Problem Sheet 1

Please Write your solutions clearly and try to simplify all final answers.

1. Three vectors in a flat two dimensional plane $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are

$$\mathbf{u} = \begin{pmatrix} -2 \\ -1 \end{pmatrix} \qquad \mathbf{v} = \begin{pmatrix} 4 \\ -3 \end{pmatrix} \qquad \mathbf{w} = \begin{pmatrix} -3 \\ 6 \end{pmatrix}$$

   (a) Are any of these vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ perpedicular to each other? Determine if any pairs of vectors taken from these three vectors are perpendicular to each other [10]

   (b) Determine the distance between each pair of vectors taken from the same set of three vectors, $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$, and hence find which two of vectors are closest to each other. [10]

   Hint you should find that there are three pairs of vectors to compare and you may find it useful to calculate the norm of the difference between pairs of vectors; *e.g.* $||\mathbf{u} - \mathbf{v}||$.

2. Two vectors in a flat two dimensional plane, $\mathbf{a}$ and $\mathbf{b}$, are given by

$$\mathbf{a} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

   (a) Use Gramm Schmidt orthogonalization to construct a vector $\mathbf{b}'$ based on $\mathbf{b}$, which is perpendicular to $\mathbf{a}$. [10]

   (b) Express the following vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in terms of the vectors $\mathbf{a}$ and $\mathbf{b}'$ where

$$\mathbf{u} = \begin{pmatrix} -2 \\ -1 \end{pmatrix} \qquad \mathbf{v} = \begin{pmatrix} 4 \\ -3 \end{pmatrix} \qquad \mathbf{w} = \begin{pmatrix} -3 \\ 6 \end{pmatrix}$$

   [15]

   Hint you should express your answers in the form $\mathbf{u} = \alpha\mathbf{a} + \beta\mathbf{b}'$ where α and β are scalar quantities.

3. A line in a flat two dimensional plane passes through the origin and has a direction defined by the vector $\mathbf{a} = \left(\begin{smallmatrix} -3 \\ 1 \end{smallmatrix}\right)$.

   (a) Find vectors in this line which are closest to the points $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ where

$$\mathbf{u} = \begin{pmatrix} -2 \\ -1 \end{pmatrix} \qquad \mathbf{v} = \begin{pmatrix} 4 \\ -3 \end{pmatrix} \qquad \mathbf{w} = \begin{pmatrix} -3 \\ 6 \end{pmatrix}$$

   [15]

(b) Which vector, $\mathbf{u}$, $\mathbf{v}$ or $\mathbf{w}$, is closest to the line through the origin defined by $\mathbf{a}$? [10]

4. Show that the set P_3 of polynomials with degree 3 or less satisfies the vector space axioms for closure under addition and scalar multiplication. *Hint, consider general vectors/scalars in this vector space.* [10]

5. Normally in $\mathbb{R}^2$ scalar multiplication is defined as [10]

$$k\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} ka_1 \\ ka_2 \end{pmatrix}$$

If we redefine scalar multiplication to be

$$k\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ ka_1 \end{pmatrix}$$

would $\mathbb{R}^2$ still be a vector space with this redefined scalar multiplication? *Hint, consider vector space axiom 10.*

6. Is the set of vectors V a vector subspace of $\mathbb{R}^2$ if we define a general vector, $\mathbf{a}$, in V with [10]

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \qquad \text{where} \quad a_1 \geq 0 \qquad \text{and} \quad a_2 \geq 0$$

7. Challenge Problem *optional.*

Identify the vector space $\mathbb{R}^n$ which is isomorphic to the set of polynomials P_5 and write down an isomorphic vector to the general polynomial, $p(x)$ in P_5

$$p(x) = p_0 + p_1 x + p_2 x^2 + p_3 x^3 + p_4 x^4 + p_5 x^5$$

8. Challenge Problem *optional.*

Determine if the following transformations are linear transformations:

(a)

$$T : \mathbb{R}^3 \to \mathbb{R}^2 \qquad \text{where} \quad T\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_1 a_2 \\ a_1 a_3 \end{pmatrix}$$

(b)

$$T : P_3 \to M_{22} \qquad \text{where} \quad T\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3\right) = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_1 - a_0 \end{pmatrix}$$

Problem Sheet 2

Please Write your solutions clearly and try to simplify all final answers.

1. A vector space, $F[0,1]$, is the set of all functions that give real values for $0 \leq x \leq 1$. The inner product is defined for two general vectors $\mathbf{a} = a(x)$ and $\mathbf{b} = b(x)$ in $F[0,1]$ by

$$\langle \mathbf{a}, \mathbf{b} \rangle = \int_0^1 a(x)b(x)dx$$

Three functions are defined by $\mathbf{f} = f(x) = x$, $\mathbf{g} = g(x) = x^2 - 1$ and $\mathbf{h} = h(x) = x^3$.

   (a) Determine the distances [16]

      i. $||\mathbf{f} - \mathbf{g}||$

      ii. $||\mathbf{f} - \mathbf{h}||$

   (b) Based on the distances you have calculated comment on whether $f(x)$ is a better approximation for $g(x)$ or $h(x)$. [4]

2. The following vectors span W a vector subspace of $\mathbb{R}^4$

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \qquad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix} \qquad \mathbf{v}_3 = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 2 \end{pmatrix}$$

   (a) How would you describe the vector subspace W defined by these three vectors? [4]

   (b) Given that the inner product for $\mathbb{R}^4$ and the vector subspace W is deined by the normal dot product, use the Gram-Schmidt orthogonalization procedure to derive the following set of orthogonal vectors from $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ [18]

$$\mathbf{p}_1 = \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \qquad \mathbf{p}_2 = \begin{pmatrix} 3 \\ 3 \\ 1 \\ 1 \end{pmatrix} \qquad \mathbf{p}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

   (c) Find the vectors in the subspace W which are closest to the following vectors in $\mathbb{R}^4$; [12]

$$\text{(i)} \quad \mathbf{q}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \qquad \text{(ii)} \quad \mathbf{q}_2 = \begin{pmatrix} 6 \\ 2 \\ -2 \\ -2 \end{pmatrix} \qquad \text{(iii)} \quad \mathbf{q}_3 = \begin{pmatrix} 2 \\ -3 \\ 4 \\ -1 \end{pmatrix}$$

(d) Find the distance between each of the vectors $\mathbf{q}_1$, $\mathbf{q}_2$ and $\mathbf{q}_3$ and the closest points to them in the subspace W. [6]

3. Consider P_2, a vector space of polynomials of degree up to 2 where the inner product is defined as in Question 1; $\langle \mathbf{f}, \mathbf{g} \rangle = \int_0^1 f(x)g(x)dx$. The following polynomial vectors form an orthogonal basis set for P_2;

$$\mathbf{p}_1 = 1 \qquad \mathbf{p}_2 = x - \frac{1}{2} \qquad \mathbf{p}_3 = x^2 - x + \frac{1}{6}$$

(a) Find the inner products $\langle \mathbf{p}_1, \mathbf{p}_2 \rangle$, $\langle \mathbf{p}_1, \mathbf{p}_3 \rangle$ and $\langle \mathbf{p}_2, \mathbf{p}_3 \rangle$ to confirm that these functions are orthogonal to each other. *Hint: what is the value of the inner product of orthogonal vectors?* [20]

(b) A general polynomial vector in P_2 is given by

$$\mathbf{a} = a(x) = a_0 + a_1 x + a_2 x^2$$

project this general polynomial onto the orthogonal basis set $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$. [20]
Hint: your answer should be that $\mathbf{a}$ *is equal to a linear sum of the basis vecotrs; e.g.* $\mathbf{a} = a(x) = \lambda_0 \mathbf{p}_0 + \lambda_1 \mathbf{p}_1 + \lambda_2 \mathbf{p}_2$ *where the* λ_i *terms are scalars. Also note that this question is much easier to solve by using the isomorphism with* $\mathbb{R}^3$ *as explained below;*
Isomorphism with $\mathbb{R}^3$

$$a_0 + a_1 x + a_2 x^2 \quad \begin{array}{c} \xrightarrow{T} \\ \xleftarrow[T^{-1}]{} \end{array} \quad \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$$

Transforming the question we get

$$\mathbf{a} = a(x) = a_0 + a_1 x + a_2 x^2 = \lambda_0 \mathbf{p}_0 + \lambda_1 \mathbf{p}_1 + \lambda_2 \mathbf{p}_2$$

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \lambda_0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} -1/2 \\ 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1/6 \\ -1 \\ 1 \end{pmatrix}$$

which can be reorganized to give a familiar type of equation;

$$\begin{pmatrix} 1 & -1/2 & 1/6 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$$

Note the λ_i values you calculate in the isomorphic vector space $\mathbb{R}^3$ will be the same as the λ_i values you require in the polynomial vector space P_2.
Finally note the λ_i values you calculate will be equations in terms of a_i values.

4. Challenge Question *optional*

   (a) Find the best linear fit to the function $f(x) = \sqrt{x}$ in the range $0 \leq x \leq 1$ using the orthogonal basis $\{\mathbf{p}_1,\ \mathbf{p}_2\} = \{1, x - \frac{1}{2}\}$. Use the inner product as defined in Question 1; $\langle \mathbf{f}, \mathbf{g} \rangle = \int_0^1 f(x)g(x)dx$

   Answer $\frac{4}{5}x + \frac{4}{15}$

   (b) Improve your fit to the function $f(x) = \sqrt{x}$ in the range $0 \leq x \leq 1$ by adding a quadratic term with the basis function $\mathbf{p}_3 = x^2 - x + \frac{1}{6}$. Note this is quite long and is *optional*! If you get stuck move on to the Gram-Schmidt orthogonalization in the next part and come back to this question.

   Answer: $\frac{-4}{7}x^2 + \frac{48}{35}x + \frac{6}{35}$

   (c) Use the Gram-Schmidt orthogonalization procedure to derive the orthogonal set of basis functions $\{\mathbf{p}_1,\ \mathbf{p}_2, \mathbf{p}_3\}$ used above for the polynomial vector space P_2 in the range $0 \leq x \leq 1$. Build up the basis from the functions $\{\mathbf{v}_1,\ \mathbf{v}_2, \mathbf{v}_3\}$;

$$\mathbf{v}_1 = 1 \qquad\qquad \mathbf{v}_2 = x \qquad\qquad \mathbf{v}_3 = x^2$$

   Here you should start from $\mathbf{p}_1 = \mathbf{v}_1$ and then find $\mathbf{p}_2$ from $\mathbf{v}_2$.

Problem Sheet 3

Please Write your solutions clearly and try to simplify all final answers.

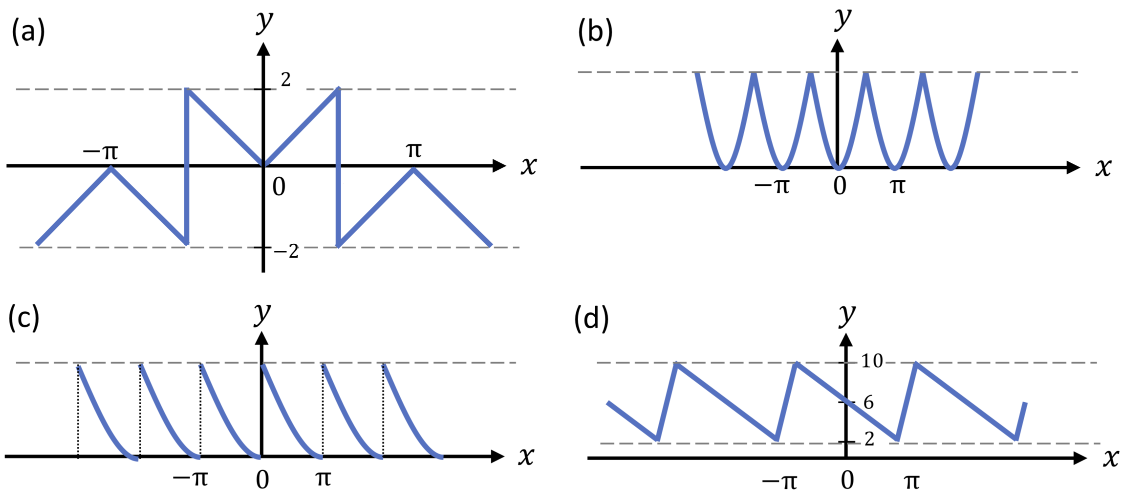

Figure 1: Periodic functions for question 1

1. Consider each of the functions shown in Figure 1. Determine by inspection whether the following coefficients of the Fourier series corresponding to each function are zero or non-zero. [24]

   - a_0, the constant term.
   - the a_k terms where $k \geq 1$, the cosine term coefficients.
   - the b_k terms where $k \geq 1$, the sine term coefficients.

   Hint; you do not need to consider each of the a_i and b_i terms individually. You need to indicate if all the a_i terms, for example, must be zero or if some of them must be non-zero by inspection of each function.

2. The function, $f(x)$, shown in Figure 2 is defined by;

$$
\begin{aligned}
f(x) &= 1 && \text{for} \quad -\pi < x < -\frac{\pi}{2} \\
f(x) &= 0 && \text{for} \quad -\frac{\pi}{2} < x < \frac{\pi}{2} \\
f(x) &= -1 && \text{for} \quad \frac{\pi}{2} < x < \pi
\end{aligned}
$$

   (a) Show that the Fourier series corresponding to this function is given by [20]

$$f(x) = \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \left((-1)^k - cos\frac{k\pi}{2} \right) \sin(kx)$$

   Hint; by inspection you should be able to establish that some of the a_k and b_k terms are equal to zero so that you do not need to explicitly calculate them.

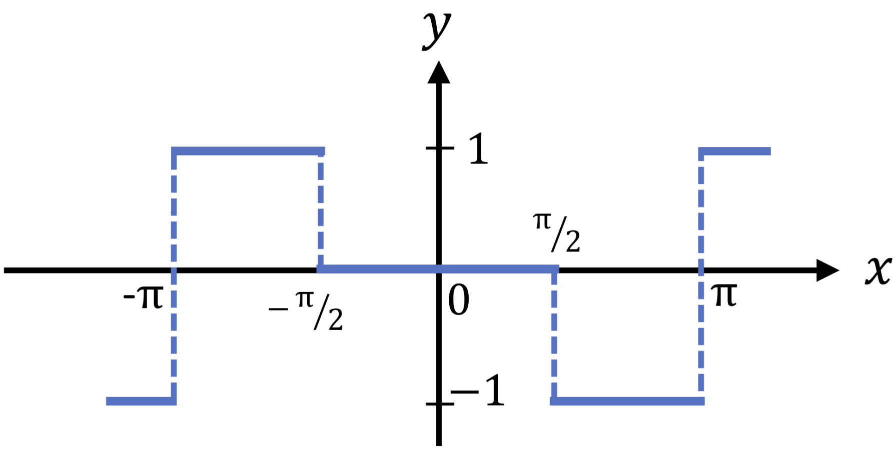

Figure 2: Function for question 2

(b) Write down $f(x)$ explicitly up to and including the $\sin 4x$ term. [6]

(c) What value is the Fourier series expected to converge to at for the following values of x; [4]

i. $x = -\frac{\pi}{2}$

ii. $x = \frac{\pi}{2}$

iii. $x = \pi$

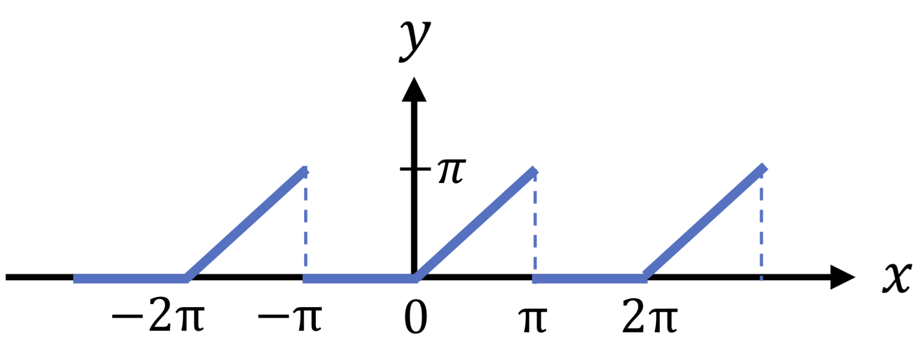

Figure 3: Function for question 3

3. Figure 3 shows a periodic function $f(x)$ that is defined by

$$f(x) = 0 \qquad \text{for} \qquad -\pi < x < 0$$
$$f(x) = x \qquad \text{for} \qquad 0 < x < \pi$$

(a) Show that the coefficients of the Fourier series for $f(x)$ are [40]

$$a_0 = \frac{\pi}{2} \qquad a_k = \begin{cases} -\frac{2}{k^2\pi} & : \quad (\text{for } k \text{ is odd}) \\ 0 & : \quad (k \text{ is even and } k \neq 0) \end{cases}$$

$$b_k = -\frac{1}{k}(-1)^k$$

(b) Write out the Fourier series explicitly for terms up to and including $k = 3$. [6]

4. Challenge Question *optional*

A periodic function is defined by

$$f(x) = x^2 \qquad \text{for} \qquad -L \leq x \leq L$$
$$f(x + 2L) = f(x)$$

(a) Sketch the shape of the function over the interval $-3L \leq x \leq 3L$.

(b) show that the Fourier series of this function is given by

$$f(x) = \frac{L^2}{3} + \frac{4L^2}{\pi^2} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \cos\left(\frac{k\pi x}{L}\right)$$

(c) If we differentiate the function $f(x)$ and divide by 2 we obtain a new function $g(x)$

$$g(x) = \frac{f'(x)}{2} = x \qquad \text{for} \qquad -L \leq x \leq L$$
$$g(x + 2L) = g(x)$$

Sketch the shape of the function over the interval $-3L \leq x \leq 3L$.

(d) Differentiate the Fourier series for $f(x)$ and divide by 2 to show that the Fourier series for $g(x)$ is given by

$$g(x) = -\frac{2L}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \sin\left(\frac{k\pi x}{L}\right)$$

(e) Write out the first four terms of the Fourier series for $g(x)$ explicitly, the terms where $k = 1, 2, 3, 4$.

Problem Sheet 4

Please Write your solutions clearly and try to simplify all final answers.

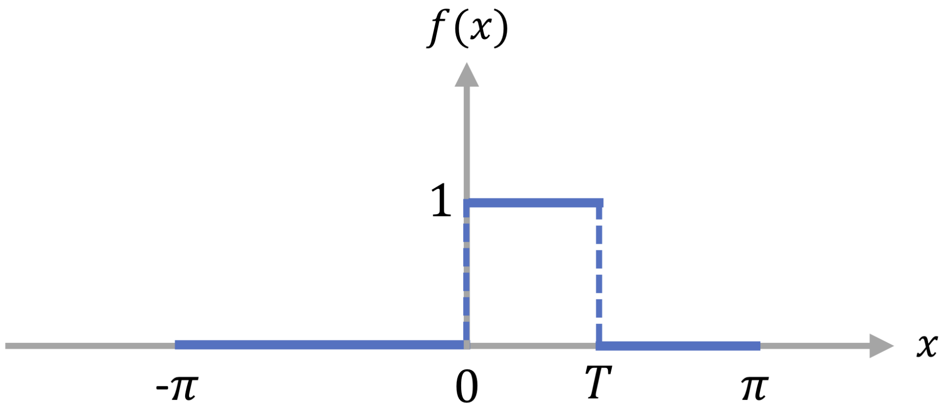

Figure 4: Regular pulse of width T where $T \leq \pi$ for Question 1

1. A function, $f(x)$, with a regular pulse, or pulsetrain, is shown in Figure 4. $f(x)$ is defined by;

$$\begin{aligned} f(x) &= 0 \quad &\text{for} \quad & -\pi < x < 0 \\ f(x) &= 1 \quad &\text{for} \quad & 0 < x < T \\ f(x) &= 0 \quad &\text{for} \quad & T < x < \pi \qquad [T \leq \pi] \end{aligned}$$

   (a) Calculate the complex Fourier series for $f(x)$ and show that [20]

$$f(x) = \frac{T}{2\pi} + \frac{i}{2\pi} \sum_{k=-\infty, k\neq 0}^{\infty} \frac{1}{k}(e^{-ikT} - 1)e^{ikx}$$

   (b) write out the terms of the complex Fourier series from $k = -3$ to $k = 3$. [5]

   (c) Use your c_k values to calculate a_k and b_k values for the real Fourier series for $f(x)$. Hence show that the real Fourier series can be written as [20]

$$f(x) = \frac{T}{2\pi} + \sum_{k=1}^{\infty} \left(\frac{\sin(kT)}{k\pi} \cos(kx) + \frac{1 - cos(kT)}{k\pi} sin(kx) \right)$$

   (d) Consider the case where $T = \pi$.

      i. calculate the a_k values for this case [5]

      ii. with the aid of a quick sketch explain why the a_k values have the values you calculated [5]

2. Use the results of Parseval's theorem to calculate the time-averaged power, P, in a purely resistive circuit when a time varying voltage $V(t)$ is applied to a resistance, $R = 5\Omega$. The driving voltage $V(t)$ is well represented by the following Fourier series; [15]

$$V(t) = 8.6 + 5.4\cos(\omega t) + 1.8\cos(2\omega t) - 0.3\cos(3\omega t)$$
$$+ 2.3\sin(\omega t) - 0.7\sin(2\omega t) + 0.2\sin(3\omega t)$$

Note that power can be calculated with V^2/R, but Parseval's theorem should be used to calculate the time-averaged power.

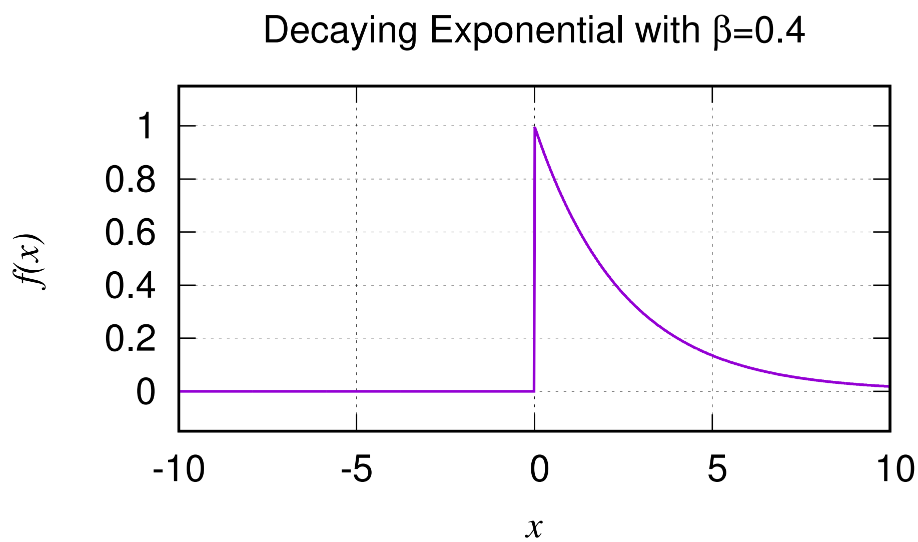

Figure 5: The decaying exponential function for Question 3

3. A decaying exponential function $f(x)$ is shown in Figure 5 and defined by

$$f(x) = 0 \qquad \text{for} \quad x < 0$$
$$f(x) = e^{-\beta x} \qquad \text{for} \quad x > 0 \qquad \beta \in \mathbb{R} \qquad \beta > 0$$

Show that the Fourier transform, $g(k)$, of this function, $f(x)$ is given by the equation [30]

$$g(k) = \frac{1}{2\pi}\left(\frac{1}{\beta + ik}\right)$$

4. Challenge Question *optional*

For this question you need to do a bit of research in books/ on the web. *You should not write more than a paragraph for each answer.*

(a) What is the 'Discrete Fourier Transform' and how is it different from the 'Fourier Transform'.

(b) What is the 'Fast Fourier Transform'.

(c) If we perform a discrete Fourier transform on a set of real numbered data do we need to store all the c_k data to be able to perform the inverse Fourier transform and regenerate the orginal set of data?

(d) If we have 10,001 sampled data points sampled at regular time intervals how many frequencies would you expect in the Fourier transform of the data?