Lecture 5:
The electric dipole and electrostatic energy

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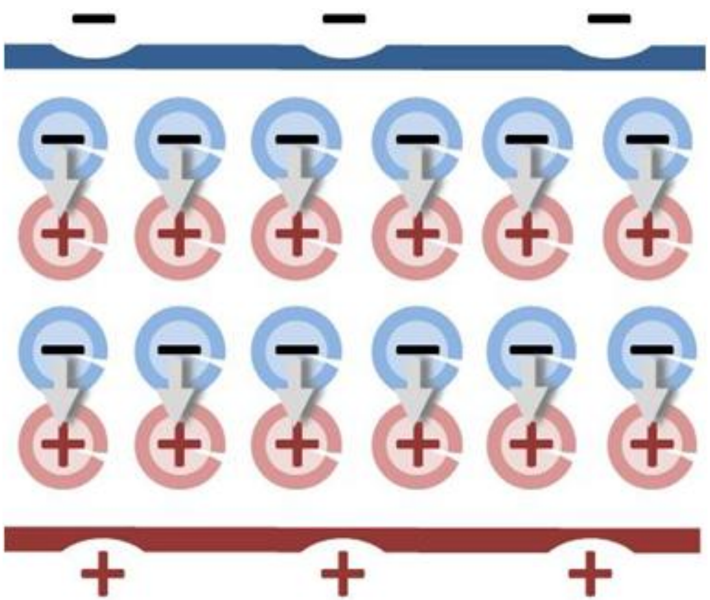
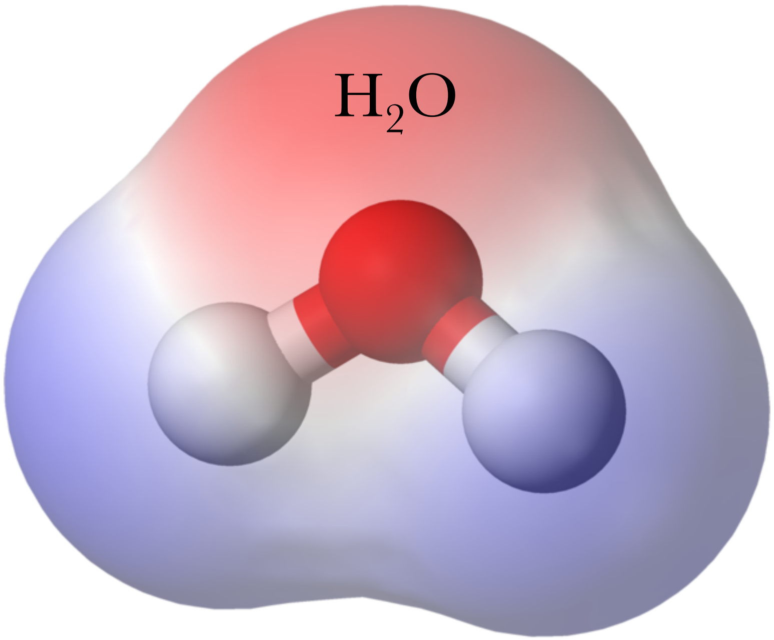
What is a dipole?

good approximation for:

- antennas
- polar molecules
- atoms in external fields

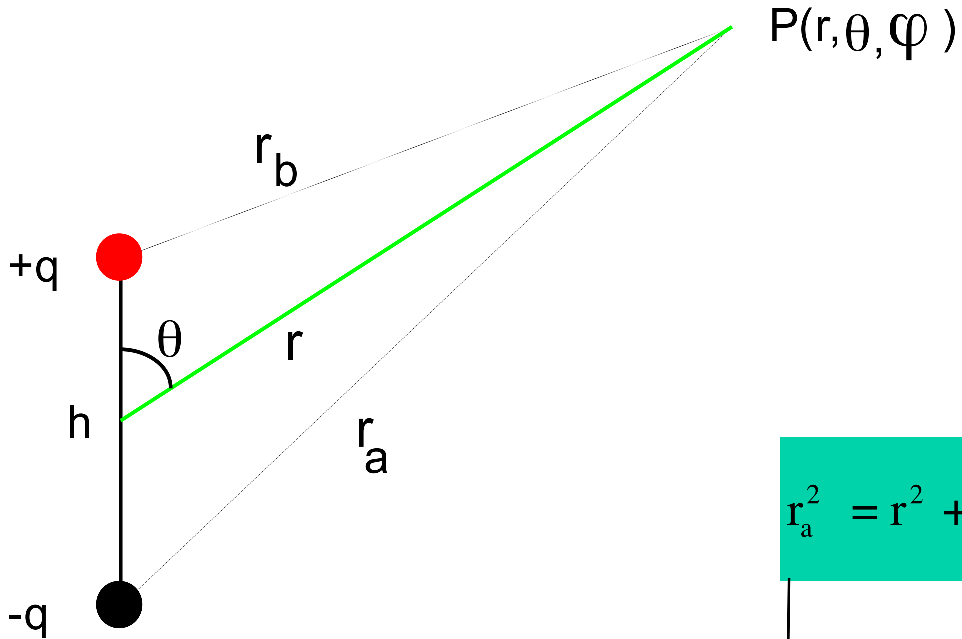


atoms get polarised in an external field (electron cloud slightly separates from the nucleus)



A polar molecule has the electrons orbiting around one part of the molecule (O^-) more than the other (H^+)

What is a dipole?



Potential at P(r,θ,φ) is

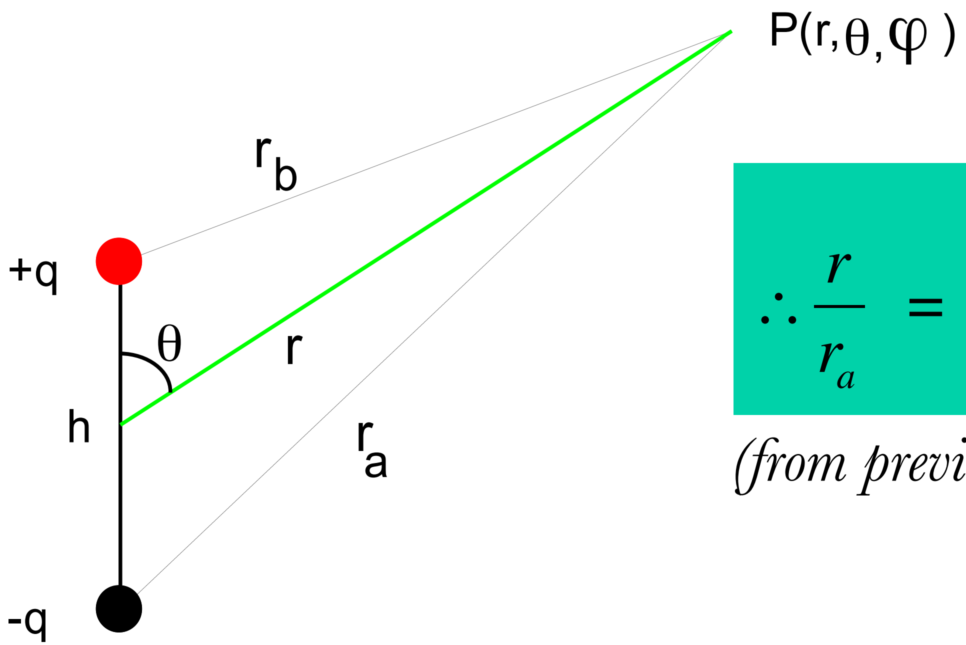
$$\psi_p = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r_b} - \frac{1}{r_a}\right]$$

Using the geometrical identity

$$r_a^2 = r^2 + \left(\frac{h}{2}\right)^2 - rh\cos(\pi - \theta) = r^2 + \left(\frac{h}{2}\right)^2 + rh\cos\theta$$

$$\therefore \frac{r_a^2}{r^2} = 1 + \left(\frac{h}{2r}\right)^2 + \frac{h}{r}\cos\theta$$

What is a dipole?



$$\therefore \frac{r}{r_a} = \left[1 + \left(\frac{h}{2r}\right)^2 + \frac{h}{r}\cos\theta\right]^{-\frac{1}{2}}$$

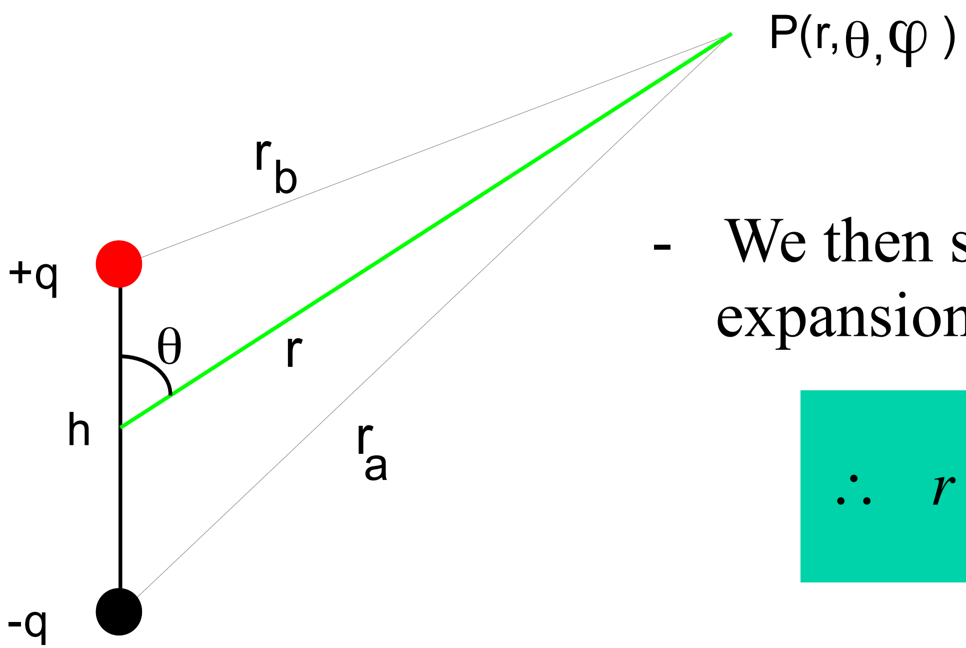
(from previous slide)

- now we make the assumption of being far from the dipole ($r >> h$)
- we can then make a Taylor expansion of the square root ($x = h/r$):

$$\frac{1}{\sqrt{1+x}} \approx 1 - \frac{1}{2}x + \frac{1}{8}x^2 \longrightarrow \frac{r}{r_a} \approx 1 - \frac{1}{2}\left(\left(\frac{h}{2r}\right)^2 + \frac{h}{r}\cos\theta\right) + \frac{1}{8}\left(\frac{h^2}{r^2}\cos^2\theta\right)$$

Potential of a dipole

- The same reasoning applies for r_b (try as an exercise...)



- We then stop at the first order of the expansion in h/r and neglect constant terms

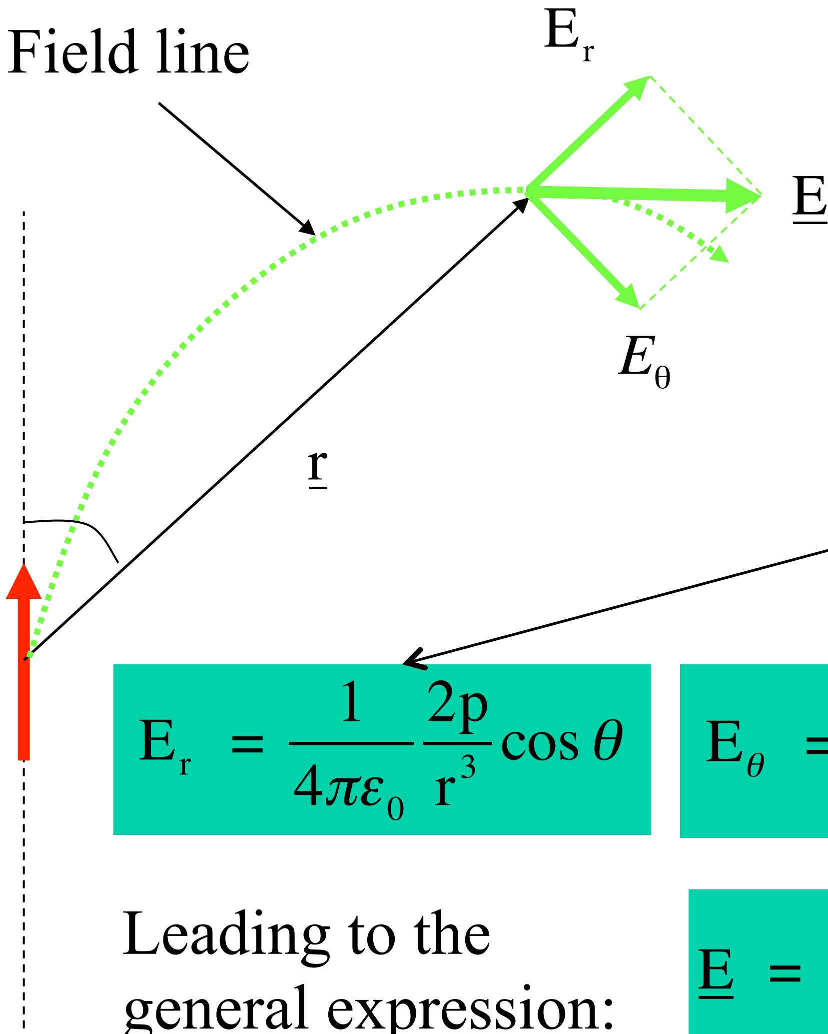
$$\therefore \quad r\left[\frac{1}{r_b} - \frac{1}{\mathrm{r_a}}\right] = \frac{\mathrm{h}}{\mathrm{r}}\cos\theta$$

- This leads to as simple formula for the potential:

$$\therefore \psi_p = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}$$

where p = qr is called the *dipole moment*. The potential is then proportional to p

Electric field of a dipole



Remember that $\vec{E} = -\nabla\psi$

This operation in spherical coordinates reads:

$$\underline{E} = -\left(\frac{\partial\psi}{\partial r}\underline{\hat{r}} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\underline{\hat{\theta}} + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\varphi}\underline{\hat{\varphi}}\right)$$

$$E_r = \frac{1}{4\pi\varepsilon_0}\frac{2p}{r^3}\cos\theta$$

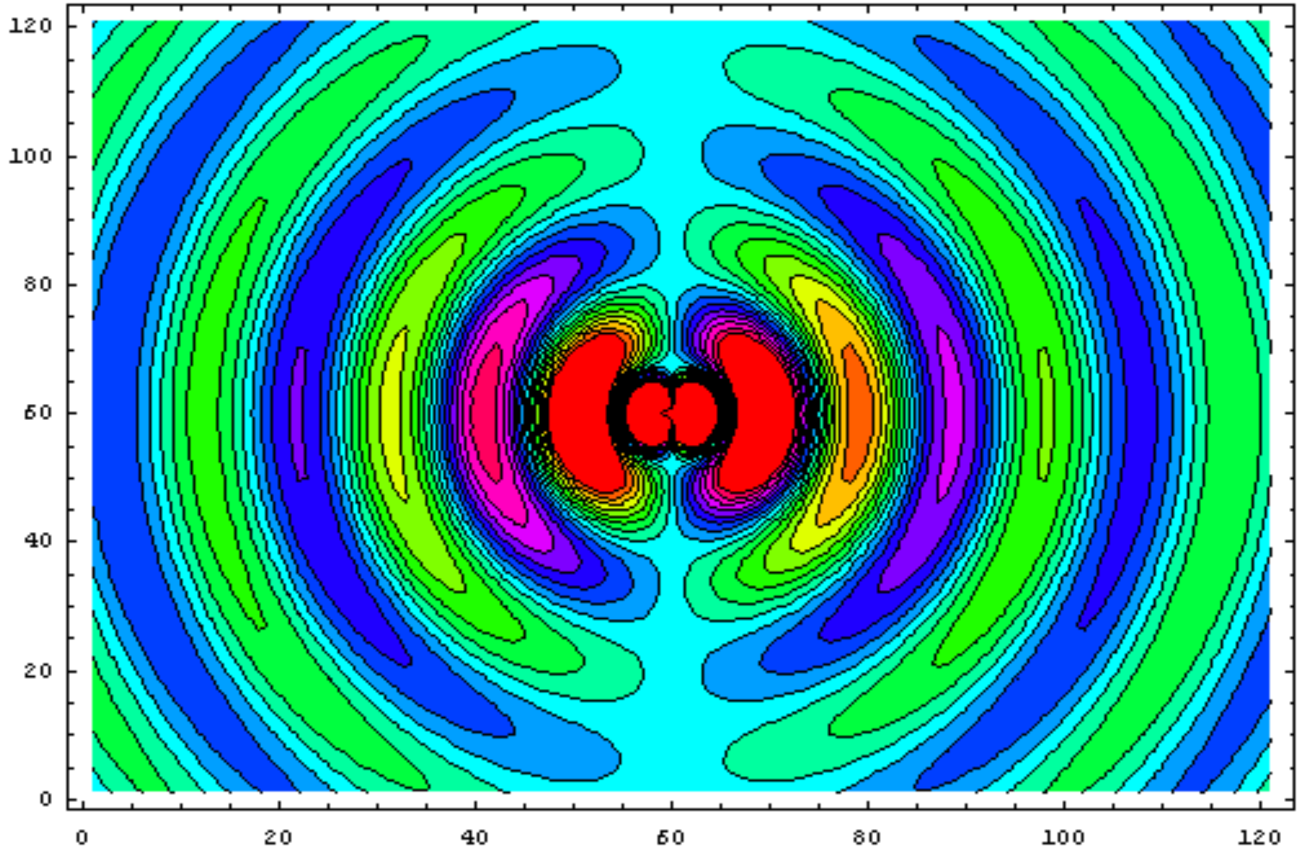
$$E_\theta = \frac{1}{4\pi\varepsilon_0}\frac{p}{r^3}\sin\theta$$

$$E_\varphi = 0$$

Leading to the general expression:

$$\underline{E} = \frac{1}{4\pi\varepsilon_0}\left[\frac{2p\cos\theta}{r^3}\underline{\hat{r}} + \frac{p\sin\theta}{r^3}\underline{\hat{\theta}}\right]$$

Electric field of a dipole



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Electrostatic energy

Assume that you want to estimate the energy required to bring together an ensemble of point-like charges.

The energy required to bring two charges together is:

$$\int q_1 \underline{E} . \, \underline{\mathrm{d}\ell} = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r_{12}}$$

The superposition principle allows us to go in "steps" , meaning that the total work done will be the sum of the works required to move one charge at a time.

We need to be careful though of not double-counting!

(i.e., the charges will not induce self-potentials, and each pair of charges has to be counted only once!)

Electrostatic energy

The work done for two charges will then be:

$$W = \frac{q_1 q_2}{4\pi\varepsilon_0 r} = q_1\psi$$

It can be demonstrated that double-counting is avoided if we include a ½ in the summation:

$$W = \frac{1}{2}\sum_{i=1}^{N} q_i \psi_i$$

If we have a continuous distribution of charges (ρ): $$W = \frac{1}{2}\int_V \rho\psi \, dV$$

If we have the surface of a conductor (σ): $$W = \frac{1}{2}\int_s \sigma\psi \, ds$$

Electrostatic energy

But, for a conductor, the potential is constant on the surface, so we can take it out of the integral:

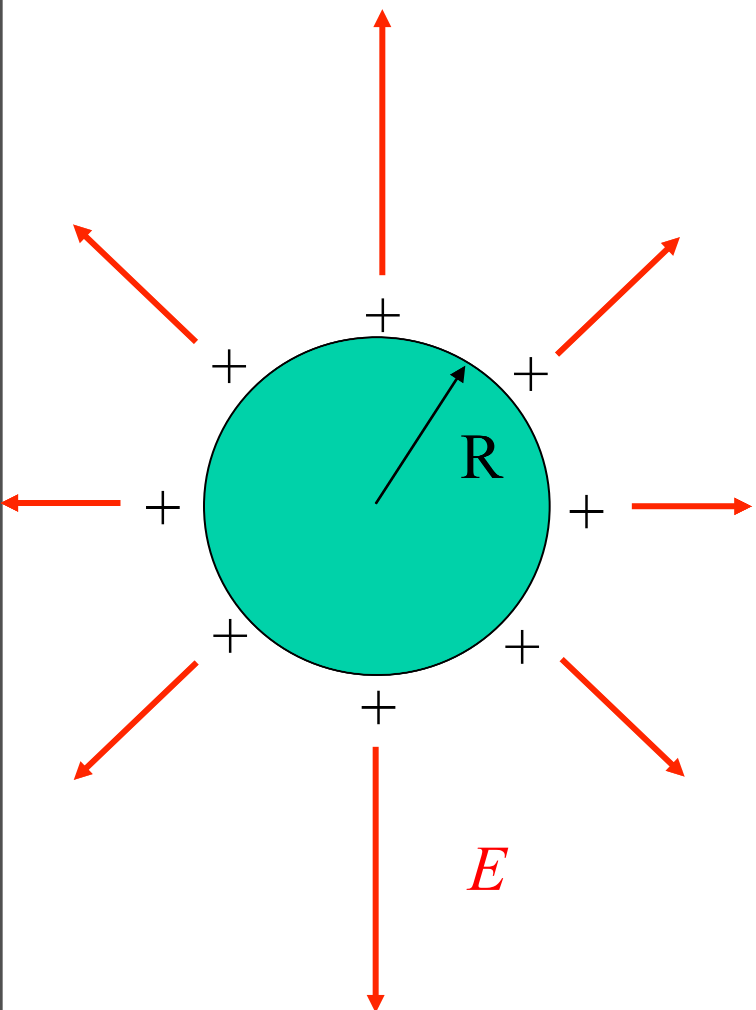
$$W = \frac{1}{2}\psi\int_s \sigma \ \mathrm{dS} = \frac{1}{2}\psi Q$$

Moreover, we can relate the electrostatic energy to the electric field (see Jackson for a demonstration):

$$\therefore W = \frac{1}{2}\varepsilon_0\int_V E^2 \ \mathrm{dV}$$

So that, the energy per unit volume is: $U = \frac{1}{2}\varepsilon_0 E^2$

Example



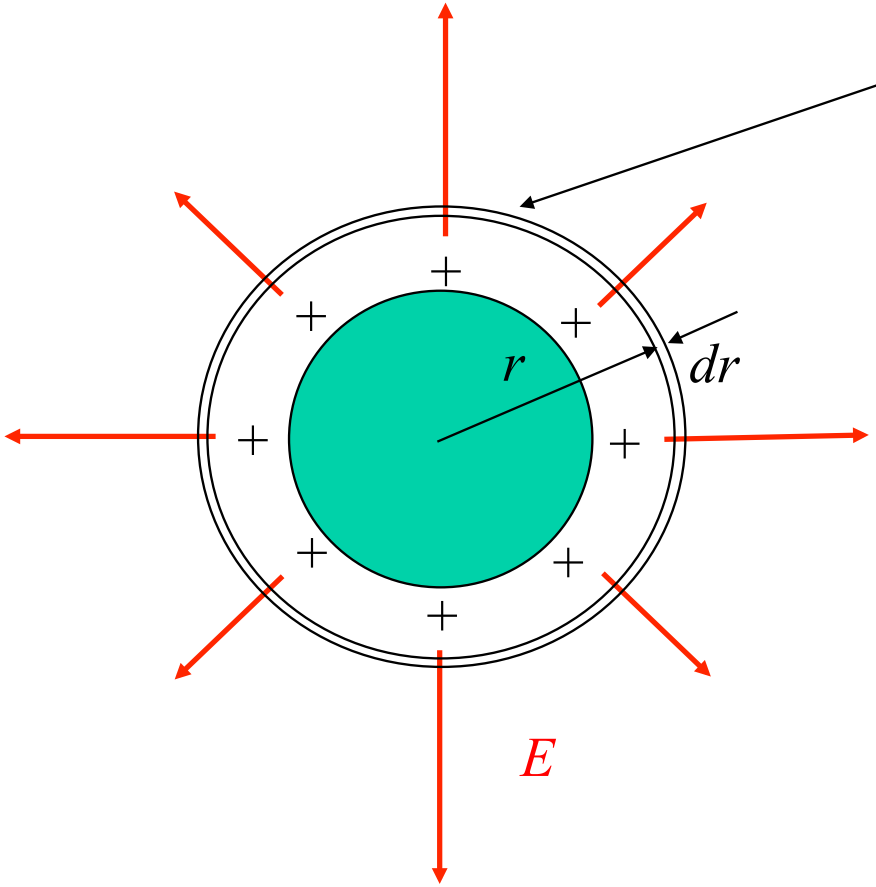
A conducting sphere of radius R, holds a charge Q.

Calculate the work done (W.D) in assembling this charge on the sphere and show that this energy is stored in the electric field around the sphere.

$$W.D = \frac{1}{2}\psi Q = \frac{1}{2}\cdot\frac{1}{4\pi\varepsilon_0}\cdot\frac{Q}{R}\cdot Q$$

$$= \frac{1}{8\pi\varepsilon_0}\frac{Q^2}{R}$$

Example



Spherical shell radius *r* thickness *dr. Energy in electric field in this shell is:-*

$$\frac{1}{2}\varepsilon_o E^2 dV = \frac{1}{2}\varepsilon_o \left[\frac{Q}{4\pi\varepsilon_o r^2}\right]^2 4\pi r^2 dr$$

Total *Energy in electric field is therefore:-*

$$\int_R^\infty \frac{1}{2}\varepsilon_o \left[\frac{Q}{4\pi\varepsilon_o r^2}\right]^2 4\pi r^2\, dr$$

$$\int_R^\infty \frac{1}{2}\varepsilon_o \left[\frac{Q}{4\pi\varepsilon_o r^2}\right]^2 4\pi r^2\, dr = \int_R^\infty \frac{Q^2}{8\pi\varepsilon_o r^2} dr = \left[-\frac{Q^2}{8\pi\varepsilon_o r}\right]_R^\infty = \frac{Q^2}{8\pi\varepsilon_o R}$$