ASSIGNMENT 2

1(b)

$$\psi = \underbrace{C(t)\,x}\,\underbrace{e^{-\gamma x^2}}$$

$u = Cx \qquad \frac{du}{dx} = C$

$v = e^{-\gamma x^2} \qquad \frac{dv}{dx} = -2\gamma x\, e^{-\gamma x^2}$

$$\frac{\partial \psi}{\partial x} = C e^{-\gamma x^2} + (Cx)(-2\gamma x)\, e^{-\gamma x^2}$$

$$\frac{\partial \psi}{\partial x} = C e^{-\gamma x^2} - 2C\gamma x^2 e^{-\gamma x^2}$$

1(c)

2(a) $T(r,t)$ NOT DEPENDING ON θ, ϕ $\quad \frac{\partial T}{\partial \theta} = 0$

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\underbrace{\left(r^2 \frac{\partial T}{\partial r}\right)}_{\text{PRODUCT RULE}}$$

2.1 Conservation Equation

1D $$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0$$

ρ - density

v - velocity

3D $$\frac{\partial \rho}{\partial t} + \nabla \cdot \underline{F} = 0$$

$$\frac{\partial \rho}{\partial t} = - \nabla \cdot \underline{F}$$

$\underline{F} = \rho \underline{v}$ Flow vector

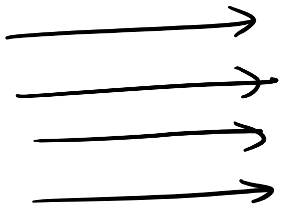
- charge/current
- energy
- fluids
- people

2.2 Advection Equation

$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \underline{v} + \underline{v} \cdot (\nabla \rho) = 0$$

"Incompressible flow" approx.

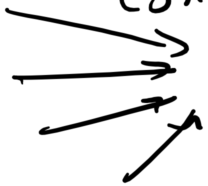
$$\nabla \cdot \underline{v} \approx 0$$



$\underline{v}$

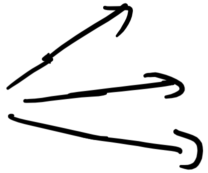
Laminar flow

Compression



$\nabla \cdot \underline{v} < 0$

Expansion



$\nabla \cdot v > 0$

2.3 HEAT EQUATION

FOURIER LAW

RATE OF HEAT FLOW PER UNIT AREA (W/m^2) $= -K\frac{dT}{dx}$ 1D

ENERGY CONSERVATION $= -K(\nabla T)$ 3D

$$\frac{\partial T}{\partial t} = D(\nabla^2 T)$$

2.4 WAVE EQUATION

$$\nabla^2 \Phi = \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} \quad 3D$$

Φ – SOUND → PRESSURE

– LIGHT → ELECTRIC FIELD

– STRING → DISPLACEMENT

– SEISMIC

2.5 LAPLACE/POISSON

$$\nabla^2 \Phi = 0 \quad \text{(LAPLACE)}$$

$$\neq 0 \quad \text{(POISSON)}$$

2.6 SCHRÖDINGER

3. Classification of PDEs

Order

Linearity

Homogeneity

Hyperbolic, Elliptical, Parabolic

Kahoot PHY2006 – 5. Classification of 2^{nd} order PDEs

Order, Linearity, Homogeneity

ORDER

$$\frac{\partial^{(n)} u}{\partial x^{(n)}}$$

Same definitions as for ODEs

Dependent variable $u(x, y)$

- Order – number of times u differentiated
- Linear – $u^n, \left(\frac{\partial u}{\partial x}\right)^n \ldots \quad n = 1$
- Homogeneity – all terms contain u RHS = 0

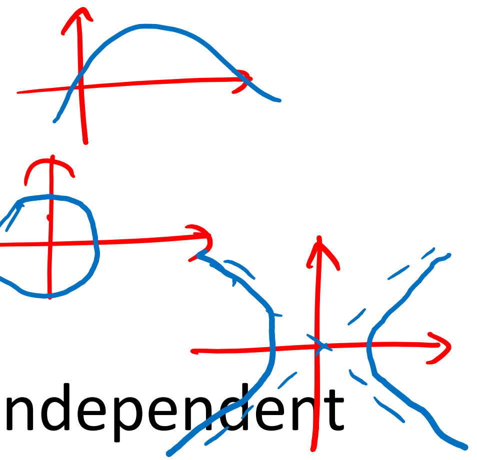
	ORDER	LINEAR	HOMOGENEOUS
$(x^2 + y^2)\frac{\partial u}{\partial x} + 2y\frac{\partial^3 u}{\partial x^2 \partial y} - 3u = 0$	3	L	H
$u\left(\frac{\partial u}{\partial x}\right)^2 + x\frac{\partial u}{\partial y} = 0$	1	NL	H
$\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 + \frac{\partial^2 u}{\partial y^2} - x^2 - y^2 = 0$	2	NL	I

Types of 2nd order PDEs

- Consider to $ax^2 + bxy + cy^2 + dx + ey + f = 0$. In x,y space this describes

$b^2 - 4ac$

- $= 0$ • Parabola $c = 0, b = 0, c = 1$ $y = -x^2 - dx - f$
- < 0 • Ellipse $b, d, e = 0$ $a = c = f = 1$ $x^2 + y^2 = 1$
- > 0 • Hyperbola $b, d, e = 0$ $a = -c = f$ $x^2 - y^2 = 1$



- A 2nd order linear PDE with one dependent variable $u(x, y)$ and two independent variables x, y be generally written in the form
- $A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + F = 0$
- $B^2 - 4AC > 0$ HYPERBOLIC
- $B^2 - 4AC < 0$ ELLIPTICAL
- $B^2 - 4AC = 0$ PARABOLIC
- Type of PDE influences the method for solving

Types of 2nd order PDEs - Examples

- Laplace $1\frac{\partial^2\Phi}{\partial x^2} + 1\frac{\partial^2\Phi}{\partial y^2} = 0$
- Heat $\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2}$
- Wave $v^2\frac{\partial^2 u}{\partial x^2} \cancel{=} - \cancel{\frac{1}{v^2}}\frac{\partial^2 u}{\partial t^2} = 0$
- $1\frac{\partial^2 u}{\partial x^2} \cancel{=} - x\frac{\partial^2 u}{\partial y^2} = 0$
- $\frac{\partial^2 u}{\partial x^2} + 3xy\frac{\partial^2 u}{\partial x \partial y} + 2u = 0$

A	B	C	$B^2 - 4AC$	Type
1	0	1	-4	ELLIPTICAL
D	0	0	0	PARABOLIC
v^2	0	-1	$4v^2$	HYPERBOLIC ($v=0$ PARABOLIC)
1	0	$-x$	$4x$	MIXED $x=0$ PARABOLIC $x>0$ HYPERBOLIC $x<0$ ELLIPTICAL
1	$3xy$	0	$9x^2y^2$	MIXED HYPERBOLIC $x=0$, $y=0$ PARABOLIC

4. Analytical Solutions to PDEs

4.1 Initial/Boundary Conditions

4.2 Method of Characteristics

4.3 Separation of Variables + Fourier Solutions

$$\frac{dN}{dt} = -\lambda N \qquad t = 0 \quad N = N_0$$

$$N = N_0 e^{-\lambda t}$$

Initial / Boundary Conditions

- Unique solution to ODE requires: ORDER $\longrightarrow$ NO. CONSTRAINTS
- PDEs give infinite solutions. Unique solution requires ***well-posed problem***
  - Too few constraints – unique solution not possible
  - Too many constraints – no solution or unphysical solution

$$\frac{d^2x}{dt^2} + k^2x = 0$$

$$x = A\cos kt + B\cos kt$$

2 CONSTRAINTS

$t=0$ $\quad x=0$ $\quad \frac{dx}{dt}=v$

- **Constraints of a PDE** representing physical problem are

INITIAL CONDITIONS (IC): TIME $t=0$, $t=t_0$

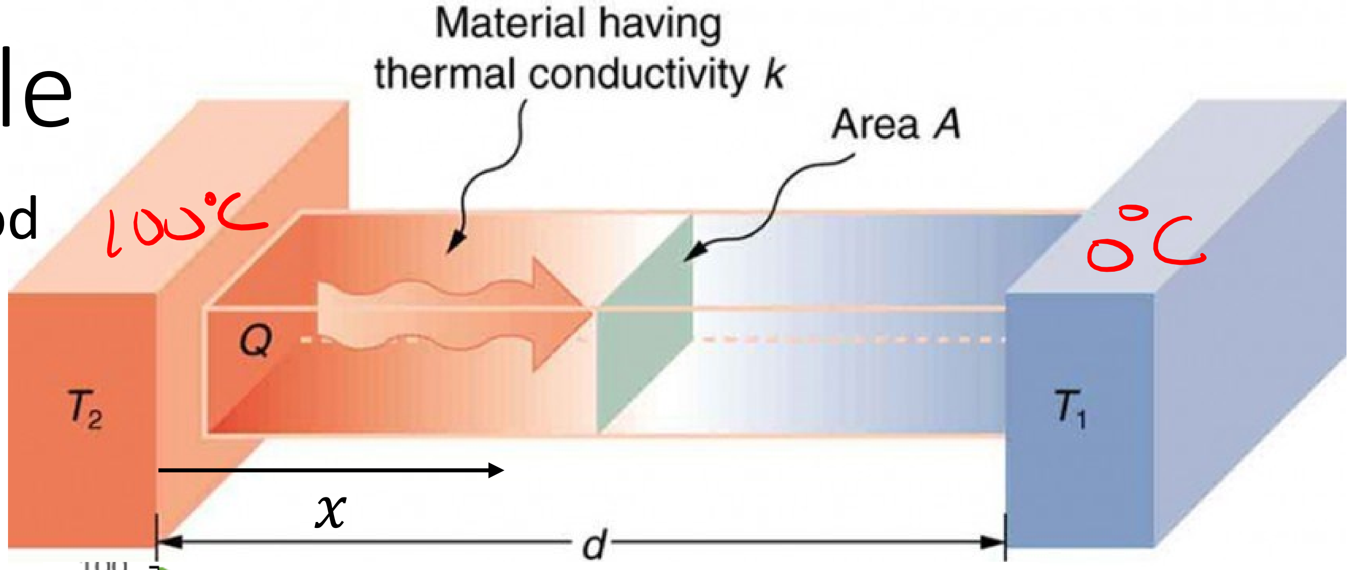
BOUNDARY CONDITIONS (BC): PHYSICAL DIMENSIONS $x, y \ldots$

$x = x_0$ $\quad u, \frac{\partial u}{\partial t} = \text{CONST.}$

- Types of BCs
  - Dirichlet – fixed values of dependent variable $u(x_0, y_0)$
  - Neumann – fixed values of its derivative $u'(x_0, y_0)$ $= \frac{\partial y}{\partial t} = \text{CONST.}$

Heat Equation Example

- Consider a straight, well insulated metal rod with its ends in good thermal contact with heat reservoirs of temperature T_1 and T_2.



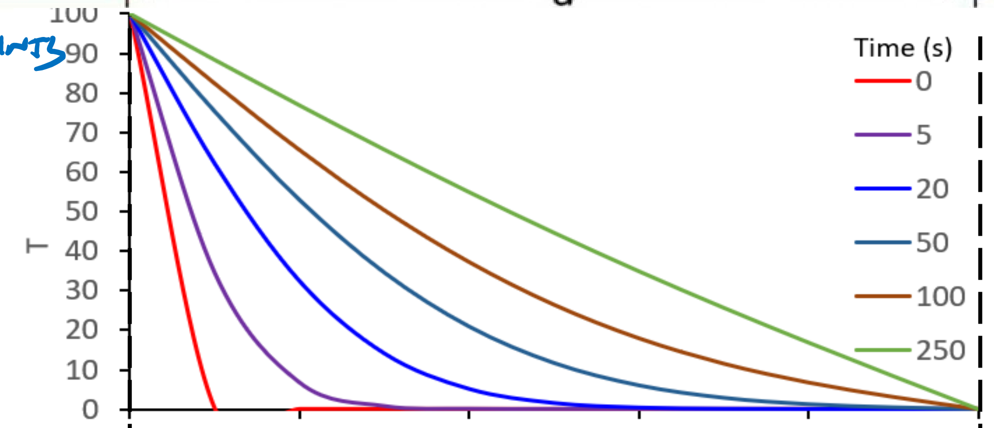
$$\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2}$$

ORDER 1 + 2 = 3 CONSTRAINTS

- Example 1

BC $T(0,t) = 100$ (1) $T(d,t) = 0$ (2)

IC $T(x,0) = 0$ (3)

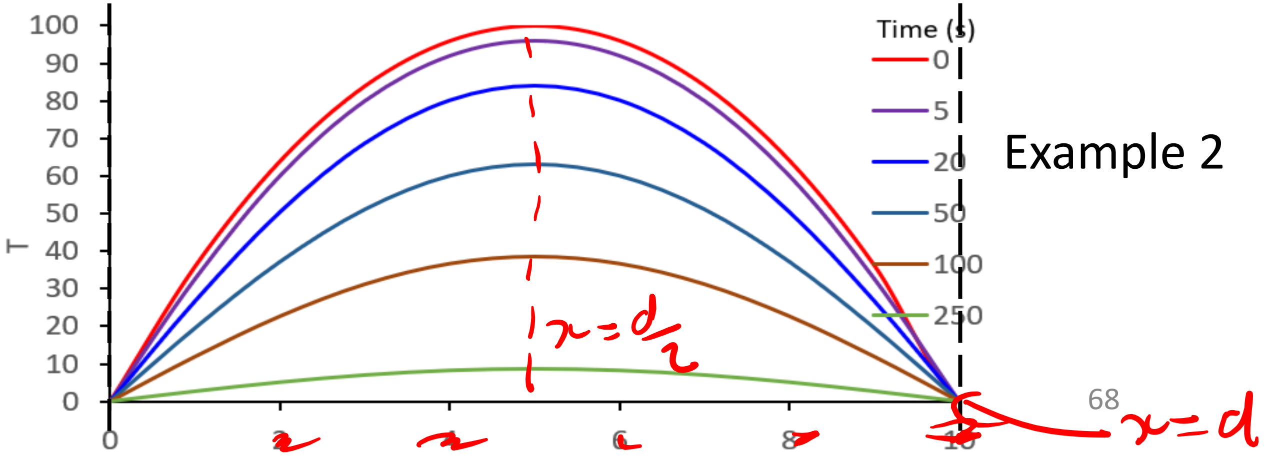


Example 1

- Example 2

BC $T(0,t) = 0$ $T(d,t) = 0$

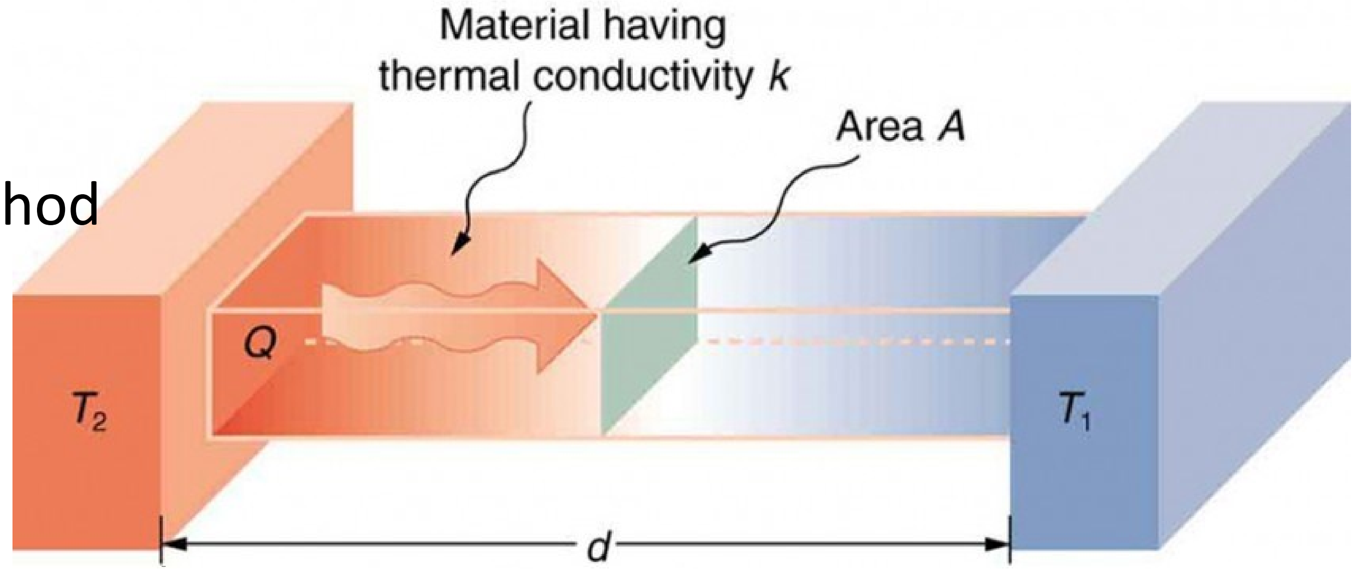
IC $T(x,0) = 400\frac{x}{d}\left(1 - \frac{x}{d}\right)$

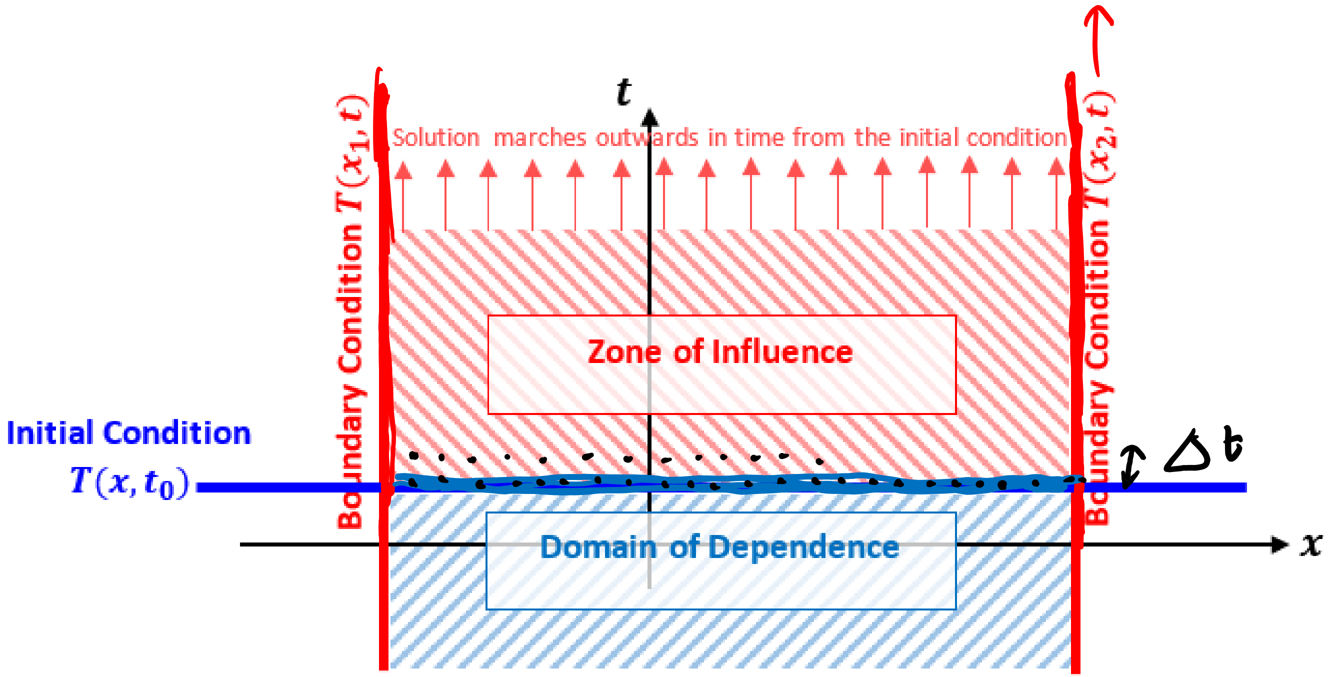


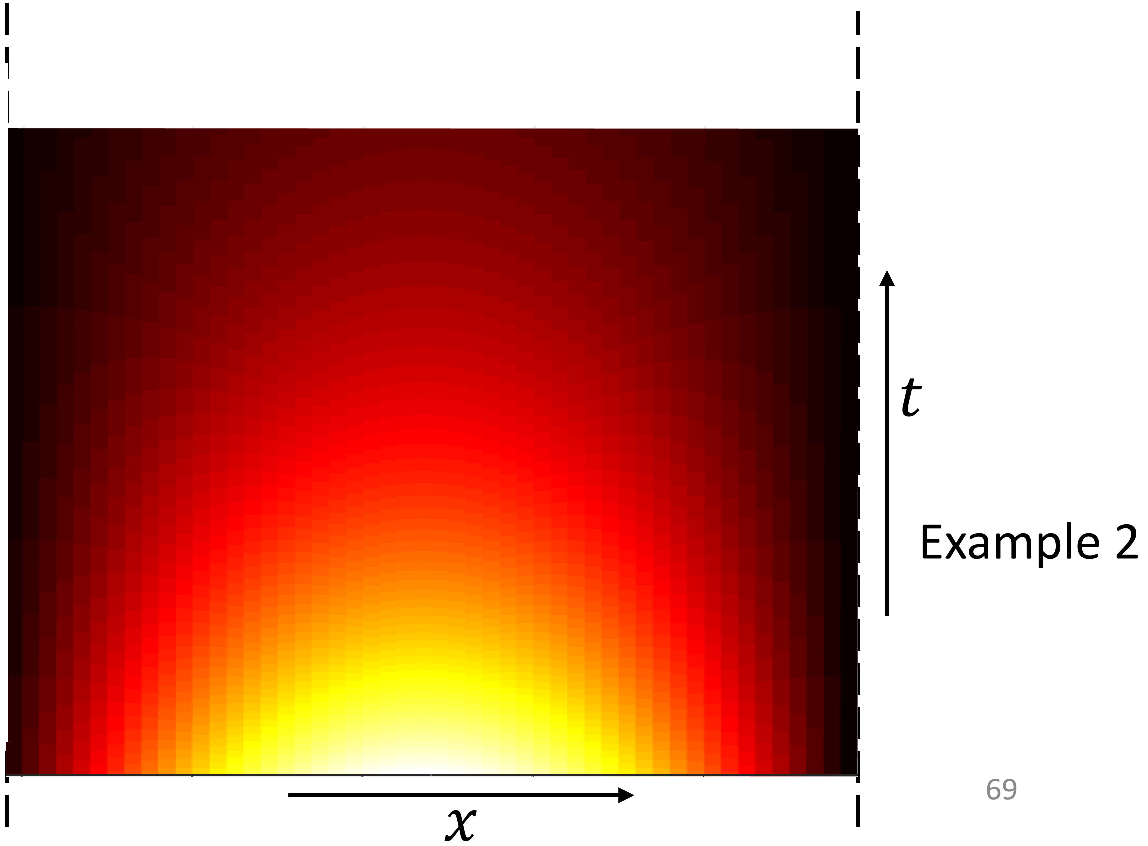
Example 2

Parabolic PDEs

- Type of equation determines solution method
- E.g. Heat Equation $\frac{\partial T}{\partial t} = D\,\frac{\partial^2 T}{\partial x^2}$
- Solutions obtained
  marching forward in time



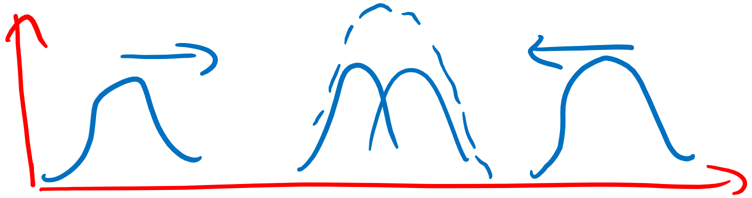


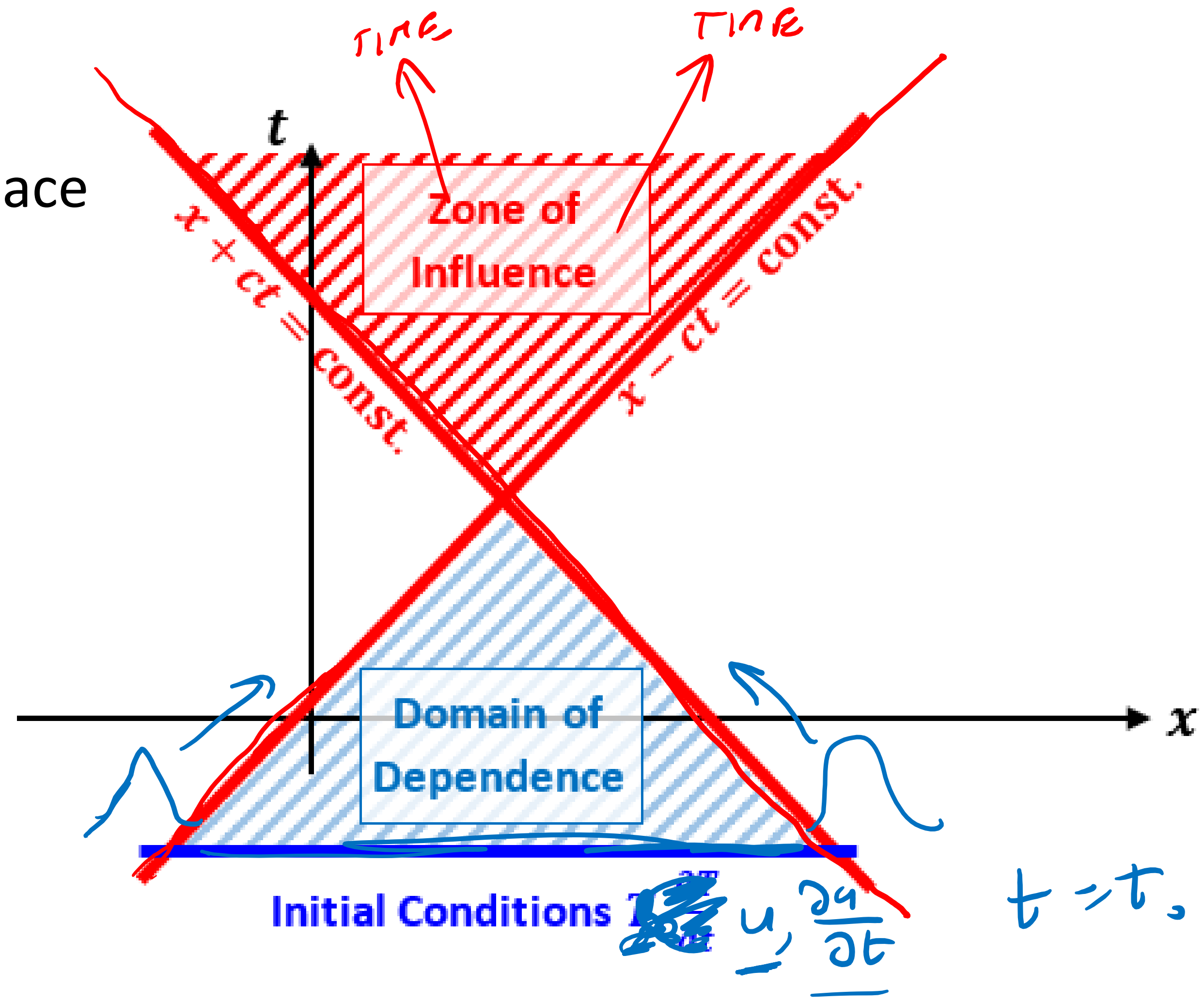


Hyperbolic PDEs

- E.g. Wave Equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{v^2}\dfrac{\partial^2 u}{\partial t^2}$
  ORDER $\underbrace{}_{2}$ $\underbrace{}_{2}$
- Velocity of forward and backward waves determine boundaries of solutions in (x,t) space
- Forward travelling wave ($x - ct = \text{const.}$)
- Backward travelling wave ($x + ct = \text{const.}$)

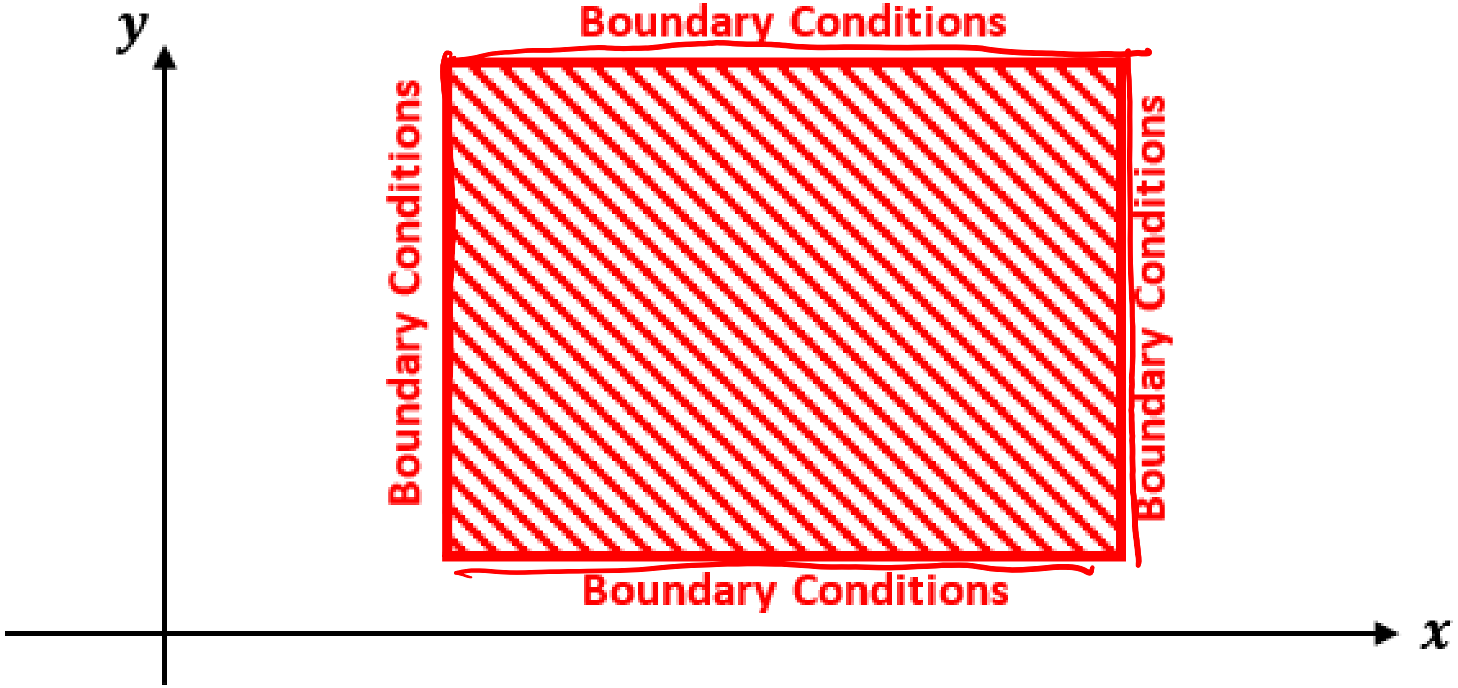
- Solutions obtained
  marching forward in time





Elliptical PDEs

- E.g. Laplace's Equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$
- No time dependence – static problem, no ICs
- BCs only, generally these enclose a space
- Change in BC, ***instantaneously*** influences all other points



4.2 Method of Characteristics

ORDER = 1

2 VARIABLES - x, t

INITIAL CONDITION $u(x,0) = f(x)$

Method of Characteristics

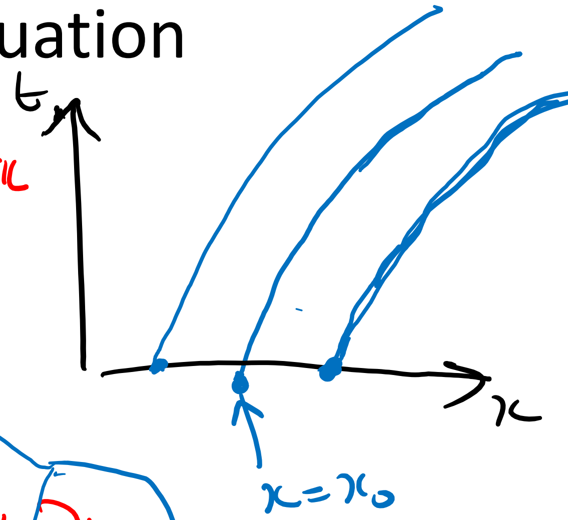
- Used to solve PDEs by ***marching forward in time***
- ***Characteristic*** –trajectory in (x,t) space along which a solution is propagated
- We will use to solve 1st order, linear PDEs such as advection equation

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = f$$

CHARACTERISTIC

where v, f may be a functions of x, t, u

$x = f(t)$



- Comparing to the total differential

$$\frac{du}{dt} = \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial t} + \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t} = \frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x}$$

reduces the problem to solving 2 ODEs

$$\frac{dx}{dt} = v \qquad\qquad \frac{du}{dt} = f$$

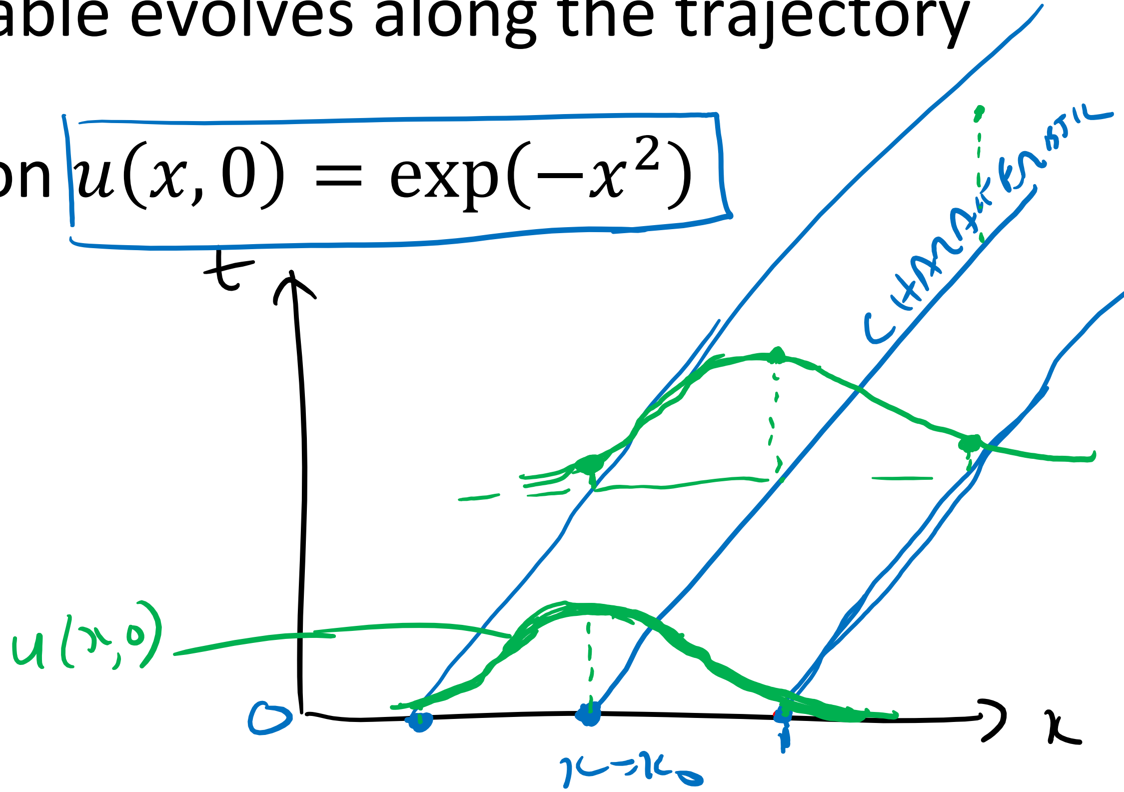
Method of Characteristics

- $\dfrac{dx}{dt} = v$ defines the trajectory or ***characteristic*** of a solution through (x, t) space
- $\dfrac{du}{dt} = f$ determines how the dependent variable evolves along the trajectory
- Example $\dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial x} = 0$ with initial condition $\boxed{u(x,0) = \exp(-x^2)}$

$t=0 \quad x=x_0$

$$\frac{dx}{dt} = c$$

$$\boxed{x = ct + x_0}$$

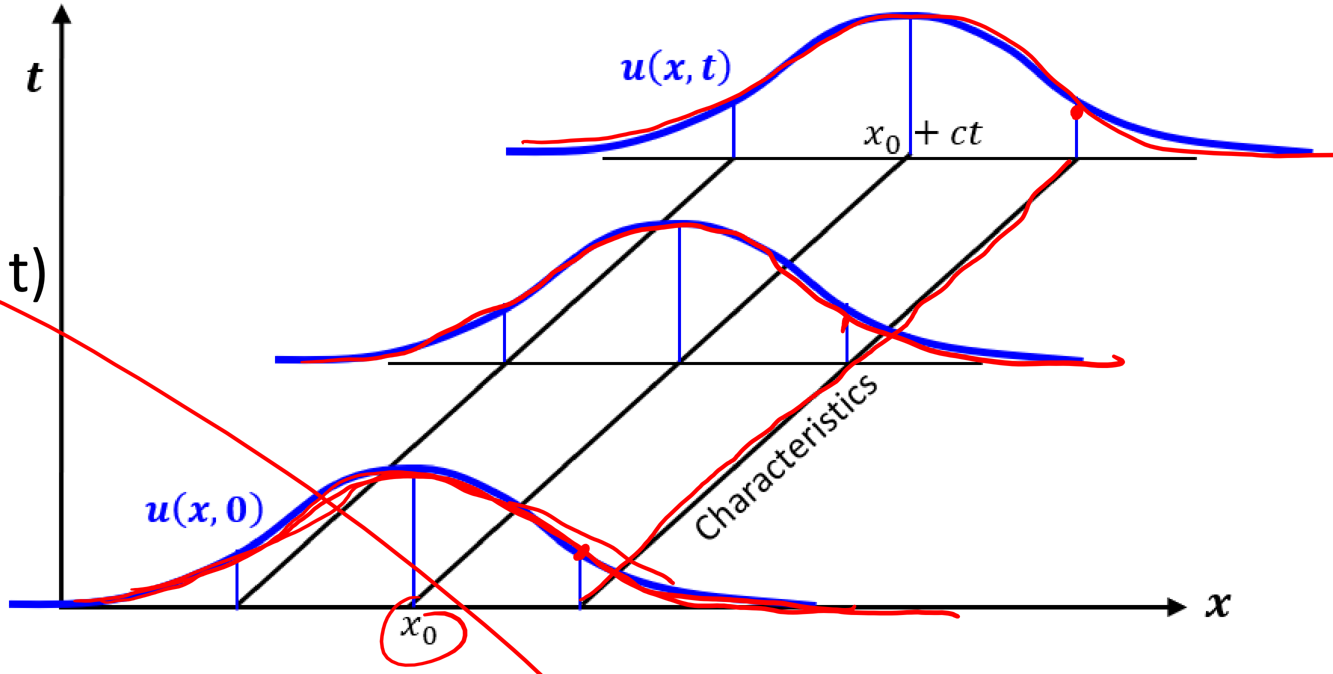


- Characteristic is a straight line

Example 1

- $\boxed{x = ct + x_0}$ CHARACTERISTIC
- $\frac{du}{dt} = 0$ (remember u is a function of x and t)

$u(x_0,t) = C(x_0)$



- For each initial position x_0 along the x axis we have a distribution

$u(x_0, 0) = \underbrace{\exp(-x_0^2) = C(x_0)}$

- As u is unchanged along a characteristic $\underline{u(x_0, t) = \exp(-x_0^2)}$
- $u(x_0, t)$ is a solution along a characteristic defined by x_0. To obtain the solution for all characteristics $u(x, t)$, then $x_0 = x - ct$ is substituted in

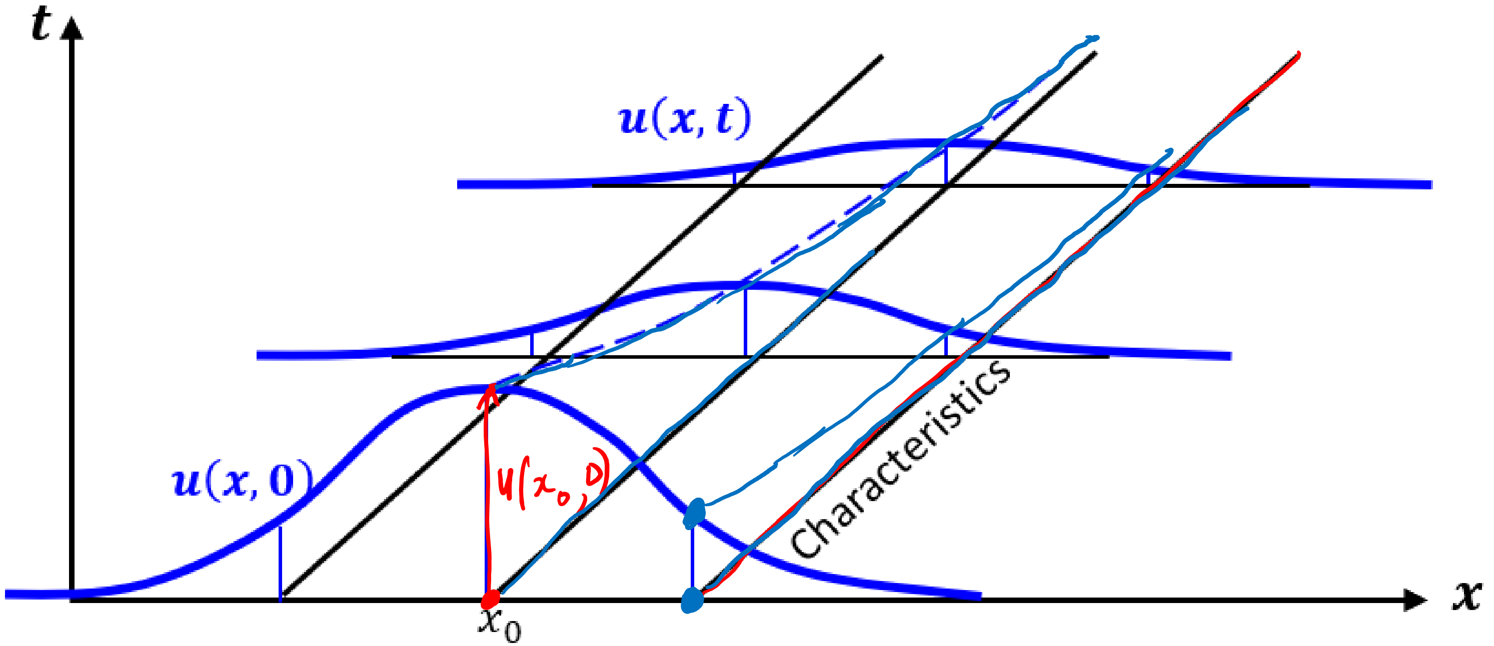
$u(x,t) = \exp(-x_0^2) = \exp(-(x-ct)^2)$ SOLITON WAVE

Example 2 – u is not constant along a characteristic

- $\dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial x} = -u$
- $\dfrac{dx}{dt} = c$
- $x = ct + x_0$

$\dfrac{du}{dt} = -u$

$u(x_0, t) = C(x_0)\,\boxed{e^{-t}}$

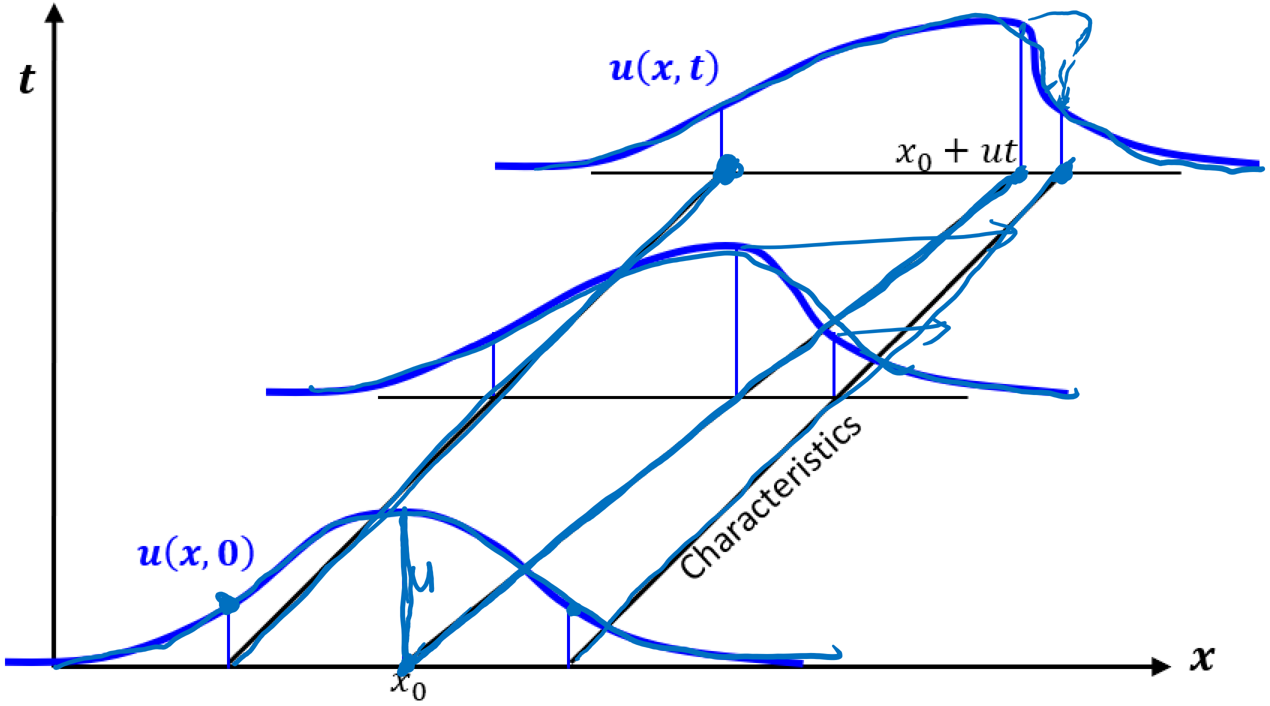


- From the initial conditions
- $u(x_0, 0) = C(x_0)\exp(0) = C(x_0) = \exp(-x_0^2)$

$u(x,t) = \exp(-x_0^2)\,e^{-t} = \underbrace{\exp\left(-(x-ct)^2\right)}_{\text{GAUSSIAN}}\underbrace{e^{-t}}_{\text{EXP DECAY}}$

Example 3 – Wave with amplitude dependent velocity

- $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$ $\quad \frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x} = \frac{du}{dt}$
- Parts of the wave travel at different speeds depending on amplitude $v = u$.
- Amplitude remains constant $\frac{du}{dt} = 0$
- $\boxed{\frac{dx}{dt} = u}$
- $x = ut + x_0$
- Giving characteristics $\boxed{x_0 = x - ut}$
- From $\frac{du}{dt} = 0$, $u(x_0, 0) = C(x_0)$
- Initial "wave" = $\exp(-x_0^2)$
- $u(x, t) = \exp(-(x - ut)^2)$



Additional Question 1

1. Using the method of characteristics, find the solution to the following first order partial differential equation:

$$\frac{\partial u}{\partial t} + xt^2 \frac{\partial u}{\partial x} = 0 \ ,$$

subject to the initial condition $u(x,0) = x^2 \sin(2x)$.

- $\frac{du}{dt} = 0$
- $u(x_0, t) = C(x_0)$
- $u(x_0, 0) = x_0^2 \sin(2x_0) = C(x_0)$
- $u(x,t) = x_0^2 \sin(2x_0)$
- $u(x,t) = x^2 \exp\left(-\frac{2}{3}t^3\right) \sin\left(2x \exp\left(-\frac{1}{3}t^3\right)\right)$

- $\frac{dx}{dt} = xt^2$
- $\int \frac{dx}{x} = \int t^2 dt$
- $\ln x = \frac{1}{3}t^3 + A$
- $t = 0, \ x = x_0$
- $A = \ln x_0$
- $\ln x_0 = \ln x - \frac{1}{3}t^3$
- $x_0 = x \exp\left(-\frac{1}{3}t^3\right)$

Additional Question 2

1.

$$\frac{\partial u}{\partial t} + (x+t)\frac{\partial u}{\partial x} = t^2$$

with initial condition $u(x,0) = x^8$.

- $\frac{du}{dt} = t^2$
- $u(x_0, t) = \frac{1}{3}t^3 + C(x_0)$
- $u(x_0, 0) = x_0^8 = C(x_0)$
- $u(x,t) = \frac{1}{3}t^3 + x_0^8$

- $u(x,t) = \frac{1}{3}t^3 + [(x+t+1)e^{-t} - 1\,]^8$

- $\frac{dx}{dt} = x + t \quad \frac{dx}{dt} - x = t$
- Multiply by I.F. = e^{-t} and integrate
- $\int \frac{d}{dt}[xe^{-t}]dx = \int te^{-t}dt$
- Integrate RHS by parts
- $u = t \quad \frac{dv}{dt} = e^{-t}$
- $\frac{du}{dt} = 1 \quad v = -e^{-t}$
- $xe^{-t} = uv - \int v\frac{du}{dt}dt + A$
- $xe^{-t} = -te^{-t} - e^{-t} + A$
- $t = 0,\ x = x_0, x_0 = A - 1$
- $xe^{-t} = -(t+1)e^{-t} + 1 + x_0$
- $x_0 = (x+t+1)e^{-t} - 1$

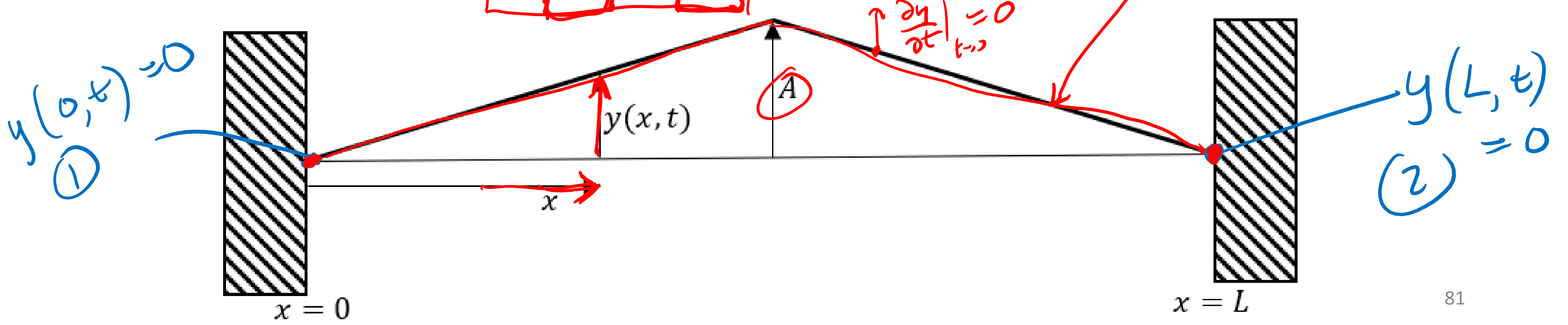
4.3 Separation of Variables

PDEs in which variables are separable

- Some linear PDEs can be solved by this process:
  1. Write solution as a **product of single variable functions**, e.g. $y(x,t) = f(x)g(t)$
  2. Substitute into PDE and **separate variables** on either side of the equation
  3. For equality to hold for all values, each side must equal a **separation constant**
  4. Solve the resulting ODEs
  5. Apply initial conditions and boundary condition find coefficients
  6. Obtain the most general possible (Fourier series solution)

IC $t=0$ ③ $\left.\frac{\partial y}{\partial t}\right|_{t=0} = 0$ ④

Example: Waves on a wire $\boxed{\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}}$ $\quad c^2 = T/\mu$



PDEs in which variables are separable

1. $y(x,t) = \underbrace{f(x)g(t)}$

2. $\frac{1}{c^2}\frac{\partial^2}{\partial t^2}[f(x)g(t)] = \frac{\partial^2}{\partial x^2}[f(x)g(t)]$

$$\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}$$

$$\frac{1}{c^2} f(x) \frac{d^2 g}{dt^2} = g(t) \frac{d^2 f}{dx^2}$$

SEPARATE x, t DIVIDE BY fg

$$\frac{1}{c^2} \frac{1}{g(t)} \frac{d^2 g(t)}{dt^2} = \frac{1}{f(x)} \frac{d^2 f(x)}{dx^2}$$

3. $\underbrace{\frac{1}{c^2}\frac{d^2 g}{dt^2}\frac{1}{g(t)}}_{t} = \underbrace{\frac{d^2 f}{dx^2}\frac{1}{f(x)}}_{x} = -k^2$ (separation constant)

c - WAVE VELOCITY

4. ODEs

$$\frac{d^2 f}{dx^2} + k^2 f = 0 \qquad \frac{d^2 g}{dt^2} + c^2 k^2 g = 0$$

$$f(x) = C_1 \sin(kx) + C_2 \cos(kx) \qquad g(t) = C_3 \sin(ckt) + C_4 \cos(ckt)$$

PDEs in which variables are separable

5. Apply Boundary Conditions $y(0,t) = y(L,t) = 0$

$f(x=0) = 0 = C_1 \sin(0) + C_2 \cos(0)$

$C_2 = 0$

$$f(L) = 0 = C_1 \underbrace{\sin(kL)}_{=0} + 0$$

$C_1 = 0$ (TRIVIAL)

$$\boxed{k = \frac{n\pi}{L}}$$

n - INTEGER

$f(x) = C_1 \sin\left(\frac{n\pi}{L} x\right)$

$\times$

$g(t) = C_3 \sin\left(c\frac{n\pi}{L} t\right) + C_4 \cos\left(c\frac{n\pi}{L} t\right)$

$y(x,t)$

$$y(x,t) = f(x)g(t) = \sin\left(\frac{n\pi}{L}x\right)\left[a_n \underbrace{\sin\left(\frac{n\pi}{L}ct\right)}_{=0} + b_n \underbrace{\cos\left(\frac{n\pi}{L}ct\right)}_{=1}\right]$$

$t=0$

$a_n = C_1 C_3$

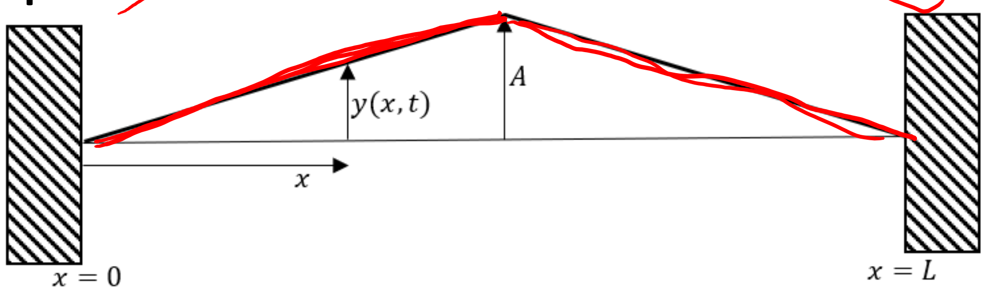
$b_n = C_1 C_4$

Most general solution – Fourier coefficients

6. Initial Conditions $y(x,0) = b_n \sin\left(\frac{n\pi}{L}x\right)$

But What if $y(x,0)$ is not sinusoidal? For example

$$y(x,0) = \frac{2Ax}{L} \text{ for } 0 < x < \frac{L}{2}$$
$$y(x,0) = 2A\left(1 - \frac{x}{L}\right) \text{ for } \frac{L}{2} < x < L$$

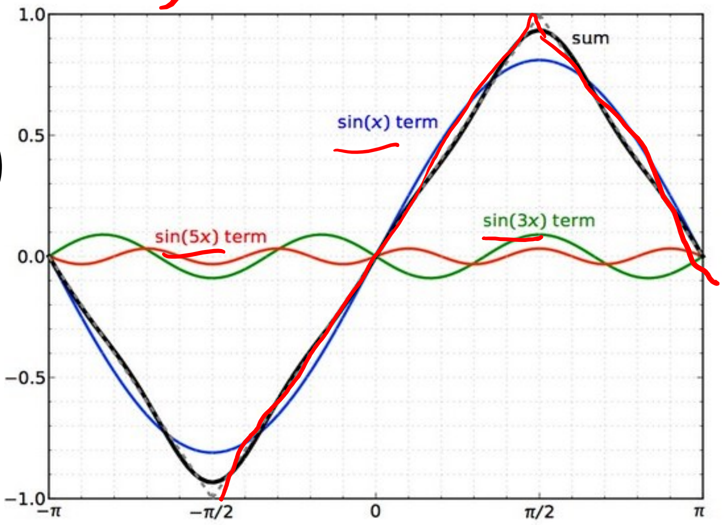


A superposition of sine/cosine wave solutions gives the most general solution.

$$y(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left(a_n \sin\left(\frac{n\pi}{L}ct\right) + b_n \cos\left(\frac{n\pi}{L}ct\right)\right)$$

This is a Fourier Series!

Coefficients found by Fourier methods(see Tom's notes)



$t = 0$ MULTIPLY BY $\sin\left(\frac{m\pi x}{L}\right)$ m - INTEGER

INTEGRATE IN x FROM $0 \rightarrow L$

Variable Separation – Heat Equation

$$\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2}$$

1. $T(x,t) = f(x)g(t)$
2. $f(x)\frac{dg}{dt} = D\, g(t)\frac{d^2 f}{dx^2}$ $\quad \div$ BY $f(x)\, g(t)$
3. $\frac{1}{g}\frac{dg}{dt} = \frac{D}{f}\frac{d^2f}{dx^2} = -k^2$
4.

$\frac{dg}{dt} = -k^2 g$ $\qquad \int \frac{dg}{g} = \int -k^2 dt$ $\qquad \frac{d^2 f}{dx^2} + \frac{k^2}{D} f = 0$

$f(x) = B\sin\left(\frac{k}{\sqrt{D}}x\right) + C\cos\left(\frac{k}{\sqrt{D}}x\right)$

$g(t) = A\exp(-k^2 t)$

$$T(x,t) = \exp(-k^2 t)\left(B' \sin\left(\frac{k}{\sqrt{D}}x\right) + C' \cos\left(\frac{k}{\sqrt{D}}x\right)\right)$$

$AB = B'$

$AC = C'$