QUEEN'S UNIVERSITY BELFAST

PHY2006
Exam Time Table
Code PHY2006

Answer Books A, B and C.

Any calculator, except one with a programmable memory, may be used in this examination.

Level 2
Examination contributing to the Degrees of Bachelor of Science (BSc) and Master in Science (MSci)

PHY2006
Mathematical Physics

Friday, 16th August 2019 9:30 AM - 12:30 PM

Examiners: Prof P Browning
Dr P van der Burgt
and the Internal Examiners

Answer ALL QUESTIONS in Section A
Answer ONE QUESTION in Section B
Answer ONE QUESTION in Section C

Use a separate answer book for each Section
You have THREE HOURS to complete this paper.

SECTION A

A.1 Show whether the following second order PDEs are Hyberbolic, Parabolic or Elliptic

(a) $\dfrac{\partial u}{\partial t} = D\,\dfrac{\partial^2 u}{\partial x^2}$

(b) $\dfrac{\partial^2 u}{\partial t^2} = D\,\dfrac{\partial^2 u}{\partial x^2}$

(c) $i\hbar\,\dfrac{\partial \psi}{\partial t} = -\dfrac{\hbar^2}{2m}\dfrac{\partial^2 \psi}{\partial x^2} + V\psi$

[10]

A.2 Find the solution to the following first order partial differential equation;

$$\frac{\partial u}{\partial t} + t\sqrt{x}\frac{\partial u}{\partial x} = 0$$

with initial condition $u(x,0) = \exp(2x) \quad (\equiv \mathrm{e}^{2x})$. [10]

A.3 The temperature of a rod satisfies the heat equation (with constant conductivity D)

$$\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2}$$

Using the method of separation of variables, *i.e.* letting $T(x,t) = Y(x)\Theta(t)$, show that the heat equation can be written in the form

$$\frac{1}{D\Theta}\frac{d\Theta}{dt} = \frac{1}{Y}\frac{d^2Y}{dx^2} = A$$

where A is an arbitrary constant. [10]

A.4 (a) Use the Gram-Schmidt orthogonalization procedure to make a function $g'(x)$ from $g(x) = 1$ which is orthogonal to $f(x) = x$ in the vector space where the inner product $< f(x)|g(x) >$ is defined by

$$< f(x)|g(x) >= \int_0^1 f(x)g(x)\;dx$$

(b) Use the Gram-Schmidt orthogonalization procedure to generate another function $h'(x)$ from $h(x) = x^2$ which is orthogonal to both $f(x)$ and $g'(x)$. [10]

A.5 Consider by inspection each of the four periodic functions shown in Figure 1 and the Fourier series which could be used to represent them. For each of these four functions you should indicate for the equivalent Fourier series

/CONTINUED

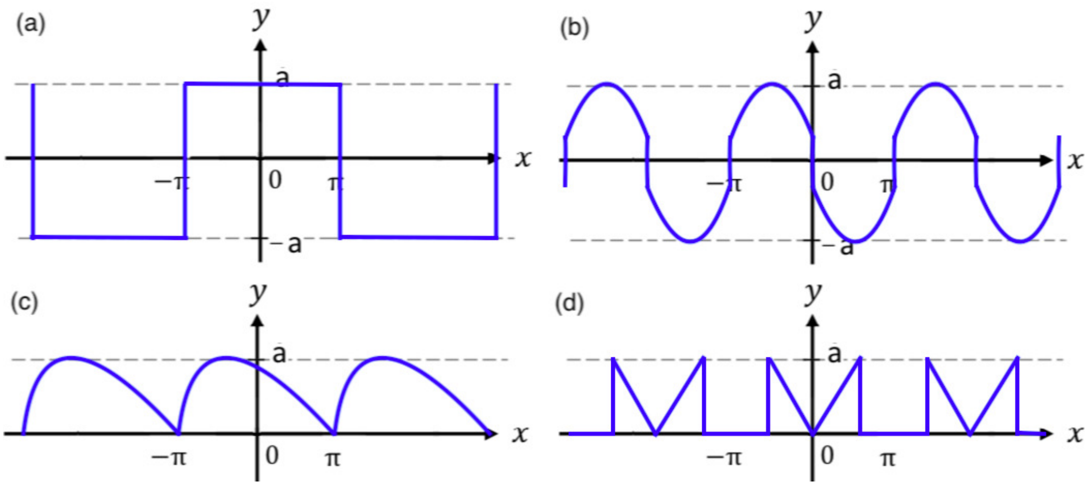

Figure 1: Four periodic functions for question **A.5**

- If a_0 is zero or non-zero.
- If all the a_k values (for $k > 0$) are zero or not.
- If all the b_k values are zero or not.

[10]

A.6 Calculate the Fourier transform of the function $f(x)$ where

$$\begin{aligned} f(x) &= \mathrm{e}^{x} && x < 0 \\ f(x) &= \mathrm{e}^{-x} && x \geq 0 \end{aligned}$$

[10]

/CONTINUED

SECTION B

B.1 Consider the function

$$u(x,t) = \frac{a}{\sqrt{t}} \exp\left(-x^2/4t\right)$$

where a is an arbitrary constant.

(a) Evaluate the following derivatives

i. $\frac{\partial u}{\partial t}$

ii. $\frac{\partial^2 u}{\partial x^2}$

(b) Hence, for what value of a is $u(x,t)$ given above a solution to the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

(c) Draw two rough sketch graphs;

i. sketch two plots for $u(x,t)$ against x; one plot should have $t = 1$ and the other should have $t = 4$. Draw both plots on the same graph and label the plots carefully with $t = 1$ and $t = 4$.

ii. sketch two plots for $u(x,t)$ against t; one plot should have $x = 1$ and the other should have $x = 2$. Draw both plots on the same graph and label the plots carefully with $x = 1$ and $x = 2$.

Hint: you may find it useful to write out each expression for $u(x,t)$ with a fixed value of x or t.

[20]

B.2 The Lagrangian for a single particle of mass m is defined in Cartesian coordinates as

$$L = \frac{1}{2} m \left[\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right] - V$$

where V is the potential energy of the particle. **Here, in this question**, $V = mgz$.

Re-write the Lagrangian in terms of a new set of variables

(a) Polar Coordinates – $(x,y) \Rightarrow (r,\theta)$

$$x = r\cos\theta \ , \qquad y = r\sin\theta$$

i. Find $\dot{x}$ and $\dot{y}$

[Hint: Use $\dfrac{dx}{dt} = \dfrac{\partial x}{\partial r}\dfrac{dr}{dt} + \dfrac{\partial x}{\partial \theta}\dfrac{d\theta}{dt}$ and similarly for $y(r,\theta)$ and $z(r,\theta)$]

ii. Write out the resulting expression for $\dot{x}^2 + \dot{y}^2$ and find L.
[Simplify your result as much as possible. Note: without loss of generality, you can set $z = 0$.]

(b) Cylindrical Coordinates – $(x, y, z) \Rightarrow (r, \theta, z)$

$$x = r\cos\theta\ , \quad y = r\sin\theta\ , \quad z = z$$

i. Find $\dot{x}$, $\dot{y}$ and $\dot{z}$

ii. Write out the resulting expression for $\dot{x}^2 + \dot{y}^2 + \dot{z}^2$ and find L.

(c) Spherical Coordinates – $(x, y, z) \Rightarrow (r, \theta, \phi)$

$$x = r\cos\theta\ , \quad y = r\sin\theta\cos\phi\ , \quad z = r\sin\theta\sin\phi$$

i. Find $\dot{x}$, $\dot{y}$ and $\dot{z}$

ii. Write out the resulting expression for $\dot{x}^2 + \dot{y}^2 + \dot{z}^2$ and find L.

[20]

SECTION C

C.1 The function $f(x)$ is defined by

$$
\begin{aligned}
f(x) &= 0 && -\pi \le x \le -\frac{\pi}{2} \\
f(x) &= -x && -\frac{\pi}{2} < x < \frac{\pi}{2} \\
f(x) &= 0 && \frac{\pi}{2} \le x \le \pi \\
f(x) &= f(x+2\pi)
\end{aligned}
$$

(a) Draw a sketch of the function $f(x)$. [4]

(b) Is $f(x)$ even, odd, or neither? [1]

(c) Indicate if any of the terms a_0, a_k and b_k of the Fourier series expansion of $f(x)$ are expected to be zero by inspection of the sketch of $f(x)$. [3]

(d) Determine the values of the terms a_0, a_k and b_k of the Fourier series expansion of $f(x)$. It is not necessary to explicitly calculate any terms that you have determined to be zero by inspection. [9]

(e) Determine the values a_0, a_1, a_2, a_3, a_4, b_1, b_2, b_3 and b_4. [3]

C.2 (a) A subspace of $\mathbb{R}^5$ is defined by the vectors

$$
\mathbf{a} = \begin{pmatrix} 1 \\ 4 \\ 2 \\ -2 \\ 0 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 0 \\ -4 \\ 4 \\ -4 \\ 1 \end{pmatrix} \qquad \mathbf{c} = \begin{pmatrix} 3 \\ -1 \\ -8 \\ 4 \\ -5 \end{pmatrix}
$$

i. Show that $\mathbf{a}$ and $\mathbf{b}$ are orthogonal. [3]

ii. Use the Gram-Schmidt orthogonalization procedure to make a vector $\mathbf{c}'$ from $\mathbf{c}$ orthogonal to both $\mathbf{a}$ and $\mathbf{b}$. [7]

(b) Waves of similar intensities at 400 Hz and 500 Hz are combined (added together). The combined signal is used to modulate a high frequency 'carrier wave' of 120 kHz.

i. Let us assume that the signals at 400 Hz and 500 Hz can be represented by $f(t) = A\cos(2\pi at)$ where A is the amplitude and a is the frequency in Hz. Now assuming that the amplitudes, A, for the two signals are similar the combined signal, $s(t)$, is given by

$$s(t) = A(\cos(2\pi 400t) + \cos(2\pi 500t))$$

Predict what the Fourier transform of this combined signal will look like and draw a rough sketch of what you expect it to be. [2]

/CONTINUED

ii. When the high frequency 120 kHz carrier wave is modulated by the combined 400 and 500 Hz the overall signal, $cw(t)$, is given by

$$cw(t) = (10A + s(t))\cos(2\pi \cdot 120 \times 10^3 \cdot t)$$

$$\text{where} \qquad (10A + s(t)) = 10A + A(\cos(2\pi 400t) + \cos(2\pi 500t))$$

Predict what the Fourier transform of this combined signal will look like and draw a rough sketch of what you expect it to be. Note, it is not necessary to mathematically calculate the Fourier transform, you may find it useful to use the formula

$$\cos(A)\cos(B) = \frac{1}{2}(\cos(A - B) + \cos(A + B))$$

to help predict the Fourier transform. [8]

END OF EXAMINATION