RELATIVISTIC MECHANICS

Why do we need relativistic mechanics?

The answer lies in nuclear or elementary particle collisions and decays where particles are generally travelling with speeds that make relativistic corrections necessary, *i.e.* where the relativistic factor γ deviates from unity. This, as we know occurs for velocities greater than about $0.1c$. In many instances particles are travelling at speeds very close to the speed of light such that very large γ factors are possible.

e.g. The Large Hadron Collider at CERN has accelerated proton beams to an energy of 4 Tev. Colliding two such beams leads to interactions at a collision energy of 8 TeV. You should be able to show that this corresponds to a factor of $\gamma{\sim}10^4$.

How do we solve problems in relativistic mechanics?

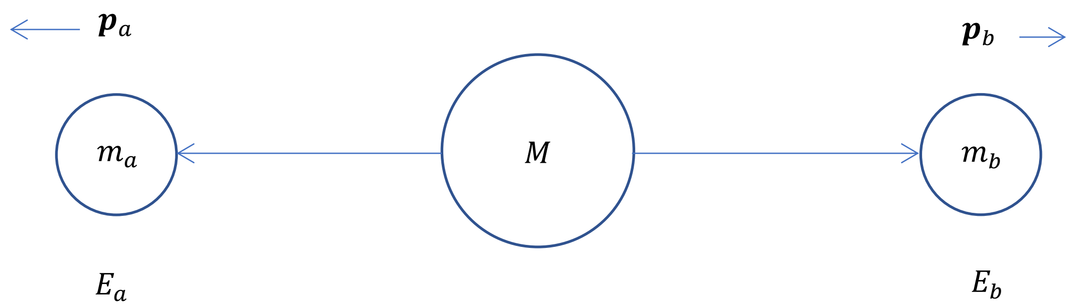
Whilst rest mass is not necessarily conserved in nuclear collision or decay processes, we have a strict conservation law for $\boldsymbol{P}$ in all 4-components.

This gives us a very powerful way of solving such problems, allowing us to calculate parameters such as energy, momentum and velocity of particles post collision.

We will consider a decay process to outline the approach. The approach is equally valid for a collision-type process.

Example Problem:

A particle of rest mass M decays into two product particles with rest masses m_a and m_b respectively.



To find the energies of the two product particles

Recall:

$$\boldsymbol{P} = (p_x, p_y, p_z, \frac{iE}{c})$$

The 4-vector for momentum is conserved in all of its 4-components.

Consider $P_1 = p_x$

Let the decay occur along the x-axis

Then conservation of P_1:

$$0 = p_a - p_b \qquad \text{----- (3.1)}$$

Consider $P_4 = \frac{iE}{c}$

Then conservation of P_4:

$$M = E_a + E_b \qquad \text{----- (3.2)}$$

Note that we have again omitted c's, in this case the c^2 from the LHS of eqn (3.2)

We showed in chapter 2 that due to the invariance of the scalar product $\boldsymbol{P}.\boldsymbol{P}$ we can show that:

$$E^2 = p^2c^2 + {m_0}^2c^4$$

Applying this equation to each of the product particles gives us:

$${E_a}^2 = {p_a}^2 + {m_a}^2 \qquad \text{----- (3.3)}$$

and

$${E_b}^2 = {p_b}^2 + {m_b}^2 \qquad \text{----- (3.4)}$$

Subtracting (3.4) from (3.3) and using eqn (3.1) gives:

$${E_a}^2 - {E_b}^2 = {m_a}^2 - {m_b}^2$$

$$(E_a + E_b)(E_a - E_b) = {m_a}^2 - {m_b}^2$$

Substitute from eqn (3.2):

$$M(E_a - E_b) = {m_a}^2 - {m_b}^2$$

$$(E_a - E_b) = \frac{{m_a}^2 - {m_b}^2}{M}$$

Substituting for E_b from (3.2):

$$2E_a = M + \frac{{m_a}^2 - {m_b}^2}{M}$$

$$E_a = \frac{M^2 + {m_a}^2 - {m_b}^2}{2M}$$

----- (3.5)

This gives us a value for the total energy of product particle a in terms of the rest masses of the parent and product particles.

Since the RHS of this equation has dimensionality of mass, we must multiply by c^2 to regain the correct expression.

$$E_a = \frac{(M^2 + {m_a}^2 - {m_b}^2)c^2}{2M}$$

From eqn (3.2) and substituting from (3.5):

$$E_b = M - E_a = M - \frac{M^2 + {m_a}^2 - {m_b}^2}{2M}$$

$$E_b = \frac{(M^2 + {m_b}^2 - {m_a}^2)c^2}{2M}$$

----- (3.6)

[Note – we could also have deduced this from (3.5) from the symmetry of the decay.]

Having found the energies of the two particles, we can now find other properties of particles a and b as required.

<u>To find the momenta of the two product particles</u>

Substituting for E_a from (3.5) in eqn (3.3) gives us p_a.

[Note - be careful to replace the c's correctly in the final expression]

p_b can be obtained similarly using (3.4) and (3.6), or, more readily, by substituting the expression obtained for p_a into eqn (3.1).

To find the velocities of the two product particles

On the face of it the logical approach would be to take the value obtained for p_a say, and use this to find u_a

However, recall that

$$\boldsymbol{p} = m\boldsymbol{u}$$

so that

$$p_a = \gamma(u) m_a u_a$$

This leads to complications in the algebra as u appears in the γ factor as well as explicitly in the equation, leading to quadratic expressions in u

It is therefore better to consider the 4th rather than the 1st component of $\boldsymbol{P}$ in obtaining an expression for u

Recall

$$P_4 = i\gamma(u) m_0 c = \frac{iE}{c}$$

or, just

$$E = \gamma(u) m_0 c^2$$

Knowing E_a from (3.5) we can thus obtain an expression for u_a from the γ factor.

Similarly for u_b using (3.6).

What if the problem involves a decaying particle that is travelling with some finite velocity?

As is usually the case we first seek to solve the problem in the frame where the solution will be the simplest to evaluate. In this case it would be where the decaying particle is at rest – *i.e.* we follow the steps as shown above. When the velocities, say, of the product particles have been obtained, applying the reverse LT gives the velocities in the original frame where the parent particle is in motion.

Example

Consider the decay of a K meson of rest mass 498 MeV into two π mesons, each having rest mass 135 MeV. Calculate the total energy, the kinetic energy and the velocity of the product π mesons.

From equation (3.5):

$$E_a = \frac{M^2 + {m_a}^2 - {m_b}^2}{2M}$$

Then given the symmetry in this problem, where the two product particles have the same rest mass, the energy of each π meson is given by:

$$E_\pi = \frac{M_K}{2} = 249\ MeV$$

To calculate the kinetic energy of each π meson:

$$T_\pi = E_\pi - m_\pi = 249 - 135 = 114\ MeV$$

To calculate the velocity of each π meson:

$$E = \gamma(u) m_0 c^2$$

So

$$E_\pi = \gamma(u_\pi) m_\pi$$

$$\gamma(u_\pi) = \frac{E_\pi}{m_\pi} = \frac{249}{135}$$

By putting in the expression for $\gamma(u_\pi)$ it can be shown that:

$$u_\pi = 0.84c$$

Example Problem involving LT

Doppler Shift for Light

Consider some inertial frame S in which light at frequency f is detected.

Let the source of the light be travelling in the x direction, and consider a beam of light travelling in the x-y plane at an angle θ with respect to the x-axis.

Since, to date, we have only considered particle properties, let us view this problem from the particle-like photon representation of light.

The photon as a particle has interesting properties.

For example, its rest mass $m_0 = 0$ and its velocity is c

We have shown in chapter 2 that:

$$\boldsymbol{P}.\boldsymbol{P} = -{m_0}^2c^2$$

But this means that since $m_0 = 0$ we must also have that $\boldsymbol{P}.\boldsymbol{P} = 0$, and hence the 4-vector for momentum for a photon has zero length, *i.e.* $|\boldsymbol{P}| = 0$.

Of course this does not mean that $\boldsymbol{P} = \boldsymbol{0}$

$$\boldsymbol{P} = (\boldsymbol{p}, \frac{iE}{c})$$

i.e. the photon possesses both energy and momentum. Therefore:

$$\boldsymbol{P}.\boldsymbol{P} = p^2 - \frac{E^2}{c^2}$$

or, and dropping c's:

$$p^2 = E^2$$

Hence, and from Einstein:

$$E = p = hf$$

where h is Planck's constant.

To understand the Doppler shift in frequency for light we must transform into some other inertial frame S', defined in the usual way.

Of course we know how the 4-vector for momentum transforms, *i.e.* it transforms in exactly the same way as all other 4-vectors, namely:

$$P_\mu{}' = a_{\mu\nu}P_\nu$$

Since P_4 relates to energy, which we know is related to the frequency of a photon, let is transform the 4th component of the 4-vector:

$$P_4{}' = a_{4\nu}P_\nu = \sum_{\nu=1}^{4} a_{4\nu}P_\nu = a_{41}P_1 + a_{42}P_2 + a_{43}P_3 + a_{44}P_4$$

Hence we obtain:

$$E' = \gamma(E - vp_x)$$

where we have again dropped the c's.

Given the direction of travel of the photon, as defined above:

$$\boldsymbol{p} = (pcos\theta, psin\theta, 0)$$

Therefore:

$$hf' = \gamma(hf - vpcos\theta)$$

But we have seen that $p = hf$

Hence:

$$f' = \gamma f(1 - vcos\theta)$$

------ (3.7)

If we let the transformation take us into a frame S' where the source of the light is at rest then the frequency of the light in that frame will be the <u>proper frequency</u> f_0 of the light. *i.e.* $f' = f_0$

Therefore, the Doppler shifted frequency as measured in frame S:

$$f = \frac{f_0}{\gamma(1 - vcos\theta)}$$

A very simple result.

Expanding γ and replacing c's:

$$f = f_0 \frac{(1 - \frac{v^2}{c^2})^{1/2}}{(1 - \frac{v}{c}cos\theta)}$$

For $\theta = 0$ – Blue shift

$$f = f_0 \left(\frac{c+v}{c-v}\right)^{1/2}$$

For $\theta = \pi$ – Red shift

$$f = f_0 \left(\frac{c-v}{c+v}\right)^{1/2}$$

For $\theta = \frac{\pi}{2}$

$$f = \frac{f_0}{\gamma}$$

Examples involving Waves

We have seen how we can solve problems involving light (or more generally electromagnetic waves travelling at speed c) by invoking the photon, or particle, concept of light. However, we might wish to consider how more generally we can deal with relativistic problems involving waves.

To do this let us define a 4- vector for frequency, also known as the Wave Vector:

4-FREQUENCY

$$\boldsymbol{N} = f(\frac{c}{\omega}\hat{\boldsymbol{n}}, i)$$

combining the three kinematic characteristics of a wave at a given event.

f - wave frequency
ω - wave, or phase, velocity
$\hat{\boldsymbol{n}}$ - unit vector normal to the wave surface in the direction of propogation

The 4-vector $\boldsymbol{N}$ fully determines the transformation properties of these three wave characteristics, *via*

$$N_\mu{}' = a_{\mu\nu}N_\nu$$

We can express the frequency 4-vector explicitly as:

$$N_\mu = (\frac{c}{\omega}fn_x, \frac{c}{\omega}fn_y, \frac{c}{\omega}fn_{z,}if)$$

The de Broglie Equation

In the greatest success story for 19th century physics, Maxwell eloquently developed a theory which demonstrated that light could be described as an electromagnetic wave. This was consistent with the understanding that wave theory was required to explain interference and diffraction phenomena. However, in 1905 Einstein showed that the explanation for the results of certain experiments (in this case the photoelectric effect) required a particle model of light.

In 1924 Louis de Broglie postulated that any particle of energy E has an associated wave of frequency $f = \frac{E}{h}$ travelling in the same direction. This became known as the concept of wave-particle duality.

This concept of wave-particle duality can be quantified neatly through a 4-vector equation known as the de Broglie equation:

$$c\boldsymbol{P} = h\boldsymbol{N} \qquad \text{------ (3.8)}$$

where the 4-vectors $\boldsymbol{P}$ and $\boldsymbol{N}$ encapsulate the particle-like and wave-like properties respectively.

Rewriting equation (3.8) as:

$$c(m\boldsymbol{u}, imc) = h(\frac{c}{\omega}f\hat{\boldsymbol{n}}, if)$$

In 4-vector equations, each of the four components on LHS and RHS must satisfy the equality condition. Hence, for example, equating the time-like 4th components:

$$mc^2 = hf \qquad \text{------ (3.9)}$$

equating Einstein's two famous equations for energy of a particle of relativistic mass m and a photon of frequency f

Equating the space-like components 1-3:

$$\boldsymbol{u} \propto \hat{\boldsymbol{n}} \qquad \text{------ (3.10)}$$

and

$$mu = \frac{hf}{\omega}$$

substituting from (3.9)

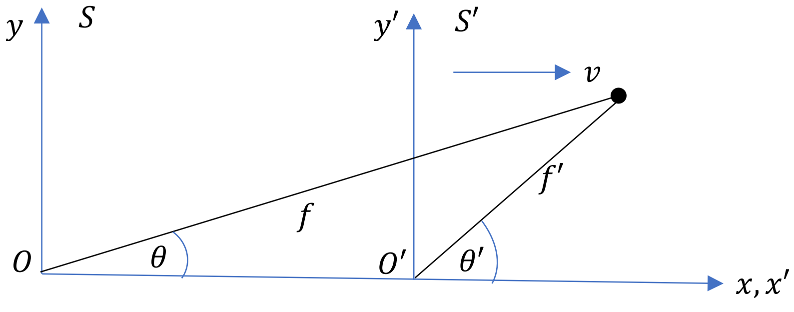
$$u\omega = c^2 \qquad \text{------ (3.11)}$$

i.e. particle can only be thought of as moving along with its associated wave if it is moving at the speed of light,

$$u = \omega = c$$

Example – Doppler Shift for a Wave

Adopting the same conditions as considered in the earlier derivation involving the particle model of light.



Then:

$$\hat{\boldsymbol{n}} = (cos\theta, sin\theta, 0)$$

4-frequency:

$$N_\mu = (\frac{c}{\omega} f n_x, \frac{c}{\omega} f n_y, \frac{c}{\omega} f n_{z,} if)$$

Transformation property:

$$N_\mu{}' = a_{\mu\nu} N_\nu$$

To derive a Doppler shift we are interested in how frequency of a wave transforms between the inertial frames S and S'

Hence transform the time-like 4th component of N_μ

$$N_4{}' = a_{4\nu} N_\nu = \sum_{\nu=1}^{4} a_{4\nu} N_\nu = a_{41} N_1 + a_{42} N_2 + a_{43} N_3 + a_{44} N_4$$

$$if' = \gamma(if - i\frac{v}{c} f \frac{c}{\omega} n_x)$$

$$f' = \gamma f(1 - \frac{v}{\omega} cos\theta)$$

which is the Doppler shift equation for the frequency of a wave with phase velocity ω

Note that for $\omega = c$ this is identical to the expression derived in equation (3.7) for the Doppler shift for light (with c replaced).

Example – Wave Aberration Formula

Considering the same scenario as above, we can also investigate how the direction of the wave transforms from the inertial frame S to the frame S'

Transform the space-like components of N_μ

From N_1 :

$$\frac{c}{\omega'} f' cos\theta' = \gamma f \frac{c}{\omega} cos\theta - \gamma \frac{v}{c} f$$

Note that all three of the parameters characterising the properties of the wave, f, ω and θ are transformed.

Simplifying:

$$\frac{c}{\omega'} f' cos\theta' = \gamma f (\frac{c}{\omega} cos\theta - \frac{v}{c}) \quad \text{------ (3.12)}$$

From N_2 :

$$\frac{c}{\omega'} f' sin\theta' = f \frac{c}{\omega} sin\theta \quad \text{------ (3.13)}$$

Divide (3.13) by (3.12):

$$tan\theta' = \frac{\frac{c}{\omega} sin\theta}{\gamma(\frac{c}{\omega} cos\theta - \frac{v}{c})}$$

Simplifying:

$$tan\theta' = \frac{sin\theta}{\gamma(cos\theta - \frac{v\omega}{c^2})}$$

This is the general form of the wave aberration formula.

For light, putting $\omega = c$ we get:

$$tan\theta' = \frac{sin\theta}{\gamma(cos\theta - \frac{v}{c})}$$

This is known as the stellar aberration formula.

A classical form of this equation (missing the γ factor) was derived by James Bradley in 1729 to explain the small movement of a star during the course of a year, as the Earth orbited the Sun.

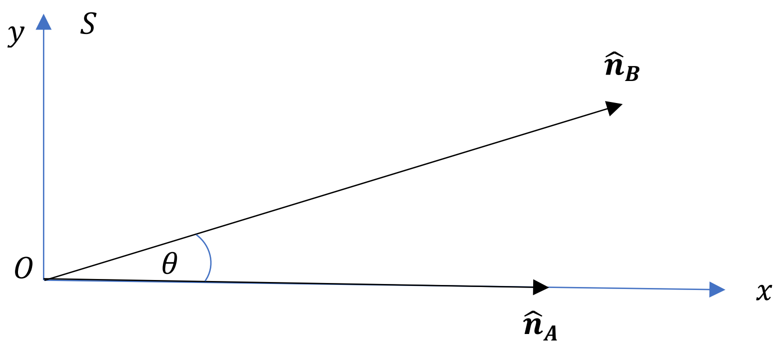
Collisions Involving Photons

When considering collision involving photons, the Lorentz invariance of the scalar product of 4-momenta vectors can prove very useful.

Scalar product of momenta of 2 photons, labelled A and B

Consider two photons, A and B, with 4-momenta $\boldsymbol{P_A}$ and $\boldsymbol{P_B}$, frequencies f_A and f_B , and velocities that make an angle θ with each other.

We can choose our axes in such a way that photon A travels along the x-axis, and photon B travels at an angle θ to the x-axis in the x-y plane.



Hence:

$$\hat{\boldsymbol{n}}_{\boldsymbol{A}} = (1,0,0) \quad and \quad \hat{\boldsymbol{n}}_{\boldsymbol{B}} = (cos\theta, sin\theta, 0)$$

Relating the 4-momentum vector of a photon to its wave properties *via* the de Broglie equation:

$$c\boldsymbol{P} = h\boldsymbol{N} \quad or \quad \boldsymbol{P} = \frac{h}{c}\boldsymbol{N}$$

Recall:

$$\boldsymbol{N} = (\frac{c}{\omega}fn_x, \frac{c}{\omega}fn_y, \frac{c}{\omega}fn_{z,}if)$$

Hence we have:

$$\boldsymbol{P_A} = \frac{h}{c}f_A(\frac{c}{\omega_A}, 0\,, 0\,, i)$$

and

$$\boldsymbol{P_B} = \frac{h}{c}f_B(\frac{c}{\omega_B}cos\theta, \frac{c}{\omega_B}sin\theta\,, 0\,, i)$$

Therefore, taking the scalar product of the two 4-momenta:

$$\boldsymbol{P_A}.\boldsymbol{P_B} = \frac{h^2}{c^2}f_Af_B\left[\frac{c^2}{\omega_A\omega_B}cos\theta + 0 + 0 - 1\right]$$

Since we are dealing with photons, $\omega_A = \omega_B = c$

$$\boldsymbol{P_A}.\boldsymbol{P_B} = -\frac{h^2}{c^2}f_Af_B(1 - cos\theta)$$

Scalar product of the momentum of a photon with the momentum of a particle

Consider a photon of frequency f, and a particle of rest mass m_0 travelling with velocity $\boldsymbol{u}$

Let $\boldsymbol{P}$ be the 4-momentum of the photon, and again representing it in terms of its wave properties *via* the de Broglie equation:

$$\boldsymbol{P} = \frac{h}{c} f\left(\frac{c}{\omega}\boldsymbol{\hat{n}}, i\right)$$

Let $\boldsymbol{Q}$ be the 4-momentum of the particle:

$$\boldsymbol{Q} = \left(\boldsymbol{p}, \frac{iE}{c}\right) = m_0 \gamma(u)(\boldsymbol{u}, ic)$$

As usual with a problem of this kind we transform to the simplest frame making use of the invariance property of the scalar product.

Hence transform to the frame of reference where the particle is at rest. [*note that we couldn't use this trick with the two photons, as photons do not have a rest frame*.]

Then:

$$\boldsymbol{Q}' = m_0(0, 0, 0, ic)$$

and

$$\boldsymbol{P}' = \frac{h}{c} f'\left(\frac{c}{\omega'} n'_x, \frac{c}{\omega'} n'_y, \frac{c}{\omega'} n'_z, i\right)$$

where f' is the frequency of the photon in the rest frame of the particle.

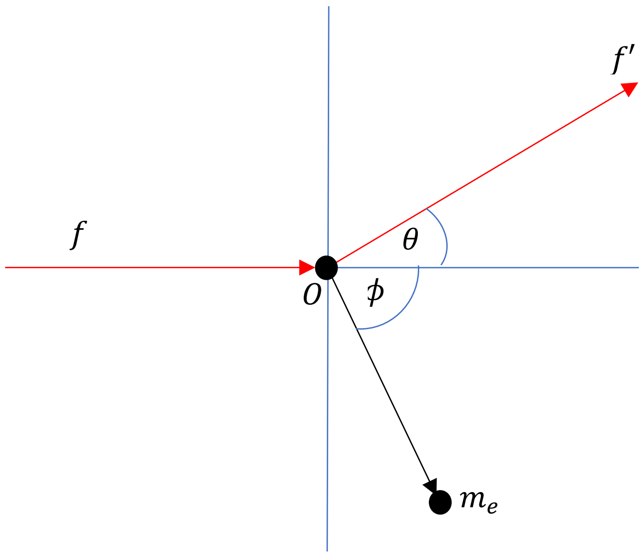
Hence:

$$\boldsymbol{P}.\boldsymbol{Q} = \boldsymbol{P}'.\boldsymbol{Q}' = \frac{h}{c} m_0 f'[0 + 0 + 0 - c]$$

$$\boldsymbol{P}.\boldsymbol{Q} = -h m_0 f'$$

Example Problem – Compton Scattering

The Compton Effect, discovered experimentally and theoretically explained by Arthur Compton in the early 1920's, is the partial absorption of high energy EM radiation by an atom, which can be described as the scattering of an energetic photon from a loosely bound valence electron. The collision results in the high energy recoil of the electron with the photon scattered though some angle with a loss in energy (hence reduced frequency, *i.e.* longer wavelength).



A photon of frequency f collides with an electron of rest mass m_e , which can be assumed to be stationary. The photon is scattered through an angle θ with a reduced frequency f', while the electron recoils through an angle ϕ

Let $\boldsymbol{P}$ and $\boldsymbol{P}'$ be the respective pre and post collision momenta of the photon

Let $\boldsymbol{Q}$ and $\boldsymbol{Q}'$ be the respective pre and post collision momenta of the electron

4-momentum is conserved in the collision (and recall that this is the case for all 4 components):

$$\boldsymbol{P} + \boldsymbol{Q} = \boldsymbol{P}' + \boldsymbol{Q}'$$

We wish to find the change in frequency of the scattered photon. Here we are less interested in what happens to the electron post-collision, hence we will isolate and try to eliminate $\boldsymbol{Q}'$

Thus re-arranging the above equation:

$$\boldsymbol{Q'} = \boldsymbol{P} + \boldsymbol{Q} - \boldsymbol{P'}$$

Taking the scalar product of $\boldsymbol{Q'}$ with itself:

$$\boldsymbol{Q'}.\boldsymbol{Q'} = (\boldsymbol{P} + \boldsymbol{Q} - \boldsymbol{P'}).(\boldsymbol{P} + \boldsymbol{Q} - \boldsymbol{P'})$$

Expanding:

$$\boldsymbol{Q'}.\boldsymbol{Q'} = \boldsymbol{P}.\boldsymbol{P} + \boldsymbol{P}.\boldsymbol{Q} - \boldsymbol{P}.\boldsymbol{P'} + \boldsymbol{Q}.\boldsymbol{P} + \boldsymbol{Q}.\boldsymbol{Q} - \boldsymbol{Q}.\boldsymbol{P'} - \boldsymbol{P'}.\boldsymbol{P} - \boldsymbol{P'}.\boldsymbol{Q} + \boldsymbol{P'}.\boldsymbol{P'}$$

But we know for the photon that $\boldsymbol{P}.\boldsymbol{P} = \boldsymbol{P'}.\boldsymbol{P'} = \boldsymbol{0}$, and for the electron we must have that $\boldsymbol{Q'}.\boldsymbol{Q'} = \boldsymbol{Q}.\boldsymbol{Q}$

Therefore, simplifying:

$$2\boldsymbol{P}.\boldsymbol{Q} - 2\boldsymbol{P'}.\boldsymbol{Q} - 2\boldsymbol{P}.\boldsymbol{P'} = 0 \qquad \text{------ (3.14)}$$

But we know from our earlier results:

$$\boldsymbol{P}.\boldsymbol{Q} = -hm_e f$$

$$\boldsymbol{P'}.\boldsymbol{Q} = -hm_e f'$$

$$\boldsymbol{P}.\boldsymbol{P'} = -\frac{h^2}{c^2} ff'(1 - cos\theta)$$

Substituting into eqn (3.14):

$$-hm_e f + hm_e f' + \frac{h^2}{c^2} ff'(1 - cos\theta) = 0$$

Simplifying:

$$\frac{h}{c^2} ff'(1 - cos\theta) = m_e(f - f')$$

Since $f = {}^{c}/_{\lambda}$:

$$\frac{h}{c^2}\frac{c^2}{\lambda\lambda'}(1 - cos\theta) = cm_e(\frac{1}{\lambda} - \frac{1}{\lambda'})$$

$$h\frac{1}{\lambda\lambda'}(1 - cos\theta) = cm_e\frac{(\lambda' - \lambda)}{\lambda\lambda'}$$

$$\lambda' - \lambda = \frac{h}{cm_e}(1 - cos\theta)$$

which is the usual expression for the change in wavelength due to Compton scattering.