Design and Analysis of Algorithms

CSE 5311

Lecture 11 Red-Black Trees

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Reviewing: Binary Search Trees

- ***Binary Search Trees*** **(BSTs) are an important data structure for dynamic sets**
  - Each node has at most two children
- Each node contains:
  - key and data
  - left: points to the left child
  - right: points to the right child
  - p(parent): point to parent
- Binary-search-tree property:
  - y is a node in the left subtree of x: $y.key \le x.key$
  - y is a node in the right subtree of x: $y.key \ge x.key$
  - Height: ***h***

Review: Inorder Tree Walk

- An *inorder walk* prints the set in sorted order:

  TreeWalk(x)

      TreeWalk(left[x]);

      print(x);

      TreeWalk(right[x]);

  – Easy to show by induction on the BST property
  – *Preorder tree walk*: print root, then left, then right
  – *Postorder tree walk*: print left, then right, then root

Review: BST Search

```
TreeSearch(x, k)
    if (x = NULL  or  k = key[x])
        return x;
    if (k < key[x])
        return TreeSearch(left[x], k);
    else
        return TreeSearch(right[x], k);
```

Review: Sorting With BSTs

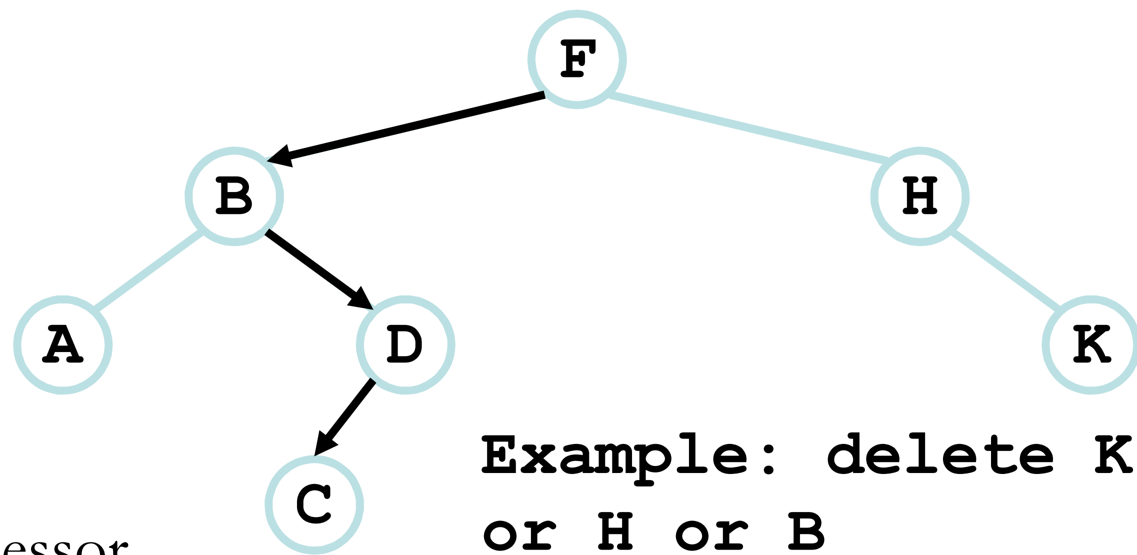
- **Basic algorithm:**
  - Insert elements of unsorted array from 1..*n*
  - Do an inorder tree walk to print in sorted order
- **Running time:**
  - Best case: $\Omega(n \lg n)$ (it's a comparison sort)
  - Worst case: $O(n^2)$
  - Average case: $O(n \lg n)$ (it's a quicksort!)

Review: More BST Operations

- **Minimum:**
  - Find leftmost node in tree
- **Successor:**
  - x has a right subtree: successor is minimum node in right subtree
  - x has no right subtree: successor is first ancestor of x whose left child is also ancestor of x
    - Intuition: As long as you move to the left up the tree, you're visiting smaller nodes.
- **Predecessor: similar to successor**

Review: More BST Operations

- **Delete:**
  - x has no children:
    - ➢Remove x
  - x has one child:
    - ➢Splice out x
  - x has two children:
    - ➢Swap x with successor
    - ➢Perform case 1 or 2 to delete it



Example: delete K or H or B

Red-Black Trees

- ***Red-black trees*:**
  - Binary search tree with an additional attribute for its nodes: color which can be **red** or **black**
  - "Balanced" binary search trees guarantee an O(lgn) running time
  - Constrains the way nodes can be colored on any path from the root to a leaf:

Ensures that no path is more than twice as long as any other path

$\Rightarrow$ the tree is balanced

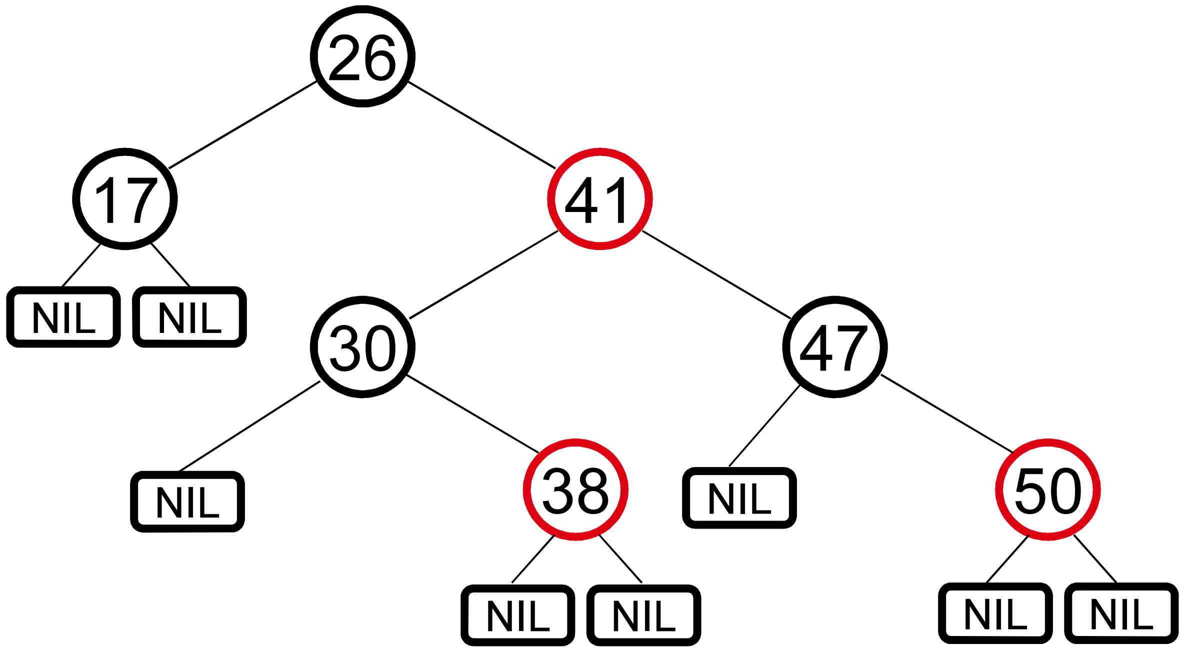
Red-Black Properties (**Satisfy the binary search tree property**)

- **The *red-black properties*:**
  1. Every node is either red or black
  2. Every leaf (NULL pointer) is black
     - Note: this means every "real" node has 2 children
  3. If a node is red, both children are black
     - Note: can't have 2 consecutive reds on a path
  4. Every path from node to descendent leaf contains the same number of black nodes
  5. The root is always black

***black-height:* # black nodes on path to leaf**

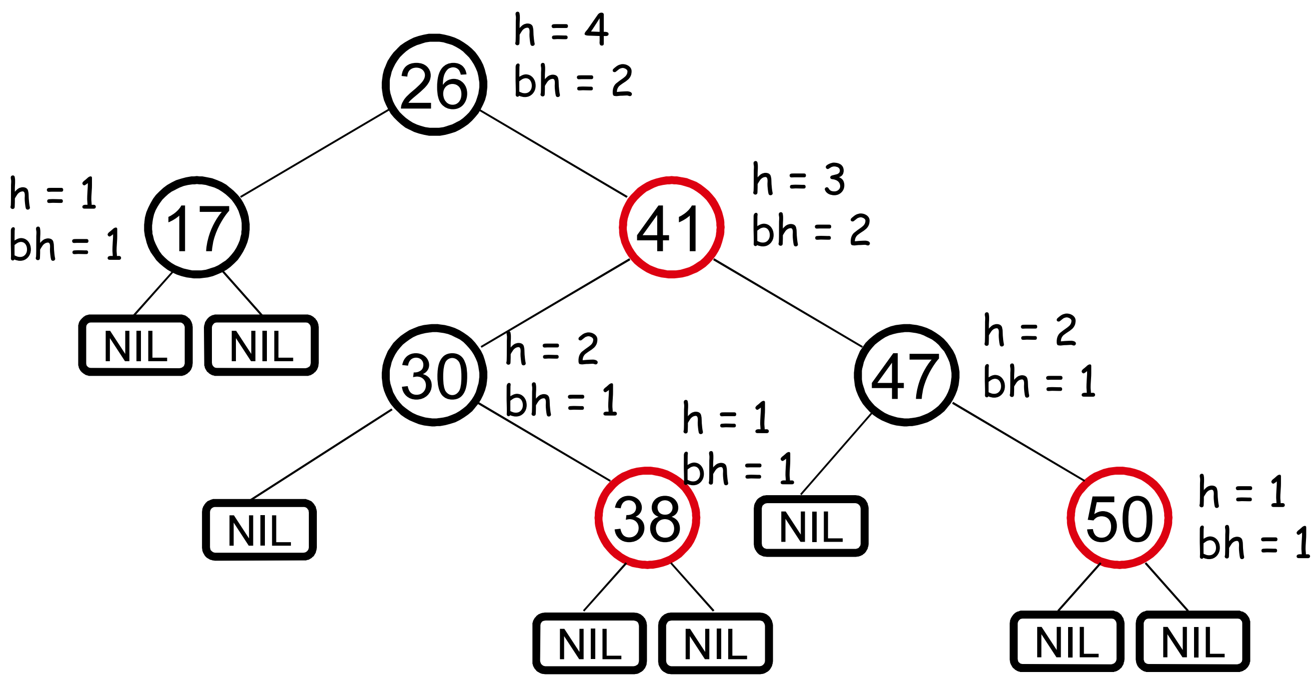
Label example with *h* and bh values

Example: RED-BLACK-TREE



- For convenience we use a sentinel NIL[T] to represent all the NIL nodes at the leafs
  - NIL[T] has the same fields as an ordinary node
  - Color[NIL[T]] = BLACK
  - The other fields may be set to arbitrary values

Black-Height of a Node



- **Height of a node:** the number of edges in the **longest** path to a leaf
- **Black-height** of a node x: bh(x) is the number of black nodes (including NIL) on the path from x to a leaf,

<u>not counting x</u>

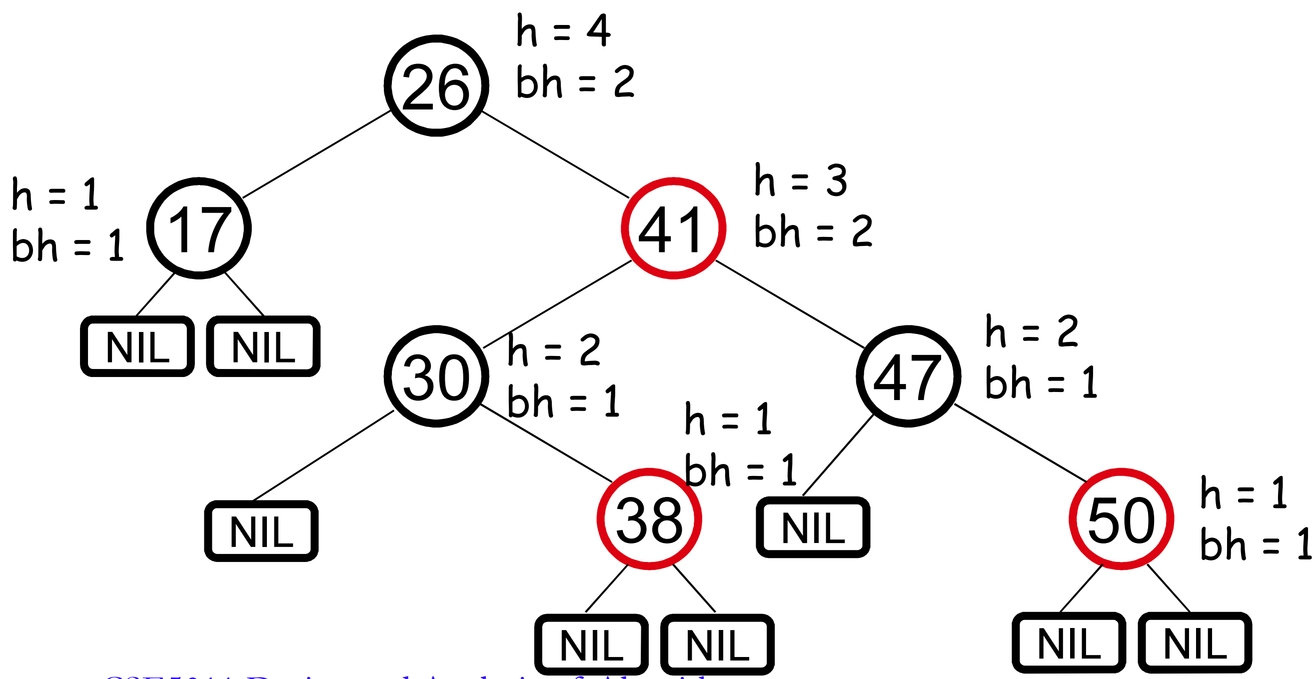
Height of Red-Black Trees

- ***What is the minimum black-height of a node with height h?***
- A: a height-h node has black-height $\geq h/2$
- **Theorem:** A red-black tree with n internal nodes has height $h \leq 2 \lg(n + 1)$
- ***How do you suppose we'll prove this?***
- ***Need to prove two claims first!!!***

Claim 1

4. If a node is **red**, then both its children are **black**
   - **No two consecutive red nodes on a simple path from the root to a leaf**

- Any node x with height h(x) has bh(x) ≥ h(x)/2
- **Proof**
  - By property 4, at most h/2 **red** nodes on the path from the node to a leaf
  - Hence at least h/2 are **black**

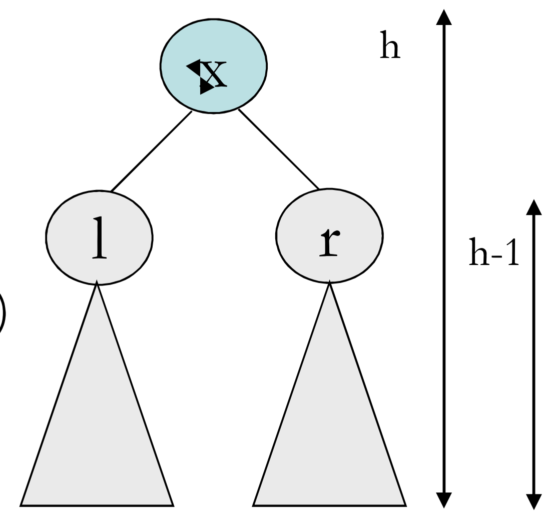


Claim 2

- **A subtree rooted at a node *x* contains at least $2^{bh(x)} - 1$ internal nodes**
- **Proof:**
  - Proof by induction on height *h*
  - Base step: *x* has height 0 (i.e., NULL leaf node)
    - ***What is bh(x)?***
    - A: 0
    - So…subtree contains $2^{bh(x)} - 1$
      $= 2^0 - 1$
      $= 0$ internal nodes (TRUE)

Claim 2: cont'd

- **Inductive proof that subtree at node *x* contains at least $2^{bh(x)}$ - 1 internal nodes**
  - Inductive step: *x* has positive height and 2 children
    - ➢ Each child has black-height of bh(*x*) (if the child is **red**) or bh(*x*)-1 (if the child is **black**)
    - ➢ The height of a child = (height of *x*) - 1
    - ➢ So the subtrees rooted at each child contain at least $2^{bh(x)-1} - 1$ internal nodes
    - ➢ Thus subtree at *x* contains $(2^{bh(x)-1} - 1) + (2^{bh(x)-1} - 1) + 1$ $= 2 \bullet 2^{bh(x)-1} - 1 = 2^{bh(x)} - 1$ nodes



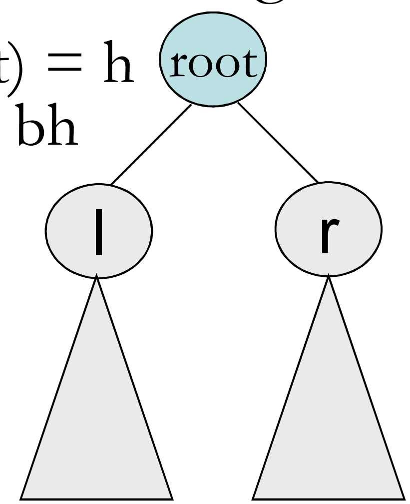
bh(l)≥bh(x)-1

bh(r)≥bh(x)-1

Height of Red-Black-Trees

Lemma: A red-black tree with n internal nodes has height at most 2 lg(n + 1).

height(root) = h
bh(root) = bh



Proof:

$$n \geq 2^{bh} - 1 \geq 2^{h/2} - 1$$

number **n** of internal nodes

since bh ≥ h/2

- Add 1 to both sides and then take logs:

$$n + 1 \geq 2^{bh} \geq 2^{h/2}$$

$$\lg(n + 1) \geq h/2 \Rightarrow$$

$$h \leq 2 \lg(n + 1)$$

RB Trees: Worst-Case Time

- **So we've proved that a red-black tree has O(lg *n*) height**
- **Corollary: These operations take O(lg *n*) time:**
  - Minimum(), Maximum()
  - Successor(), Predecessor()
  - Search()
- **Insert() and Delete():**
  - Will also take O(lg *n*) time
  - But will need special care since they modify tree
  - We have to guarantee that the modified tree will still be a red-black tree

Red-Black Tree

- **Recall binary search tree**
  - Key values in the left subtree <= the node value
  - Key values in the right subtree >= the node value
- **Operations:**
  - insertion, deletion
  - Search, maximum, minimum, successor, predecessor.
  - O(h), h is the height of the tree.

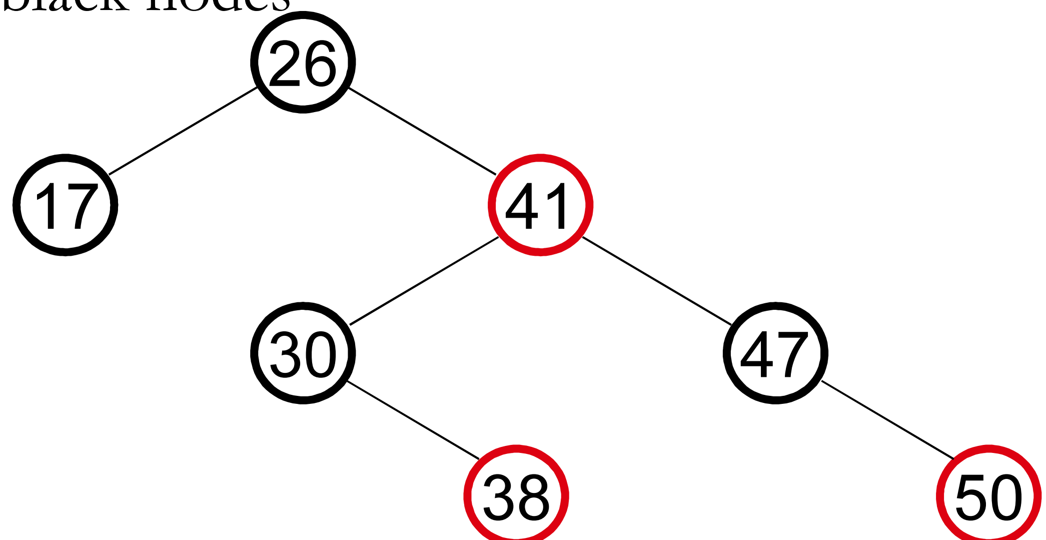
Red-black trees

- **Definition: a binary tree, satisfying:**
  1. Every node is red or black
  2. The root is black
  3. Every leaf is NIL and is black
  4. If a node is red, then both its children are black
  5. For each node, all paths from the node to descendant leaves contain the same number of black nodes.
- **Purpose: keep the tree balanced.**
- **Other balanced search tree:**
  – AVL tree, 2-3-4 tree, Splay tree, Treap

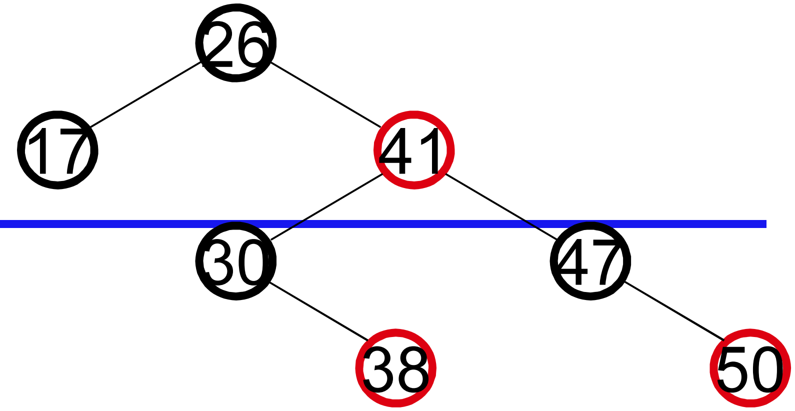
INSERT

INSERT: what color to make the new node?

- Red? Let's insert 35!
  - Property 4 is violated: if a node is red, then both its children are black
- Black? Let's insert 14!
  - Property 5 is violated: all paths from a node to its leaves contain the same number of black nodes



DELETE



DELETE: what color was the node that was removed? **Black?**

1. Every node is either **red** or **black**

OK!

2. The root is **black**

Not OK! If removing the root and the child that replaces it is **red**

3. Every leaf (NIL) is **black**

OK!

4. If a node is red, then both its children are black

Not OK! Could change the black heights of some nodes

Not OK! Could create two red nodes in a row

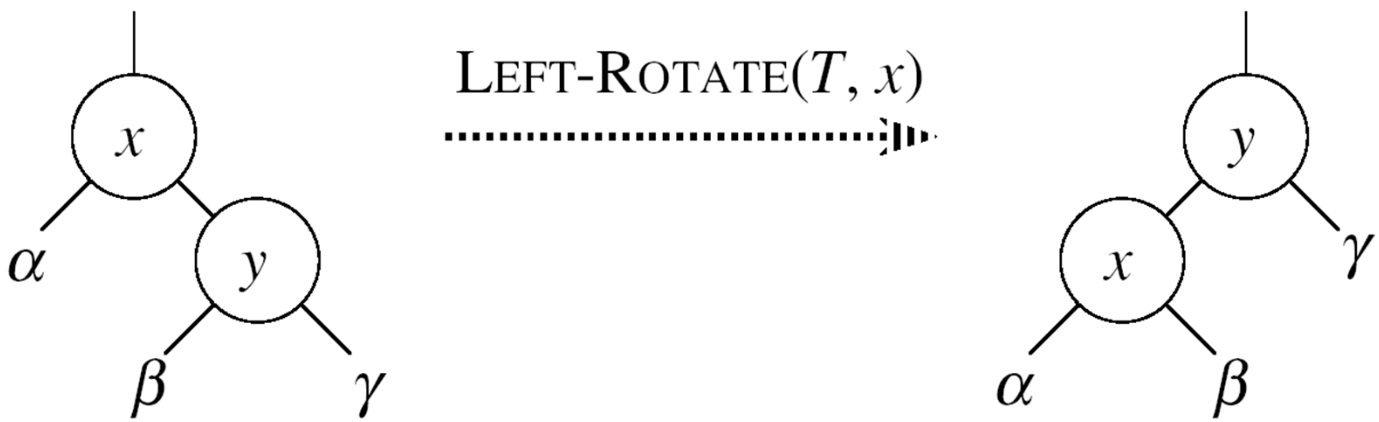
5. For each node, all paths from the node to descendant leaves contain the same number of black nodes

Rotations

- Operations for re-structuring the tree after insert and delete operations on red-black trees
- **Rotations take a red-black-tree and a node within the tree and:**
  - Together with some node re-coloring they help restore the red-black-tree property
  - Change some of the pointer structure
  - **Do not** change the binary-search tree property
- **Two types of rotations:**
  - Left & right rotations

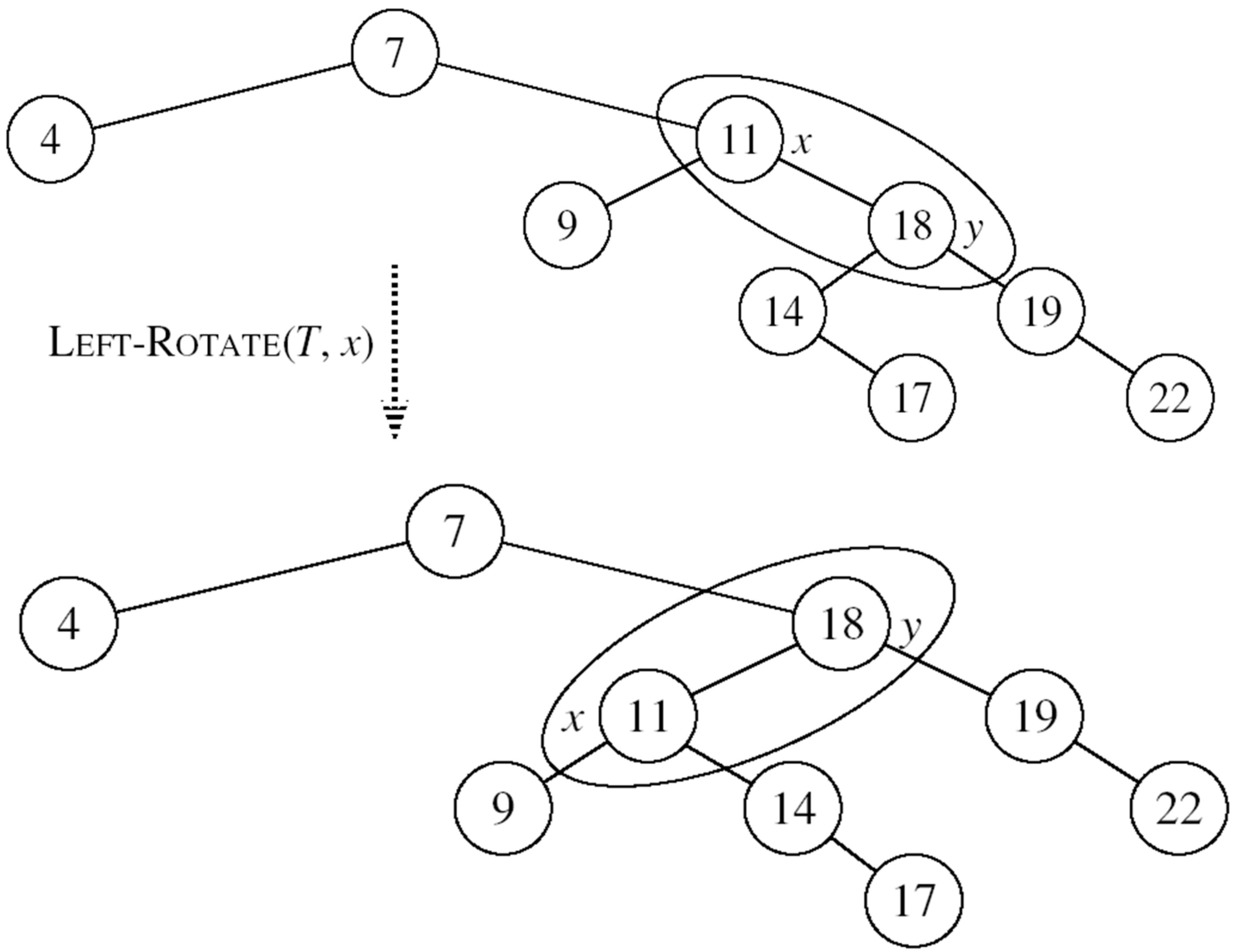
Left Rotations

- **Assumptions for a left rotation on a node x:**
  - The right child of x (y) is not NIL



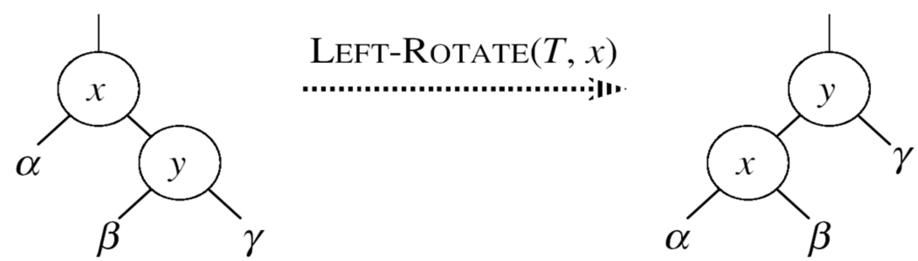
- **Idea:**
  - Pivots around the link from x to y
  - Makes y the new root of the subtree
  - x becomes y's left child
  - y's left child becomes x's right child

Example: LEFT-ROTATE



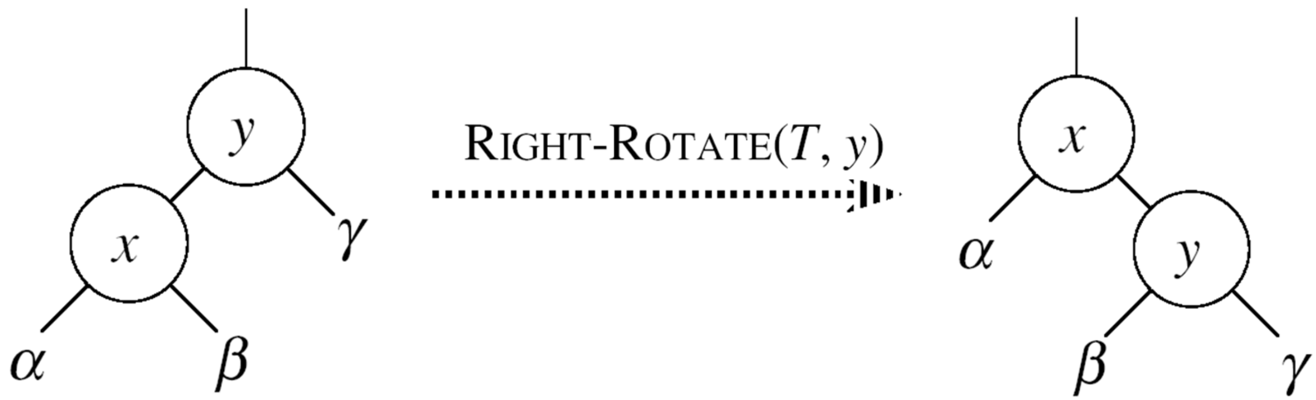
LEFT-ROTATE(T, x)

```
y ← right[x]                     ► Set y
right[x] ← left[y]               ► y's left subtree becomes x's right subtree
if left[y] ≠ NIL
    then p[left[y]] ← x          ► Set the parent relation from left[y] to x
p[y] ← p[x]                      ► The parent of x becomes the parent of y
if p[x] = NIL
    then root[T] ← y
    else if x = left[p[x]]
            then left[p[x]] ← y
            else right[p[x]] ← y
left[y] ← x                      ► Put x on y's left
p[x] ← y                         ► y becomes x's parent
```



Right Rotations

- **Assumptions for a right rotation on a node x:**
  - The left child of y (x) is not NIL



- **Idea:**
  - Pivots around the link from y to x
  - Makes x the new root of the subtree
  - y becomes x’s right child
  - x’s right child becomes y’s left child

Insertion

- **Goal:**
  - Insert a new node *z* into a red-black-tree
- Idea:
  - Insert node *z* into the tree as for an ordinary binary search tree
  - Color the node **red**
  - Restore the red-black-tree properties
    - Use an auxiliary procedure RB-INSERT-FIXUP

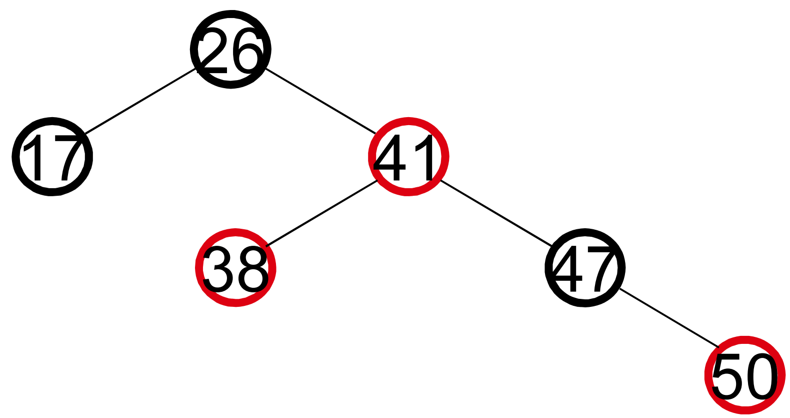
RB Properties Affected by Insert

1. Every node is either **red** or **black** OK!
2. The root is **black** If z is the root $\Rightarrow$ not OK
3. Every leaf (NIL) is **black** OK!
4. If a node is red, then both its children are black

If p(z) is red $\Rightarrow$ not OK
z and p(z) are both red

OK!

5. For each node, all paths from the node to descendant leaves contain the same number of black nodes

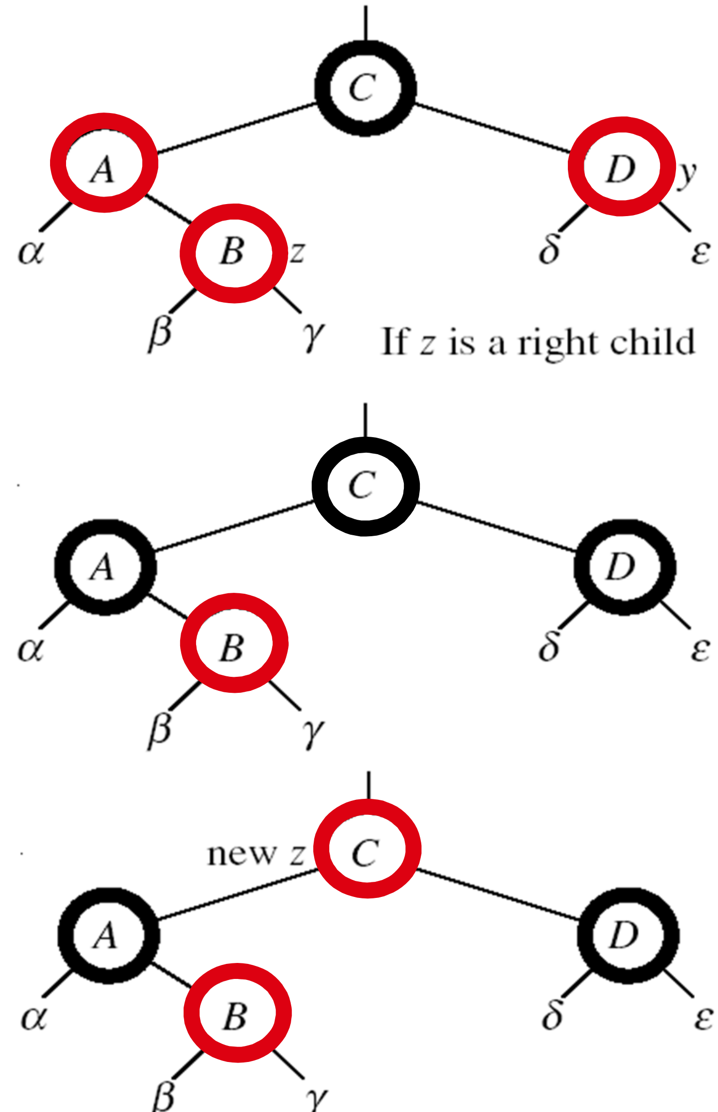


RB-INSERT-FIXUP – Case 1

z's "uncle" (y) is **red**

Idea: (z is a right child)

- p[p[z]] (z's grandparent) must be black: z and p[z] are both red
- Color p[z] **black**
- Color y **black**
- Color p[p[z]] **red**
- z = p[p[z]]
  - Push the **"red"** violation up the tree



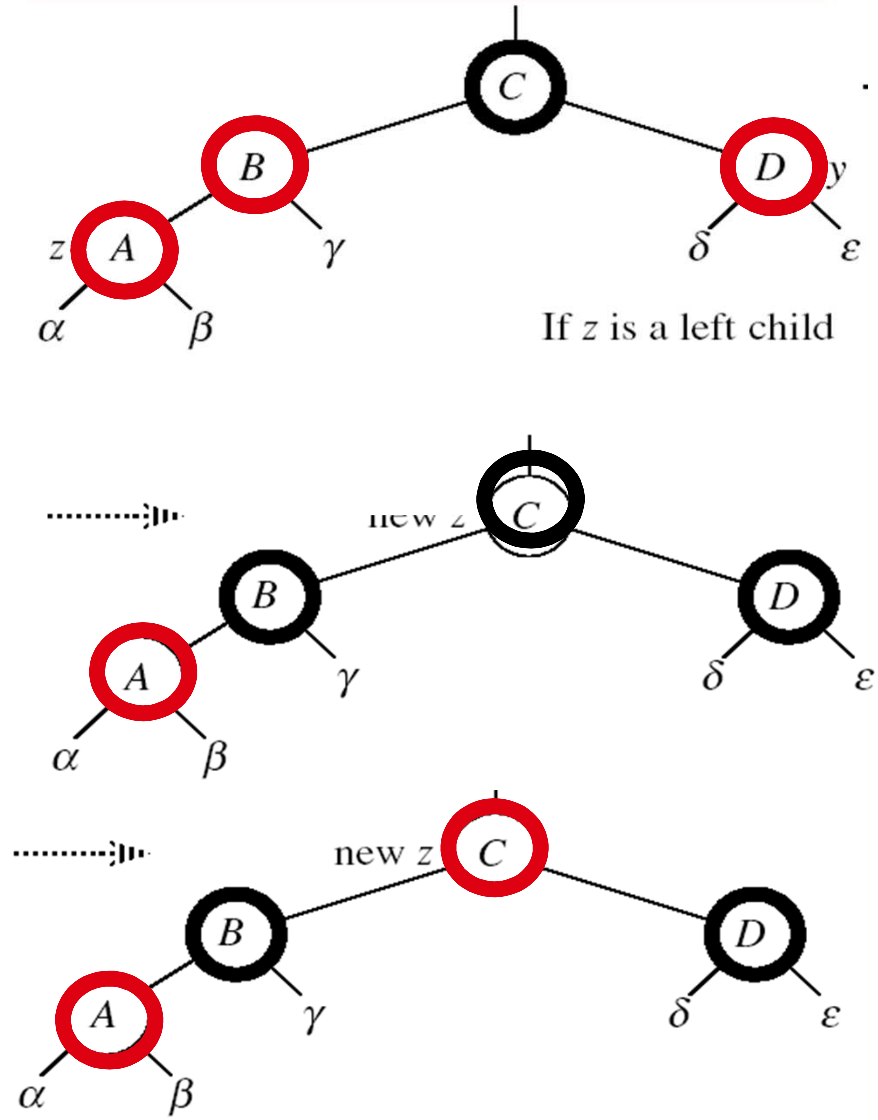
RB-INSERT-FIXUP – Case 1

z's "uncle" (y) is **red**

Idea: (<u>z is a left child</u>)

- p[p[z]] (z's grandparent) must be black: z and p[z] are both red

- color p[z] ← **black**
- color y ← **black**
- color p[p[z]] ← **red**
- z = p[p[z]]
  - Push the **"red"** violation up the tree



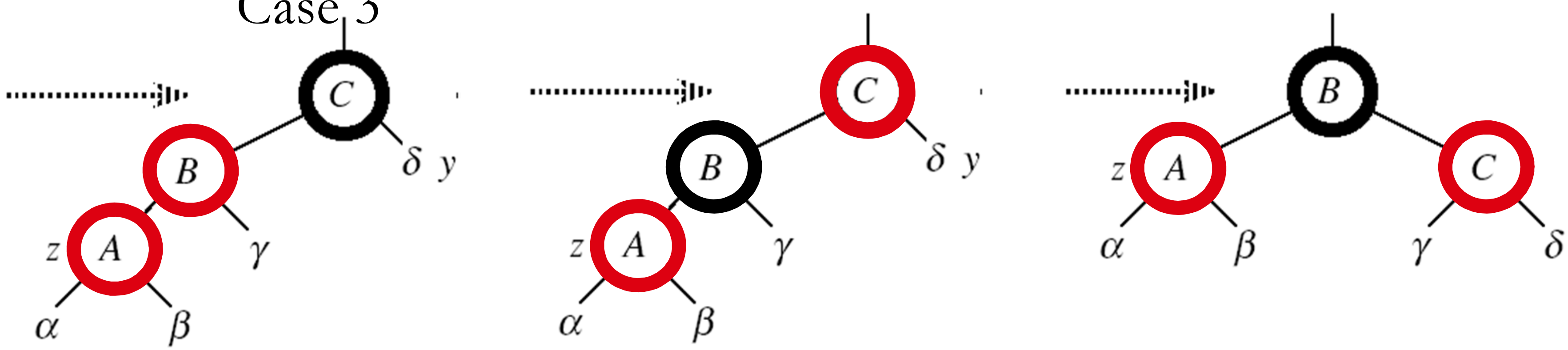
RB-INSERT-FIXUP – Case 3

Case 3:

- z's "uncle" (y) is **black**
- z is a left child

Idea:

- color p[z] ← **black**
- color p[p[z]] ← **red**
- RIGHT-ROTATE(T, p[p[z]])
- No longer have 2 reds in a row
- p[z] is now black



RB-INSERT-FIXUP – Case 2

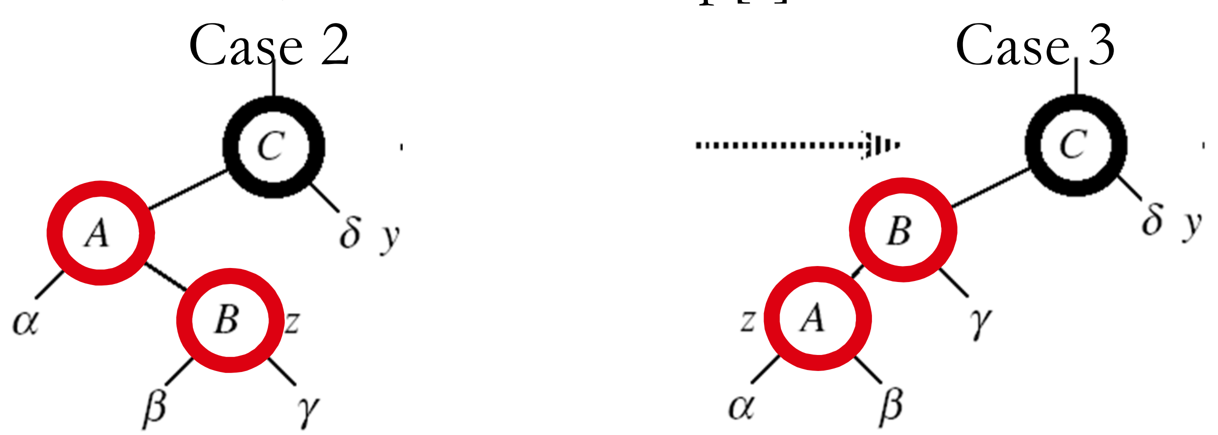
Case 2:

- z's "uncle" (y) is **black**
- z is a right child

Idea:

- z ← p[z]
- LEFT-ROTATE(T, z)

⇒ now z is a left child, and both z and p[z] are red ⇒ case 3



RB-INSERT-FIXUP(T, z)

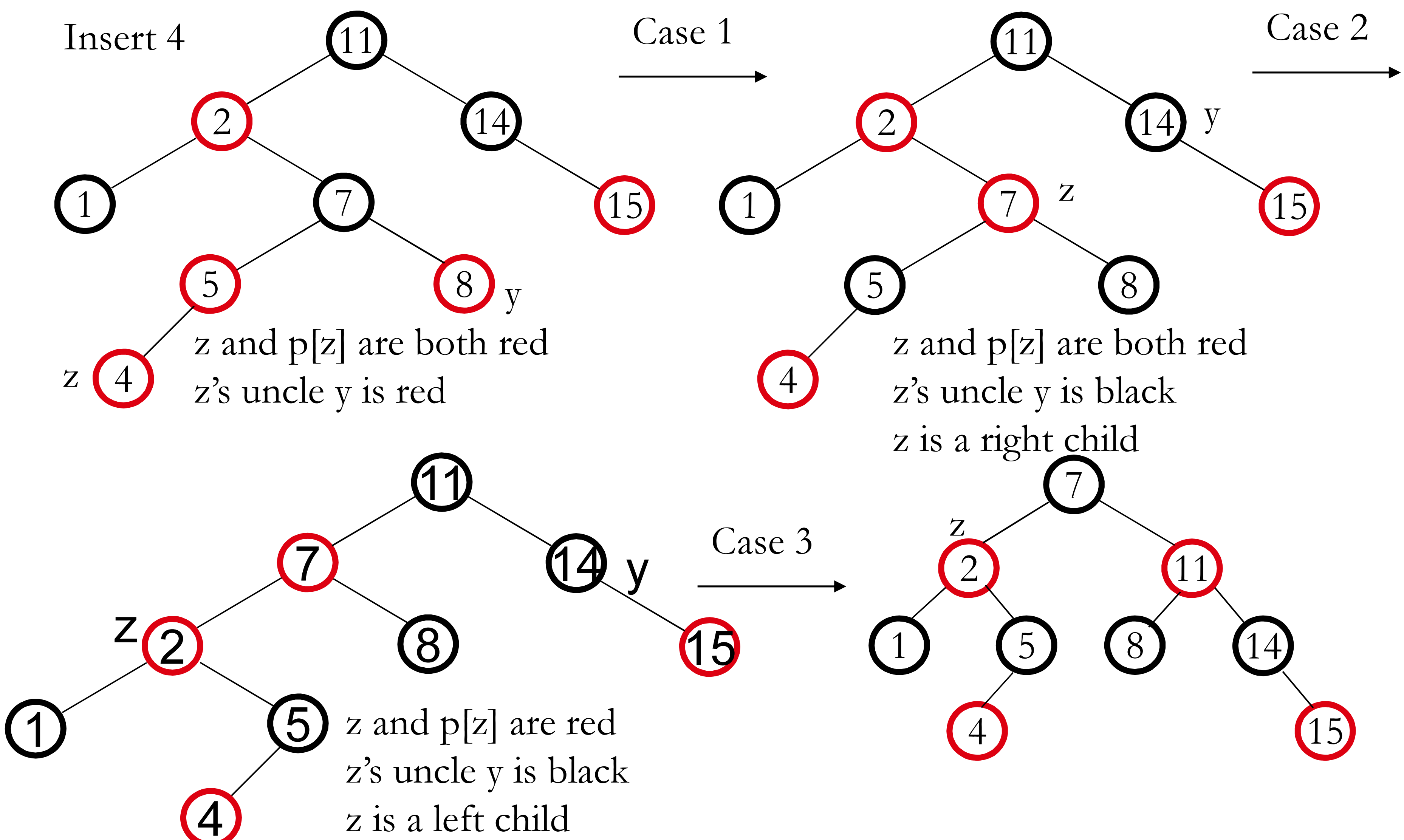
```
while color[p[z]] = RED
      do if p[z] = left[p[p[z]]]
            then y ← right[p[p[z]]]
                  if color[y] = RED
                     then Case1
                     else if z = right[p[z]]
                              then Case2
                                   Case3
      else (same as then clause with "right" and "left" exchanged)
color[root[T]] ← BLACK
```

The while loop repeats only when case1 is executed: O(lgn) times

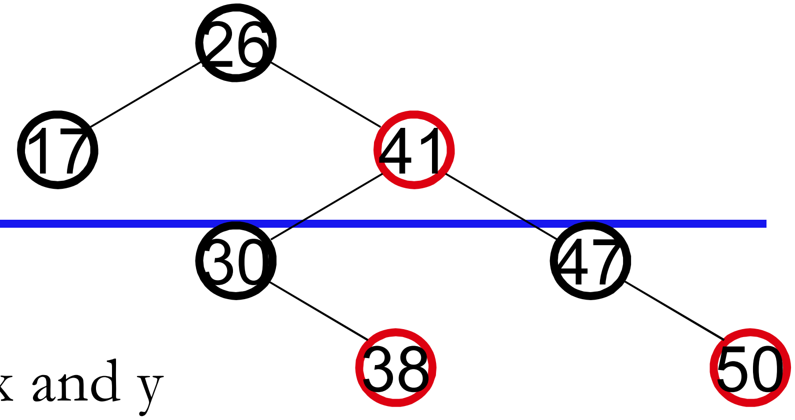
Set the value of x's "uncle"

We just inserted the root, or The red violation reached the root

Example


Insert 4
Case 1
Case 2
Case 3
11
2
14
1
7
15
5
8
y
z
4
z and p[z] are both red
z's uncle y is red
z and p[z] are both red
z's uncle y is black
z is a right child
z and p[z] are red
z's uncle y is black
z is a left child

RB-INSERT(T, z)




```
y ← NIL
x ← root[T]
```

- Initialize nodes x and y
- Throughout the algorithm y points to the parent of x

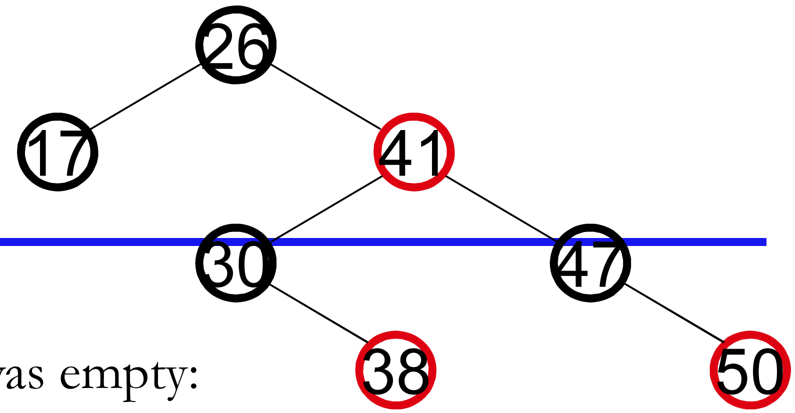
```
while x ≠ NIL
      do y ← x
         if key[z] < key[x]
            then x ← left[x]
            else x ← right[x]
```

- Go down the tree until reaching a leaf
- At that point y is the parent of the node to be inserted

```
p[z] ← y
```

- Sets the parent of z to be y

RB-INSERT(T, z)




```
if y = NIL
    then root[T] ← z
    else if key[z] < key[y]
            then left[y] ← z
            else right[y] ← z
left[z] ← NIL
right[z] ← NIL
color[z] ← RED
RB-INSERT-FIXUP(T, z)
```

The tree was empty: set the new node to be the root

Otherwise, set z to be the left or right child of y, depending on whether the inserted node is smaller or larger than y's key

Set the fields of the newly added node

Fix any inconsistencies that could have been introduced by adding this new red node

Analysis of RB-INSERT

- Inserting the new element into the tree O(lgn)
- RB-INSERT-FIXUP
  - The while loop repeats only if CASE 1 is executed
  - The number of times the while loop can be executed is O(lgn)
- Total running time of RB-INSERT: O(lgn)

Red-Black Trees - Summary

- **Operations on red-black-trees:**
  - SEARCH O(h)
  - PREDECESSOR O(h)
  - SUCCESOR O(h)
  - MINIMUM O(h)
  - MAXIMUM O(h)
  - INSERT O(h)
  - DELETE O(h)
- **Red-black-trees guarantee that the height of the tree will be O(lgn)**

Problems

- What is the ratio between the longest path and the shortest path in a red-black tree?

  - The shortest path is at least bh(root)
  - The longest path is equal to h(root)
  - We know that $h(root) \leq 2bh(root)$

  - Therefore, the ratio is ≤ 2

Problems

- What red-black tree property is violated in the tree below? How would you restore the red-black tree property in this case?
  - Property violated: if a node is red, both its children are black
  - Fixup: color 7 black, 11 red, then right-rotate around 11