The Travelling Salesman Problem

October 2, 2020

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Imagine that you're a salesperson. You start from home. Along the way, you have to make a number of stops before you end up back home. However, the order in which you make the stops matters. The problem of finding the optimal order here is what's known as **the travelling salesman problem**.

Problem Definition

- **Input** : A complete undirected graph with non-negative edge costs.
- **Output** : A minimum cost tour i.e. a cycle that visits all the vertices exactly once.

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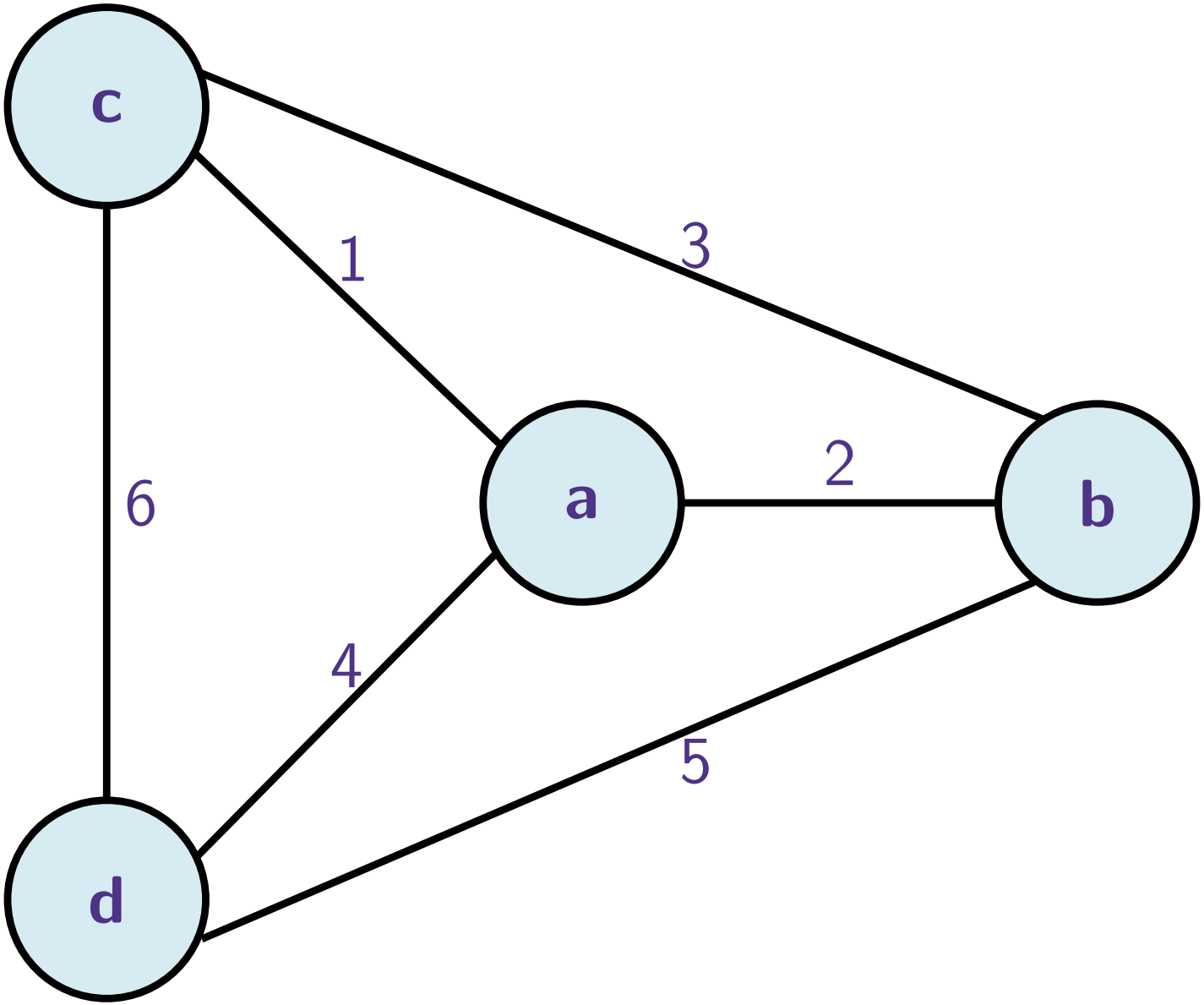
Naive Algorithm



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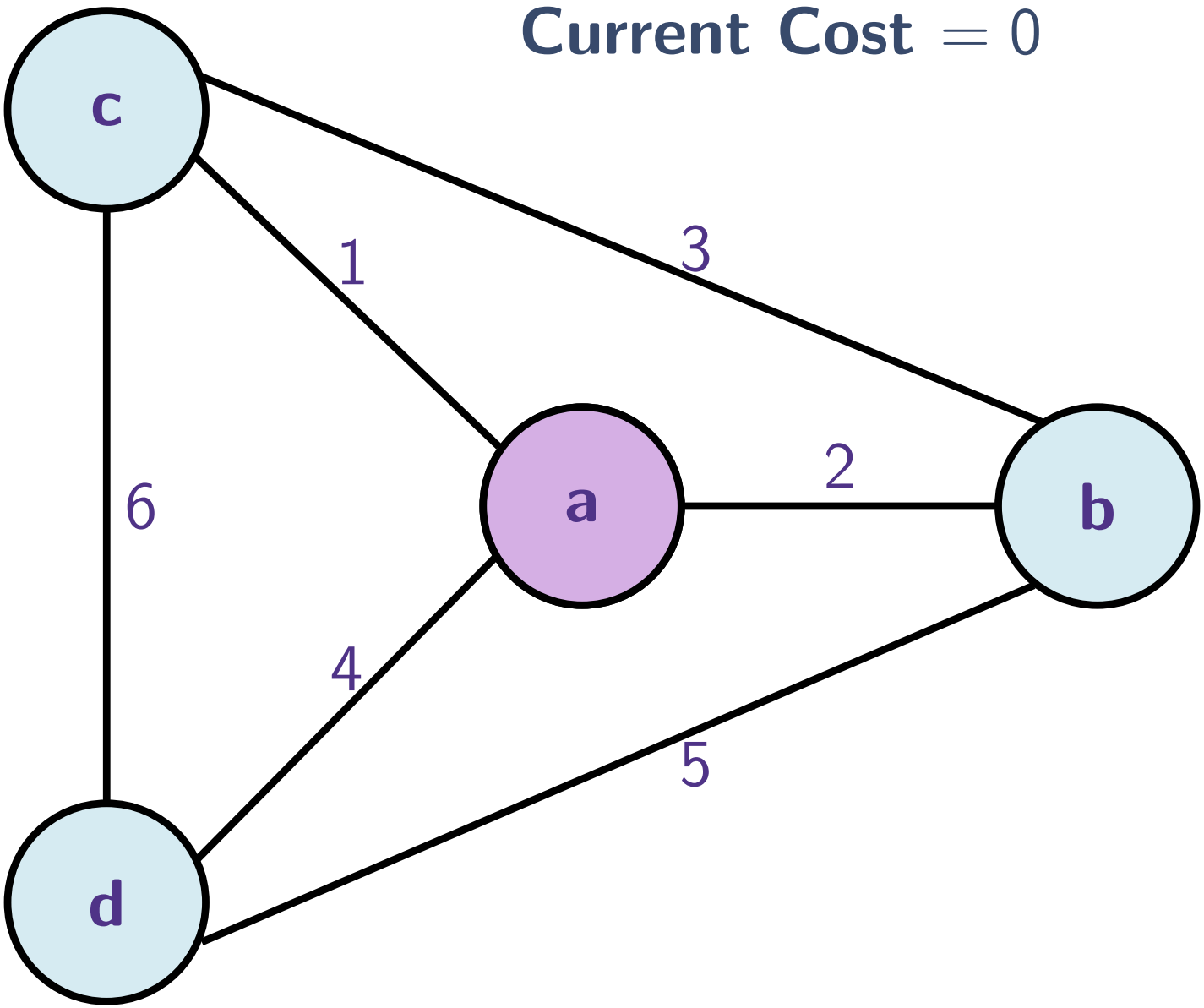

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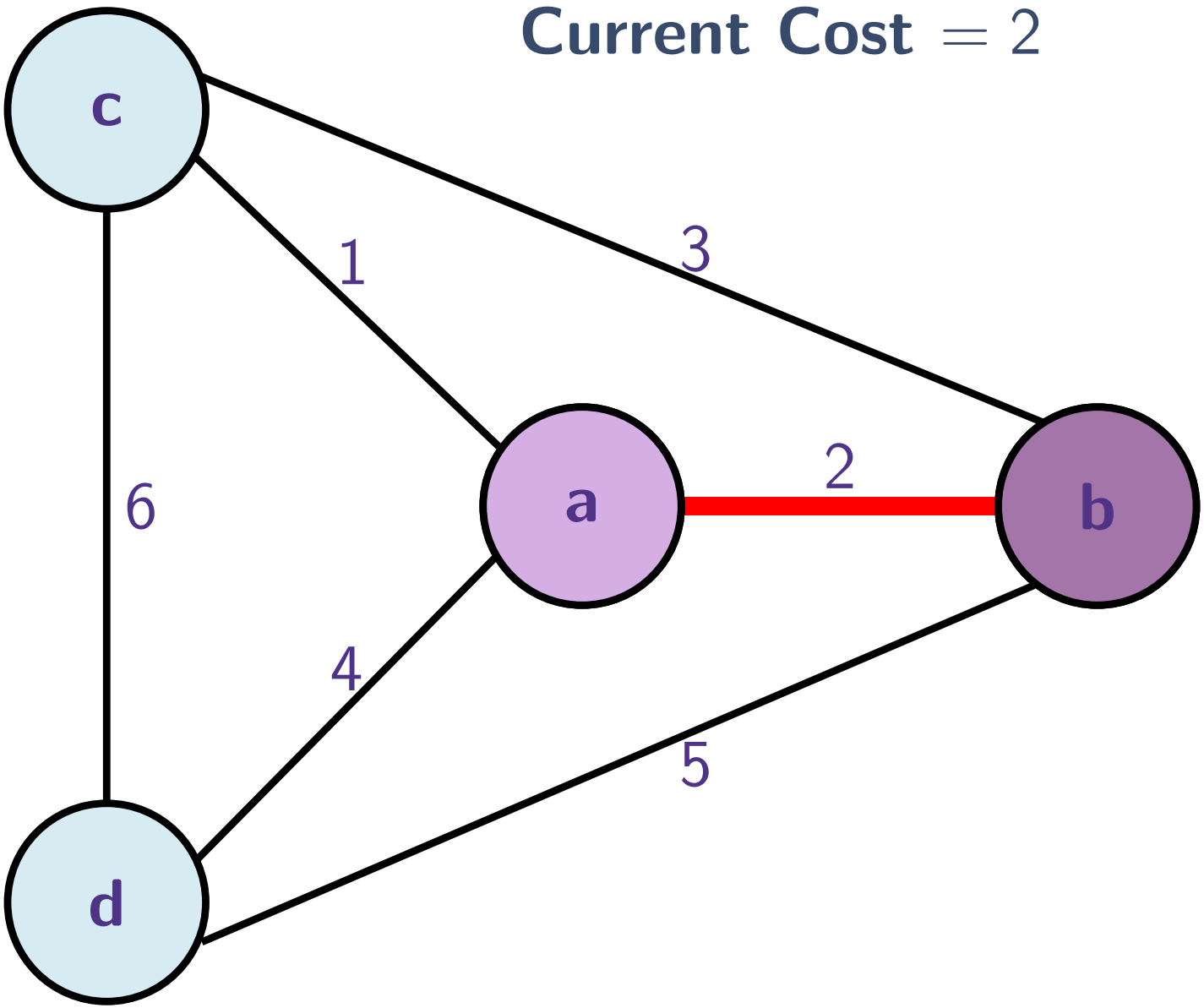

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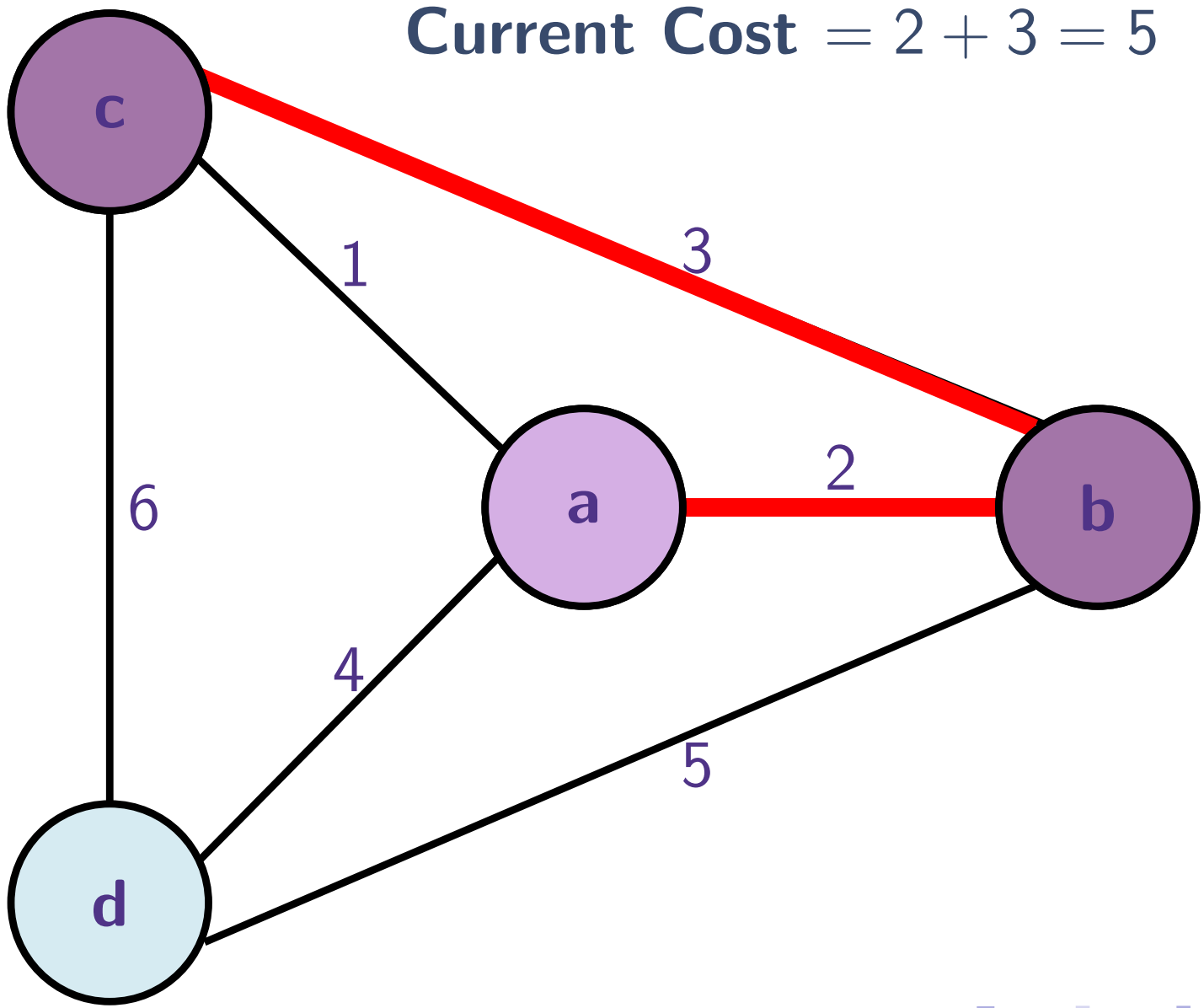

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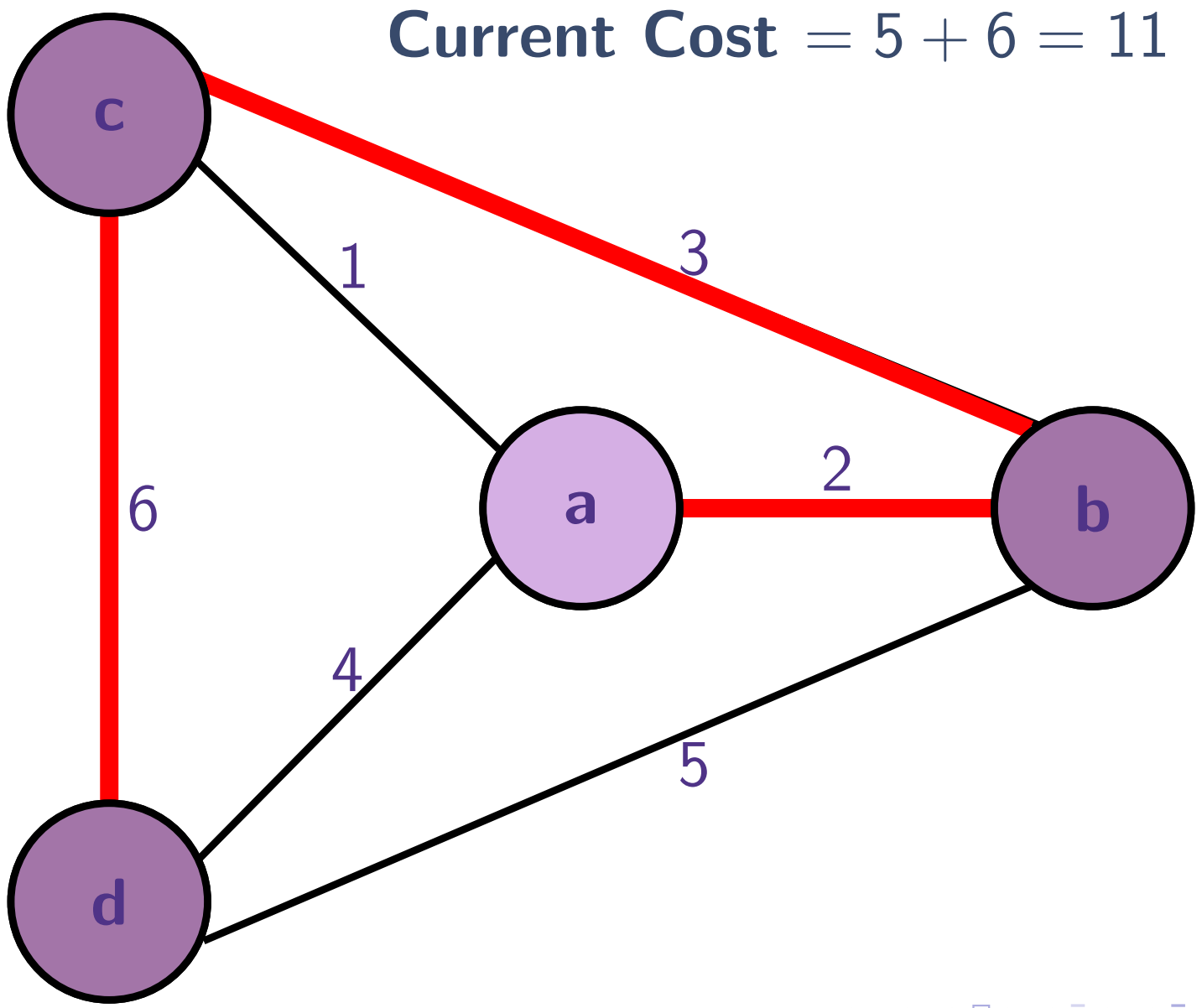

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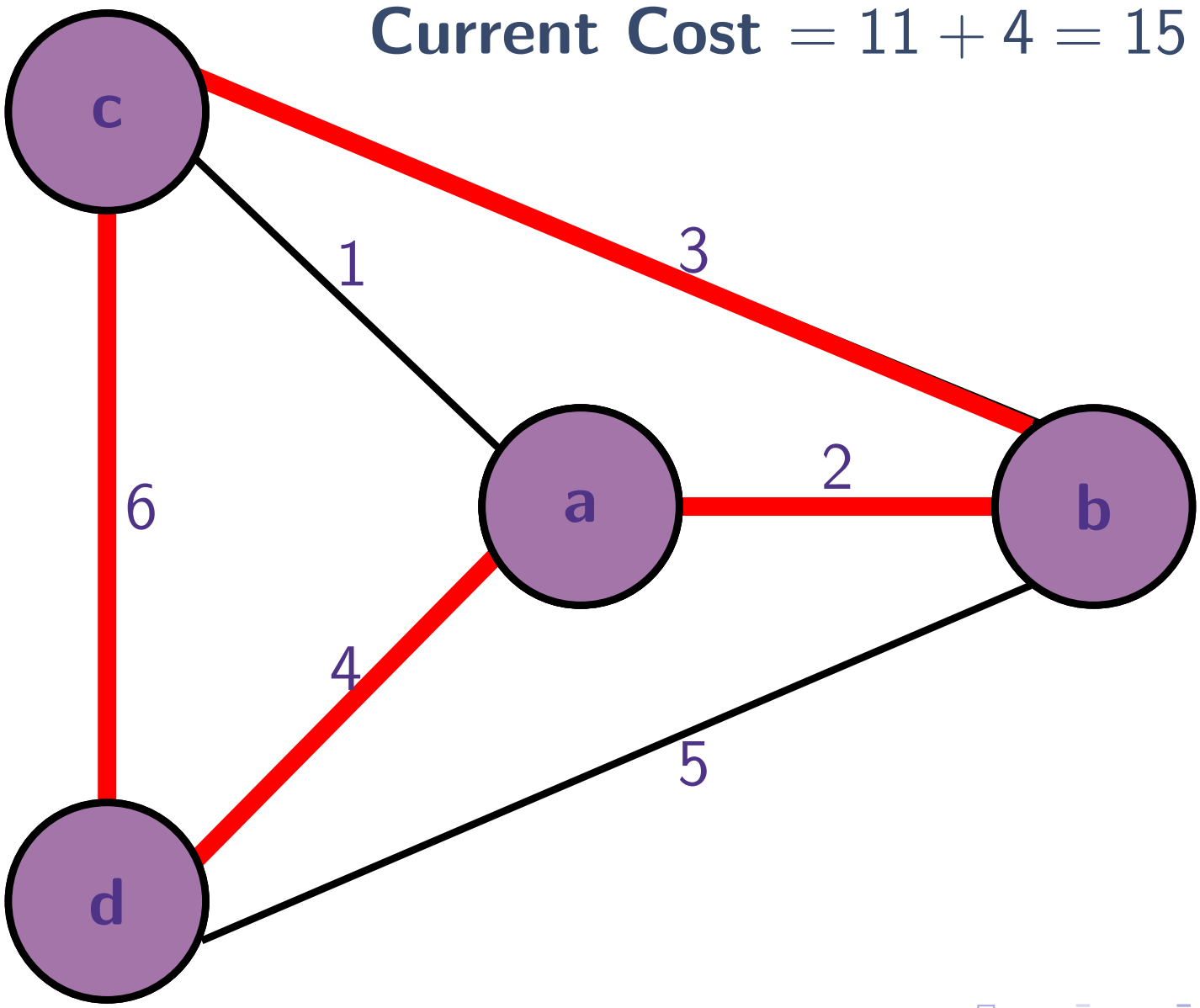
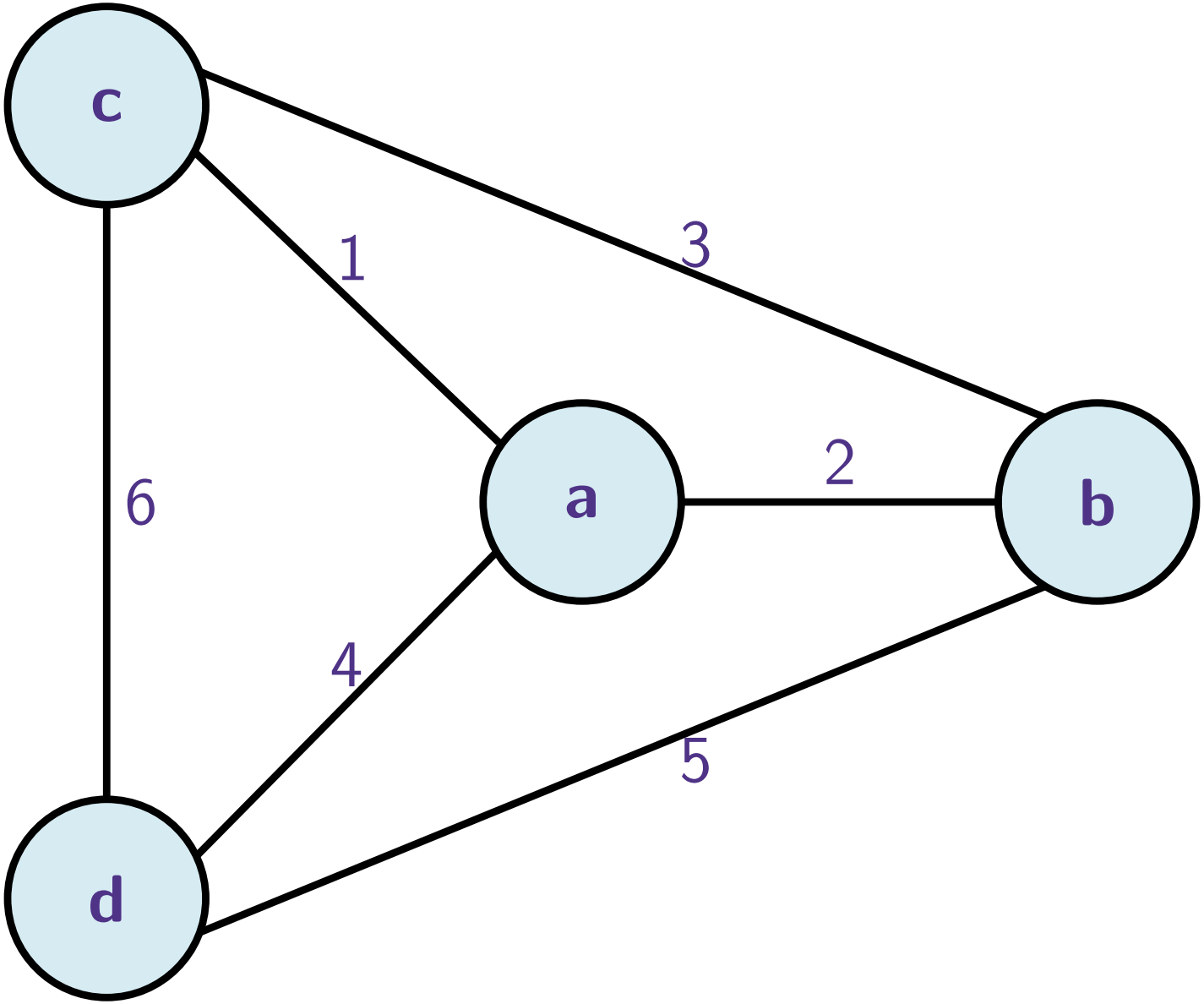


Another Tour



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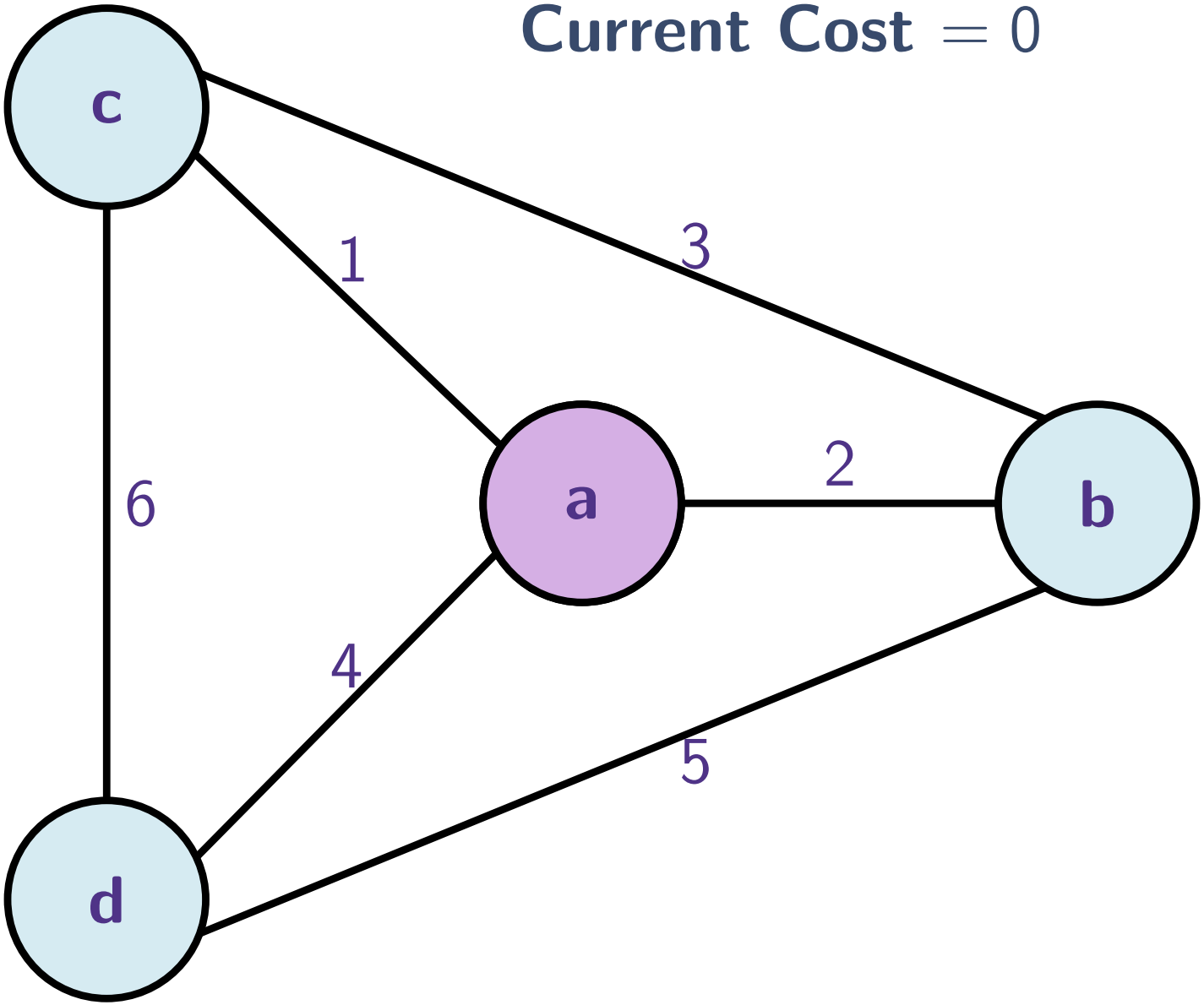

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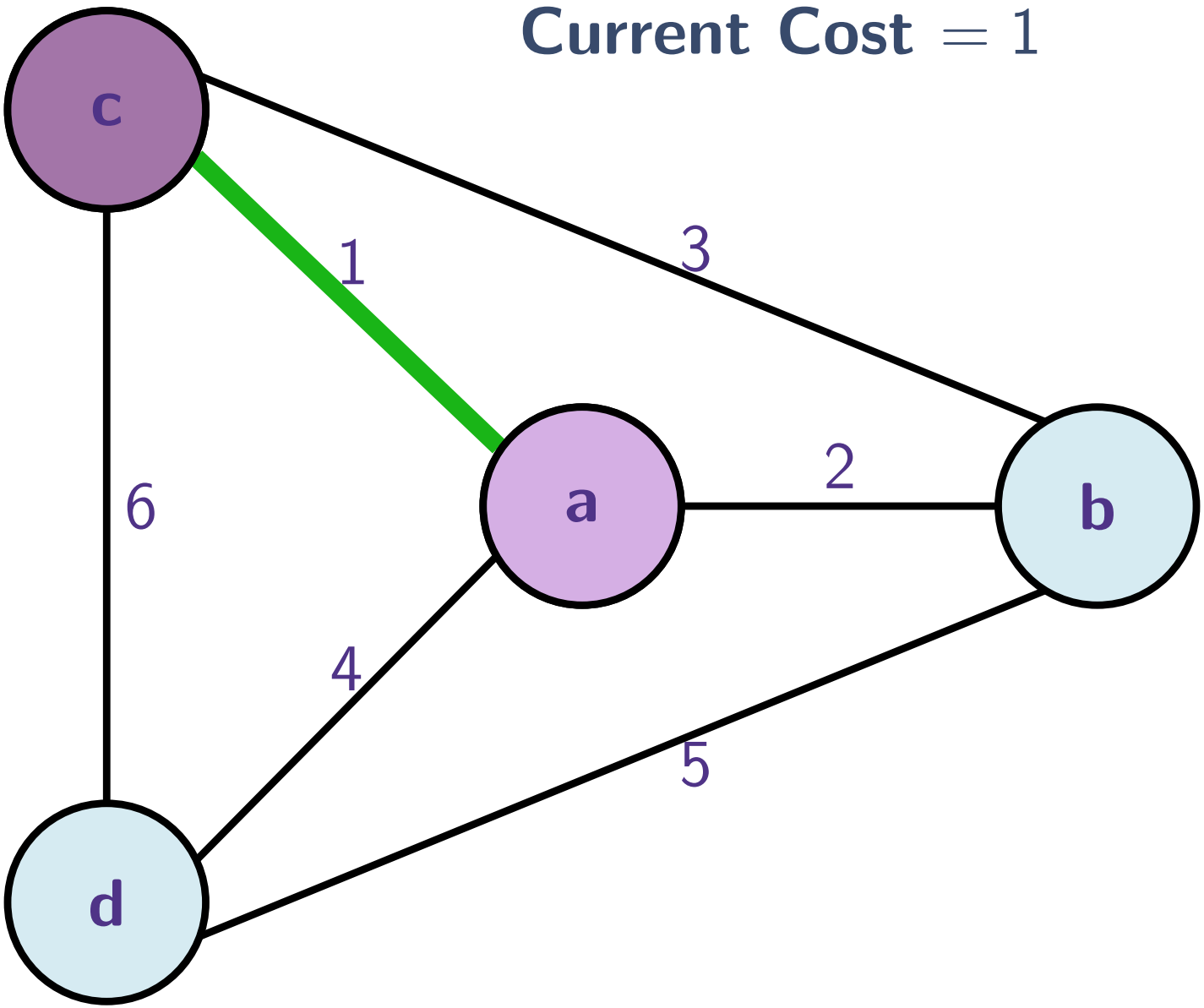

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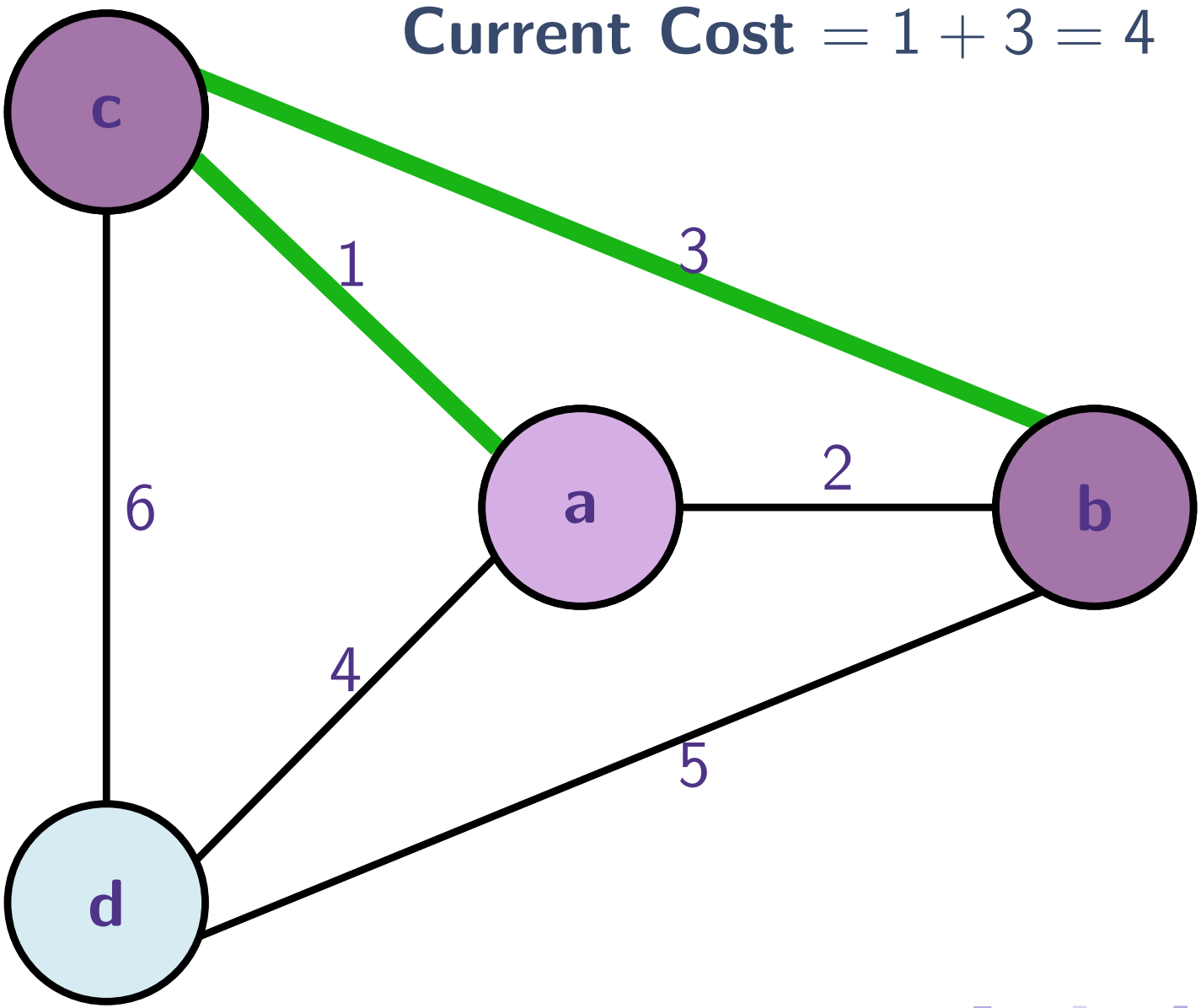

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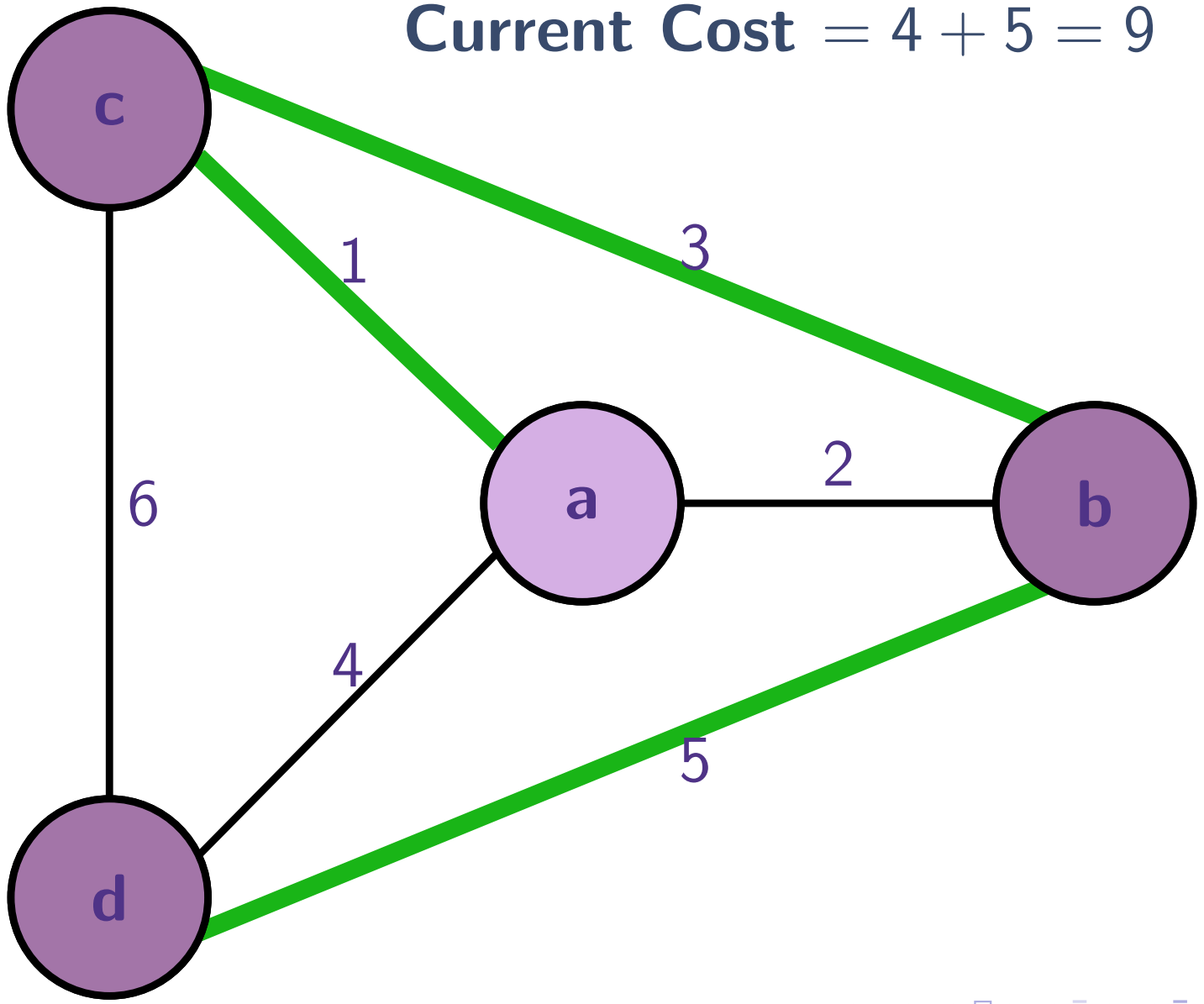

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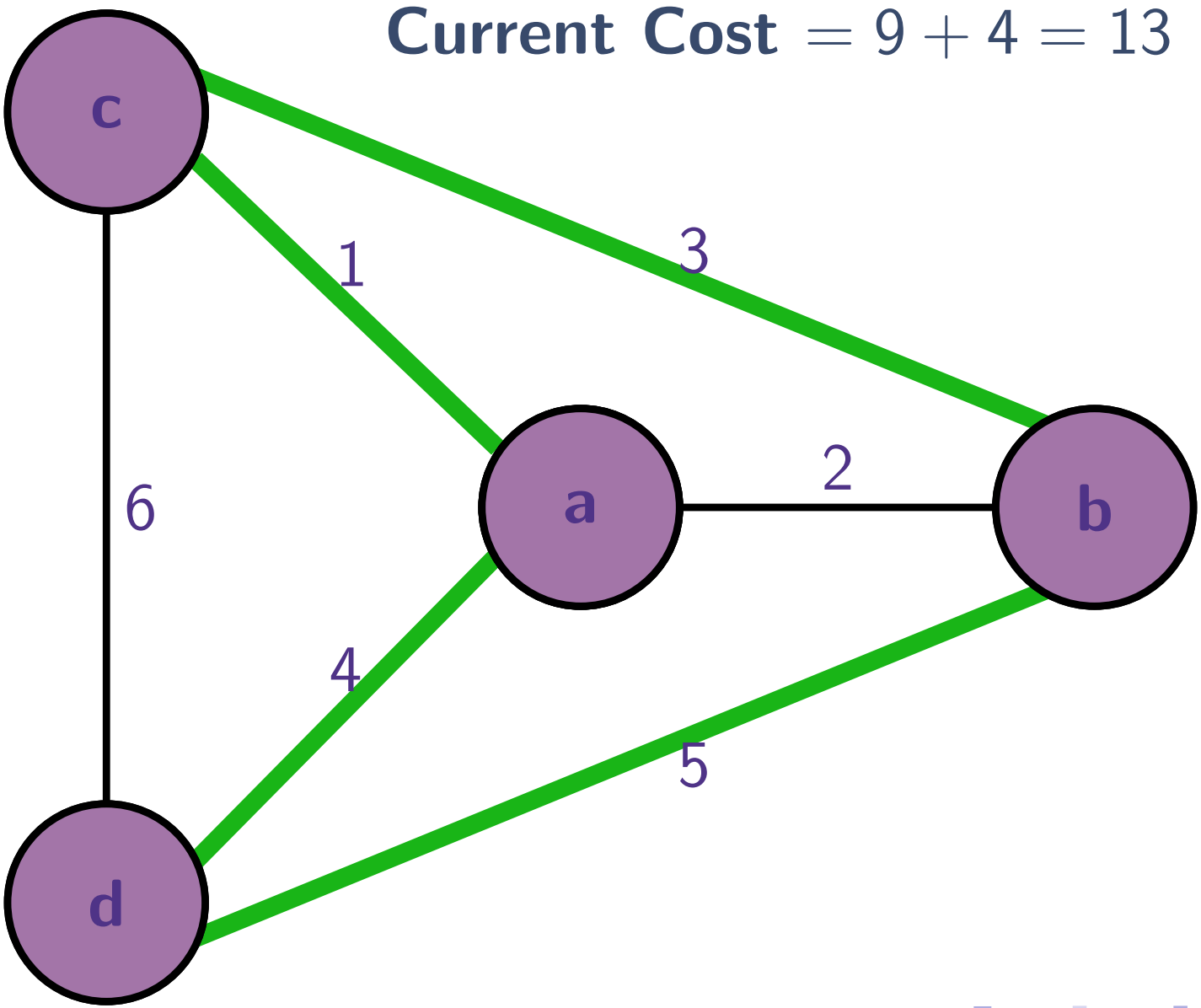

Travelling Salesman Problem

A Question

How do you find such a tour?

Brute Force Algorithm

The Brute Force approach:

- Look at all possible tours in the graph.
- Compute their costs.
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Brute Force Algorithm Complexity

However

- In a graph on n vertices, there are (n-1)! such tours.
- Computing the cost of a tour takes linear time.
- **Brute-Force Algorithm running time:**
  number of tours $\times$ cost of computing one tour
  $= (n-1)! \times O(n) = O(n!)$

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An Efficient Algorithm?

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Can we do better?

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We can. But a polynomial time algorithm doesn't seem likely.

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- Edmonds' Conjecture equivalent to $P \neq NP$.

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- Edmonds' Conjecture equivalent to $P \neq NP$.
- *The Traveling Salesman Problem* is NP-Complete!

Coping with NP-Completeness

We can

- Solve TSP exactly, but take a really long time for it.
- Solve it only approximately, but do it fast.
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Hard to even approximate

There is a catch.

Inapproximability of TSP

Unless P $=$ NP, there does not exist a polynomial time α approximation algorithm for travelling salesman problem.

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Metric TSP

Triangle inequality holds shortest path between vertices = the one hop path between them.

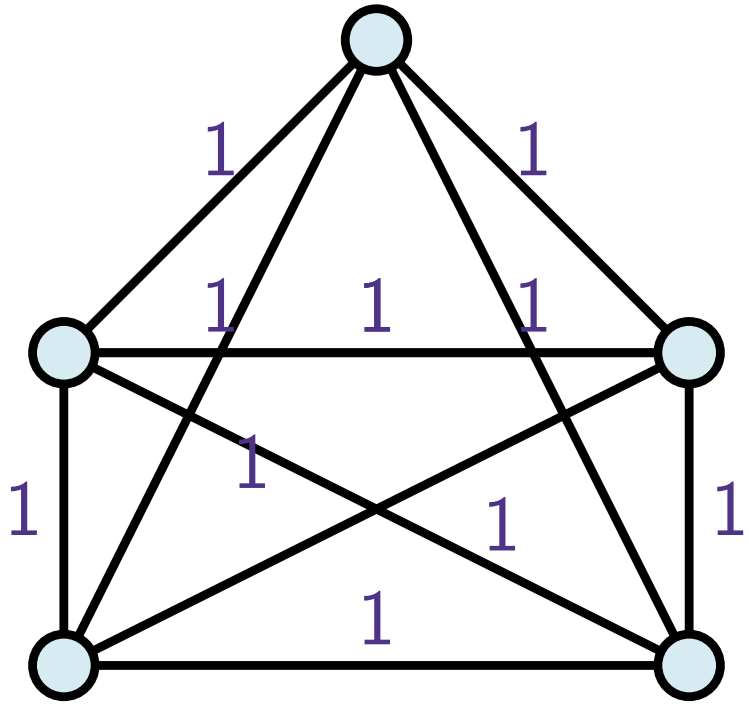

Figure: A Metric TSP instance.

Approximation algorithm for metric TSP

- Still a NP-Complete.
- But there are good approximation algorithms.
  - The MST Heuristic
  - Christofide's Algorithm

Acknowledgement

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