Learning Objectives:

- Introduce the notion of *polynomial-time reductions* as a way to relate the complexity of problems to one another.
- See several examples of such reductions.
- 3SAT as a basic starting point for reductions.

14
Polynomial-time reductions

Consider some of the problems we have encountered in Chapter 12:

1. The *3SAT* problem: deciding whether a given 3CNF formula has a satisfying assignment.
2. Finding the *longest path* in a graph.
3. Finding the *maximum cut* in a graph.
4. Solving *quadratic equations* over n variables $x_0, \ldots, x_{n-1} \in \mathbb{R}$.

All of these problems have the following properties:

- These are important problems, and people have spent significant effort on trying to find better algorithms for them.
- Each one of these is a *search* problem, whereby we search for a solution that is "good" in some easy to define sense (e.g., a long path, a satisfying assignment, etc.).
- Each of these problems has a trivial exponential time algorithm that involve enumerating all possible solutions.
- At the moment, for all these problems the best known algorithm is not much faster than the trivial one in the worst case.

In this chapter and in Chapter 15 we will see that, despite their apparent differences, we can relate the computational complexity of these and many other problems. In fact, it turns out that the problems above are *computationally equivalent*, in the sense that solving one of them immediately implies solving the others. This phenomenon, known as **NP** *completeness*, is one of the surprising discoveries of theoretical computer science, and we will see that it has far-reaching ramifications.

This chapter: A non-mathy overview

This chapter introduces the concept of a *polynomial time reduction* which is a central object in computational complexity and this book in particular. A polynomial-time reduction is a way to *reduce* the task of solving one problem to another. The way we use reductions in complexity is to argue that if the first problem is hard to solve efficiently, then the second must also be hard. We see several examples for reductions in this chapter, and reductions will be the basis for the theory of **NP** *completeness* that we will develop in Chapter 15.

All the code for the reductions described in this chapter is available on the following Jupyter notebook.

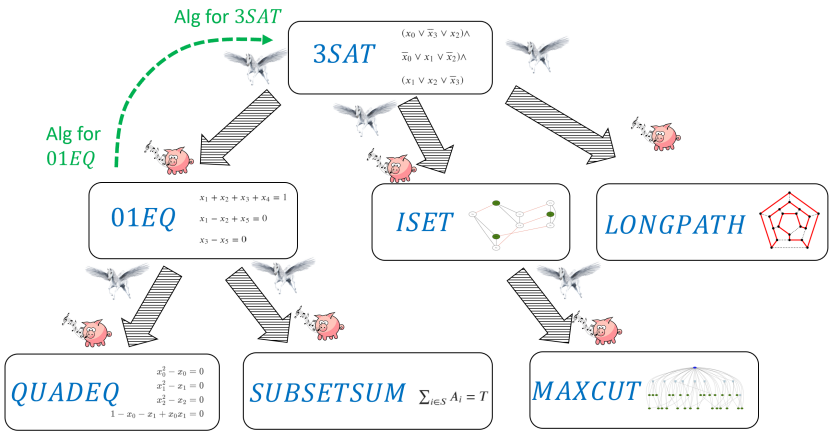

Figure 14.1: In this chapter we show that if the 3*SAT* problem cannot be solved in polynomial time, then neither can the *QUADEQ*, *LONGESTPATH*, *ISET* and *MAXCUT* problems. We do this by using the *reduction paradigm* showing for example "if pigs could whistle" (i.e., if we had an efficient algorithm for *QUADEQ*) then "horses could fly" (i.e., we would have an efficient algorithm for 3*SAT*.)

In this chapter we will see that for each one of the problems of finding a longest path in a graph, solving quadratic equations, and finding the maximum cut, if there exists a polynomial-time algorithm for this problem then there exists a polynomial-time algorithm for the 3SAT problem as well. In other words, we will *reduce* the task of solving 3SAT to each one of the above tasks. Another way to interpret these results is that if there *does not exist* a polynomial-time algorithm for 3SAT then there does not exist a polynomial-time algorithm for these other problems as well. In Chapter 15 we will see evidence (though no proof!) that all of the above problems do not have polynomial-time algorithms and hence are *inherently intractable*.

14.1 FORMAL DEFINITIONS OF PROBLEMS

For reasons of technical convenience rather than anything substantial, we concern ourselves with *decision problems* (i.e., Yes/No questions) or

in other words *Boolean* (i.e., one-bit output) functions. We model the problems above as functions mapping $\{0,1\}^*$ to $\{0,1\}$ in the following way:

3SAT. The *3SAT problem* can be phrased as the function $3SAT : \{0,1\}^* \to \{0,1\}$ that takes as input a 3CNF formula φ (i.e., a formula of the form $C_0 \wedge \cdots \wedge C_{m-1}$ where each C_i is the OR of three variables or their negation) and maps φ to 1 if there exists some assignment to the variables of φ that causes it to evalute to *true*, and to 0 otherwise. For example

$$3SAT\,("(x_0 \vee \overline{x}_1 \vee x_2) \wedge (x_1 \vee x_2 \vee \overline{x_3}) \wedge (\overline{x}_0 \vee \overline{x}_2 \vee x_3)") = 1 \quad (14.1)$$

since the assignment $x = 1101$ satisfies the input formula. In the above we assume some representation of formulas as strings, and define the function to output 0 if its input is not a valid representation; we use the same convention for all the other functions below.

Quadratic equations. The *quadratic equations problem* corresponds to the function $QUADEQ : \{0,1\}^* \to \{0,1\}$ that maps a set of quadratic equations E to 1 if there is an assignment x that satisfies all equations, and to 0 otherwise.

Longest path. The *longest path problem* corresponds to the function $LONGPATH : \{0,1\}^* \to \{0,1\}$ that maps a graph G and a number k to 1 if there is a simple path in G of length at least k, and maps (G,k) to 0 otherwise. The longest path problem is a generalization of the well-known Hamiltonian Path Problem of determining whether a path of length n exists in a given n vertex graph.

Maximum cut. The *maximum cut problem* corresponds to the function $MAXCUT : \{0,1\}^* \to \{0,1\}$ that maps a graph G and a number k to 1 if there is a cut in G that cuts at least k edges, and maps (G,k) to 0 otherwise.

All of the problems above are in **EXP** but it is not known whether or not they are in **P**. However, we will see in this chapter that if either *QUADEQ* , *LONGPATH* or *MAXCUT* are in **P**, then so is $3SAT$.

14.2 POLYNOMIAL-TIME REDUCTIONS

Suppose that $F, G : \{0,1\}^* \to \{0,1\}$ are two Boolean functions. A *polynomial-time reduction* (or sometimes just *"reduction"* for short) from F to G is a way to show that F is "no harder" than G, in the sense that a polynomial-time algorithm for G implies a polynomial-time algorithm for F.

Definition 14.1 — Polynomial-time reductions. Let $F, G : \{0,1\}^* \to \{0,1\}$. We say that F *reduces to* G, denoted by $F \leq_p G$ if there is a polynomial-time computable $R : \{0,1\}^* \to \{0,1\}^*$ such that for every $x \in \{0,1\}^*$,

$$F(x) = G(R(x)) \ . \tag{14.2}$$

We say that F and G have *equivalent complexity* if $F \leq_p G$ and $G \leq_p F$.

The following exercise justifies our intuition that $F \leq_p G$ signifies that "F is no harder than G.

Solved Exercise 14.1 — Reductions and P. Prove that if $F \leq_p G$ and $G \in \mathbf{P}$ then $F \in \mathbf{P}$.

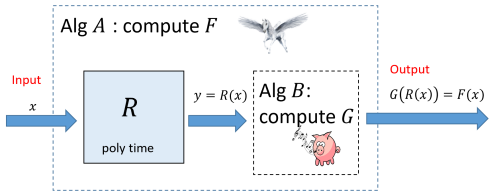

Figure 14.2: If $F \leq_p G$ then we can transform a polynomial-time algorithm B that computes G into a polynomial-time algorithm A that computes F. To compute $F(x)$ we can run the reduction R guaranteed by the fact that $F \leq_p G$ to obtain $y = F(x)$ and then run our algorithm B for G to compute $G(y)$.

As usual, solving this exercise on your own is an excellent way to make sure you understand Definition 14.1.

Solution:

Suppose there was an algorithm B that compute G in time $p(n)$ where p is its input size. Then, (14.2) directly gives an algorithm A to compute F (see Fig. 14.2). Indeed, on input $x \in \{0,1\}^*$, Algorithm A will run the polynomial-time reduction R to obtain $y = R(x)$ and then return $B(y)$. By (14.2), $G(R(x)) = F(x)$ and hence Algorithm A will indeed compute F.

We now show that A runs in polynomial time. By assumption, R can be computed in time $q(n)$ for some polynomial q. In particular, this means that $|y| \leq q(|x|)$ (as just writing down y takes $|y|$ steps). Computing $B(y)$ will take at most $p(|y|) \leq p(q(|x|))$ steps. Thus the total running time of A on inputs of length n is at most the time to compute y, which is bounded by $q(n)$, and the time to compute $B(y)$, which is bounded by $p(q(n))$, and since the composition of two polynomials is a polynomial, A runs in polynomial time.

Big Idea 21 A *reduction* $F \leq_p G$ shows that F is "no harder than G" or equivalently that G is "no easier than F".

14.2.1 Whistling pigs and flying horses

A reduction from F to G can be used for two purposes:

- If we already know an algorithm for G and $F \leq_p G$ then we can use the reduction to obtain an algorithm for F. This is a widely used tool in algorithm design. For example in Section 12.1.4 we saw how the *Min-Cut Max-Flow* theorem allows to reduce the task of computing a minimum cut in a graph to the task of computing a maximum flow in it.

- If we have proven (or have evidence) that there exists *no polynomial-time algorithm* for F and $F \leq_p G$ then the existence of this reduction allows us to conclude that there exists no polynomial-time algorithm for G. This is the "if pigs could whistle then horses could fly" interpretation we've seen in Section 9.4. We show that if there was an hypothetical efficient algorithm for G (a "whistling pig") then since $F \leq_p G$ then there would be an efficient algorithm for F (a "flying horse"). In this book we often use reductions for this second purpose, although the lines between the two is sometimes blurry (see the bibliographical notes in Section 14.9).

The most crucial difference between the notion in Definition 14.1 and the reductions we saw in the context of *uncomputability* (e.g., in Section 9.4) is that for relating time complexity of problems, we need the reduction to be computable in *polynomial time*, as opposed to merely computable. Definition 14.1 also restricts reductions to have a very specific format. That is, to show that $F \leq_p G$, rather than allowing a general algorithm for F that uses a "magic box" that computes G, we only allow an algorithm that computes $F(x)$ by outputting $G(R(x))$. This restricted form is convenient for us, but people have defined and used more general reductions as well (see Section 14.9).

In this chapter we use reductions to relate the computational complexity of the problems mentioned above: 3SAT, Quadratic Equations, Maximum Cut, and Longest Path, as well as a few others. We will reduce 3SAT to the latter problems, demonstrating that solving any one of them efficiently will result in an efficient algorithm for 3SAT. In Chapter 15 we show the other direction: reducing each one of these problems to 3SAT in one fell swoop.

Transitivity of reductions. Since we think of $F \leq_p G$ as saying that (as far as polynomial-time computation is concerned) F is "easier or equal in difficulty to" G, we would expect that if $F \leq_p G$ and $G \leq_p H$, then it would hold that $F \leq_p H$. Indeed this is the case:

Solved Exercise 14.2 — Transitivity of polynomial-time reductions. For every $F, G, H : \{0,1\}^* \to \{0,1\}$, if $F \leq_p G$ and $G \leq_p H$ then $F \leq_p H$.

■

Solution:

If $F \leq_p G$ and $G \leq_p H$ then there exist polynomial-time computable functions R_1 and R_2 mapping $\{0,1\}^*$ to $\{0,1\}^*$ such that for every $x \in \{0,1\}^*$, $F(x) = G(R_1(x))$ and for every $y \in \{0,1\}^*$, $G(y) = H(R_2(y))$. Combining these two equalities, we see that for every $x \in \{0,1\}^*$, $F(x) = H(R_2(R_1(x)))$ and so to show that $F \leq_p H$, it is sufficient to show that the map $x \mapsto R_2(R_1(x))$ is computable in polynomial time. But if there are some constants c, d such that $R_1(x)$ is computable in time $|x|^c$ and $R_2(y)$ is computable in time $|y|^d$ then $R_2(R_1(x))$ is computable in time $(|x|^c)^d = |x|^{cd}$ which is polynomial.

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14.3 REDUCING 3SAT TO ZERO ONE AND QUADRATIC EQUATIONS

We now show our first example of a reduction. The *Zero-One Linear Equations problem* corresponds to the function $01EQ : \{0,1\}^* \to \{0,1\}$ whose input is a collection E of linear equations in variables $x_0, \ldots, x_{n-1}$, and the output is 1 iff there is an assignment $x \in \{0,1\}^n$ of $0/1$ values to the variables that satisfies all the equations. For example, if the input E is a string encoding the set of equations

$$\begin{aligned} x_0 + x_1 + x_2 &= 2 \\ x_0 + x_2 &= 1 \\ x_1 + x_2 &= 2 \end{aligned} \tag{14.3}$$

then $01EQ(E) = 1$ since the assignment $x = 011$ satisfies all three equations. We specifically restrict attention to linear equations in variables $x_0, \ldots, x_{n-1}$ in which every equation has the form $\sum_{i \in S} x_i = b$ where $S \subseteq [n]$ and $b \in \mathbb{N}$.[1]

[1] If you are familiar with matrix notation you may note that such equations can be written as $Ax = \mathbf{b}$ where A is an $m \times n$ matrix with entries in $0/1$ and $\mathbf{b} \in \mathbb{N}^m$.

If we asked the question of whether there is a solution $x \in \mathbb{R}^n$ of *real numbers* to E, then this can be solved using the famous *Gaussian elimination* algorithm in polynomial time. However, there is no known efficient algorithm to solve $01EQ$. Indeed, such an algorithm would imply an algorithm for $3SAT$ as shown by the following theorem:

Theorem 14.2 — Hardness of $01EQ$. $3SAT \leq_p 01EQ$

Proof Idea:

A constraint $x_2 \vee \overline{x}_5 \vee x_7$ can be written as $x_2 + (1 - x_5) + x_7 \geq 1$. This is a linear *inequality* but since the sum on the left-hand side is at most three, we can also turn it into an *equality* by adding two new variables y, z and writing it as $x_2 + (1 - x_5) + x_7 + y + z = 3$. (We will use fresh such variables y, z for every constraint.) Finally, for every variable x_i we can add a variable x'_i corresponding to its negation by

adding the equation $x_i + x'_i = 1$, hence mapping the original constraint $x_2 \vee \overline{x}_5 \vee x_7$ to $x_2 + x'_5 + x_7 + y + z = 3$. The main **takeaway technique** from this reduction is the idea of adding *auxiliary variables* to replace an equation such as $x_1 + x_2 + x_3 \geq 1$ that is not quite in the form we want with the equivalent (for $0/1$ valued variables) equation $x_1 + x_2 + x_3 + u + v = 3$ which is in the form we want.

⋆

```python
def SAT2ZOE(ϕ):
    # Reduce 3SAT to 0/1 equations
    n = numvars(ϕ)
    E = ""
    for i in range(n): # add vars for negations
        E+= f"x{subscript(i)} + x{subscript(n+i)} = 1\n"
    m = 2*n
    for literals in getclauses(ϕ):
        # map each clause to equation
        def var(lit): # map literal to variable
            return f"x{subscript(n+int(lit[2:]))}" if lit[0]== "¬" else f"x{subscript(lit[1:])}"
        E+= " + ".join([var(lit) for lit in literals])
        E+= f" + x{subscript(m)} + x{subscript(m+1)} = 3\n"
        m += 2
    return Equation(E)
```

```
SAT2ZOE("(x0 ∨ ¬x3 ∨ x2 ) ∧ (x0 ∨ x1 ∨ ¬x2 ) ∧ (x1 ∨ x2 ∨ ¬x3 )")
```

$$
\begin{aligned}
x_0 + x_4 &= 1 \\
x_1 + x_5 &= 1 \\
x_2 + x_6 &= 1 \\
x_3 + x_7 &= 1 \\
x_0 + x_7 + x_2 + x_8 + x_9 &= 3 \\
x_0 + x_1 + x_6 + x_{10} + x_{11} &= 3 \\
x_1 + x_2 + x_7 + x_{12} + x_{13} &= 3
\end{aligned}
$$

Figure 14.3: Left: Python code implementing the reduction of $3SAT$ to $01EQ$. Right: Example output of the reduction. Code is in our repository.

Proof of Theorem 14.2. To prove the theorem we need to:

1. Describe an algorithm R for mapping an input φ for $3SAT$ into an input E for $01EQ$.

2. Prove that the algorithm runs in polynomial time.

3. Prove that $01EQ(R(\varphi)) = 3SAT(\varphi)$ for every 3CNF formula φ.

We now proceed to do just that. Since this is our first reduction, we will spell out this proof in detail. However it straightforwardly follows the proof idea.

Algorithm 14.3 — $3SAT$ to $01EQ$ reduction.

Input: 3CNF formula φ with n variables $x_0, \ldots, x_{n-1}$ and m clauses.

Output: Set E of linear equations over $0/1$ such that $3SAT(\varphi) = 1$ -iff $01EQ(E) = 1$.

Let E's variables be $x_0, \ldots, x_{n-1}, x'_0, \ldots, x'_{n-1}, y_0, \ldots, y_{m-1}, z_0, \ldots, z_{m-1}$.
for $i \in [n]$ **do**
    add to E the equation $x_i + x'_i = 1$
end for
for j∈[m] **do**
    Let j-th clause be $w_0 \vee w_1 \vee w_2$ where w_0, w_1, w_2 are literals.
    for $a \in [3]$ **do**
        if w_a is variable x_i **then**
            set $t_a \leftarrow x_i$
        end if
        if w_a is negation $\neg x_i$ **then**
            set $t_a \leftarrow x'_i$
        end if
    end for
    Add to E the equation $t_0 + t_1 + t_2 + y_j + z_j = 3$.
end for
return E

The reduction is described in Algorithm 14.3, see also Fig. 14.3. If the input formula has n variable and m steps, Algorithm 14.3 creates a set E of $n+m$ equations over $2n+2m$ variables. Algorithm 14.3 makes an initial loop of n steps (each taking constant time) and then another loop of m steps (each taking constant time) to create the equations, and hence it runs in polynomial time.

Let R be the function computed by Algorithm 14.3. The heart of the proof is to show that for every 3CNF φ, $01EQ(R(\varphi)) = 3SAT(\varphi)$. We split the proof into two parts. The first part, traditionally known as the **completeness** property, is to show that if $3SAT(\varphi) = 1$ then $O1EQ(R(\varphi)) = 1$. The second part, traditionally known as the **soundness** property, is to show that if $01EQ(R(\varphi)) = 1$ then $3SAT(\varphi) = 1$. (The names "completeness" and "soundness" derive viewing a solution to $R(\varphi)$ as a "proof" that φ is satisfiable, in which case these conditions corresponds to completeness and soundness as defined in Section 11.1.1. However, if you find the names confusing you can simply think of completeness as the "1-instance maps to 1-instance"

property and soundness as the "0-instance maps to 0-instance" property.)

We complete the proof by showing both parts:

- **Completeness:** Suppose that $3SAT(\varphi) = 1$, which means that there is an assignment $x \in \{0,1\}^n$ that satisfies φ. If we use the assignment $x_0, \ldots, x_{n-1}$ and $1 - x_0, \ldots, 1 - x_{n-1}$ for the first $2n$ variables of $E = R(\varphi)$ then we will satisfy all equations of the form $x_i + x'_i = 1$. Moreover, for every $j \in [n]$, if $t_0 + t_1 + t_2 + y_j + z_j = 3(*)$ is the equation arising from the jth clause of φ (with t_0, t_1, t_2 being variables of the form x_i or x'_i depending on the literals of the clause) then our assignment to the first $2n$ variables ensures that $t_0 + t_1 + t_2 \geq 1$ (since x satisfied φ) and hence we can assign values to y_j and z_j that will ensure that the equation $(*)$ is satisfied. Hence in this case $E = R(\varphi)$ is satisfied, meaning that $01EQ(R(\varphi)) = 1$.

- **Soundness:** Suppose that $01EQ(R(\varphi)) = 1$, which means that the set of equations $E = R(\varphi)$ has a satisfying assignment $x_0, \ldots, x_{n-1}$, $x'_0, \ldots, x'_{n-1}$, $y_0, \ldots, y_{m-1}$, $z_0, \ldots, z_{m-1}$. Then, since the equations contain the condition $x_i + x'_i = 1$, for every $i \in [n]$, x'_i is the negation of x_i, and morover, for every $j \in [m]$, if C has the form $w_0 \vee w_1 \vee w_2$ is the j-th clause of C, then the corresponding assignment x will ensure that $w_0 + w_1 + w_2 \geq 1$, implying that C is satisfied. Hence in this case $3SAT(\varphi) = 1$.

■

14.3.1 Quadratic equations

Now that we reduced $3SAT$ to $01EQ$, we can use this to reduce $3SAT$ to the *quadratic equations* problem. This is the function $QUADEQ$ in which the input is a list of n-variate polynomials $p_0, \ldots, p_{m-1} : \mathbb{R}^n \to \mathbb{R}$ that are all of degree at most two (i.e., they are *quadratic*) and with integer coefficients. (The latter condition is for convenience and can be achieved by scaling.) We define $QUADEQ(p_0, \ldots, p_{m-1})$ to equal 1 if and only if there is a solution $x \in \mathbb{R}^n$ to the equations $p_0(x) = 0$, $p_1(x) = 0$, ..., $p_{m-1}(x) = 0$.

For example, the following is a set of quadratic equations over the variables x_0, x_1, x_2:

$$\begin{aligned} x_0^2 - x_0 &= 0 \\ x_1^2 - x_1 &= 0 \\ x_2^2 - x_2 &= 0 \\ 1 - x_0 - x_1 + x_0 x_1 &= 0 \end{aligned} \tag{14.4}$$

You can verify that $x \in \mathbb{R}^3$ satisfies this set of equations if and only if $x \in \{0,1\}^3$ and $x_0 \vee x_1 = 1$.

Theorem 14.4 — Hardness of quadratic equations.

$$3SAT \leq_p QUADEQ \tag{14.5}$$

Proof Idea:

Using the transitivity of reductions (Solved Exercise 14.2), it is enough to show that $01EQ \leq_p QUADEQ$, but this follows since we can phrase the equation $x_i \in \{0,1\}$ as the quadratic constraint $x_i^2 - x_i = 0$. The **takeaway technique** of this reduction is that we can use *non-linearity* to force continuous variables (e.g., variables taking values in $\mathbb{R}$) to be discrete (e.g., take values in $\{0,1\}$).

⋆

Proof of Theorem 14.4. By Theorem 14.2 and Solved Exercise 14.2, it is sufficient to prove that $01EQ \leq_p QUADEQ$. Let E be an instance of $01EQ$ with variables $x_0, \ldots, x_{m-1}$. We map E to the set of quadratic equations E' that is obtained by taking the linear equations in E and adding to them the n quadratic equations $x_i^2 - x_i = 0$ for all $i \in [n]$. (See Algorithm 14.5.) The map $E \mapsto E'$ can be computed in polynomial time. We claim that $01EQ(E) = 1$ if and only if $QUADEQ(E') = 1$. Indeed, the only difference between the two instances is that:

- In the $01EQ$ instance E, the equations are over variables $x_0, \ldots, x_{n-1}$ in $\{0,1\}$.
- In the $QUADEQ$ instance E', the equations are over variables $x_0, \ldots, x_{n-1} \in \mathbb{R}$ but we have the extra constraints $x_i^2 - x_i = 0$ for all $i \in [n]$.

Since for every $a \in \mathbb{R}$, $a^2 - a = 0$ if and only if $a \in \{0,1\}$, the two sets of equations are equivalent and $01EQ(E) = QUADEQ(E')$ which is what we wanted to prove.

■

Algorithm 14.5 — $3SAT$ to $01EQ$ reduction.

Input: Set E of linear equations over n variables $x_0, \ldots, x_{n-1}$.
Output: Set E' of quadratic eqations ovar m variables $w_0, \ldots, w_{m-1}$ such that there is an $0/1$ assignment $x \in \{0,1\}^n$

satisfying the equations of E -iff there is an assignment $w \in \mathbb{R}^m$ satisfying the equations of E'.
That is, $O1EQ(E) = QUADEQ(E')$.
Let $m \leftarrow n$.
Variables of E' are set to be same variable $x_0, \ldots, x_{n-1}$ as E.
for every equation $e \in E$ **do**
    Add e to E'
end for
for $i \in [n]$ **do**
    Add to E' the equation $x_i^2 - x_i = 0$.
end for
return E'

14.4 THE INDEPENDENT SET PROBLEM

For a graph $G = (V, E)$, an independent set (also known as a *stable set*) is a subset $S \subseteq V$ such that there are no edges with both endpoints in S (in other words, $E(S, S) = \emptyset$). Every "singleton" (set consisting of a single vertex) is trivially an independent set, but finding larger independent sets can be challenging. The *maximum independent set* problem (henceforth simply "independent set") is the task of finding the largest independent set in the graph. The independent set problem is naturally related to *scheduling problems*: if we put an edge between two conflicting tasks, then an independent set corresponds to a set of tasks that can all be scheduled together without conflicts. The independent set problem has been studied in a variety of settings, including for example in the case of algorithms for finding structure in protein-protein interaction graphs.

As mentioned in Section 14.1, we think of the independent set problem as the function $ISET : \{0,1\}^* \to \{0,1\}$ that on input a graph G and a number k outputs 1 if and only if the graph G contains an independent set of size at least k. We now reduce 3SAT to Independent set.

Theorem 14.6 — Hardness of Independent Set. $3SAT \leq_p ISET$.

Proof Idea:

The idea is that finding a satisfying assignment to a 3SAT formula corresponds to satisfying many local constraints without creating any conflicts. One can think of "$x_{17} = 0$" and "$x_{17} = 1$" as two conflicting events, and of the constraints $x_{17} \vee \overline{x}_5 \vee x_9$ as creating a conflict between the events "$x_{17} = 0$", "$x_5 = 1$" and "$x_9 = 0$", saying that these three cannot simultaneously co-occur. Using these ideas, we can we can think of solving a 3SAT problem as trying to schedule non-conflicting events, though the devil is, as usual, in the details. The **takeaway technique** here is to map each clause of the original formula into a *gadget* which is a small subgraph (or more generally "subinstance") satisfying some convenient properties. We will see these "gadgets" used time and again in the construction of polynomial-time reductions.

⋆

Algorithm 14.7 — $3SAT$ to IS reduction.

Input: $3SAT$ formula φ with n variables and m clauses.
Output: Graph $G = (V, E)$ and number k, such that G has an independent set of size k -iff φ has a satisfying assignment.

That is, $3SAT(\varphi) = ISET(G, k)$,
Initialize $V \leftarrow \emptyset, E \leftarrow \emptyset$
for every clause $C = y \vee y' \vee y''$ of φ **do**
    Add three vertices $(C, y), (C, y'), (C, y'')$ to V
    Add edges $\{(C, y), (C, y')\}, \{(C, y'), (C, y'')\}, \{(C, y''), (C, y)\}$ to E.
end for
for every distinct clauses C, C' in φ **do**
    for every $i \in [n]$ **do**
        if C contains literal x_i and C' contains literal $\overline{x}_i$ **then**
            Add edge $\{(C, x_i), (C, \overline{x}_i)\}$ to E
        end if
    end for
end for
return $G = (V, E)$

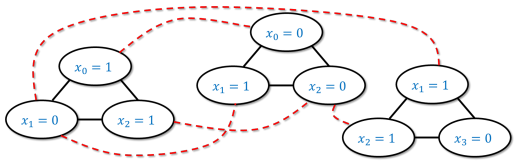


Figure 14.4: An example of the reduction of $3SAT$ to *ISET* for the case the original input formula is $\varphi = (x_0 \vee \overline{x}_1 \vee x_2) \wedge (\overline{x}_0 \vee x_1 \vee \overline{x}_2) \wedge (x_1 \vee x_2 \vee \overline{x}_3)$. We map each clause of φ to a triangle of three vertices, each tagged above with "$x_i = 0$" or "$x_i = 1$" depending on the value of x_i that would satisfy the particular literal. We put an edge between every two literals that are *conflicting* (i.e., tagged with "$x_i = 0$" and "$x_i = 1$" respectively).

Proof of Theorem 14.6. Given a 3SAT formula φ on n variables and with m clauses, we will create a graph G with $3m$ vertices as follows. (See Algorithm 14.7, see also Fig. 14.4 for an example and Fig. 14.5 for Python code.)

- A clause C in φ has the form $C = y \vee y' \vee y''$ where y, y', y'' are *literals* (variables or their negation). For each such clause C, we will

add three vertices to G, and label them (C, y), (C, y'), and (C, y'') respectively. We will also add the three edges between all pairs of these vertices, so they form a *triangle*. Since there are m clauses in φ, the graph G will have $3m$ vertices.

- In addition to the above edges, we also add an edge between every pair vertices of the form (C, y) and (C', y') where y and y' are *conflicting* literals. That is, we add an edge between (C, y) and (C', y') if there is an i such that $y = x_i$ and $y' = \overline{x}_i$ or vice versa.

The algorithm constructing of G based on φ takes polynomial time since it involves two loops, the first taking $O(m)$ steps and the second taking $O(m^2 n)$ steps (see Algorithm 14.7). Hence to prove the theorem we need to show that φ is satisfiable if and only if G contains an independent set of m vertices. We now show both directions of this equivalence:

Part 1: Completeness. The "completeness" direction is to show that if φ has a satisfying assignment x^*, then G has an independent set S^* of m vertices. Let us now show this.

Indeed, suppose that φ has a satisfying assignment $x^* \in \{0,1\}^n$. Then for every clause $C = y \vee y' \vee y''$ of φ, one of the literals y, y', y'' must evaluate to *true* under the assignment x^* (as otherwise it would not satisfy φ). We let S be a set of m vertices that is obtained by choosing for every clause C one vertex of the form (C, y) such that y evaluates to true under x^*. (If there is more than one such vertex for the same C, we arbitrarily choose one of them.)

We claim that S is an independent set. Indeed, suppose otherwise that there was a pair of vertices (C, y) and (C', y') in S that have an edge between them. Since we picked one vertex out of each triangle corresponding to a clause, it must be that $C \neq C'$. Hence the only way that there is an edge between (C, y) and (C, y') is if y and y' are conflicting literals (i.e. $y = x_i$ and $y' = \overline{x}_i$ for some i). But then they can't both evaluate to *true* under the assignment x^*, which contradicts the way we constructed the set S. This completes the proof of the completeness condition.

Part 2: Soundness. The "soundness" direction is to show that if G has an independent set S^* of m vertices, then φ has a satisfying assignment $x^* \in \{0,1\}^n$. Let us now show this.

Indeed, suppose that G has an independent set $S*$ with m vertices. We will define an assignment $x^* \in \{0,1\}^n$ for the variables of φ as follows. For every $i \in [n]$, we set x^*_i according to the following rules:

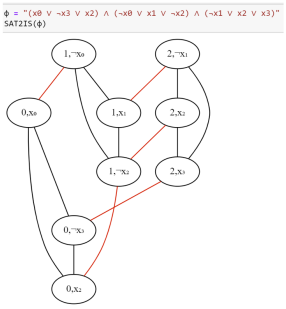
- If S^* contains a vertex of the form (C, x_i) then we set $x^*_i = 1$.
- If S^* contains a vertex of the form $(C, \overline{x_i})$ then we set $x^*_i = 0$.

- If S^* does not contain a vertex of either of these forms, then it does not matter which value we give to x^*_i, but for concreteness we'll set $x^*_i = 0$.

The first observation is that x^* is indeed well defined, in the sense that the rules above do not conflict with one another, and ask to set x^*_i to be both 0 and 1. This follows from the fact that S^* is an *independent set* and hence if it contains a vertex of the form (C, x_i) then it cannot contain a vertex of the form $(C', \overline{x_i})$.

We now claim that x^* is a satisfying assignment for φ. Indeed, since S^* is an independent set, it cannot have more than one vertex inside each one of the m triangles $(C, y), (C, y'), (C, y'')$ corresponding to a clause of φ. Hence since $|S^*| = m$, it must have exactly one vertex in each such triangle. For every clause C of φ, if (C, y) is the vertex in S^* in the triangle corresponding to C, then by the way we defined x^*, the literal y must evaluate to *true*, which means that x^* satisfies this clause. Therefore x^* satisfies all clauses of φ, which is the definition of a satisfying assignment.

This completes the proof of Theorem 14.6 ■




```python
# Reduction SAT to Independent Set
def SAT2IS(ϕ):
    n = numvars(ϕ)
    G =Graph() #(engine='neato')

    def nname(c,v): return f"({c},{v})"

    c = 0
    # nodes[literal] = vertices corresponding to this literal
    nodes = {f"x{i}":[] for i in range(n)}
    nodes.update(  {f"¬x{i}":[] for i in range(n)})

    c=0
    for C in getclauses(ϕ):
        for u in C:
            G.node(nname(c,u),f"{c},{subscript(u)}",fontsize="12")
            negu = u[1:] if u[0]=="-" else "-"+u
            # add conflicting edges
            for v in nodes[negu]: G.edge(nname(c,u),v,color="red")
            nodes[u].append(nname(c,u))
        # add triangle edges
        G.edges([[nname(c,C[0]), nname(c,C[1])],[nname(c,C[1]), nname(c,C[2])],[nname(c,C[0]), nname(c,C[2])]])
        c += 1

    return G
```

Figure 14.5: The reduction of 3SAT to Independent Set. On the right-hand side is *Python* code that implements this reduction. On the left-hand side is a sample output of the reduction. We use black for the "triangle edges" and red for the "conflict edges". Note that the satisfying assignment $x^* = 0110$ corresponds to the independent set $(0, \neg x_3), (1, \neg x_0), (2, x_2)$.

14.5 SOME EXERCISES AND ANATOMY OF A REDUCTION.

Reductions can be confusing and working out exercises is a great way to gain more comfort with them. Here is one such example. As usual, I recommend you try it out yourself before looking at the solution.

Solved Exercise 14.3 — Vertex cover. A *vertex cover* in a graph $G = (V, E)$ is a subset $S \subseteq V$ of vertices such that every edge touches at least one vertex of S (see **??**). The *vertex cover problem* is the task to determine, given a graph G and a number k, whether there exists a vertex cover in the graph with at most k vertices. Formally, this is the function $VC : \{0,1\}^* \to \{0,1\}$ such that for every $G = (V, E)$ and $k \in \mathbb{N}$,

$VC(G,k) = 1$ if and only if there exists a vertex cover $S \subseteq V$ such that $|S| \leq k$.

Prove that $3SAT \leq_p VC$.

Solution:

The key observation is that if $S \subseteq V$ is a vertex cover that touches all vertices, then there is no edge e such that both s's endpoints are in the set $\overline{S} = V \setminus S$, and vice versa. In other words, S is a vertex cover if and only if $\overline{S}$ is an independent set. Since the size of $\overline{S}$ is $|V| - |S|$, we see that the polynomial-time map $R(G,k) = (G, n - k)$ (where n is the number of vertices of G) satisfies that $VC(R(G,k)) = ISET(G,k)$ which means that it is a reduction from independent set to vertex cover.

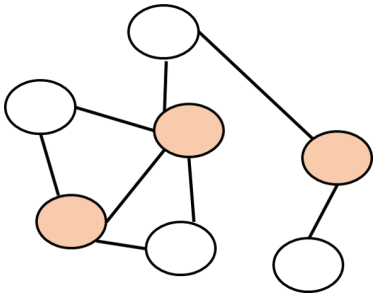


Figure 14.6: A *vertex cover* in a graph is a subset of vertices that touches all edges. In this 7-vertex graph, the 3 filled vertices are a vertex cover.

Solved Exercise 14.4 — Clique is equivalent to independent set. The maximum clique problem corresponds to the function $CLIQUE : \{0,1\}^* \to \{0,1\}$ such that for a graph G and a number k, $CLIQUE(G,k) = 1$ iff there is a S subset of k vertices such that for *every* distinct $u, v \in S$, the edge u, v is in G. Such a set is known as a *clique*.

Prove that $CLIQUE \leq_p ISET$ and $ISET \leq_p CLIQUE$.

Solution:

If $G = (V,E)$ is a graph, we denote by $\overline{G}$ its *complement* which is the graph on the same vertices V and such that for every distinct $u, v \in V$, the edge $\{u,v\}$ is present in $\overline{G}$ if and only if this edge is *not* present in G.

This means that for every set S, S is an independent set in G if and only if S is a *clique* in $\overline{G}$. Therefore for every k, $ISET(G,k) = CLIQUE(\overline{G},k)$. Since the map $G \mapsto \overline{G}$ can be computed efficiently, this yields a reduction $ISET \leq_p CLIQUE$. Moreover, since $\overline{\overline{G}} = G$ this yields a reduction in the other direction as well.

14.5.1 Dominating set

In the two examples above, the reduction was almost "trivial": the reduction from independent set to vertex cover merely changes the number k to $n - k$, and the reduction from independent set to clique flips edges to non-edges and vice versa. The following exercise requires a somewhat more interesting reduction.

Solved Exercise 14.5 — Dominating set. A *dominating set* in a graph $G = (V,E)$ is a subset $S \subseteq V$ of vertices such that for every $u \in V \setminus S$ is a neighbor in G of some $s \in S$ (see Fig. 14.7). The *dominating set problem*

is the task, given a graph $G = (V, E)$ and number k, of determining whether there exists a dominating set $S \subseteq V$ with $|S| \leq k$. Formally, this is the function $DS : \{0,1\}^* \to \{0,1\}$ such that $DS(G, k) = 1$ iff there is a dominating set in G of at most k vertices.

Prove that $ISET \leq_p DS$.

Solution:

Since we know that $ISET \leq_p VC$, using transitivity, it is enough to show that $VC \leq_p DS$. As Fig. 14.7 shows, a dominating set is not the same thing as a vertex cover. However, we can still relate the two problems. The idea is to map a graph G into a graph H such that a vertex cover in G would translate into a dominating set in H and vice versa. We do so by including in H all the vertices and edges of G, but for every edge $\{u, v\}$ of G we also add to H a new vertex $w_{u,v}$ and connect it to both u and v. Let ℓ be the number of isolated vertices in G. The idea behind the proof is that we can transform a vertex cover S of k vertices in G into a dominating set of $k + \ell$ vertices in H by adding to S all the isolated vertices, and moreover we can transform every $k + \ell$-sized dominating set in H into a vertex cover in G. We now give the details.

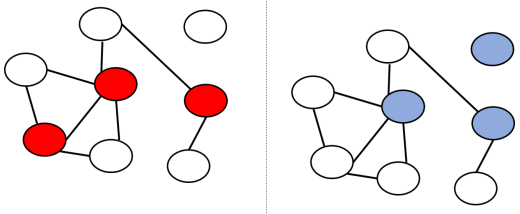


Figure 14.7: A dominating set is a subset S of vertices such that every vertex in the graph is either in S or a neighbor of S. The figure above are two copies of the same graph. The red vertices on the left are a vertex cover that is not a dominating set. The blue vertices on the right are a dominating set that is not a vertex cover.

Description of the algorithm. Given an instance (G, k) for the vertex cover problem, we will map G into an instance (H, k') for the dominating set problem as follows (see Fig. 14.8 for Python implementation):

Algorithm 14.8 — VC to DS reduction.

Input: Graph $G = (V, E)$ and number k.

Output: Graph $H = (V', E')$ and number k', such that G has a vertex cover of size k -iff H has a dominating set of size k'

1: That is, $DS(H, k') = ISET(G, k)$,
2: Initialize $V' \leftarrow V, E' \leftarrow E$
3: **for** every edge $\{u, v\} \in E$ **do**
4: Add vertex $w_{u,v}$ to V'
5: Add edges $\{u, w_{u,v}\}$, $\{v, w_{u,v}\}$ to E'.
6: **end for**
7: Let $\ell \leftarrow$ number of isolated vertices in G
8: **return** $(H = (V', E')\ ,\ k + \ell)$

Algorithm 14.8 runs in polynomial time, since the loop takes $O(m)$ steps where m is the number of edges, with each step can be implemented in constant or at most linear time (depending on the

representation of the graph H). Counting the number of isolated vertices in an n vertex graph G can be done in time $O(n^2)$ if G is represented in the adjacency matrix representation and $O(n)$ time if it is represented in the adjacency list representation. Regardless the algorithm runs in polynomial time.

To complete the proof we need to prove that for every G, k, if H, k' is the output of Algorithm 14.8 on input (G, k), then $DS(H, k') = VC(G, k)$. We split the proof into two parts. The *completeness* part is that if $VC(G, k) = 1$ then $DS(H, k') = 1$. The *soundness* part is that if $DS(H, k') = 1$ then $VC(G, k) = 1$.

Completeness. Suppose that $VC(G, k) = 1$. Then there is a vertex cover $S \subseteq V$ of at most k vertices. Let I be the set of isolated vertices in G and ℓ be their number. Then $|S \cup I| \leq k + \ell$. We claim that $S \cup I$ is a dominating set in H'. Indeed for every vertex v of H' there are three cases:

- **Case 1:** v is an isolated vertex of G. In this case v is in $S \cup I$.
- **Case 2:** v is a non-isolated vertex of G and hence there is an edge $\{u, v\}$ of G for some u. In this case since S is a vertex cover, one of u, v has to be in S, and hence either v or a neighbor of v has to be in $S \subseteq S \cup I$.
- **Case 3:** v is of the form $w_{u,u'}$ for some two neighbors u, u' in G. But then since S is a vertex cover, one of u, u' has to be in S and hence S contains a neighbor of v.

We conclude that $S \cup I$ is a dominating set of size at most $k' = k + \ell$ in H' and hence under the assumption that $VC(G.k) = 1$, $DS(H', k') = 1$.

Soundness. Suppose that $DS(H, k') = 1$. Then there is a dominating set D of size at most $k' = k + \ell$ in H. For every edge $\{u, v\}$ in the graph G, if D contains the vertex $w_{u,v}$ then we remove this vertex and add u in its place. The only two neighbors of $w_{u,v}$ are u and v, and since u is a neighbor of both $w_{u,v}$ and of v, replacing $w_{u,v}$ with v maintains the property that it is a dominating set. Moreover, this change cannot increase the size of D. Thus following this modification, we can assume that D is a dominating set of at most $k + \ell$ vertices that does not contain any vertices of the form $w_{u,v}$.

Let I be the set of isolated vertices in G. These vertices are also isolated in H and hence must be included in D (an isolated vertex must be in any dominating set, since it has no neighbors). We let $S = D \setminus I$. Then $|S| \leq I$. We claim that S is a vertex cover in G. Indeed, for every edge $\{u, v\}$ of G, either the vertex $w_{u,v}$ or

one of its neighbors must be in S by the dominating set property. But since we ensured S doesn't contain any of the vertices of the form $w_{u,v}$, it must be the case that either u or v is in S. This shows that S is a vertex cover of G of size at most k, hence proving that $VC(G,k) = 1$.

A corollary of Algorithm 14.8 and the other reduction we have seen so far is that if $DS \in \mathbf{P}$ (i.e., dominating set has a polynomial-time algorithm) then $3SAT \in \mathbf{P}$ (i.e., $3SAT$ has a polynomial-time algorithm). By the contra-positive, if $3SAT$ does *not* have a polynomial-time algorithm then neither does dominating set.

Figure 14.8: Python implementation of the reduction from vertex cover to dominating set, together with an example of an input graph and the resulting output graph. This reduction allows to transform a hypothetical polynomial-time algorithm for dominating set (a "whistling pig") into a hypothetical polynomial-time algorithm for vertex-cover (a "flying horse").

14.5.2 Anatomy of a reduction

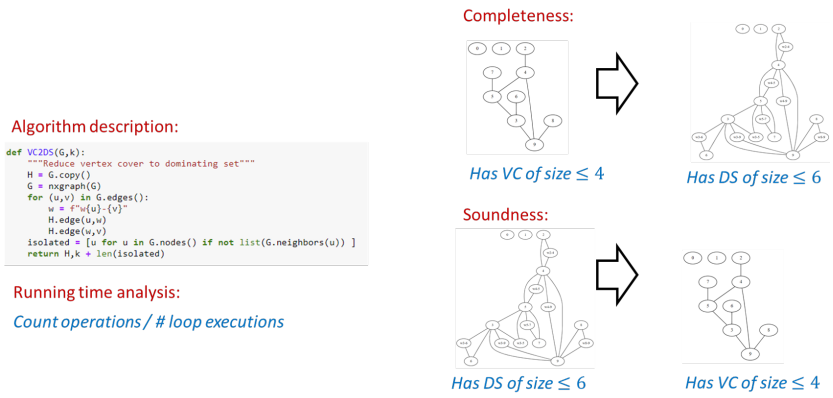


Figure 14.9: The four components of a reduction, illustrated for the particular reduction of vertex cover to dominating set. A reduction from problem F to problem G is an algorithm that maps an input x for F into an input $R(x)$ for G. To show that the reduction is correct we need to show the properties of *efficiency*: algorithm R runs in polynomial time, *completeness*: if $F(x) = 1$ then $G(R(x)) = 1$, and *soundness*: if $F(R(x)) = 1$ then $G(x) = 1$.

The reduction of Solved Exercise 14.5 gives a good illustration of the anatomy of a reduction. A reduction consists of four parts:

- **Algorithm description:** This is the description of *how* the algorithm maps an input into the output. For example, in Solved Exercise 14.5 this is the description of how we map an instance (G,k) of the *vertex cover* problem into an instance (H,k') of the *dominating set* problem.

- **Algorithm analysis:** It is not enough to describe *how* the algorithm works but we need to also explain *why* it works. In particular we need to provide an *analysis* explaining why the reduction is both *efficient* (i.e., runs in polynomial time) and *correct* (satisfies that $G(R(x)) = F(x)$ for every x). Specifically, the components of analysis of a reduction R include:
  - **Efficiency:** We need to show that R runs in polynomial time. In most reductions we encounter this part is straightforward, as the reductions we typically use involve a constant number of nested loops, each involving a constant number of operations. For example, the reduction of Solved Exercise 14.5 just enumerates over the edges and vertices of the input graph.
  - **Completeness:** In a reduction R demonstrating $F \leq_p G$, the *completeness* condition is the condition that for every $x \in \{0,1\}^*$, if $F(x) = 1$ then $G(R(x)) = 1$. Typically we construct the reduction to ensure that this holds, by giving a way to map a "certificate/solution" certifying that $F(x) = 1$ into a solution certifying that $G(R(x)) = 1$. For example, in Solved Exercise 14.5 we constructed the graph H such that for every vertex cover S in G, the set $S \cup I$ (where I is the isolated vertices) would be a dominating set in H.
  - **Soundness:** This is the condition that if $F(x) = 0$ then $G(R(x)) = 0$ or (taking the contrapositive) if $G(R(x)) = 1$ then $F(x) = 1$. This is sometimes straightforward but can often be harder to show than the completeness condition, and in more advanced reductions (such as the reduction $SAT \leq_p ISET$ of Theorem 14.6) demonstrating soundness is the main part of the analysis. For example, in Solved Exercise 14.5 to show soundness we needed to show that for *every* dominating set D in the graph H, there exists a vertex cover S of size at most $|D| - \ell$ in the graph G (where ℓ is the number of isolated vertices). This was challenging since the dominating set D might not be necessarily the one we "had in mind". In particular, in the proof above we needed to modify D to ensure that it does not contain vertices of the form $w_{u,v}$, and it was important to show that this modification still maintains the property that D is a dominating set, and also does not make it bigger.

Whenever you need to provide a reduction, you should make sure that your description has all these components. While it is sometimes tempting to weave together the description of the reduction and its analysis, it is usually clearer if you separate the two, and also break down the analysis to its three components of efficiency, completeness, and soundness.

14.6 REDUCING INDEPENDENT SET TO MAXIMUM CUT

We now show that the independent set problem reduces to the *maximum cut* (or "max cut") problem, modeled as the function *MAXCUT* that on input a pair (G, k) outputs 1 iff G contains a cut of at least k edges. Since both are graph problems, a reduction from independent set to max cut maps one graph into the other, but as we will see the output graph does not have to have the same vertices or edges as the input graph.

Theorem 14.9 — Hardness of Max Cut. $ISET \leq_p MAXCUT$

Proof Idea:

We will map a graph G into a graph H such that a large independent set in G becomes a partition cutting many edges in H. We can think of a cut in H as coloring each vertex either "blue" or "red". We will add a special "source" vertex s^*, connect it to all other vertices, and assume without loss of generality that it is colored blue. Hence the more vertices we color red, the more edges from s^* we cut. Now, for every edge u, v in the original graph G we will add a special "gadget" which will be a small subgraph that involves u,v, the source s^*, and two other additional vertices. We design the gadget in a way so that if the red vertices are not an independent set in G then the corresponding cut in H will be "penalized" in the sense that it would not cut as many edges. Once we set for ourselves this objective, it is not hard to find a gadget that achieves it— see the proof below. Once again the **takeaway technique** is to use (this time a slightly more clever) gadget.

⋆

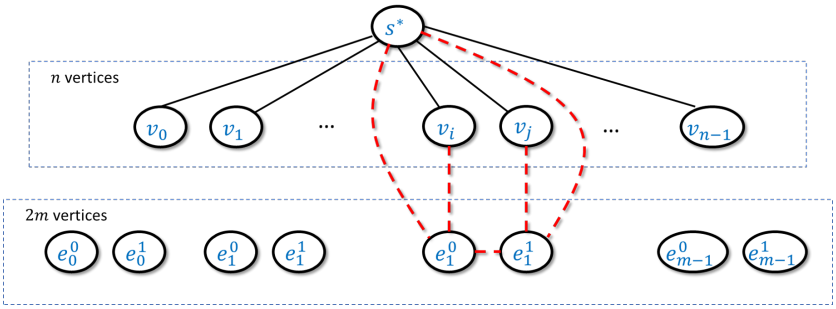


Figure 14.10: In the reduction of *ISET* to *MAXCUT* we map an n-vertex m-edge graph G into the $n + 2m + 1$ vertex and $n + 5m$ edge graph H as follows. The graph H contains a special "source" vertex s^*,n vertices $v_0, \ldots, v_{n-1}$, and $2m$ vertices $e_0^0, e_0^1, \ldots, e_{m-1}^0, e_{m-1}^1$ with each pair corresponding to an edge of G. We put an edge between s^* and v_i for every $i \in [n]$, and if the t-th edge of G was (v_i, v_j) then we add the five edges $(s^*, e_t^0), (s^*, e_t^1), (v_i, e_t^0), (v_j, e_t^1), (e_t^0, e_t^1)$. The intent is that if cut at most one of v_i, v_j from s^* then we'll be able to cut 4 out of these five edges, while if we cut both v_i and v_j from s^* then we'll be able to cut at most three of them.

Proof of Theorem 14.9. We will transform a graph G of n vertices and m edges into a graph H of $n + 1 + 2m$ vertices and $n + 5m$ edges in the following way (see also Fig. 14.10). The graph H contains all vertices

of G (though not the edges between them!) and in addition H also has:

* A special vertex s^* that is connected to all the vertices of G

* For every edge $e = \{u, v\} \in E(G)$, two vertices e_0, e_1 such that e_0 is connected to u and e_1 is connected to v, and moreover we add the edges $\{e_0, e_1\}, \{e_0, s^*\}, \{e_1, s^*\}$ to H.

Theorem 14.9 will follow by showing that G contains an independent set of size at least k if and only if H has a cut cutting at least $k + 4m$ edges. We now prove both directions of this equivalence:

Part 1: Completeness. If I is an independent k-sized set in G, then we can define S to be a cut in H of the following form: we let S contain all the vertices of I and for every edge $e = \{u, v\} \in E(G)$, if $u \in I$ and $v \notin I$ then we add e_1 to S; if $u \notin I$ and $v \in I$ then we add e_0 to S; and if $u \notin I$ and $v \notin I$ then we add both e_0 and e_1 to S. (We don't need to worry about the case that both u and v are in I since it is an independent set.) We can verify that in all cases the number of edges from S to its complement in the gadget corresponding to e will be four (see Fig. 14.11). Since s^* is not in S, we also have k edges from s^* to I, for a total of $k + 4m$ edges.

Part 2: Soundness. Suppose that S is a cut in H that cuts at least $C = k + 4m$ edges. We can assume that s^* is not in S (otherwise we can "flip" S to its complement $\overline{S}$, since this does not change the size of the cut). Now let I be the set of vertices in S that correspond to the original vertices of G. If I was an independent set of size k then we would be done. This might not always be the case but we will see that if I is not an independent set then it's also larger than k. Specifically, we define $m_{in} = |E(I, I)|$ be the set of edges in G that are contained in I and let $m_{out} = m - m_{in}$ (i.e., if I is an independent set then $m_{in} = 0$ and $m_{out} = m$). By the properties of our gadget we know that for every edge $\{u, v\}$ of G, we can cut at most three edges when both u and v are in S, and at most four edges otherwise. Hence the number C of edges cut by S satisfies $C \leq |I| + 3m_{in} + 4m_{out} = |I| + 3m_{in} + 4(m - m_{in}) = |I| + 4m - m_{in}$. Since $C = k + 4m$ we get that $|I| - m_{in} \geq k$. Now we can transform I into an independent set I' by going over every one of the m_{in} edges that are inside I and removing one of the endpoints of the edge from it. The resulting set I' is an independent set in the graph G of size $|I| - m_{in} \geq k$ and so this concludes the proof of the soundness condition.

■

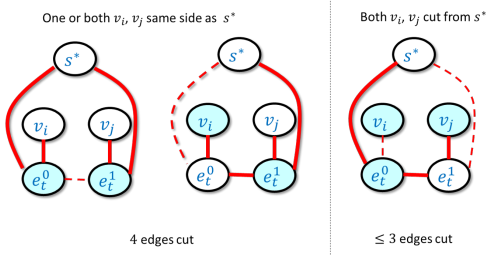
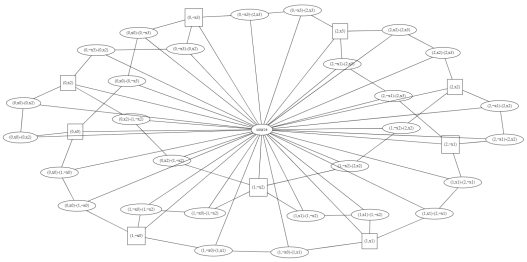


Figure 14.11: In the reduction of independent set to max cut, for every $t \in [m]$, we have a "gadget" corresponding to the t-th edge $e = \{v_i, v_j\}$ in the original graph. If we think of the side of the cut containing the special source vertex s^* as "white" and the other side as "blue", then the leftmost and center figures show that if v_i and v_j are not both blue then we can cut four edges from the gadget. In contrast, by enumerating all possibilities one can verify that if both u and v are blue, then no matter how we color the intermediate vertices e_t^0, e_t^1, we will cut at most three edges from the gadget. The figure above contains only the gadget edges and ignores the edges connecting s^* to the vertices $v_0, \ldots, v_{n-1}$.



```python
# Reduction IS to MAXCUT
def IS2MAXCUT(G):
    G =nxgraph(G)
    H =Graph(engine='sfdp')
    s ="source"
    H.node(s) # create source node

    for v in G.nodes():
        H.node(v,shape="square")
        H.edge(s,v)

    j =0
    for (u,v) in G.edges():
        g1 = "e"+str(j)+"a"
        g2 = "e"+str(j)+"b"
        H.node(g1,label=u+"-"+v)
        H.node(g2,label=u+"-"+v)
        # add gadget
        H.edges([(s,g1),(s,g2),(g1,g2),(u,g1),(v,g2)])
        j +=1


    return H
```

Figure 14.12: The reduction of independent set to max cut. On the right-hand side is Python code implementing the reduction. On the left-hand side is an example output of the reduction where we apply it to the independent set instance that is obtained by running the reduction of Theorem 14.6 on the 3CNF formula $(x_0 \vee \overline{x}_3 \vee x_2) \wedge (\overline{x}_0 \vee x_1 \vee \overline{x}_2) \wedge (\overline{x}_1 \vee x_2 \vee x_3)$.

14.7 REDUCING 3SAT TO LONGEST PATH

Note: This section is still a little messy; feel free to skip it or just read it without going into the proof details. The proof appears in Section 7.5 in Sipser's book.

One of the most basic algorithms in Computer Science is Dijkstra's algorithm to find the *shortest path* between two vertices. We now show that in contrast, an efficient algorithm for the *longest path* problem would imply a polynomial-time algorithm for 3SAT.

Theorem 14.10 — Hardness of longest path.

$$3SAT \leq_p LONGPATH \tag{14.6}$$

Proof Idea:

To prove Theorem 14.10 need to show how to transform a 3CNF formula φ into a graph G and two vertices s, t such that G has a path of length at least k if and only if φ is satisfiable. The idea of the reduction is sketched in Fig. 14.13 and Fig. 14.14. We will construct a graph that contains a potentially long "snaking path" that corresponds to all variables in the formula. We will add a "gadget" corresponding to each clause of φ in a way that we would only be able to use the gadgets if we have a satisfying assignment.

⋆

```python
def TSAT2LONGPATH(ϕ):
    """Reduce 3SAT to LONGPATH"""
    def var(v): # return variable and True/False depending
     ↪  if positive or negated
        return int(v[2:]),False if v[0]=="¬" else
         ↪  int(v[1:]),True
    n = numvars(ϕ)
    clauses = getclauses(ϕ)
    m = len(clauses)
    G =Graph()
```

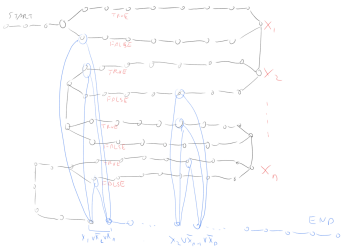


Figure 14.13: We can transform a 3SAT formula φ into a graph G such that the longest path in the graph G would correspond to a satisfying assignment in φ. In this graph, the black colored part corresponds to the variables of φ and the blue colored part corresponds to the vertices. A sufficiently long path would have to first "snake" through the black part, for each variable choosing either the "upper path" (corresponding to assigning it the value `True`) or the "lower path" (corresponding to assigning it the value `False`). Then to achieve maximum length the path would traverse through the blue part, where to go between two vertices corresponding to a clause such as $x_{17} \vee \overline{x}_{32} \vee x_{57}$, the corresponding vertices would have to have been not traversed before.

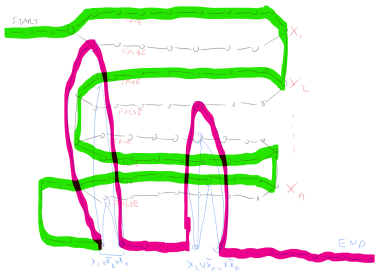


Figure 14.14: The graph above with the longest path marked on it, the part of the path corresponding to variables is in green and part corresponding to the clauses is in pink.

```
    G.edge("start","start_0")
    for i in range(n): # add 2 length-m paths per variable
        G.edge(f"start_{i}",f"v_{i}_{0}_T")
        G.edge(f"start_{i}",f"v_{i}_{0}_F")
        for j in range(m-1):
            G.edge(f"v_{i}_{j}_T",f"v_{i}_{j+1}_T")
            G.edge(f"v_{i}_{j}_F",f"v_{i}_{j+1}_F")
        G.edge(f"v_{i}_{m-1}_T",f"end_{i}")
        G.edge(f"v_{i}_{m-1}_F",f"end_{i}")
        if i<n-1:
            G.edge(f"end_{i}",f"start_{i+1}")
    G.edge(f"end_{n-1}","start_clauses")
    for j,C in enumerate(clauses): # add gadget for each
     ↪  clause
        for v in enumerate(C):
            i,sign = var(v[1])
            s = "F" if sign else "T"
            G.edge(f"C_{j}_in",f"v_{i}_{j}_{s}")
            G.edge(f"v_{i}_{j}_{s}",f"C_{j}_out")
        if j<m-1:
            G.edge(f"C_{j}_out",f"C_{j+1}_in")
    G.edge("start_clauses","C_0_in")
    G.edge(f"C_{m-1}_out","end")
    return G, 1+n*(m+1)+1+2*m+1
```

Proof of Theorem 14.10. We build a graph G that "snakes" from s to t as follows. After s we add a sequence of n long loops. Each loop has an "upper path" and a "lower path". A simple path cannot take both the upper path and the lower path, and so it will need to take exactly one of them to reach s from t.

Our intention is that a path in the graph will correspond to an assignment $x \in \{0,1\}^n$ in the sense that taking the upper path in the i^{th} loop corresponds to assigning $x_i = 1$ and taking the lower path corresponds to assigning $x_i = 0$. When we are done snaking through all the n loops corresponding to the variables to reach t we need to pass through m "obstacles": for each clause j we will have a small gadget consisting of a pair of vertices s_j, t_j that have three paths between them. For example, if the j^{th} clause had the form $x_{17} \vee \overline{x}_{55} \vee x_{72}$ then one path would go through a vertex in the lower loop corresponding to x_{17}, one path would go through a vertex in the upper loop corresponding to x_{55} and the third would go through the lower loop corresponding to x_{72}. We see that if we went in the first stage according to a satisfying assignment then we will be able to find a free vertex to travel from s_j to t_j. We link t_1 to s_2, t_2 to s_3, etc and link t_m to t. Thus

a satisfying assignment would correspond to a path from s to t that goes through one path in each loop corresponding to the variables, and one path in each loop corresponding to the clauses. We can make the loop corresponding to the variables long enough so that we must take the entire path in each loop in order to have a fighting chance of getting a path as long as the one corresponds to a satisfying assignment. But if we do that, then the only way if we are able to reach t is if the paths we took corresponded to a satisfying assignment, since otherwise we will have one clause j where we cannot reach t_j from s_j without using a vertex we already used before.

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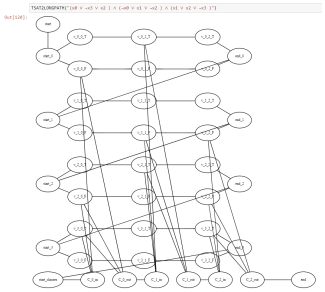

Figure 14.15: The result of applying the reduction of $3SAT$ to $LONGPATH$ to the formula $(x_0 \vee \neg x_3 \vee x_2) \wedge (\neg x_0 \vee x_1 \vee \neg x_2) \wedge (x_1 \vee x_2 \vee \neg x_3)$.

14.7.1 Summary of relations

We have shown that there are a number of functions F for which we can prove a statement of the form "If $F \in \mathbf{P}$ then $3SAT \in \mathbf{P}$". Hence coming up with a polynomial-time algorithm for even one of these problems will entail a polynomial-time algorithm for $3SAT$ (see for example Fig. 14.16). In Chapter 15 we will show the inverse direction ("If $3SAT \in \mathbf{P}$ then $F \in \mathbf{P}$") for these functions, hence allowing us to conclude that they have *equivalent complexity* to $3SAT$.

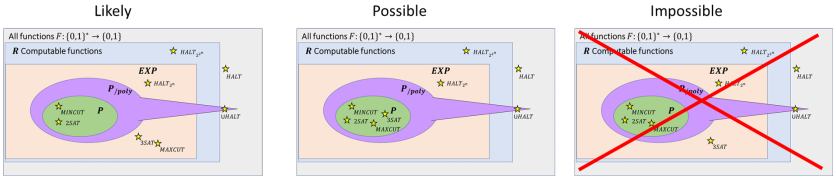

Figure 14.16: So far we have shown that $\mathbf{P} \subseteq \mathbf{EXP}$ and that several problems we care about such as $3SAT$ and $MAXCUT$ are in $\mathbf{EXP}$ but it is not known whether or not they are in $\mathbf{EXP}$. However, since $3SAT \leq_p MAXCUT$ we can rule out the possiblity that $MAXCUT \in \mathbf{P}$ but $3SAT \notin \mathbf{P}$. The relation of $\mathbf{P_{/poly}}$ to the class $\mathbf{EXP}$ is not known. We know that $\mathbf{EXP}$ does not contain $\mathbf{P_{/poly}}$ since the latter even contains uncomputable functions, but we do not know whether ot not $\mathbf{EXP} \subseteq \mathbf{P_{/poly}}$ (though it is believed that this is not the case and in particular that both $3SAT$ and $MAXCUT$ are not in $\mathbf{P_{/poly}}$).

Chapter Recap

- The computational complexity of many seemingly unrelated computational problems can be related to one another through the use of *reductions*.
- If $F \leq_p G$ then a polynomial-time algorithm for G can be transformed into a polynomial-time algorithm for F.
- Equivalently, if $F \leq_p G$ and F does *not* have a polynomial-time algorithm then neither does G.
- We've developed many techniques to show that $3SAT \leq_p F$ for interesting functions F. Sometimes we can do so by using *transitivity* of reductions: if $3SAT \leq_p G$ and $G \leq_p F$ then $3SAT \leq_p F$.

14.8 EXERCISES

14.9 BIBLIOGRAPHICAL NOTES

Several notions of reductions are defined in the literature. The notion defined in Definition 14.1 is often known as a *mapping reduction*, *many to one reduction* or a *Karp reduction*.

The *maximal* (as opposed to *maximum*) independent set is the task of finding a "local maximum" of an independent set: an independent set S such that one cannot add a vertex to it without losing the independence property (such a set is known as a *vertex cover*). Unlike finding a *maximum* independent set, finding a *maximal* independent set can be done efficiently by a greedy algorithm, but this local maximum can be much smaller than the global maximum.

Reduction of independent set to max cut taken from these notes. Image of Hamiltonian Path through Dodecahedron by Christoph Sommer.

We have mentioned that the line between reductions used for algorithm design and showing hardness is sometimes blurry. An excellent example for this is the area of *SAT Solvers* (see [Gom+08]). In this field people use algorithms for SAT (that take exponential time in the worst case but often are much faster on many instances in practice) together with reductions of the form $F \leq_p SAT$ to derive algorithms for other functions F of interest.