Lecture 10:

generalised Faraday's and Ampere's law

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QUEEN'S UNIVERSITY BELFAST

Electrostatic and magnetostatic fields

Static fields are <u>unchanging in time.</u>
The time differential of an unchanging quantity is zero.

$$\frac{d\vec{E}}{dt} = 0$$

Electrostatic fields have divergence:

In differential form:

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

Charge density

In integral form:

Enclosed charge

$$\oint_S \vec{E} \cdot \hat{n}\, da = \oint_S \vec{E} \cdot \vec{da} = \frac{q_{enc}}{\epsilon_0}$$

Component of electric field normal to surface

Magnetostatic fields are **<u>solenoidal</u>** (divergence-free):

$$\nabla \cdot \vec{B} = 0$$

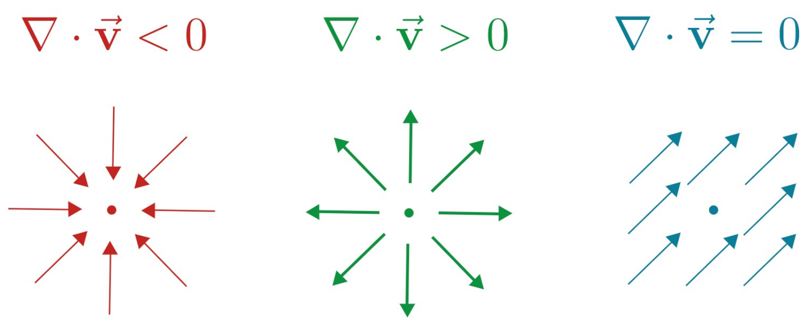
$$\oint_S \vec{B} \cdot \hat{n}\, da = \oint_S \vec{B} \cdot \vec{da} = 0$$

Component of magnetic field normal to surface

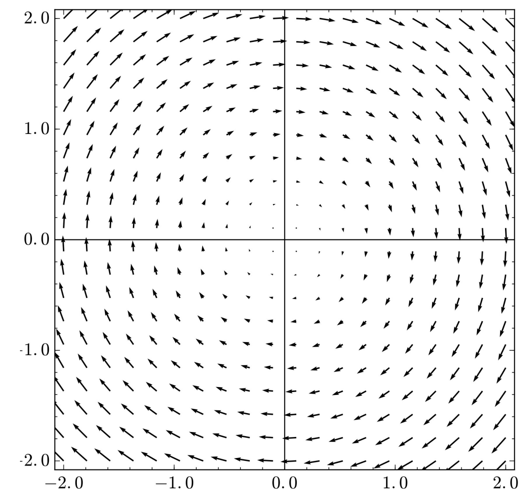
Reminder on divergence

Divergence is a property that we can calculate for vector fields measured **at a point**.

It represents the flow into and out from the point. If there is more flow out of a point we have positive divergence – "a source" (positive charge). Alternatively we have "a sink" (negative charge).



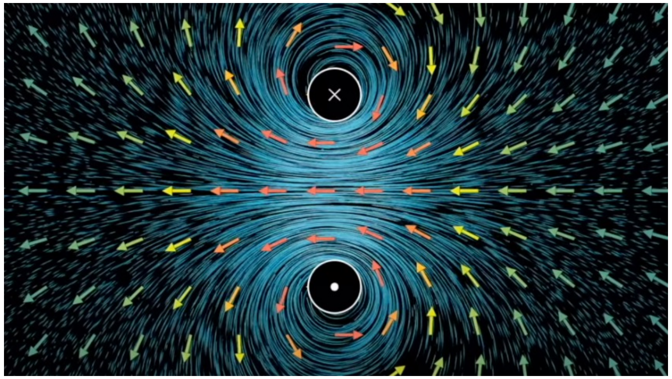
From Khan Academy: Divergence



From Wikipedia: Solenoidal vector field

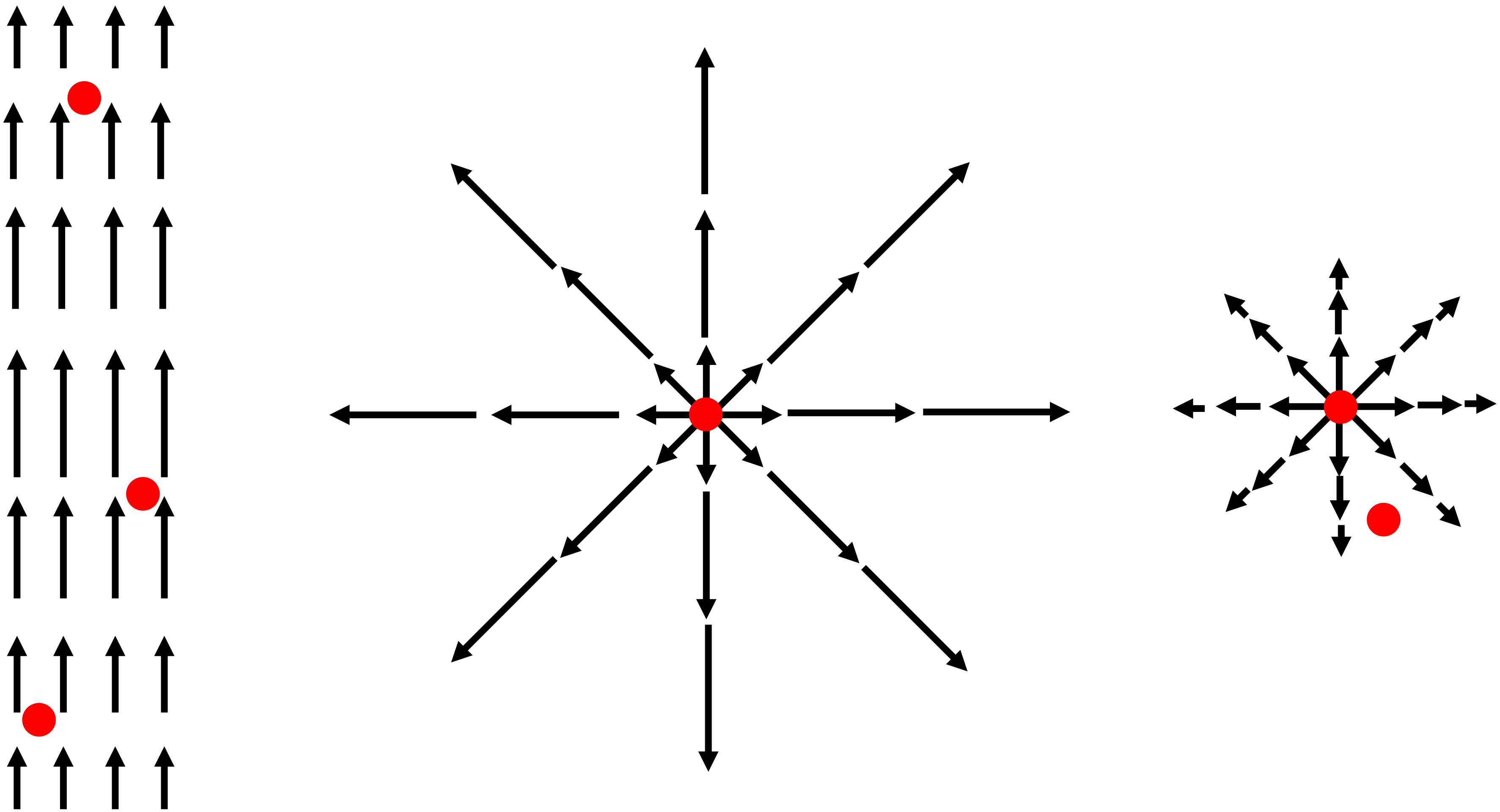
If we have zero divergence everywhere then the fluid density is constant.

Just like magnetic field likes through a current loop.



From Youtube 3Blue1Brown: Divergence and Curl

Does it have divergence?



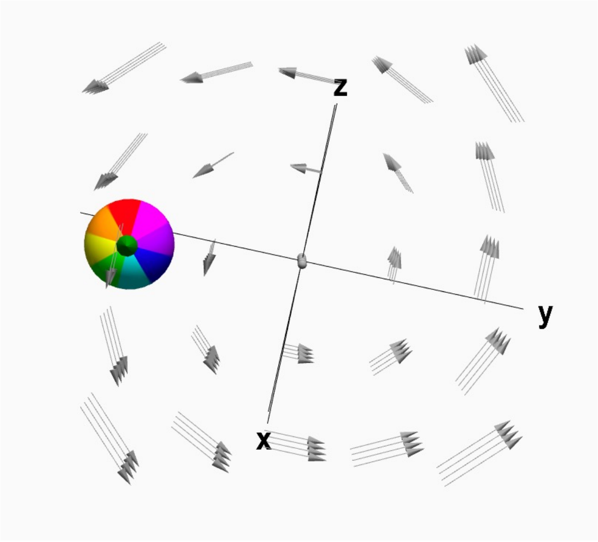
From Fleish – A student's guide to Maxwell's equations

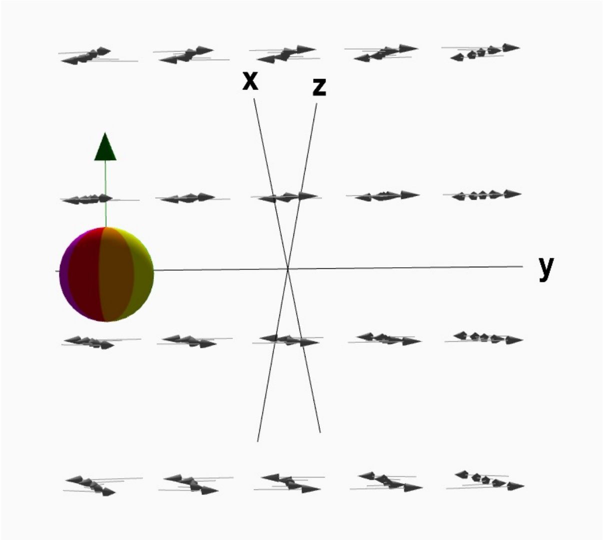
Reminder on curl

Curl is also calculated at a point and represents the circulation or rotation of the vector field.

It's easy to picture macroscopic circulation but curl measures microscopic circulation. To imagine this picture a beach ball fixed in place in the field. If the beach ball rotates around its centre, curl is non-zero.

(Note: Often fields with macroscopic circulation also have curl)

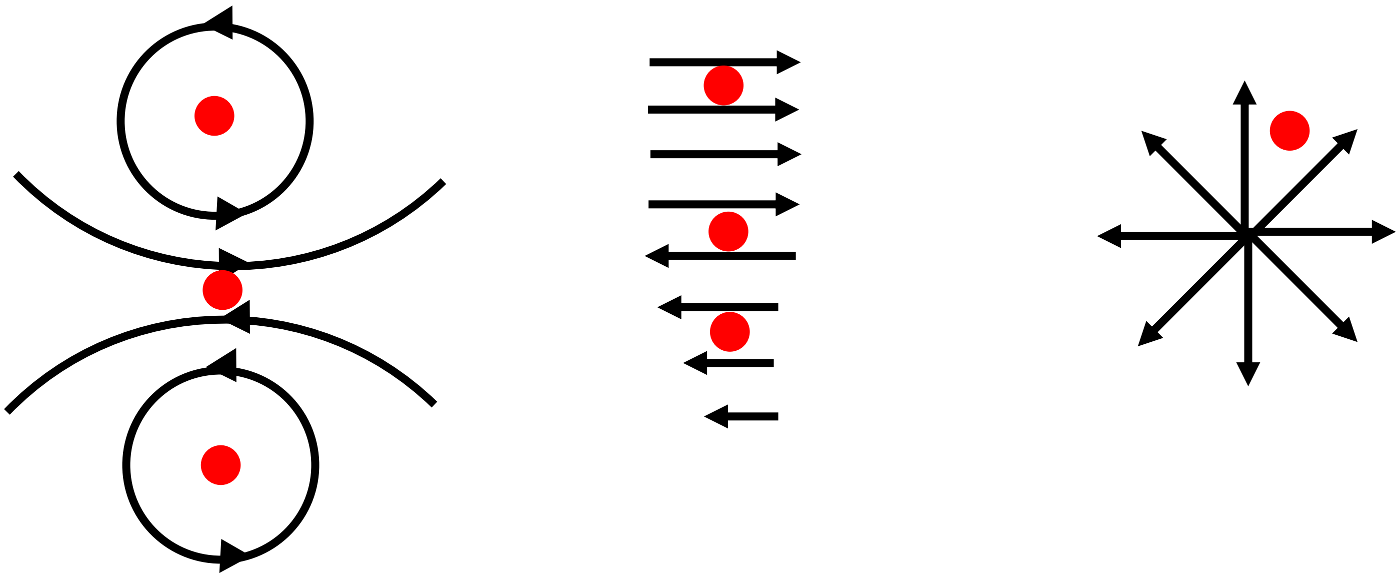




Curl is a vector and has direction which you can work out.

From Math Insight: Curl Idea

Does it have curl?



From Fleish – A student's guide to Maxwell's equations

Faraday's observations

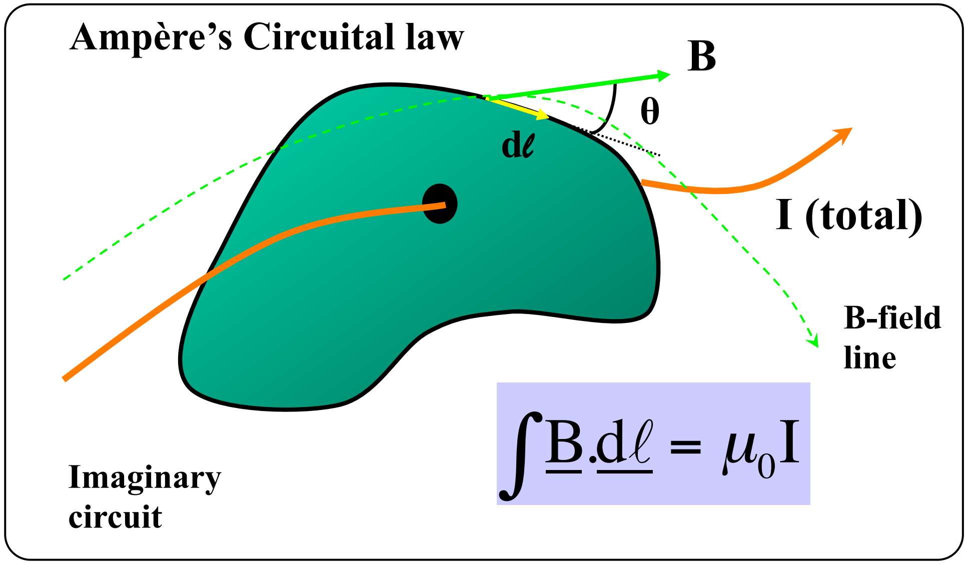
Electrostatic fields are **irrotational** (curl-free): $\nabla \times \vec{E} = 0$

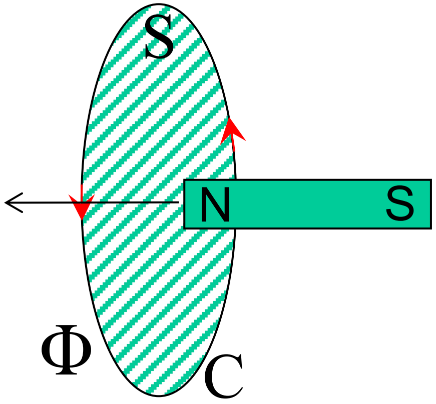
This also makes them **conservative**

In integral form that reads:

$$\oint \underline{E} \, . \, \mathrm{d}\underline{\ell} = 0$$

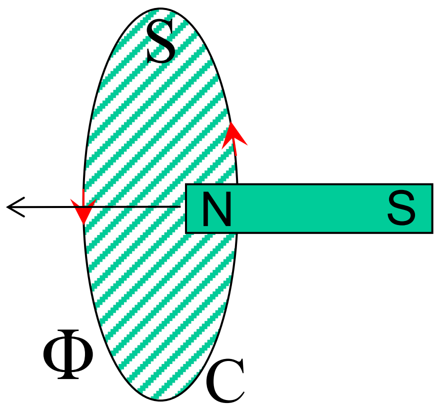
Component of electric field parallel with contour





However, Faraday noticed that **moving** a magnetic field into an electric circuit **induced** an electric field such that $\oint \underline{E} \, . \, \mathrm{d}\ell \neq 0$

Time-varying flux of magnetic fields



The flux of magnetic field is defined as:

Surface integral

Unit surface normal

$$\Phi = \int_S \underline{\mathrm{B}} \,.\, \mathrm{d}\underline{\mathcal{S}}$$

Magnetic flux

Magnetic field

Component of magnetic field normal to surface

We can calculate how the flux changes with time using the following equation.

$$\frac{d\Phi}{dt} = \int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}$$

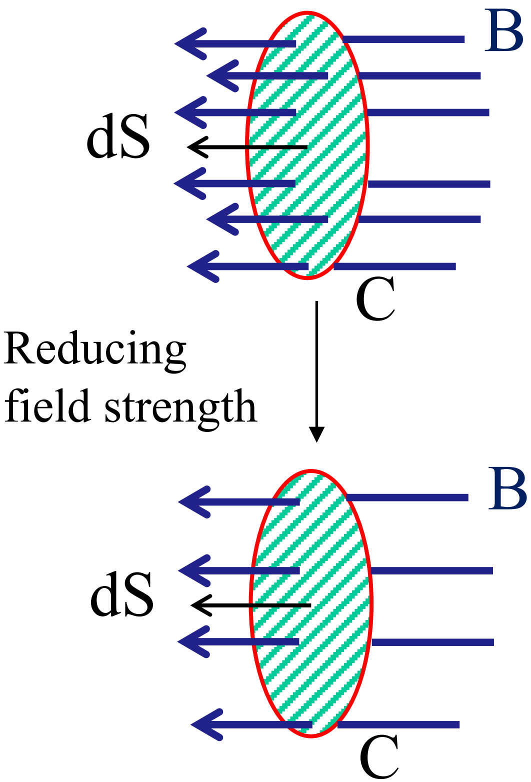
Note: If the effective area or your loop is changing with time but the magnetic field is constant within the loop it can be easier to consider the rate of change of the area directly.

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \mathbf{B}\frac{\mathrm{d}A_{\mathrm{effective}}}{\mathrm{d}t}$$

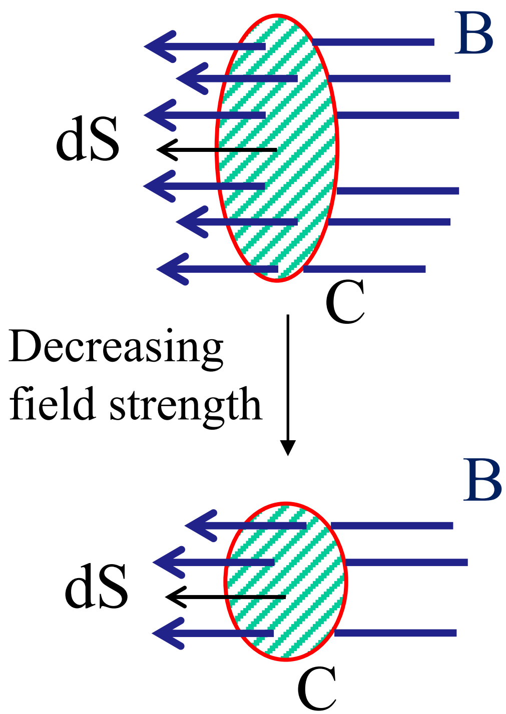
Time-varying flux of magnetic fields

How can we vary the magnetic flux:

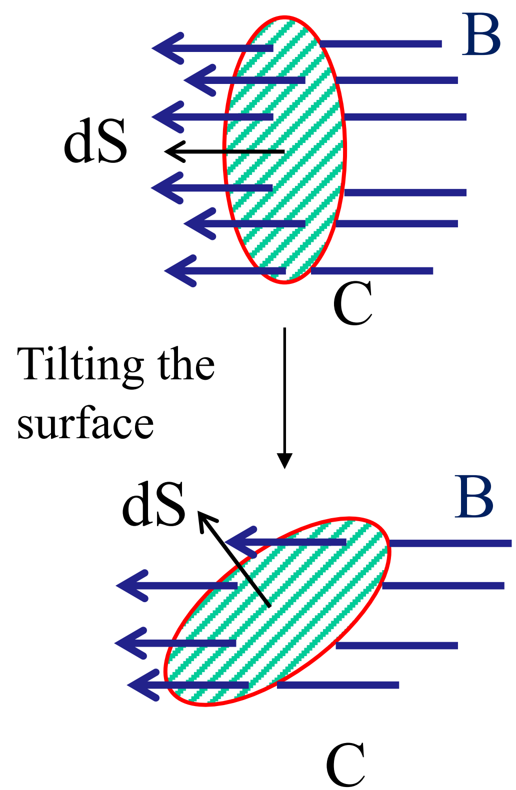
a) changing field



b) changing surface area

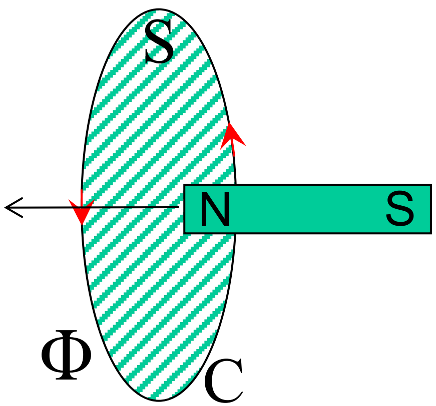


c) changing surface orientation



Electromotive force

We can relate this changing flux to the integral of our induced electric field along the loop.



Line integral

Changing magnetic flux

$$\oint_C \vec{E} \cdot d\vec{\ell} = -\frac{\partial \Phi}{\partial t}$$

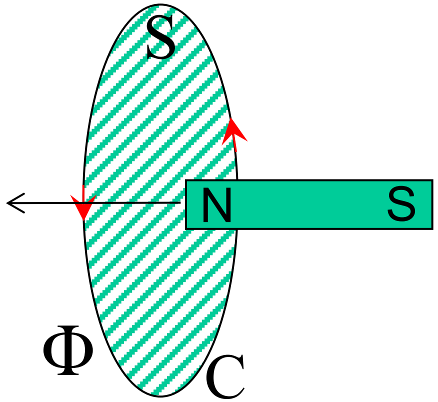
Component of electric field along contour

Unit surface normal

This line integral is also often called the electromotive force.

This is not really a force but a force per unit charge over a specified distance – this is **work per unit charge**.

Lenz's law: the important minus sign

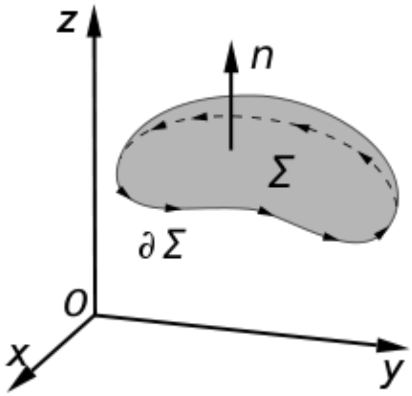


For a changing magnetic flux through a surface, the **induced** electric field acts to <u>**oppose the change**</u>.

We can write: $$\oint_C \vec{E} \cdot d\vec{\ell} = -\frac{\partial \Phi}{\partial t} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}$$

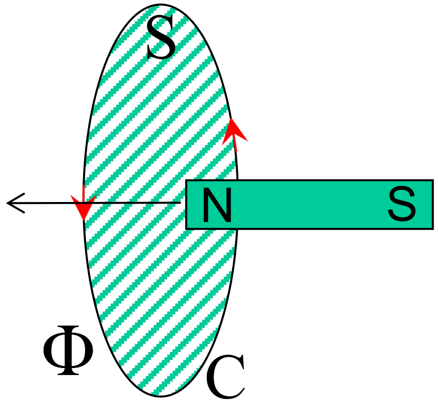
Maxwell-Faraday law (integral form)

Be careful with sign direction:



The direction of the surface normal and dl need to be chosen consistently to ensure the signs are correct.

Lenz's law: the important minus sign



For a changing magnetic flux through a surface, the **induced** electric field acts to **<u>oppose the change</u>**.

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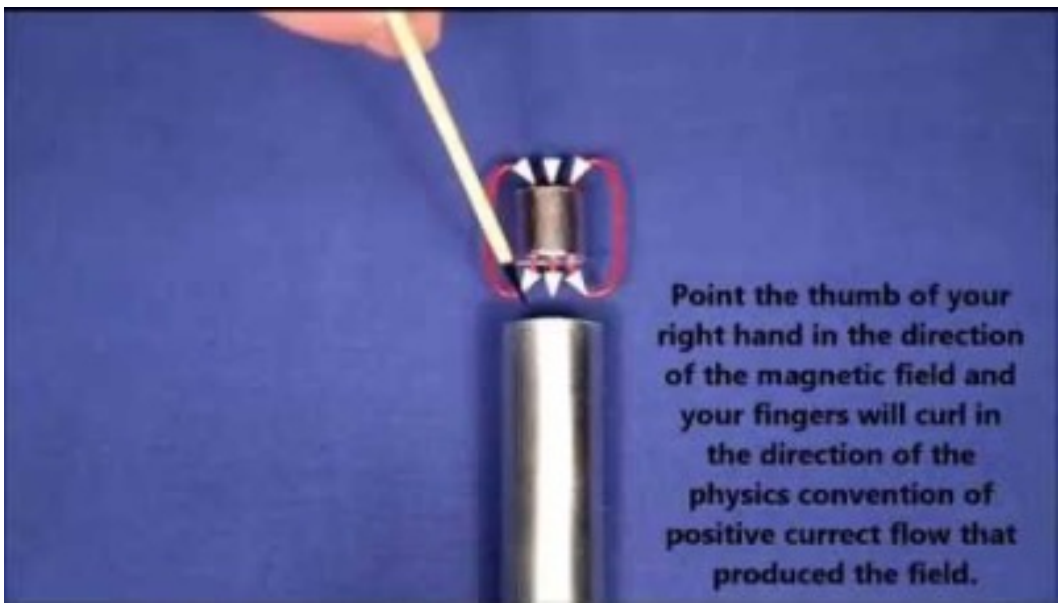
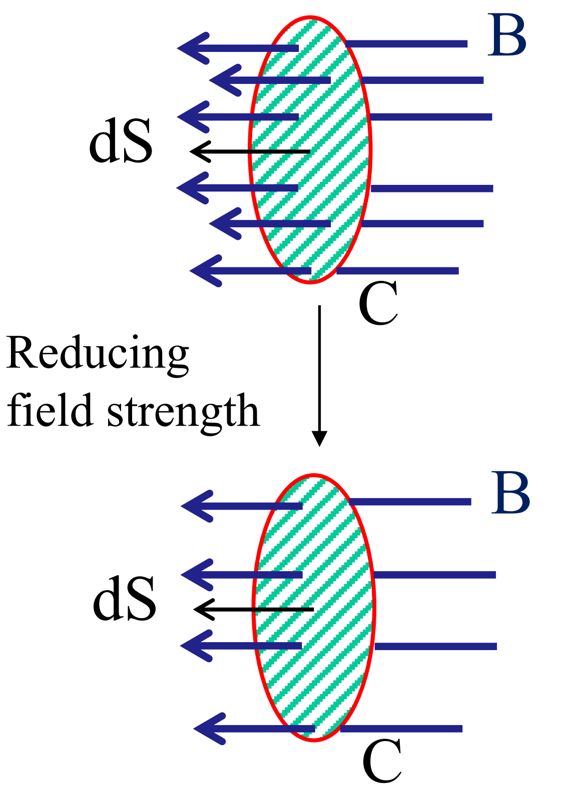


From Youtube: Wayne Schmidt

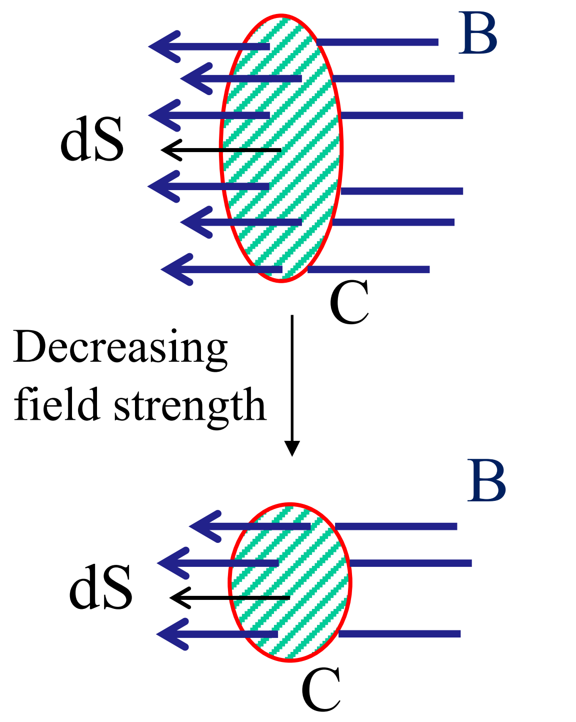
Which direction is the induced Electric field?

Considering the changing flux situations from before?

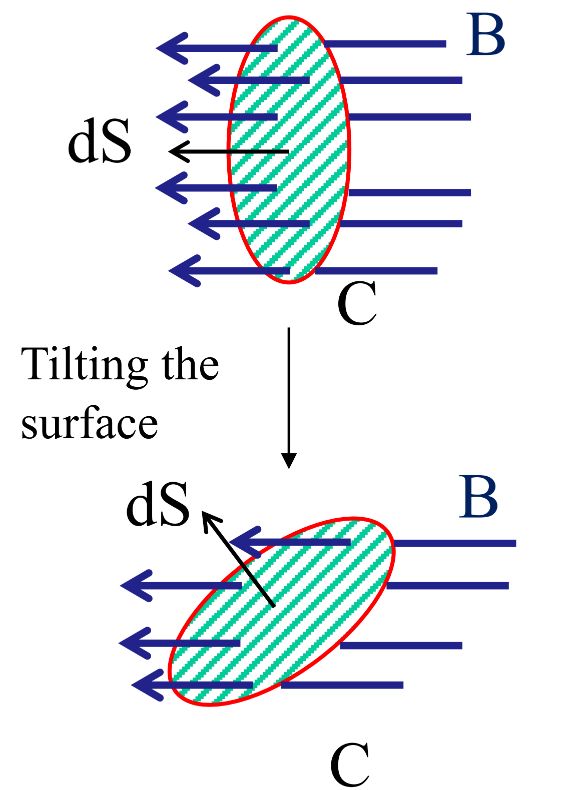
a) changing field



b) changing surface area



c) changing surface orientation



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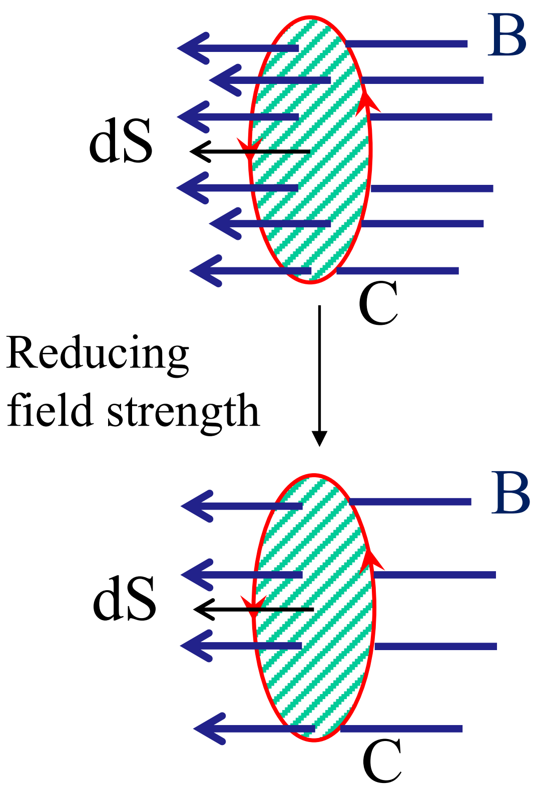

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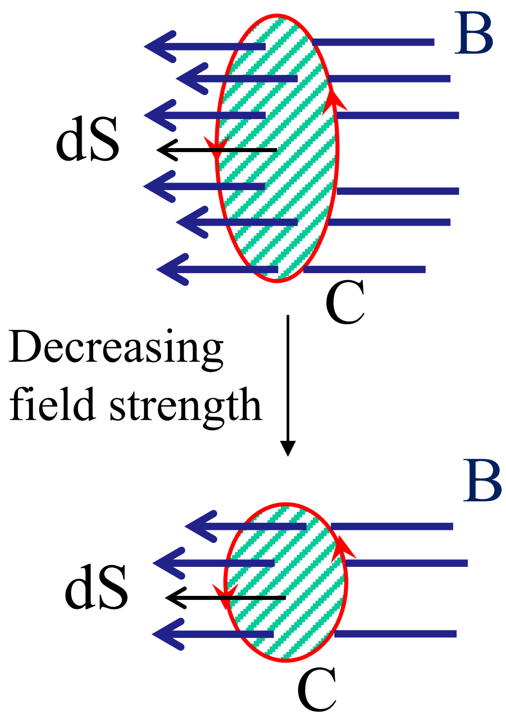

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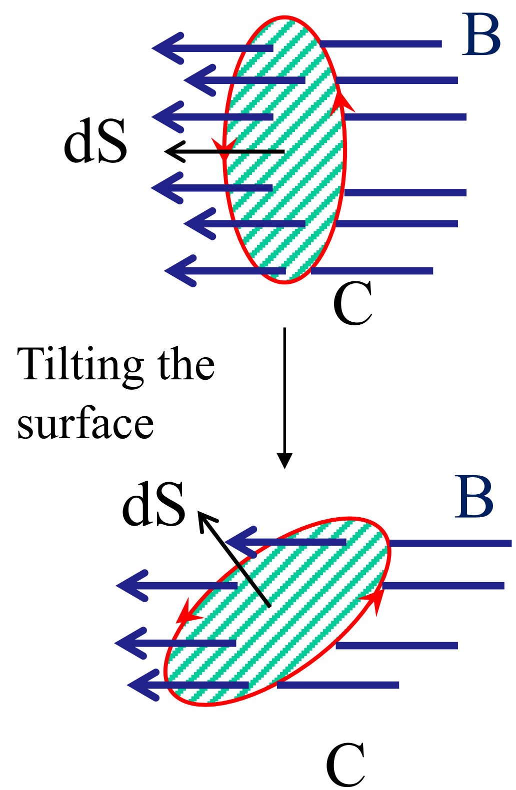

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Time-varying flux of magnetic fields

$$\oint_C \vec{E}\cdot d\vec{\ell} = -\frac{\partial\Phi}{\partial t} = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{S}$$

Maxwell-Faraday law (integral form):

Kelvin-Stokes' theorem:
The line integral of a vector field, *F*, over a loop is equal to the flux of its curl through the enclosed surface.

$$\oint_C \vec{F}\cdot d\vec{l} = \int_S (\nabla\times\vec{F})\cdot d\vec{S}$$

Use Kelvin-Stokes' theorem to convert the line integral to the curl of the electric field:

$$\int_S \nabla\times\underline{E}\ .\ \underline{d}S = -\int_S \frac{\partial \underline{B}}{\partial t}\ .\ \underline{d}S$$

Maxwell-Faraday law (differential form):

From this we see the curl of the electric field is:

$$\nabla\times\underline{E} = -\frac{\partial \underline{B}}{\partial t}$$

If the magnetic field is unchanging then the curl is zero.

Electric fields in terms of potentials

The magnetic field is defined in terms of a vector potential: $\underline{B} = \nabla \times \underline{A}$

Substituting this in the new expression for the curl of E leads to:

$$\nabla \times \underline{E} = -\frac{\partial}{\partial t}(\nabla \times \underline{A}) \xrightarrow{\text{Curl is distributive}} \nabla \times \left(E + \frac{\partial}{\partial t}\underline{A}\right) = 0$$

Now, it is a mathematical identity that:

$$\nabla \times (-\nabla \psi) = 0$$

This is needed here because our gradient should point to the increasing field

That implies that:

$$\underline{E} = -\frac{\partial \underline{A}}{\partial t} - \nabla \psi \quad \leftarrow \quad \textit{due to electrostatic field}$$

$\uparrow$

due to changing

B - field

for time-independent magnetic fields, we retrieve the electrostatic form.

Maxwell's equation so far

$$\nabla \cdot \underline{D} = \rho_F \qquad \nabla \cdot \underline{B} = 0 \qquad \nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$$

with the material equations:

$$\underline{B} = \mu \underline{H} = \mu_0 \underline{H} + \mu_0 \underline{M} \qquad \underline{D} = \varepsilon \underline{E} = \varepsilon_0 \underline{E} + \underline{P}$$

and the Lorentz force:

$$\frac{d\underline{p}}{dt} = q\left(\underline{E} + \underline{v} \times \underline{B}\right)$$

If time-varying B fields induce an electric field, can we also expect the opposite (time-varying electric field inducing a magnetic field)?

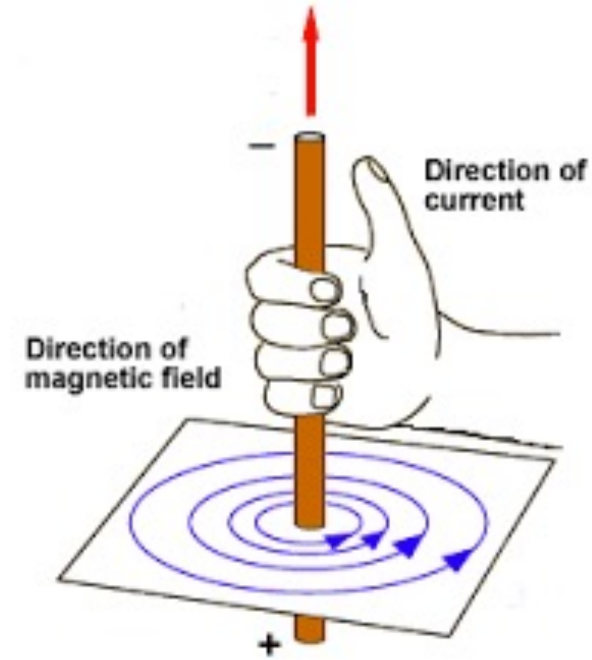
In other words, should we also refine the equation: $\nabla \times \vec{B} = \mu_0 \vec{J}$?

Ampere's law

We have seen that, in magnetostatics: $\nabla \times \vec{B} = \mu_0 \vec{J}$ ← Current density

If we take the divergence of this equation:

$$\nabla \cdot \left(\nabla \times \vec{B} \right) = \mu_0 \nabla \cdot \vec{J}$$



The left-hand side is identically zero (divergence of the curl of a vector is always zero, regardless of the vector).

So: $\nabla \cdot \left(\nabla \times \vec{B} \right) = \mu_0 \nabla \cdot \vec{J}$

strictly implies: $\boxed{\nabla \cdot \vec{J} = 0}$

Charge continuity

The divergence of J is related to the rate of change of the charge density via the charge continuity equation:

$$\nabla \cdot \underline{J} + \frac{d\rho}{dt} = 0 \longrightarrow \frac{d\rho}{dt} = 0$$

$$\nabla \cdot \underline{J} = 0$$

While the charge continuity equation is generally applicable constant current density indicates this is valid **ONLY** for steady currents (magnetostatics)

But what if the current isn't constant, and the change in charge density isn't zero?

$$\frac{d\rho}{dt} \neq 0$$

Reminder - charge continuity equation:

Charge is locally conserved:

$$\therefore \int_S \underline{J} \cdot \underline{dS} = -\int_V \frac{d\rho}{dt}\, dV$$

Flux of current through the surface

Rate of change of charge density in volume

Gauss' divergence theorem: *The total sources of vector field, **A**, within a volume equals the flux through an enclosing surface.*

$$\int_V (\nabla \cdot \mathbf{A})\mathrm{d}V = \int_S \mathbf{A} \cdot \mathrm{d}\mathbf{S}$$

Apply Gauss' theorem to the LHS:

$$\int_V \nabla \cdot \underline{J}\ dV = -\int \frac{d\rho}{dt}\, dV$$

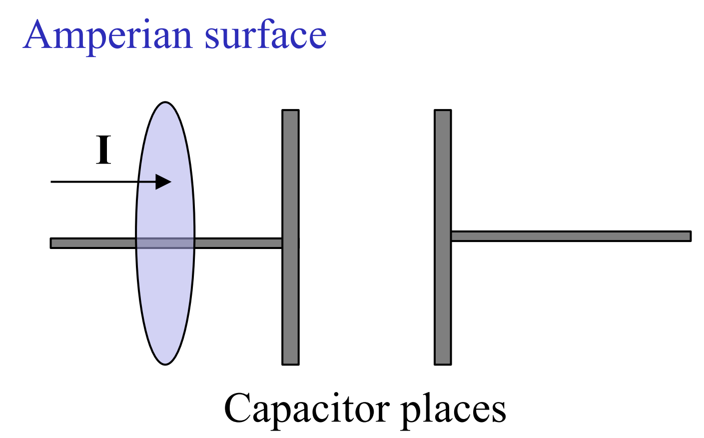
So: $$\boxed{\frac{d\rho}{dt} + \nabla \cdot \vec{J} = 0}$$

Charging capacitor

You're familiar with capacitors which are common components of circuits. Here, we have current flowing into a break in the circuit.

If we have a constant input current we can use our magnetostatics to try to calculate the magnetic field.

Based on our previous work we draw the Amperian loop around the wire and the closed surface as a flat disk.



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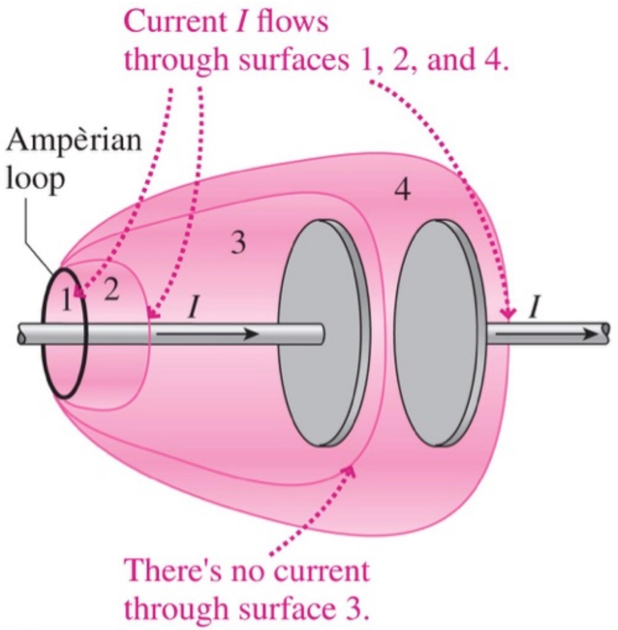
Based on our previous work we draw the Amperian loop around the wire and the closed surface as a flat disk.

But Ampere's law should give the same answer even if we choose a different surface enclosed by the same loop.

Here we can form lots of surfaces with no current.



From Dr. C. L. Davis, Uni. Louisville

Displacement current

Starting with the continuity equation and Gauss' law:

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \qquad\qquad \nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

We can rearrange Gauss' law to express the charge density in terms of the electric field.

$$\epsilon_0 \nabla \cdot \vec{E} = \rho$$

And substitute this into the continuity equation in place of charge density.

$$\boxed{\nabla \cdot \vec{J} + \epsilon_0 \frac{\partial}{\partial t}(\nabla \cdot \vec{E}) = 0}$$

Because the del operator is commutative, we can rewrite this as:

$$\boxed{\nabla \cdot \left(\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right) = 0}$$

Displacement current

If we look back to Ampere's law: $\nabla \times \vec{B} = \mu_0 \vec{J}$

We can now generalise Ampere's law to include both constant currents and changing charge density using our formulation of the charge continuity equation.

physical currents Changing charge density

$$\nabla \times \vec{B} = \mu_0 \underbrace{\left(\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right)}_{\text{total “current”}}$$

We can think of these as two different currents:

J, which relates to physical currents in the system.

And rate of change of charge with time.

This second term is often confusingly called ***displacement current, J_D***

Ampere's law in vacuum and in material

Maxwell-Ampere law in a vacuum.

$$\boxed{\nabla \times \vec{B} = \mu_0 \left(\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right)}$$

We can also express this in terms of ***H*** and ***D***. If we start again in the material.

Continuity equation

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

Gauss' law

$$\nabla \cdot \vec{D} = \rho_f$$

$$\nabla \cdot \vec{J} + \frac{\partial}{\partial t}(\nabla \cdot \vec{D}) = 0$$

$$\nabla \cdot \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) = 0$$

Displacement field, *D*, is determined using the density of 'free charge' or the charges that have made the volume not charge neutral.

$$\rho = \rho_f + \rho_b$$

$$-\nabla \cdot \vec{P}$$

Bound charges responsible for polarization field

Ampere's law in vacuum and in material

This time combining this with the general form of Ampere's law with *H*.

$$\nabla \cdot \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) = 0$$

$$\nabla \times \vec{H} = \vec{J}_f$$

free current

We find the generalized form of the Maxwell-Ampere law:

Generalised Maxwell-Ampere law

$$\boxed{\nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}}$$

J_f is often called 'free current' as in currents due to the propagation of charges rather than magnetic dipole generated currents of bound electrons.

For a nice proof of this, read "Proof of equivalence" section of the Wikipedia entry on Ampere's circuital law.

Full Maxwell's equations – microscopic form

Microscopic form = including all charges

Integral form	**Differential form**
Relates fields in a region of space to fields on a boundary – useful for systems with symmetry	Calculating the local field at a point

Gauss' law:

$$\oint_S \vec{E}\cdot \mathrm{d}\vec{S} = \frac{1}{\epsilon_0}\int_V \rho \mathrm{d}V \qquad \nabla\cdot\vec{E} = \frac{\rho}{\epsilon_0}$$

$$\oint_S \vec{B}\cdot \mathrm{d}\vec{S} = 0 \qquad \nabla\cdot\vec{B} = 0$$

Faraday's law:

$$\oint_C \vec{E}\cdot \mathrm{d}\vec{l} = -\frac{\mathrm{d}}{\mathrm{d}t}\int \vec{B}\cdot \mathrm{d}\vec{S} \qquad \nabla\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Ampere's law:

$$\oint_C \vec{B}\cdot \mathrm{d}\vec{l} = \mu_0\int \vec{J}\cdot \mathrm{d}\vec{S} + \mu_0\epsilon_0\frac{\mathrm{d}}{\mathrm{d}t}\int \vec{E}\cdot \mathrm{d}\vec{S} \qquad \nabla\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t}$$

Relating microscopic and macroscopic forms

Microscopic form of Maxwell's equations includes all charges/currents but we know that in materials this can get complicated.

The Macroscopic forms help us to deal with these situations where we may have contributions to total charge from bound charges.

In magnetic materials the bound currents, J_b, can act as a source of magnetic fields.

$$\vec{J}_b = \nabla \times \vec{M}$$ Magnetisation of the material

Changing polarization of the material also provides a source of current as this involves movement of charges (currents):

$$\vec{J}_P = \frac{\partial \vec{P}}{\partial t}$$

Total current is therefore: $\vec{J} = \vec{J}_f + \vec{J}_b + \vec{J}_P$

Microscopic Ampere-Maxwell law is:

$$\boxed{\nabla \times \vec{B} = \mu_0 \left(\vec{J}_f + \vec{J}_b + \vec{J}_P + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right)}$$

Derivation following from Fleisch

Relating microscopic and macroscopic forms

$$\boxed{\nabla \times \vec{B} = \mu_0 \left(\vec{J}_f + \vec{J}_b + \vec{J}_P + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right)}$$

$$\vec{J}_b = \nabla \times \vec{M} \qquad \vec{J}_P = \frac{\partial \vec{P}}{\partial t}$$

Sub in bound and polarization currents:

$$\boxed{\frac{1}{\mu_0} \nabla \times \vec{B} = \vec{J}_f + \nabla \times \vec{M} + \frac{\partial \vec{P}}{\partial t} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}}$$

Gather terms:

$$\boxed{\nabla \times \left(\frac{\vec{B}}{\mu_0} - \vec{M} \right) = \vec{J}_f + \left(\frac{\partial (\epsilon_0 \vec{E} + \vec{P})}{\partial t} \right)}$$

$$\nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$$

Derivation following from Fleisch

Full Maxwell's equations – macroscopic form

Macroscopic form = equations in matter/simplified for accommodating bound charge

Integral form	**Differential form**
Relates fields in a region of space to fields on a boundary – useful for systems with symmetry	Calculating the local field at a point

Gauss' law:

$$\oint_S \vec{D} \cdot \mathrm{d}\vec{S} = \int_V \rho_f \mathrm{d}V$$

free charge density

$$\nabla \cdot \vec{D} = \rho_f$$

$$\oint_S \vec{B} \cdot \mathrm{d}\vec{S} = 0$$

$$\nabla \cdot \vec{B} = 0$$

Faraday's law:

$$\oint_C \vec{E} \cdot \mathrm{d}\vec{l} = -\frac{\mathrm{d}}{\mathrm{d}t} \int \vec{B} \cdot \mathrm{d}\vec{S}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Ampere's law:

$$\oint_C \vec{H} \cdot \mathrm{d}\vec{l} = \int \vec{J}_f \cdot \mathrm{d}\vec{S} + \frac{\mathrm{d}}{\mathrm{d}t} \int \vec{D} \cdot \mathrm{d}\vec{S}$$

free current

$$\nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$$

Relating integral and differential forms

Relating the integral and differential forms using:

Gauss' divergence theorem:

The divergence of a vector field, F, within a volume, V, is equal to the total flux of that vector through an enclosed surface.

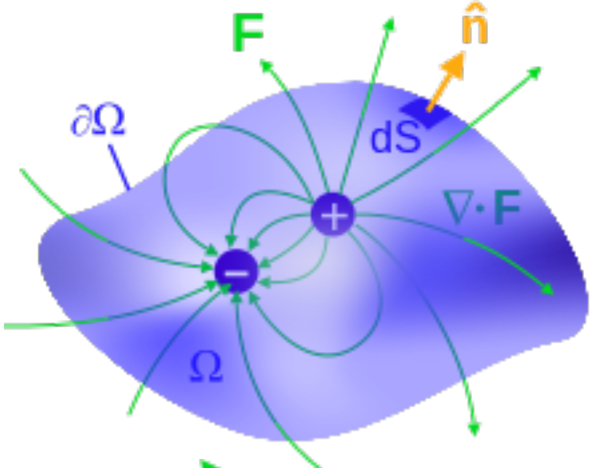
$$\int_V (\nabla \cdot \vec{F})\mathrm{d}V = \oint_S \vec{F} \cdot \mathrm{d}\vec{S}$$

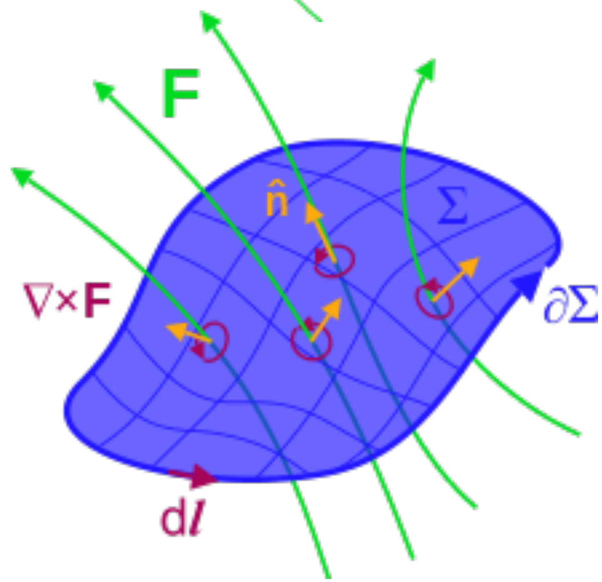
Kelvin-Stoke's theorem:

The line integral of a vector field, F, over a loop is equal to the flux of its curl through the enclosed surface.

$$\oint_C \vec{F} \cdot \vec{dl} = \int_S (\nabla \times \vec{F}) \cdot \vec{dS}$$

Note these apply to vector fields that are continuous and have continuous derivatives.





From Wikipedia – Maxwell's equations