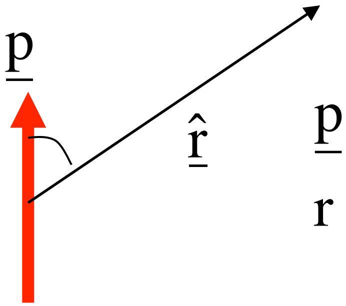
Lecture 6:
Electric fields in dielectric media

Dr. Gianluca Sarri

QUEEN'S UNIVERSITY BELFAST

Recap on the dipole



$\underline{p} = q\underline{h}$

$r >> h$

$$\psi_p = \frac{\underline{p} \cdot \hat{\underline{r}}}{4\pi\varepsilon_0 r^2} = \frac{p\cos\theta}{4\pi\varepsilon_o r^2}$$

$$\underline{E} = -\left(\frac{\partial\psi}{\partial r}\hat{\underline{r}} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\underline{\theta}} + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\varphi}\hat{\underline{\varphi}} \right)$$

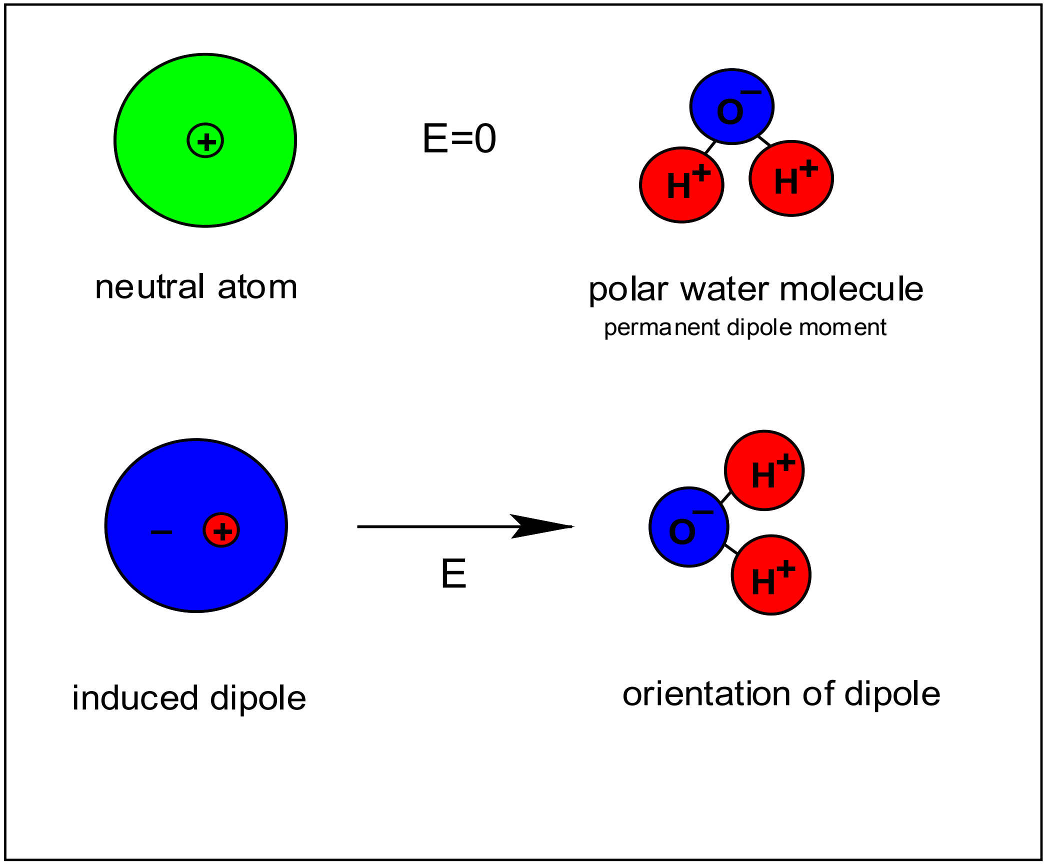
The electrostatic energy in the case of a continuous distribution of charges is

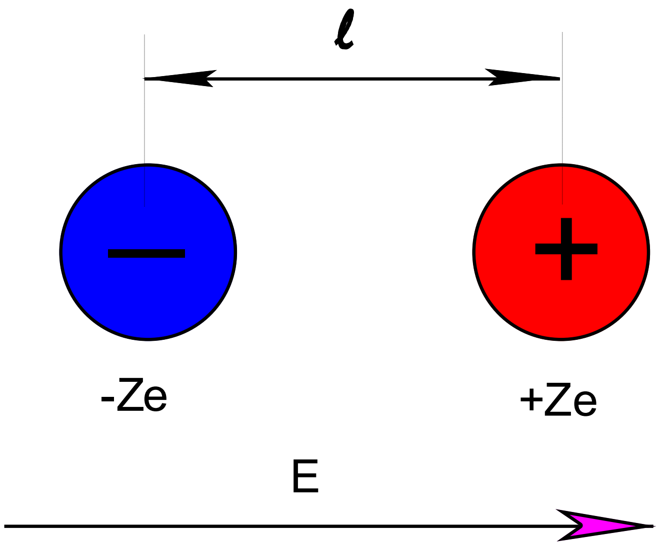
$$W = \frac{1}{2}\int_V \rho\psi \, dV$$

The electrostatic *energy density* (energy / volume) is: $U = \frac{1}{2}\varepsilon_0\varepsilon_r E^2$

Dipoles in dielectrics

In a dielectric, charges are not free to move (electrons bound to the nucleus). Nonetheless, the charges will orient themselves following the external field. This is **polarisation**





We then generate a series of tiny dipoles, each with:

$$\underline{p} = Ze\underline{\ell}$$

Dipoles in dielectrics

In a medium, we have so many atoms that we can approximate a **continuous distribution of dipoles**

The number of dipoles inside the material can be expressed as:

$$\underline{P} = \underline{p}\,N = NeZ\underline{\ell}$$ *(assuming all dipoles are identical!)*

The dipole moment of a small element of volume is: $\underline{P} = \underline{p}\,dV$

The potential induced by this tiny element is:

$$d\psi = \frac{-1}{4\pi\varepsilon_0}\frac{\underline{P}\,.\,\hat{\underline{r}}}{r^2}\,dV$$ *(compare with the case of a single dipole)*

Dipoles in dielectrics

In order to express that relation a bit more elegantly we can use vectorial identities:

$$\left(\frac{-\hat{\underline{r}}}{r^2}\right) = \nabla\left(\frac{1}{r}\right) \quad \rightarrow \quad \psi_p = \frac{1}{4\pi\varepsilon_0}\int_v \underline{P} \cdot \nabla\left(\frac{1}{r}\right) dV$$

and: $\nabla \cdot \psi \underline{A} = \psi \nabla \cdot \underline{A} + \underline{A} \cdot \nabla \psi$ $\rightarrow$ $\psi_p = \frac{1}{4\pi\varepsilon_0}\int \nabla \cdot \left(\frac{\underline{P}}{r}\right) dV - \frac{1}{4\pi\varepsilon_0}\int \frac{(\nabla \cdot \underline{P})}{r} dV$

finally, using the divergence theorem:

$$\psi_p = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\underline{P} \cdot \underline{dS}}{r} - \frac{1}{4\pi\varepsilon_0}\int_V \frac{\nabla \cdot \underline{P}}{r} dV$$

The physical interpretation of polarisation

$$\psi_p = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\underline{P}\,.\,\underline{dS}}{r} - \frac{1}{4\pi\varepsilon_0}\int_V \frac{\nabla\,.\,\underline{P}}{r}\,dV$$

POISSON'S EQUIVALENT DISTRIBUTION

If we look at the above equation we see that the potential at any point, due to the effects of Polarisation, can be calculated in terms of a surface charge density σ_b and a volume charge density $\sigma_{b.}$ Each of these is related to P as follows

$$\sigma_b = \underline{P}\,.\,\underline{d\hat{S}} = P_n$$

This is a bound surface charge equal to the component of P that is normal to the surface

$$\rho_b = -\nabla\,.\,\underline{P}$$

This is a bound volume charge equal to the negative of the divergence of P .

Electric field in a dielectric

The electric field in a dielectric will then be the sum of two contributions: the external electric field (E) and the field induced by the polarisation of the material (P).

$$\underline{D} = \varepsilon \underline{E} = \varepsilon_0 \underline{E} + \underline{P}$$

Generally speaking, e is not a number but, rather, a matrix (a tensor). This is because polarisation might be different along different axis (think of a crystal for example)

$$\underline{D} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{pmatrix} \underline{E}$$

Electric field in a dielectric

Only if the material is **linear** (its response is not a function of the intensity of the applied electric field)
homogenous (its response is the same across the material) and
isotropic (its response does not depend on the orientation of the electric field),

then the dielectric constant is a number ($\varepsilon = \varepsilon_0\varepsilon_r$).
Reshuffling the equation, we can define another quantity, which directly relates the external electric field to the polarisation:

$$\underline{D} = \varepsilon_0\varepsilon_r\underline{E} = \varepsilon_0\underline{E} + \underline{P}$$

$$\underline{P} = \varepsilon_0\left(\varepsilon_r - 1\right)\underline{E} = \varepsilon_0\chi\underline{E}$$

➔ χ is called the material *susceptibility*

Examples

Some examples for ε_r:

Vacuum:	1
Air:	1.00058986
Paper:	3.85
Water:	80.1 !! For visible light: 1.77
Glass (Silicon dioxide):	3.9
Rubber:	7

Gauss' law in a material

Starting from: $\underline{\mathrm{D}} = \varepsilon_0 \varepsilon_r \underline{\mathrm{E}} = \varepsilon_0 \underline{\mathrm{E}} + \underline{\mathrm{P}}$

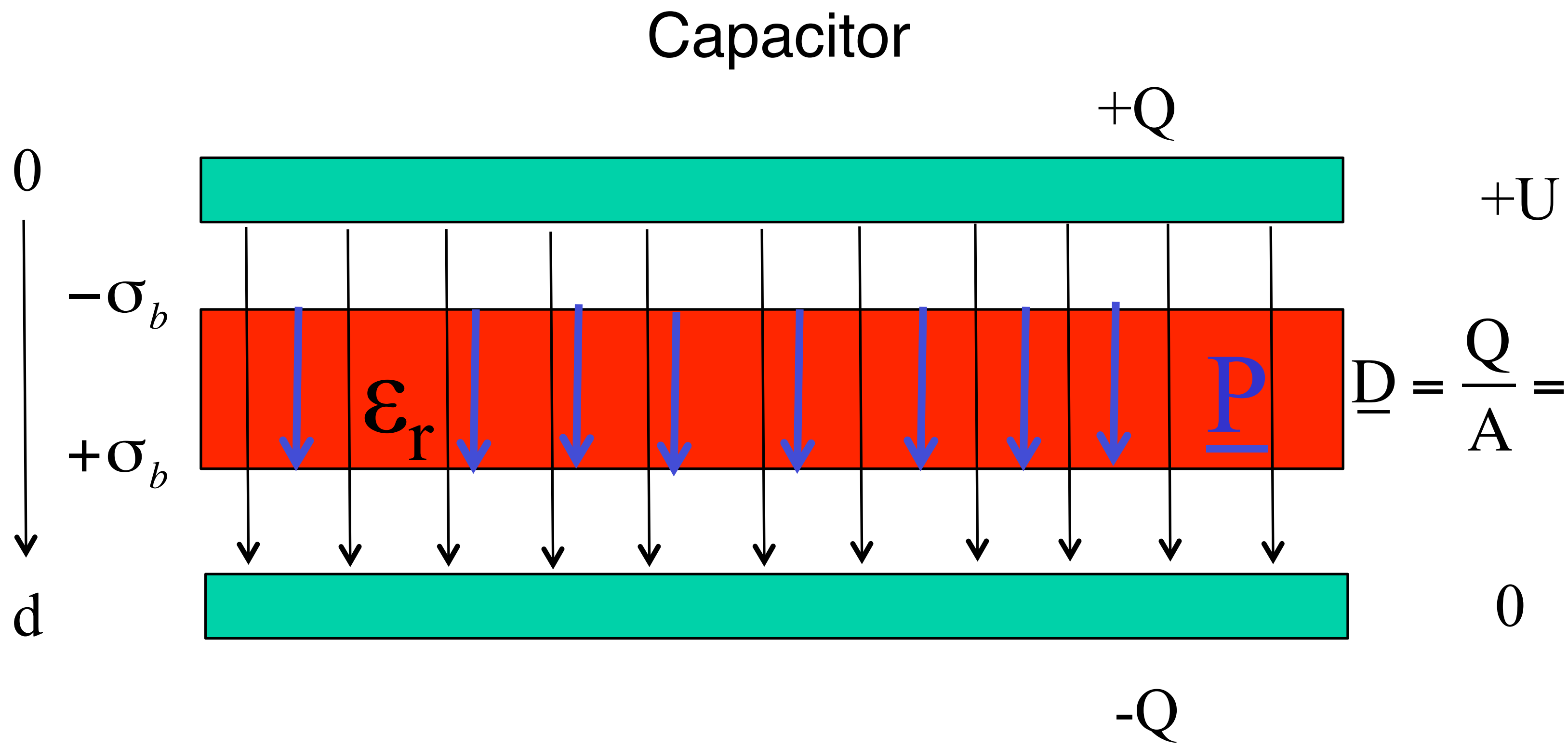
if we take the divergence of it we get: $\nabla \cdot \underline{\mathrm{D}} = \varepsilon_0 \nabla \cdot \underline{\mathrm{E}} + \nabla \cdot \underline{\mathrm{P}}$

and we know that the divergence of E relates to total charges (ρ) whilst the divergence of P relates to minus the bound charges (ρ_B):

$$\left.\begin{array}{l} \varepsilon_0 \nabla \cdot \vec{E} = \rho \\ \nabla \cdot \vec{P} = -\rho_B \end{array}\right. \longrightarrow \nabla \cdot \vec{D} = \rho - \rho_B = \rho_F$$

This is the generalised Gauss' law in the presence of a medium

Example



$$\underline{D} = \frac{Q}{A} = \sigma$$

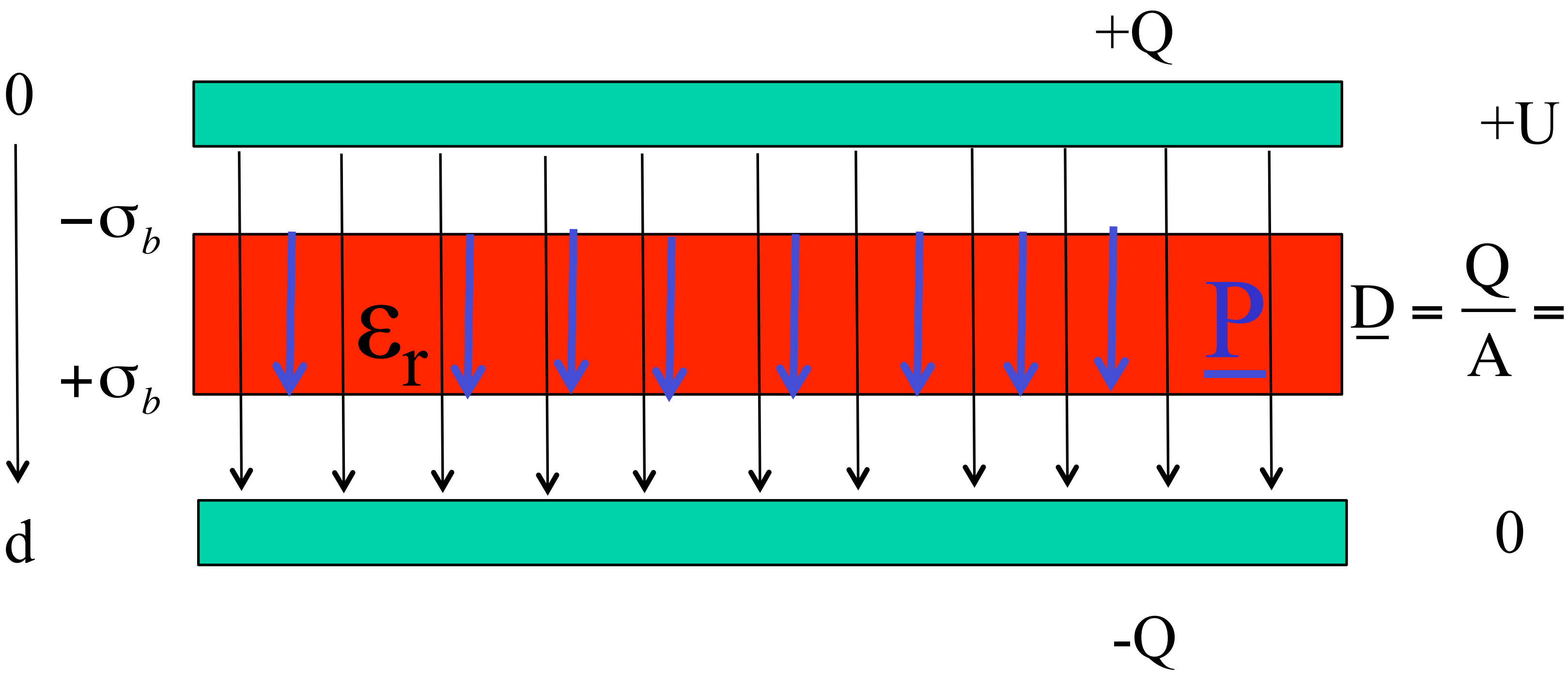
$$U = -\int_0^d E\,dx \neq Ed$$

$$\boxed{\underline{D} = \varepsilon_0 \underline{E} + \underline{P}}$$

So E field in the air gap is greater than that in the dielectric

Example

Capacitor



$$\underline{D} = \frac{Q}{A} = \sigma$$

To find E we use

$$D = \varepsilon_0 \varepsilon_r E$$

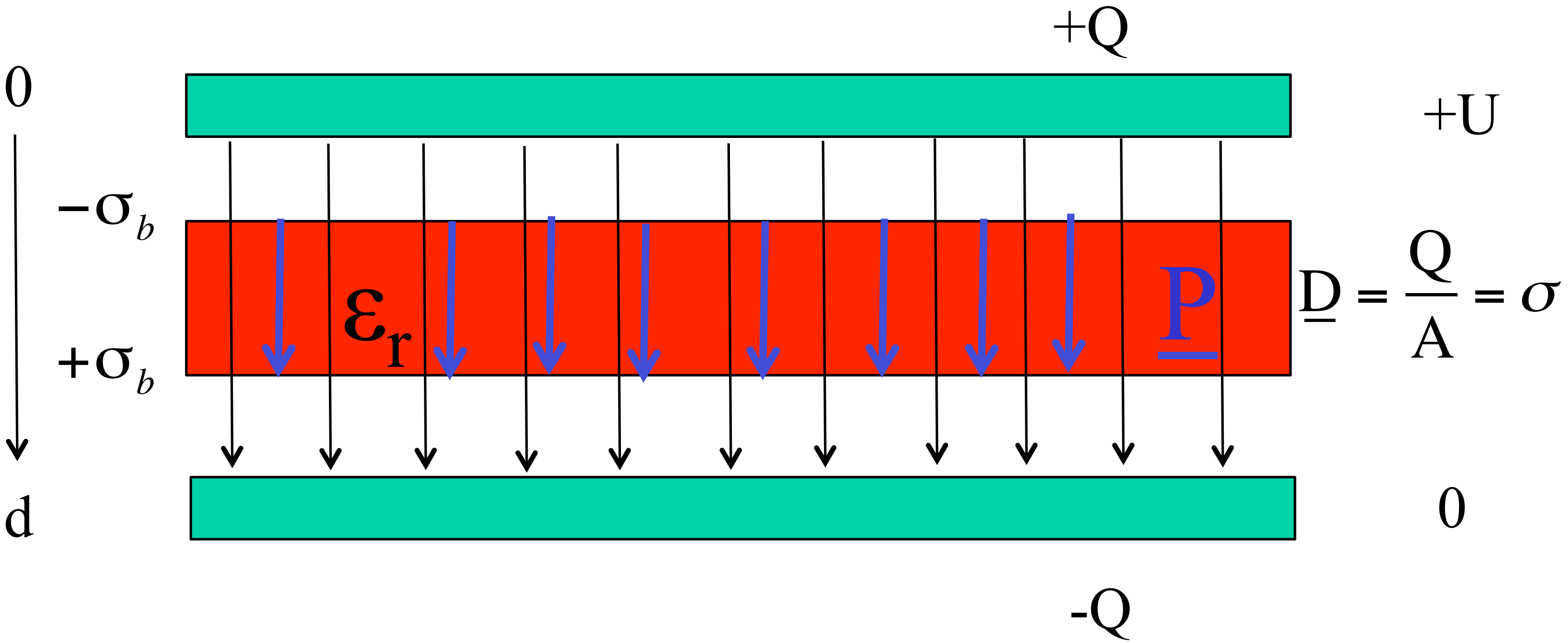
In the air gap

$$E_{air} = \frac{D}{\varepsilon_0} = \frac{\sigma}{\varepsilon_0}$$

$$E_{slab} = \frac{D}{\varepsilon_0 \varepsilon_r} = \frac{\sigma}{\varepsilon_0 \varepsilon_r}$$

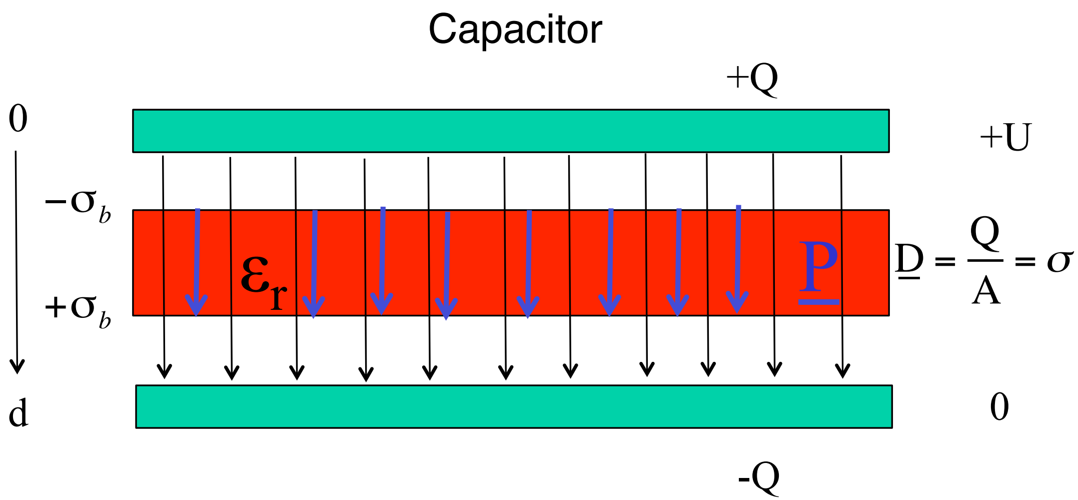
Example

Capacitor



$$\underline{P} = \underline{D} - \varepsilon_0 \underline{E} = \sigma\left(1 - \frac{1}{\varepsilon_r}\right) \qquad \sigma_b = \sigma\left(1 - \frac{1}{\varepsilon_r}\right) = P$$

Example



Without the dielectric, the capacitance of the capacitor is defined as:

$$C = \frac{Q}{\psi} = \varepsilon_0 \frac{A}{d}$$

If we insert a dielectric in the middle: $C' = \frac{Q}{\psi'} = \varepsilon_r \varepsilon_0 \frac{A}{d} = \varepsilon_r C >$ C