Lecture 8:

Magnetic field: general properties

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Source of magnetostatic fields

We have seen that the source of a magnetostatic field is a steady current:

Ampere's law

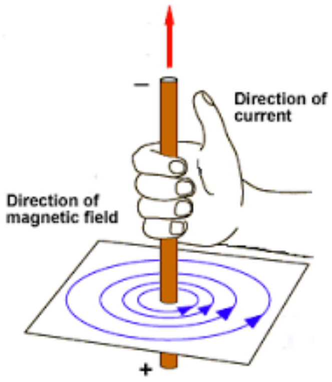
$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I$$

In differential form this becomes:
(see Jackson for derivation)

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

This is a fundamental difference with the electrostatic case. The curl of the field is not zero, meaning that **the field is not conservative** or, equivalently, **we cannot define a scalar magnetic potential.**

Another consequence of this formula is that magnetic field lines are always in the form of closed loops around the current that has generated them:



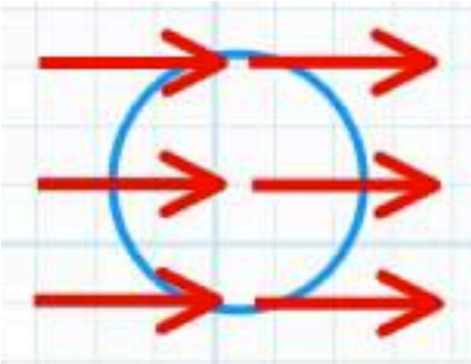
Divergence of the magnetic field

What about the divergence of the field?

It is an empirical fact that **the divergence of B is always zero**

$$\nabla \cdot \vec{B} = 0$$

in other words, the flux of magnetic field through a closed surface is **always** zero.



This implies that, for any surface, the amount of magnetic field entering it is the same as the one leaving it.

This has a fundamental consequence:

Magnetic monopoles do not exist in Nature!

The vector potential

Can we define some form of potential, even if it cannot be a scalar?

- It is a mathematical identity that the divergence of the curl of a vector is always zero, regardless of the specific choice of the vector:

$$\nabla \cdot \nabla \times \vec{A} = 0 \qquad \qquad \forall \vec{A}$$

By looking at the divergence of B, we can then say that:

$$\nabla \cdot \vec{\mathrm{B}} = 0 \;\; , \;\; \nabla \cdot \nabla \times \vec{\mathrm{A}} = 0 \quad \rightarrow \quad \vec{\mathrm{B}} = \nabla \times \vec{\mathrm{A}}$$

A magnetic field can then always be expressed as the curl of another vector, which is then called the **vector potential**
(compare with the scalar electrostatic potential!)

Vector potential choice

We might recall that the electrostatic potential was not uniquely defined, since adding any constant to it, would not change the resulting electric field.

A similar property applies to the vector potential. Let us assume that we define a different vector potential: $\vec{A}' = \vec{A} - \nabla\Omega$, where Ω is any **scalar function**. In this case the curl of A' is:

$$\nabla \times \vec{A}' = \nabla \times \vec{A} - \nabla \times \nabla\Omega$$

but the curl of a gradient is always zero (why?) and, therefore the second term on the right hand side is always zero, for any choice of Ω.

A and A' produce the same magnetic filed and are, thus, undistinguishable!

Vector potential choice

What about the divergence of A and A' though?

$$\vec{A}' = \vec{A} - \nabla\Omega \qquad \rightarrow \qquad \nabla\cdot\vec{A}' = \nabla\cdot\vec{A} - \nabla^2\Omega$$

Now, the second term is not exactly zero anymore!

We have then a certain degree of freedom in choosing A. We can choose it with the gradient of an arbitrary function summed to it, implying that we can construct A so that its divergence is anything we want!

This freedom is called in physics **Gauge Freedom**

Vector potential choice

There are different possible choices for A that are routinely used in electromagnetism. Here we focus on a specific one. We choose A such that $\vec{B} = \nabla \times \vec{A}$ and $\underline{\underline{\nabla \cdot \vec{A} = 0}}$ → this is the Gauge choice!

This choice is particularly useful, because it allows us to find A starting from the currents:

$$\nabla \times \vec{B} = \nabla \times \nabla \times \vec{A} = \mu_0 \vec{J}$$

It is a mathematical identity that:

$$\nabla \times \nabla \times \vec{A} = \nabla\left(\cancel{\nabla \cdot \vec{A}}\right) - \nabla^2 \vec{A} \quad \rightarrow \quad \boxed{\nabla^2 \vec{A} = -\mu_0 \vec{J}}$$

our gauge choice!

Vector potential choice

This equation is quite similar to what we have found for the electrostatic case:

Magnetostatics: $\nabla^2 \vec{A} = -\mu_0 \vec{J}$

Electrostatics: $\nabla^2 \psi = -\dfrac{\rho}{\varepsilon_0}$

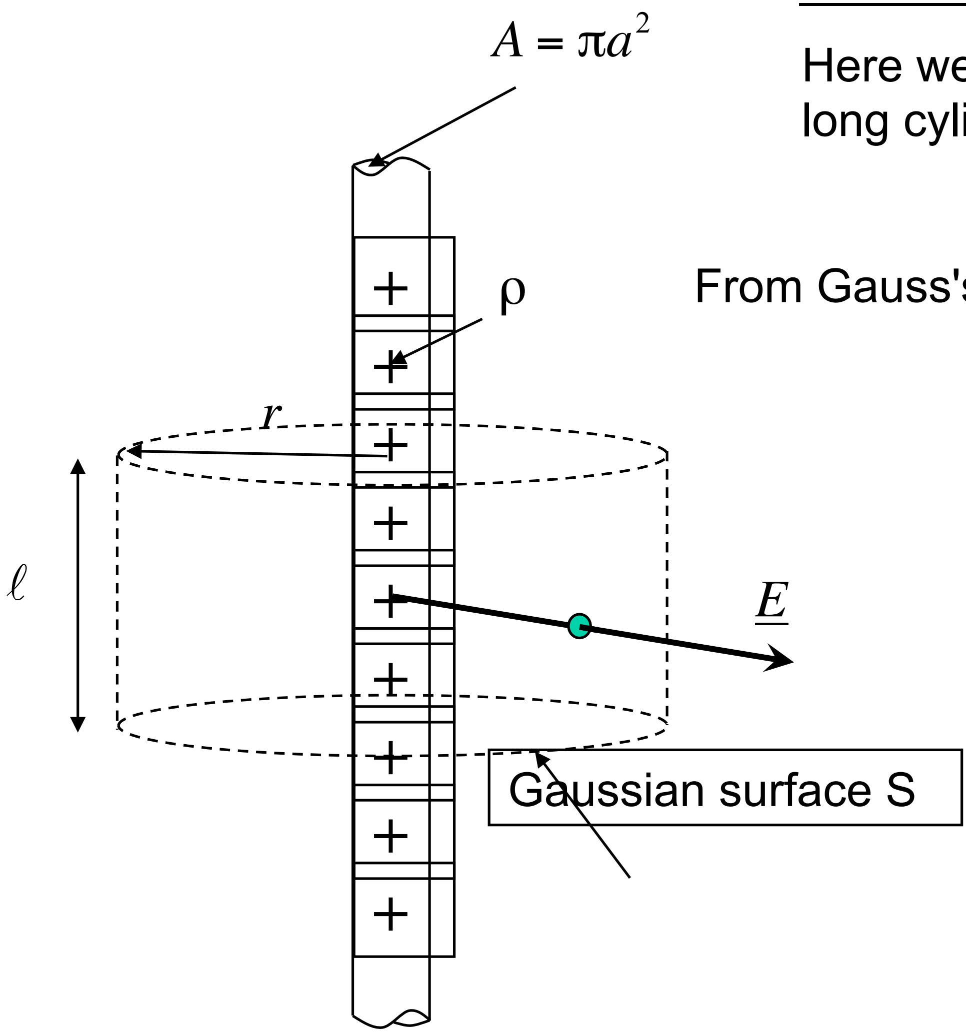
The formal solution of these two equations is then very similar:

$$\psi = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho}{r}\, \mathrm{dV}$$

$$\underline{A} = \frac{\mu_0}{4\pi} \int \frac{\underline{J}}{r}\, \mathrm{dV}$$

Example

Example: Electrostatic Problem



OUTSIDE

Here we are calculating the E-field from a long cylinder of volume charge density ρ

From Gauss's law $\int_S \underline{E} \cdot \underline{dS} = \int_V \frac{\rho dV}{\varepsilon_0}$

$$E 2\pi r \ell = \rho \pi a^2 \ell / \varepsilon_0$$

$$E = \frac{\rho \pi a^2}{2\pi\varepsilon_0 r} = \frac{-d\psi}{dr}$$

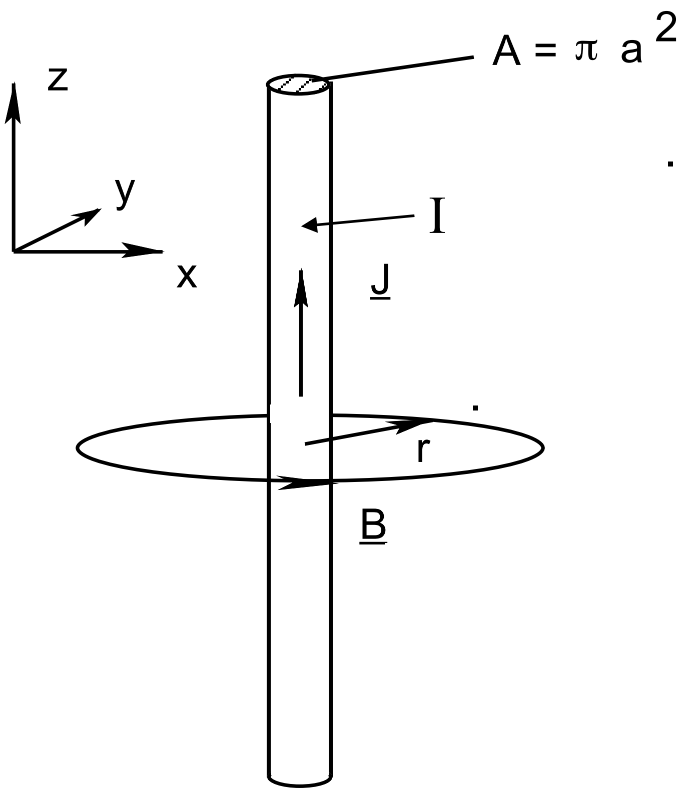
$$\psi = \frac{-\pi a^2 \rho}{2\pi\varepsilon_0} \int \frac{dr}{r}$$

$$\psi = -\frac{\pi a^2 \rho}{2\pi\varepsilon_0} \ln r$$

Example

Example

$$\underline{J} = J_z = I / \pi a^2$$



Comparing solution with electrostatic case

$$\ldots \quad \psi = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho dV}{r} \quad \text{AND} \quad A_z = \frac{\mu_0}{4\pi}\int\frac{J_z dV}{r}$$

$$\rho / \varepsilon_0 \Rightarrow \mu_0 J_z$$

$$\therefore A_z = \frac{-\pi a^2}{2\pi}\mu_0 J_Z \ln r$$

or

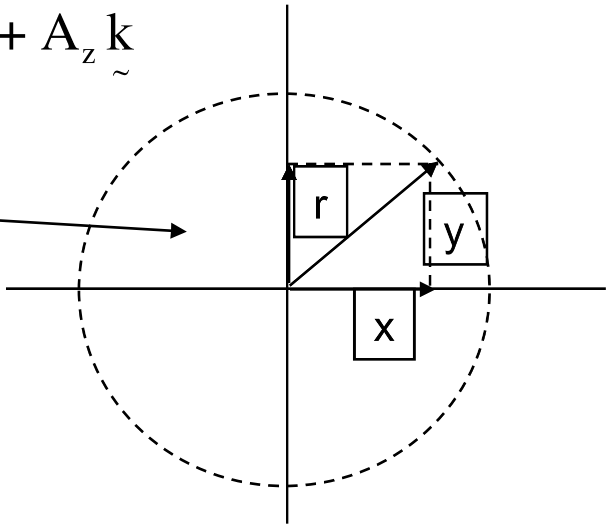
$$A_z = \frac{-\mu_0 I_z}{2\pi}\ln r$$

and since $J_x = J_y = 0$ then $A_x = A_y = 0$

Example

Now $\underline{B} = \nabla \times \underline{A}$ and $\underline{A} = 0\,\underset{\sim}{i} + 0\,\underset{\sim}{j} + A_z\,\underset{\sim}{k}$

With $r = \left(x^2 + y^2\right)^{\frac{1}{2}}$



This leads to

$$B_x = -\frac{I\,\mu_0}{2\pi}\,\frac{y}{r^2}$$

$$B_y = \frac{I\,\mu_0}{2\pi}\,\frac{x}{r^2}$$

$$B_z = 0$$

Students should fill in the details here. The procedure is similar to that which follows

Hence $B = \dfrac{\mu_0}{2\pi}\,\dfrac{I}{r}$

This circulates around the wire!