PHY2003 ASTROPHYSICS I

Lecture 7. Orbital mechanics

Keplers First Law

The orbit of each planet is an ellipse with the Sun at one focus.

a =semi-major axis, b =semi-minor axis, e =eccentricity

$$b^2 = a^2(1 - e^2)$$

In polar coordinates, r =radius vector, θ =radius angle.

$$r = \frac{a(1 - e^2)}{1 + e\cos\theta}$$

If one integrates around an orbit, the average distance of a planet from the Sun is $\bar{r} = a$.

At perihelion, $\theta = 0$, so

$$r = \frac{a(1 - e^2)}{(1 + e)} = \frac{a(1 + e)(1 - e)}{(1 + e)} = a(1 - e)$$

At aphelion, $\theta = \pi$ so $r = a(1 + e)$.

Example: The eccentricity of the Earth's orbit is 0.017. Calculate the relative change in Solar flux at the Earth between perihelion and aphelion

Keplers Second Law

The radius vector to a planet sweeps out equal areas in equal times.

$$\frac{dA}{dt} = constant$$

Note: This means that orbiting bodies move faster at perihelion and slower at aphelion.

Newton's derivation of K2:

A body is in orbit at position r with total velocity v and a tangential velocity component v_t.

During a small amount of time Δt the radius vector sweeps out an angle:

$$\delta\theta \simeq v_t \Delta t / r$$

The area swept out by the radius vector is:

$$\Delta A \simeq r v_t \Delta t / 2$$

As $\Delta t \to 0$,

$$dA/dt = rv_t/2 = r^2(d\theta/dt)/2$$

But angular momentum per unit mass $H = r^2\omega = r^2(d\theta/dt)$.

Therefore K2 is due to the conservation of angular momentum:

$$\boxed{\frac{dA}{dt} = \frac{H}{2} = \text{constant}}$$

Keplers Third Law

The squares of the orbital periods are proportional to the cubes of the mean distance from the Sun.

$$P^2 \propto a^3$$

If the period (P) is measured in years, the resulting semimajor axis (a) calculated will be in astronomical units (au). If the period (P) is measured in seconds, then the semimajor axis fo the orbit will be in meters.

Newton's derivation of K3

Consider two bodies of mass m_1 and m_2 in orbit about the center of mass at distances r_1 and r_2. They will both orbit with the same period P.

Therefore the orbital velocity for each body is given by $v = 2\pi r/P$

The centripetal force on each body is given by

$$F_1 = m_1 v_1^2/r_1 = 4\pi^2 m_1 r_1/P^2$$

$$F_2 = m_2 v_2^2/r_2 = 4\pi^2 m_2 r_2/P^2$$

The forces must balance, $F_1 = F_2$, therefore

$$r_1/r_2 = m_2/m_1$$

This then defines the position of the center of mass.

The distance between the two bodies is given by

$$\begin{aligned}
a &= r_1 + r_2 \\
a &= r_1(1 + r_2/r_1) \\
a &= r_1(1 + m_1/m_2) \\
a &= \frac{r_1}{m_2}(m_1 + m_2)
\end{aligned}$$

The centripetal forces are individually balanced by the gravitational force.

$$F_1 = F_2 = F_{grav}$$

$$F_1 = 4\pi^2 m_1 r_1/P^2 = \frac{Gm_1 m_2}{a^2}$$

$$\begin{aligned}
P^2 &= 4\pi^2 m_1 r_1 \frac{a^2}{Gm_1 m_2} \\
P^2 &= 4\pi^2 a^2 \frac{r_1}{Gm_2}
\end{aligned}$$

Substitute for r_1:

$$\boxed{P^2 = \frac{4\pi^2 a^3}{G(m_1 + m_2)}}$$

Note: In the equation above semimajor axis (a) is in meters and orbital period (P) is in seconds.

For Solar orbits, $m << M_\odot$, so

$$P^2 \simeq \frac{4\pi^2 a^3}{GM_\odot}$$

Note: In the equation above semimajor axis (a) is in meters and orbital period (P) is in seconds.

Example: The Galilean natural satellite Io orbits Jupiter at a distance of 4.2×10^5km with an orbital period of 1.77 days. What is the mass of Jupiter?

Orbital and Escape Velocities

For a circular orbit, circumference is $2\pi a$. So the relative orbital velocity is given by

$$v = \frac{2\pi a}{P} = 2\pi a \sqrt{\frac{G(m_1 + m_2)}{4\pi^2 a^3}}$$

$$\boxed{v = \sqrt{\frac{G(m_1 + m_2)}{a}}}$$

For anything in orbit about the Sun:

$$v \simeq \sqrt{\frac{GM_\odot}{a}}$$

The escape velocity of an object is given by equating the gravitational and kinetic energies, giving $v = 0$ at $d = \infty$.

$$\frac{1}{2} m v_{esc}^2 = \frac{GM_\odot m}{d}$$

$$\boxed{v_{esc} = \sqrt{\frac{2GM_\odot}{d}}}$$

For an object in a circular orbit, $d = a$ and

$$v_{esc} = \sqrt{2} v_{orb}$$

Example: What is the Earth's average orbital velocity, and how fast must a spacecraft be travelling at Earth's orbit to escape from the Solar system?