DEFINITIONS

Frames of Reference:

A frame of reference is a conventional standard of rest relative to which measurements can be made and experiments described.

Inertial Frames:

An inertial frame is one in which spatial relations, as determined by rigid scales at rest in the frame, are Euclidean and in which there exists a universal time in terms of which free particles remain at rest or continue to move with constant speed along straight lines.

(*i.e.* an inertial frame is a frame within which free particles obey Newton's first law.)

(Euclidean, *e.g.* a straight line joining two points is a geodesic - shortest distance between the two points.)

Axiom 1

Any frame in uniform motion relative to a given inertial frame is itself inertial.

(Clearly must be true in Newton's theory.)

Axiom 2

All inertial frames are spatially homogenous and isotropic.

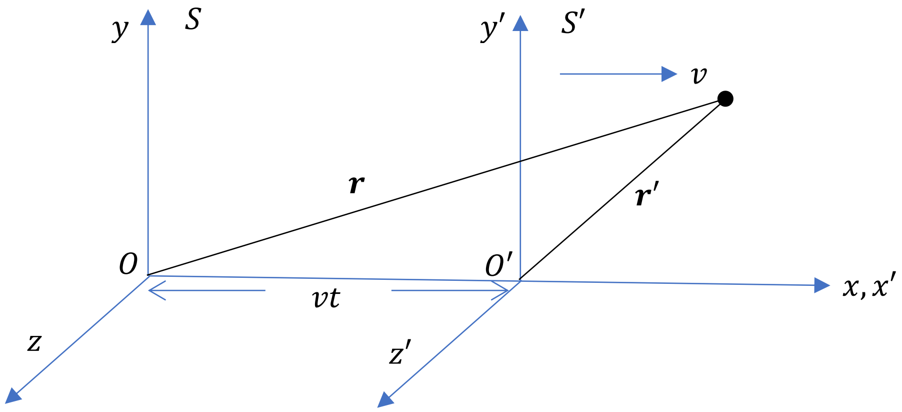
(*i.e.* the outcome of an experiment is the same whenever its initial conditions differ only by a translation (homogeneity) and rotation (isotropy) in some inertial frame).

Axiom 3

All inertial frames are temporally homogenous.

(*i.e.* identical experiments at different times give identical results)

GALILEAN TRANSFORMATION



Let frame S' move with constant velocity v in the x-direction with respect to frame S. Note that these are inertial frames. Also, that there is no loss of generality in considering v to be in the x-direction as we can always rotate frames to make this true. However, the velocity $\boldsymbol{u}$ of a particle moving in the frames should generally have components in all three directions, *i.e.* $\boldsymbol{u} = (u_x, u_y, u_z)$. In this course we will always set-up the frames in this way.

Let the position vectors in S and S' respectively be:

$$\boldsymbol{r} = (x, y, z) \quad \text{and} \quad \boldsymbol{r}' = (x', y', z')$$

At time $t = 0$ let $\boldsymbol{r} = \boldsymbol{r}' = (0,0,0)$, *i.e.* the origins O and O' are overlapped.

Then at some later time t:

$$x' = x - vt$$
$$y' = y$$
$$z' = z$$
$$t' = t \qquad \text{(universality, or absoluteness of time)}$$

These are the Galilean Transformation equations

Velocity transformation:

Velocity:

$$\boldsymbol{u} = \frac{d\boldsymbol{r}}{dt} \quad \text{and} \quad \boldsymbol{u}' = \frac{d\boldsymbol{r}'}{dt'}$$

$$\frac{dx'}{dt'} = \frac{dx'}{dt} = \frac{dx}{dt} - v \qquad \text{or} \qquad u'_x = u_x - v$$

$$\frac{dy'}{dt'} = \frac{dy'}{dt} = \frac{dy}{dt} \qquad \text{or} \qquad u'_y = u_y$$

$$\frac{dz'}{dt'} = \frac{dz'}{dt} = \frac{dz}{dt} \qquad \text{or} \qquad u'_z = u_z$$

Acceleration transformation:

Acceleration:

$$\boldsymbol{a} = \frac{d\boldsymbol{u}}{dt} \quad \text{and} \quad \boldsymbol{a}' = \frac{d\boldsymbol{u}'}{dt'}$$

$$\frac{du'_x}{dt'} = \frac{du'_x}{dt} = \frac{du_x}{dt} \qquad \text{or} \qquad a'_x = a_x$$

$$\text{similarly} \qquad a'_y = a_y$$

$$\text{and} \qquad a'_z = a_z$$

i.e. acceleration is identical in both frames

Definition:

A property is <u>invariant</u> if it is identical in all inertial frames

(*i.e.* independent of v)

Hence acceleration $\boldsymbol{a}$ is invariant under a Galilean transformation

Also, in Newtonian mechanics mass m is invariant

Newton's second law:

$$\boldsymbol{F} = \frac{d}{dt}(m\boldsymbol{u})$$

$$\boldsymbol{F} = m\,\frac{d\boldsymbol{u}}{dt}$$

$$\boldsymbol{F} = m\boldsymbol{a}$$

Since both m and $\boldsymbol{a}$ are invariant, force $\boldsymbol{F}$ must also be invariant

Thus, in Newtonian mechanics the laws of mechanics are identical in all inertial frames, or 'all inertial observers are equivalent as regards the laws of mechanics.'

EINSTEIN'S POSTULATES

1. The laws of Physics are identical in all inertial frames.

    - The principle of relativity -

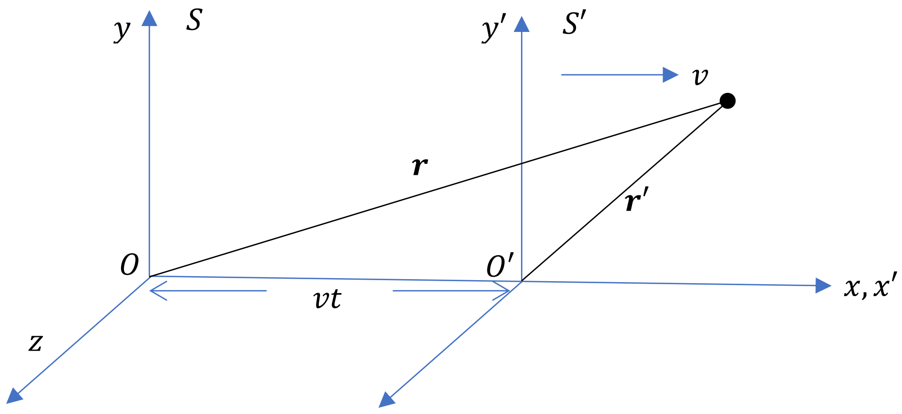
2. There exists an inertial frame in which light signals in vacuum travel rectilinearly at constant speed c, in all directions, independently of the motion of the source.

Combining these two axioms:

Light signals in vacuo are propogated rectilinearly with the same speed c in all inertial frames.

- Law of light propogation -

LORENTZ TRANSFORMATION (LT)



As before, at time $t = t' = 0$ let the origins O and O' overlap, $\boldsymbol{r} = \boldsymbol{r}' = (0,0,0)$.

At this time a pulse of light is produced at the common origin.

By Einstein's second axiom both observers (in S and S' respectively) must see the light propagating outwards from their individual origins with the same speed, c.

[Note that this constraint arising from Einstein's postulates replaces the constraint of universality of time, meaning that t can no longer be assumed to be equal to t']

i.e. the observers must see a wavefront as a sphere of radius r and r' respectively.

i.e. in S $$r = ct$$

Therefore $$r^2 - c^2t^2 = 0$$

or $$x^2 + y^2 + z^2 - c^2t^2 = 0$$

Similarly in S' $$x'^2 + y'^2 + z'^2 - c^2t'^2 = 0$$

Note that these expressions (for S and S') are the same (both = 0), *i.e.* the expressions are <u>invariant</u> wrt transformation from S to S'.

Since, as always, the frame S' is moving only in the x-direction, then

$$y' = y$$
$$z' = z$$

Hence $$c^2t^2 - x^2 = c^2t'^2 - x'^2 \tag{1}$$

As for the Galilean set of transformations we wish to relate x' to x and t.

Note that the transformation must be linear, due to the spatial and temporal homogeneity requirement.

Also, we have the same constraint as before:

i.e. at $x' = 0$, $x = vt$

Hence assume a solution of the form:

$$x' = \gamma(x - vt) \tag{2}$$

where γ is an invariant quantity. This satisfies both conditions.

The inverse transformation, from S' to S, then becomes:

$$x = \gamma'(x' + vt') \tag{3}$$

where we cannot assume that $\gamma = \gamma'$

Rearranging (3) to eliminate t':

$$t' = \gamma\{t - \tfrac{x}{v}\left(1 - \tfrac{1}{\gamma\gamma'}\right)\} \tag{4}$$

Substituting for t' from (4) and x' from (2) in (1):

$$c^2t^2 - x^2 = c^2\gamma^2\{t^2 - \frac{2}{v}\left(1 - \frac{1}{\gamma\gamma'}\right)xt + \frac{1}{v^2}\left(1 - \frac{1}{\gamma\gamma'}\right)^2 x^2\} - \gamma^2(x^2 - 2vxt + v^2t^2)$$

Equating the coefficients of t^2:

$$c^2 = c^2\gamma^2 - v^2\gamma^2$$

Therefore

$$\gamma^2 = \frac{1}{\left(1 - \frac{v^2}{c^2}\right)}$$

i.e.

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

Equating coefficients of xt:

$$0 = -2\frac{c^2\gamma^2}{v}\left(1 - \frac{1}{\gamma\gamma'}\right) + 2v\gamma^2$$

therefore

$$v^2 = c^2\left(1 - \frac{1}{\gamma\gamma'}\right)$$

and

$$\gamma' = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} = \gamma$$

Eliminating x' from (1) and (2) can show that:

$$t' = \gamma(t - \frac{v}{c^2}x)$$

Trivially

$$y' = y$$

and

$$z' = z$$

Hence the set of Lorentz transformation equations may be expressed as:

$$\boxed{\begin{gathered} x' = \gamma(x - vt) \\ y' = y \\ z' = z \\ t' = \gamma(t - \frac{v}{c^2}x) \end{gathered}}$$

where

$$\boxed{\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}}$$

CONSEQUENCES OF LT

1. Simultaneity

Suppose two events occur at points x_1 and x_2 in S and are simultaneous, *i.e.* $t_1 = t_2$

To an observer in the frame S' these two events will be observed at times t'_1 and t'_2

$$\therefore \qquad \Delta t' = t_2' - t_1' = \gamma \frac{v}{c^2} (x_2 - x_1) \neq 0$$

- Failure of simultaneity at a distance -

2. Lorentz Contraction

In S, let $L = x_2 - x_1$ be the distance between two points (say the length of a rod)

To an observer in the frame S' the ends of the rod will have positions x_2' and x_1'

$$\therefore \qquad L' = x_2' - x_1' = \gamma\{x_2 - x_1 - v(t_2 - t_1)\}$$

But since the observer in S' is moving with velocity v with respect to the rod it is essential that positions x_2' and x_1' are measured simultaneously

i.e. require that $\qquad t_2' = t_1'$

Transforming

$$\gamma\left(t_2 - \frac{v}{c^2}x_2\right) = \gamma\left(t_1 - \frac{v}{c^2}x_1\right)$$

$$t_2 - t_1 = \frac{v}{c^2}(x_2 - x_1)$$

and

$$L' = \gamma\left(L - \frac{v^2}{c^2}L\right) = \gamma L(1 - \frac{v^2}{c^2})$$

i.e.

$$L' = {}^{L}\!/_{\gamma}$$

i.e. $\quad L' < L \quad$ – hence a contraction

Note that the length of a body as measured in the frame of reference where its velocity is zero, *i.e.* its rest frame, is known as its proper length or rest length, and denoted L_0

Also, as $\quad v \to c, \quad \gamma \to \infty, \quad L' \to 0$

3. Time Dilation

Consider a clock at rest in the frame S at a position $x = x_0$

Consider two separate events (say ticks of the clock) occurring at times t_1 and t_2

An observer in the frame S' sees these two events occurring at times t_1' and t_2'

where

$$t_1' = \gamma\left(t_1 - \frac{v}{c^2}x_0\right)$$

and

$$t_2' = \gamma\left(t_2 - \frac{v}{c^2}x_0\right)$$

$$\therefore \qquad \Delta t' = t_2' - t_1' = \gamma(t_2 - t_1)$$

i.e. $\Delta t' > \Delta t$ - hence a time dilation

The time interval between two ticks of a clock appears longer in the moving frame.

Hence a clock moving uniformly with velocity v in an inertial frame of reference goes slow by a factor $1/\gamma$ relative to a standard clock at rest in that frame.

A clock in its rest frame is said to go at its rest rate

Note that as $v \to c$, $\gamma \to \infty$, and the clock rate $\to 0$

RELATIVISTIC MASS

The <u>*rest mass*</u> m_0 *of a particle is the mass of the particle as measured in its rest frame, i.e. the frame where the particle has a velocity* $u = 0$

Consider a particle of rest mass m_0 moving with a finite velocity u

According to Einstein's equation relating energy and mass

$$E = mc^2$$

where m is the <u>total mass</u> or <u>relativistic mass</u> of the particle

Now energy can be expressed as force x distance

$$E = Fx$$

Differentiating

$$\frac{dE}{dt} = Fu$$

or

$$\frac{d}{dt}(mc^2) = u\frac{d}{dt}(mu)$$

Integrate wrt t

$$c^2\int dm = \int u\,d(mu)$$

Multiply both sides by m

$$c^2\int m\,dm = \int mu\,d(mu)$$

Integrating

$$\frac{1}{2}c^2m^2 = \frac{1}{2}(mu)^2 + K$$

When $u = 0,\ \ m = m_0$

$$\Longrightarrow \qquad K = \frac{1}{2}c^2m_0^2$$

Therefore

$$m = m_0\left(1 - \frac{u^2}{c^2}\right)^{-1/2}$$

Note that the factor $\left(1 - \frac{u^2}{c^2}\right)^{-1/2}$ has the same form as γ with v, the velocity of the moving frame, replaced by u, the velocity of a moving particle.

We can thus refer to this factor as $\gamma(u)$, the (u) denoting that it is a function of u.

Hence $$m = \gamma(u)m_0$$

It is very important to note that $m(u)$, *i.e.* that the relativistic mass is a function of the particle velocity.

Hence Newton's second law

$$F = \frac{d}{dt}(mu) \neq m\frac{du}{dt}$$

Also note that as $u \to c, \quad \gamma(u) \to \infty,$ and the relativistic mass $m \to \infty$

VELOCITY TRANSFORMATION

As defined: $\boldsymbol{r} = (x, y, z)$ and $\boldsymbol{r}' = (x', y', z')$

Define: $\boldsymbol{u} = (u_x, u_y, u_z)$ and $\boldsymbol{u}' = (u'_x, u'_y, u'_z)$

$$\boldsymbol{u} = \frac{d\boldsymbol{r}}{dt} \quad \text{and} \quad \boldsymbol{u}' = \frac{d\boldsymbol{r}'}{dt'}$$

$$u_x = \frac{dx}{dt} \quad \text{and} \quad u'_x = \frac{dx'}{dt'}$$

Now

$$u'_x = \frac{dx'}{dt'} = \gamma\left(\frac{dx}{dt'} - v\frac{dt}{dt'}\right)$$

$$= \gamma\frac{dt}{dt'}\left(\frac{dx}{dt} - v\right)$$

But

$$t' = \gamma\left(t - \frac{v}{c^2}x\right)$$

Therefore

$$\frac{dt'}{dt} = \gamma\left(1 - \frac{v}{c^2}u_x\right)$$

Therefore

$$u'_x = \frac{u_x - v}{\left(1 - \frac{v}{c^2}u_x\right)}$$

Similarly

$$u'_y = \frac{u_y}{\gamma\left(1 - \frac{v}{c^2}u_x\right)}$$

and

$$u'_z = \frac{u_z}{\gamma\left(1 - \frac{v}{c^2}u_x\right)}$$