PHY2003 ASTROPHYSICS I

Lecture 6. Inside the Sun

Hydrostatic equilibrium

The Sun is trying to collapse from self-gravity. At a distance r from the center, the mass within r is given by

$$m(r) = \int_0^r \rho(r')4\pi r'^2 dr'$$

This gives an inward gravitational acceleration

$$a_g = -\frac{Gm(r)}{r^2}$$

Now consider the pressure on a parcel of gas at r, width Δr and area ΔA

$$F_p = [P_{outer} - P_{inner}]\,\Delta A$$

P_{inner} must be $> P_{outer}$, otherwise the star will collapse! So there must be a pressure gradient $dP/dr < 0$

The pressure on the outer surface of the gas parcel is

$$P_{outer} = P(r + \Delta r) = P(r) + (\frac{dP}{dr})\Delta r$$

$$F_p = \left[P(r) + (\frac{dP}{dr})\Delta r - P(r)\right]\Delta A = (\frac{dP}{dr})\Delta r \Delta A$$

As the mass $\Delta M = \rho(r)\Delta V = \rho(r)\Delta A \Delta r$, acceleration is given by

$$a_p = (\frac{dP}{dr})\Delta r \Delta A / \Delta M = \frac{dP}{dr}\frac{1}{\rho(r)}$$

In equilibrium, $a_p = a_g$

$$\frac{dP}{dr}\frac{1}{\rho(r)} = -\frac{Gm(r)}{r^2}$$

The Equation of Hydrostatic Equilibrium:

$$\boxed{\frac{dP}{dr} = -\frac{Gm(r)\rho(r)}{r^2}}$$

Example: From spectroscopic measurements, the photosphere has an average density of $\simeq 10^{-4}$ kg/m^3; what is the pressure gradient in the photosphere?

Average interior conditions

Take the EHE and multiply both sides by $4\pi r^3$ and integrate from center to surface:

$$\int_0^{R_\odot} 4\pi r^3 \frac{dP}{dr} dr = \int_0^{R_\odot} \frac{Gm(r)}{r^2}\rho(r) 4\pi r^3 dr$$

LHS Integrate by parts:

$$LHS = \left[P(r)4\pi r^3\right]_0^{R_\odot} - 3\int_0^{R_\odot} P(r)4\pi r^2 dr$$

First term is effectively zero ($P \simeq 0$ at $r = R_\odot$)

For the Second term, substitute $\overline{P}$ for $P(r)$

$$3\int_0^{R_\odot} \overline{P}4\pi r^2 dr = 3\overline{P}\int_0^{R_\odot} 4\pi r^2 dr = 3\overline{P}V$$

where V is volume.

RHS is gravitational potential energy U_g of the Sun:

$$RHS = \int_0^{R_\odot} \frac{Gm(r)}{r^2}\rho(r)4\pi r^3 dr = \int_0^{R_\odot} \frac{Gm(r)}{r}\rho(r)4\pi r^2 dr$$

$$RHS = \int_0^{R_\odot} \frac{Gm(r)}{r} dm = U_g$$

$$0 - 3\overline{P}V = U_g$$

$$\boxed{\overline{P} = -\frac{U_g}{3V}}$$

(This is one form of the *Virial Theorem*, which gives a relationship between the kinetic and potential energies of a gravitationally bound system of particles).

Using the ideal gas equation $\overline{P} = \overline{n}k\overline{T}$, $\overline{n} = \overline{\rho}/\overline{m}$. $\overline{m}$ (the mean molecular weight of the Sun) = 0.61 $\cdot$ $\mathrm{m}_{\mathrm{Hydrogen}}$.

$$\overline{T} = \frac{\overline{P}\,\overline{m}}{k\overline{\rho}},\ \overline{P} = -\frac{U_g}{3V},\ U_g \simeq -\frac{GM_\odot^2}{R_\odot}$$

$$\boxed{\overline{T} \simeq \frac{GM_\odot\overline{m}}{3kR_\odot}}$$

Example: Calculate the average pressure and temperature of the Sun.

Plasma values at the Solar core

From the EHE:

$$\frac{dP}{dr} = -\rho(r)\frac{Gm(r)}{r^2} = -\rho(r)\frac{G}{r^2}\,\frac{4}{3}\pi r^3\overline{\rho}$$

Approximate $\rho(r)$ by $\overline{\rho}$:

$$\frac{dP}{dr} \simeq -\overline{\rho}^2\,G\,\frac{4}{3}\pi r$$

Integrate this from core to surface:

$$\int_{P_c}^{0} dP \simeq -\overline{\rho}^2\,G\,\frac{4}{3}\pi \int_0^{R_\odot} r dr$$

$$-P_c \simeq -\overline{\rho}^2\,G\,\frac{4}{3}\pi\frac{R_\odot^2}{2}$$

$$\boxed{P_c \simeq \frac{2}{3}\pi\,G\overline{\rho}^2\,R_\odot^2}$$

To estimate the temperature, use ideal gas equation:

$$T_c = \frac{P_c}{n_c k} \simeq \frac{P_c}{\overline{n} k}$$

$$\boxed{T_c \simeq \frac{P_c \overline{m}}{\overline{\rho}\, k}}$$

The Sun's structure: The Core - $0 \leq r \leq 0.25R_{\odot}$.

Gravitational collapse cannot generate the Sun's luminosity: The gravitational PE of the Sun is

$$U_{\odot} \simeq GM_{\odot}^2/R_{\odot} \equiv 3.8 \times 10^{41} \text{ J}$$

So the lifetime of the Sun under this process would be

$t_{\odot} \simeq U_g/L_{\odot} \simeq 10^{15}$ sec $\simeq 3 \times 10^7$ years. But the Sun is $\sim 4.5 \times 10^9$ years old.

The most likely explanation for the Suns luminosity is nuclear fusion, where $H \rightarrow He + \gamma$. The most common form of this reaction is the proton-proton $(p - p)$ chain.

$$\begin{aligned} {}^1_1\text{H} + {}^1_1\text{H} &\rightarrow {}^2_1\text{D} + e^+ + \nu_e \\ {}^1_1\text{H} + {}^2_1\text{D} &\rightarrow {}^3_2\text{He} + \gamma \\ {}^3_2\text{He} + {}^3_2\text{He} &\rightarrow {}^4_2\text{He} + {}^1_1\text{H} + {}^1_1\text{H} \end{aligned}$$

The result of this reaction is that 4 hydrogen atoms create one helium atom, with 2 neutrinos and 6 photons carrying an associated release of energy of 26MeV $\equiv 4 \times 10^{-12}$ J

Therefore the number of p-p chains per second occurring is $\sim 3.8 \times 10^{26} J/4 \times 10^{-12} J$ or $\simeq 10^{38}$.

In one second 4×10^{38} atoms of Hydrogen are transformed into Helium, or $4 \times 10^{38} \times 1.66 \times 10^{-27} = 6.6 \times 10^{11}$ kg. Therefore the Sun has so far used $6.6 \times 10^{11} \times 4.5 \times 10^{9} \times 3.2 \times 10^{7} \simeq 9 \times 10^{28}$kg of H.

This is $\sim 5\%$ of its present H-mass. Therefore this is a viable energy source, and implies that the Sun has not grossly changed since formation.

Example: Estimate the flux of solar neutrinos through each square cm of your body.

The Sun's structure: The Radiative Zone

$0.25 \leq r \leq 0.7 R_{\odot}$

Region in which most energy is transported through radiation.

Photons follow a random-walk through scattering by ions.

D is distance traveled, N is number of steps, l is length of one step:

$$D^2 = Nl^2$$

For the Sun

$$N = R_\odot^2/l^2$$

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The time taken for a photon to move one step is l/c.

The time taken for a photon to leave the Sun is

$$T = Nl/c = R_\odot^2/lc$$

The Sun is very opaque, $l \simeq 10^{-3}$m, so $T \simeq 50,000$ years!

The Sun's structure: The Convective Zone

$0.7 \leq r \leq 1.0 R_\odot$

Region in which most energy is transported through convection.

Top of the convective zone is the photosphere - granules are individual convective bubbles of gas (but remember size!).

Observational tests

(a) Solar Oscillations

Regions of the photosphere are seen to rise and fall on a 5-minute period. This is caused by granulation sending sound waves throughout Sun, setting up standing waves and resonances. Astronomers have now measured thousands of different wave frequencies to accuracies of $\pm 0.01\%$. The resulting passage of sound waves allows determination of internal properties – helioseismology.

(b) Solar Neutrinos

The neutrinos released by the p-p chain are of low energy and are difficult to detect. However, a small amount of ${}^3_2\mathrm{He}$ undergoes a different reaction that releases energetic neutrinos:

$$\begin{aligned}
{}^3_2\mathrm{He} + {}^4_2\mathrm{He} &\rightarrow {}^7_4\mathrm{Be} + \gamma \\
{}^7_4\mathrm{Be} + p &\rightarrow {}^8_5\mathrm{B} + \gamma \\
{}^8_5\mathrm{B} &\rightarrow {}^8_4\mathrm{Be} + e^+ + \nu_e + \gamma \\
{}^8_4\mathrm{Be} &\rightarrow {}^4_2\mathrm{He} + {}^4_2\mathrm{He}
\end{aligned}$$

For 40 years, experiments detecting neutrinos only detected 25-50% of predicted numbers. In 2001, this was solved by experiments that found the total number of electron neutrinos leaving the Sun is as predicted, but roughly half change into a different type.