PHY2006
Exam Time Table
Code PHY2006

Use lined, single-sided A4 paper
with a black or blue pen.
Write your student number
at the top of every page.

Any non-graphical calculator, except those with pre-programmable memory, may be used in this examination

LEVEL 2
Examination contributing to the Degrees of Bachelor of Science (BSc) and Master in Science (MSci)

PHY2006 - EXAM
Mathematical Physics
Wednesday, 12th August 2020, 09.30 - 13.30

Examiners: Prof S Matthews, Dr P van der Burgt
and the Internal Examiners
Dr J Greenwood (j.greenwood@qub.ac.uk)

Answer ALL SIX questions in Section A for 10 marks each.
Answer ONE question in Section B for 20 marks.
Answer ONE question in Section C for 20 marks.
You have FOUR hours to complete and upload this paper.

Contact the module coordinator if you have queries/problems at t.field@qub.ac.uk and copy to mpts@qub.ac.uk

By submitting the work, you are declaring that:

1. The submission is your own original work and no part of it has been submitted for any other assignments;
2. You understand that collusion and plagiarism in an exam are major academic offences, for which a range of penalties may be imposed, as outlined in the Procedures for Dealing with Academic Offences.

SECTION A

Answer ALL questions from Section A

A.1 (a) Use the Gram-Schmidt orthogonalization procedure to make a function $g'(x)$ from $g(x) = x$ which is orthogonal to $f(x) = 1$ in the vector space where the inner product $< f(x)|g(x) >$ is defined by

$$< f(x)|g(x) >= \int_0^2 f(x)g(x)\,dx$$

Note that the limits for integration over x are from 0 to 2.

(b) Use the Gram-Schmidt orthogonalization procedure to generate another function $h'(x)$ from $h(x) = x^2$ which is orthogonal to both $f(x)$ and $g'(x)$. [10]

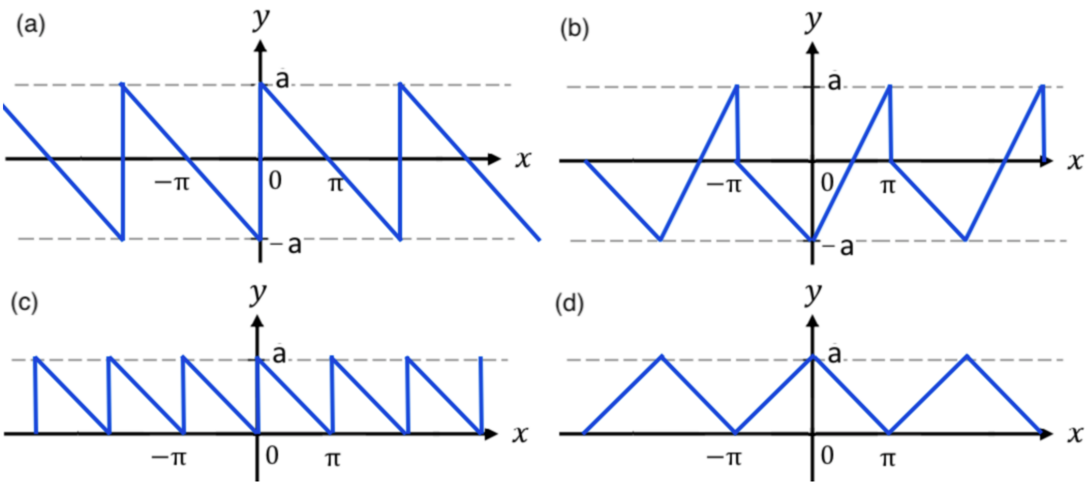

Figure 1: Four periodic functions for question **A.2**

A.2 Consider by inspection each of the four periodic functions shown in Figure 1 and the Fourier series which could be used to represent them. For each of these four functions you should indicate for the equivalent Fourier series;

- If a_0 is zero or non-zero.
- If all the a_k values (for $k > 0$) are zero or if some of them will be non-zero.
- If all the b_k values are zero or if some of them will be non-zero.

[10]

A.3 Calculate the Fourier transform of the function $f(x)$ where

$$f(x) = 0 \qquad x < 0$$
$$f(x) = \mathrm{e}^{-x} \qquad x \geq 0$$

[10]

A.4 Consider the following function

$$u(x,t) = \frac{a}{\sqrt{t}} \exp\left(-\frac{x^2}{4t}\right)$$

Show that it is a solution to the diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

[10]

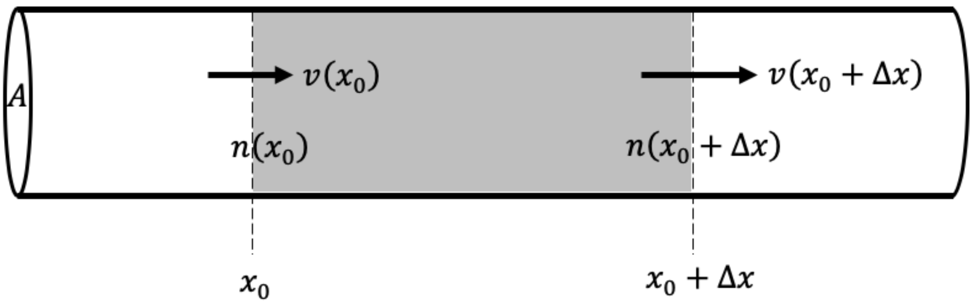

Figure 2: Pipe diagram for question **A.5**

A.5 Consider a fluid moving in a pipe of uniform cross sectional area A as shown in Figure 2

The number density of fluid particles at a distance x along the pipe at a time t is $n(x,t)$ and the velocity of the fluid is $v(x,t)$

(a) Write down the expressions for the number of particles entering and leaving the volume $A\Delta x$ in a time Δt.

(b) Show that as Δx, $\Delta t \to 0$ the continuity equation is

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nv) = 0$$

(c) Show how the linear advection equation is obtained if the flow is incompressible.

$$\frac{\partial n}{\partial t} + v\frac{\partial n}{\partial x} = 0$$

[10]

A.6 Using the method of characteristics, find the solution to the following first order partial differential equation

$$\frac{\partial u}{\partial t} + x^2 t\frac{\partial u}{\partial x} = 0$$

subject to the initial condition $u(x,0) = x^2 \sin{(2x)}$ [10]

SECTION B

Answer ONE question from Section B

B.1 The function $f(x)$ is defined by

$$
\begin{aligned}
f(x) &= -x^2 && -\pi \leq x \leq 0 \\
f(x) &= x^2 && 0 \leq x < \pi \\
f(x) &= f(x+2\pi)
\end{aligned}
$$

(a) Draw a sketch of the function $f(x)$. [5]

(b) Is $f(x)$ even, odd, or neither? [1]

(c) Indicate if any of the terms a_0, a_k and b_k of the Fourier series expansion of $f(x)$ are expected to be zero by inspection of the sketch of $f(x)$ and briefly explain your reasoning. [3]

(d) Determine the terms a_0, a_k and b_k of the Fourier series expansion of $f(x)$. *Note* that it is not necessary to explicitly calculate any terms that you have determined to be zero by inspection and your answer for non-zero terms may be an equation that depends on k, for example. [8]

(e) Determine the numerical values of the terms a_0, a_1, a_2, a_3, a_4, b_1, b_2, b_3 and b_4. [3]

B.2 (a) A two dimensional planar subspace within an $\mathbb{R}^3$ vector space is defined by two vectors $\mathbf{a}$ and $\mathbf{b}$

$$
\mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ 7 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}
$$

i. Perform a calculation to demonstrate that the vectors $\mathbf{a}$ and $\mathbf{b}$ are not orthogonal. [1]

ii. Use Gramm-Schmidt orthogonalisation to calculate from $\mathbf{b}$ a modified vector $\mathbf{b}'$, which is perpendicular to $\mathbf{a}$ [3]

iii. Determine $\mathbf{c}'$, $\mathbf{d}'$ and $\mathbf{e}'$ which are the vectors which lie in the plane defined by $\mathbf{a}$ and $\mathbf{b}'$ that are closest to the vectors $\mathbf{c}$, $\mathbf{d}$ and $\mathbf{e}$. [9]

$$
\mathbf{c} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} \qquad \mathbf{d} = \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} \qquad \mathbf{e} = \begin{pmatrix} -1 \\ -8 \\ 3 \end{pmatrix}
$$

iv. Which of the vectors $\mathbf{c}'$, $\mathbf{d}'$ and $\mathbf{e}'$ lies closest to the plane defined by $\mathbf{a}$ and $\mathbf{b}'$? [3]

(b) Consider carefully the inner product calculation method below where the functions $f(x)$ and $g(x)$ are squared inside the integral;

$$
< f(x)|g(x) >= \int_0^1 (f(x))^2 \, (g(x))^2 \; dx
$$

Use the Gram-Schmidt orthogonalization procedure to generate a function $g'(x)$ from $g(x) = x$ which is orthogonal to both $f(x) = 1$ using the inner product defined above. [4]

SECTION C

Answer ONE question from Section C

C.1 Consider Laplaces equation in 2D polar coordinates (r,θ)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}=0$$

(a) By letting $\phi(r,\theta)=R(r)T(\theta)$ and using the separation of variables method, show that this partial differential equation can be decomposed into the following ordinary differential equations

$$\frac{d^2T}{d\theta^2}+n^2T=0 \qquad\qquad r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}-n^2R=0$$

where n^2 is a separation constant. [7]

(b) Obtain solutions to both these equations when $n=0$ and $n\neq 0$. (Hint: use the substitution $u=\ln r$ to solve for R). [10]

(c) If the only values of n which are allowed are positive integers, write down the most general solution of $\phi(r,\theta)$ [3]

C.2 The SpaceX Falcon 9 rocket had a total mass of M_0= 550,000 kg just before launch. During the 1st stage, it expels burnt fuel at a speed of $u=2770$ m/s and loses mass at a rate of $L=2,500$ kg/s. At the end of the 1st stage the rocket's mass reduces to $M=150,000$ kg. The evolution of the velocity v and mass m of the rocket are governed by the following differential equation

$$\frac{dv}{dm}=\frac{g}{L}-\frac{u}{m}$$

(a) Solve this differential equation to show that the velocity of the rocket at the end of its 1st stage is given by

$$v_f=u\ln\frac{M_0}{M}-\frac{g}{L}\left(M_0-M\right)$$

Calculate this velocity assuming g =9.81 ms^{-2} [6]

(b) Using the Euler method, taking five steps with a step size of $\Delta m=-80,000$ kg, complete the table below and determine a numerical solution to the equation in part (a).

i	m (kg)	v_i (m s^{-1})	$\left(\frac{dv}{dm}\right)_i$(m s^{-1} kg^{-1})
0	550,000	0	-0.00111
1	470,000		
2	390,000		
3	310,000		
4	230,000		
5	150,000		

[9]

(c) What is the global error of this numerical calculation? Roughly how many steps would be required for the Euler method to attain a solution which is within 25 m/s of the analytical solution. [5]

END OF EXAMINATION