Use lined, single-sided A4 paper
with a black or blue pen.
Write your student number
at the top of every page.

Any non-graphical calculator, except those with pre-programmable memory, may be used in this examination

LEVEL 2
Examination contributing to the Degrees of Bachelor of Science (BSc) and Master in Science (MSci)

PHY2004 - EXAM
Electricity, Magnetism and Optics
Saturday, 15th August 2020, 09.30 - 13.30

Examiners: Prof S Matthews, Dr P van der Burgt
and the Internal Examiners
Dr J Greenwood (j.greenwood@qub.ac.uk)

Answer ALL TEN questions in Section A for 4 marks each.
Answer TWO questions in Section B for 20 marks each.
Answer ONE question in Section C for 20 marks.
You have FOUR hours to complete and upload this paper.

Contact the module coordinator if you have queries/problems at g.sarri@qub.ac.uk and copy to mpts@qub.ac.uk

By submitting the work, you are declaring that:

1. The submission is your own original work and no part of it has been submitted for any other assignments;
2. You understand that collusion and plagiarism in an exam are major academic offences, for which a range of penalties may be imposed, as outlined in the Procedures for Dealing with Academic Offences.

THE QUEEN'S UNIVERSITY OF BELFAST
DEPARTMENT OF PHYSICS AND ASTRONOMY

PHYSICAL CONSTANTS

Speed of light in a vacuum	$c = 3.00\times10^{8}\ \mathrm{ms^{-1}}$
Permeability of a vacuum	$\mu_0 = 4\pi\times10^{-7}\ \mathrm{Hm^{-1}}$ $\approx 1.26\times10^{-6}\ \mathrm{Hm^{-1}}$
Permittivity of a vacuum	$\varepsilon_0 = 8.85\times10^{-12}\ \mathrm{Fm^{-1}}$
Elementary charge	$e = 1.60\times10^{-19}\ \mathrm{C}$
Electron charge	$= -1.60\times10^{-19}\ \mathrm{C}$
Planck Constant	$h = 6.63\times10^{-34}\ \mathrm{Js}$
Reduced Planck Constant	$\hbar = 1.05\times10^{-34}\ \mathrm{Js}$
Rydberg Constant for hydrogen	$R_\infty = 1.097\times10^{7}\ \mathrm{m^{-1}}$
Unified atomic mass unit	$1u = 1.66\times10^{-27}\ \mathrm{kg}$ $1u = 931\ \mathrm{MeV}$
1 electron volt (eV)	$= 1.60\times10^{-19}\ \mathrm{J}$
Mass of electron	$m_e = 9.11\times10^{-31}\ \mathrm{kg}$
Mass of proton	$m_p = 1.67\times10^{-27}\ \mathrm{kg}$
Mass of neutron	$m_n = 1.67\times10^{-27}\ \mathrm{kg}$
Molar gas constant	$R = 8.31\ \mathrm{JK^{-1}mol^{-1}}$
Boltzmann constant	$k = 1.38\times10^{-23}\ \mathrm{JK^{-1}}$
Avogadro constant	$N_A = 6.02\times10^{23}\ \mathrm{mol^{-1}}$
Gravitational constant	$G = 6.67\times10^{-11}\ \mathrm{Nm^{2}kg^{-2}}$
Acceleration of free fall on the Earth's surface	$g = 9.81\ \mathrm{ms^{-2}}$

MATHEMATICAL IDENTITIES

In answering the questions on this paper you may make use of any of the following:

Divergence theorem

$$\int_V \nabla \cdot \underline{E}\, dV = \oint_S \underline{E} \cdot d\underline{S}$$

Stoke's Theorem

$$\int_S \nabla \times \underline{E} \cdot d\underline{S} = \oint \underline{E} \cdot d\underline{\ell}$$

Identities

$$\nabla \times \nabla \left(Scalar \right) = \underline{0}$$

$$\nabla \cdot \nabla \times \left(Vector \right) = 0$$

$$\nabla \times \left(\nabla \times \underline{E} \right) = \nabla \left(\nabla \cdot \underline{E} \right) - \nabla^2 \underline{E}$$

$$\nabla \left(\psi\, \underline{E} \right) = \psi\, \nabla \cdot \underline{E} + \underline{E} \cdot \nabla \psi$$

$$\nabla \cdot \left(\underline{E} \times \underline{H} \right) = \underline{H} \cdot \left(\nabla \times \underline{E} \right) - \underline{E} \cdot \left(\nabla \times \underline{H} \right)$$

Material Equations

$$\underline{J} = \sigma \underline{E} \qquad \underline{B} = \mu \underline{H} \qquad \underline{D} = \varepsilon \underline{E}$$

Poynting Vector

$$\underline{S} = \underline{E} \times \underline{H}$$

Trigonometric identities

$$cos(\, A \pm B\,) = cos\, A\, cos\, B \mp sin\, A\, sin\, B$$

$$sin(\, A \pm B\,) = sin\, A\, cos\, B \pm sin\, B\, cos\, A$$

$$cos\, A + cos\, B = 2\, cos\left(\frac{A+B}{2} \right) cos\left(\frac{A-B}{2} \right)$$

$$sin\, A - sin\, B = 2\, cos\left(\frac{A+B}{2} \right) sin\left(\frac{A-B}{2} \right)$$

SECTION A

Use a section A answer book

Answer ALL 10 questions in this section

Full explanations of your answers are required to attain full marks

1. Using Gauss' law, derive the expression for the electrostatic field generated by a uniformly charged sphere (volumetric charge density ρ, dielectric constant ε_0) of radius a, outside ($r > a$) and inside ($r < a$) the sphere. Show that the field is continuous at $r = a$. **[4]**

2. Consider an infinitely long cylinder of radius a, relative permeability μ_r, and carrying a steady current I, surrounded by vacuum. Calculate the magnetization $\vec{M}$ outside ($r > a$) and inside ($r < a$) the cylinder. Is it continuous across the surface of the cylinder? **[4]**

3. Figure 1 below shows a thin insulating ring holding a charge per unit length λ on its circumference. The ring sits in a uniform magnetic field, which is orthogonal to the plane of the ring. The magnetic field reduces linearly with time t, according to $B = B_o(1 - t/T)$, over a time interval T.

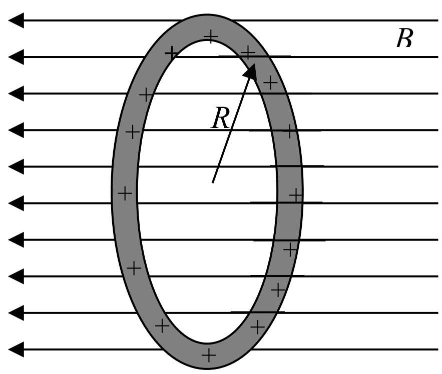

Fig. 1

Re-draw the diagram and indicate the direction of the electric field, at the ring circumference, due to the time-varying magnetic field. Calculate the amplitude of the electric field. **[4]**

[SECTION A CONTINUED OVERLEAF]

4. Consider the circuit in Figure 2:

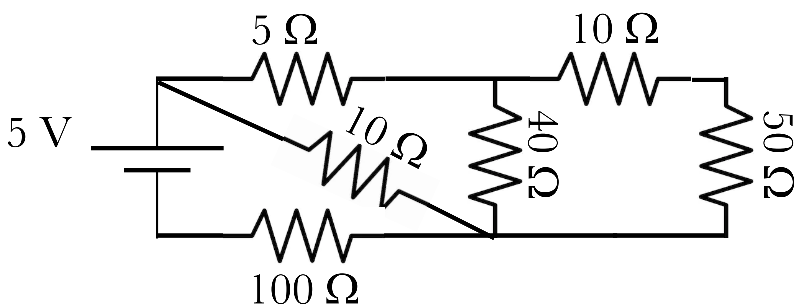

Fig. 2

What is the equivalent resistance of the whole circuit? What is the power dissipated? **[4]**

5. Show that, in electrostatics, $\nabla \times \vec{E} = 0$ and that this expression allows for the definition of a scalar potential. Discuss how this implies that the electrostatic force is conservative. **[4]**

6. By considering the electric and magnetic energy densities in a volume V bounded by a surface S, the rate of change of the total electromagnetic energy W is described by the Poynting's Theorem

$$\int_S \underline{E} \times \underline{H} \cdot d\underline{S} + \int_V \underline{E} \cdot \underline{J}\, dV = -\frac{\partial W}{\partial t} \qquad \text{Eq.1}$$

Explain, in detail, the physical meaning of each term in Eq. 1. Which term is related to the Poynting vector? What are the SI units of this vector? **[4]**

7. Figure 3 shows the cross section of an air-cored solenoid of radius *R* carrying a current *I* and a number of circular lines indicating the vector potential field.

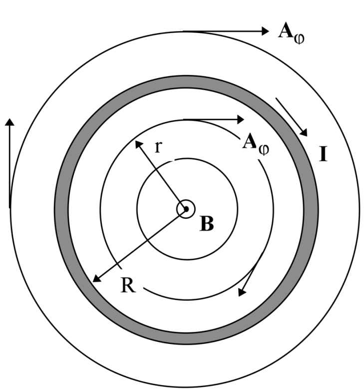

Fig. 3

[SECTION A CONTINUED OVERLEAF]

Show that the vector potential A_φ is given by the following expressions:

$$A_\varphi = \frac{\mu_0 n I r}{2} \qquad r \le R, \qquad\qquad A_\varphi = \frac{\mu_0 n I R^2}{2r} \qquad r \ge R$$

[4]

8. Describe the states of polarisations of the following electromagnetic waves, where $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ represents the unit vector along the x and y axis.

(a) $\vec{\mathbf{E}} = \mathrm{E}_0\cos(kz - \omega t)\hat{\boldsymbol{x}} + \mathrm{E}_0\sin(kz - \omega t)\hat{\boldsymbol{y}}$

(b) $\vec{\mathbf{E}} = \mathrm{E}_0\sin(kz - \omega t)\hat{\boldsymbol{x}} - 2\mathrm{E}_0\sin(kz - \omega t)\hat{\boldsymbol{y}}$

(c) $\vec{\mathbf{E}} = \mathrm{E}_0\cos(kz - \omega t)\hat{\boldsymbol{x}} + \mathrm{E}_0\sin\left(kz - \omega t + \frac{\pi}{2}\right)\hat{\boldsymbol{y}}$

(d) $\vec{\mathbf{E}} = \mathrm{E}_0\cos(kz - \omega t)\hat{\boldsymbol{x}} + \mathrm{E}_0\sin\left(kz - \omega t + \frac{\pi}{4}\right)\hat{\boldsymbol{y}}$

[4]

9. An electromagnetic plane wave is incident from air onto a crystal (n=1.4) with a planar interface. Sketch the curves of phase change of the reflected beam for s- and p-polarisations respectively, for incident angles ranging from 0 to 90 degrees. **[4]**

10 With respect to the following diagram (Fig.4), explain how a laser beam is generated. Describe the roles of the labelled elements. **[4]**

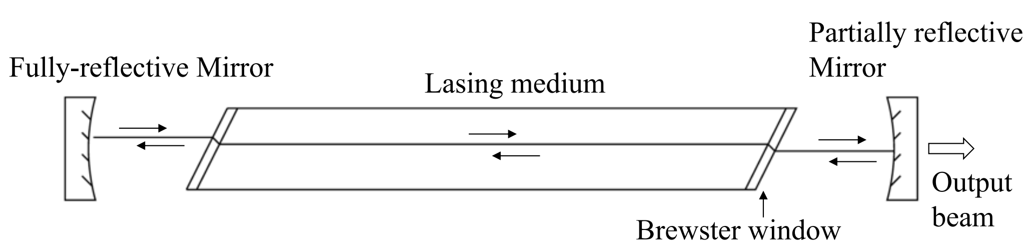

Fig.4

SECTION B

Use a Section B answer book

Answer TWO questions from this section

11. Figure 5 shows a uniformly charged, infinitely long cylinder, of charge density ρ, radius a, and permittivity ε_0. The cylinder is sheathed by a conducting skin that is maintained at zero potential.

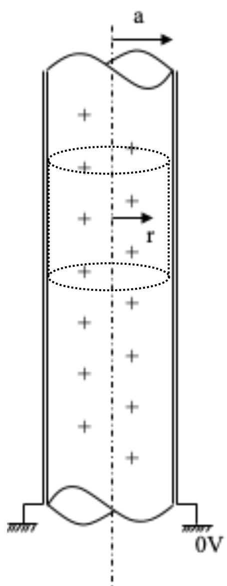

Fig. 5

(a) Use Gauss's law to show that the electric field inside the cylinder is given by:

$$E_r = \frac{\rho r}{2\varepsilon_0}$$ **[6]**

(b) Hence calculate the potential at radial distance $r < a$ **[5]**

(c) Using cylindrical co-ordinates, show that the potential satisfies Poisson's equation. **[4]**

(d) If the cylinder's material had a relative permittivity ε_r, how would that affect the electric field and potential calculated for questions (a) and (b)? What would be the polarisation $\vec{P}$ of the material? Would the new potential still satisfy Poission's equation? **[5]**

SECTION B

12. Let us assume a magnetized cylinder (magnetization $\vec{M}$) of radius *R*, height *h*, and mass *m*, aligned along the vertical axis $\hat{z}$, as in Figure 6.

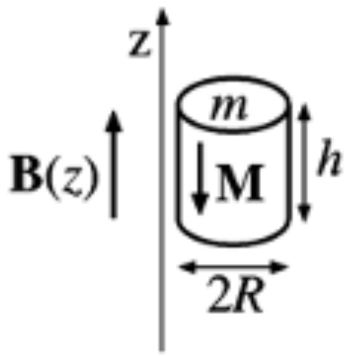

Fig. 6

The cylinder is in the presence of an external magnetic field aligned along *z* of the form (assume *a* and *b* > 0):

$$\vec{B} = \frac{a}{b+z}\hat{z}$$

(a) Calculate the magnetic dipole moment of the magnet $\vec{m}$. By using the fact that the force felt by a magnet in an external magnetic field is given by: $\vec{F} = \nabla\left(\vec{m}\cdot\vec{B}\right)$, calculate the force felt by the magnet. What is the direction of the force? **[8]**

(b) By assuming that the magnet is also subject to a constant gravitational field $\vec{g}$, pointing downwards, calculate the equilibrium position of the magnet. **[6]**

(c) Is the equilibrium stable or unstable? Motivate your answer. **[6]**

SECTION B

13. Half of a rectangular circuit of resistance *R* and sides *a* and 4*a* is immersed in an external constant magnetic field, as in Figure 7.

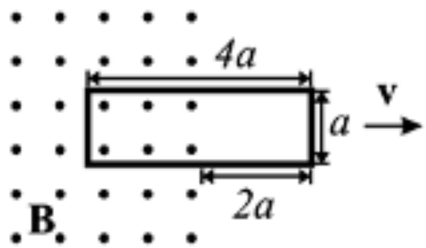

Fig. 7

The circuit is pulled away from the region of magnetic field with a constant velocity *v*.

(a) Show that the temporal derivative of the flux of the magnetic field is given by:

$$\frac{\partial \Phi(B)}{\partial t} = -Bav$$

And calculate the current circulating in the circuit. In what direction does the current flow? **[7]**

(b) Using the result in (a), calculate the overall force felt by the circuit as it is pulled away. What is the direction of the force for each arm of the circuit? **[7]**

(c) What is the work done to extract the circuit completely from the magnetic field region? **[6]**

SECTION B

14. **(a)** Four point-like charges (q = 5 nC, h = 1 cm, and d = 5 cm) are in vacuum and aligned along the x-axis as in Figure 8.

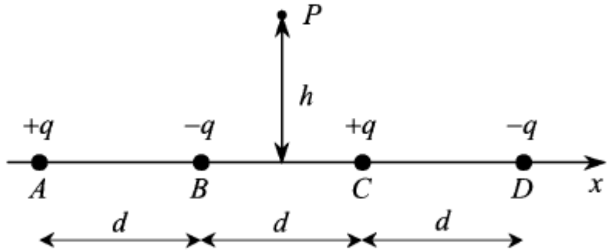

Fig. 8

Calculate the electrostatic potential and the electric field generated by the four charges at the point P. What is the direction of the field? **[6]**

(b) Two long concentric conducting cylinders of radii a and b ($a<b$) carry equal and opposite charge per unit length λ (positive on the inner conductor). The space between the two conductors is filled with a dielectric of permittivity ε_r (Fig. 9).

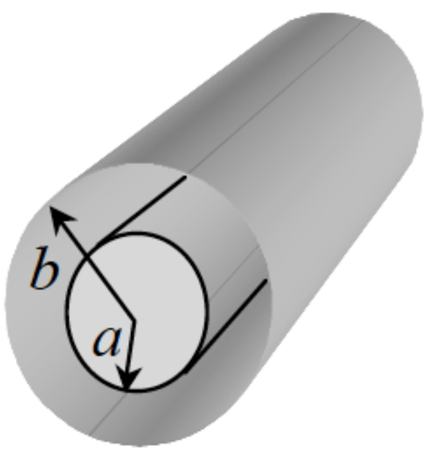

Fig.9

(i) Sketch the distribution of bound and free charges on the conductors and the dielectric. **[2]**

[QUESTION 14 CONTINUED OVERLEAF]

(ii) Calculate the amplitude of the electric field $\vec{E}$, polarisation $\vec{P}$, and electric displacement field $\vec{D}$ for the three regions: $r > b$, $b > r > a$, and $r < a$. **[7]**

(iii) Calculate the potential difference between the two conductors and the capacitance per unit length of the whole system. **[5]**

SECTION C

Use a Section C answer book

Answer ONE question from this section

15 (i) A plane wave of unpolarised light is incident from water ($n_1 = 1.33$) to air ($n_2 = 1$).

(a) Calculate the critical angle. **[2]**

(b) Derive the Brewster angle formula: $\tan\theta_b = \frac{n_2}{n_1}$ **[2]**

(c) Starting from normal incidence, the angle of incidence gradually increases, approaching to a grazing angle near 90 degree. Describe how the intensity of the reflected light changes during this procedure, for both s- and p- polarisations. Sketch the curves of reflectance against the incident angles for both polarisations. Label the axes. **[6]**

(ii) Briefly describe the phenomenon of Faraday Effect. Explain how the applied magnetic field induces Faraday Effect. Faraday Effect can be exploited in optical isolators to block the back-reflection light, describe the working principles. Illustrate your answers with diagrams. **[7]**

(iii) Polarised filters can be used to improve image quality during a haze day. Explain the working principle. **[3]**

SECTION C

16 A two-level system has an energy difference of $\Delta\mathrm{E}$= 2.5 eV. Assuming the emission only has natural broadening and the frequency width of the emission line is 7.5×10^7 Hz.

(i) What is the population ratio N_2/N_1 at T = 300 K? At what temperature will the population ratio be doubled from the room temperature? **[3]**

(ii) What is the Einstein coefficient A_{21}? **[2]**

(iii) An incoming photon causes a stimulated emission of the system. What is the frequency of the incoming photon? **[2]**

(iv) Describe the procedures of spontaneous emission and stimulated emission, respectively, and specify the different properties of light emitted from the two mechanisms. **[6]**

(v) Sketch the diagrams of a 3-level and a 4-level lasing system, respectively. Describe the working principles and explain why a 4-level system is considered to be more effective in achieving population inversion than a 3-level system. **[7]**

END OF THE EXAMINATION PAPER