PHY2006

Any calculator, except one with pre-programmable memory, may be used in this examination.

Section A, B & C answer books

LEVEL 2
Examination contributing to the Degrees of Bachelor of Science (BSc) and Master in Science (MSci)

PHY2006
Mathematical Physics

Duration: 3 Hours plus additional 30 Minutes for upload of work

Friday 15 January 2021
09:30 AM – 1:00 PM

Examiners: Prof S Matthews, Dr F. Peters
and the internal examiners
Dr S Sim (s.sim@qub.ac.uk)

Answer ALL questions in Section A for 10 marks each.
Answer ONE question from Section B for 20 marks.
Answer ONE question from Section C for 20 marks.

If you have any problems or queries, contact the School Office at mpts@qub.ac.uk or 028 9097 1907, and the module coordinator T.Field@qub.ac.uk

SECTION A
Answer ALL questions in this section.

A.1 Use the Gram-Schmidt orthogonalization to make three orthogonal functions in the vector space where the inner product $< f(x)|g(x) >$ is defined by

$$< f(x)|g(x) >= \int_{-1}^{1} f(x)g(x)\ dx$$

Note that the limits for integration over x are from -1 to 1.

(a) Use the Gram-Schmidt orthogonalization procedure to generate a function $g'(x)$ from $g(x) = x^2$ which is orthogonal to $f(x)$ where $f(x) = 1$.

(b) Use the Gram-Schmidt orthogonalization procedure to generate a function $h'(x)$ from $h(x) = x^4$ which is orthogonal to both $f(x)$ and $g'(x)$.

[10]

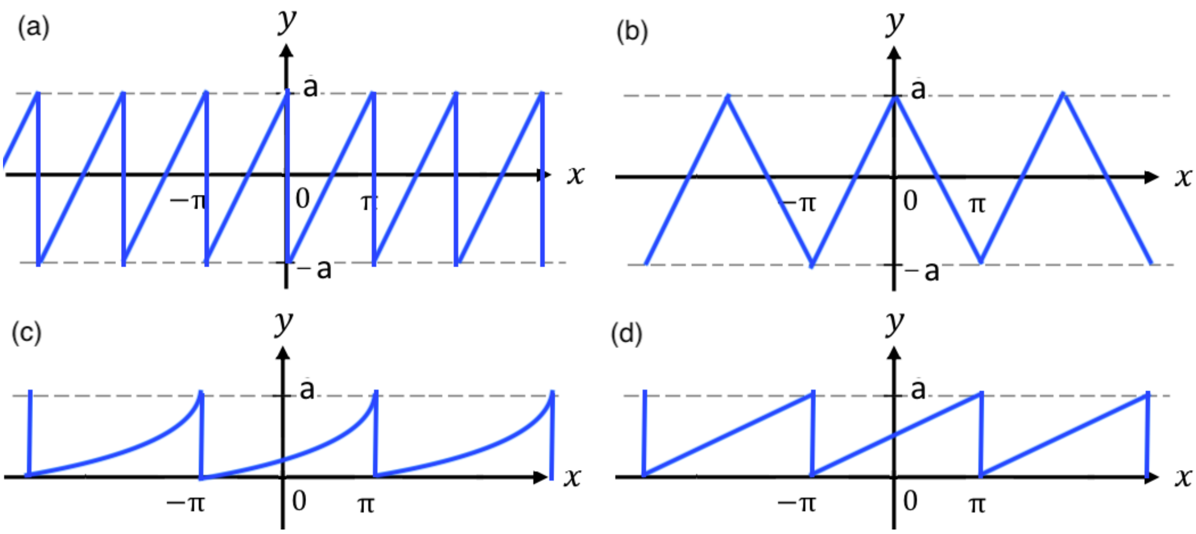

Figure 1: Four periodic functions for question **A.2**

A.2 Consider by inspection each of the four periodic functions shown in Figure 1 and the Fourier series which could be used to represent them. For each of these four functions you should indicate for the equivalent Fourier series

- If a_0 is zero or non-zero.
- If all the a_k values (for $k > 0$) are zero or or if at least some of them will be non-zero.
- If all the b_k values are zero or or if at least some of them will be non-zero.

[10]

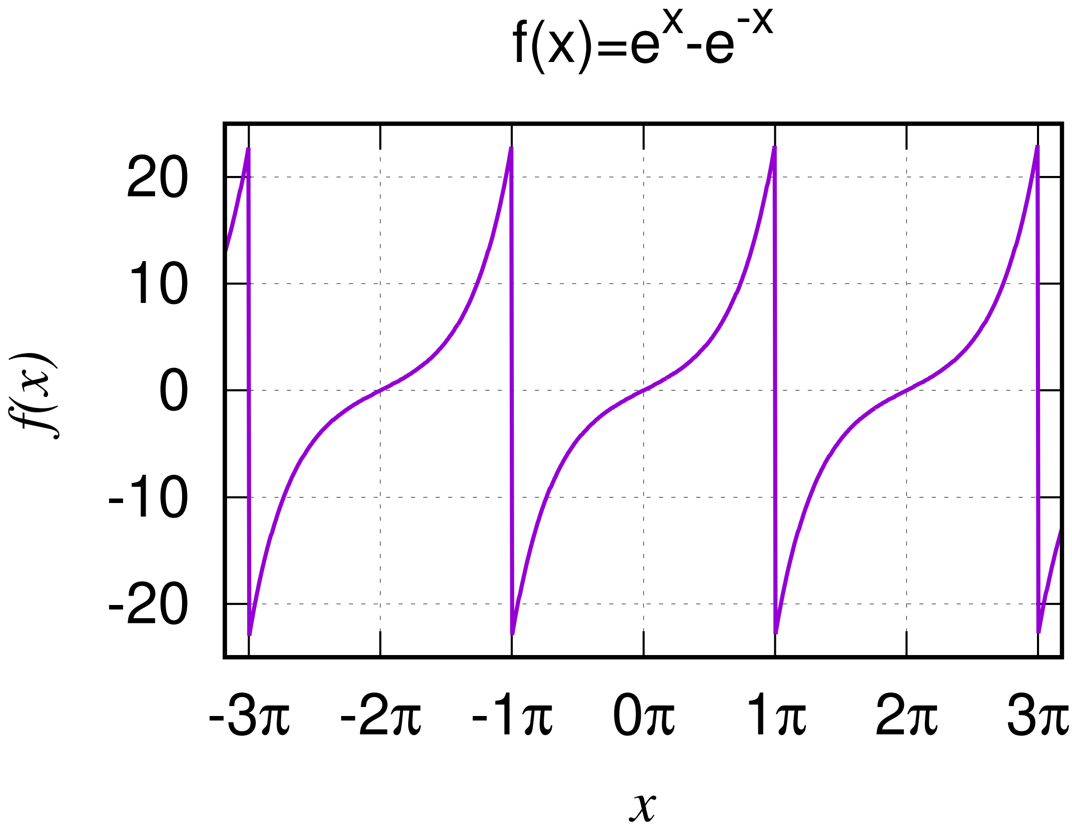

Figure 2: Function in question **A.3**

A.3 The function shown in figure 2 is defined by

$$f(x) = \mathrm{e}^{x} - \mathrm{e}^{-x} \qquad -\pi < x \leq \pi$$
$$f(x) = f(x + 2\pi)$$

Calculate an expression for the c_k coefficients of the the complex Fourier series that represents this function for $k \neq 0$ and simplify it.

Hint: Note that for integer values of k

$$\mathrm{e}^{ik\pi} = \mathrm{e}^{-ik\pi} = (-1)^k$$

[10]

A.4 Solve the following second order differential equation using a power series solution.

$$x\frac{d^2u}{dx^2} + u = 0$$
$$u(x) = \sum_{n=0}^{\infty} a_n x^n$$

Determine the coefficients a_n up to $n = 4$. **[10]**

A.5 Characterise the following partial differential equation in terms of the following:

- Order
- Linear : Non-linear
- Homogeneous : Inhomogeneous
- Elliptical : Parabolic : Hyperbolic : Mixed

$$\frac{\partial^2 (xu)}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = -xu$$

[10]

A.6 Using the method of characteristics, find the solution to the following first order partial differential equation;

$$\frac{\partial u}{\partial t} + xt\frac{\partial u}{\partial x} = -1$$

subject to the initial condition $u(x, 0) = \exp(-x^2)$ **[10]**

SECTION B
Answer ONE question from this section

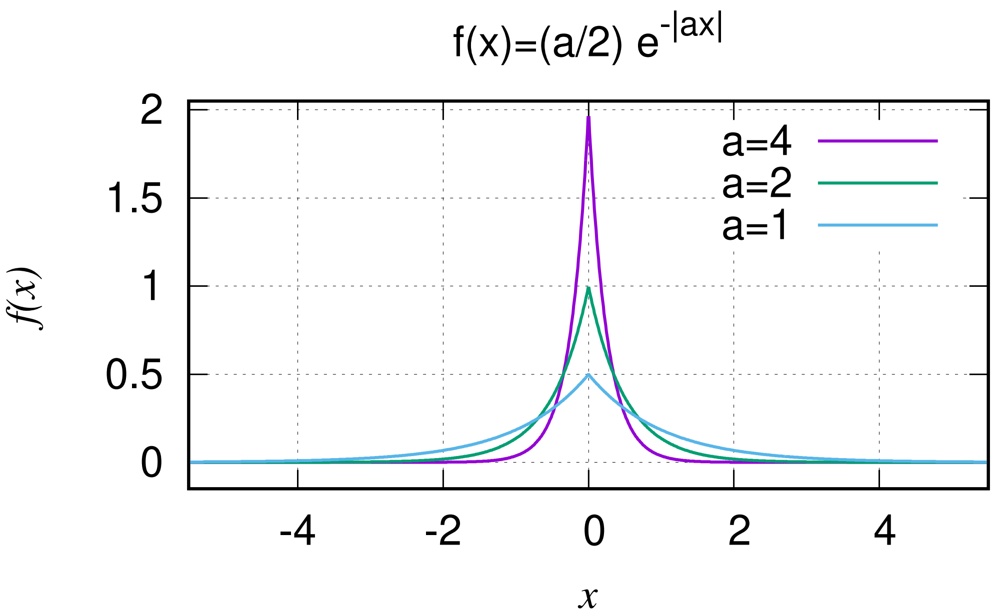

Figure 3: Function in question **B.1**

B.1 The function $f(x) = (a/2)\mathrm{e}^{-|ax|}$ shown in figure 3 with several values of a can be defined by

$$f(x) = \frac{a}{2}\mathrm{e}^{ax} \qquad\qquad x < 0$$
$$f(x) = \frac{a}{2}\mathrm{e}^{-ax} \qquad\qquad x \geq 0$$

(a) Evaluate the integral of $f(x)$ from minus infinity to plus infinity to show that

$$\int_{-\infty}^{\infty} f(x) = 1$$

[4]

(b) Calculate the Fourier transform of $f(x)$, the function $g(k)$ and reduce $g(k)$ to its simplest terms. **[8]**

(c) Calculate the value of $g(0)$, *viz.* the value of the function $g(k)$ when $k = 0$. **[2]**

(d) Determine all the values of k which satisfy the following conditions;

(i) $g(k) = \dfrac{g(0)}{2}$ (ii) $g(k) = \dfrac{g(0)}{5}$ (iii) $g(k) = \dfrac{g(0)}{10}$

[2]

(e) hence, or otherwise, sketch $g(k)$ for $a = 1$ **[2]**

(f) What would be the shape of $g(k)$ in the limit where a tends to infinity? **[2]**

B.2 A subspace in $\mathbb{R}^4$ is defined by the vectors

$$\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 2 \\ -2 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} -1 \\ 3 \\ -3 \\ 4 \end{pmatrix} \qquad \mathbf{c} = \begin{pmatrix} -2 \\ 7 \\ 6 \\ 1 \end{pmatrix}$$

(a) Write down the type of subspace that the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ define, **[1]**

(b) Use the Gramm-Schmidt orthogonalization procedure to make a vector $\mathbf{b}'$ from vector $\mathbf{b}$ which is perpendicular to $\mathbf{a}$. **[3]**

(c) Use the Gram-Schmidt orthogonalization procedure to make a vector $\mathbf{c}'$ from $\mathbf{c}$ which is orthogonal to both $\mathbf{a}$ and $\mathbf{b}'$. **[6]**

(d) Determine the vectors $\hat{\mathbf{p}}$ and $\hat{\mathbf{q}}$, which are the closest vectors within the vector subspace to the vectors $\mathbf{p}$ and $\mathbf{q}$;

$$\mathbf{p} = \begin{pmatrix} 13 \\ -1 \\ 1 \\ 7 \end{pmatrix} \qquad \mathbf{q} = \begin{pmatrix} 1 \\ 0 \\ 12 \\ 0 \end{pmatrix}$$

[6]

(e) Which of the vectors $\mathbf{p}$ and $\mathbf{q}$ is closest to the subspace and determine the distance between this closest point and the subspace. **[2]**

(f) What angle do you expect between the vector $(\hat{\mathbf{p}} - \mathbf{p})$ and the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{b}'$ and $\mathbf{c}'$? Briefly explain your reasoning. **[2]**

SECTION C
Answer ONE question from this section

C.1 An insulated, straight aluminium rod of length 1 m and uniform cross-sectional area has its ends ($x = 0, 1$) in perfect thermal contact with a heat reservoir at a temperature of 0 °C. The temperature $T(x, t)$ of the rod is governed by the one-dimensional heat equation where $D = 10^{-4}$ m^2 s^{-1} is the heat diffusivity of aluminium.

$$\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2}$$

(a) Using the separation of variables method, obtain two ordinary differential equations in x and t, and hence obtain a solution for $T(x, t)$. **[6]**

(b) By applying the boundary conditions and using the principle of superposition, show that the most general solution of $T(x, t)$ is

$$T(x,t) = \sum_{n=1}^{\infty} A_n sin(n\pi x)\exp(-(n\pi)^2 Dt)$$

[6]

(c) If the initial temperature distribution in °C is

$$T(x, 0) = 387.5\; x(1 - x)$$

Determine values for the coefficients A_n for $n = 1, 2, 3$. **[8]**

$$\int_0^1 sin(n\pi x)sin(m\pi x)dx = \begin{cases} 0 & n \neq m \\ \frac{1}{2} & n = m \end{cases}$$

$$\int_0^1 x(1-x)\; sin(m\pi x)dx = \frac{2}{(m\pi)^3}(1 - (-1)^m)$$

C.2 A wire of length $L = 1$ m and density per unit length $\mu = 0.01$ kg m^{-1}, is attached rigidly at both ends and placed under tension $T = 1$ N. At a position x along the wire, it is displaced a distance $y(x,t)$ from equilibrium at time t. The motion of the wire is governed by the wave equation

$$\frac{\mu}{T}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}$$

(a) The following is a Taylor expansion to third order for a function $u(x)$

$$u(x+\Delta x) = u(x) + \Delta x u'(x) + \frac{\Delta x^2}{2!}u''(x) + \frac{\Delta x^3}{3!}u'''(x)$$

By considering $u(x+\Delta x)$ and $u(x-\Delta x)$, obtain an approximate expression for the second order derivative $u''(x)$. To what order of Δx does the error in this expression depend? **[6]**

(b) To obtain a solution to the wave equation, $y(x,t)$ can be represented on a grid $y_{m,n}$ and solved numerically. Show that the wave equation can be written as the following finite difference equation

$$y_{m,n+1} = r(y_{m+1,n} + y_{m-1,n}) + 2y_{m,n}(1-r) - y_{m,n-1}$$

where m, n are the grid indices for the variables x, t and

$$r = \frac{T}{\mu}\left(\frac{\Delta t}{\Delta x}\right)^2$$

[4]

(c) The wire is pulled a distance $y = 1$ from equilibrium at its midpoint and released from rest so that

$$y(x,0) = \begin{cases} 2x & \text{for } 0 < x < 1/2 \\ 2(1-x) & \text{for } 1/2 < x < 1 \end{cases}$$

A numerical solution based on the grid shown below (where $\Delta x = 0.25$, $\Delta t = 0.01$) can be obtained using these initial conditions by "marching forward in time". Explaining your methodology, fill the first 3 rows of this grid with the values of $y_{m,n}$ for $n = 0, 1, 2$.

t \ x	0.0	0.25	0.50	0.75	1.0
0.00					
0.01					
0.02					

As the wire is originally at rest you may assume that $y_{m,-1} = y_{m,0}$ **[10]**

END OF EXAMINATION