CSE 462 | ALGORITHM ENGINEERING SESSIONAL

The Subset Sum Problem

Group 6

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INTRODUCTION

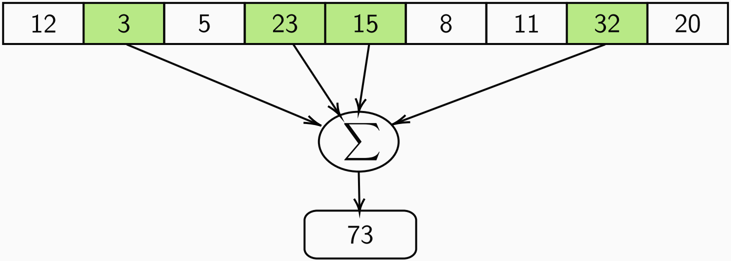
The Subset Sum Problem

Let's assume we're given a set of integers. Can we find a subset that sums up to a given target integer?

Input Set:

12	3	5	23	15	8	11	32	20

Selection of Subset:



The Subset Sum Problem

Formal definition of the problem:
Given a Multiset of integers, $S = \{x_1, x_2, x_3, \cdots, x_n\}$ and a target sum W, does there exist a subset $S' \subseteq S$ such that $\sum_{x \in S'} x = W$?

EXACT EXPONENTIAL ALGORITHMS

Exact Exponential Algorithms

Exact algorithms are algorithms that always solve an optimization problem to optimality.

Unless $\mathsf{P} = \mathsf{NP}$, an exact algorithm for an NP-hard optimization problem cannot run in worst-case polynomial time.

BRUTE-FORCE ALGORITHM

Brute-Force Algorithm

A brute-force algorithm is a method of problem-solving in which every possible scenario is examined and the best one is chosen.

For the subset sum problem, a brute-force algorithm would require:

- **Finding all possible subsets of S.** (Total $= 2^n$)
- **Finding the sum of the elements of every subset.** ($\forall S' \subseteq S$, find $\sum_{x \in S'} x$)
- **Checking if any of these sums equals target sum W.** (Check if $\exists S' \subseteq S$ for which $\sum_{x \in S'} x = W$?)

Brute-Force Algorithm

$$S = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline\end{array} \qquad W = \boxed{10}$$

Brute-Force Algorithm

$S = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline\end{array}$ $\qquad W = \boxed{10}$

$s_0 = \boxed{}$

Brute-Force Algorithm

$S = \boxed{1 \mid 3 \mid 6}$ $W = \boxed{10}$

$s_1 = \boxed{1}$

$s_0 = \boxed{}$ $s_2 = \boxed{3}$

$s_3 = \boxed{6}$

Brute-Force Algorithm

$S = \boxed{1 \mid 3 \mid 6}$ $\quad W = \boxed{10}$

$s_0 = \boxed{}$

$s_1 = \boxed{1}$ $\quad s_4 = \boxed{1 \mid 3}$

$s_2 = \boxed{3}$ $\quad s_5 = \boxed{1 \mid 6}$

$s_3 = \boxed{6}$ $\quad s_6 = \boxed{3 \mid 6}$

Brute-Force Algorithm

$S = \boxed{1 \mid 3 \mid 6}$ $\quad W = \boxed{10}$

$s_1 = \boxed{1}$ $\quad s_4 = \boxed{1 \mid 3}$

$s_0 = \boxed{}$ $\quad s_2 = \boxed{3}$ $\quad s_5 = \boxed{1 \mid 6}$ $\quad s_7 = \boxed{1 \mid 3 \mid 6}$

$s_3 = \boxed{6}$ $\quad s_6 = \boxed{3 \mid 6}$

Brute-Force Algorithm

$S = \boxed{1} \boxed{3} \boxed{6}$ $\quad W = \boxed{10}$

$s_0 = \boxed{}$ 0

$s_1 = \boxed{1}$ 1

$s_2 = \boxed{3}$ 3

$s_3 = \boxed{6}$ 6

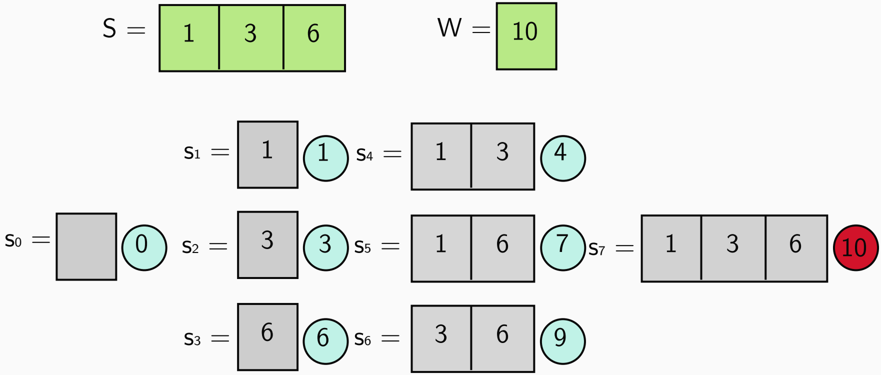
$s_4 = \boxed{1} \boxed{3}$ 4

$s_5 = \boxed{1} \boxed{6}$ 7

$s_6 = \boxed{3} \boxed{6}$ 9

$s_7 = \boxed{1} \boxed{3} \boxed{6}$ 10

Brute-Force Algorithm



Brute-Force Algorithm

Algorithm 1: BruteForceSubsetSum(S,i,r,W)

Input: Set S, Index of element (i), Remaining sum(r), Target (W)

Output: A decision on whether there is a subset whose sum is W.

```
if i == 0 then
    return (r == 0);
else
    if BruteForceSubsetSum(S,i-1,r,W) == true then
        return true ;
    else if (r − w_i ≥ 0) then
        return (BruteForceSubsetSum(S,i-1,r-w_i,W) == true) ;
    else
        return false ;
    end
end
```

Brute-Force Algorithm Time-Complexity

S	1	3	6

Target sum = 10

Subsets of ***S***

S_0	$\{ \}$		S_1	$\{1\}$	1
S_2	$\{3\}$	3	S_3	$\{6\}$	6
S_4	$\{1, 3\}$	4	S_5	$\{1, 6\}$	7
S_6	$\{3, 6\}$	9	S_7	$\{1, 3, 6\}$	10

Here, n = 3. By checking all $2^3 = 8$ subsets, the solution is S_7

- For, input set of size n, there are 2^n possible subsets.
- Each subset can be checked in $O(n)$ time.
- Overall, complexity of brute-force algorithm is exponential, $O(2^n n)$

BETTER EXPONENTIAL ALGORITHMS

Better Exponential Algorithms

- Backtracking Algorithm
- Branch and Bound Algorithm
- Dynamic Programming Algorithm

SUBSET SUM USING BACKTRACKING ALGORITHM

Backtracking

- Search the solution space tree in depth first manner.
- Upon reaching a candidate that cannot be a valid solution, backtrack.
- Search tree for subset problem is a binary tree of 2^n leaves mapping to 2^n subsets. (Leaves represent members of solution space.)
- With effective bounding functions, large instances can be solved.
  Examples of Bounding Functions:
  - When sum of a node equals target sum, terminate.
  - When sum of a node exceeds target sum, backtrack.

Backtracking Time-complexity

- There are 2^n leaf nodes.
- Forward and backward moves through the tree takes $O(1)$ time.
- So, Subset Sum Backtracking with Bounding Functions has complexity of $O(2^n)$
- Better than brute force solution that takes $O(2^n n)$.

BRANCH AND BOUND ALGORITHM

Branch and Bound Algorithm

- Let, $S = x_1, x_2, ...x_n$ such that $x_1 \geq x_i;\ for\ i = 2, 3...n$ and W is the target sum.
- There are 2^n possible subsets.
- There is a binary tree rooted at the empty subset with starting level -1 for convenience.
- Each node in level i is divided into a left and right branch, to either exclude x_i from the subset so far, or include x_i in the subset so far.

Branch and Bound Algorithm

- For each node, calculate two values: Sum So Far, SSF and Maximum Potential Sum, $MPS = SSF + \sum_{j=i+1}^{n-1} x_j$
- If $SSF > W$ then there is no node beneath current node that can add up to W. No point in creating those nodes.
- If $MPS < W$ then there is no node beneath that node that can add up to W. No point in creating those nodes.
- If $SSF = W$ or $MPS = W$, then this subset should be recorded in the list of solutions.
- At a node on level i, when $SSF < W < MPS$ the potential for a solution exists and the two branches to exclude or include x_{i+1} should be created and investigated.
- Complexity: $O(2^n)$ [1]

Dynamic Programming Algorithm

Dynamic Programming Algorithm attempts to solve a problem by combining solutions of the sub-problems. [2]

Steps in Dynamic Programming Algorithm for Subset Sum problem:

- For set S (where $|S| = n$) and target sum W, make a table T with $n + 1$ rows and $W + 1$ columns.
- Populate first row with 0s, and the top left cell with 1.
- For every other cell T[i,j], copy the value of previous row T[i-1,j].
- Check if current column index (j) exceeds or equals value of ith element (w_i).
- If so, populate cell with $max(T[i-1, j], T[i-1, j-w_i])$.

Problem: Given a set $S = \{1, 3, 7, 9, 12\}$ and $W = 11$, is there a subset $S' \in S$, such that $\sum_{x \in S'} x = W$?

For $S' = \{1\}, sum = 1 < W$
$i = 1, j = 0, w_i = 1, j < w_i$
So, $T[i, j] = T[i-1, j]$
$T[1, 0] = T[0, 0] = 1$

	0	1	2	3	4	5	6	7	8	9	10	11
$\emptyset$	1	0	0	0	0	0	0	0	0	0	0	0
1	1											
3												
7												
9												
12												

Dynamic Programming

For $S' = \{1\}, sum = 1 < W$

$i = 1, j = 1, w_i = 1, j = w_i$

So, $T[i, j] = T[i-1, j] \mid\mid T[i-1, j-w_i]$

$T[1, 1] = T[0, 1] \mid\mid T[0, 0] = 1$

	0	1	2	3	4	5	6	7	8	9	10	11
$\emptyset$	1	0	0	0	0	0	0	0	0	0	0	0
1	1	1										
3												
7												
9												
12												

Dynamic Programming

Complete table:

	0	1	2	3	4	5	6	7	8	9	10	11
$\emptyset$	1	0	0	0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0	0	0	0
3	1	1	0	1	1	0	0	0	0	0	0	0
7	1	1	0	1	1	0	0	1	1	0	1	1
9	1	1	0	1	1	0	0	1	1	1	1	1
12	1	1	0	1	1	0	0	1	1	1	1	1

As $T[5, 11] = 1$, there exists a subset $S' \in S$ where $\sum_{x \in S'} x = W = 11$.
Now to get the subset S' -

	0	1	2	3	4	5	6	7	8	9	10	11
$\emptyset$	1	0	0	0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0	0	0	0
3	1	1	0	1	1	0	0	0	0	0	0	0
7	1	1	0	1	1	0	0	1	1	0	1	1
9	1	1	0	1	1	0	0	1	1	1	1	1
12	1	1	0	1	1	0	0	1	1	1	1	1

Dynamic Programming

	0	1	2	3	4	5	6	7	8	9	10	11
∅	1	0	0	0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0	0	0	0
3	1	1	0	1	1	0	0	0	0	0	0	0
7	1	1	0	1	1	0	0	1	1	0	1	1
9	1	1	0	1	1	0	0	1	1	1	1	1
12	1	1	0	1	1	0	0	1	1	1	1	1

Dynamic Programming

	0	1	2	3	4	5	6	7	8	9	10	11
∅	1	0	0	0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0	0	0	0
3	1	1	0	1	1	0	0	0	0	0	0	0
7	1	1	0	1	1	0	0	1	1	0	1	1
9	1	1	0	1	1	0	0	1	1	1	1	1
12	1	1	0	1	1	0	0	1	1	1	1	1

So, $7 \in S'$, $S' = \{7\}$

	0	1	2	3	4	5	6	7	8	9	10	11
∅	1	0	0	0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0	0	0	0
3	1	1	0	1	1	0	0	0	0	0	0	⊗
7	1	1	0	1	1	0	0	1	1	0	1	1
9	1	1	0	1	1	0	0	1	1	1	1	1
12	1	1	0	1	1	0	0	1	1	1	1	1

$S' = \{7\}$

	0	1	2	3	4	5	6	7	8	9	10	11
$\emptyset$	1	0	0	0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0	0	0	0
3	1	1	0	1	1	0	0	0	0	0	0	0
7	1	1	0	1	1	0	0	1	1	0	1	1
9	1	1	0	1	1	0	0	1	1	1	1	1
12	1	1	0	1	1	0	0	1	1	1	1	1

Dynamic Programming

So, $3 \in S'$, $S' = \{7, 3\}$

	0	1	2	3	4	5	6	7	8	9	10	11
∅	1	0	0	0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0	0	0	0
3	1	1	0	1	1	0	0	0	0	0	0	0
7	1	1	0	1	1	0	0	1	1	0	1	1
9	1	1	0	1	1	0	0	1	1	1	1	1
12	1	1	0	1	1	0	0	1	1	1	1	1

$S' = \{7, 3\}$

	0	1	2	3	4	5	6	7	8	9	10	11
∅	1	0	0	0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0	0	0	0
3	1	1	0	1	1	0	0	0	0	0	0	0
7	1	1	0	1	1	0	0	1	1	0	1	1
9	1	1	0	1	1	0	0	1	1	1	1	1
12	1	1	0	1	1	0	0	1	1	1	1	1

So, $1 \in S'$, $S' = \{7, 3, 1\}$

	0	1	2	3	4	5	6	7	8	9	10	11
∅	1	$\otimes$	0	0	0	0	0	0	0	0	0	0
1	1	1	0	0	$\otimes$	0	0	0	0	0	0	0
3	1	1	0	1	1	0	0	0	0	0	0	$\otimes$
7	1	1	0	1	1	0	0	1	1	0	1	1
9	1	1	0	1	1	0	0	1	1	1	1	1
12	1	1	0	1	1	0	0	1	1	1	1	1

$S' = \{7, 3, 1\}$

	0	1	2	3	4	5	6	7	8	9	10	11
∅	1	0	0	0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0	0	0	0
3	1	1	0	1	1	0	0	0	0	0	0	0
7	1	1	0	1	1	0	0	1	1	0	1	1
9	1	1	0	1	1	0	0	1	1	1	1	1
12	1	1	0	1	1	0	0	1	1	1	1	1

Algorithm 2: DPSubsetSum(n,W)

Input: Number of elements (n), Target (W)

Output: A decision on whether there is a subset whose sum is W.

```
for j = 1 to W do T[0,j] = 0
T[0,0] = 1
for i = 1 to n do
    for j = 0 to W do
        T[i,j] = T[i-1,j]
        if j ≥ w_i then
            T[i,j] = max{T[i-1,j],T[i-1,j-w_i]}
    end
end
return T[n,W]
```

DPSubsetSum Algorithm Time Complexity

- Since the table formed has n+1 rows and W+1 columns, the algorithm runs in $O(nW)$.
- If W is represented in binary, its size is $log_2 W$.
- W is thus exponential in input size.
- But if input is given in unary, then $O(nW)$ is polynomial in size of input. So it is a pseudopolynomial algorithm.
- Better than Brute-Force approach that takes $O(2^n n)$

VARIATION OF SUBSET SUM PROBLEM

Polynomial Time Variation

SSP is solvable in polynomial time if the sequence $\{a_1, a_2, \cdots, a_n\}$ is restricted to be an arithmetic progression. [3]
The sequence can be specified concisely by the triple (a_1, n, j).

Expanded set from triple, $T = \{a, a + j, \cdots, a + (n - 1)j\}$

- If $S \subseteq T$ with $|S| = k$, let $SS_k = \sum_{s \in S} s$

Polynomial Time Variation

- If $S \subseteq T$ with $|S| = k$, let $SS_k = \sum_{s \in S} s$
- $SS_k = ka + mj, m \in Z^{\geq}$

Polynomial Time Variation

- If $S \subseteq T$ with $|S| = k$, let $SS_k = \sum_{s \in S} s$
- $SS_k = ka + mj, m \in Z^{\geq}$
- $c_k = MIN\{SS_k\}$ (Leftmost k elements)
  $d_k = MAX\{SS_k\}$ (Rightmost k elements)

Polynomial Time Variation

- For any k, SS_k can take any of the values from $c_k, c_k + j, c_k + 2j, \cdots, d_k - 2j, d_k - j, d_k$

Polynomial Time Variation

- For any k, SS_k can take any of the values from $c_k, c_k + j, c_k + 2j, \cdots, d_k - 2j, d_k - j, d_k$
- Increasing k causes c_k to increase

Polynomial Time Variation

- For any k, SS_k can take any of the values from $c_k, c_k + j, c_k + 2j, \cdots, d_k - 2j, d_k - j, d_k$
- Increasing k causes c_k to increase
- To achieve t from SS_k, $ka \equiv t \ (mod\ j)$ must hold

Polynomial Time Variation

- For any k, SS_k can take any of the values from $c_k, c_k + j, c_k + 2j, \cdots, d_k - 2j, d_k - j, d_k$
- Increasing k causes c_k to increase
- To achieve t from SS_k, $ka \equiv t\ (mod\ j)$ must hold
- Let k_1 be the lowest k with $d_k \geq t$

Polynomial Time Variation

- For any k, SS_k can take any of the values from $c_k, c_k + j, c_k + 2j, \cdots, d_k - 2j, d_k - j, d_k$
- Increasing k causes c_k to increase
- To achieve t from SS_k, $ka \equiv t \ (mod \ j)$ must hold
- Let k_1 be the lowest k with $d_k \geq t$
- k_1 can be found using binary search

Polynomial Time Variation

- For any k, SS_k can take any of the values from $c_k, c_k + j, c_k + 2j, \cdots, d_k - 2j, d_k - j, d_k$
- Increasing k causes c_k to increase
- To achieve t from SS_k, $ka \equiv t \ (mod\ j)$ must hold
- Let k_1 be the lowest k with $d_k \geq t$
- k_1 can be found using binary search
- Solution exists if and only if $c_{k_1} \leq t$

Polynomial Time Variation

Algorithm 3: PolynomialTimeSubsetSum(a,n,j,t)

Input: First element (a), increment value (j), in(r), Target (W)

Output: A decision on whether there is a subset whose sum is W.

Other polynomial time variation

- Low-Density Subset Sum Problem belongs to the class P.

Given a set $A = \{a_i : 1 < i < n\}$ of positive integers and positive integer M, find a subset of A that has sum equal to M.

The density of these a_i is defined by,

$$d = \frac{n}{log_2(max_i\ a_i)} \tag{1}$$

- [4] converts the problem to one of finding a particular short vector v in a lattice, and then uses a lattice based reduction algorithm to find v.
- Then for "almost all" problems of density $d < 0.645$, it is proved that lattice based reduction algorithm locates v in polynomial time.

Other polynomial time variation

- A sub-problem of the problem Subset Sum in which $s_1, \cdots, s_k$ are the members of increasing geometric progression belongs to the class P. [5]

CONCLUSION

Conclusion

- We have discussed exact exponential algorithms.
- We have presented Brute-force approach for the Subset sum problem.
- We have presented some better exact algorithms along with their time complexities.
- We have discussed the Dynamic Programming algorithm for Subset sum problem in detail.
- We have shown some polynomial time variations of Subset Sum problem.

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THANK YOU!