PHY2006 Assignment 3 – Analytical Solutions to Partial Differential Equations

Deadline for Submission 6pm, Monday 11 Oct 2021

1. Using the method of characteristics, find the solution to the following first order PDE

$$\frac{\partial u}{\partial t} + (1 - x)\frac{\partial u}{\partial x} = -u$$

subject to the initial condition $u(x, 0) = \exp(-x^2)$

[35]

2. A straight aluminium bar with uniform cross section and length 1 m is heated uniformly to a temperature of 100°C. At $t = 0$ the ends at $x = 0,1$ are placed in good thermal contact with a heat reservoir at a temperature of 0°C. If the bar is insulated along its sides, then heat is only conducted along the bar (x direction) so that it is described by the 1D Heat Equation

$$\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2}$$

(a) Use the separation of variables method with a solution of the form $T(x, t) = X(x)Y(t)$ to obtain the following ordinary differential equations where $-\lambda^2$ is the separation constant

$$\frac{dY}{dt} + \lambda^2 DY = 0 \qquad \frac{d^2X}{dx^2} + \lambda^2 X = 0$$

[10]

(b) Carefully explaining each step, derive the most general solution to the heat equation

$$T(x, t) = \sum_{n=1}^{\infty} \exp\left(-\lambda_n^2 Dt\right)(A_n \sin(\lambda_n x) + B_n \cos(\lambda_n x))$$

[15]

(c) Using the boundary conditions as described at the start of the question, obtain expressions for B_n and λ_n

[15]

(c) Using the initial temperature distribution along the bar, determine values for the coefficients A_n for $n = 1 - 4$ and hence write down an approximate solution for $T(x, t)$ for this problem and determine the temperature at the centre of the bar 10 minutes later.

($D = 10^{-4}$ m^{2}s^{-1} for aluminium)

$\int_0^1 \sin(m\pi x)\sin(n\pi x) = \frac{1}{2}$ for $m = n$

$\int_0^1 \sin(m\pi x)\sin(n\pi x) = 0$ for $m \neq n$

[25]

Extra Question

$$\frac{\partial u}{\partial t} + (x+u)\frac{\partial u}{\partial x} = 0$$

subject to the initial condition $u(x,0) = x$

(i) Obtain an expression for u in terms of x_0, where $x = x_0$ when $t = 0$

(ii) Substitute your answer from (i) into an expression for $\frac{dx}{dt}$ and hence obtain u in terms of x, t