Lecture 3:
Electrostatic potential

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Have we defined the electric field yet?

- We have related the electric field to its source via the Gauss' law:

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

- Unfortunately, this is not enough, since a field can be (almost*) uniquely defined if its curl and divergence are known everywhere in space.

- We thus look for an equation of the kind: $\nabla \times \vec{E} = ?$

* *apart from the gradient of a scalar function* ψ *so that* $\nabla^2 \psi = 0$

Have we defined the electric field yet?

- Before continuing, let's just remind ourselves that we are still neglecting magnetic fields. Another way of saying it is that we only have forces depending on the *position* of the particles:

$$\vec{F} = q(\vec{E} + \cancel{\vec{v} \times \vec{B}})$$ ➔ Electrostatic assumption!

- In this case, we can express Coulomb's Law as the gradient of a scalar function (see Jackson paragraph 1.5 for the derivation):

$$\vec{F} = -q\ \nabla\Phi$$ ➔ meaning that: $$\vec{E} = -\ \nabla\Phi$$

- It is a general property that, for any scalar function:

$$\nabla \times \nabla\Phi = 0 \qquad \forall\Phi$$ ➔ $$\boxed{\nabla \times \vec{E} = 0}$$

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Electrostatic fields

- We thus have the divergence and curl of the electrostatic field:

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \qquad \nabla \times \vec{E} = 0$$

These are the Maxwell's equation for the electrostatic field!

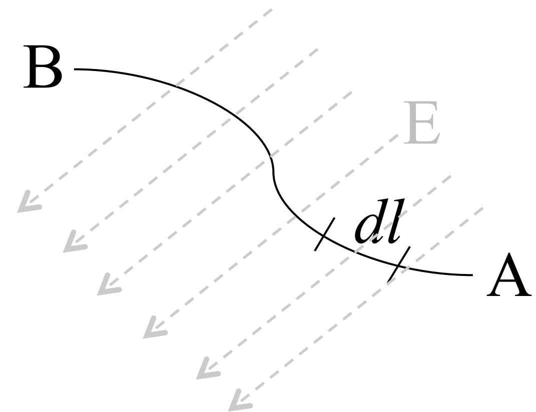
- What is the physical meaning of Φ?

- Let us consider the work (energy spent/earned) to be done while moving a charged particle in an electrostatic field....

Electrostatic potential

- The work between points A and B will be given, by definition, by:

$$W = -\int_A^B \vec{F}\cdot d\vec{l} = -q\int_A^B \vec{E}\cdot d\vec{l}$$



- But $\vec{E} = -\nabla\Phi$, so:

$$W = q\int_A^B \nabla\Phi\cdot d\vec{l} = q\int_A^B d\Phi = q\,[\Phi(B) - \Phi(A)]$$

$q\Phi$ is then the potential energy of the electrostatic field!

Electrostatic potential

- with $q\Phi$ being the potential energy, the function $\mathbf{\Phi}$ is called the electrostatic potential!

(*careful… electrostatic potential and electrostatic potential energy are two different things!*)

- From this simple argument, we notice two very important properties:

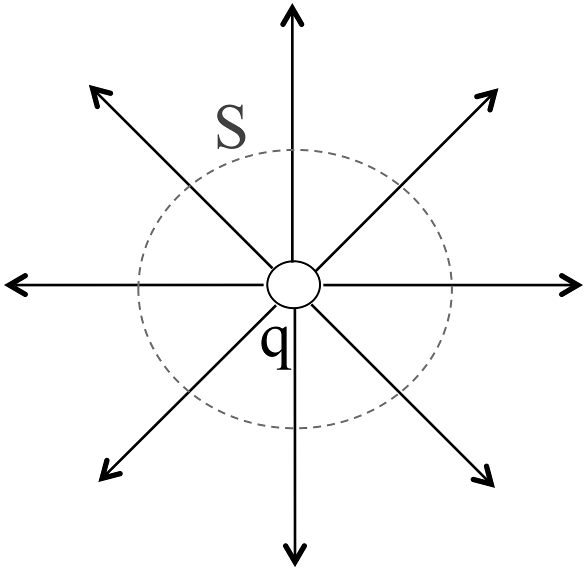
1. The work done on a particle in an electrostatic field depends only on the initial and final position, not on the path!
2. The work done to move a particle around a closed loop is always zero (particular case of point 1)

Fields that have this property are called conservative

Recap on electrostatic equations

- **Divergence of the electrostatic field:** $$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$
- **Curl of the electrostatic field:** $$\nabla \times \vec{E} = 0$$
- **Electrostatic force:** $$\vec{F} = q\vec{E}$$
- **Electrostatic potential:** $$\vec{E} = -\nabla\Phi$$

Example: point-like source



$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \quad \rightarrow \quad \oint_S \vec{E} \cdot \vec{n} \; da = q / \varepsilon_0$$

This formula does not depend on the specific choice of S, so we might as well use the simplest one: the surface of sphere!

$$E \; 4\pi r^2 = q / \varepsilon_0 \quad \rightarrow \quad E = \frac{q}{4\pi\varepsilon_0 r^2}$$

(Exercise: prove that $\nabla \times \vec{E} = 0$)

The electrostatic potential is then: $\boxed{\Phi(r) = -\frac{q}{4\pi\varepsilon_0 r}}$