Lecture 13:
Composition of angular momenta and spin

We have seen in the last lecture that the eigenevalue of the total angular momentum J^2 is $j(j+1)$. This corresponds to $2j+1$ possible values of the eigenvalue for the operator J_3: $|m| \leq j$. It is now important to see how the angular momenta combine with each other. The following considerations are general, and do not depend on the particular nature of the angular momentum or the particle considered. They thus apply to the sum of the total angular momenta of two particles, or to the sum of spin and orbital angular momentum of a single particle, and so on.

Calling j_1 and j_2 the quantum numbers of the two momenta to be added, we thus see that the overall momentum will have a number of eigenfunctions equal to: $(2j_1+1)(2j_2+1)$.

It is also straightforward to see that the maximum total angular momentum is j_1+j_2, whereas the minimum one is $|j_1-j_2|$.

As an example, let us consider $j_1 = 1$ and $j_2 = 1$. The total number of states of the composition of these two states is: $(2j_1+1)(2j_2+1) = 9$. For $J = 2$, there are 5 states, 3 for $J = 1$, and 1 for $J = 0$.

Let us now describe the spin of a particle. A particular useful representation of the spin operators is given in terms of Pauli's matrices:

$$s_1 = \frac{1}{2}\sigma_1 \quad , \quad s_2 = \frac{1}{2}\sigma_2 \quad , \quad s_3 = \frac{1}{2}\sigma_3 \tag{1}$$

with

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad , \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad , \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{2}$$

where it is intended that these matrices operate on a 2×2 vectorial space. Each vector is of the form:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = |1/2, 1/2\rangle \tag{3}$$

In the right hand-side of the equation, the first number denotes the angular momentum j, whereas the second denotes the projection of the momentum along z, m. In other words, this state has a total momentum $j = 1/2$, oriented along positive z. Conversely, a particle with a total momentum $j = 1/2$, oriented along negative z can be represented as:

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1/2, -1/2\rangle \tag{4}$$

The Pauli's matrices obey commutation rules that are very similar to those of the orbital angular momentum:

$$[\frac{1}{2}\sigma_i, \frac{1}{2}\sigma_j] = i\epsilon_{ijk}\sigma_k \tag{5}$$

Moreover, they obey the important relation:

$$\sigma_i^2 = 1 \quad i = x, y, z \tag{6}$$

Let us now see what happens if we combine two particles with spin 1/2: $j_1 = j_2 = 1/2$. As we have mentioned above, a particle with $j = 1/2$ has two possible states $m = -1/2$ or $m = +1/2$. These two possible states are usually represented in the following compact notation:

$$\begin{pmatrix}1\\0\end{pmatrix} = |1/2, 1/2\rangle \quad , \quad \begin{pmatrix}0\\1\end{pmatrix} = |1/2, -1/2\rangle \tag{7}$$

From the general considerations above, we expect the composition of these two states to result into $(2j_1 + 1)(2j_2 + 1) = 4$ states; three states with overall spin 1 and one with overall spin 0. We know that these states must be a linear combination of the following base:

$$\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2 , \begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2 , \begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2 , \begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2 , \tag{8}$$

We will show here the full derivation of these states [1].

The operator s_{1+} is the operator that raises the spin number by 1, analogously to L_+ for the orbital angular momentum (see Lecture 11):

$$s_{1+} = \frac{1}{2}(\sigma_{1x} + i\sigma_{1y}) = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}_1 \tag{9}$$

The operator s_{1-} is, instead:

$$s_{1-} = \frac{1}{2}(\sigma_{1x} - i\sigma_{1y}) = \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}_1 \tag{10}$$

The total spin operator can thus be written as:

$$s_{tot}^2 = \frac{3}{2} + s_{1+}s_{2-} + s_{1-}s_{2+} + 2s_{1z}s_{2z} \tag{11}$$

It is easy to find that:

$$s_{tot}^2\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2 = 2\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2 \tag{12}$$

and

$$s^2_{tot}\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2 = 2\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2 \tag{13}$$

These two states are thus eigenfunctions of the total spin, with eigenvector 1. On the other hand, the other two states of the base are not eigenfunctions:

$$s^2_{tot}\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2 = \begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2 + \begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2 \tag{14}$$

and

$$s^2_{tot}\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2 = \begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2 + \begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2 \tag{15}$$

However, the following linear combinations are eigenvalues of the total spin:

$$|par\rangle = \left[\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2 + \begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2\right]/\sqrt{2} \tag{16}$$

and

$$|anti\rangle = \left[\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2 - \begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2\right]/\sqrt{2} \tag{17}$$

since they give:

$$s^2_{tot}|par\rangle = 2|par\rangle \quad , \quad s^2_{tot}|anti\rangle = 0 \tag{18}$$

According to our general predictions, we have found three states with eigenvalue 1 and one state with eigenvalue 0. For an eigenvalue equal to 1, we have a state corresponding to a spin "up" (both spins oriented along positive z), a spin "down" (both spins oriented along negative z) and a combination of the two spin which effectively results in the spins being "parallel" (this last state has no classical equivalent). For an eigenvalue of 0, we have only a combination of spins which is equivalent of having the two spins "anti-parallel" (also this state has no classical equivalent).

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[1] This derivation goes a little beyond the scope of this introductory course on Quantum Mechanics, and it is presented here only for the interested student. For the rest, it is only important to remember the final results of this derivation