Lecture 16:
Bell inequality and Quantum Entanglement

The probabilistic nature of Quantum Mechanics is probably its most uncomfortable aspect despite the uncountable experimental confirmations. It is really possible that our Universe is so dominated by chance?
This question tormented the very same fathers of Quantum Mechanics, up to the point that it was suggested that Quantum Mechanics is not an exact theory, but only a statistical average over some exact hidden variables that we have no knowledge of (a situation comparable to thermodynamics, where all macroscopic variables arise from an average over the microscopic atom trajectories). The main arguments for this assumption are the conservation of locality and causality, apparently obvious properties of our macroscopic world.

However, J. S. Bell formulated a theory for hidden variables and mathematically demonstrated that this theory can not reproduce, by any means, Quantum Mechanics. This mathematical demonstration has then been validated by numerous experiments, confirming that, even though uncomfortable, the Universe we live in is governed by probability.

The example brought forward by J. S. Bell is the following: let us assume we have two particles, each with spin 1/2, that are in a state with total spin 0:

$$\psi_{\text{tot}} = \frac{1}{\sqrt{2}}[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle] \tag{1}$$

where $\uparrow$ and $\downarrow$ are just compact notations to indicate spin "up" and spin "down", respectively. We can assume that these two particles arise from the decay of a parent particle with total angular momentum $J = 0$ and that they travel in opposite directions. Let us also suppose that two systems A and B measure the components of the spin $(\vec{a}\cdot\sigma_A)$ and $(\vec{b}\cdot\sigma_B)$ along the randomly chosen axis $\vec{a}$ and $\vec{b}$. Let us start from the case where $\vec{a} = \vec{b}$. If we are measuring $(\vec{a}\cdot\sigma_A) = 1$ then $(\vec{b}\cdot\sigma_B)$ must be equal to -1 and viceversa. Since locality and causality assure us that the measurement of $(\vec{a}\cdot\sigma_A)$ cannot dynamically affect the measurement of $(\vec{b}\cdot\sigma_B)$, it seems like we are reaching a paradox. Quantum Mechanics should in fact tell us that the measurements of $(\vec{b}\cdot\sigma_B)$ should be ± 1, which with a probability 1/2, independently from the measurement of $(\vec{a}\cdot\sigma_A)$. One might then think that these two measurements are somehow predetermined. In other words, the probabilistic nature of Quantum Mechanics is only due to the fact that we lack of knowledge of additional, "hidden" variables, a statistical average over which appears to give a probabilistic aspect to our understanding of Nature.

This argument is, however, wrong. Since the two events (the measurements of $(\vec{a} \cdot \sigma_A)$ and $(\vec{b} \cdot \sigma_B)$ are not connected causally, the information of the measurements of A is not usable by the observer B since the latter can not have any access to this piece of information before his own measurement has been performed. For B, the measurement of $(\vec{b} \cdot \sigma_B)$ still gives a random series of values 1 or -1, each with a probability 1/2.

However, it still remains the fact that, when the two measurements are confronted *a posteriori*, there is still an exact correlation between the two. It might seem then that the measurement of B induces a sudden collapse [1] of the wavefunction detected by A. This is an obvious violation of locality and causality.

The answer to this paradox is found if we consider that Quantum Mechanics gives an exact prediction of the correlation between these two measurements:

$$F(\vec{a}, \vec{b}) = \langle (\vec{a} \cdot \sigma_A)(\vec{b} \cdot \sigma_B) \rangle = \overline{R(\vec{a} \cdot \sigma_A) R(\vec{b} \cdot \sigma_B)} \tag{2}$$

where $R(\vec{a} \cdot \sigma_A) = \pm 1$ and $R(\vec{b} \cdot \sigma_B) = \pm 1$ represent the possible results of the two measurements. The result is:

$$F(\vec{a}, \vec{b}) = -\cos\theta \quad \text{with } \theta \text{ the angle between the two vectors} \tag{3}$$

The problem is now well defined. The question is: is there any theory with hidden variables that is able to reproduce Quantum Mechanics entirely?

The answer to this question is no, and it was first demonstrated by J. S. Bell in 1960.

Let us assume that λ is a hidden parameter for our theory with hidden variables (THV). The predictions for the measurements aforementioned will be:

$$A(\vec{a}, \lambda) = \pm 1 \quad B(\vec{b}, \lambda) = \pm 1 \tag{4}$$

The correlation between these two measurements in THV is:

$$F(\vec{a}, \vec{b}) = \int \mathrm{d}\lambda P(\lambda) A(\vec{a}, \lambda) B(\vec{b}, \lambda) \tag{5}$$

with $P(\lambda)$ being the statistical probability for the values of λ:

$$P(\lambda) \geq 0 \ \ \forall \lambda \ \ , \quad \int \mathrm{d}\lambda P(\lambda) = 1 \tag{6}$$

Let us consider the case where $\vec{a} = -\vec{b}$:

$$F(\vec{a}, \vec{b}) = -\int \mathrm{d}\lambda P(\lambda) A(\vec{a}, \lambda) A(\vec{b}, \lambda) \tag{7}$$

We then have:

$$F(\vec{a},\vec{b}) - F(\vec{a},\vec{c}) = -\int \mathrm{d}\lambda P(\lambda)[A(\vec{a},\lambda)A(\vec{b},\lambda) - A(\vec{a},\lambda)A(\vec{c},\lambda)] \tag{8}$$

which is equal to:

$$F(\vec{a},\vec{b}) - F(\vec{a},\vec{c}) = \int \mathrm{d}\lambda P(\lambda)A(\vec{a},\lambda)A(\vec{b},\lambda)[A(\vec{b},\lambda)A(\vec{c},\lambda) - 1] \tag{9}$$

So that:

$$|F(\vec{a},\vec{b}) - F(\vec{a},\vec{c})| \le \int \mathrm{d}\lambda P(\lambda)[1 - A(\vec{b},\lambda)A(\vec{c},\lambda)] = 1 + F(\vec{b},\vec{c}) \tag{10}$$

Any THV must then satisfy the inequality (*Bell's inequality*):

$$|F(\vec{a},\vec{b}) - F(\vec{a},\vec{c})| \le 1 + F(\vec{b},\vec{c}) \tag{11}$$

It is easy to see that, in general, Quantum Mechanics does not satisfy this relation. Any THV is thus inconsistent with Quantum Mechanics explaining elegantly that no additional hidden variables are required to reproduce the results of Quantum Mechanics.

---

[1] Regarding wave collapse, Erwin Schrödinger stated: "if we should go on with this damned wave function collapse, then I am sorry that I ever got involved"