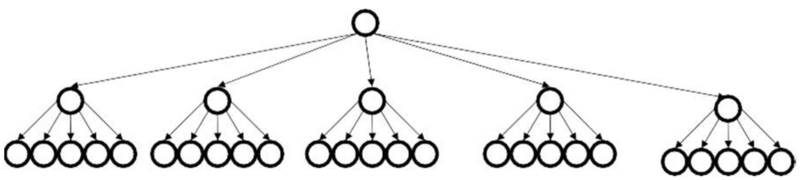
M-ary Search Tree

Suppose, *somehow*, we devised a search tree with maximum branching factor *M*:

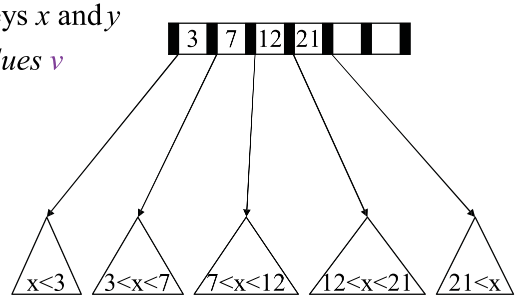


Complete tree has height:

B-Trees

How do we make an *M*-ary search tree work?

- Each **node** has (up to) M-1 keys.
- Order property:
  - subtree between two keys *x* and *y* contain leaves with *values* *v* such that $x < v < y$



B-Tree Structure Properties

Root (special case)

– has between 2 and M children (or root could be a leaf)

Internal nodes

– store up to M-1 keys

– have between ceil(M/2) and M children

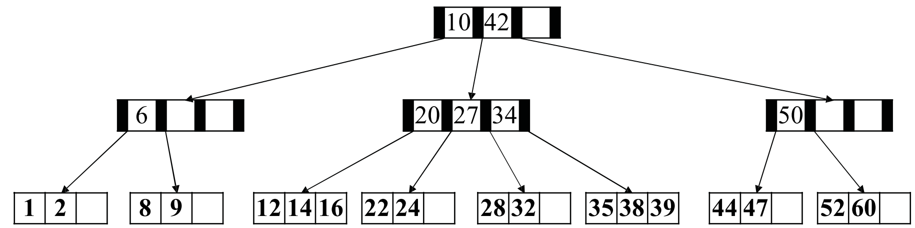
Leaf nodes

– store between ceil((M - 1) / 2) and M-1 sorted keys

– all at the same depth

B-Tree: Example

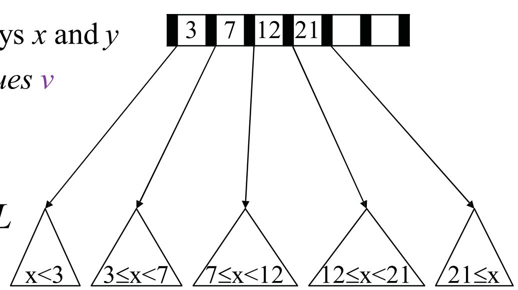
B-Tree with `M = 4`



B+ Trees

In a B+ tree, the internal nodes have no data – only the leaves do!

- Each internal node still has (up to) M-1 keys:
- Order property:
  - subtree between two keys x and y contain leaves with *values* v such that $x \le v < y$
  - Note the "≤"
- Leaf nodes have up to L sorted keys.



B+ Tree Structure Properties

Root (special case)

– has between 2 and M children (or root could be a leaf)

Internal nodes

– store up to M-1 keys

– have between ceil(M/2) and M children

Leaf nodes

– where data is stored

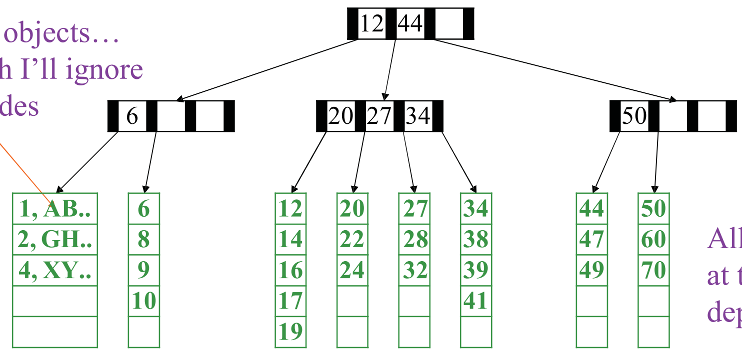
– all at the same depth

– contain between ceil(L/2) and L data items

B+ Tree: Example

B+ Tree with $M = 4$ (# pointers in internal node)
and $L = 5$ (# data items in leaf)

Data objects… which I'll ignore in slides



All leaves at the same depth

Definition for later: "neighbor" is the next sibling to the left or right.

Disk Friendliness

What makes B+ trees disk-friendly?

1. **Many keys stored in a node**
   - All brought to memory/cache in one disk access.
2. Internal nodes contain *only* keys;
   Only leaf nodes contain keys and actual *data*
   - Much of tree structure can be loaded into memory irrespective of data object size
   - Data actually resides in disk

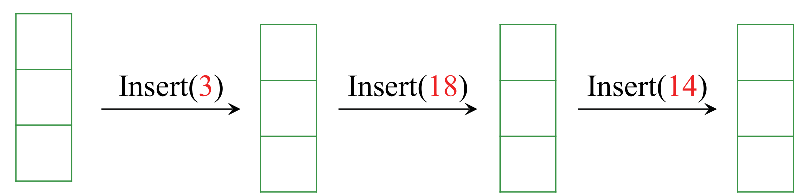
B+ trees vs. AVL trees

Suppose again we have $n = 2^{30} \approx 10^9$ items:

- Depth of AVL Tree
- Depth of B+ Tree with M = 256, L = 256

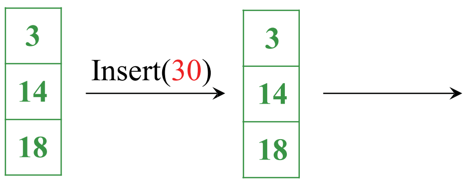
Great, but how to we actually make a B+ tree and keep it balanced…?

Building a B+ Tree with Insertions

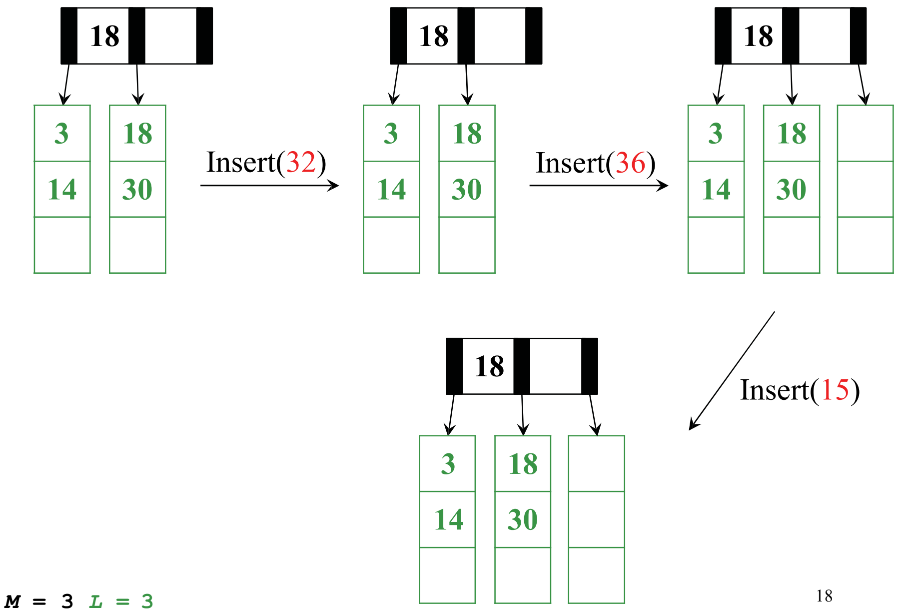


The empty B-Tree

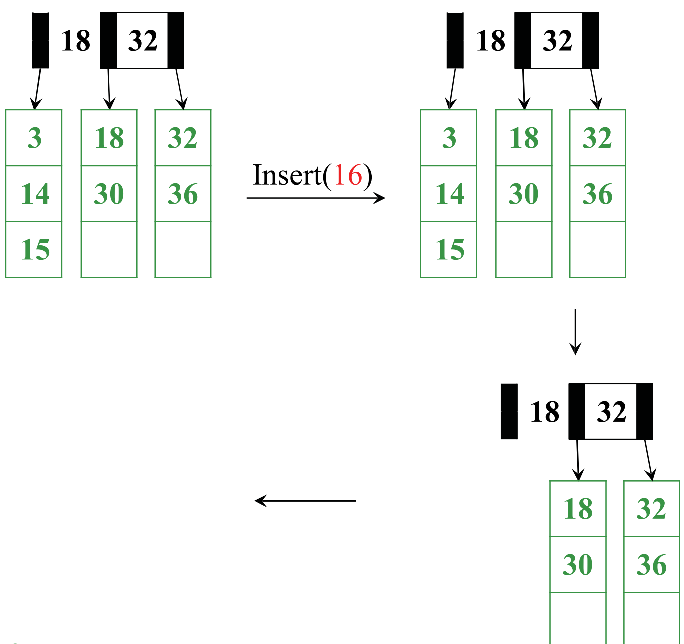
$M = 3$ $L = 3$



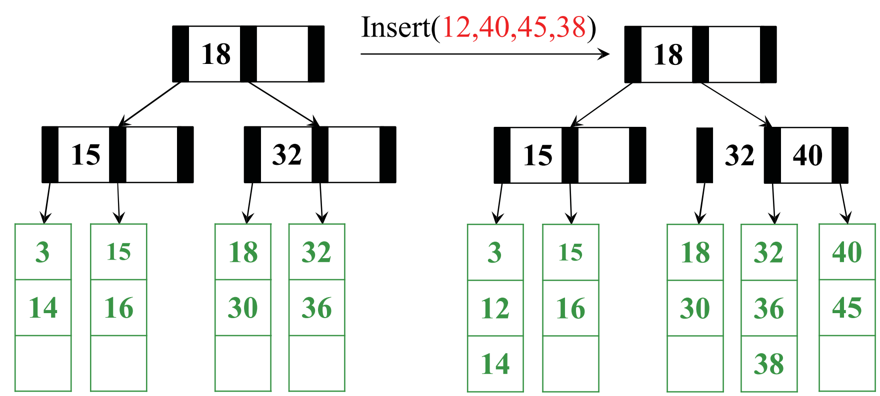
M = 3 L = 3



M = 3 L = 3



M = 3 L = 3



M = 3 L = 3

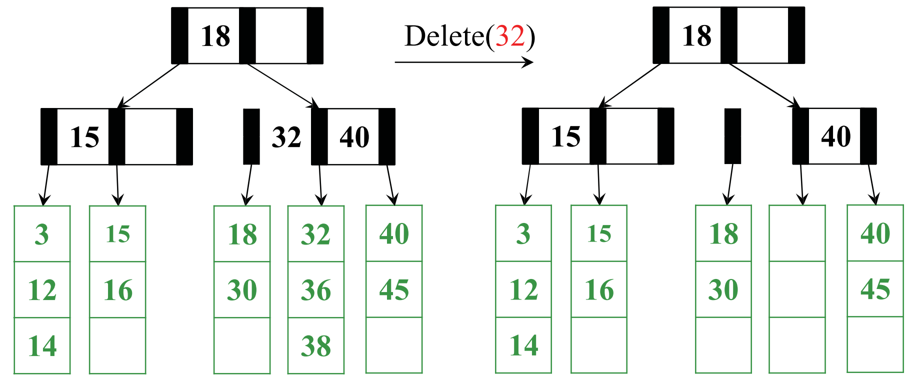
Insertion Algorithm

1. Insert the key in its leaf in sorted order
2. If the leaf ends up with L+1 items, **overflow**!
   - Split the leaf into two nodes:
     - original with ceil `(L+1)/2` items
     - new one with ceil `(L+1)/2` items
   - Add the new child to the parent
   - If the parent ends up with `M+1` children, **overflow**!

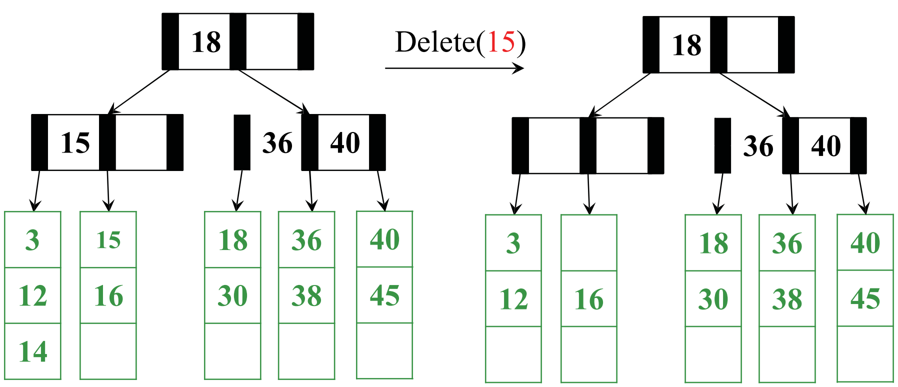
3. If an internal node ends up with M+1 children, **overflow**!
   - Split the node into two nodes:
     - original with ceil `(M+1)/2` children
     - new one with ceil `(M+1)/2` children
   - Add the new child to the parent
   - If the parent ends up with `M+1` items, **overflow**!
4. Split an overflowed root in two and hang the new nodes under a new root
5. Propagate keys up tree.

This makes the tree deeper!

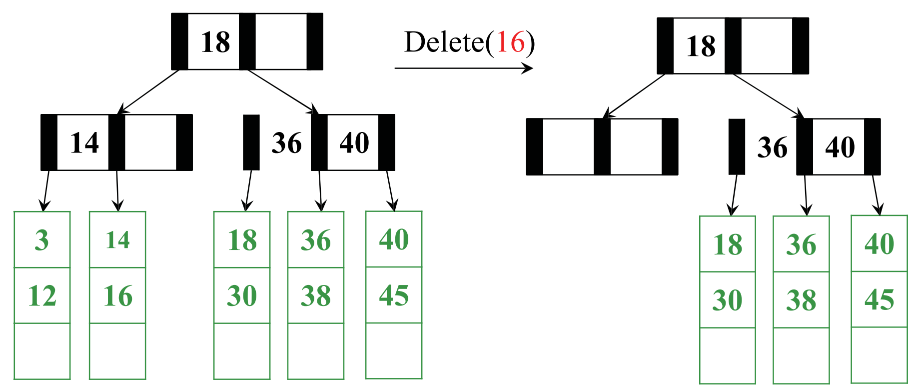
And Now for Deletion…



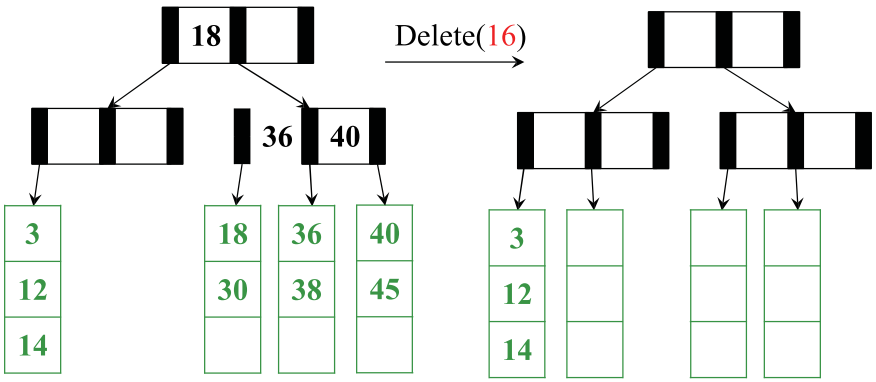
`M = 3 L = 3`



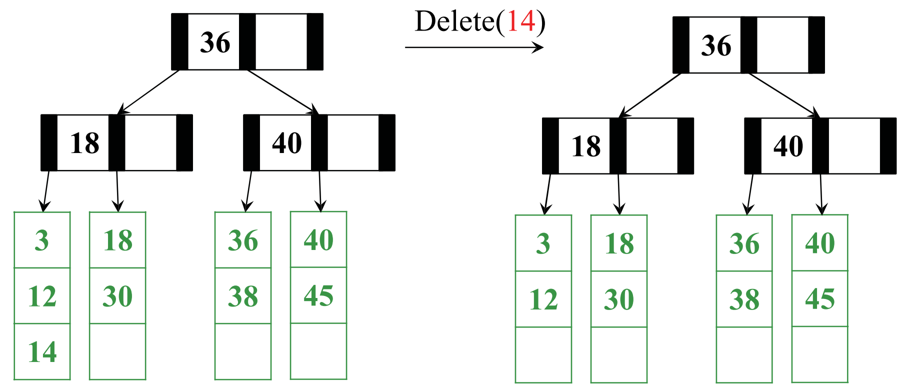
`M = 3 L = 3`



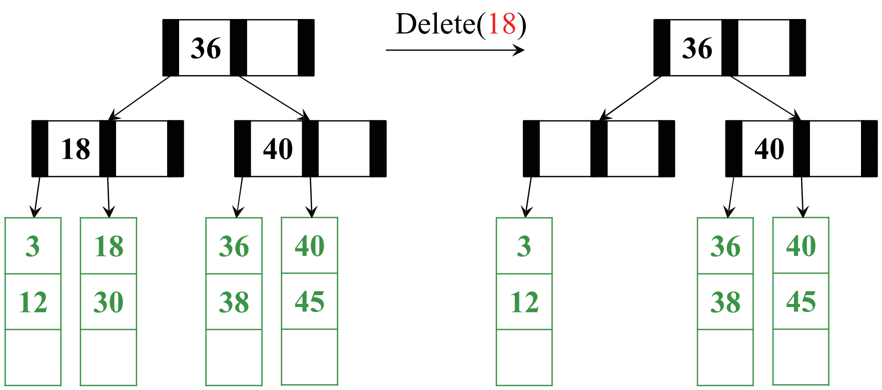
`M = 3 L = 3`



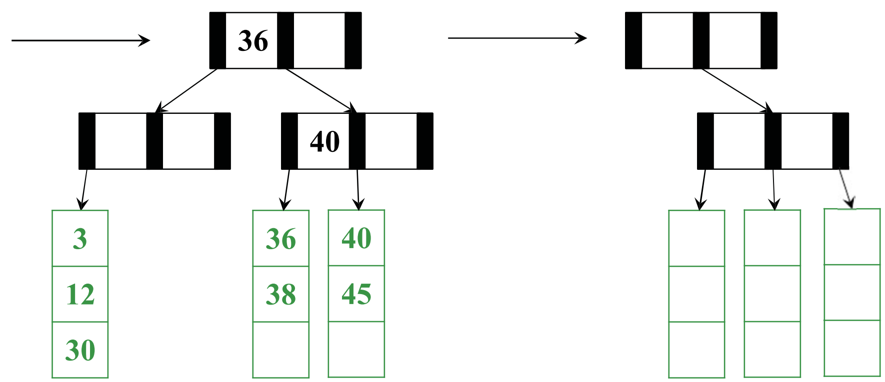
M = 3 *L* = 3



M = 3 *L* = 3



M = 3 *L* = 3



M = 3 *L* = 3

Deletion Algorithm

1. Remove the key from its leaf

2. If the leaf ends up with fewer than ceil(`L / 2`) items, **underflow**!
   - Adopt data from a neighbor; update the parent
   - If adopting won't work, delete node and merge with neighbor
   - If the parent ends up with fewer than ceil(`M / 2`) children, **underflow**!

Deletion Slide Two

3. If an internal node ends up with fewer than ceil(`M / 2`) children, **underflow**!
   - Adopt from a neighbor; update the parent
   - If adoption won't work, merge with neighbor
   - If the parent ends up with fewer than ceil(`M / 2`) children, **underflow**!

4. If the root ends up with only one child, make the child the new root of the tree

This reduces the height of the tree!

5. Propagate keys up through tree.

Thinking about B+ Trees

- B+ Tree insertion can cause (expensive) splitting and propagation
- B+ Tree deletion can cause (cheap) adoption or (expensive) deletion, merging and propagation

Tree Names You Might Encounter

FYI:

- B-Trees with `M = 3`, `L = x` are called **2-3 trees**
  - Nodes can have 2 or 3 keys
- B-Trees with `M = 4`, `L = x` are called **2-3-4 trees**
  - Nodes can have 2, 3, or 4 keys