QUEEN'S UNIVERSITY BELFAST

PHY2001
Exam Time Table
Code PHY2001

Answer Books A, B and C.

Any calculator, except one with pre-programmable memory, may be used in this examination.

LEVEL 2
Examination contributing to the Degrees of Bachelor of Science (BSc) and Master in Science (MSci)

PHY2001
Quantum and Statistical Physics

Tuesday, 6th August 2019 9:30 AM - 12:30 PM

Examiners: Professor P Browning
Dr P van der Burgt
and the Internal Examiners

Answer ALL TEN questions in Section A for 4 marks each.
Answer TWO questions in Section B for 20 marks each.
Answer ONE question in Section C for 20 marks.
Use a separate answer book for each Section.

Use a separate answer book for each Section.
You have THREE HOURS to complete this paper.

THE QUEEN'S UNIVERSITY OF BELFAST

SCHOOL OF MATHS AND PHYSICS

PHYSICAL CONSTANTS

Speed of light in a vacuum	$c = 3.00 \times 10^{8}\ \mathrm{ms^{-1}}$
Permeability of a vacuum	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{Hm^{-1}}$ $\approx 1.26 \times 10^{-6}\ \mathrm{Hm^{-1}}$
Permittivity of a vacuum	$\varepsilon_0 = 8.85 \times 10^{-12}\ \mathrm{Fm^{-1}}$
Elementary charge	$e = 1.60 \times 10^{-19}\ \mathrm{C}$
Electron charge	$= -1.60 \times 10^{-19}\ \mathrm{C}$
Planck Constant	$h = 6.63 \times 10^{-34}\ \mathrm{Js}$
Reduced Planck Constant	$\hbar = 1.05 \times 10^{-34}\ \mathrm{Js}$
Rydberg Constant for hydrogen	$R_\infty = 1.097 \times 10^{7}\ \mathrm{m^{-1}}$
Unified atomic mass unit	$1u = 1.66 \times 10^{-27}\ \mathrm{kg}$ $1u = 931\ \mathrm{MeV}$
1 electron volt (eV)	$= 1.60 \times 10^{-19}\ \mathrm{J}$
Mass of electron	$m_e = 9.11 \times 10^{-31}\ \mathrm{kg}$
Mass of proton	$m_p = 1.67 \times 10^{-27}\ \mathrm{kg}$
Mass of neutron	$m_n = 1.67 \times 10^{-27}\ \mathrm{kg}$
Molar gas constant	$R = 8.31\ \mathrm{JK^{-1}mol^{-1}}$
Boltzmann constant	$k = 1.38 \times 10^{-23}\ \mathrm{JK^{-1}}$
Avogadro constant	$N_A = 6.02 \times 10^{23}\ \mathrm{mol^{-1}}$
Gravitational constant	$G = 6.67 \times 10^{-11}\ \mathrm{Nm^{2}kg^{-2}}$
Acceleration of free fall on the Earth's surface	$g = 9.81\ \mathrm{ms^{-2}}$

SECTION A

Use a section A answer book

Answer ALL 10 questions in this section

Full explanations of your answers are required to attain full marks

1 In a deuteron, a proton and a neutron are very weakly bound by the strong nuclear force with an average distance between the particles of about 5 fm. Due to the uncertainty principle, the particles have a minimum momentum. Determine the minimum kinetic energy of the particles. **[4]**

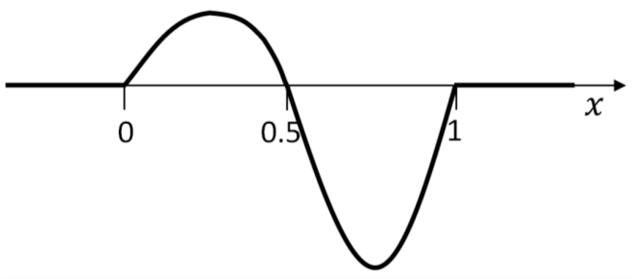
2 A particle is described by a wavefunction

$\psi = Ax(1+i)$ for $-1 < x < +1$

$\psi = 0$ for $x > +1,\ x < -1$

Find the value of the normalisation constant $\boldsymbol{A}$. **[4]**

3 Write down an equation for the quantum mechanical expectation value for a particle's position ($\langle x \rangle$). If a particle is described by the following wavefunction, qualitatively estimate $\langle x \rangle$?



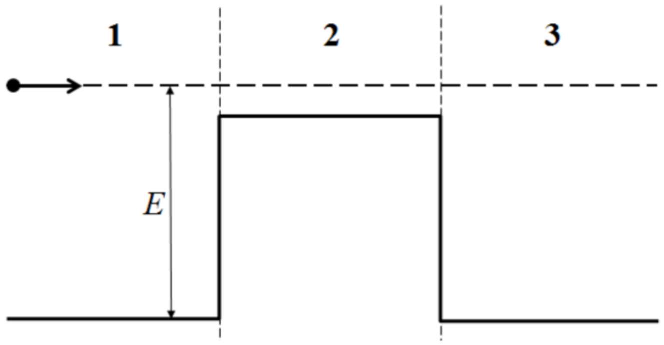
[4]

4 QLED display screens use sized nanoparticles to produce particular colour of light. Explain how different nanoparticle sizes can be used to produce red, green and blue emission. **[4]**

CONTINUED…

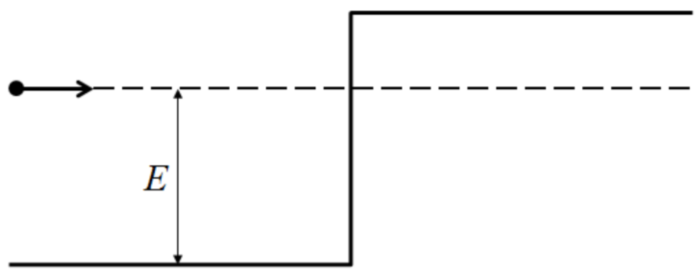
SECTION A

5 A particle of energy E approaches a potential barrier as shown below. Under what conditions is there 100% probability that the particle is transmitted past the barrier?



[4]

6 The diagram below shows a particle of total energy E approaching a barrier with a potential energy greater than E. Which of the following particles penetrates deeper into the barrier? Justify your answer.



(a) E= 5 eV electron
(b) E= 5 eV proton
(c) E= 10 eV electron
(d) E= 10 eV proton **[4]**

7 Describe two physical situations in which quantum mechanical tunneling is important. **[4]**

CONTINUED…

SECTION A

8 Briefly explain what is meant by the terms *distinguishable* and *indistinguishable* as applied to particles, and discuss the relevance of these concepts to the study of *statistical mechanics*. **[4]**

9 Consider a system of classical distinguishable particles with access to two energy levels. The lower energy level has energy $\varepsilon_1 = 0$ J and degeneracy $g_1 = 1$, and the upper energy level has energy $\varepsilon_2 = 2.9 \times 10^{-20}$ J and degeneracy $g_2 = 5$. Calculate the temperature at which the occupation probabilities of the two levels are the same. **[4]**

10 Sketch a graph showing the Fermi-Dirac distribution function versus energy for a system in which the temperature is much lower than the Fermi temperature. On the same sketch, draw a second Fermi-Dirac distribution function for a different system of fermions that has the same Fermi level but higher temperature. Label your axes and clearly indicate which of the functions you plot corresponds to the higher temperature case. **[4]**

CONTINUED…

SECTION B

Use a Section B answer book

Answer TWO questions from this section

11 **(a)** A particle of mass m is trapped in an infinite 1-dimensional potential well in the region $-\frac{a}{2} < x < \frac{a}{2}$. Obtain the odd and even parity eigenfunction solutions of the Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + (V - E)\psi = 0$$

for this system and hence obtain an expression for the ground state energy. **[8]**

(b) The ground state eigenfunction is given by $\psi = A\cos\left(\frac{\pi x}{a}\right)$.

(i) Explain why the operator $\hat{p} = -i\hbar\frac{\partial}{\partial x}$ is required to evaluate an expectation value <p> for the momentum of the particle.

(ii) Calculate <p> and explain why the answer is still consistent with the particle's kinetic energy in the ground state. **[8]**

(c) Describe qualitatively how the eigenfunctions for the allowed states in a finite potential well differ from those for the infinite well. **[4]**

CONTINUED…

SECTION B

12 A particle of mass m is confined in a one dimensional well with potential energy

$$V = \frac{1}{2}m\omega^2 x$$

(a) For this potential, show that $\psi = A\exp(-\gamma x^2)$ is a solution of the Schrödinger wave equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + (V - E)\psi = 0$$

and hence obtain expressions for γ and the energy of this state. **[9]**

(b) Obtain an expression for the normalization constant $A.$ **[6]**

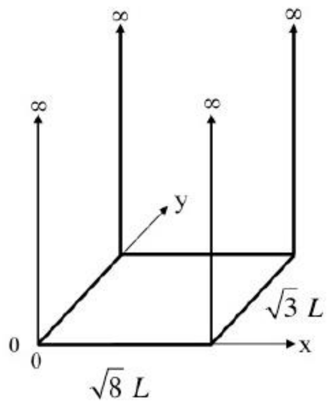
(c) Contrast the probability density for this state compared to that expected for a classical particle moving in this well. **[5]**

[Hint: $\int_0^\infty \exp(-\beta x^2) = \frac{1}{2}\sqrt{\frac{\pi}{\beta}}$]

CONTINUED…

SECTION B

13 An electron is bound inside a two-dimensional infinite quantum well with sides of length $\sqrt{8}L$ and $\sqrt{3}L$ as shown in the figure below.



$$V(x,y) = 0 \quad \text{for} \quad 0 < x < \sqrt{8}\,L,\ 0 < y < \sqrt{3}\,L$$
$$= \infty \quad \text{elsewhere}$$

(a) Using the time-independent Schrödinger wave equation, derive an expression for the wavefunction $\psi(x, y)$ of the electron in the 2D potential well. **[12]**

(b) Derive an expression for the allowed energies of the electron in the 2D potential well. **[4]**

(c) Tabulate the energies for the first three excited energy levels, clearly listing the appropriate combinations of quantum numbers, and the degree of degeneracy. **[4]**

CONTINUED…

SECTION B

14 **(a)** The equation below is the radial Schrödinger wave equation for the hydrogen atom. Without going into full mathematical detail, explain how this equation is obtained from the full three-dimensional Schrödinger wave equation and explain what each of the terms inside the square brackets represent.

$$\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}+\frac{2\mu}{\hbar^2}\left[E+\frac{e^2}{4\pi\epsilon_0 r}-\frac{l(l+1)\hbar^2}{2\mu r^2}\right]R=0$$

[7]

(b) Explain how the quantum number l is related to the orbital angular momentum of the electron. **[3]**

(c) Make a sketch of the hydrogen energy-level diagram, identifying the allowed eigenvalues and the eigenstates corresponding to different l values for n up to 4. Indicate the degeneracies of each of the eigenstates.
(Hint: Ground state energy of hydrogen atom = –13.6 eV) **[5]**

(d) The normalized wavefunction for the ground state of the hydrogen atom is

$$\psi_{1s}(r)=\frac{1}{\pi^{1/2}a_0^{3/2}}\ \exp(-r/a_0)$$

where a_0 is the Bohr radius. If an electron occupies this state, what is its most likely radial position? **[5]**

CONTINUED…

SECTION C

Use a Section C answer book

Answer ONE question from this section

15 Consider a system of N spin-1/2 distinguishable particles that occupy a volume, V, and interact with a uniform external magnetic field that points in the z-direction, $\boldsymbol{B} = (0,0,B)$.

(a) Given that the energy of interaction between a magnetic dipole $\boldsymbol{\mu}$ and a magnetic field is

$$E = -\boldsymbol{\mu}\,.\boldsymbol{B},$$

show that the partition function associated with a single spin-1/2 particle in the system is

$$Z = \exp\left(\frac{\mu_z B}{k_B T}\right) + \exp\left(\frac{-\mu_z B}{k_B T}\right)$$

where μ_z is the magnitude of the z-component of the magnetic dipole moment associated with the particle's spin. **[6]**

(b) Hence, or otherwise, show that the magnetization of the complete system can be expressed as

$$M = \frac{N\mu_z}{V}\frac{\exp\left(\frac{\mu_z B}{k_B T}\right) - \exp\left(\frac{-\mu_z B}{k_B T}\right)}{\exp\left(\frac{\mu_z B}{k_B T}\right) + \exp\left(\frac{-\mu_z B}{k_B T}\right)}$$

[5]

(c) Show that the magnetization of the system is proportional to T^{-1} at very high temperatures. **[5]**

(d) Explain how the magnetization will depend on T for very low temperatures. **[4]**

[The magnetization, M, of a system can be defined as the net magnetic dipole moment per unit volume.]

CONTINUED…

SECTION C

16 **(a)** By considering quantum states in a three-dimensional infinite potential well, or otherwise, prove that the density of states in wavenumber space, $g(k)$, for a three-dimensional system occupying a volume V is given by

$$g(k)\,\mathrm{d}k = G\frac{Vk^2}{2\pi^2}\,\mathrm{d}k$$

where G is a degeneracy factor. **[8]**

(b) For a classical, non-relativistic dilute gas, the probability distribution in velocity-space is given by the Maxwell-Boltzmann distribution

$$p(v)\,\mathrm{d}v = 4\pi G\left(\frac{m}{2\pi k_B T}\right)^{3/2} v^2 \exp\left(-\frac{mv^2}{2k_B T}\right)\mathrm{d}v$$

where m is the particle mass, v is the particle speed and T is the temperature.

(i) Without detailed calculation, outline the physics that determines the v-dependent terms in this equation, and the connection to the result in part (a). **[4]**

(ii) Use the Maxwell-Boltzmann formula to *estimate* the number of helium-4 atoms moving with speed in the range 25 ms^{-1} to 26 ms^{-1} in a pure helium-4 gas with a total of 3×10^{26} particles and a temperature of 10 K. **[5]**

(iii) The calculation in part **(ii)** does not involve the volume of the system. Explain whether or not the number of particles in the given speed range would actually be different if the gas were to occupy either a very large or very small volume. **[3]**

[You may assume that the mass of a helium-4 atom is 4 a.m.u. and that the degeneracy factor for helium-4 particles is $G = 1$.]

END OF EXAMINATION