We now show that Algorithm PTAS_ISP is an $1+\epsilon$-approximation algorithm. Let O be a set of maximum independent set obtained by an oprimal algorithm. We can observe that V_i for $i = 0, \cdots, k$ is a partition of V. Then by peigion hole principle, there exists some $jin\{0, 1, \cdots, k$ such that V_j contains at most $\frac{|O|}{k+1}$ vertices in O. Then the following relation holds.

$$|O| \le |S_j| + |O \cap V_j|$$

Since $|S_m| \ge |S_j|$ and $|O \cap V_j| \le \frac{|O|}{k+1}$,

$$\begin{aligned}
|O| &\le |S_j| + |O \cap V_i| \\
&\le |S_m| + \frac{|O|}{k+1} \\
(k+1)|O| - |O| &\le (k+1)|S_m| \\
\frac{|O|}{S_m} &\le \frac{k+1}{k} \\
&= 1 + \frac{1}{k} \\
&\le 1 + \epsilon \\
\frac{OPT}{ALG} &\le 1 + \epsilon
\end{aligned}$$

4.4 The Traveling Salesman Problem (TSP)

4.4.1 Approximibility of TSP

We first present the inapproximability of Traveling Salesman Problem (TSP) as in the following theorem.

Theorem 4.1 *There is no constant factor approximation algorithm for Traveling Salesman Problem (TSP), unless P=NP.*

Proof Assume for a contradiction that there is an approximation algorithm with a constant approximation ratio ρ. We will show that we can design a polynomial time algorithm for solving Hamiltonian Cycle problem using the approximation algorithm for TSP. Let $G = (V, E)$ be a graph with n vertices. We construct an instance G' of TSP problem from G as follows. Let V corresponds to the set of cities and let $d(u, v)$ be the distance between two cities u and v in V. We define $d(u, v)$ as follows.

$$d(u,v) = \begin{cases} 1 & \text{if } (u,v) \in E \\ n\rho & \text{if } (u,v) \notin E \end{cases}$$

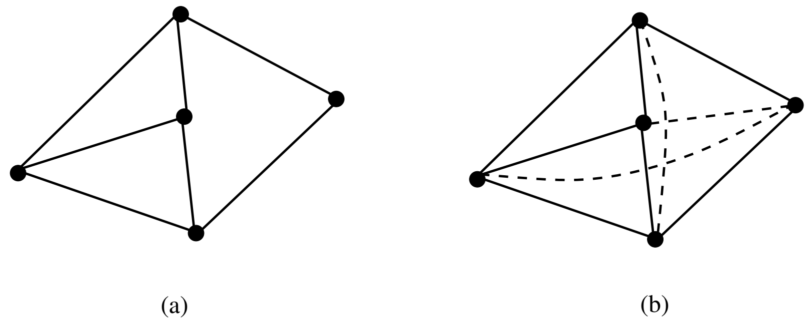

Fig. 4.5 (a) An instance of Hamiltonian cycle problem and (b) an instance of Traveling Salesman problem where each solid edge has weight 1 and each dotted edge has weight $n\rho$.

Then G' is a weighted complete graph on which we run the approximation algorithm for TSP. One can observe that if G has a Hamiltonian cycle, then the optimal solution for the TSP problem in G' is n, otherwise the optimal solution is larger than $n\rho$. Then G has a hamiltonian cycle if and only if the approximation algorith gives a solution of $n\rho$ or less. Thus we have solved the Hamiltonian cycle in polynomial time. But this is impossible since Hamiltonian cycle is NP-complete. □

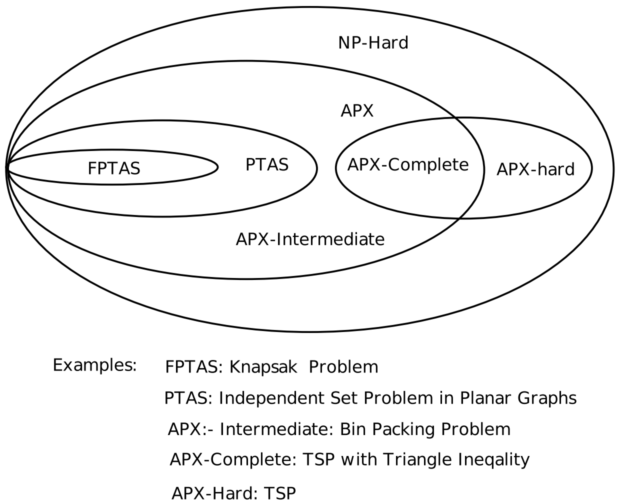

Fig. 4.6 Relationship among NP-hard, APX-hard, APX, PTAS and FPTAS

However a constant factor approximation can be achieved for a special case of TSP problem where every pair of vertices are connected and the distances between pair of cities satisfy the triangle inequality. That is, for any three cities c_u, c_v and c_w, we have

$$d((c_u, c_v)) + d((c_v, c_w)) \geq d((c_u, c_w))$$

.

4.4.2 2-Approximation Algorithm

The 2-approximation algorithm works in three steps. In the first step a minimum spanning tree T is constructed. In the second step, In the second step an Eulerian Circuit E, that is a traversal of T that starts and ends at the same city and traverses each edge exactly once in each direction, is constructed. In the third step a tour $\mathcal{T}$ from E is constructed by bypassing some vertices which have already been visited. Let $C(H)$ denote the total weight of the edges in a subgraph H of G. Let $\mathcal{T}^*$ be the optimal tour in G.

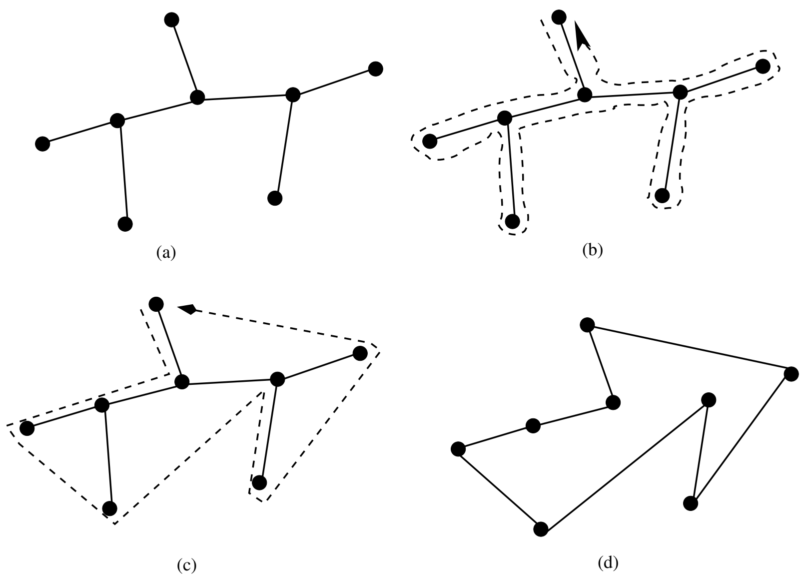

Fig. 4.7 Illustration for 2-approximation algorithm for TSP.

If we delete an edge from $\mathcal{T}^*$ we get a spannig tree of G. Thus

$$C(T) < C(\mathcal{T}^*).$$

Clearly $C(E) = 2C(T)$ since Eulerian circuit visits each edge of T exactly once in each direction. Triangule inequality ensures that weight does not increase by bypassing the edges. Hence

$$\begin{aligned} C(\mathcal{T}) &\le C(E) \\ &= 2C(T) \\ &< 2C(\mathcal{T}^*) \\ \frac{C(\mathcal{T})}{C(\mathcal{T}^*)} &< 2. \end{aligned}$$

Time complexity $O(nlogm)$

4.4.3 $\frac{3}{2}$-Approximation Algorithm

Christofides modified the Minimum Spanning Tree based algorithm to improve the approximation ratio from 2 to $\frac{3}{2}$ as follows. In the first step a minimum spanning tree T is constructed. To make an Eulerian circuit, Christofides did not double the edges of T. Instead a minimum weight perfect matching M of odd degree vertices in T is found. Then $T \cup M$ is a Eulerian circuit E. In the third step a tour $\mathcal{T}$ from E is constructed by bypassing some vertices which have already been visited.

Let $\mathcal{T}^*$ be the optimal tour in G.

If we delete an edge from $\mathcal{T}^*$ we get a spannig tree of G. Thus

$$C(T) < C(\mathcal{T}^*).$$

Let S be the set of odd degree vertices in T. Clearly $|S|$ is an even number.

$$C(E) = C(T) + C(M)$$

Again

$$C(\mathcal{T}) \le C(E)$$

$$\begin{aligned} C(\mathcal{T}) &\le C(E) \\ &= C(T) + C(M) \\ &< C(\mathcal{T}^*) + C(\mathcal{M}) \end{aligned}$$

Let $\mathcal{T}_{\mathcal{S}}^*$ be an optimal TSP tour among the vertices in S. Then clearly $\mathcal{T}_{\mathcal{S}}^* \le \mathcal{T}^*$. Since $|S|$ is even, $\mathcal{T}_{\mathcal{S}}^*$ can be partitioned into two perfect matichings M_1 and M_2 of S. Then $C(\mathcal{T}_{\mathcal{S}}^*) = C(M_1) + C(M_2)$.

Since M is a minimum weight perfect matching, $C(M) \le C(M_1)$ and $C(M) \le C(M_2)$. Hence

$$\begin{aligned} C(M) &\le \tfrac{1}{2}(C(M_1) + C(M_2)) \\ &= \tfrac{1}{2}C(\mathcal{T}_{\mathcal{S}}^*) \\ &\le \tfrac{1}{2}C(\mathcal{T}^*) \end{aligned}$$

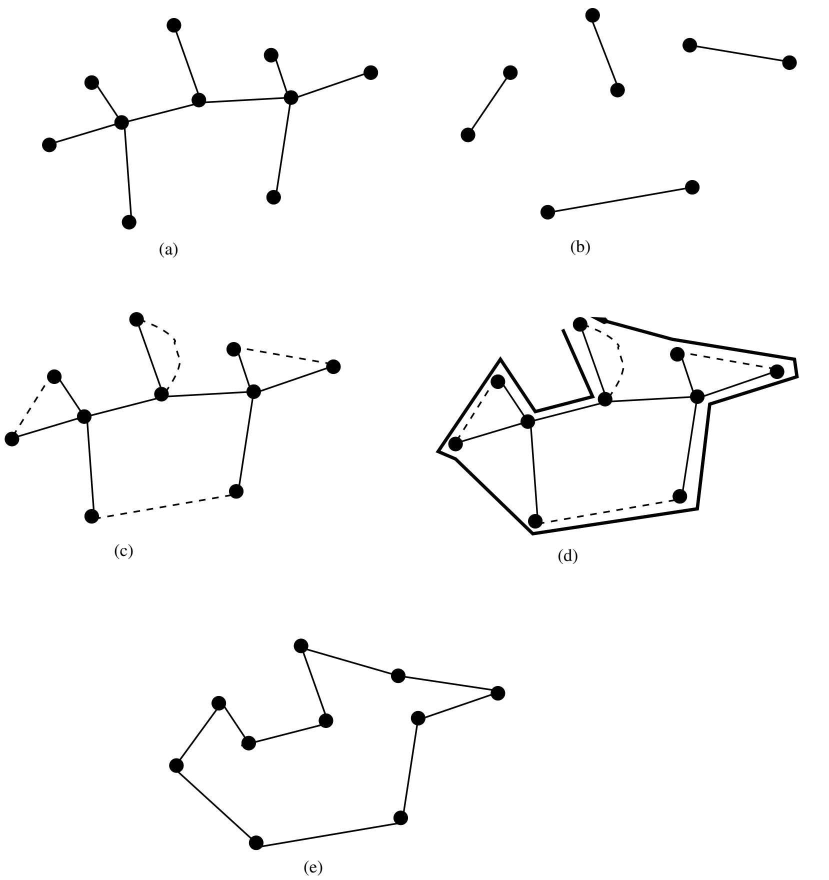

Fig. 4.8 Illustration for $\frac{3}{2}$-approximation algorithm for TSP.

$$
\begin{aligned}
C(\mathcal{T}) &< C(\mathcal{T}^*) + C(\mathcal{M}) \\
&\leq C(\mathcal{T}^*) + \tfrac{1}{2}C(\mathcal{T}^*) \\
&= \tfrac{3}{2}C(\mathcal{T}^*) \\
\frac{C(\mathcal{T})}{C(\mathcal{T}^*)} &< \tfrac{3}{2}.
\end{aligned}
$$

Time complexity $O(n^3)$ due to matching.