Exact Algorithms for the Feedback Arc Set Problem

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Table of Contents

A Recap

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- Let us first recap what feedback arc sets are.

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Definition (Feedback Arc Set)

Given a directed graph $G = (V, A)$, a feedback arc set $F \subseteq A$ is a set of arcs such that $G - F$ is acyclic.

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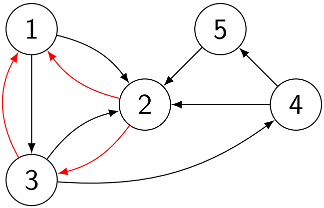

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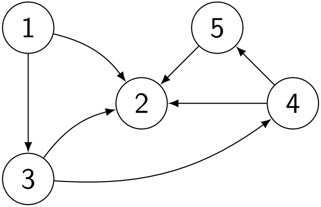

Recap: FEEDBACK ARC SET is Hard!

- From our previous presentation, we know that FEEDBACK ARC SET is *NP*-hard.
- So, unless $P = NP$, there do not exist polynomial time algorithms that solve FEEDBACK ARC SET exactly.
- Any exact algorithm for FEEDBACK ARC SET must contend with the fact that it might take exponential time on some input instances.

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- Understanding how to design such algorithms is the goal of this presentation.

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- First, a small discussion on brute-force search algorithms (bad running times but are a starting point for designing better algorithms).
- Next, modify one of these brute-force algorithms with dynamic programming (significantly better running time but exponential space).
- After that, fix the space complexity issue using divide-and-conquer (polynomial space but worse running time than dynamic programming).
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Brute-Force Algorithms for FEEDBACK ARC SET

- Why do we care about brute-force algorithms?
- Although brute-force algorithms usually have very bad running times and are only feasible on the smallest of input instances, they can often be a launchpad for more sophisticated exact algorithms.
- By understanding what makes brute-force algorithms inefficient, we can sometimes avoid unnecessary computation and end up with algorithms that have better guaranteed running times.
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NAIVEFEEDBACKARCSET(G)

Algorithm 1: NAIVEFEEDBACKARCSET(G)

input : A directed graph $G = (V, A)$
output: A smallest possible set $F \subseteq A$ such that $G - F$ is acyclic

$m \leftarrow \infty$;
$F^* \leftarrow \varnothing$;
foreach $F \subseteq A$ **do**
  $G' \leftarrow G - F$;
  if G' *is acyclic* **and** $|F| < m$ **then**
    $m \leftarrow |F|$;
    $F^* \leftarrow F$;
  end
end
return F^*

Running Time of NAIVEFEEDBACKARCSET(G)

- How bad is this algorithm?
- This algorithm always has to look at every possible subset of the arcs.
- Therefore, this algorithm has a running time of $\mathcal{O}^*(2^m)$ where m is the number of arcs in the graph. For dense graphs, $m = \Theta(n^2)$ and so, the running time is $\mathcal{O}^*(2^{n^2})$.
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A Better Brute-Force Algorithm

- We can come up with a slightly cleverer algorithm by using that fact that every directed acyclic graph has a topological ordering.

Definition (Topological Ordering)

A topological ordering is a permutation of the vertices in which for every arc (u, v), u comes before v in the permutation.

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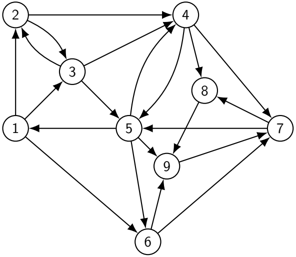

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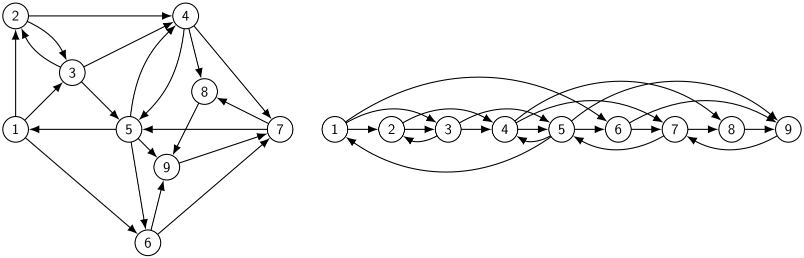

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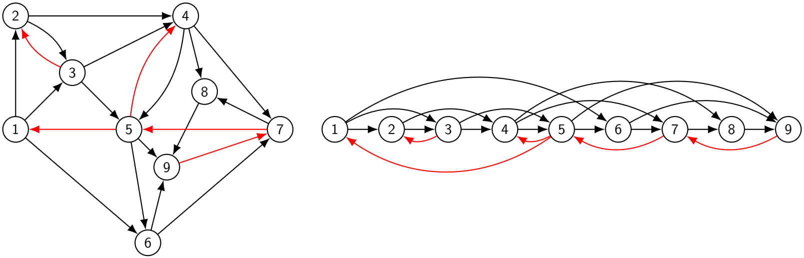

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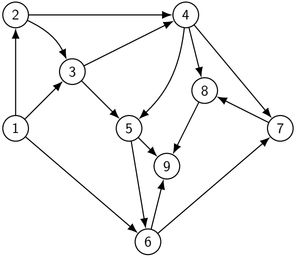

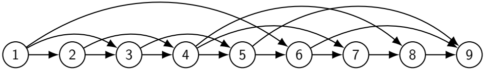

A Better Brute-Force Algorithm

More formally, we have the following theorem.

Theorem

Let $G = (V, A)$ be a directed graph with $V = \{v_1, v_2, \cdots, v_n\}$ and π be a permutation of the numbers $1, 2, \cdots, n$. Let $F = \{(v_{\pi(i)}, v_{\pi(j)}) \in A : \pi(i) > \pi(j)\}$. Then $G - F$ is acyclic.

A Better Brute-Force Algorithm

- This gives us the idea for another algorithm for Feedback Arc Set .
- For every possible permutation of the vertices, count the number of "backward" arc that results in.
- Pick a permutation that results in the least number of "backward" arcs.

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PermutationFeedbackArcSet(G)

Algorithm 2: PermutationFeedbackArcSet(G)

input : A directed graph $G = (V, A)$
output: A smallest possible set $F \subseteq A$ such that $G - F$ is acyclic

$m \leftarrow \infty$;
$F^* \leftarrow \varnothing$;
foreach *permutation* π *of the numbers* $1, 2, \cdots, |V|$ **do**
  $F \leftarrow \{(v_{\pi(i)}, v_{\pi(j)}) \in A : \pi(i) > \pi(j)\}$;
  if $|F| < m$ **then**
    $m \leftarrow |F|$;
    $F^* \leftarrow F$;
  end
end
return F^*

Analyzing PermutationFeedbackArcSet(G)

- The running time of PermutationFeedbackArcSet(G) is $\mathcal{O}^*(n!)$ (since it looks at every possible permutation of the vertices).
- Better than before but still not good enough.
- But it does offer us with a very important insight.

The Insight

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FEEDBACK ARC SET can be thought of as finding a permutation of the vertices with the minimum "cost".

- Therefore, we might be able to exploit techniques used in solving *other* optimal permutation or sequencing problems (The TSP for example).
- In particular, we consider the dynamic programming approach which has been very successful in solving such problems.

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A DP Algorithm for Feedback Arc Set

- As promised, now we are going to use the insight from the previous slides and give a dynamic programming algorithm for the Feedback Arc Set problem.
- We are essentially going to mimic the idea used in the classic Held-Karp algorithm (1962) for solving the Traveling Salesperson Problem [1].

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- We are essentially going to mimic the idea used in the classic Held-Karp algorithm (1962) for solving the TRAVELING SALESPERSON PROBLEM [1].

Setting Up the Dynamic Program

- Let us first formally write down what we want.
- We have a directed graph $G = (V, A)$ with $V = \{v_1, v_2, \cdots, v_n\}$.
- What we want is a permutation of the vertices that minimizes the number of "backward" arcs.
- In other words, we want a permutation π of the numbers $1, 2, \cdots, n$ that minimizes the cardinality of the following set.

$$F = \{(v_{\pi(i)}, v_{\pi(j)}) \in A : \pi(i) > \pi(j)\}$$

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- To design a dynamic programming algorithm, we must first define the sub-problems.
- In our formulation, we will have one sub-problem per each subset of the vertices.

The Sub-problems

For every non-empty $S \subseteq V$, let $OPT[S]$ be the size of a minimum feedback arc set of the graph induced by the vertices of S.

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- We now know that the value of $OPT[S]$ corresponds to a permutation of the vertices in S that results in the least number of backward arcs.
- By conditioning on the *last* vertex that appears in such a permutation, we can express the value of $OPT[S]$ in the following way.

The Recurrence

$$OPT[S] = \min_{v \in S}\{OPT[S - \{v\}] + c(v, S - \{v\})\}$$

where $c(v, S - \{v\})$ is the number arcs going from v to $S - \{v\}$.

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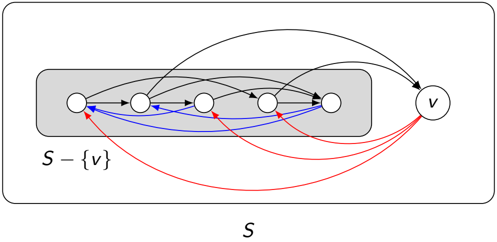

Figure 1: Number of blue arcs $= OPT[S - \{v\}]$, number of red arcs$=c(v, S - \{v\})$.

Setting Up the Dynamic Program

- The size of a minimum feedback arc set of the graph is therefore $OPT[V]$.
- A recurrence like the one shown in the previous slide can be transformed into a dynamic programming algorithm by solving sub-problems in order of their sizes.
- The following algorithm can be attributed to Lawler (1964) [2].

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DP-FEEDBACKARCSET(G)

Algorithm 3: DP-FEEDBACKARCSET(G)

input : A directed graph $G = (V, A)$
output: The size of a smallest possible set $F \subseteq A$ such that $G - F$ is acyclic

foreach $v \in V$ **do**
  $OPT[\{v\}] \leftarrow 0$;
end
for $i \leftarrow 2$ **to** n **do**
  foreach $S \subseteq V$ *with* $|S| = i$ **do**
    $OPT[S] \leftarrow \min_{v \in S}\{OPT[S - \{v\}] + c(v, S - \{v\})\}$;
  end
end
return $OPT[V]$

Analyzing the Running Time

- The recipe to analyzing the running time of any dynamic programming algorithm is simple.
- We first count the number of sub-problems.
- Then we tally up the work done per sub-problem.

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Analyzing the Running Time

- The number of sub-problems of size k is $\binom{n}{k}$.
- Given the adjacency matrix of the graph, a sub-problem of size k can be solved in $O(k^2)$ time.
- Therefore, the running time of our algorithm is:
- $O\left(\sum_{k=0}^{n} k^2\binom{n}{k}\right) = O\left(\sum_{k=0}^{n}\left(n\binom{n-1}{k-1} + n(n-1)\binom{n-2}{k-2}\right)\right) = O(n^2 2^n) = O^*(2^n)$.

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Analyzing the Running Time

Theorem

DP-FEEDBACKARCSET(G) *runs in* $\mathcal{O}^*(2^n)$ *time.*

Analyzing the Space Complexity

- This is a significant improvement!
- This algorithm has a downside, however.
- The OPT table has an entry for each subset of V.
- Therefore, the space complexity of DP-FEEDBACKARCSET(G) is $\Omega(2^n)$.

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Analyzing the Space Complexity

- Ideally, we want our algorithms to use only polynomial space.
- So, next we will see an algorithm that uses only polynomial space.
- This reduction in space complexity is not free, however.
- We are going to have to contend with a larger running time in exchange for it.

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Trading Time for Space: Divide and Conquer!

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- The idea is to use another very versatile algorithm design paradigm: **divide and conquer**.

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- For a set $S \subseteq V$, let $OPT(S)$ be the number of backward arcs in an optimal permutation of the vertices in S.
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- The idea is to condition on the first half of the vertices in an optimal permutation of S.

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The Recurrence!

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$$OPT(S) = \min_{\substack{S' \subseteq S \\ |S'| = \left\lceil \frac{|S|}{2} \right\rceil}} \{OPT(S') + OPT(S - S') + c(S - S', S')\}$$

where $c(S - S', S)$ is the number of arcs going from $S - S'$ to S'.

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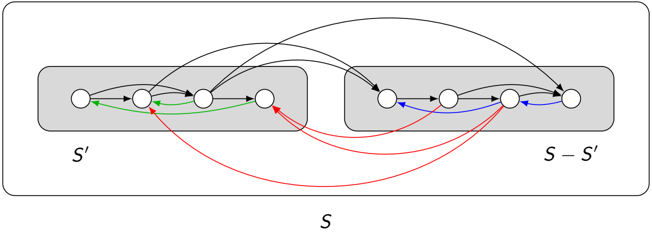

Figure 2: $OPT(S')$ =number of green arcs, $OPT(S - S')$ =number of blue arcs, $c(S - S', S')$ =number of red arcs.

Divide-and-Conquer for Feedback Arc Set

- The size of a minimum feedback arc set is $OPT(V)$.
- We can now use the recurrence from the previous slide to design a recursive algorithm for the Feedback Arc Set problem.

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D&C-FEEDBACKARCSET(G)

Algorithm 4: D&CFEEDBACKARCSET(G)

input : A directed graph $G = (V, A)$
output: The size of a smallest possible set $F \subseteq A$ such that $G - F$ is acyclic

Function $OPT(S)$
  if $|S| = 1$ **then**
    return 0
  end
  return $\min_{\substack{S' \subseteq S \\ |S'| = \lceil \frac{|S|}{2} \rceil}} \{OPT(S') + OPT(S - S') + c(S - S', S')\};$
end
return $OPT(V)$

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- This algorithm requires only polynomial space.
- This is because on each recursion level, we use only polynomial space and the depth of the recursion tree is lg n.

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$$OPT(S) = \min_{\substack{S' \subseteq S \\ |S'| = \lceil \frac{|S|}{2} \rceil}} \{OPT(S') + OPT(S - S') + c(S - S', S')\}$$

- For a fixed subset S of size k, the number of subsets S' of S that we have try is bounded above by 2^k.
- After fixing such an S', we must then compute $c(S - S', S')$. Given the adjacency matrix of the graph, this takes $O(k^2)$ time.
- If $T(n)$ is the running time on a graph with n vertices, then:

The Running Time

$$T(n) \leq 2^n \left(T\left(\left\lceil \frac{n}{2} \right\rceil \right) + T\left(\left\lfloor \frac{n}{2} \right\rfloor \right) + cn^2 \right)$$

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&\approx 2^n \cdot 2T\left(\frac{n}{2} \right) + 2^n cn^2
\end{aligned}
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Approximating *floor* and *ceiling* to *exact* value.

Analyzing the Time Complexity

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&\quad \cdots + 2^{n+\frac{n}{2}+\cdots+\frac{n}{2^{\lg n}}} \cdot 2^{\lg n} c \left(\frac{n}{2^{\lg n}}\right)^2
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Expanding and using the fact that $\lg n$ substitutions are needed to reach $T(1)$.

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\end{aligned}
$$

There are $\lg n + 1$ terms containing cn^2 and $2^{n+\frac{n}{2}+\cdots+\frac{n}{2^{\lg n}}} cn^2$ is greater than any of those.
(Note: We are being a bit loose, as a tighter analysis will not result in a better $\mathcal{O}^*$ complexity.)

Analyzing the Time Complexity

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&< 2^{n+\frac{n}{2}+\cdots+\frac{n}{2^{\lg n}}} \cdot 2^{\lg n}\left(2T(1) + cn^2(\lg n + 1)\right)\\
&= 2^{n(1+\frac{1}{2}+\frac{1}{4}+\cdots)} n\left(2T(1) + cn^2(\lg n + 1)\right)
\end{aligned}$$

Replacing the finite sum with an infinite sum.

Analyzing the Time Complexity

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&= \mathcal{O}^*(4^n)
\end{aligned}
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Analyzing the Running Time

Theorem

D&C-FEEDBACKARCSET(G) *runs in* $\mathcal{O}^*(4^n)$ *time.*

Product of Time and Space

- The time and space complexities of our divide-and-conquer algorithm are $\mathcal{O}^*(4^n)$ and $\mathcal{O}^*(1)$.
- The time and space complexities of our dynamic programming algorithm are $\mathcal{O}^*(2^n)$ and $\mathcal{O}^*(2^n)$.
- In both cases, $\mathit{TIME} \times \mathit{SPACE} = \mathcal{O}^*(4^n)$.
- The dynamic programming algorithm saves a lot of time in exchange for space.
- The divide and conquer algorithm goes to the other extreme and uses only a polynomial amount of space but does worse in the time complexity department.

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Product of Time and Space

- It is possible to try a hybrid of both dynamic programming and divide-and-conquer and get a balance of both space and time.
- The idea is to start with divide and conquer first, stop as soon as the sub-problem sizes drop below a certain amount and use dynamic programming after that.
- $TIME \times SPACE$ is still $\mathcal{O}^*(4^n)$ in this hybrid approach.
- However, using an idea by Koivisto and Parviainen (2010) [3], it is possible to get $TIME \times SPACE = \mathcal{O}^*(3.93^n)$.

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- The decision version of FEEDBACK ARC SET (when parameterized by the size of the feedback arc set desired) is fixed parameter tractable.
- The running time of the best known such algorithm is $O\left((k+1)! \cdot 4^k \cdot k^3 \cdot n(n+m)\right)$ [4] (2008) (where k is the size of the feedback arc set being asked for).
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Poly-time algorithms for Restricted Instances

- Now we are going to talk about the final topic of this presentation: poly-time algorithms for FEEDBACK ARC SET on restricted inputs.
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- The undirected version of FEEDBACK ARC SET, which we can aptly call FEEDBACK EDGE SET, is clearly in P.
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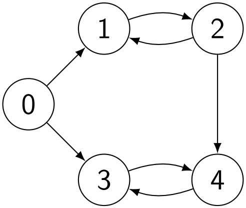
Given a directed graph $G = (V, A)$, a dijoin is a subset F of its arcs such that if we add the reversed version of all the arcs in F to G, G will be strongly-connected.



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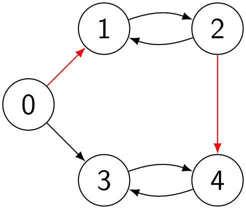

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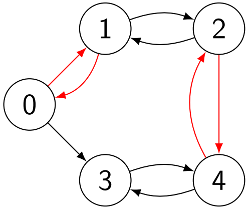

Poly-time algorithms for Restricted Instances

- The key to solving Feedback Arc Set on Planar Digraphs in polynomial time is the following two results:

Theorem

Finding a minimum dijoin of a directed graph can be done in polynomial time [6, 7, 8] (1981, 1995, 2005).

Theorem

Finding a minimum feedback arc set of a planar digraph is equivalent to finding a minimum dijoin of its dual [9].

- Combining these two results, we get a polynomial time algorithm for Feedback Arc Set on Planar Digraphs.

Poly-time algorithms for Restricted Instances

- The key to solving FEEDBACK ARC SET ON PLANAR DIGRAPHS in polynomial time is the following two results:

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Finding a minimum dijoin of a directed graph can be done in polynomial time [6, 7, 8] (1981, 1995, 2005).

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Finding a minimum feedback arc set of a planar digraph is equivalent to finding a minimum dijoin of its dual [9].

- Combining these two results, we get a polynomial time algorithm for FEEDBACK ARC SET ON PLANAR DIGRAPHS.

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