the 'Triangle Inequality'.

4.4. **Normalization of vectors .** If a vector, $\mathbf{u}$ is multiplied by a scalar equal to one divided by its norm, $1/||\mathbf{u}||$, then the resulting vector will have unit length and point in the direction of $\mathbf{u}$.

The process of making a vector have unit length is 'Normalization' and normalized vectors are sometimes identified with a 'hat'; for example $\hat{\mathbf{u}}$. Thus,

$$\hat{\mathbf{u}} = \frac{1}{||\mathbf{u}||}\mathbf{u}$$

4.5. **Norm angle and distance.** The distance between two vectors $\mathbf{u}$ and $\mathbf{v}$ is written as $d(\mathbf{u}, \mathbf{v})$ and defined by

$$d(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}|| = \sqrt{\langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle}$$

The angle, θ, between two vectors, $\mathbf{u}$ and $\mathbf{v}$, is defined by

$$\begin{aligned} \langle \mathbf{u}, \mathbf{v} \rangle &= ||\mathbf{u}||\ ||\mathbf{v}|| \cos\theta \\ \theta &= \cos^{-1}\left(\frac{\langle \mathbf{u}, \mathbf{v} \rangle}{||\mathbf{u}||\ ||\mathbf{v}||} \right) \end{aligned}$$

Note that if the inner product between two vectors is equal to zero, $\langle \mathbf{u}, \mathbf{v} \rangle = 0$, the the angle between the two vectors is equal to $\frac{\pi}{2}$ or $90°$. We say that two vectors with an angle of $90°$ between them are 'Orthogonal'. Orthogonality is an important concept in linear algebra and we will be considering it in more detail in Section 5, for example.

Note that we can prove Pythagoras' theorem for orthogonal vectors.

$$\begin{aligned} ||\mathbf{u} + \mathbf{v}||^2 &= \langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle \\ &= \langle \mathbf{u}, \mathbf{u} \rangle + 2\langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle \\ &= ||\mathbf{u}||^2 + ||\mathbf{v}||^2 \end{aligned}$$

Note that to get to the final line of the proof above we rely on $\langle \mathbf{u}, \mathbf{v} \rangle = 0$, which is true for orthogonal vectors.

4.6. **Inner product calculation examples.** We have seen that the inner product is used to calculate the norm and hence to measure the length of vectors, to normalize vectors, and to measure the angle and distance between vectors.

Now we will look at how the inner product is calculated for different vector spaces.

4.6.1. *Inner product calculation:* $\mathbb{R}^n$. For column vectors in $\mathbb{R}^n$ the inner product can be calculated with the 'dot product' described Section 4, which should be familiar to us.

$$\mathbf{u}\cdot\mathbf{v} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = u_1v_1 + u_2v_2 + \ldots + u_nv_n$$

The inner product can also be redefined to be a weighted dot product:

$$\mathbf{u}\cdot\mathbf{v} = z_1u_1v_1 + z_2u_2v_2 + \ldots + z_nu_nv_n$$

here the z_i terms are the weights; the weights z_i are positive real numbers. So, for example, in $\mathbb{R}^3$, an example of a weighted inner product is

$$\mathbf{u}\cdot\mathbf{v} = 2u_1v_1 + 4u_2v_2 + u_3v_3$$

Note that by weighting the terms in the inner product the lengths and angles will, in general, change.

4.6.2. *Inner product calculation: matrices.* The inner product between two matrices $\mathbf{a}$ and $\mathbf{b}$ is calculated with

$$\langle \mathbf{a}, \mathbf{b}\rangle = \mathrm{Tr}(\mathbf{a}^{\mathrm{T}}\mathbf{b})$$

Thus, the inner product is the trace of the transpose of matrix $\mathbf{a}$ multiplied by matrix $\mathbf{b}$. For example, for 2×2 matrices $\mathbf{a}$ and $\mathbf{b}$

$$\mathbf{a} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$$

$$\langle \mathbf{a}, \mathbf{b}\rangle = \mathrm{Tr}(\mathbf{a}^{\mathrm{T}}\mathbf{b}) = \mathrm{Tr}\begin{pmatrix} a_1 & a_3 \\ a_2 & a_4 \end{pmatrix}\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \mathrm{Tr}\begin{pmatrix} a_1b_1 + a_3b_3 & a_1b_2 + a_3b_4 \\ a_2b_1 + a_4b_3 & a_2b_2 + a_4b_4 \end{pmatrix}$$
$$= a_1b_1 + a_3b_3 + a_2b_2 + a_4b_4$$

Note that if we had transformed these 2×2 matrices into column vectors in $\mathbb{R}^4$ using the isomorphism in Section 3.9 we would have got exactly the same result by finding the inner product of the isomorphic vectors in $\mathbb{R}^4$. The equivalent calculation in $\mathbb{R}^4$ is;

$$\langle \mathbf{a}, \mathbf{b}\rangle = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4$$

4.6.3. *Inner product calculation: polynomials up to order n.* For polynomial functions up to order n there are several different ways to calculate the inner product.

Given general polynomials $\mathbf{a} = a(x)$ and $\mathbf{b} = b(x)$;

$$\mathbf{a} = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n \qquad \mathbf{b} = b_0 + b_1x + b_2x^2 + \ldots + b_nx^n$$

the inner product can be calculated with

$$\langle \mathbf{a}, \mathbf{b} \rangle = a_0 b_0 + a_1 b_1 + a_2 b_2 + \ldots + a_n b_n$$

This method of calculating the inner product is equivalent to calculating the inner product of the isomorphic column vector in $\mathbb{R}^n$.

A second method for calculating the inner product of polynomials is the method described below for real-valued functions in Section 4.6.4. This method is more general than just for polynomials so it is described there for polynomials and other functions.

4.6.4. *Inner product calculation: for real-valued continuous functions .* Recall that $F[a, b]$ is the set of functions, such as $\mathbf{f} = f(x)$ and $\mathbf{g} = g(x)$ which give real values for in the range $[a, b]$ as described in Section 3.6. A common way of defining the inner product for $F[a, b]$ is

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_a^b f(x) g(x) dx$$

This inner product method can be weighted with a 'weighting function', $w(x)$, where $w(x) \geq 0$ for all x in the range $[a, b]$. As with the weighting described in Section 4.6.1, weighting will, of course, change distances, angles and orthogonality by modifying the inner product. Thus, two functions which are orthogonal when there is no weighting will not normally be orthogonal if a weighting function is added to the inner product.

Note that we can calculate the distance between two functions with

$$d((\mathbf{f} - \mathbf{g}), (\mathbf{f} - \mathbf{g})) = ||\mathbf{f} - \mathbf{g}|| = \sqrt{\langle \mathbf{f} - \mathbf{g}, \mathbf{f} - \mathbf{g} \rangle} \quad = \sqrt{\int_a^b (f(x) - g(x))^2 dx}$$

4.6.5. *Inner product calculation: for complex continuous functions .* If $\mathbf{y} = y(x)$ and $\mathbf{z} = z(x)$ are members of $C[a, b]$, the set of functions which give complex values in the range $x = [a, b]$ then then inner product from Section 4.6.4 needs to be modified to

$$\langle \mathbf{y}, \mathbf{z} \rangle = \int_a^b y(x)^* z(x) dx$$

This modification is required because of the axiom that $\langle \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{z}, \mathbf{y} \rangle^*$.

Note that in quantum mechanics the integral

$$\langle \psi | \psi \rangle = \int_{all-space} \psi^* \psi d\tau$$

is used to normalize the wavefunction, and in Dirac notation this integral may be written as $\langle \psi | \psi \rangle$.

5. Orthogonality, Basis and Fitting

(return to contents)

The inner product and norm allow us to measure angles and distances between vectors. In this section we will use these tools to develop basis sets and see how we they can be used to describe vectors. This work on basis sets will lead us to methods to fit functions, which is the foundation of the Fourier analysis.

5.1. **Basis sets of vectors .** A set of vectors, $B = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$, in a vector space, V, is a basis set if every vector, u, in V can be represented in a unique way by a linear sum of the vectors in B.

$$\mathbf{u} = k_1\mathbf{b}_1 + k_2\mathbf{b}_2 + \ldots + k_n\mathbf{b}_n = \sum_{i=1}^{n} k_i\mathbf{b}_i$$

There are two axioms for basis sets;

(1) *Linear independence:* The vectors in B are linearly independent, which implies that

$$k_1\mathbf{b}_1 + k_2\mathbf{b}_2 + \ldots + k_n\mathbf{b}_n = \mathbf{0} \qquad \Rightarrow \quad k_1 = k_2 = \ldots = k_n = 0$$

Thus, the only linear sum of basis vectors that gives the zero vector is a sum of the vectors where all the scalar coefficients k_i are zero.

(2) *Span the vector space:* As noted above for all vectors $\mathbf{u}$ in V

$$\mathbf{u} = \sum_{i=1}^{n} k_i\mathbf{b}_i$$

Note the the dimensionality of a vector space, V, is equal to the number of vectors in a basis set of V. So, for example, the basis set of a vector space in $\mathbb{R}^n$ will have n vectors and we say that it is n-dimensional. that used to describe any vector in a vector space.

So, for example, the normal basis set in $\mathbb{R}^3$ is $\{\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}\}$ where

$$\hat{\mathbf{i}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \hat{\mathbf{j}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \hat{\mathbf{k}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Now if $\mathbf{a}$ is a general vector in $\mathbb{R}^3$ it can be expressed as a sum of the vectors in the basis set; $\hat{\mathbf{i}}$,$\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$;

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$$

Generally the vectors in a basis set are chosen to be **orthogonal** and **normalized**, as can be seen for the basis set $\{\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}\}$. One reason for choosing an orthogonal and normalized basis set can be seen if we measure the

length of the vector $\mathbf{a}$ above by finding its norm. First note that because they are normalized

$$\langle\hat{\mathbf{i}},\hat{\mathbf{i}}\rangle = \langle\hat{\mathbf{j}},\hat{\mathbf{j}}\rangle = \langle\hat{\mathbf{k}},\hat{\mathbf{k}}\rangle = 1$$

and because they are orthogonal

$$\langle\hat{\mathbf{i}},\hat{\mathbf{j}}\rangle = \langle\hat{\mathbf{i}},\hat{\mathbf{k}}\rangle = \langle\hat{\mathbf{j}},\hat{\mathbf{k}}\rangle = 0$$

so

$$\begin{aligned}
||\mathbf{a}|| &= \sqrt{\langle\mathbf{a},\mathbf{a}\rangle} \\
&= \sqrt{\langle(a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}),(a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}})\rangle} \\
&= \sqrt{a_1a_1\langle\hat{\mathbf{i}},\hat{\mathbf{i}}\rangle + a_1a_2\langle\hat{\mathbf{i}},\hat{\mathbf{j}}\rangle + \ldots + a_2a_2\langle\hat{\mathbf{j}},\hat{\mathbf{j}}\rangle + \ldots + a_3a_3\langle\hat{\mathbf{k}},\hat{\mathbf{k}}\rangle} \\
&= \sqrt{a_1^2 + a_2^2 + a_3^2}
\end{aligned}$$

Above the dots indicate other terms with the inner product between two different basis set vectors which go to zero because the basis set vectors are orthogonal to each other.

Finally, orthogonal and normalized basis sets are often described as **orthonormal**.

5.2. **Linear dependence.** If a set of vectors are 'linearly dependent' then at least one of the vectors can be constructed by addition and subtraction of mulitples of the others vectors. Thus, a set of vectors $\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_n\}$ is linearly dependent if for one or more of the vectors $\mathbf{v}_i$

$$\mathbf{v}_i = \sum_{j=1\ (j\neq i)}^{n} k_j\mathbf{v}_j$$

Basis sets are 'linearly independent' and have just enough vectors to span a vectors space.

5.3. **Gram-Schmidt orthogonalization process.** If we have set of linearly independent vectors $\{\mathbf{w}_1,\mathbf{w}_2,\ldots,\mathbf{w}_n\}$ which span a vector space we can use the Gram-Schmidt orthoganlization process to convert them into an *orthogonal* set of vectors $\{\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_n\}$. The inner product is key here because the inner product between two orthogonal vectors is zero.

The first vector is trivial;

$$\mathbf{p}_1 = \mathbf{w}_1$$

The second vector $\mathbf{p}_2$ then needs to be orthogonal to p_1 and we make it from $\mathbf{w}_2$ and a scaled contribution of $\mathbf{p}_1$.

$$\mathbf{p}_2 = \mathbf{w}_2 + m_1\mathbf{p}_1$$

where m_1 is a scalar chosen so that $\mathbf{p}_1$ and $\mathbf{p}_2$ are orthogonal. The orthogonality means, of course, that $\langle \mathbf{p}_1, \mathbf{p}_2 \rangle = 0$ so

$$\begin{aligned} 0 &= \langle \mathbf{p}_1, \mathbf{p}_2 \rangle \\ &= \langle \mathbf{p}_1, (\mathbf{w}_2 + m_1 \mathbf{p}_1) \rangle \\ &= \langle \mathbf{p}_1, \mathbf{w}_2 \rangle + m_2 \langle \mathbf{p}_1, \mathbf{p}_1 \rangle \\ \Rightarrow \; m_2 &= -\frac{\langle \mathbf{p}_1, \mathbf{w}_2 \rangle}{\langle \mathbf{p}_1, \mathbf{p}_1 \rangle} = -\frac{\langle \mathbf{p}_1, \mathbf{w}_2 \rangle}{||\mathbf{p}_1||^2} \end{aligned}$$

Thus, we can calculate $\mathbf{p}_2$ with

$$\mathbf{p}_2 = \mathbf{w}_2 - \frac{\langle \mathbf{p}_1, \mathbf{w}_2 \rangle}{||\mathbf{p}_1||^2} \mathbf{p}_1$$

similarly for $\mathbf{p}_3$

$$\mathbf{p}_3 = \mathbf{w}_3 + m_1 \mathbf{p}_1 + m_2 \mathbf{p}_2$$

and

$$\mathbf{p}_3 = \mathbf{w}_3 - \frac{\langle \mathbf{p}_1, \mathbf{w}_3 \rangle}{||\mathbf{p}_1||^2} \mathbf{p}_1 - \frac{\langle \mathbf{p}_2, \mathbf{w}_3 \rangle}{||\mathbf{p}_2||^2} \mathbf{p}_2$$

The general case for $\mathbf{p}_n$ is given by

$$\mathbf{p}_n = \mathbf{w}_n - \sum_{i=1}^{n-1} \frac{\langle \mathbf{p}_i, \mathbf{w}_n \rangle}{||\mathbf{p}_i||^2} \mathbf{p}_i \tag{5.1}$$

Note that, once an orthogonal basis set has been generated, each vector can be normalised with the method in Section 4.4 to give an orthonormal basis set.

5.4. **Examples of Gram-Schmidt orthogonalization.**

5.4.1. *Orthogonalization of vectors in* $\mathbb{R}^3$. Given the set of vectors

$$\mathbf{w}_1 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \qquad \mathbf{w}_2 = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} \qquad \mathbf{w}_3 = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$$

following the method of Equation 5.1 we find $\mathbf{p}_1$ and $\mathbf{p}_2$

$$\mathbf{p}_1 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \qquad \mathbf{p}_2 = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} - \frac{(4+2+4)}{(1+9)} \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$$

Now the same method can be applied to find $\mathbf{p}_3$

$$\mathbf{p}_3 = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} - \frac{(3+0+0)}{(1+9)} \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} - \frac{(9+2+0)}{(9+1+1)} \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3-0.3-3 \\ 2-0-1 \\ 0-.9--1 \end{pmatrix} = \begin{pmatrix} -0.3 \\ 1 \\ 0.1 \end{pmatrix}$$

An important note here is that the vector $\mathbf{p}_3$ can be taken to be

$$\mathbf{p}_3 = \begin{pmatrix} -3 \\ 10 \\ 1 \end{pmatrix}$$

by multiplication of the result of the calculation by 10. The requirement for the set of orthogonal vectors is that they are orthogonal to each other and multiplication by a scalar does not change the direction or the orthogonality of a vector. When an orthonormal set of vectors are being generated, of course, the vectors must all have length 1 so the results cannot be multiplied by scalars.

we can check the result by checking the orthogonality of $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$;

5.4.2. *Orthogonalization of polynomial in range* $[-1, 1]$. First we need to define the inner product, $\langle \mathbf{f}, \mathbf{g} \rangle$, in the vector space of polynomial functions over the range $[-1, 1]$.

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_{-1}^{1} f(x)g(x)dx$$

Now given a set of initial vectors

$$\mathbf{w}_1 = 1 \qquad \mathbf{w}_2 = x \qquad \mathbf{w}_3 = x^2$$

the orthogonalized vectors can be found with Equation 5.1.

$$\begin{aligned} \mathbf{p}_1 &= 1 \\ \mathbf{p}_2 &= x - \frac{\int_{-1}^{1} x dx}{\int_{-1}^{1} dx} 1 = x - \frac{[x^2/2]_{-1}^{1}}{[x]_{-1}^{1}} 1 \\ &= x - \frac{1}{2}\frac{(1-1)}{(1--1)} 1 = x - 0 = x \end{aligned}$$

Now the same method can be applied to find $\mathbf{p}_3$

$$\begin{aligned} \mathbf{p}_3 &= x^2 - \frac{\int_{-1}^{1} x^2 dx}{\int_{-1}^{1} dx} 1 - \frac{\int_{-1}^{1} x^3 dx}{\int_{-1}^{1} x^2 dx} x \\ &= x^2 - \frac{[x^3/3]_{-1}^{1}}{[x]_{-1}^{1}} 1 - \frac{[x^4/4]_{-1}^{1}}{[x^3/3]_{-1}^{1}} x \\ &= x^2 - \frac{1}{3}\frac{(1--1)}{(1--1)} 1 - \frac{3}{4}\frac{(1-1)}{(1--1)} x \\ &= x^2 - \frac{1}{3} \end{aligned}$$

so to summarize in the range $[-1, 1]$ an orthogonal set of vectors in P_2 is given by

$$\mathbf{p}_1 = 1 \qquad \mathbf{p}_2 = x \qquad \mathbf{p}_3 = x^2 - \frac{1}{3}$$

These are, in fact, the first three Legendre polynomials.

5.5. **Projection onto an orthogonal basis set .** Above in Section 5.1 we saw how an arbitrary vector in $\mathbb{R}^3$ could be expressed as a sum of the orthonormal basis set vectors $\{\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}\}$. Here, we will consider how to project an arbirtrary vector, $\mathbf{a}$, onto an orthogonal basis set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$. The process is similar to Gram Schmidt orthogonalization described in Section 5.3. What we want to do here is to find the scalar constants $k_1, k_2, \ldots, k_n$ such that.

$$\mathbf{a} = k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + \ldots + k_n\mathbf{u}_n$$

Now if we calculate the inner product of $\mathbf{a}$ with each of the basis vectors $\mathbf{u}_i$ each of the scalars k_i can be determined.

$$\begin{aligned}\langle \mathbf{a}, \mathbf{u}_i\rangle &= \langle (k_1\mathbf{u}_1 + \ldots + k_i\mathbf{u}_i + \ldots + k_n\mathbf{u}_n), \mathbf{u}_i\rangle \\ &= k_1\langle \mathbf{u}_1, \mathbf{u}_i\rangle + \ldots + k_i\langle \mathbf{u}_i, \mathbf{u}_i\rangle + \ldots k_n\langle \mathbf{u}_n, \mathbf{u}_i\rangle \\ &= k_i\langle \mathbf{u}_i, \mathbf{u}_i\rangle \\ \Rightarrow\ k_i &= \frac{\langle \mathbf{a}, \mathbf{u}_i\rangle}{\langle \mathbf{u}_i, \mathbf{u}_i\rangle} = \frac{\langle \mathbf{a}, \mathbf{u}_i\rangle}{||\mathbf{u}_i||^2}\end{aligned}$$

Note that above because the basis set is orthogonal all the inner products between different basis set vectors are zero, which can be expressed as $\langle \mathbf{u}_i, \mathbf{u}_j\rangle = 0$ if $i \neq j$. Thus

$$\mathbf{a} = \frac{\langle \mathbf{a}, \mathbf{u}_1\rangle}{||\mathbf{u}_1||^2}\mathbf{u}_1 + \frac{\langle \mathbf{a}, \mathbf{u}_2\rangle}{||\mathbf{u}_2||^2}\mathbf{u}_2 + \ldots + \frac{\langle \mathbf{a}, \mathbf{u}_n\rangle}{||\mathbf{u}_n||^2}\mathbf{u}_n \tag{5.2}$$

If the basis set is orthonormal, $\{\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, \ldots, \hat{\mathbf{u}}_n\}$, then $||\mathbf{u}_i|| = 1$ so the equation above simplifies to

$$\mathbf{a} = \langle \mathbf{a}, \hat{\mathbf{u}}_1\rangle\hat{\mathbf{u}}_1 + \langle \mathbf{a}, \hat{\mathbf{u}}_2\rangle\hat{\mathbf{u}}_2 + \ldots + \langle \mathbf{a}, \hat{\mathbf{u}}_n\rangle\hat{\mathbf{u}}_n$$

Finally, it is worth noting that in this process we find an exact representation of the vector $\mathbf{a}$ in terms of the basis vectors $\mathbf{u}_i$ because the basis vectors span the vector space and every vector can be represented as a linear sum of basis set vectors.

In the next section we consider what happens if, by contrast, the set of orthogonal vectors that we project a vector $\mathbf{a}$ onto does not span the vector space.

5.6. **Fitting: The Best Approximation Theorem.** The 'Best Approximation Theorem' is a way to find approximations to vectors. If we have a vector $\mathbf{u}$ in a vector space V then we might want to find an approximation to $\mathbf{u}$ in W, a vector subspace of V. The best approximation to $\mathbf{u}$ in the vector subspace W is written $\hat{\mathbf{u}}$. *Note here that the notation is confusing.* In this context $\hat{\mathbf{u}}$ is the closest vector to $\mathbf{u}$ in the vector subspace, but elsewhere $\hat{\mathbf{u}}$ is a unit vector.

For example, V could be the vector space $\mathbb{R}^3$ of three dimensional position vectors in x, y and z. The vector subspace could be the $\mathbb{R}^2$ subspace of

vectors in the xy plane. If we have a vector $\mathbf{p}$ in V then $\hat{\mathbf{u}}$ would be the vector in W which is closest to $\mathbf{p}$.

The 'Best Approximation Theorem' states that if we have a vector $\mathbf{u}$ in a vector space V then the closest approximation, or closest vector, to $\mathbf{u}$ in a vector subspace W is given by $\hat{\mathbf{u}}$, which is the orthogonal projection of $\mathbf{u}$ onto W. In Section 5.5 we saw how to project a vector onto a set of orthogonal basis vectors. Exactly the same method can be used here. The vector to approximate is projected onto an orthogonal basis set $\{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n\}$ of the subspace W.

$$\hat{\mathbf{u}} = \frac{\langle \mathbf{u}, \mathbf{p}_1 \rangle}{||\mathbf{p}_1||^2}\mathbf{p}_1 + \frac{\langle \mathbf{u}, \mathbf{p}_2 \rangle}{||\mathbf{p}_2||^2}\mathbf{p}_2 + \ldots + \frac{\langle \mathbf{u}, \mathbf{p}_n \rangle}{||\mathbf{p}_n||^2}\mathbf{p}_n$$

The same method can be applied to find the best fit to a function. If we have a function $f(x)$ in the vector space of all real valued functions in the range $a \leq x \leq b$, $F[a, b]$, then we might want to approximate $f(x)$ by a polynomial up to degree n in the same range $P_n[a, b]$. Another application of this method is finding the best fit of a function to a series of experimental data points.

5.6.1. *Fitting:$f(x) = e^x$ with a polynomial for $-1 \leq x \leq 1$.* To fit the function $\mathbf{f} = f(x) = e^x$ with a quadratic polynomial in the x range $[-1, 1]$ we can use the orthogonal functions derived above in Section 5.4.2; $\{\mathbf{p}_1 = 1, \mathbf{p}_2 = x, \mathbf{p}_3 = x^2 - \frac{1}{3}\}$

$$\begin{aligned}\hat{\mathbf{f}} &= \frac{\langle \mathbf{f}, \mathbf{p}_1 \rangle}{||\mathbf{p}_1||^2}\mathbf{p}_1 + \frac{\langle \mathbf{f}, \mathbf{p}_2 \rangle}{||\mathbf{p}_2||^2}\mathbf{p}_2 + \frac{\langle \mathbf{f}, \mathbf{p}_3 \rangle}{||\mathbf{p}_3||^2}\mathbf{p}_3 \\ &= \frac{\int_{-1}^{1} e^x dx}{\int_{-1}^{1} dx}1 + \frac{\int_{-1}^{1} xe^x dx}{\int_{-1}^{1} x^2 dx}(x) + \frac{\int_{-1}^{1}(x^2 - 1/3)e^x dx}{\int_{-1}^{1}(x^2 - 1/3)^2 dx}(x^2 - \frac{1}{3})\end{aligned}$$

taking these integrals one by one;

$$\int_{-1}^{1} e^x dx = [e^x]_{-1}^{1} = e - \frac{1}{e} = 2.35$$

$$\int_{-1}^{1} dx = [x]_{-1}^{1} = 1 + 1 = 2$$

$$\int_{-1}^{1} xe^x dx = [xe^x]_{-1}^{1} - \int_{-1}^{1} e^x dx = [xe^x - e^x]_{-1}^{1} = 0 + \frac{2}{e} = 0.736$$

$$\int_{-1}^{1} x^2 dx = [x^3/3]_{-1}^{1} = 1/3 + 1/3 = 2/3 = 0.667$$

$$\int_{-1}^{1} (x^2 - 1/3)e^x dx = \int_{-1}^{1} x^2 e^x dx - \frac{1}{3}\int_{-1}^{1} e^x dx = \int_{-1}^{1} x^2 e^x dx - \frac{2.35}{3}$$

$$\begin{aligned}\int_{-1}^{1} x^2 e^x dx &= [x^2 e^x]_{-1}^{1} - \int_{-1}^{1} 2xe^x dx = [x^2 e^x - 2xe^x]_{-1}^{1} + \int_{-1}^{1} 2e^x dx \\ &= [x^2 e^x - 2xe^x + 2e^x]_{-1}^{1} = [(x^2 - 2x + 2)e^x]_{-1}^{1} \\ &= e - \frac{5}{e} = 0.879\end{aligned}$$

$$\begin{aligned}\int_{-1}^{1} (x^2 - 1/3)^2 dx &= \int_{-1}^{1} x^4 - 2x^2/3 + 1/9 dx = [x^5/5 - 2x^3/9 + x/9]_{-1}^{1} \\ &= \frac{2}{5} - \frac{4}{9} + \frac{2}{9} = 0.178\end{aligned}$$

Now subsituting these numbers in for the integrals;

$$\begin{aligned}\hat{\mathbf{f}} &= \frac{2.35}{2}1 + \frac{0.736}{0.667}x + \frac{0.879 - 2.35/3}{0.178}(x^2 - \frac{1}{3}) \\ &= 1.18 + 1.10x + 0.537(x^2 - \frac{1}{3}) = 0.996 + 1.10x + 0.537x^2\end{aligned}$$

Thus in the range $[-1, 1]$ the best quadratic fit to e^x is $0.996+1.10x+0.537x^2$

5.6.2. *Meaning of the best fit.* The nature of the best fits obtained 'Best Approximation Theorem' depends completely on the inner product. So, for example, the fit obtained above for e^x was made with an inner product that gives fits by the method of least squares.

6. Fourier Series with real numbers

(return to contents)

Fourier series are introduced here. We should be familiar with Taylor and McLauren series where polynomials can be used to fit functions. Fourier series can also be used to fit function, but they are made up of sine and cosine functions. There are many ways to describe Fourier series. Here they are introduced using the 'language' of the linear algebra we have considered to this point.

6.1. **Hilbert Spaces.** A Hilbert space is a vector space with an inner product that allows length and angle to be calculated, and it is also 'complete'. The exploration of 'completeness' is beyond the present scope. An example of a 'set' that is not complete, however, is the set of rational numbers, which are numbers that can be defined by a/b where a and b are both integers. If the set of rational numbers was complete then all limits of rational number sequences would also be rational. The limit, however, of the sum $1/0! + 1/1! + 1/2! + 1/3! + 1/4! \ldots$ is the irrational number e, which cannot be represented as a rational number. One way of thinking about it is that the set of rational number has holes in it. If these holes are filled in by the addition of irrational numbers then we have the set of real numbers, which is complete.

Hilbert spaces are often assumed to have infinite dimension because most common examples of Hilbert space have infinite dimension, but in general Hilbert spaces are not necessarily infinitely dimensional. When a Hilbert space is infinite dimensional the inner product is always finite.

6.2. **Hilbert Space of $\mathbf{L}^2$ Periodic Functions.** An L^2 space is a space of functions, which are 'square integrable' over a particular range, S, which means that the integral is not infinite.

$$\langle \mathbf{f}, \mathbf{f} \rangle = \int_S f(x)^* f(x) dx < \infty$$

These functions form an infinite Hilbert space with the inner product $\langle \mathbf{f}, \mathbf{g} \rangle$. defined by

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_S f(x)^* g(x) dx$$

So for example if S is the range $[-\infty, \infty]$ for all the functions in L^2

$$\langle \mathbf{f}, \mathbf{f} \rangle = \int_{-\infty}^{\infty} f(x)^* f(x) dx < \infty$$

Here the set of functions that we are concerned with initially are square integrable in the range $[-\pi, \pi]$ and repeat so that $f(x + 2\pi) = f(x)$. Later we will be concerned with more general ranges $[-L, L]$ and repetition $f(x + 2L) = f(x)$

6.3. **Fourier Theorem / An Orthogonal Basis Set Of Sine And Cosine Functions For $\mathbf{L}^2$.** Fourier's theorem is generally stated as 'any periodic function can be represented exactly by a sum of single frequency waves'.

In the language of linear algebra we may state that the set of functions below form an orthogonal basis set of periodic L^2 functions where $f(x+2\pi) = f(x)$ and the inner product is defined in the x-range $[-\pi, \pi]$

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_{-\pi}^{\pi} f(x)^* g(x) dx$$

The orthogonal basis set has the function $\mathbf{C}_0$ and an infinite number of functions $\mathbf{C}_k$ and $\mathbf{S}_k$ where $k \in \mathbb{Z}$ and $1 \leq k < \infty$. The functions are

$$\mathbf{C}_0 = 1 \qquad \mathbf{C}_k = \cos(kx) \qquad \mathbf{S}_k = \sin(kx)$$

Now using the method of Section 5.5 we can project any periodic L^2 function $f(x)$ where $f(x + 2\pi) = f(x)$ onto these basis functions to find a linear sum functions that represent $f(x)$. With enough terms the function $f(x)$ should be fitted exactly.

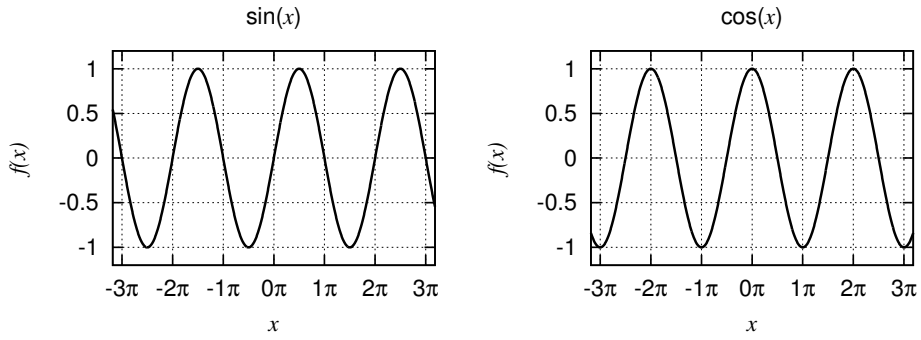

FIGURE 4. $\sin(x)$ and $cos(x)$ functions

6.4. **Sine And Cosine.** It is helpful to remember the form of the sine and cosine functions, which are shown in Figure 4. Note that

- $\cos(-A) = \cos(A)$, cosine functions are **even**.
- $\sin(-A) = -\sin(-A)$, sine functions are **odd**.
- for $k \in \mathbb{Z}$ note that $\sin(k\pi) = 0$ and $\cos(k\pi) = (-1)^k$

6.5. **Orthogonality Of The Basis Set Of Sine And Cosine Functions.** The orthogonality of the basis set $\{\mathbf{C}_0, \mathbf{C}_k, \mathbf{S}_k\}$ is demonstrated before projection of functions onto this basis set is considered. If these functions are orthogonal then the inner product between any two different basis functions

must be zero.

$$\langle \mathbf{C}_0, \mathbf{C}_k \rangle = \int_{-\pi}^{\pi} \cos(kx)dx = \frac{1}{k}\left[\sin(kx)\right]_{-\pi}^{\pi} = \frac{1}{k}(0-0) = 0$$

$$\text{Note that} \quad \sin(0) = \sin(k\pi) = 0 \quad (k \in \mathbb{Z})$$

$$\langle \mathbf{C}_0, \mathbf{S}_k \rangle = \int_{-\pi}^{\pi} \sin(kx)dx = -\frac{1}{k}\left[\cos(kx)\right]_{-\pi}^{\pi} = -\frac{1}{k}(\cos(k\pi) - \cos(-k\pi)) = 0$$

$$\text{Note that} \quad \cos(-a) = \cos(a) \quad \Rightarrow \quad \cos(k\pi) - \cos(-k\pi) = 0$$

Now we have shown that $\mathbf{C}_0$ is orthogonal to the other functions. Before continuing it is helpful to recall some trigonometric identities;

$$\cos(A)\cos(A) = \cos^2(A) = \frac{1}{2}(1 + \cos(2A))$$

$$\sin(A)\sin(A) = \sin^2(A) = \frac{1}{2}(1 - \cos(2A))$$

$$\sin(A)\cos(A) = \frac{1}{2}\sin(2A)$$

$$\cos(A)\cos(B) = \frac{1}{2}(\cos(A+B) + \cos(A-B))$$

$$\sin(A)\sin(B) = \frac{1}{2}(\cos(A-B) - \cos(A+B))$$

$$\sin(A)\cos(B) = \frac{1}{2}(\sin(A+B) + \sin(A-B))$$

So now for functions with different frequencies where $k \neq m$

$$\langle \mathbf{C}_k, \mathbf{C}_m \rangle = \int_{-\pi}^{\pi} \cos(kx)\cos(mx)dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos((k+m)x) + \cos((k-m)x)dx$$

$$= \left[\frac{\sin((k+m)x)}{k+m} + \frac{\sin((k-m)x)}{k-m}\right]_{-\pi}^{\pi} = 0$$

$$\langle \mathbf{S}_k, \mathbf{S}_m \rangle = \int_{-\pi}^{\pi} \sin(kx)\sin(mx)dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos((k-m)x) - \cos((k+m)x)dx$$

$$= \left[\frac{\sin((k-m)x)}{k-m} - \frac{\sin((k+m)x)}{k+m}\right]_{-\pi}^{\pi} = 0$$

$$\langle \mathbf{S}_k, \mathbf{C}_m \rangle = \int_{-\pi}^{\pi} \sin(kx)\cos(mx)dx = \frac{1}{2}\int_{-\pi}^{\pi} \sin((k+m)x) + \sin((k-m)x)dx$$

$$= \left[-\frac{\cos((k+m)x)}{k+m} - \frac{\cos((k-m)x)}{k-m}\right]_{-\pi}^{\pi} = 0$$

So the functions with different frequencies are orthogonal.

Now testing the orthogonality of $\mathbf{S}_k$ and $\mathbf{C}_k$, which have the same frequency.

$$\langle \mathbf{S}_k, \mathbf{C}_k \rangle = \int_{-\pi}^{\pi} \sin(kx)\cos(kx)dx = \frac{1}{2}\int_{-\pi}^{\pi} \sin(2kx)dx = \left[-\frac{\cos(2kx)}{2k}\right]_{-\pi}^{\pi} = 0$$

Thus all the basis functions are orthogonal to each other.

Finally, it is helpful here to find the inner product of each basis function with itself.

$$\langle \mathbf{C}_0, \mathbf{C}_0 \rangle = \int_{-\pi}^{\pi} dx = [x]_{-\pi}^{\pi} = \pi + \pi = 2\pi \qquad \Rightarrow ||\mathbf{C}_0||^2 = 2\pi$$

$$\begin{aligned}\langle \mathbf{C}_k, \mathbf{C}_k \rangle &= \int_{-\pi}^{\pi} \cos^2(kx)dx = \frac{1}{2}\int_{-\pi}^{\pi}(1+\cos(2kx))dx \\ &= \frac{1}{2}\left[x + \frac{\sin(2kx)}{2k}\right]_{-\pi}^{\pi} = \frac{1}{2}(\pi+\pi) = \pi \qquad \Rightarrow ||\mathbf{C}_k||^2 = \pi\end{aligned}$$

$$\begin{aligned}\langle \mathbf{S}_k, \mathbf{S}_k \rangle &= \int_{-\pi}^{\pi} \sin^2(kx)dx = \frac{1}{2}\int_{-\pi}^{\pi}(1-\cos(2kx))dx \\ &= \frac{1}{2}\left[x - \frac{\sin(2kx)}{2k}\right]_{-\pi}^{\pi} = \frac{1}{2}(\pi+\pi) = \pi \qquad \Rightarrow ||\mathbf{S}_k||^2 = \pi\end{aligned}$$

At this point the basis functions could be normalized by dividing them by their norms, but that is not how we will proceed here.

6.6. **Calculation Of Fourrier Series.** Now periodic a function, $\mathbf{f} = f(x)$, can be projected onto the this orthogonal basis set to obtain the 'Fourier Series' which corresponds to $f(x)$.

To project $\mathbf{f} = f(x)$ onto the orthogonal basis set Equaiton 5.2 from Section 5.5 is used;

$$\mathbf{f} = \frac{\langle \mathbf{f}, \mathbf{C}_0 \rangle}{||\mathbf{C}_0||^2}\mathbf{C}_0 + \sum_{k=1}^{\infty}\frac{\langle \mathbf{f}, \mathbf{C}_k \rangle}{||\mathbf{C}_k||^2}\mathbf{C}_k + \sum_{k=1}^{\infty}\frac{\langle \mathbf{f}, \mathbf{S}_k \rangle}{||\mathbf{S}_k||^2}\mathbf{S}_k$$

If we put in the values of the norm squared for each function from above, $||\mathbf{C}_0||^2 = 2\pi$ and $||\mathbf{C}_k||^2 = ||\mathbf{S}_k||^2 = \pi$

$$\mathbf{f} = \frac{\langle \mathbf{f}, \mathbf{C}_0 \rangle}{2\pi}\mathbf{C}_0 + \sum_{k=1}^{\infty}\frac{\langle \mathbf{f}, \mathbf{C}_k \rangle}{\pi}\mathbf{C}_k + \sum_{k=1}^{\infty}\frac{\langle \mathbf{f}, \mathbf{S}_k \rangle}{\pi}\mathbf{S}_k$$

Now if we put the functions into the expression above we get the 'normal' equation for a Fourier series corresponding to $\mathbf{f} = f(x)$,

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + \sum_{k=1}^{\infty} b_k \sin(kx)$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx \qquad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx$$

In fact, there are several different ways of expressing this equation and in different books there are different conventions. Particular care should be taken in calculations with the a_0 because in this definition of $f(x)$ the constant term a_0 is divided by a factor of 2. Note, however, that we could

have defined a_0 differently and here an arbitrary convention is set which may be different elsewhere.

So to calculate a Fourier series for a function $f(x)$ it is necessary to perform a series of integrations to calculate the $\{a_0, a_k, b_k\}$ values. Let us consider these constants;

- a_0 is a constant term, which shifts the function above or below the x-axis. So if, by inspection, it is apparent that the function lies equally above and below the x-axis then the a_0 term must be equal to zero. Note that the other a_k and b_k terms add sine and cosine functions that are equally above and below the x-axis so any shift above or below the x-axis must be due to a_0.
- the a_k terms are coefficients for the **even** cosine terms.
- the b_k terms are coefficients for the **odd** sine terms.

Here, of course, 'even' means 'even function' for which $f(-x) = f(x)$ and 'odd' means 'odd function' for which $f(-x) = -f(x)$. Thus, if a function $f(x)$ is an even function then it must only have cosine functions in its Fourier series and all the b_k must be equal to zero. Similarly if $f(x)$ is odd then all the a_k terms must be equal to zero. It is, however, not always immediately obvious if a function is odd because if it is shifted above or below the x-axis by a non-zero a_0 term then it will not follow the definition that $f(-x) = -f(x)$. It is necessary, therefore, by inspection to imagine the x-axis running through the centre of the function and then decide if it is odd.

If a function is odd then we know by inspection that all the a_k terms must be zero, but we only know that some of b_k terms are non-zero. Similarly for even functions we know by inspection that all the b_k terms must be zero, but we only know that some of a_k terms are non-zero.

It is very useful to be able to use symmetry arguments to determine if some of the $\{a_0, a_k, b_k\}$ terms are zero without having to calculate them.

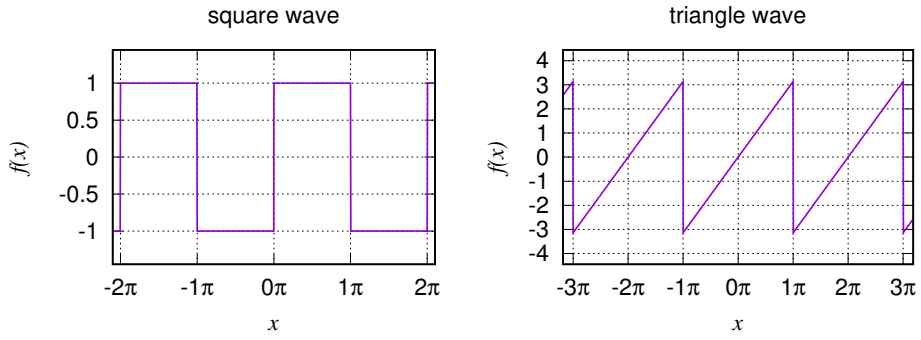

FIGURE 5. square wave and triangle or sawtooth functions

6.7. **Examples: Square Wave And Triangle/Sawtooth Functions.** Figure 5 shows square wave, $f(x)$, and triangle wave or sawtooth, $g(x)$,

functions, which are defined by

$$\begin{aligned} f(x) &= -1 \qquad -\pi < x < 0 \qquad g(x) = x \qquad -\pi < x < \pi \\ f(x) &= 1 \qquad\quad 0 < x < \pi \qquad g(x+2\pi) = g(x) \\ f(x+2\pi) &= f(x) \end{aligned}$$

6.7.1. *Square wave.* By inspection of the square wave, a_0 is zero as the function is equally above and below the x-axis. The function is odd so that all the a_k terms must be zero. Thus, we only need to calculate the b_k terms; The $\{b_k\}$ terms are given by

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx$$

Note that integrals can, of course, be split up so

$$\begin{aligned} b_k &= \frac{1}{\pi}\int_{-\pi}^{0} -\sin(kx)dx + \frac{1}{\pi}\int_{0}^{-\pi}\sin(kx)dx \\ &= \frac{1}{\pi}\left[\frac{1}{k}\cos(kx)\right]_{-\pi}^{0} - \frac{1}{\pi}\left[\frac{1}{k}\cos(kx)\right]_{0}^{\pi} = \frac{1}{k\pi}\Big((1-(-1)^k) - ((-1)^k - 1)\Big) \\ &= \begin{cases} \frac{4}{k\pi} & : \quad \text{(for } k \text{ is odd)} \\ 0 & : \quad \text{(for } k \text{ is even)} \end{cases} \end{aligned}$$

Thus with these b_k terms;

$$\begin{aligned} f(x) &= \frac{4}{\pi}\sum_{k=1,odd}^{\infty}\frac{\sin(kx)}{k} \\ &= \frac{4}{\pi}\left[\frac{\sin(x)}{1} + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \frac{\sin(7x)}{7} + \ldots\right] \end{aligned}$$

6.7.2. *Triangle/sawtooth wave.* By inspection of the triangular wave, a_0 is zero as the function is equally above and below the x-axis. The function is odd so that all the a_k terms must be zero. Thus, we only need to calculate the b_k terms;

$$\begin{aligned} b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(kx)dx \qquad \text{integrate by parts} \\ &= \frac{1}{\pi}\Big[-\frac{x}{k}\cos(kx)\Big]_{-\pi}^{\pi} + \frac{1}{\pi}\int_{-\pi}^{-\pi}\frac{\cos(kx)}{k}dx \\ &= \frac{1}{\pi}\left[-\frac{x}{k}\cos(kx) + \frac{1}{k^2}sin(kx)\right]_{-\pi}^{\pi} = \frac{1}{k\pi}\Big((-\pi(-1)^k + 0) - (\pi(-1)^k + 0)\Big) \\ &= \begin{cases} \frac{2}{k} & : \quad \text{(for } k \text{ is odd)} \\ \frac{-2}{k} & : \quad \text{(for } k \text{ is even)} \end{cases} \end{aligned}$$

Thus with these b_k terms;

$$\begin{aligned} f(x) &= -\frac{2}{\pi}\sum_{k=1}^{\infty}(-1)^k\frac{\sin(kx)}{k} \\ &= 2\left[\frac{\sin(x)}{1} - \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} - \frac{\sin(4x)}{4} + \dots\right] \end{aligned}$$

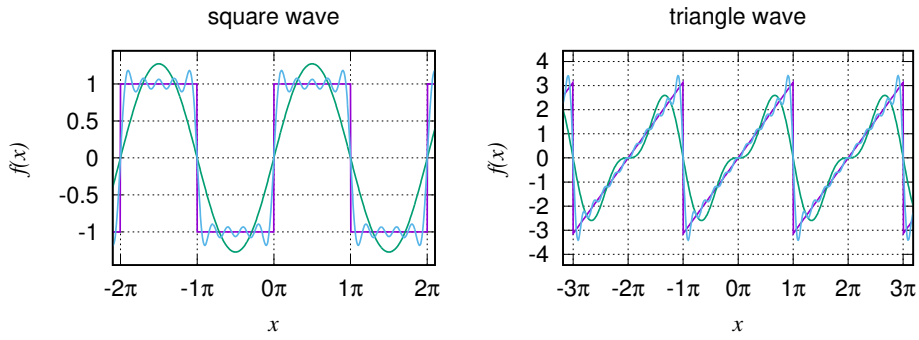

FIGURE 6. square wave and triangle or sawtooth functions. The Fourier series with terms up to and including $k = 2$ are shown in green and the Fourier series with terms up to and including $k = 10$ is shown in blue. The original functions are shown in purple.

6.7.3. *Fourier solutions.* Figure 6 shows the Fourier series calculated above for square and triangle waves with terms up to and including $k = 2$ and terms up to and including $k = 10$.

6.7.4. *Discontinuities in f(x).* Note that functions $f(x)$ in L^2 may have discontinuities. For example, consider the square wave shown above. The value of the Fourier series will normally converge halfway between the two limits at discontinuities. Thus expect

(1) $x = -\pi, 0, \pi : f(x)$ converge at 0

6.8. **Fourier Series For $[-L, L]$.** If a periodic function is defined by

$$\begin{aligned} f(x) \qquad\qquad & \text{for} \qquad -L \le x \le L \\ f(x+2L) &= f(x) \end{aligned}$$

we can still find a Fourier Series to fit it, but we need to modify the equations.

Note, of course, that the function has the range $-L \le x \le L$ and so the frequency of the sine and cosine functions needs to be modified

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(k\frac{\pi x}{L}\right) + \sum_{k=1}^{\infty} b_k \sin\left(k\frac{\pi x}{L}\right)$$

and the modified equations for $\{a_0, a_k, b_k\}$ are;

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx$$

$$a_k = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(k\frac{\pi x}{L}\right)dx \qquad b_k = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(k\frac{\pi x}{L}\right)dx$$

7. Fourier Series with complex numbers

(return to contents)

Fourier series can be made with sums of the function e^{ikx}. In this section we consider complex Fourier series, which are the next step towards the 'Fourier Transform', which we will consider in the next Section.

7.1. Complex Fourier Series For A Real Function $f(x)$.

We have seen above that a real function $f(x)$ in the range $[-\pi, \pi]$ can be represented with a Fourier series of sine and cosine functions;

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + \sum_{k=1}^{\infty} b_k \sin(kx)$$

Now to make a complex Fourier series we need to replace the sine and cosine terms with e^{ikx} terms.

7.1.1. *The relationship between $e^{i\theta}$ and sine and cosine functions.*

It is important to remember that

$$\begin{aligned} e^{i\theta} &= \cos(\theta) + i\sin(\theta) \\ e^{-i\theta} &= \cos(-\theta) + i\sin(-\theta) = \cos(\theta) - i\sin(\theta) \end{aligned}$$

and that if we combine terms of $e^{i\theta}$ and $e^{-i\theta}$ we can obtain $\cos(\theta)$ and $sin(\theta)$ functions;

$$\begin{aligned} \frac{1}{2}(e^{i\theta} + e^{-i\theta}) &= \frac{1}{2}\left(\cos(\theta) + i\sin(\theta) + \cos(-\theta) + i\sin(-\theta)\right) \\ &= \frac{1}{2}\left(\cos(\theta) + \cos(\theta) + i\sin(\theta) - i\sin(\theta)\right) = \frac{1}{2}2\cos(\theta) = \cos(\theta) \\ \frac{-i}{2}(e^{i\theta} - e^{-i\theta}) &= \frac{-i}{2}\left(\cos(\theta) + i\sin(\theta) - \cos(-\theta) - i\sin(-\theta)\right) \\ &= \frac{-i}{2}\left(\cos(\theta) - \cos(\theta) + i\sin(\theta) + i\sin(\theta)\right) = \frac{-i}{2}2i\sin(\theta) = \sin(\theta) \end{aligned}$$

Thus

$$\cos(\theta) = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) \qquad \sin(\theta) = \frac{-i}{2}(e^{i\theta} - e^{-i\theta})$$

Now, if have a complex number $c = \frac{1}{2}(a - ib)$ then

$$\begin{aligned} c &= \frac{1}{2}(a-ib) \qquad c^* = \frac{1}{2}(a+ib) \\ ce^{i\theta} + c^* e^{-i\theta} &= \frac{1}{2}(a-ib)e^{i\theta} + \frac{1}{2}(a+ib)e^{-i\theta} = \frac{1}{2}(ae^{i\theta} - ibe^{i\theta} + ae^{-i\theta} + ibe^{-i\theta}) \\ &= \frac{1}{2}\left(a(e^{i\theta} + e^{-i\theta}) - ib(e^{i\theta} - e^{-i\theta})\right) \\ &= a\frac{1}{2}(e^{i\theta} + e^{-i\theta}) + b\frac{-i}{2}(e^{i\theta} - e^{-i\theta}) \\ &= a\cos(\theta) + b\sin(\theta) \end{aligned}$$

Thus if we define c_k with

$$c_k = \frac{1}{2}(a_k - ib_k)$$

we can generate the sine and cosine terms of a Fourier Series with

$$\begin{aligned} c_k e^{ikx} + c_k^* e^{-ikx} &= \frac{1}{2}\left(a_k(e^{ikx} + e^{-ikx}) - ib_k(e^{ikx} - e^{-ikx})\right) \\ &= a_k\cos(kx) + b_k\sin(kx) \end{aligned}$$

7.1.2. *The complex Fourier series.* From the working above the equation for the Fourier Series that corresponds to a real function $f(x)$

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k\cos(kx) + \sum_{k=1}^{\infty} b_k\sin(kx)$$

is equivalent to

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left(c_k e^{ikx} + c_k^* e^{-ikx}\right)$$

and notice that if we apply the definition above for c_k to c_0 and take $b_0 = 0$ then

$$\begin{aligned} c_k &= \frac{1}{2}(a_k - ib_k) \qquad\qquad \Rightarrow \qquad c_0 = \frac{1}{2}(a_0 - ib_0) \\ \Rightarrow \quad c_0 &= \frac{a_0}{2} \end{aligned}$$

So that the Fourier series can be written as

$$f(x) = c_0 + \sum_{k=1}^{\infty}\left(c_k e^{ikx} + c_k^* e^{-ikx}\right)$$

If we define $c_{-k} = c_k^*$ then instead of summing k from 1 to ∞ with terms of e^{ikx} and e^{-ikx} we can instead have a summation for k between $-\infty$ and ∞ with a single term of e^{ikx} thus;

$$f(x) = c_0 + \sum_{k=-\infty, k\neq 0}^{\infty} c_k e^{ikx}$$

Finally, putting $k = 0$ into e^{ikx} gives $e^{i0x} = 1$ so the c_0 term can be included in the summation from $-\infty$ to ∞. Thus

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx} \qquad \text{when} \qquad f(x) \qquad \text{is a real valued function}$$

$$\text{where} \qquad c_k = \begin{cases} \frac{1}{2}(a_k - ib_k) & : \quad (\text{for } k \geq 1) \\ \frac{1}{2}a_0 & : \quad (\text{for } k = 0) \\ \frac{1}{2}(a_{|k|} + ib_{|k|}) & : \quad (\text{for } k \leq -1) \end{cases}$$

$$\text{and thus} \qquad c_{-k} = c_k^*$$

Thus to summarize we have found that a real function, $f(x)$, can be represented by an infinite sum of e^{ikx} terms with complex coefficients c_k. Note that the complex c_k coefficients contain the 'information' of a_k and b_k. In fact, the c_k coefficients may be easier to understand in some ways than the a_k and b_k coefficients because these complex c_k coefficients can be thought of as having **magnitude** and **phase**. If we consider that $[r, \theta]$ representation of complex numbers on an Argand diagram the magnitude, $r = |c_k|$, of each complex term represents the magnitude of the 'k-wave' in the Fourier series and the argument, $\theta = \text{Arg}(c_k)$, represents the phase of the k-wave.

7.1.3. *Conversion Between Complex Fourier Series A Real Fourier Series For Real Functions.* We have seen above how to calculate a_k and b_k coefficients for real functions and below in Section 7.2 we will see how to calculate c_k coefficients. Here we consider how to convert between real and complex Fourier series.

Above we have seen for a **real** function that

$$c_k = \begin{cases} \frac{1}{2}(a_k - ib_k) & : \quad (\text{for } k \geq 1) \\ \frac{1}{2}a_0 & : \quad (\text{for } k = 0) \\ \frac{1}{2}(a_{|k|} + ib_{|k|}) & : \quad (\text{for } k \leq -1) \end{cases}$$

Hence

$$\begin{aligned} a_0 &= 2c_0 \\ a_k &= 2\text{Re}(c_k) = c_k + c_{-k} \qquad && \text{Re}(z) \equiv \text{the real part of z} \\ b_k &= -2\text{Im}(c_k) = i(c_k - c_{-k}) \qquad && \text{Im}(z) \equiv \text{the imaginary part of z} \end{aligned}$$

Note again that these equations are valid for real-valued functions. Complex valued functions must be represented by complex Fourier series and cannot be represented by sums of sine and cosine functions.

7.2. **Complex Fourier Series For A Complex Function $f(x)$.** We have seen that a real function can be represented by a sum of e^{ikx} terms in the range $[-\pi, \pi]$, but the complex Fourier series can also represent complex

functions $f(x)$ which return a complex number z for each input value of x. For a complex function, $f(x)$, in the range $[-\pi, \pi]$;

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx} \qquad \text{when} \qquad f(x) \in \mathbb{C} \qquad \text{for} \qquad x \in [-\pi, \pi]$$

$$\text{where} \qquad c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(e^{ikx}\right)^* f(x) dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikx} f(x) dx$$

Note that c_0, the constant term, may need to be calculated separately from the other c_k terms.

Note that the above equation also holds for the real valued functions in the range $[-\pi, \pi]$, which form a vector subspace of these complex valued functions. Note also that $c_{-k} = c_k^*$ holds for real-valued functions, but does not hold in general for complex-valued functions.

In the language of linear algebra the set of functions e^{ikx} are an orthogonal basis set for the vector space of L^2 complex valued functions, $f(x)$, in the range $[-\pi, \pi]$ and the equation above is the projection of $f(x)$ onto this orthogonal basis set.

7.2.1. *Orthogonality of the e^{ikx} basis functions.* The orthogonality fo the e^{ikx} functions can be confirmed by calculating the inner product between $\mathbf{E}_m = E_m(x) = e^{imx}$ and $\mathbf{E}_n = E_n(x) = e^{inx}$ for $m, n \in \mathbb{Z}$;

$$\begin{aligned}
\langle \mathbf{E}_m, \mathbf{E}_n \rangle &= \int_{-\pi}^{\pi} \left(E_m(x)\right)^* E_n(x) dx = \int_{-\pi}^{\pi} \left(e^{imx}\right)^* e^{inx} dx = \int_{-\pi}^{\pi} e^{-imx} e^{inx} dx \\
&= \int_{-\pi}^{\pi} e^{i(n-m)x} dx \\
&= \begin{cases} \left[\frac{e^{i(n-m)x}}{(n-m)}\right]_{-\pi}^{\pi} = \frac{(-1)^{n-m} - (-1)^{n-m}}{(n-m)} = 0 & : \quad (\text{for } m \neq n) \\ \int_{-\pi}^{\pi} dx = [x]_{-\pi}^{\pi} = \pi - (-\pi) = 2\pi & : \quad (\text{for } m = n) \end{cases}
\end{aligned}$$

Thus, the e^{ikx} functions are orthogonal and $||e^{ikx}||^2 = 2\pi$.

7.2.2. *Kronecker Delta.* A sometimes useful piece of notation is the Kronecker delta, δ_{mn}, which is defined by

$$\delta_{mn} = \begin{cases} 0 & : \quad (\text{for } m \neq n) \\ 1 & : \quad (\text{for } m = n) \end{cases}$$

Thus from above we can write that

$$\langle \mathbf{E}_m, \mathbf{E}_n \rangle = \int_{-\pi}^{\pi} e^{i(n-m)x} dx = 2\pi \delta_{mn}$$

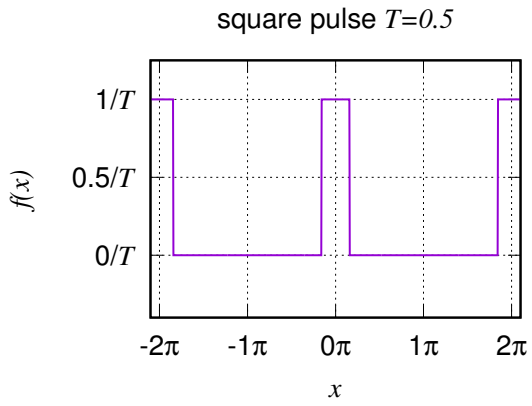

FIGURE 7. General square wave function, $f(x)$, with $T = 0.5$.

7.2.3. *Complex Fourier Series For Square Pulse.* A regular square pulse function, $f(x)$, is described by

$$\begin{aligned} f(x) &= 0 && -\pi \leq x < -T \\ f(x) &= \frac{1}{T} && -T < x < T \\ f(x) &= 0 && T < x \leq \pi \\ f(x + 2\pi) &= f(x) \end{aligned}$$

and is shown in Figure 7 for the case that $T = 0.5$. This function is defined so that as T gets smaller the height of the pulse $1/T$ increases to maintain a constant 'area' of pulse.

To caclulate the complex Fourier series for this function we find the c_k values by integration, but note that we have to calculate c_k and c_0 separately.

$$\begin{aligned}
c_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikx} f(x)dx = \frac{1}{2\pi}\int_{-T}^{T} \frac{1}{T} e^{-ikx} dx = \frac{1}{2T\pi}\left[\frac{e^{-ikx}}{-ik}\right]_{-T}^{T} \\
&= \frac{i}{2kT\pi}(e^{-ikT} - e^{ikT}) = \frac{i}{2kT\pi}(\cos(kT) - i\sin(kT) - \cos(kT) - i\sin(kT)) \\
&= \frac{-i^2 2}{2kT\pi}\sin(kT) = \frac{1}{kT\pi}\sin(kT) \\
c_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_{-T}^{T} \frac{1}{T} dx \\
&= \frac{1}{2T\pi}\left[x\right]_{-T}^{T} = \frac{1}{2T\pi}(T - -T) = \frac{2T}{2T\pi} = \frac{1}{\pi}
\end{aligned}$$

hence

$$f(x) = \frac{1}{\pi} + \sum_{k=-\infty, k\neq 0}^{\infty} \frac{1}{kT\pi}\sin(kT)e^{ikx}$$

Some Fourier series for regular square pulses with $T = 0.3, 0.5, 1$ are shown in Figure 8 along with plots of the c_k values, which are real. The lowest two panels show a single square pulse and its Fourier transform. Fourier transforms are discussed in Section 8.

7.3. **Complex Fourier Series In The Range $[-L, L]$.** If the range of x is generalized to $[-L, L]$ then, as in Setion 6.8, the complex basis functions need to be slightly modified;

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ik\pi x/L} \qquad \text{when} \qquad f(x) \in \mathbb{C} \qquad \text{for} \qquad x \in [-L, L]$$

where $$c_k = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-ik\pi x/L}dx$$

7.4. **Why Use The Complex Fourier Series?** A natural question at this point is whether it is better to use a complex Fourier series or a real Fourier series to represent a real-valued function. A complex Fourier series must be used with a complex-valued function, but either can be used for with real-valued functions. It is a mater of preference, but a very good reason to use a complex Fourier series is because maths with complex exponentials is generally quicker and more straightforward than maths with sine and cosine functions. For example, compare the length of Section 6.5, where the orthogonality of the sine and cosine functions is demonstrated, with the length of the equivalent demonstration for complex exponentials in Section 7.2.1.

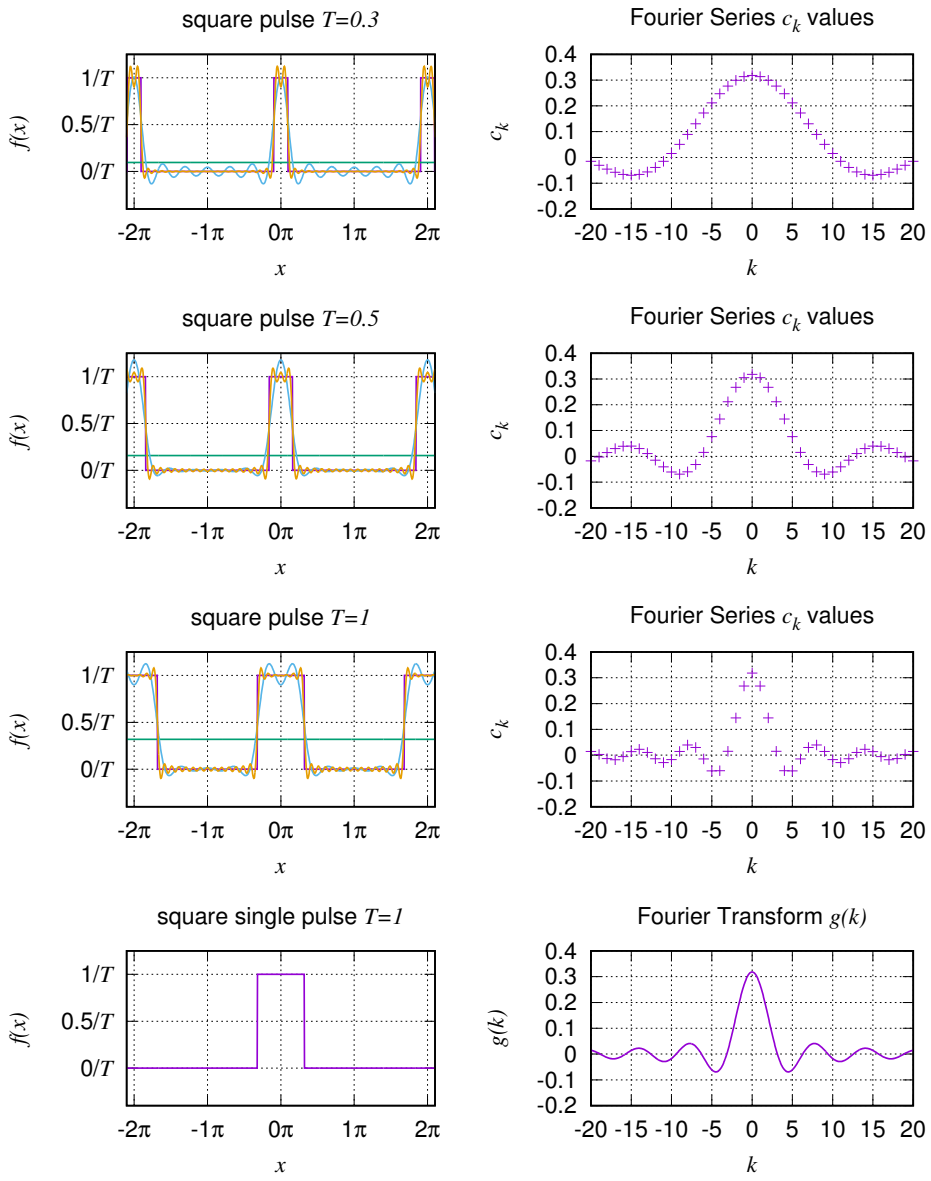

FIGURE 8. Square pulse functions with $T = 0.3, 0.5, 1$ are shown with Fourier series with just the $k = 0$ term, green, $|k| \leq 6$ terms, blue, and $|k| \leq 20$ terms, light brown. The c_k values for each Fourier series is also shown for $-20 \leq k \leq 20$. The lowest two panels show the single pulse function and its Fourier transform.

7.5. **Parseval's Theorem.** Parseval's Theorem presents a way to use the Fourier series to calculate 'power' or 'energy' of a signal and also the amount

of power at particular frequencies. Here we start with a general expression of Parseval's theorem and then move on to consider applications for signals.

The general expression of Parseval's Theorem can be derived by taking the inner product of two L^2 functions, $\mathbf{f} = f(x)$ and $\mathbf{g} = g(x)$, which are periodic with a period of 2π and have complex Fourier series coefficients, f_k and g_k respectively over the complex basis functions $\mathbf{E}_k = e^{ikx}$;

$$\mathbf{f} = \sum_{k=-\infty}^{\infty} f_k \mathbf{E}_k \qquad \mathbf{g} = \sum_{k=-\infty}^{\infty} g_k \mathbf{E}_k$$

$$\begin{aligned}\langle \mathbf{f}, \mathbf{g} \rangle &= \langle \sum_{k=-\infty}^{\infty} f_k \mathbf{E}_k, \sum_{k'=-\infty}^{\infty} g_{k'} \mathbf{E}_{k'} \rangle \\ &= \sum_{k=-\infty}^{\infty} \sum_{k'=-\infty}^{\infty} f_k^* g_{k'} \langle \mathbf{E}_k, \mathbf{E}_{k'} \rangle\end{aligned}$$

Now, of course, we saw in Section 7.2.1 that the $\mathbf{E}_k$ functions are orthogonal so $\langle \mathbf{E}_k, \mathbf{E}_{k'} \rangle = 0$ if $k \neq k'$ and that if $k = k'$ and $\langle \mathbf{E}_k, \mathbf{E}_{k'} \rangle = 2\pi$. Thus,

$$\begin{aligned}\langle \mathbf{f}, \mathbf{g} \rangle &= \sum_{k=-\infty}^{\infty} \sum_{k'=-\infty}^{\infty} f_k^* g_{k'} \langle \mathbf{E}_k, \mathbf{E}_{k'} \rangle \\ &= 2\pi \sum_{k=-\infty}^{\infty} f_k^* g_k\end{aligned}$$

and hence

$$\langle \mathbf{f}, \mathbf{f} \rangle = 2\pi \sum_{k=-\infty}^{\infty} f_k^* f_k = 2\pi \sum_{k=-\infty}^{\infty} |f_k|^2 \qquad = \int_{-\pi}^{\pi} (f(x))^* f(x) dx$$

Going back to using c_k for the complex coefficients of the Fourier series then if the function is real then

$$c_{-k} = c_k^* \qquad c_k = \frac{1}{2}(a_k + ib_k) \qquad \Rightarrow \quad |c_k|^2 = |c_{-k}|^2 = \frac{1}{4}(a_k^2 + b_k^2)$$

so

$$\begin{aligned}\int_{-\pi}^{\pi} (f(x))^* f(x) dx = \langle \mathbf{f}, \mathbf{f} \rangle &= 2\pi \sum_{k=-\infty}^{\infty} |c_k|^2 = 2\pi \left[c_0^2 + 2 \sum_{k=1}^{\infty} |c_k|^2 \right] \\ &= \frac{\pi}{2} a_0^2 + \pi \sum_{k=1}^{\infty} a_k^2 + b_k^2\end{aligned}$$

Parseval's Theorem is useful for calculating the power with AC voltage waveforms. For example, power can be calculated with V^2/R and so if the Fourier Series of a voltage waveform, $V(t)$, is known then to get a time averaged power, P, we need to integrate the power over 2π time and then

divide by 2π time;

$$\begin{aligned} P &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|V(t)|^2}{R}dt = \frac{1}{2\pi R}\int_{-\pi}^{\pi}|V(t)|^2dt = \frac{1}{2\pi R}2\pi\sum_{k=-\infty}^{\infty}|c_k|^2 \\ &= \frac{1}{R}\sum_{k=-\infty}^{\infty}|c_k|^2 \\ &= \frac{1}{4R}a_0^2 + \frac{1}{2R}\sum_{k=1}^{\infty}a_k^2 + b_k^2 \end{aligned}$$

Note that the sometimes the quantity of $|c_k|^2$ is referred to as the spectral power of the k-wave.

8. The Fourier Transform

(return to contents)

8.1. **Calculation of the Fourier transform.** For most of the discussion of Fourier series we have considered periodic functions with a period of 2π and sometimes the equations have been extended to consider functions with a period of $2L$. In the Fourier Transform the range is extended to $[-\infty, \infty]$ and so functions, $\mathbf{f} = f(x)$, that are not periodic may be considered, provided as above that they are square integrable and part of L^2. Thus, the functions must satisfy;

$$\langle \mathbf{f}, \mathbf{f} \rangle < \infty \qquad \Rightarrow \qquad \int_{-\infty}^{\infty} (f(x))^* f(x) dx < \infty$$

In Fourier series over a length $2L$ the complex basis functions are $e^{ik\pi x/L}$, where the factor $k\pi/L$ becomes smaller and smaller as L increases. In the limit where L goes to infinity the gap between adjacent k-waves becomes zero and the function $f(x)$ is transformed onto a continuous function of k-waves; $g(k)$.

The equation for the Fourier series over for functions $f(x)$ which are periodic over 2π

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx} \qquad \text{when} \qquad f(x) \in \mathbb{C} \qquad \text{for} \qquad x \in [-\pi, \pi]$$

$$\text{where} \qquad c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx$$

becomes

$$f(x) = \int_{-\infty}^{\infty} g(k) e^{ikx} dk \qquad \text{when} \qquad f(x) \in \mathbb{C} \qquad \text{for} \qquad x \in [-\infty, \infty]$$

$$\text{where} \qquad g(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

Now x is position and k is wavenumber or 'angular spatial frequency' and $k = \frac{2\pi}{\lambda}$, where λ is wavelength.

Thus, the Fourier transform here changes a function in space, x, to a function in 'reciprocal space', k.

For functions in time, $f(t)$, the Fourier transform is similar

$$f(t) = \int_{-\infty}^{\infty} g(\omega) e^{i\omega t} d\omega \qquad \text{when} \qquad f(t) \in \mathbb{C} \qquad \text{for} \qquad t \in [-\infty, \infty]$$

$$\text{where} \qquad g(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

Now t is time and ω is angular frequency and $\omega = 2\pi s$, where s is frequency. Here, s is used for frequency instead of f to avoid confusion with the function $f(t)$. Here a function in time is converted into a function in 'reciprocal time'.

You should be aware that although in maths and physics most people use *omega*, angular frequency, in engineering it is more common to use frequency, s, instead of angular frequency. If we work with frequency then

$$f(t) = \int_{-\infty}^{\infty} g(s)e^{i2\pi st}ds \qquad \text{when} \qquad f(t) \in \mathbb{C} \qquad \text{for} \qquad t \in [-\infty, \infty]$$

$$\text{where} \qquad g(s) = \int_{-\infty}^{\infty} f(t)e^{-i2\pi st}dt$$

Note that there is now a term of 2π in the exponential, but the term of $\frac{1}{2\pi}$ is removed from the integral for the Fourier transform and it makes the two integrals more symmetrical.

Also note that the process of finding the continuous function in reciprocal space or reciprocal time is generally called the Fourier transform whereas if a function in space or time is generated from a function in reciprocal space or time then this process is called the 'inverse Fourier transform'.

8.1.1. *Notation For The Fourier Transform.* We have already seen that the notation $\hat{\mathbf{f}}$ can mean that $\hat{\mathbf{f}}$ is a unit vector or that $\hat{\mathbf{f}}$ is the closest point to $\mathbf{f}$ in a vector subspace. It turns out that sometimes that Fourier transform of a function $f(x)$ can be represented by $\hat{f}(k)$. Here this convention is not adopted and instead the Fourier transform of a function is written as $\bar{f}(k)$. There is, of course, also a risk with this notation as the 'over-bar' is sometimes used for the complex conjugate. Here, the complex conjugate is *always* indicated with z^* is the complex conjugate of z.

The actual Fourier transform can be represented by $\mathcal{F}$, $\mathfrak{F}$ and other fancy F symbols. Here $\mathcal{F}$ is used. For example,

$$\bar{f}(k) = \mathcal{F}f(x) \qquad\qquad f(x) = \mathcal{F}^{-1}\bar{f}(k)$$
$$\bar{f}(\omega) = \mathcal{F}f(t) \qquad\qquad f(t) = \mathcal{F}^{-1}\bar{f}(\omega)$$

where $\mathcal{F}^{-1}$ represents the inverse Fourier transform.

8.2. **Parseval's Theorem For Continuous Functions.** Parseval's Theorem was introduced above in Section 7.5 for Fourier series. Here, the definition is extended to cover the cotinuous functions that are obtained with the Fourier transform.

In general, expression of Parseval's Theorem can be derived by taking the inner product for two L^2 functions, $\mathbf{f} = f(x)$ and $\mathbf{g} = g(x)$, which fourier transform to $\bar{\mathbf{f}} = \bar{f}(x)$ and $\bar{\mathbf{g}} = \bar{g}(x)$;

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_{-\infty}^{\infty} f(x)g(x)dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\bar{f}(k)\right)^* \bar{g}(k)dx$$

and for the inner product of a function with itself, which is Parseval's theorem;

$$\langle \mathbf{f}, \mathbf{f} \rangle = \int_{-\infty}^{\infty} f(x)f(x)dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\bar{f}(k)\right)^* \bar{f}(k)dx$$

8.3. **The Dirac Delta Function.** Dirac introduced the Delta function to represent the mass of the nucleus of the charge of an electron, which is confined to an infinitessimally small volume. The function can be defined in one dimension by

$$\delta(x) = 0 \qquad \text{for} \qquad x \neq 0$$
$$\int_{-\infty}^{\infty} \delta(x)dx = 1$$

Thus the value of the Dirac delta is zero everywhere except where $x = 0$, but the integral is equal to 1. At $x = 0$ the function is either infinite or undefined.

A mathematical property of Dirac delta is seen in the equations below;

$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)$$
$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)$$

thus it is sometimes described as a 'sifting' function which extracts the value of a function at $x = 0$ or $x = a$. Note the the limits of the integrals above do not need to be in the range from $-\infty$ to ∞, but can be any limits provided that the $x = 0$ or $x = a$ is included in the range of the integral.

Thus, the fourier transform of the Dirac delta in time, $\delta(t)$ is given by

$$g(s) = \int_{-\infty}^{\infty} \delta(t)e^{-i2\pi st}dt = e^{-i2\pi s0} = 1$$

The physical significance of this is imporant. The Dirac delta represents a sudden impulse and, as a function of frequency s, the Fourier transform $g(s) = 1$. This implies that all frequecies are equally represented in the impulse.

The Dirac delta is very similar to the Kronecker delta, δ_{mn} mentioned in Section 7.2.2. The Kronecker delta is useful with discrete sums, whereas the Dirac delta is useful with continuous functions.

8.4. **Convolution.** Ideally when we measure physically observable things, like optical spectra, we would prefer to have instruments that would behave like the Dirac delta functions. For example, in optical spectroscopy we would like to measure at each wavelength only light with exactly the wavelength we are intererested in with no contribution from light with slightly higher and slightly lower wavelength. Another example is in photography, we would like to have completely sharp images where there was no blurring of with light from slightly differents points on the object contributing to the signal at a single point in the image.

In reality, however, measurement systems are not perfect and some instrumental function will describe the 'resolution' of the apparatus. If this

function is $f(\epsilon)$ then signal at any point, x, has a relative contribution $f(\epsilon)$ from neighbouring points $x-\epsilon$. Thus the observed signal $h(x)$ is a convolution of the true signal $g(x)$ with the instrumental $f(\epsilon)$ function, which can be written as the integral;

$$\begin{aligned} h(x) &= \int_{-\infty}^{\infty} f(\epsilon)g(x-\epsilon)d\epsilon \\ &= (f*g)(x) \end{aligned}$$

Note that $*$ represents the convolution operation.

The Convolution Theorem states that

$$(f*g)(x) = \mathcal{F}^{-1}\left(\mathcal{F}f(x)\mathcal{F}g(x)\right)$$

and

$$(\mathcal{F}f*\mathcal{F}g)(k) = \mathcal{F}\left(f(x)g(x)\right)$$

Thus, convolution of two functions can be determined by taking the inverse Fourier transform of the product of the Fourier transforms of the two functions. Similarly the convolution of two Fourier transformed functions can be determined by taking the Fourier transform of the product of the two functions.

Now consider the case above where the measured signal, $h(x)$, was a product of an instrumental function, $f(x)$, and the true signal, $g(x)$. If we know $f(x)$ then we can **deconvolute** the signals to obtain $g(x)$, the true signal.

$$\begin{aligned} h(x) = (f*g)(x) &= \mathcal{F}^{-1}\left(\mathcal{F}f(x)\mathcal{F}g(x)\right) \\ \Rightarrow \qquad \mathcal{F}h(x) &= \mathcal{F}f(x)\mathcal{F}g(x) \\ \Rightarrow \qquad \mathcal{F}g(x) &= \frac{\mathcal{F}h(x)}{\mathcal{F}f(x)} \\ \Rightarrow \qquad g(x) &= \mathcal{F}^{-1}\left(\frac{\mathcal{F}h(x)}{\mathcal{F}f(x)}\right) \end{aligned}$$

Finally, convolution with the Dirac delta, $\delta(x-\epsilon)$, is worth considering.

$$h(x) = \int_{-\infty}^{\infty} f(\epsilon)\delta(\epsilon - x)d\epsilon = f(x)$$

Thus the convolution of a function with the Dirac delta leaves the function unchanged. Also note that the Fourier transform of the Dirac delta is unity, $\mathcal{F}\delta(x) = 1$, which confirms that any function, $f(x)$, is left unchanged by convolution with the Dirac delta because the Fourier transform of $f(x)$ will simply be multiplied by 1 at every frequency and remain unchanged.

8.5. **AM Radio.** The aim of radio transmission is to send a signal, normally an audio signal, over large distances. In AM and FM radio broadcasting the audio signal is shifted up in frequency to frequencies close to a carrier

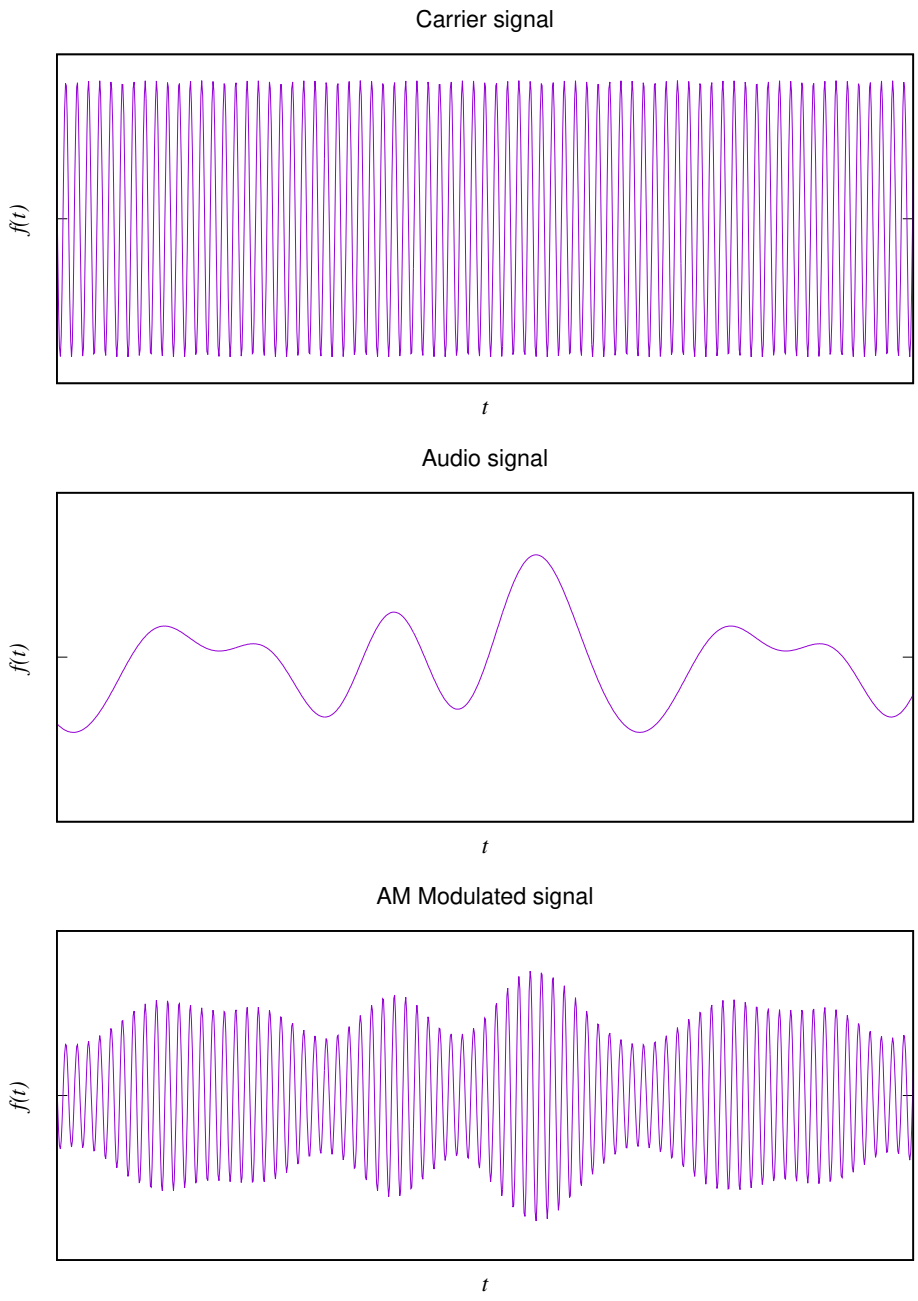

FIGURE 9. Key signals in AM radio.

signal frequency. Audio signal range is typically from about 200 Hz to 20 kHz. Carrier signals range from hundreds of kilohertz to hundreds of megahertz. The audio signal is encoded by AM, 'Amplitude Modulation', or FM, 'Frequency Modulation'. If our audio signal is $f(t)$ and our carrier

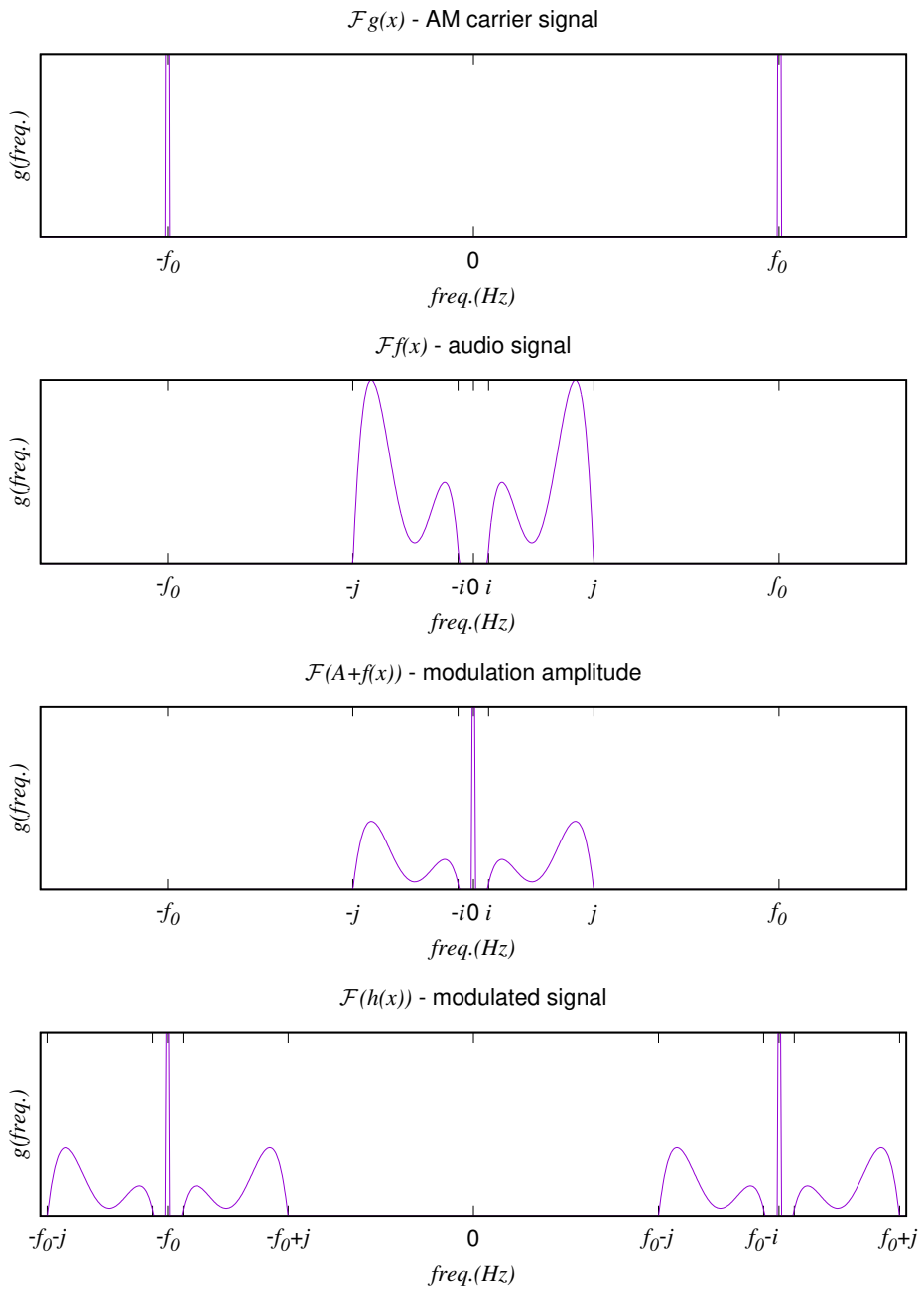

FIGURE 10. Fourier transforms of key signals in AM radio.

signal at frequency f_0 is $g(t) = A\cos(2\pi f_0 t)$ then the modulated signal in these two schemes, $h(t)$ can be represented mathematically by

AM $\quad h(t) = (f(t) + A)\cos(2\pi f_0 t)$ $\quad$ FM $\quad h(t) = A\cos(2\pi(f_0 + f(t))t)$

The AM signal is an interesting case to investigate with Fourier analysis. The AM signals is shown schematically in Figure 9. The top panel shows the carrier wave, which is modulated by the audio signal in the middle panel to give the overall modulated signal shown in the bottom panel. In reality the frequency of the carrier wave is, of course, orders of magnitude higher than the frequency of the audio signal.

Now we can calculate the Fourier transform of the AM broadcast signal, $\mathcal{F}h(x)$, with the convolution theorem;

$$\begin{aligned} & h(t) = (f(t) + A)\cos(2\pi f_0 t) \\ \Rightarrow \qquad & \mathcal{F}h(t) = \mathcal{F}(f(t) + A) * \mathcal{F}\cos(2\pi f_0 t) \end{aligned}$$

Figure 10 shows the Fourier transforms of these signals and the convolution. The Fourier transform of carrier signal, $\mathcal{F}g(t)$ is just like a double Dirac delta at $-f_0$ and f_0 in the frenquency domain because it is a single frequency signal. The Fourier transform of the audio signal, $\mathcal{F}f(t)$ will be some function in the audio frequency range between about 200 Hz to 20 kHz. Now the Fourier transform of the total amplitude function, $(f(t) + A)$, will be the Fourier transform of the audio signal plus a peak at zero frequency because the frequency of the A term is zero, because it is a constant.

The Fourier transform of the signal shown in the bottom panel of Figure 10 is calculated by convolution of the Fourier transform of the modulation amplitude with the Fourier transform of the carrier signal. Figure 10 is schematic because, of course, the carrier frequency, f_0, will be orders of magnitude higher than the audio signal which lies between frequencies i and j in the diagram. Note that the upshifted audio signal lies in two band each side of the carrier frequencies with frequencies between $f_0 + i$ and $f_0 + j$ above the carrier signal and $f_0 - i$ and $f_0 - j$ below the carrier signal. Now one technical point is that AM radio stations broadcast one of the upshifted audio side bands, which has the useful audio information and don't waste power on broadcasting the carrier wave, because it has no useful information.

References

Centre for Plasma Physics, School of Mathematics and Physics,, Queen's University Belfast

Email address: t.field@qub.ac.uk