Lecture 11:
Three-dimensional systems: the angular momentum

The classical expression for the angular momentum $\vec{L}$ is:

$$\vec{L} = \vec{r} \times \vec{p} \tag{1}$$

In quantum mechanics, the operator angular momentum is defined in the same way, provided that the quantum expression for the momentum ($\vec{p} = -i\hbar\nabla$) is used:

$$\vec{L} = -i\hbar\vec{r} \times \nabla \tag{2}$$

If we break this into cartesian components we get:

$$\begin{cases} L_x = L_1 = yp_z - zp_y = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \\ L_y = L_2 = zp_x - xp_z = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \\ L_z = L_3 = xp_y - yp_x = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \end{cases} \tag{3}$$

It is useful to define the anti-symmetric tensor (*Ricci's tensor*):

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if} \quad (ijk) = (123) \quad \text{or any even permutation} \\ -1 & \text{if} \quad (ijk) = (213) \quad \text{or any odd permutation} \\ 0 & \qquad\qquad\qquad \text{otherwise} \end{cases} \tag{4}$$

The i^{th} component of the angular momentum can then defined, in a more compact way, as:

$$L_i = \sum_{j,k} \epsilon_{ijk} x_j p_k = \epsilon_{ijk} x_j p_k \tag{5}$$

Whereby the second expression arises from neglecting the summation symbol, implying that summation is performed for repeated indices. This convention is widely used in physics.
It is straightforward to note that, since the operator momentum and position are Hermitian, the angular momentum is also Hermitian.
The fundamental commutators for the angular momentum are:

$$[L_i, x_j] = i\hbar\epsilon_{ijk}x_k \quad , \quad [L_i, p_j] = i\hbar\epsilon_{ijk}p_k \quad , \quad [L_i, L_j] = i\hbar\epsilon_{ijk}L_k \tag{6}$$

It is useful to define the square of angular momentum:

$$L^2 = L_1^2 + L_2^2 + L_3^2 \tag{7}$$

An important property of this operator is that it commutes with every component of the angular momentum:

$$[L^2, L_i] = 0 \quad \text{for} \quad i = 1, 2, 3 \tag{8}$$

Due to these commutation rules, and the additional hypothesis that the modulus of an operator be positive, it follows that the eigenvalues of the angular momentum are always quantised.
It is implicit here that we can substitute the notation x, y, z for the components with $1, 2, 3$. As a matter of notation, we note here that the angular momentum has the same dimension as the Planck's constant. In the following treatment we will thus omit the omnipresent $\hbar$.
It is useful to define the following operators:

$$\begin{aligned} L_+ &= L_1 + iL_2 \\ L_- &= L_1 - iL_2 \end{aligned} \tag{9}$$

The fundamental commutators of these new operators are:

$$[L_+, L_-] = 2L_3 \quad , \quad [L_3, L_+] = L_+ \quad , \quad [L_3, L_-] = -L_- \tag{10}$$

We can also express the operator L^2 in terms of these new operators:

$$L^2 = L_- L_+ + L_3^2 + L_3 \tag{11}$$

However, a particle in Quantum Mechanics does not have only orbital angular momentum L but might also have, as many experiments indicate, an intrinsic angular momentum, the spin. The total angular momentum of a particle is thus the sum of the orbital angular momentum L and the spin S: $J = L + S$. All the commutation rules we have written so far for the orbital angular momentum L apply also for the total angular momentum J.
It is useful now to study the possible eigenvalues of J^2 and, in particular, how do they relate with those of one component of the total angular momentum J_3. Let us assume that ψ_3 is a normalised eigenfunction of the operator J_3, with eigenvalue m:

$$J_3 \psi_3 = m \psi_3 \tag{12}$$

If we use the commutation rules in Eq. 10, we have:

$$J_3 J_+ \psi_3 = (J_+ J_3 + J_+) \psi_3 = (m + 1) J_+ \psi_3 \tag{13}$$

This means that the state $J_+\psi_3$ is itself an eigenfunction of J_3 with eigenvalue $(m+1)$. Analogously, we have for J_-:

$$J_3 J_- \psi_3 = (J_- J_3 - J_-)\psi_3 = (m-1)J_-\psi_3 \tag{14}$$

We thus see that the operators J_+ and J_- increase or decrease the value of the angular momentum by 1, respectively.

Due to the commutation rules, it is straightforward to see that all the wavefunctions $J_\pm\psi_3$ are also eigenfunctions of the total angular momentum J^2.

Let us now assume that ψ_j is an eigenfunction of J^2 with the maximum eigenvalue for J_3, which we call j. Classically, this corresponds to the situation where the angular momentum of our particle is aligned along z (or axis 3, for our notation). Since j is the maximum value of the angular momentum along z, it follows that $J_+\psi_j = 0$ (if this were not true, it would imply that there exists a state with an eigenvalue for J_3 equal to $j+1$, contradicting the hypothesis that j is the maximum eigenvalue for J_3). We can thus write:

$$J^2\psi_j = (J_- J_+ + J_3^2 + J_3)\psi_j = j(j+1)\psi_j \tag{15}$$

Starting from the state ψ_j we can iteratively apply the operator J_- to build a series of states with an eigenvalue for J_3 equal to: j, $j-1$, $j-2$, ... All these states still have the same eigenvalue $j(j+1)$ for the operator J^2. Analogously, the operator L^2 has eigenvalue $\ell(\ell+1)$ and the operator s^2 (spin) has eigenvalue $s(s+1)$.

Among these states there will exist a state that has a minimum eigenvalue for the operator J_3. This state will arise from applying the operator J_- a number of times n. For this particular state, which we denote as ψ_n, we must have:

$$J_-\psi_n = 0, \quad , \quad J_3\psi_n = (j-n)\psi_n \tag{16}$$

For this state, we can write:

$$J^2\psi_n = (J_+ J_- + J_3^2 + J_3)\psi_n = [(j-n)^2 - (j-n)]\psi_n \tag{17}$$

However, it is also true that:

$$J^2\psi_n = j(j+1)\psi_n \tag{18}$$

By equating Eq. 17 and Eq. 18, we obtain:

$$(j-n)^2 - (j-n) = j(j+1) \quad \rightarrow \quad n = 2j \tag{19}$$

We thus obtain a fundamental result in Quantum Mechanics: since $n \geq 0$, the total angular momentum is always quantised with eigenvalues j that are either integer or semi-integer, independently from the details of the dynamics.

It can also be demonstrated that the orbital angular momentum L can have only integer eigenvalues ℓ. It then follows that the operator spin S can only have integer or semi-integer eigenvalues.