PHY2006 Differential Equation Solutions - Mock questions

SECTION A

1. Given Taylor's theorem

$$f(x_0 + \Delta x) = f(x_0) + f'(x_0)\Delta x + \frac{\Delta x^2}{2!} f''(x_0) + \frac{(\Delta x)^2}{3!} f'''(x_0) + \cdots$$

Write down the time dependent Schrödinger equation as a finite difference equation in terms of $\Delta t, \Delta x, \Psi_m^{(n)}$ where m and n are indicies of a discretised grid in x and t.

$$i\hbar \frac{\partial \Psi}{\partial t} = V\Psi - \frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2}$$

[10]

2. Using the method of characteristics, find the solution to the following first order partial differential equation:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = \frac{t}{u}$$

subject to the initial condition $u(x, 0) = \exp(-x^2)$

[10]

3. The 2D Laplace equation is

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

If this is to be solved numerically using a finite difference method by discretising ϕ using a grid in (x, y) space where the points are equally spaced, i.e. $\Delta x = \Delta y$:

(a) Write down a finite difference equation which relates a value of $\phi_m^{(n)}$ on this grid to surrounding points (where m and n are indicies of points in x and y).

(b) Explain how a relaxation technique can be used to obtain a numerical solution.

[A second order differential of a function $f(x)$ at a point $x = x_0$ can be expressed in the following form as $\Delta x \to 0$

$$f''(x_0) = \frac{f(x_0+\Delta x) - 2f(x_0) + f(x_0 - \Delta x)}{(\Delta x)^2}$$]

[10]

SECTION C

1. Consider the following differential equation with initial condition $y(0) = 1$

$$y' + 2y = 2 - e^{-4t}$$

(a) Use the Euler method to determine a numerical solutions for y at $t = 0.5$ using a step size of $\Delta t = 0.1$. In doing so, complete the following table, to 3 decimal places

i	t_i	y_i	y_i'
0	0	1	-1
1	0.100	0.900	-0.470
2	0.200	…	…
3	0.300	…	…
4	0.400	…	…
5	0.500	…	…

[9]

(b) Given the analytic solution

$$y = \frac{1}{2}(e^{-4t} - e^{-2t}) + 1$$

determine the percentage error of the numerical calculation for $y(0.5)$. **[3]**

(c) If the step size was reduced to $\Delta t = 0.05$, estimate the new percentage error? **[2]**

(d) Suggest an alternative method for numerically calculating $y(0.5)$ which is more accurate than the simple Euler method for the same Δt. Write down a finite difference equation showing how this could be implemented. **[6]**

2. The Saturn V rocket had a total mass of $M_0 = 3{,}000{,}000$ kg just before launch. During its 1st stage, burning fuel was expelled at a speed of $u = 2600$ m/s so that it lost mass at a rate of $L = 13{,}500$ kg/s. At the end of the 1st stage the rocket's mass had reduced to $M = 800{,}000$ kg. The evolution of the velocity v and mass m of the rocket are governed by the following differential equation

$$\frac{dv}{dm} = \frac{g}{L} - \frac{u}{m}$$

(a) Show that the velocity of the rocket at the end of its 1st stage is given by

$$v_f = u \ln \frac{M_0}{M} - \frac{g}{L}(M_0 - M)$$

[6]

(b) Using the Euler method, taking five steps with a step size of $\Delta m = -440{,}000$ kg, complete the table below and determine a numerical solution to problem in part (a).

i	m (kg)	v_i (m s^{-1})	$\left(\frac{dv}{dm}\right)_i$ (m s^{-1} kg^{-1})
0	3,000,000	0	-0.00014
1	2,560,000	61.6	-0.000289
2	2,120,000	…	…
3	1,680,000	…	…
4	1,240,000	…	…
5	800,000	…	…

[9]

(c) What is the global error of this numerical calculation? Roughly how many steps would be required for the Euler method to attain a solution which is within 50 m/s of the analytical solution. **[5]**

3. Consider the following differential equation with initial condition $y(0) = -1$

$$y' - y = e^t \sin t$$

It has the following analytical solution

$$y(t) = -e^t \cos t$$

(a) Use the Euler method to determine a numerical solution for y at $t = 5$ using a step size of $\Delta t = 1$. In doing so, complete the following table, to 2 decimal places

i	t_i	y_i	y_i'	$y(t)$
0	0	-1.00	-1.00	-1.00
1	1	-2.00	0.29	-1.47
2	2	…	…	…
3	3	…	…	…
4	4	…	…	…
5	5	…	…	…

where $y(t)$ is the analytical solution. **[8]**

(b) Draw a rough graph of y_i and $y(t)$ against t and hence explain why the Euler method is a poor approximation in this instance. **[6]**

(c) In order to solve the differential equation more accurately, the midpoint approximation can be used. Show that with this approximation, a numerical solution can be calculated using the following equation

$$y_{i+1} = y_i\left(1 + \Delta t + \frac{(\Delta t)^2}{2}\right) + \Delta t e^{t_{i+1/2}} \sin t_{i+\frac{1}{2}} + \frac{(\Delta t)^2}{2} e^{t_i} \sin t_i$$

[6]