PHY2006 Assignment 4 –Numerical Solutions to Differential Equations

Deadline for Submission 6pm, Monday 18 Oct 2021

1. An energetic proton in the solar wind is moves in a region of space where the Earth's magnetic field has a magnitude of $B = 1.044 \times 10^{-5}$ T in the positive z direction. The proton has a velocity of $\dot{x} = 10^6$ m/s in the positive x direction ($\dot{y} = \dot{z} = 0$) as it passes through the origin at $t = 0$. The Lorentz force on the proton results in the following equations of motion

$$(1)\ m\frac{d\dot{x}}{dt} = eB\dot{y} \qquad (2)\ m\frac{d\dot{y}}{dt} = -eB\dot{x} \qquad (3)\ m\frac{d\dot{z}}{dt} = 0$$

While these are ODEs (the dependent variables x and y are each a function of t only) equations (1) and (2) are coupled since x, y are present in both.

(a)(i) By differentiating equation (1) and doing a substitution, solve these equations analytically using the initial conditions to obtain expressions for the components of the velocity $\dot{x}, \dot{y}, \dot{z}$ as a function of ω, t where $\omega = \frac{eB}{m}$. **[25]**

(ii) Using initial conditions again, obtain expressions for x, y, z as a function of t. Using Excel (or other software) plot out the motion of the proton in the x, y plane (make the scales on each axis the same length and plot points every 10^{-4} s until $t = 6 \times 10^{-3}$ s). **[20]**

(b)(i) Using the Euler method write down finite difference versions of equations (1) and (2) in terms of $\dot{x}_{i+1}$, $\dot{x}_i$, $\dot{y}_{i+1}, \dot{y}_i, \omega$ and Δt. Similarly write down expressions for x_{i+1} and y_{i+1} in terms of $x_i, y_i, \dot{x}_i, \dot{y}_i$ and Δt. **[15]**

(ii) Using Excel (or other) complete the following table using your finite difference equations with $\Delta t = 2\pi/10\omega$.

i	t_i	$\dot{x}_i$	$\dot{y}_i$	x_i	y_i
0	0	10^6	0	0	0
1	Δt	…	…	…	…
2	$2\Delta t$	…	…	…	…
3	$3\Delta t$	…	…	…	…
⋮	⋮	⋮	⋮	⋮	⋮
10	$10\Delta t$	…	…	…	…

[20]

(iii) Plot the numerical and analytical solution on the same x, y graph. **[10]**

(iv) For the numerical solution, what value of Δt is needed so that after one revolution the proton is within 500 m of the origin? **[10]**

Extra Question

Consider the following differential equation with initial conditions, $t = 0, y = 1$

$$\frac{dy}{dt} + 2y = 2 - e^{-4t}$$

(a) Obtain the analytical solution of this 1st order ODE.

(b) Using the explicit Euler method write down the differential equation as a finite difference equation which can be used to obtain a numerical solution. Express it in terms of y_{i+1}, y_i, t_i and Δt.

(c) Using $\Delta t = 0.1$, complete the below table to determine the values of y_i up to $t_i = 0.5$.

(d) By comparing your analytical and numerical solutions work out the % global error at each step.

i	t_i	y_i	*$y(t_i)$ - analytical*	*% Error*
0	0	1	1	0
1	0.1	…	…	…
2	0.2	…	…	…
3	0.3	…	…	…
4	0.4	…	…	…
5	0.5	…	…	…