The Feedback Arc Set Problem

Problem Definition and Discussions on Hardness

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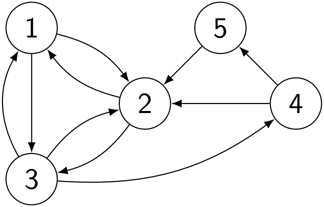

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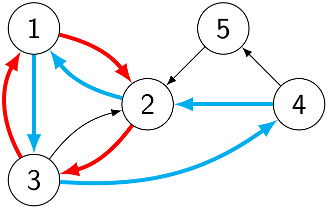

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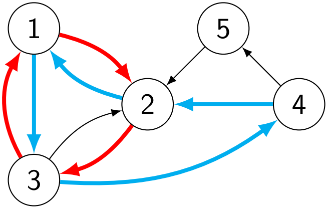

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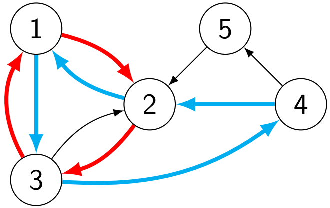

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What is a Feedback Arc Set?

A feedback arc set of a directed graph is a set of arcs whose removal leave the graph acyclic. More formally,

Definition (Feedback Arc Set)

Given a directed graph $G = (V, A)$, a feedback arc set $F \subseteq A$ is a set of arcs such that $G - F$ is acyclic.

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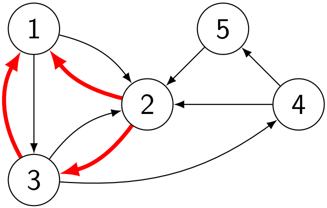

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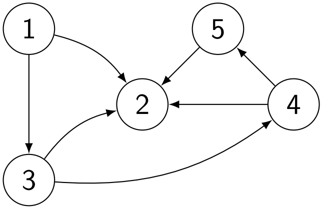

Problem Definition

- A natural thing we might want to do is the following:

Definition (The Feedback Arc Set Problem (Optimization Version))

Given a directed graph $G = (V, A)$, find a minimum cardinality feedback arc set of G.

- However, for this presentation, we will only be interested in the following decision version:

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Given a directed graph $G = (V, A)$ and an integer k, does G have a feedback arc set of size at most k?

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Motivations and Applications

- The first motivation for studying the feedback arc set problem is purely theoretical. It is an interesting problem in its own right.
- Has applications in combinatorial circuit design, where cycles can potentially lead to race conditions.
- Most problems that can be modeled as feedback vertex set problems can also be modeled as feedback arc set problems.
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Hardness Status of the Feedback Arc Set Problem

- The decision version the feedback arc set problem turns out to be *NP*-complete.
- One of the first 21 problems to be proved *NP*-complete (Karp, 1972)[1].
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Proof of *NP*-completeness

- To prove that feedback arc set is *NP*-complete, first we prove that it is in *NP*.
- Given a set (of size at most k) of arcs , we can check if it is a feedback arc set by first deleting the arcs and then running depth-first search to check if there are any cycles left in the graph.
- Clearly, this takes linear time. So, the feedback arc set problem is clearly in *NP*.

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- To do so, we show that the feedback arc set problem is at least as hard as the vertex cover problem.

Claim

$$\text{Vertex Cover} \leq_p \text{Feedback Arc Set}$$

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- In the vertex cover problem, we want to cover all the *edges* with *vertices*.
- In the feedback arc set problem, we want to cover all the *cycles* with *arcs*.
- So, in the reduction process, there should be some sort of correspondence between edges and cycles i.e. *the things we want to cover*, and vertices and arcs i.e. *the things that we want to cover with*.

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- An instance of the vertex cover problem is an undirected graph $G = (V, E)$ and an integer k.
- An instance of the feedback arc set problem is a directed graph $G' = (V', A')$ and an integer k'.
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Step 1 (Creation of "Vertex Arcs")

Split every vertex in G in two i.e. for each vertex $v \in V$, add two vertices v_0 and v_1 in G'. Then add the arc (v_0, v_1).

- We call the arc (v_0, v_1) the arc "corresponding to vertex v".

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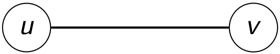

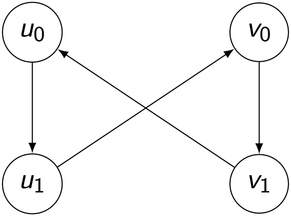

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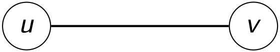

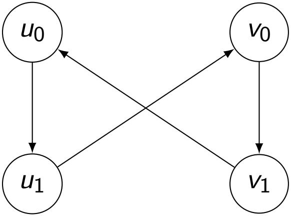

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Let $k' = k$.

Clearly, this construction is polynomial.

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- The proof is in two parts.
- First we show that if G has a vertex cover of size at most k, then G' has a feedback arc set of size at most k.
- Let S be a vertex cover of G with $|S| \leq k$.
- Then $F = \{(v_0, v_1) : v \in S\}$ is a feedback arc set of G'.
- If not, then $G' - F$ contains at least one cycle.
- That cycle uses some arc of the form (u_1, v_0).
- Any cycle that uses the arc (u_1, v_0) has to use both of the arcs (u_0, u_1) and (v_0, v_1).
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- Now we prove the opposite direction. If G' contains a feedback arc set of size at most k, then G contains a vertex cover of size at most k.
- Let F be a feedback arc set of G' with $|F| \leq k$.
- Without loss of generality, F only contains arcs of the form (v_0, v_1).
- This is because any cycle that an arc of the form (u_1, v_0) participates in also contains the arc (v_0, v_1). So, deleting the arc (v_0, v_1) instead of the arc (u_1, v_0) removes at least as many cycles (if not more).

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- The vertices corresponding to the arcs in F constitute a vertex cover in G.
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Feedback Arc Set is *NP*-complete!

Theorem

The feedback arc set problem is NP-complete.

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- We now know the feedback arc set problem is *NP*-complete on general graphs.
- But what if we restrict our attention only to special classes of graphs?.
- Does the problem become any easier?

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- Tournaments are orientations of complete graphs.
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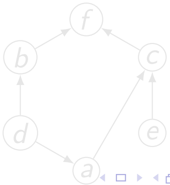
Feedback Arc Set Problem on Directed Line Graphs

- Now we turn our attention to a different class of directed graphs called directed line graphs.

Definition (Directed Line Graphs)

Given a directed graph G, we compute its directed line graph $\overline{G}$ as follows: first for every arc $a = (u, v)$ of G, we add a vertex labelled (u, v) in $\overline{G}$. Then we add an arc between the two vertices (u, v) and (w, x) of $\overline{G}$ if and only if $v = w$.



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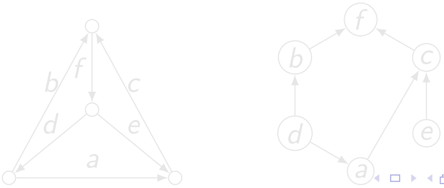

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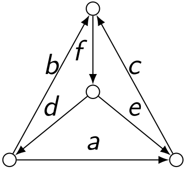

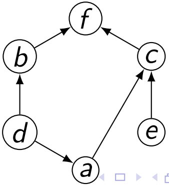

Feedback Arc Set Problem on Directed Line Graphs

- Clearly, not all directed graphs are directed line graphs.
- However, it is possible to show that even if we restrict our attention to only directed line graphs, the feedback arc set problem remains *NP*-complete (Gavril, 1977)[2].
- This is because of the following chain of reductions.

Feedback Arc set is *NP*-hard on directed line graphs

$$\text{FAS} \leq_p \text{FVS ON DIRECTED LINE GRAPHS} \leq_p \text{FAS ON DIRECTED LINE GRAPHS}$$

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- We end this section with a positive result.
- If we restrict our instances to only *planar* directed graphs, the feedback arc set problem can be solved in polynomial time (Lucchesi, 1976) [3].

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Hardness Results Summary

We summarize the results on hardness in the following table.

Problem	Status	Comments
FAS	in *NP*-complete	Reduces from vertex cover. See Karp (1972).
FAS on Undirected Graphs	in *P*	Reduces to finding spanning trees.
FAS on Tournaments	*NP*-complete	See Charbit, Thomasse and Yeo (2007).
FAS on Directed Line Graphs	*NP*-complete	See Gavril (1977)[2]
FAS on Graphs with Clique Number ≤ 3	*NP*-complete	Follows from the fact that directed line graphs have clique number at most 3.
FAS on planar directed graphs	in *P*	See Lucchesi (1976).

Problems that Feedback Arc Set Reduces To

- Now we take a look at problems that can be reduced from feedback arc set.
- We have already seen such a reduction, namely of the reduction of feedback arc set to the closely related problem feedback vertex set.

Result

$$\text{FEEDBACK ARC SET} \leq_p \text{FEEDBACK VERTEX SET}$$

- However, we could not find any other problem that feedback arc set naturally reduces to. This is because while doing reductions, FAS is not usually people's weapon of choice.

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