Lecture 10:
The harmonic oscillator, part2

It is instructive to further analyse the wavefunctions and related energy levels of the harmonic oscillator that we have obtained in the previous lecture. If we recall the definition of the spatial variable $\xi = x\sqrt{m\omega/\hbar}$ (see Eq. 3 in Lecture 9), the normalised wavefunction associated with the n^{th} energy level is:

$$\psi_n(x) = C_n H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)\exp\left(-\frac{m\omega}{2\hbar}x^2\right) \tag{1}$$

where C_n is the normalisation constant:

$$C_n = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(\frac{1}{2^n n!}\right)^{1/2} \tag{2}$$

The ground state ($n = 0$) is described by the Gaussian wavefunction and related energy:

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x^2\right) \quad , \quad E_0 = \frac{1}{2}\hbar\omega \tag{3}$$

This energy is known as *vacuum energy* or, equivalently, as *the minimum quantum fluctuation of vacuum.* It is interesting to note that this quantity has no classical counterpart and it is compatible with the Heisenberg uncertainty principle. In order to clearly see this relation, let us assume that the ground state is the one that allows for the minimum product between the uncertainties in position Δx, and momentum Δp:

$$\Delta x\Delta p \simeq \frac{\hbar}{2} \tag{4}$$

Without loss of generality, we can assume that the mean value of position and momentum are zero $\langle x\rangle = \langle p\rangle = 0$. The uncertainties in x and p are, respectively: $\Delta x = \sqrt{\langle x^2\rangle}$ and $\Delta p = \sqrt{\langle p^2\rangle}$. We recall that the one-dimensional Hamiltonian of a harmonic oscillator is:

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2 \tag{5}$$

Hence, its expectation value is:

$$\langle H\rangle = \frac{(\Delta p)^2}{2m} + \frac{m\omega^2}{2}(\Delta x)^2 \quad \rightarrow \quad \langle H\rangle = \frac{(\Delta p)^2}{2m} + \frac{m\omega^2\hbar^2}{8(\Delta p)^2} \tag{6}$$

where, in the last equation, we have used: $\Delta x \simeq \hbar\Delta p/2$ (from the uncertainty principle, Eq. 4). We are interested in the ground state, which is the state with minimum energy. We thus have to minimise the expectation value of the Hamiltonian. If we do so, in terms of $(\Delta p)^2$, we obtain:

$$\frac{1}{2m} - \frac{m\omega^2\hbar^2}{8(\Delta p)^4} = 0 \quad \rightarrow \quad (\Delta p)^2 = \frac{1}{2}m\omega\hbar \tag{7}$$

If we insert this value into Eq. 6, we re-obtain the energy of the ground state, which is:

$$E_0 = \frac{1}{2}\hbar\omega \tag{8}$$

The spectrum for the harmonic oscillator is thus discrete (as you might expect, being the system confined) and it is formally equivalent to the energy associated with n non-interacting particles. This equivalence is of fundamental importance, since it is at the core of the second quantisation formalism that allows for the modelling of harmonically oscillating systems (such as electromagnetic waves) in term of creation/annihilation operators of fundamental particles (in the case of an electromagnetic wave, the photon). A peculiar consequence of this formalism is that electromagnetic waves can be described by it only if no photon/photon interaction exists. This is not entirely correct, since modern measurements suggest that photon/photon interactions exist (and are explained by quantum electro-dynamics) and require slight modifications to the quantum theory of electromagnetic waves.
Within this context, the creation ($\hat{a}$) and annihilation ($\hat{a}^\dagger$) operators are defined, respectively, as:

$$\begin{cases} \hat{a} = \frac{1}{\sqrt{2}}\left(-\frac{\mathrm{d}}{\mathrm{d}\,\xi} + \xi\right) \\ \hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(\frac{\mathrm{d}}{\mathrm{d}\,\xi} + \xi\right) \end{cases} \tag{9}$$

(recall the definition of ξ given in Eq. 3 of Lecture 9). It is easy to see that the commutator of these two operators is $[\hat{a}, \hat{a}^\dagger] = 1$. With these two operators, the Hamiltonian of a harmonic oscillator reads:

$$H = \left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)\hbar\omega \tag{10}$$

Direct comparison of this formulation with the energy levels defined in Eq. 13 of Lecture 9, elucidates the role of these operators as the "effective number" of non-interacting particles (n) associated with the harmonic oscillator (in the case of an electromagnetic wave, the photons).
It is interesting to note how these operators work on the wavefunctions of the harmonic oscillator. It is straightforward to demonstrate that:

$$a\psi_n = \sqrt{n}\psi_{n-1} \tag{11}$$

In other words, the operator a "decreases" the level of the wavefunction. If we look at n as the effective number of particles (for example photons for an electromagnetic fields or phonons for vibrational modes) associated with the oscillation, the operator a decreases the number of those by one. This is why a is usually called the annihilation operator. Conversely, the operator $a^\dagger$ operates in the opposite way (i.e. increases the number of particles by 1):

$$a^\dagger\psi_n = \sqrt{n+1}\psi_{n+1} \tag{12}$$

and it is usually called the creation operator.