GENERALISED LORENTZ TRANSFORMATION

The LT equations in x and t are:

$$x' = \gamma(x - vt)$$

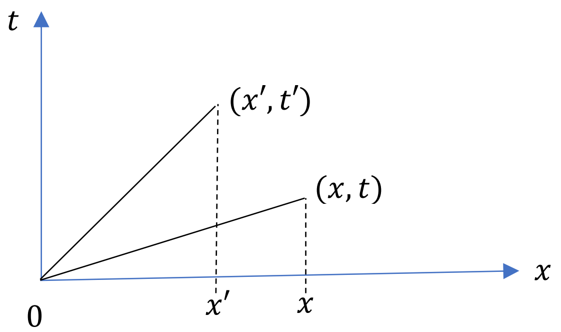
and

$$t' = \gamma(t - \frac{v}{c^2}x)$$

These linear equations may be expressed in matrix form as:

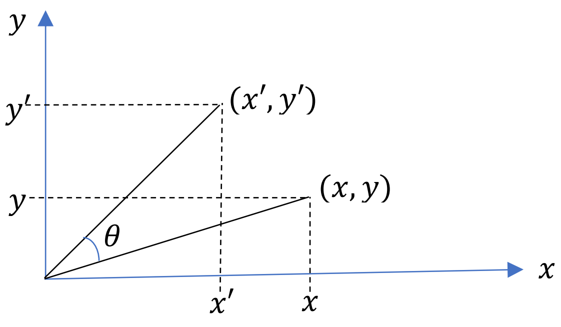
$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -v \\ -{}^{v}/_{c^2} & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$$

The matrix operator here is broadly similar to a rotation operator suggesting, for example, that the Lorentz contraction may be a consequence of rotation in space-time.



where $x' < x$

Compare this with a rotation in normal space x, y where $x^2 + y^2$ is invariant.



Clearly

$$x' = cos\theta\, x - sin\theta\, y$$

$$y' = sin\theta\, x + cos\theta\, y$$

i.e. the matrix operator is:

$$\begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}$$

Clearly here we have that

$$x'^2 + y'^2 = x^2 + y^2$$

If the Lorentz contraction of a rod is a pure rotation in space-time we might then expect that

$$x'^2 + t'^2 = x^2 + t^2$$

From LT:

$$x'^2 = \gamma^2(x^2 - 2xvt + v^2t^2) \qquad \text{------ (2.1)}$$

$$t'^2 = \gamma^2(t^2 - \frac{2xvt}{c^2} + \frac{v^2x^2}{c^4}) \qquad \text{------ (2.2)}$$

A cursory glance at these equations shows that $x'^2 + t'^2 \neq x^2 + t^2$

In order to eliminate the cross (xvt) term in these equations, multiply eqn (2) by c^2 and subtract (2) from (1):

$$x'^2 - c^2t'^2 = \gamma^2\{(x^2 - c^2t^2) - \frac{v^2}{c^2}(x^2 - c^2t^2)\}$$

$$= \gamma^2\{(x^2 - c^2t^2)(1 - \frac{v^2}{c^2})\}$$

$$= x^2 - c^2t^2$$

i.e. it is $x^2 - c^2t^2$ (not $x^2 + t^2$) that is the invariant quantity.

[But, of course, we already knew this – refer back to the derivation of the LT eqns.]

This suggests that the rotation concept is valid but in x, ict space rather than x, t space, where i is the imaginary number $\sqrt{-1}$.

This invariance property is an important result, which leads us into the formal linking of space and time.

In ordinary 3-D space the scalar product of the position vector $\boldsymbol{r} = (x, y, z)$ with itself is:

$$\boldsymbol{r}.\boldsymbol{r} = x^2 + y^2 + z^2$$

and is an invariant. For example the length of a rod will never change however you move it in 3-D space, as we noted above for rotation in x, y.

We have seen that in special relativity this 3-D invariance breaks down, so that the length of a rod will appear shorter due to the Lorentz contraction. However, we have introduced a new comparable quantity in 4-D space-time which *is invariant*, namely:

$$x^2 + y^2 + z^2 - c^2t^2$$

suggesting that we can define a <u>4-dimensional displacement vector</u>:

$$\boldsymbol{R} = (x, y, z, ict)$$

such that the scalar product of this with itself

$$\boldsymbol{R}.\boldsymbol{R} = x^2 + y^2 + z^2 - c^2t^2$$

is invariant.

Let us introduce a <u>generalized</u> co-ordinate notation, such that the 4-D displacement vector is described by:

$$\boldsymbol{R} = (X_1, X_2, X_3, X_4)$$

so that:

$$X_1 = x, \qquad X_2 = y, \qquad X_3 = z, \qquad X_4 = ict$$

Then we can express the LT equations in generalized co-ordinates as:

$$X_1{}' = \gamma(X_1 + i\frac{v}{c}X_4)$$

$$X_2{}' = X_2$$

$$X_3{}' = X_3$$

$$X_4{}' = \gamma(-i\frac{v}{c}X_1 + X_4)$$

or in matrix notation:

$$\boxed{\begin{pmatrix} X_1{}' \\ X_2{}' \\ X_3{}' \\ X_4{}' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\frac{\gamma v}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\frac{\gamma v}{c} & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix}}$$

This is the generalized form of the LT equations.

The matrix operator is known as the <u>Lorentz Transformation Matrix</u>

4-VECTORS

We have seen that under a Lorentz transformation space and time are closely linked so that we may look on time as a 4th dimension. In 4-D we consider x, y, z and ict as representing 4 orthogonal axes, allowing us to define a 4-vector for displacement as:

$$\boldsymbol{R} = (X_1, X_2, X_3, X_4) = (x, y, z, ict)$$

and to represent the LT equations in terms of this as:

$$\begin{pmatrix} X_1' \\ X_2' \\ X_3' \\ X_4' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\gamma\frac{v}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\gamma\frac{v}{c} & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix}$$

We will go on to define other 4-vectors to describe important parameters for solving relativistic problems, such as velocity, acceleration, momentum, *etc*.

Before we do this, we will introduce some <u>tensor notation</u>, which is a convenient way to represent a 4-vector approach. We will not in this course be engaging in tensor algebra, simply using the tensor notation for convenience and for its neatness of notation.

4-Displacement

The 4-Displacement vector, $\boldsymbol{R}$, defined as above, may be expressed in tensor notation as X_μ where the subscript μ denotes the fact that there are 4 components to the tensor.

$$X_\mu = (X_1, X_2, X_3, X_4) = (x, y, z, ict)$$

Lorentz Transformation

The LT equations can be expressed in tensor notation as:

$$X_\mu' = a_{\mu\nu} X_\nu \qquad \text{------ (2.3)}$$

where:

$$a_{\mu\nu} = \begin{pmatrix} \gamma & 0 & 0 & i\frac{\gamma v}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\frac{\gamma v}{c} & 0 & 0 & \gamma \end{pmatrix}$$

is the LT matrix.

Einstein Summation Rule

Repetition of a subscript within a single term of an equation indicates a summation over that subscript parameter.

e.g. the repetition of the subscript ν on the RHS of the LT expression (2.3) indicates that we must sum ν over 1 to 4, *i.e.*

$$X'_\mu = a_{\mu\nu}X_\nu = \sum_{\nu=1}^{4} a_{\mu\nu}X_\nu = a_{\mu 1}X_1 + a_{\mu 2}X_2 + a_{\mu 3}X_3 + a_{\mu 4}X_4$$

and clearly there are 4 such equations, one each for $\mu = 1, 2, 3$ and 4, thus giving the 4 LT equations.

Hence, we can see that equation (2.3) is a very neat shorthand way of representing matrix multiplication.

Exercise: *Taking $\mu = 1$, expand the above expression, and substituting for components of $a_{\mu\nu}$ and X_ν, show that you reproduce the first LT equation in generalised co-ordinates. Convert to the equation in terms of x, y, z, t*

Rank of a Tensor

The number of subscripts associated with a tensor denotes the rank of the tensor.

A scalar (*e.g.* m_0) is a zero rank tensor

A vector (*e.g.* X_μ) is a 1st rank tensor

A matrix (*e.g.* $a_{\mu\nu}$) is a 2nd rank tensor

Definition of 4-vector

In general, a 4-vector A_μ is a set of quantities (A_1, A_2, A_3, A_4) whose transformation properties are the same as those of the displacement vector X_μ

Hence all 4-vectors transform as:

$$\boxed{A_\mu{}' = a_{\mu\nu}A_\nu}$$

This is essentially the definition of a covariant tensor, *i.e.* the form is the same when subjected to a Lorentz transformation. By Einstein, the laws of physics should be unchanged under LT, *i.e.* our equations representing the laws should be covariant (compare with invariant for a scalar quantity).

If we can define 4-vectors to represent other important vector quantities such as velocity and momentum, then this condition simplifies transformations, as we know that any covariant tensor will transform as defined above. Compare this with the added complexity in 3-D when going from displacement transformation to velocity transformation, and the further increased complexity in going to acceleration.

Important Rule for Creating New 4-Vectors

If a tensor is formed by operating on a covariant tensor with an invariant quantity, then the new tensor so formed is itself covariant.

We can use this rule to define other 4-vectors, working from the one 4-vector that we have already defined, X_μ

4-VELOCITY

The velocity of a moving object is defined as the rate of change of its position with time.

Hence in order to define a 4-velocity vector we need to differentiate the 4-displacement vector with respect to time.

The question is, what time can we use? We have seen that in special relativity time changes when moving between inertial frames, *i.e.* $t' \neq t$. This time is not invariant.

The rule for creating 4-vectors requires us to operate on $\boldsymbol{R}$**,** or X_μ, with an invariant quantity. Hence, we must define an invariant time.

We do, of course, have an invariant expression which includes time, *i.e.*

$$x^2 + y^2 + z^2 - c^2t^2$$

This is also true for space and time intervals, *i.e.*

$$dx^2 + dy^2 + dz^2 - c^2dt^2$$

We can therefore define an invariant time interval $d\tau$ *via*:

$$d\tau^2 = dt^2 - \frac{1}{c^2}(dx^2 + dy^2 + dz^2)$$

The invariant time interval $d\tau$ is known as the <u>proper time</u>.

Let us define the 4-velocity of an object *via*:

$$\boldsymbol{U} = \frac{d\boldsymbol{R}}{d\tau}$$

or in tensor notation as:

$$\boxed{U_\mu = \frac{dX_\mu}{d\tau}}$$

Now

$$U_\mu = \frac{dX_\mu}{dt}\frac{dt}{d\tau}$$

------ (2.4)

Since

$$X_\mu = (x, y, z, ict)$$

$$\frac{dX_\mu}{dt} = (\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}, ic)$$

i.e.

$$\frac{dX_\mu}{dt} = (u_x, u_y, u_z, ic)$$

------ (2.5)

From definition of $d\tau$ above:

$$d\tau^2 = dt^2 - \frac{1}{c^2}(dx^2 + dy^2 + dz^2)$$

Hence

$$\left(\frac{d\tau}{dt}\right)^2 = 1 - \frac{1}{c^2}\left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right)$$

and

$$\left(\frac{d\tau}{dt}\right)^2 = \left(1 - \frac{1}{c^2}u^2\right)$$

Therefore

$$\frac{d\tau}{dt} = \left(1 - \frac{u^2}{c^2}\right)^{1/2}$$

i.e.

$$\frac{dt}{d\tau} = \left(1 - \frac{u^2}{c^2}\right)^{-1/2} = \gamma(u)$$

------ (2.6)

Substituting from (2.5) and (2.6) in (2.4):

$$\boxed{U_\mu = \gamma(u)(u_x, u_y, u_z, ic)}$$

Defining the generalised co-ordinates for 4-velocity as

$$U_\mu = (\ U_1,\ U_2,\ U_3,\ U_4)$$

Then, $U_1 = \gamma(u)u_x\ ,\ U_2 = \gamma(u)u_y\ ,\ U_3 = \gamma(u)u_z\ ,\ U_4 = \gamma(u)ic$

And we know the transformation properties, since the 4-velocity is a good 4-vector:

$$\boxed{U_\mu{}' = a_{\mu\nu}U_\nu}$$

PROOF THAT 4-VELOCITY IS A 4-VECTOR

To prove that 4-velocity as defined above is indeed a good 4-vector, we should show that it does transform in the same way as the 4-displacement under a LT.

To simplify the algebra we shall consider an object moving with velocity $\boldsymbol{u} = (u, 0{,}0)$ *i.e.* the object is moving in the same direction as the moving frame of reference. There is a loss of generality in this, but the same result can be obtained by taking the general case of $\boldsymbol{u} = (u_x, u_y, u_z)$.

Let us consider how the components of 4-velocity transform between frames.

$$U_1 = \gamma(u)u_x = \gamma(u)u$$

In a frame S'

$$U_1{}' = \gamma(u')u_x{}' \qquad \text{------ (2.7)}$$

From the 3-D velocity transformation derivation we know that:

$$u_x' = \frac{u_x - v}{(1 - \frac{v}{c^2}u_x)}$$

{<u>Aside</u>: *In order to further simplify the algebra we will, where it is convenient to do so, drop c's from our derivations. Otherwise a plethora of c, c^2, c^4 can cause confusion and lead to unnecessary mistakes. It is easy enough to replace c's in the final expression by considering dimensions. However, if you feel more comfortable leaving these in, then that's fine, but be aware that it is an added complication in your working.*}

Proceeding:

$$u_x' = \frac{u - v}{(1 - uv)} \qquad \text{------ (2.8)}$$

We now need to work out $\gamma(u')$ where

$$\gamma(u') = \left(1 - \frac{u'^2}{c^2}\right)^{-1/2} = (1 - u'^2)^{-1/2}$$

Now $u_y' = u_y = 0$, $u_z' = u_z = 0$, so that $u' = u_x'$

Hence

$$u'^2 = \frac{(u - v)^2}{(1 - uv)^2} = \frac{u^2 - 2uv + v^2}{1 - 2uv + u^2v^2}$$

$$1 - u'^2 = \frac{1 - 2uv + u^2v^2 - u^2 + 2uv - v^2}{1 - 2uv + u^2v^2}$$

$$= \frac{1 - u^2 - v^2 + u^2v^2}{(1 - uv)^2}$$

$$= \frac{(1-u^2)(1-v^2)}{(1-uv)^2}$$

Therefore

$$\gamma(u') = (1-u'^2)^{-1/2} = \frac{1-uv}{(1-u^2)^{1/2}(1-v^2)^{1/2}}$$

------ (2.9)

and

$$\gamma(u') = \gamma(u)\gamma(1-uv)$$

---- (2.10)

{*note that dimensionally* $\gamma(u')$ *has no dimensions, hence replacing* c*'s in equation (2.9) leads to:*

$$\gamma(u') = \frac{1-\frac{uv}{c^2}}{(1-\frac{u^2}{c^2})^{1/2}(1-\frac{v^2}{c^2})^{1/2}}$$

and that

$$\gamma(u) = (1-\frac{u^2}{c^2})^{-1/2}$$

etc}

Substituting from (2.8) and (2.10) in (2.7):

$$U_1{}' = \gamma(u)\gamma(1-uv)\,\frac{u-v}{(1-uv)}$$

$$= \gamma\{\gamma(u)u - \gamma(u)v\}$$

But $\quad \gamma(u)u = U_1$

and $\quad i\gamma(u) = U_4$

Hence

$$\boxed{U_1{}' = \gamma(U_1 + ivU_4)}$$

---- (2.11)

Compare with expectation from applying the LT rule:

$$U_\mu{}' = a_{\mu\nu}U_\nu$$

Expanding over $\nu = 1-4$ for $\mu = 1$:

$$U_1{}' = a_{11}U_1 + a_{12}U_2 + a_{13}U_3 + a_{14}U_4$$

$$U_1{}' = \gamma U_1 + i\frac{v}{c}\gamma U_4$$

i.e. identical to eqn (2.11) with c replaced.

It is trivial to show that

$$U_2{}' = U_2$$

and

$$U_3{}' = U_3$$

<u>Exercise</u>: Show that:

$$U_4{}' = \gamma(U_4 - ivU_1)$$

Make sure that you are confident in correctly replacing the c's in these final expressions.

4-MOMENTUM

In normal 3-D space the momentum of a moving object of mass m is denoted

$$\boldsymbol{p} = (p_x, p_y, p_z)$$

where

$$\boldsymbol{p} = m\boldsymbol{u}$$

To define a 4-D covariant tensor for 4-momentum we again take the approach of operating on a known 4-vector with an invariant quantity, in this case the rest mass m_0 of the object.

<u>Definition</u>:

$$\boxed{P_\mu = m_0 U_\mu}$$

Therefore:

$$P_\mu = m_0 \gamma(u)(u_x, u_y, u_z, ic)$$

$$= m(u_x, u_y, u_z, ic)$$

$$= (mu_x, mu_y, mu_z, imc)$$

$$\boxed{P_\mu = (p_x, p_y, p_z, \frac{iE}{c})}$$

Defining the generalised co-ordinates for 4-momentum as

$$P_\mu = (\ P_1,\ P_2,\ P_3,\ P_4)$$

Then, $P_1 = p_x\ ,\ P_2 = p_y\ ,\ P_3 = p_z\ ,\ P_4 = i\,E/c$

4-momentum transforms exactly the same as all 4-vectors:

$$\boxed{P_\mu{}' = a_{\mu\nu} P_\nu}$$

NEW NOTATION

Before going on to define 4-vectors for acceleration and force we shall introduce a new form of notation.

This involves representing the spatial part of the 4-vector, *i.e.* the first 3 components by the normal 3-D vector of that particular quantity.

This shortens the form of expressions for 4-vectors and, as we shall see in deriving the 4-vector for acceleration, simplifies manipulations.

4-Displacement:

$$\boldsymbol{R} = (x, y, z, ict)$$

In the new notation this can be written simply as:

$$\boldsymbol{R} = (\boldsymbol{r}, ict)$$

4-Velocity:

$$\boldsymbol{U} = \gamma(u)(u_x, u_y, u_z, ic)$$

becomes:

$$\boldsymbol{U} = \gamma(u)(\boldsymbol{u}, ic)$$

4-Momentum:

$$\boldsymbol{P} = (p_x, p_y, p_z, \frac{iE}{c})$$

becomes:

$$\boldsymbol{P} = (\boldsymbol{p}, \frac{iE}{c})$$

4-ACCELERATION

Acceleration is the rate of change of velocity with time.

Hence we can define a 4-acceleration vector:

$$\boxed{\boldsymbol{A} = \frac{d\boldsymbol{U}}{d\tau}}$$

Hence

$$\boldsymbol{A} = \frac{d\boldsymbol{U}}{d\tau} = \frac{d\boldsymbol{U}}{dt}\frac{dt}{d\tau} = \gamma(u)\frac{d\boldsymbol{U}}{dt}$$

Recall that 4-velocity

$$\boldsymbol{U} = \gamma(u)(u_x, u_y, u_z, ic)$$

Hence

$$\boldsymbol{A} = \gamma(u)\frac{d}{dt}\{\gamma(u)\big(u_x, u_y, u_z, ic\big)\}$$

or, in our new notation:

$$= \gamma(u)\frac{d}{dt}\{\gamma(u)(\boldsymbol{u}, ic)\}$$

$$= \gamma(u)\frac{d}{dt}\big(\gamma(u)\boldsymbol{u}, ic\gamma(u)\big)$$

Differentiating:

$$\boxed{\boldsymbol{A} = \gamma(u)\left(\gamma(u)\boldsymbol{a} + \frac{d\gamma(u)}{dt}\boldsymbol{u}, ic\frac{d\gamma(u)}{dt}\right)}$$

4-FORCE

By Newton's second law, force is the rate of change of momentum.

Hence:

$$\boxed{\boldsymbol{F} = \frac{d\boldsymbol{P}}{d\tau}}$$

$$\boldsymbol{F} = \frac{d\boldsymbol{P}}{d\tau} = \frac{d\boldsymbol{P}}{dt}\frac{dt}{d\tau} = \gamma(u)\frac{d\boldsymbol{P}}{dt}$$

Recall

$$\boldsymbol{P} = (\boldsymbol{p}, \frac{iE}{c})$$

Hence

$$\boldsymbol{F} = \gamma(u)\left(\frac{d\boldsymbol{p}}{dt}, \frac{i}{c}\frac{dE}{dt}\right)$$

Defining $\boldsymbol{f}$ as the 3-D force, $\boldsymbol{f} = (f_x, f_y, f_z)$:

$$\boxed{\boldsymbol{F} = \gamma(u)\left(\boldsymbol{f}, \frac{i}{c}\frac{dE}{dt}\right)}$$