Use lined, single-sided A4 paper
with a black or blue pen.
Write your student number
at the top of every page.

Any non-graphical calculator, except those with pre-programmable memory, may be used in this examination

LEVEL 2
Examination contributing to the Degrees of Bachelor of Science (BSc) and Master in Science (MSci)

PHY2001 - EXAM
Quantum and Statistical Physics
Friday, 7th August 2020, 09.30 - 13.30

Examiners: Prof S Matthews, Dr P van der Burgt
and the Internal Examiners
Dr J Greenwood (j.greenwood@qub.ac.uk)

Answer ALL TEN questions in Section A for 4 marks each.
Answer TWO questions in Section B for 20 marks each.
Answer ONE question in Section C for 20 marks.
You have FOUR hours to complete and upload this paper.

Contact the module coordinator if you have queries/problems at s.kar@qub.ac.uk and copy to mpts@qub.ac.uk

By submitting the work, you are declaring that:
1. The submission is your own original work and no part of it has been submitted for any other assignments;
2. You understand that collusion and plagiarism in an exam are major academic offences, for which a range of penalties may be imposed, as outlined in the Procedures for Dealing with Academic Offences.

THE QUEEN'S UNIVERSITY OF BELFAST
DEPARTMENT OF PHYSICS AND ASTRONOMY

PHYSICAL CONSTANTS

Speed of light in a vacuum	$c = 3.0 \times 10^{8}$ ms^{-1}
Permeability of a vacuum	$\mu_0 = 4\pi \times 10^{-7}$ Hm^{-1}
	$\approx 1.26 \times 10^{-6}$ Hm^{-1}
Permittivity of a vacuum	$\varepsilon_0 = 8.85 \times 10^{-12}$ Fm^{-1}
Elementary charge	$e = 1.6 \times 10^{-19}$ C
Electron charge	$= -1.6 \times 10^{-19}$ C
Planck Constant	$h = 6.63 \times 10^{-34}$ Js
Reduced Planck Constant	$\hbar = 1.05 \times 10^{-34}$ Js
Rydberg Constant for hydrogen	$R_\infty = 1.097 \times 10^{7}$ m^{-1}
Unified atomic mass unit	$1u = 931\ MeV = 1.66 \times 10^{-27}$ kg
1 electron volt (eV)	$= 1.6 \times 10^{-19}$ J
Mass of electron	$m_e = 9.11 \times 10^{-31}$ kg
Mass of proton	$m_p = 1.67 \times 10^{-27}$ kg
Mass of neutron	$m_n = 1.67 \times 10^{-27}$ kg
Molar gas constant	$R = 8.31$ JK^{-1}mol^{-1}
Boltzmann constant	$k = 1.38 \times 10^{-23}$ JK^{-1}
Avogadro constant	$N_A = 6.02 \times 10^{23}$ mol^{-1}
Gravitational constant	$G = 6.67 \times 10^{-11}$ Nm^{2}kg^{-2}
Acceleration of free fall on the Earth's surface	$g = 9.81$ ms^{-2}

CONTINUED...

SECTION A
Answer ALL 10 questions from this section
Full explanations of your answers are required to attain full marks

1. In an experiment, an electron is determined to be within 1.0 nm of a particular point. If we try to measure the electron's momentum p, what would be its minimum uncertainty? What would be the uncertainty in the electron's velocity?

[4]

2. In a measurement at the CERN accelerator (Switzerland), the electrons assume a Gaussian distribution for their density given by:

$$\rho = A\ \exp(-\lambda(x-a)^2),$$

where A and λ are constants.

(a) What is the magnitude of A assuming $\int_{-\infty}^{\infty} \rho(x)dx = 1$?

(b) Find $< x >$ for this distribution.

Hint: $\int_{-\infty}^{\infty} \exp(-\alpha u^2)du = \sqrt{\pi/\alpha}$

[4]

3. Consider the following wave function

$$\Psi(x,t) = \sqrt{\delta}\ \exp(-\delta|x|)\ \exp(-2i\alpha t)$$

where δ and α are positive real constants. What is the expectation value of x^2?

[4]

4. Consider an infinite quantum well defined as
$U = 0$ when $0 < x < a$
$U \to \infty$ when $x \leq 0$ or $x \geq a$

(a) If the width of the potential is $a = 2.0 \times 10^{-10}$ m, what is the energy (in eV) of the second excited state of an electron in this potential?

(b) Draw the wave functions for the first, second and third excited states inside the potential.

[4]

5. A nano-particle confines an electron so that when excited, a transition from the first excited state to the ground state results in emission of a photon at a wavelength of 800 nm. If the diameter of the nano-particle is reduced by 25%, what is the wavelength of the transition now? Justify your answer.

[4]

6. Discuss the nature of the wave function of a particle trapped inside a square potential well. Show that the general form of such eigenfunctions in one dimension is $\psi(x) = A\sin(kx) + B\cos(kx)$, where A and B are constants and k represents the wavenumber.

[4]

7. The tip of a scanning tunnelling microscope is placed 1.0 nm away from a conducting surface and is biased by -0.5 V with respect to the surface. If the work function of the tip is 3.9 eV, estimate the fractional change in tunnelling current if the tip moves 0.03 nm farther from the surface.

[4]

8. Explain the meaning of the terms *distinguishable* and *indistinguishable* as applied to particles. Briefly explain the relevance of these concepts to statistical mechanics.

[4]

CONTINUED...

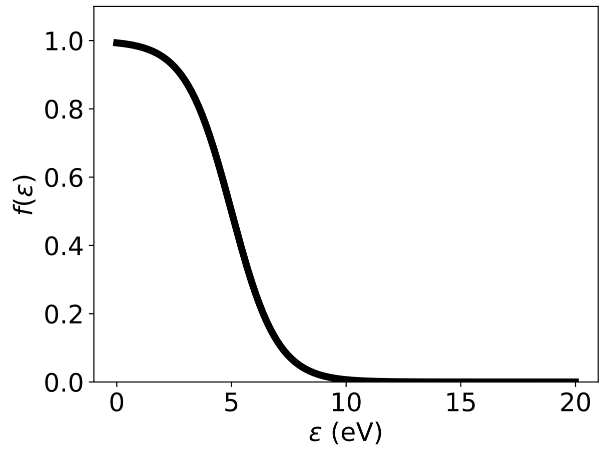
9. For a classical dilute gas, the probability distribution for particle speed is given by the Maxwell-Boltzmann distribution

$$p(v)\mathrm{d}v = 4\pi G\left(\frac{m}{2\pi k_B T}\right)^{3/2} v^2 \exp\left(-\frac{mv^2}{2k_B T}\right)\mathrm{d}v \ ,$$

where v is particle speed, m is particle mass, T is temperature and G is an appropriate degeneracy factor. Explain in *physical* terms why this distribution tends to zero at both high and low speeds.

[4]

10. The figure below shows the distribution function, $f(\varepsilon)$, for a system of particles. Explain whether the particles in this system are fermions or bosons. Estimate the temperature to which this system would need to be raised so that the dilute-gas approximation would apply, assuming that the number of particles and volume of the system remain the same. Explain your reasoning.



[4]

CONTINUED...

SECTION B

Answer <u>TWO</u> questions from this section

11. **(a)** Provide a brief description of the wave-particle duality utilized in quantum mechanics. Give arguments in terms of the de Broglie equation ($p = h/\lambda$) to explain the duality of the electron.

[3]

(b) Describe the double-slit experiment and explain wave-particle duality in this context.

[4]

(c) Give a short description of the photoelectric effect described by Einstein.

[3]

(d) Discuss the black-body radiation problem and Planck's law.

[4]

(e) Describe in your words what a stationary state means in terms of Schrödinger's equation.

[3]

(f) Explain the main implications of Born's interpretation of the wave-function.

[3]

CONTINUED...

12. **(a)** For an electron in an infinite well potential between $0 \leq x \leq L$ with a wave function given by

$$\psi(x) = \sqrt{\frac{30}{L^5}}(Lx - x^2) \ ,$$

what is the expectation value of x within the potential?

[4]

(b) An electron is in $\Psi(x,t)$ state given by the following wavefunction

$$\Psi(x,t) = A \ \exp\left(-a\left[\frac{mx^2}{\hbar} + it\right]\right)$$

Normalize this wave-function to find A.

[4]

(c) For what potential energy function, $V(x)$, does the wave-function $\Psi(x,t)$ in (b) satisfy the Schrödinger's equation?

[4]

(d) Provide a brief description of the main problems of Bohr's model of the hydrogen atom.

[4]

(e) Describe the Copenhagen interpretation of quantum mechanics.

[4]

13. Let us assume a stream of 100 non-relativistic electrons of kinetic energy E travelling along the x-axis experiences an abrupt change in potential from V_1 to V_2 at $x = a$, where $E > V_1 > V_2$.

(a) Using the time-independent Schrödinger wave equation, show that the transmission coefficient for the particles' wave function at $x = a$ is

$$T = \frac{4k_1k_2}{(k_1+k_2)^2} ,$$

where, k_1 and k_2 are the wavenumbers of the particles' wave functions in the regions $x < a$ and $x > a$ respectively. To attain full marks, the answer must carefully explain/justify all steps.

Hint: The transmission coefficient is the ratio between transmitted and incident probability fluxes, where the probability flux for a wave function $\psi(x)$ is $(\hbar k/m)|\psi(x)|^2$.

[12]

(b) Draw a suitably labelled and scaled diagram to represent appropriately the particles' wave functions in both regions.

[4]

(c) For E = 10 eV, V_1 = 5 eV and V_2 = -5 eV, calculate the wavelengths of the particle waves in both regions and the expected number of electrons that will reach the region $x > a$.

[4]

14. In Schrödinger's model, the eigenfunction for an electron orbiting around a Hydrogen nucleus can be written in the form $\psi(r,\theta,\phi) = R_{nl}(r)\ Y_{lm}(\theta,\phi)$. The radial wave function for an electron in the first excited state is given by

$$R(r) = \frac{1}{\sqrt{3}(2a_0)^{3/2}} \frac{r}{a_0} \exp\left(-\frac{r}{2a_0}\right),$$

where a_0 is the Bohr radius.

(a) Calculate the most probable and the average distances of the electron from the proton. [Note: $\int_0^\infty x^n \exp(-Kx)dx = n!/K^{n+1}$]

[10]

(b) Draw a suitably labelled sketch of the radial probability density distribution of the electron. Calculate the probability density at the peak of the graph in terms of the Bohr radius. What is the orbital angular momentum quantum number of the electron that this radial wavefunction corresponds to? Justify your answer.

[7]

(c) Calculate the energy of the electron and degree of degeneracy of the energy level in the absence of an external magnetic field. Ignore the spin of the electron.

[3]

SECTION C
Answer <u>ONE</u> question from this section

15. **(a)** Explain what is meant by the *microstate* and *macrostate* of a system of particles in statistical mechanics. Your answer should include examples of the concepts you discuss as they apply to an isolated system of weakly-interacting distinguishable particles.

[4]

(b) Consider an isolated system of 4 weakly-interacting distinguishable particles. The energy levels available to the particles have equally spaced energies of 0, ε, 2ε, 3ε, etc. Each energy level has degeneracy of two (i.e. two distinct single-particle states per energy level).

(i) If the total energy of the system is $U = 3\varepsilon$, find all the allowed distributions for the number of particles in each energy level, and calculate the statistical weight for each distribution.
Present your answer in a tabular form.

[8]

(ii) Calculate the average distribution for the number of particles in each energy level.

[3]

(iii) Calculate the entropy of this system.

[2]

(c) Explain whether your answers to parts (b)(ii) and (b)(iii) would change if the energy levels had degeneracy of three, rather than two.
Your answer does not need to include detailed calculations, but should clearly justify your reasoning.

[3]

16. **(a)** According to the Boltzmann distribution, the probability of an energy level i being populated at temperature T is given by

$$p_i = \frac{g_i}{Z} \exp\left(-\frac{\varepsilon_i}{k_B T}\right)$$

where g_i is the degeneracy and ε_i the energy of the level. Give an expression for the partition function, Z, and comment on its physical significance.

[3]

(b) The energy-levels for a one-dimensional (1D) simple harmonic oscillator are non-degenerate and have energies given by $\varepsilon_n = (n + 1/2)\hbar\omega$, where ω is a characteristic angular frequency and n is zero or a positive integer.

(i) Assume that Boltzmann statistics apply to a system of N such oscillators. Show that, at low temperatures, the number of oscillators occupying each state depends on n via

$$N \exp(-\alpha n) \ ,$$

where $\alpha = \hbar\omega / k_B T$.

[5]

(ii) For $\omega = 3.8 \times 10^{13}$ rad s^{-1}, estimate the maximum temperature for which the formula in part (b)(i) will be accurate to 1% precision (or better). Carefully explain your reasoning.

[5]

(c) **(i)** Explain what is meant by the density of states, $g(\varepsilon)$, and discuss the circumstances for which using a continuous representation of energy levels in a system is appropriate.

[4]

(ii) Calculate a value for $g(\varepsilon)$ that would be appropriate for the system discussed in part (b) when $\omega = 3.8 \times 10^{13}$ rad s^{-1}.

[3]

END OF EXAMINATION