PHY2003 ASTROPHYSICS I

Lecture 8. Orbital velocities and tidal forces

Orbital Energy

As gravity is a conservative force, and we assume that no other significant forces act on orbiting bodies, then the total energy of an orbital system must be the same at all times

Therefore the total energy TE of a gravitating two-body system is also constant, and is given at any point by

$$TE = KE + PE = (m_1 v_1^2/2) + (m_2 v_2^2/2) - (Gm_1 m_2/r) = \text{constant}$$

The total linear momentum of the two bodies must be constant ($m_1 v_1 = m_2 v_2$) and their relative velocity is $v = v_1 + v_2$.

$v = v_1(1 + v_2/v_1) = v_1(1 + m_1/m_2) = (m_1 + m_2)v_1/m_2$

$v_1 = m_2 v/(m_1 + m_2)$

$v_2 = m_1 v/(m_1 + m_2)$

Therefore the total energy can also be written as:

$$\begin{aligned} TE &= \frac{m_1}{2}\frac{m_2^2 v^2}{(m_1+m_2)^2} + \frac{m_2}{2}\frac{m_1^2 v^2}{(m_1+m_2)^2} - \frac{Gm_1 m_2}{r} \\ &= m_1 m_2 \left[\frac{m_2 v^2}{2(m_1+m_2)^2} + \frac{m_1 v^2}{2(m_1+m_2)^2} - \frac{G}{r}\right] \\ TE &= m_1 m_2 \left[\frac{v^2}{2(m_1+m_2)} - \frac{G}{r}\right] \end{aligned}$$

Elliptical orbital velocities

Considering the whole orbit and Kepler's 2nd Law:

$$\frac{dA}{dt} = \frac{A}{P} = \frac{\pi ab}{P} = \frac{H}{2}$$

$$H = \frac{2\pi ab}{P}$$

At perihelion, $v_{per} = v_t$,

$$\frac{rv_t}{2} = \frac{H}{2}$$

$$v_t = \frac{H}{r}$$

$$v_{per} = \frac{2\pi ab}{P}\frac{1}{a(1-e)} = \frac{2\pi a}{P}\frac{\sqrt{(1-e^2)}}{(1-e)}$$

$$v_{per} = \frac{2\pi a}{P}\sqrt{\frac{(1+e)}{(1-e)}}$$

Substitute into the equation for total energy:

$$TE = m_1 m_2 \left[\frac{v^2}{2(m_1+m_2)} - \frac{G}{r}\right]$$

$$TE = m_1 m_2 \left[\frac{1}{2(m_1+m_2)}\frac{4\pi^2 a^2}{P^2}\frac{(1+e)}{(1-e)} - \frac{G}{a(1-e)}\right]$$

$$TE = m_1 m_2 \left[\frac{1}{2(m_1+m_2)}4\pi^2 a^2\frac{G(m_1+m_2)}{4\pi^2 a^3}\frac{(1+e)}{(1-e)} - \frac{G}{a(1-e)}\right]$$

$$TE = m_1 m_2 \left[\frac{G}{2a}\frac{(1+e)}{(1-e)} - \frac{G}{a(1-e)}\right]$$

$$TE = \frac{Gm_1 m_2}{a(1-e)}\left[\frac{(1+e)}{2} - 1\right]$$

$$TE = \frac{Gm_1 m_2}{a(1-e)}\left[\frac{e}{2} - \frac{1}{2}\right]$$

$$\boxed{TE = -\frac{Gm_1 m_2}{2a}}$$

As the total energy TE is constant, this is true for any point in the orbit and not just perihelion. So we can equate this to the earlier expression:

$$\frac{-Gm_1m_2}{2a} = m_1m_2\left[\frac{v^2}{2(m_1+m_2)} - \frac{G}{r}\right]$$

$$\boxed{v^2 = G(m_1+m_2)\left[\frac{2}{r} - \frac{1}{a}\right]}$$

Example: What is the Earth's orbital velocity at aphelion?

Tidal Forces

The gravitational force depends on orbital distance r^{-2}

Thus small changes in distance give small changes in the gravitational force.

The far side of the Earth is pulled towards the moon with less force than the near side.

From the center of the Earth, this results in a differential acceleration. The differential force acts to stretch the Earth along the line of centers.

$$F = \frac{GMm}{r^2}$$

$$\frac{dF}{dr} = \frac{-2GMm}{r^3}$$

$$\boxed{dF = \frac{-2GMm}{r^3} dr}$$

Example: Show that the tidal acceleration of a test mass m on the Earth's surface due to the Moon is -1.1×10^{-6} m s^{-2}.

The Earth-Moon System

The yielding of oceans to tides results in energy loss through tidal friction. The tidal bulges act to slow down the rotation rate of the Earth.

As total angular momentum in the Earth-Moon system is conserved, and the Earth's (spin) angular momentum is decreasing, then the Moon's angular momentum must increase to compensate.

To relate this to the Moon's orbit, remember that the orbital velocity is proportional to the inverse square of the orbital radius:

$$v \propto \sqrt{\frac{1}{a}}$$

Angular momentum $\mathbf{L}$ is given by $\mathbf{L} = m(\mathbf{r} \times \mathbf{v})$

$$\text{So } L \propto rv \propto a\frac{1}{\sqrt{a}} \propto \sqrt{a}$$

Therefore as the Moons angular momentum increases, a increases and it moves away from the Earth.

<u>The Roche Limit</u>

Consider a moon of mass m and radius r orbiting at a distance d from a planet of mass M and radius R.

Now take a loose bit or rock of mass μ sitting on the surface of the moon.

At some orbital distance, the tidal force lifting the rock from the surface will be just balanced by the gravity of the moon.

$$F(tidal) = F(binding)$$

$$\frac{2GM\mu}{d^3}r = \frac{Gm\mu}{r^2}$$

$$d^3 = \frac{2r^3M}{m}$$

$$d = r\left(2\frac{M}{m}\right)^{1/3}$$

But $M = 4/3\pi R^3\rho_p$, and similarly for the moon.

$$d = r\left(2\frac{R^3\rho_p}{r^3\rho^m}\right)^{1/3}$$

The Roche limit is therefore approximately

$$d = R\left(2\frac{\rho_p}{\rho_m}\right)^{1/3}$$

If the densities of planet and moon are roughly the same, then

$$d \simeq 2^{1/3} \simeq 1.26R$$

An exact calculation for a single zero-strength or liquid spherical body gives a slightly larger distance than our simplified derivation above:

$$\boxed{d = 2.44R\left(\frac{\rho_p}{\rho_m}\right)^{1/3}}$$

This is the distance at which differential gravitational forces will tear a moon apart, if it has no internal tensile strength.