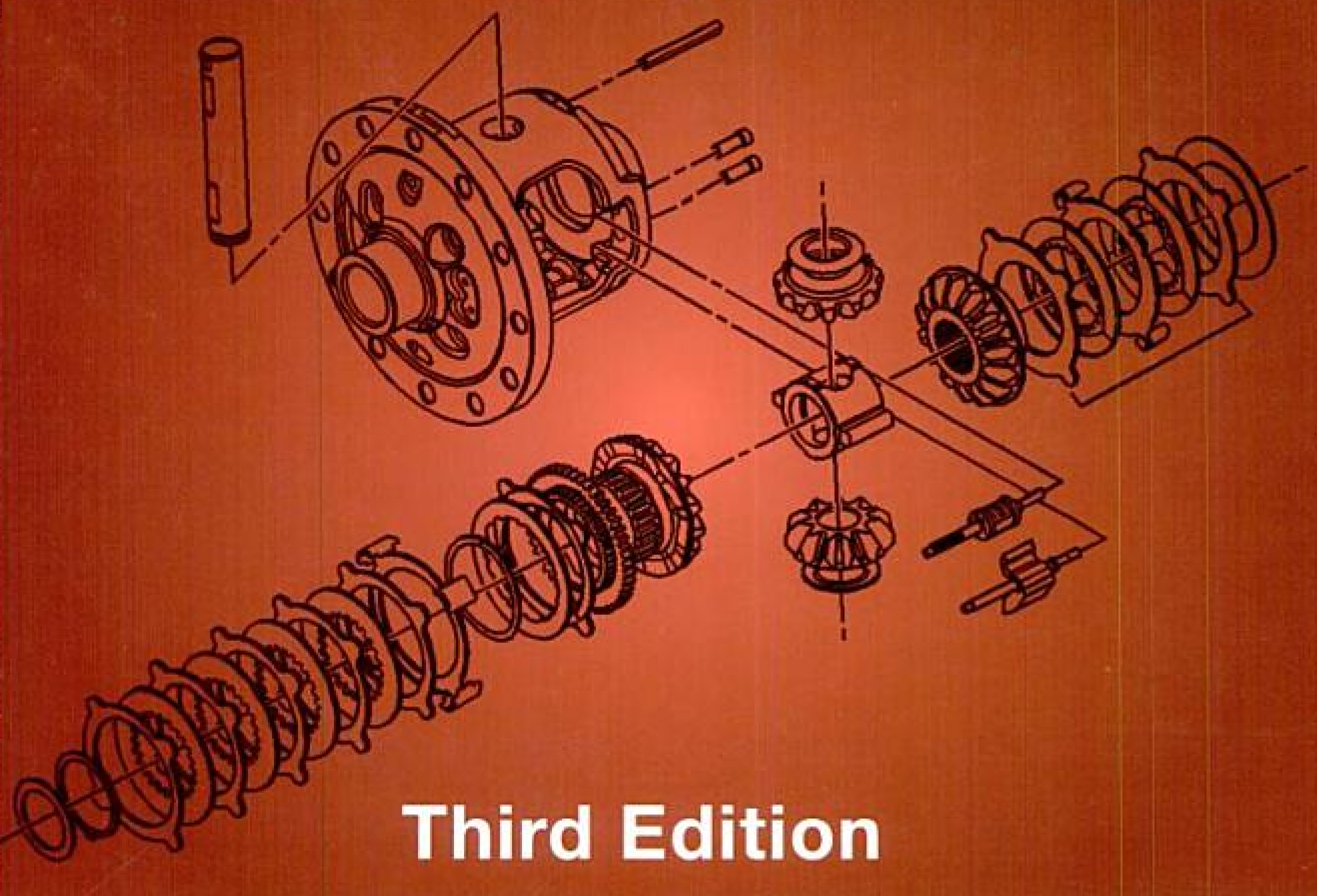

Third Edition

THEORY OF MACHINES

S S RATTAN

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PREFACE

Mechanisms and machines have considerable fascination for most students of mechanical engineering since the theoretical principles involved have immediate applications to practical problems. The main objective of writing this book has been to give a clear understanding of the concepts underlying engineering design. A sincere effort has been made to maintain the physical perceptions in the various derivations and to give the shortest comprehending solution to a variety of problems. The parameters kept in mind while writing the book are the coverage of contents, prerequisite knowledge of students, lucidity of writing, clarity of diagrams and the variety of solved and unsolved numerical problems.

The book is meant to be useful to the degree-level students of mechanical engineering as well as those preparing for AMIE and various other competitive examinations. However, diploma-level students will also find the book to be highly useful. The book will also benefit postgraduate students to some extent as it also contains advanced topics like curvature theory, analysis of rigid and elastic cam systems, complex number and vector methods, force balancing of linkages and field balancing. The salient features of the book are

- Concise and compact covering all major topics
- Presentation of concepts in a logical, innovative and lucid manner
- Evolving the basic theory from simple and readily understood principles
- A balanced presentation of the graphical and analytical approaches
- Computer programs in user-friendly C-language
- Large number of solved examples
- Summary, review questions as well as a number of unsolved problems at the end of each chapter
- An appendix containing objective-type questions
- Another appendix containing important relations and results

It is expected that the students using this book might have completed a course in applied mechanics. The book is divided broadly into two sections, kinematics and dynamics of machines. Kinematics involves study from the geometric point of view to know the displacement, velocity and acceleration of various components of mechanisms, whereas dynamics is the study of the effects of the applied and inertia forces. Chapters 1 to 11 are devoted to the study of the kinematics and the rest to that of dynamics. **Chapter 1** introduces the concepts of mechanisms and machines. **Chapters 2 and 3** describe graphical methods of velocity and acceleration analysis whereas the analytical approach is discussed in **Chapter 4**. Synthesis or designing of mechanisms is important to have the desirable motion of various components of machinery—the detail procedures for the same, both graphical and analytical, are given in **Chapter 5**. Various types of mechanisms with higher number of links are discussed in **Chapter 6**. Friction in various components of machines is very important as it affects their efficiency and is described in **Chapter 8**. Cams, belts, gears, gear trains are meant to transmit power from one shaft to another and are discussed in **chapters 7, 9, 10 and 11** respectively.

Forces are mainly of static and dynamic nature. **Chapters 12 and 13** are devoted to their effects on the components of the mechanisms. **Chapter 13** also includes the topic of flywheels which are essential components for rotary machines to regulate speeds. Speed regulation is also affected by governors which are described in **Chapter 16**. Unbalanced forces and vibrations in various components of rotating machines are mostly undesirable since the efficiency is reduced. A detailed study of these is undertaken in **chapters 14 and 18.** Brakes are essential for any moving components of machinery and are discussed in **Chapter 15**.

Moving bodies like aeroplanes, ships, two- and four-wheelers, etc., experience gyroscopic effect while taking turns. It is described in **Chapter 17**. Automatic control of machinery is very much desirable these days and an introduction of the same is given in **Chapter 19**.

The first edition of the book aimed at providing the fundamentals of the subject in a simple manner for easy comprehension by students. Simple mathematical methods were preferred instead of more elegant but less obvious methods so that those with limited mathematical skills could easily understand the expositions. However, to make the book more purposeful and acceptable to a wider section of users, the second edition also consisted of methods involving vector and complex numbers usually preferred by those who excel in mathematical skills. Such methods frequently lead to computer-aided solutions of the problems. The computer programs were rewritten in the more user-friendly C language. A Summary of each chapter was added at the end and theoretical questions were added to the exercises. One appendix containing objective-type questions was also included. All the previous figures were redrawn.

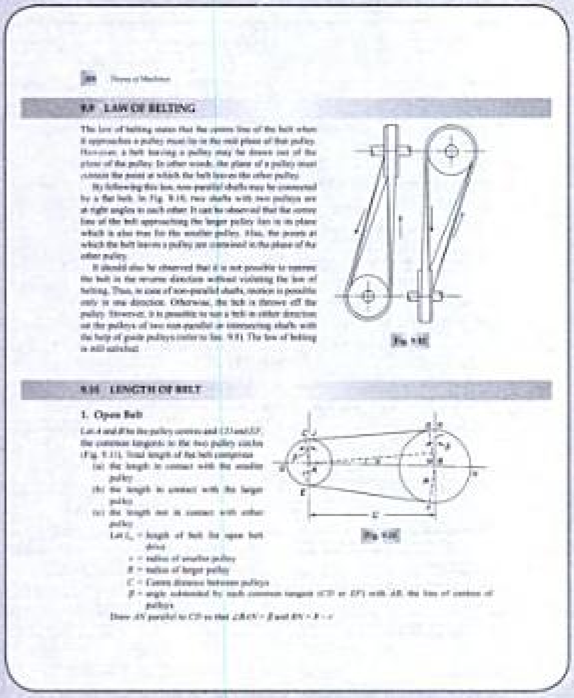
The present edition is aimed at making the book more exhaustive. Many more worked examples as well as unsolved problems have been added. Many new sections have been added in most of the chapters apart from rewriting some previous sections. Another appendix containing important relations and results has also been added. Effort has been made to remove all sorts of errors and misprints as far as possible. In spite of addition of a large amount of material, care has been taken to let the book remain concise and compact. Hints to most of the numerical problems at the end of each chapter have been provided at the publisher's website of the book for the benefit of average and weak students. Full solutions of the same are available to the faculty members at the same site. The facility can be availed by logging on to http://www.mhhe.com/rattan/tom3e.

I am grateful to all those teachers and students who pointed out errors and mistakes of the previous editions and also gave many valuable suggestions. I acknowledge the efforts of the editorial staff of Tata McGraw Hill Education Private Limited for bringing out the new edition in an excellent format.

Finally, I make an affectionate acknowledgement to my wife, Neena, and my children, Ravneet and Jasmeet, for their patience, support and putting up with it all so cheerfully. But for their sacrifice, I would not have been able to complete this work in the most satisfying way.

For further improvement of the book, readers are requested to post their comments and suggestions at ss_rattan@hotmail.com.

S S Rattan

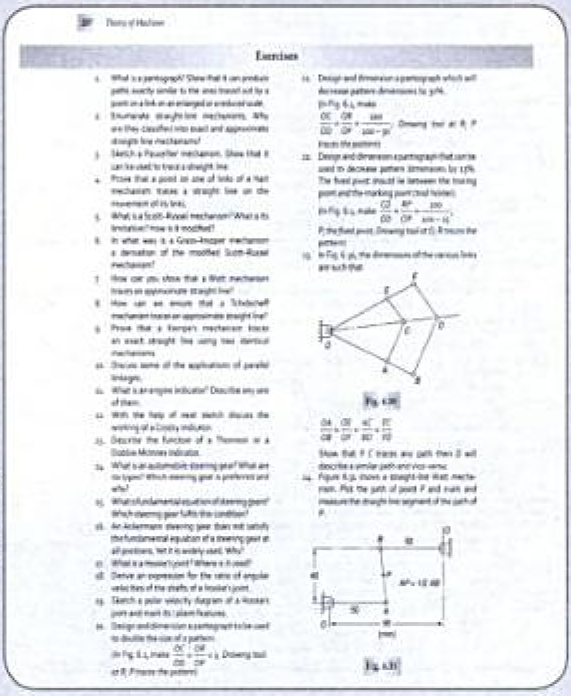


9.9 LAW OF BELTING

9.10 LENGTH OF BELT

1. Open Belt

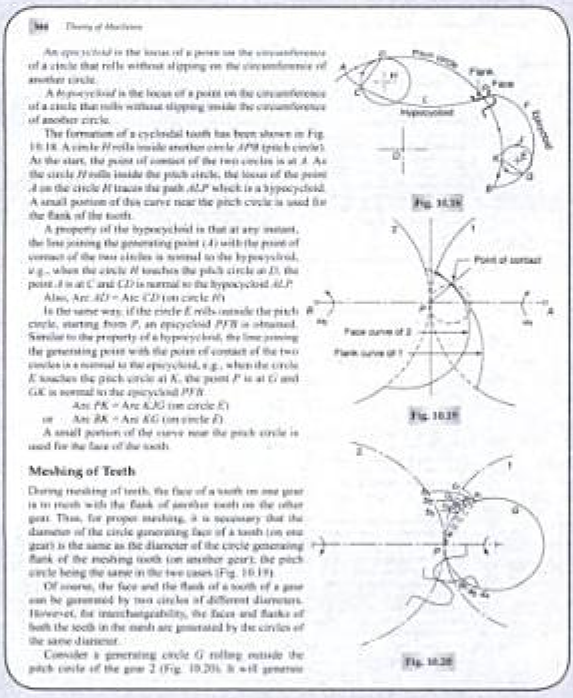
Each chapter has a concise and comprehensive treatment of topics with emphasis on fundamental concepts.



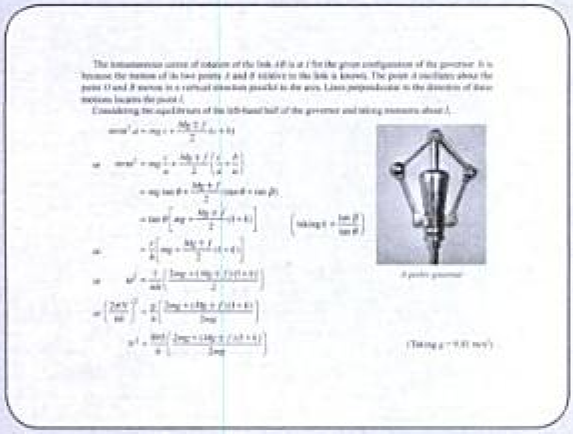
Exercises

A number of theoretical questions and unsolved exercises are given for practice to widen the horizon of comprehension of the topic.

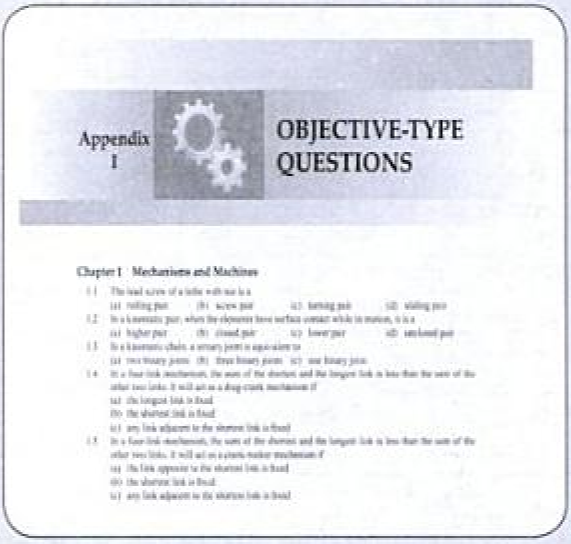
A Summary at the end of each chapter recapitulates the inferences for quick revision.



Simple diagrams are given for easy visualization of the explanations.



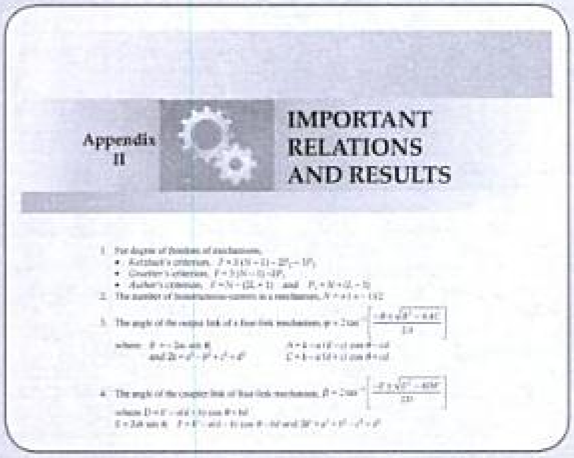
A number of photographs are given to emphasize the factual shape of various components.


Appendix I

OBJECTIVE-TYPE QUESTIONS

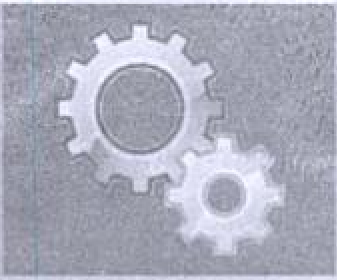
Chapter 1 Mechanisms and Machines

An Appendix containing multiple choice questions is given at the end to help students prepare for competitive examinations.


Appendix II

IMPORTANT RELATIONS AND RESULTS

An Appendix containing important relations is given for ready reference.



1 MECHANISMS AND MACHINES

Introduction

If a number of bodies are assembled in such a way that the motion of one causes constrained and predictable motion to the others, it is known as a *mechanism*. A mechanism transmits and modifies a motion. A *machine* is a mechanism or a combination of mechanisms which, apart from imparting definite motions to the parts, also transmits and modifies the available mechanical energy into some kind of desired work. Thus, a mechanism is a fundamental unit and one has to start with its study.

The study of a mechanism involves its analysis as well as synthesis. *Analysis* is the study of motions and forces concerning different parts of an existing mechanism, whereas *synthesis* involves the design of its different parts. In a mechanism, the various parts are so proportioned and related that the motion of one imparts requisite motions to the others and the parts are able to withstand the forces impressed upon them. However, the study of the relative motions of the parts does not depend on the strength and the actual shapes of the parts.

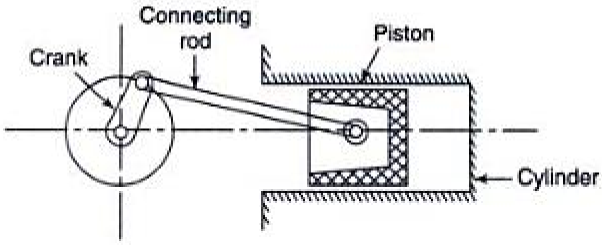


Fig. 1.1

In a reciprocating engine, the displacement of the piston depends upon the lengths of the connecting rod and the crank (Fig. 1.1). It is independent of the bearing strength of the parts or whether they are able to withstand the forces or not. Thus for the study of motions, it is immaterial if a machine part is made of mild steel, cast iron or wood. Also, it is not necessary to know the actual shape and area of the cross section of the part. Thus, for the study of motions of different parts of a mechanism, the study of forces is not necessary and can be neglected. The study of mechanisms, therefore, can be divided into the following disciplines:

Kinematics It deals with the relative motions of different parts of a mechanism without taking into consideration the forces producing the motions. Thus, it is the study, from a geometric point of view, to know the displacement, velocity and acceleration of a part of a mechanism.

Dynamics It involves the calculations of forces impressed upon different parts of a mechanism. The forces can be either static or dynamic. Dynamics is further subdivided into *kinetics* and *statics*. Kinetics is the study of forces when the body is in motion whereas statics deals with forces when the body is stationary.

1.1 MECHANISM AND MACHINE

As mentioned earlier, a combination of a number of bodies (usually rigid) assembled in such a way that the motion of one causes constrained and predictable motion to the others is known as a *mechanism*. Thus, the function of a mechanism is to transmit and modify a motion.

A *machine* is a mechanism or a combination of mechanisms which, apart from imparting definite motions to the parts, also transmits and modifies the available mechanical energy into some kind of desired work. It is neither a source of energy nor a producer of work but helps in proper utilization of the same. The motive power has to be derived from external sources.

A slider-crank mechanism (Fig. 1.2) converts the reciprocating motion of a slider into rotary motion of the crank or vice-versa. However, when it is used as an automobile engine by adding valve mechanism, etc., it becomes a machine which converts the available energy (force on the piston) into the desired energy (torque of the crank-shaft). The torque is used to move a vehicle. Reciprocating pumps, reciprocating compressors and steam engines are other examples of machines derived from the slider-crank mechanism.

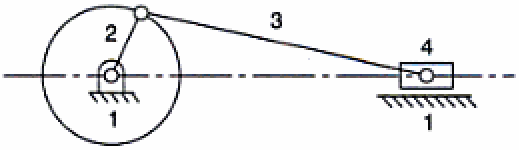


Fig. 1.2

Some other examples of mechanisms are typewriters, clocks, watches, spring toys, etc. In each of these, the force or energy provided is not more than what is required to overcome the friction of the parts and which is utilized just to get the desired motion of the mechanism and not to obtain any useful work.

1.2 TYPES OF CONSTRAINED MOTION

There are three types of constrained motion:

(i) ***Completely constrained motion*** When the motion between two elements of a pair is in a definite direction irrespective of the direction of the force applied, it is known as completely constrained motion. The constrained motion may be linear or rotary. The sliding pair of Fig. 1.3(a) and the turning pair of Fig. 1.3(b) are the examples of the completely constrained motion. In sliding pair, the inner prism can only slide inside the hollow prism. In case of a turning pair, the inner shaft can have only rotary motion due to collars at the ends. In each case the force has to be applied in a particular direction for the required motion.

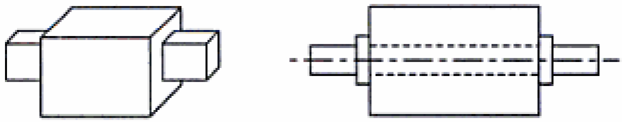

(a) (b)

Fig. 1.3

(ii) ***Incompletely constrained motion*** When the motion between two elements of a pair is possible in more than one direction and depends upon the direction of the force applied, it is known as incompletely constrained motion. For example, if the turning pair of Fig. 1.4 does not have collars, the inner shaft may have sliding or rotary motion depending upon the direction of the force applied. Each motion is independent of the other.

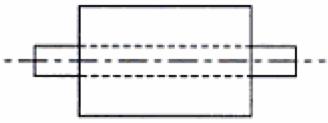
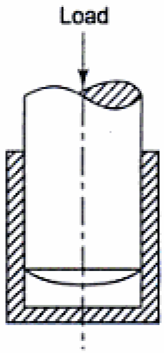


Fig. 1.4

(iii) ***Successfully constrained motion*** When the motion between two elements of a pair is possible in more than one direction but is made to have motion only in one direction by using some external means, it is a successfully constrained motion. For example, a shaft in a footstep bearing may have vertical motion apart from rotary motion (Fig. 1.5). But due to load applied on the shaft it is constrained to move in



Footstep bearing

Fig. 1.5

Kinematic Pairs according to Nature of Contact

(a) Lower Pair A pair of links having surface or area contact between the members is known as a lower pair. The contact surfaces of the two links are similar.

Examples Nut turning on a screw, shaft rotating in a bearing, all pairs of a slider-crank mechanism, universal joint, etc.

(b) Higher Pair When a pair has a point or line contact between the links, it is known as a higher pair. The contact surfaces of the two links are dissimilar.

Examples Wheel rolling on a surface, cam and follower pair, tooth gears, ball and roller bearings, etc.

Kinematic Pairs according to Nature of Mechanical Constraint

(a) Closed Pair When the elements of a pair are held together mechanically, it is known as a closed pair. The two elements are geometrically identical; one is solid and full and the other is hollow or open. The latter not only envelops the former but also encloses it. The contact between the two can be broken only by destruction of at least one of the members.

All the lower pairs and some of the higher pairs are closed pairs. A cam and follower pair (higher pair) shown in Fig. 1.7(a) and a screw pair (lower pair) belong to the closed pair category.

(b) Unclosed Pair When two links of a pair are in contact either due to force of gravity or some spring action, they constitute an unclosed pair. In this, the links are not held together mechanically, e.g., cam and follower pair of Fig. 1.7(b).

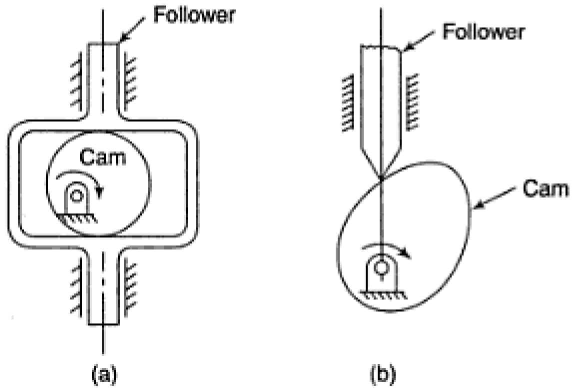


Fig. 1.7

Kinematic Pairs according to Nature of Relative Motion

(a) Sliding Pair If two links have a sliding motion relative to each other, they form a sliding pair.

A rectangular rod in a rectangular hole in a prism is a sliding pair [Fig. 1.8(a)].

(b) Turning Pair When one link has a turning or revolving motion relative to the other, they constitute a turning or revolving pair [Fig. 1.8(b)].

In a slider-crank mechanism, all pairs except the slider and guide pair are turning pairs. A circular shaft revolving inside a bearing is a turning pair.

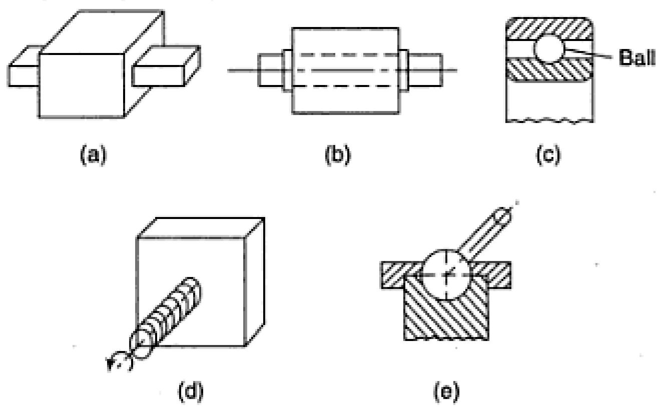


Fig. 1.8

(c) Rolling Pair When the links of a pair have a rolling motion relative to each other, they form a rolling pair, e.g., a rolling wheel on a flat surface, ball and roller bearings, etc. In a ball bearing [Fig. 1.8(c)], the ball and the shaft constitute one rolling pair whereas the ball and the bearing is the second rolling pair.

(d) Screw Pair (Helical Pair) If two mating links have a turning as well as sliding motion between them, they from a screw pair. This is achieved by cutting matching threads on the two links.

The lead screw and the nut of a lathe is a screw pair [Fig. 1.8(d)].

(e) Spherical Pair When one link in the form of a sphere turns inside a fixed link, it is a spherical pair.

The ball and socket joint is a spherical pair [Fig. 1.8(e)].

1.6 TYPES OF JOINTS

The usual types of joints in a chain are

- Binary joint
- Ternary joint
- Quaternary joint

Binary Joint If two links are joined at the same connection, it is called a binary joint. For example, Fig. 1.9 shows a chain with two binary joints named *B*.

Ternary Joint If three links are joined at a connection, it is known as a ternary joint. It is considered equivalent to two binary joints since fixing of any one link constitutes two binary joints with each of the other two links. In Fig. 1.9 ternary links are mentioned as *T*.

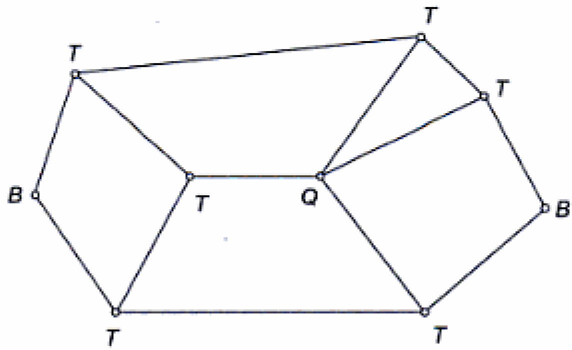


Fig. 1.9

Quaternary Joint If four links are joined at a connection, it is known as a quaternary joint. It is considered equivalent to three binary joints since fixing of any one link constitutes three binary joints. Figure 1.9 shows one quaternary joint.

In general, if n number of links are connected at a joint, it is equivalent to $(n - 1)$ binary joints.

1.7 DEGREES OF FREEDOM

An unconstrained rigid body moving in space can describe the following independent motions (Fig. 1.10):

1. Translational motions along any three mutually perpendicular axes x, y and z
2. Rotational motions about these axes

Thus, a rigid body possesses six degrees of freedom. The connection of a link with another imposes certain constraints on their relative motion. The number of restraints can never be zero (joint is disconnected) or six (joint becomes solid).

Degrees of freedom of a pair is defined as the number of independent relative motions, both translational and rotational, a pair can have.

Degrees of freedom = 6 – Number of restraints

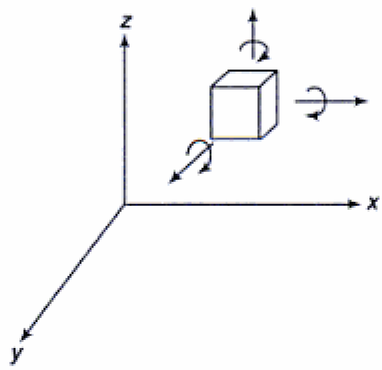


Fig. 1.10

1.8 CLASSIFICATION OF KINEMATIC PAIRS

Depending upon the number of restraints imposed on the relative motion of the two links connected together, a pair can be classified as given in Table 1.1 which gives the possible form of each class.

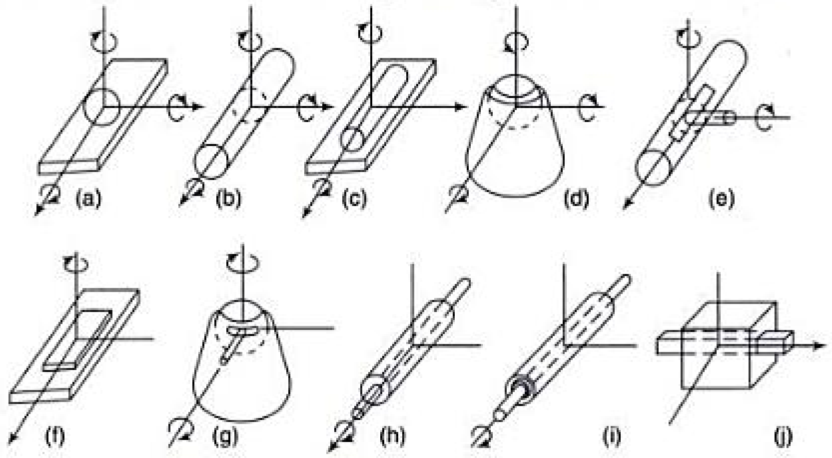


Fig. 1.11

Different forms of each class have also been shown in Fig. 1.11. Remember that a particular relative motion between two links of a pair must be independent of the other relative motions that the pair can have. A screw and nut pair permits translational and rotational motions. However, as the two motions cannot be accomplished independently, a screw and nut pair is a kinematic pair of the fifth class and not of the fourth class.

1.9 KINEMATIC CHAIN

A *kinematic chain* is an assembly of links in which the relative motions of the links is possible and the motion of each relative to the other is definite [Fig.1.12 (a), (b), and (c)].

Table 1.1

Class	Number of Restraints	Form	Restraints on Translatory motion	Restraints on Rotary motion	Kinematic pair	Fig. 1.11
I	1	1st	1	0	Sphere-plane	a
II	2	1st	2	0	Sphere-cylinder	b
		2nd	1	1	Cylinder-plane	c
III	3	1st	3	0	Spheric	d
		2nd	2	1	Sphere-slotted cylinder	e
		3rd	1	2	Prism-plane	f
IV	4	1st	3	1	Slotted-spheric	g
		2nd	2	2	Cylinder	h
V	5	1st	3	2	Cylinder (collared)	i
		2nd	2	3	Prismatic	j

Expressing the number of degrees of freedom of a linkage in terms of the number of links and the number of pair connections of different types is known as *number synthesis*. *Degrees of freedom* of a mechanism in space can be determined as follows:

Let

N = total number of links in a mechanism

F = degrees of freedom

P_1 = number of pairs having one degree of freedom

P_2 = number of pairs having two degrees of freedom, and so on

In a mechanism, one link is fixed.

Therefore,

Number of movable links = $N - 1$

Number of degrees of freedom of $(N - 1)$ movable links = $6(N - 1)$

Each pair having one degree of freedom imposes 5 restraints on the mechanism, reducing its degrees of freedom by $5P_1$.

Each pair having two degrees of freedom will impose 4 restraints, reducing the degrees of freedom of the mechanism by $4P_2$.

Similarly, other pairs having 3, 4 and 5 degrees of freedom reduce the degrees of freedom of the mechanism. Thus,

$$F = 6(N - 1) - 5P_1 - 4P_2 - 3P_3 - 2P_4 - P_5 \tag{1.1}$$

The above criterion is hardly necessary to find the degrees of freedom, as space mechanisms, especially of the zero order are not practical. Most of the mechanisms are two-dimensional such as a four-link or a slider-crank mechanism in which displacement is possible along two axes (one restraint) and rotation about only one axis (two restraints). Thus, there are three general restraints.

Therefore, for plane mechanisms, the following relation may be used to find the degrees of freedom

$$F = 3(N - 1) - 2P_1 - 1P_2 \tag{1.2}$$

This is known as *Gruebler's criterion* for degrees of freedom of plane mechanisms in which each movable link possesses three degrees of freedom. Each pair with one degree of freedom imposes two further restraints on the mechanisms, thus reducing its degrees of freedom. Similarly, each pair with two degrees of freedom reduces the degrees of freedom of the mechanism at the rate of one restraint each.

Some authors mention the above relation as *Kutzback's criterion* and a simplified relation [$F = 3\,(N-1) - 2P_1$] which is applicable to linkages with a single degree of freedom only as Gruebler's criterion. However, many authors make no distinction between Kutzback's criterion and Gruebler's criterion.

Thus, for linkages with a single degree of freedom only, $P_2 = 0$

$$F = 3(N - 1) - 2P_1 \tag{1.3}$$

Most of the linkages are expected to have one degree of freedom so that with one input to any of the links, a constrained motion of the others is obtained.

Then,

$$1 = 3(N - 1) - 2P_1$$

or

$$2P_1 = 3N - 4 \tag{1.4}$$

As P_1 and N are to be whole numbers, the relation can be satisfied only if N is even. For possible linkages made of binary links only,

$N = 4,$	$P_1 = 4$	No excess turning pair
$N = 6,$	$P_1 = 7$	One excess turning pair
$N = 8,$	$P_1 = 10$	Two excess turning pairs

and so on.

Thus, with the increase in the number of links, the number of excess turning pairs goes on increasing. Getting the required number of turning pairs from the required number of binary links is not possible. Therefore, the excess or the additional pairs or joints can be obtained only from the links having more than two joining points, i.e., ternary or quaternary links, etc.

For a six-link chain, some of the possible types are Watts six-bar chain, in which the ternary links are directly connected [Fig. 1.13(a)] and Stephenson's six-bar chain, in which ternary links are not directly connected [Fig. 1.13(b)]. Another possibility is also shown in Fig. 1.13(c). However, this chain is not a six-link chain but a four-link chain as links 1, 2 and 3 are, in fact, one link only with no relative motion of these links.

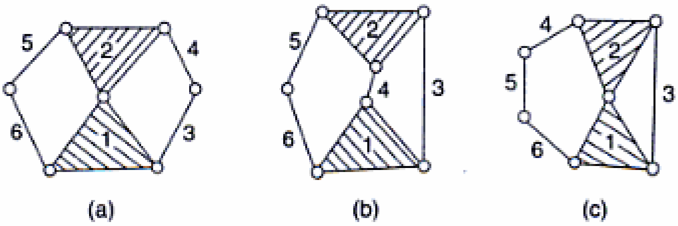


Fig. 1.13

Two excess turning pairs required for an eight-link chain can be obtained by using (apart from binary links):

four ternary links [Figs 1.14(a) and (b)]
two quaternary links [Fig.1.14(c)]
one quaternary and two ternary links [Fig. 1.14(d)].

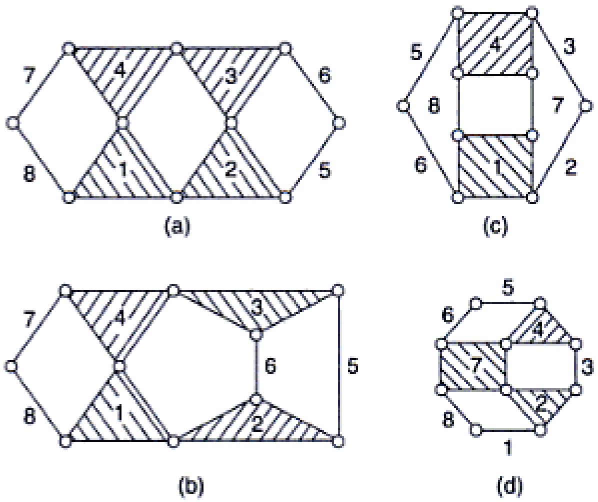


Fig. 1.14

Now, consider the kinematic chain shown in Fig. 1.15. It has 8 links, but only three ternary links. However, the links 6, 7 and 8 constitute a double pair so that the total number of pairs is again 10. The degree of freedom of such a linkage will be

$$F = 3\,(8 - 1) - 2 \times 10$$
$$= 1$$

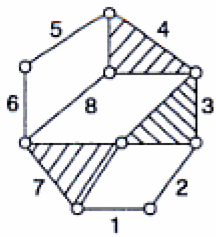


Fig. 1.15

$$F = 3(N-1) - 2P_1 - 1P_2 - F_r$$

where F_r is the number of redundant degrees of freedom. Now, as the above mechanism has a cam pair, its degree of freedom must be found from Gruebler's criterion.

Total number of links = 4

Number of pairs with 1 degree of freedom = 3

Number of pairs with 2 degrees of freedom = 1

$$F = 3(N-1) - 2P_1 - 1P_2 - F_r$$
$$= 3(4-1) - 2 \times 3 - 1 \times 1 - 1$$
$$= 1$$

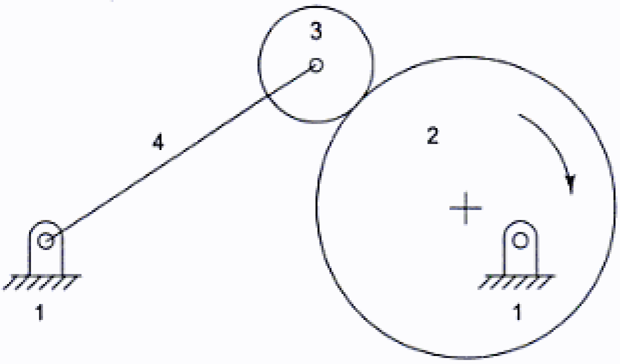


Fig. 1.17

Example 1.1 *For the kinematic linkages shown in Fig. 1.18, calculate the following:*

- *the number of binary links (N_b)*
- *the number of ternary links (N_t)*
- *the number of other (quaternary, etc.) links (N_o)*
- *the number of total links (N)*
- *the number of loops (L)*
- *the number of joints or pairs (P_1)*
- *the number of degrees of freedom (F)*

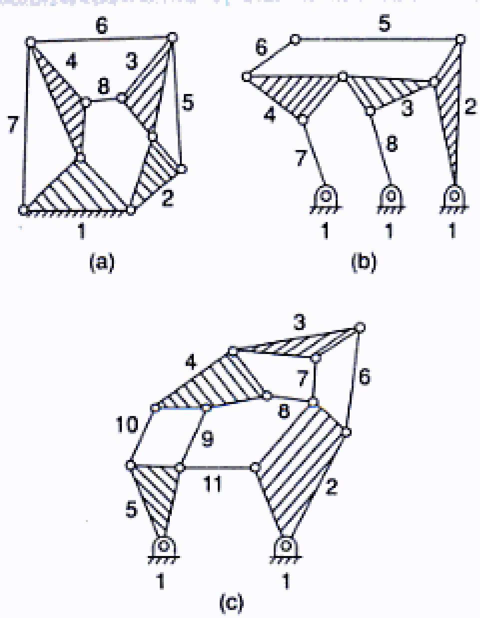


Fig. 1.18

Solution

(a) $N_b = 4$; $N_t = 4$; $N_o = 0$; $N = 8$; $L = 4$

$P_1 = 11$ by counting

or $P_1 = (N + L - 1) = 11$

$F = 3(N-1) - 2P_1$

$= 3(8-1) - 2 \times 11 = -1$

or $F = N - (2L + 1)$

$= 8 - (2 \times 4 + 1) = -1$

The linkage has negative degree of freedom and thus is a superstructure.

(b) $N_b = 4$; $N_t = 4$; $N_o = 0$; $N = 8$; $L = 3$

$P_1 = 10$ (by counting)

or $P_1 = (N + L - 1) = 10$

$F = N - (2L + 1) = 8 - (2 \times 3 + 1) = 1$

or $F = 3(N-1) - 2P_1$

$= 3(8-1) - 2 \times 10 = 1$

i.e., the linkage has a constrained motion when one of the seven moving links is driven by an external source.

(c) $N_b = 7$; $N_t = 2$; $N_o = 2$; $N = 11$

$L = 5$; $P_1 = 15$

$F = N - (2L + 1) = 11 - (2 \times 5 + 1) = 0$

Therefore, the linkage is a structure.

Example 1.2 *State whether the linkages shown in Fig. 1.19 are mechanisms with one degree of freedom. If not, make suitable changes. The number of links should not be varied by more than 1.*

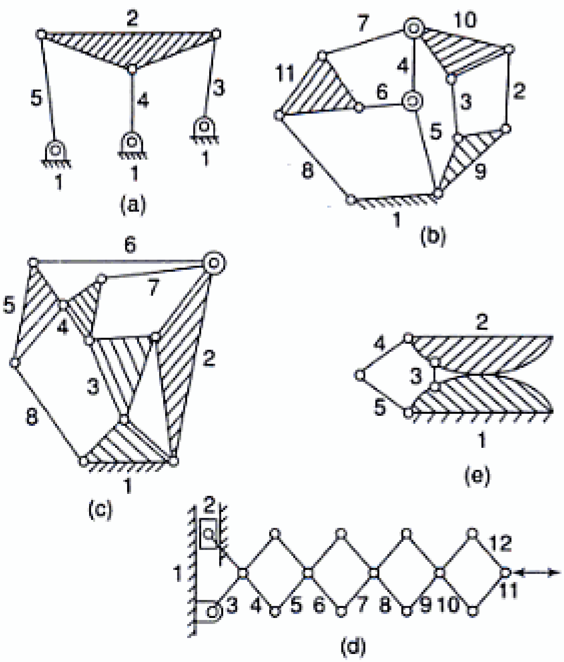


Fig. 1.19

Solution (a) The linkage has 2 loops and 5 links.

$F = N - (2L + 1) = 5 - (2 \times 2 + 1) = 0$

Thus, it is a structure. Referring Table 1.2, for a 2-loop mechanism, n should be six to have one degree of freedom. Thus, one more link should be added to the linkage to make it a mechanism of $F = 1$. One of the possible solutions has been shown in Fig. 1.20(a).

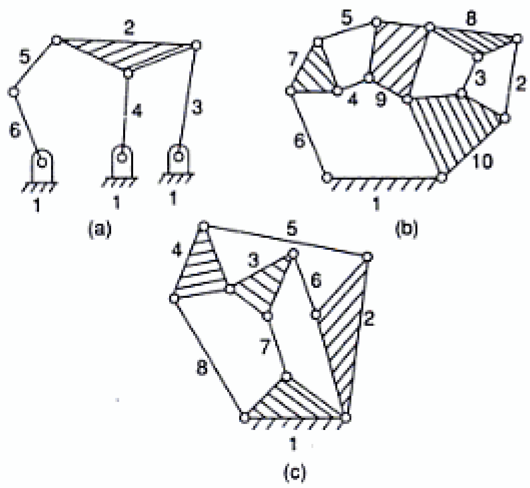


Fig. 1.20

(b) The linkage has 4 loops and 11 links. Referring Table 1.2, it has 2 degrees of freedom. With 4 loops and 1 degree of freedom, the number of links should be 10 and the number of joints 13. Three excess joints can be formed by

6 ternary links or
4 ternary links and 1 quaternary link or
2 ternary links, and 2 quaternary links, or
3 quaternary links, or
a combination of ternary and quaternary links with double joints.

Figure 1.20(b) shows one of the possible solutions.

(c) There are 4 loops and 8 links.

$F = N - (2L +) = 8 - (4 \times 2 + 1) = -1$

It is a superstructure. With 4 loops, the number of links must be 10 to obtain one degree of freedom. As the number of links is not to be increased by more than one, the number of loops has to be decreased. With 3 loops, 8 links and 10 joints, the required linkage can be designed. One of the many solutions is shown in Fig. 1.20(c).

(d) It has 5 loops and 12 links. Referring Table 1.2, it has 1 degree of freedom and thus is a mechanism.

(e) The mechanism has a cam pair, therefore, its degree of freedom must be found from Gruebler's criterion.

Total number of links = 5

Number of pairs with 1 degree of freedom = 5

Number of pairs with 2 degrees of freedom = 1

$F = 3\,(N - 1) - 2P_1 - P_2$

$= 3(5 - 1) - 2 \times 5 - 1 = 1$

Thus, it is a mechanism with one degree of freedom.

Example 1.3 *Determine the degree of freedom of the mechanisms shown in Fig. 1.21.*

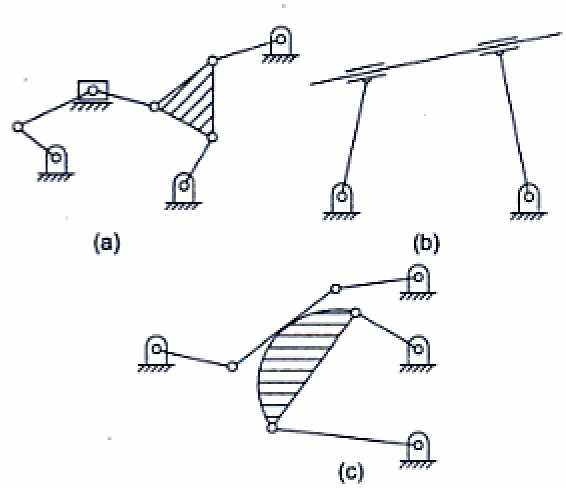


Fig. 1.21

Solution

(a) The mechanism has a sliding pair. Therefore, its degree of freedom must be found from Gruebler's criterion.

Total number of links = 8 (Fig. 1.22)

Number of pairs with 1 degree of freedom = 10

(At the slider, one sliding pair and two turning pairs)

$F = 3(N-1) - 2P_1 - P_2$

$= 3(8-1) - 2 \times 10 - 0 = 1$

Thus, it is a mechanism with a single degree of freedom.

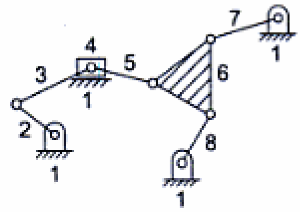


Fig. 1.22

(b) The system has a redundant degree of freedom as the rod of the mechanism can slide without causing any movement in the rest of the mechanism.

$\therefore$ effective degree of freedom

$= 3(N-1) - 2P_1 - P_2 - F_r$

$= 3(4-1) - 2 \times 4 - 0 - 1 = 0$

As the effective degree of freedom is zero, it is a locked system.

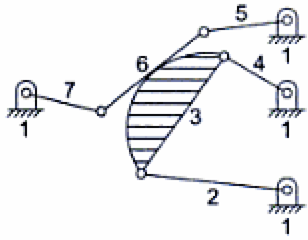


Fig. 1.23

(c) The mechanism has a cam pair. Therefore, its degree of freedom must be found from Gruebler's criterion.

Total number of links = 7 (Fig. 1.23)

Number of pairs with 1 degree of freedom = 8

Number of pairs with 2 degrees of freedom = 1

$F = 3(N-1) - 2P_1 - P_2$

$= 3(7-1) - 2 \times 8 - 1 = 1$

Thus, it is a mechanism with one degree of freedom.

Example 1.4 *How many unique mechanisms can be obtained from the 8-link kinematic chain shown in Fig. 1.24?*

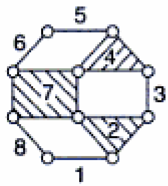


Fig. 1.24

Solution The kinematic chain has 8 links in all. A unique mechanism is obtained by fixing one of the links to the ground each time and retaining only one out of the symmetric mechanisms thus obtained.

The given kinematic chain is symmetric about links 3 or 7. Thus, identical inversions (mechanisms) are obtained if the links 2, 1, 8 or 4, 5, 6 are fixed. In addition, two more unique mechanisms can be obtained from the 8-link kinematic chain as shown in Fig. 1.25.

1. Sliding Pairs in Place of Turning Pairs

Figure 1.26(a) shows a four-link mechanism. Let the length of the link 4 be increased to infinity so that D lies at infinity. Now, with the rotation of the link 2, C will have a linear motion perpendicular to the axis of the link 4. The same motion of C can be obtained if the link 4 is replaced by a slider, and guides are provided for its motion as shown in Fig. 1.26(b). In this case, the axis of the slider does not pass through A and there is an eccentricity. Figure 1.26(c) shows a slider-crank mechanism with no eccentricity. In this way, a binary link is replaced by a slider pair.

Note that the axis of the sliding pair must be in the plane of the linkage or parallel to it.

Similarly, the turning pair at A can also be replaced by a sliding pair by providing a slider with guides at B [Fig. 1.26(d)].

In case the axes of the two sliding pairs are in one line or parallel, the two sliders along with the link 3 act as one link with no relative motion among these links. Then the arrangement ceases to be a linkage. Thus, in order to replace two turning pairs in a linkage with sliding pairs, the axes of the sliding pairs must intersect.

In the same way, the turning pairs at B and C can be replaced by sliding pairs by fixing a slider to any of the two links forming the pair [Figs 1.26(e) and (f)]. Figure 1.26(g) shows both of the turning pairs at B and C replaced by sliding pairs.

2. Spring in Place of Turning Pairs

The action of a spring is to elongate or to shorten as it becomes in tension or in compression. A similar variation in length is accomplished by two binary links joined by a turning pair. In Fig. 1.27(a), the length AB varies as OB is moved away or towards point A. Figure 1.27(b) shows a 6-link mechanism in which links 4 and 5 have been shown replaced by a spring.

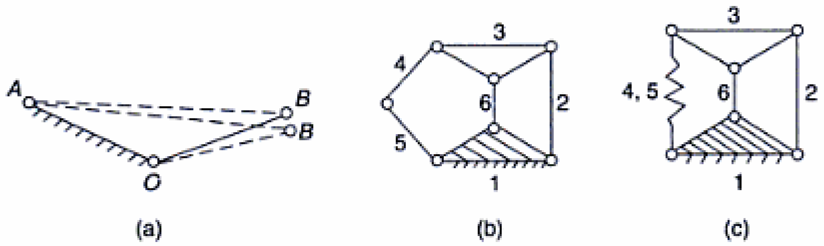


Fig. 1.27

Remember that the spring is not a rigid link but is simulating the action of two binary links joined by a turning pair. Therefore, to find the degree of freedom of such a mechanism, the spring has to be replaced by the binary links.

3. Cam Pair in Place of Turning Pair

A cam pair has two degrees of freedom. For linkages with one degree of freedom, application of Gruebler's equation yields,

$$F = 3(N-1) - 2P_1 - 1P_2$$

or $$1 = 3N - 3 - 2P_1 - 1 \times 1$$

or $$P_1 = \frac{3N-5}{2}$$

This shows that to have one cam pair in a mechanism with one degree of freedom, the number of links and turning pairs should be as below:

$N = 3, \qquad P_1 = 2$
$N = 5, \qquad P_1 = 5$
$N = 7, \qquad P_1 = 8$
$N = 9, \qquad P_1 = 11$ and so on.

A comparison of this with linkages having turning pairs only (Table 1.2) indicates that a cam pair can be replaced by one binary link with two turning pairs at each end.

Figure 1.28(a) shows link *CD* (of a four-link mechanism) with two turning pairs at its ends replaced by a cam pair. The centres of curvatures at the point of contact *X* of the two cams lie at *D* and *C*. Figures 1.28(b) and (c) show the link *BC* with turning pairs at *B* and *C* replaced by a cam pair. The centres of curvature at the point of contact *X* lie at *B* and *C* respectively. Figure 1.28(d) shows equivalent mechanism for a disc cam with reciprocating curved-face follower. The centres of curvature of the cam and the follower at the instant lie at *A* and *B* respectively.

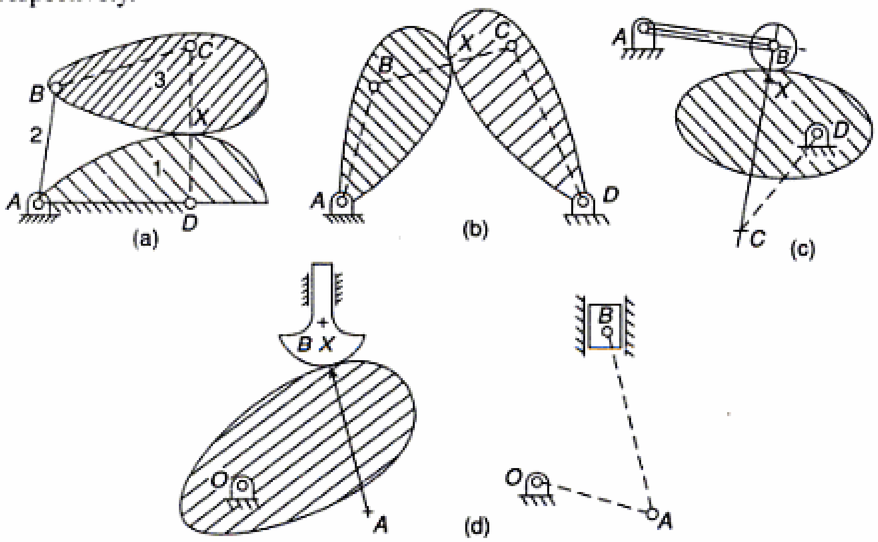


Fig. 1.28

Example 1.6 *Sketch a few slider-crank mechanisms derived from Stephenson's and Watt's six-bar chains.*

Solution Figure 1.29(a) shows a Stephenson's chain in which the ternary links are not directly connected. Thus, any of the binary links 3 or 6 can be replaced by a slider to obtain a slider-crank mechanism as shown in Fig. 1.29(b) and (c).

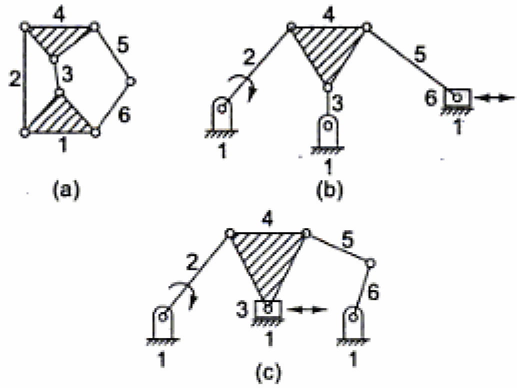


Fig. 1.29

Thus, d is less than a, b and c, i.e., it is the shortest link if a is to rotate a full circle or act as a crank. The above inequalities also suggest that out of a, b and c, whichever is the longest, the sum of that with d, the shortest link will be less than the sum of the remaining two links. Thus, the necessary conditions for the link a to be a crank is

- the shortest link is fixed, and
- the sum of the shortest and the longest links is less than the sum of the other two links.

In a similar way, it can be shown that if the link c is to rotate through a full circle, i.e., if it is to be a crank then the conditions to be realised are the same as above. Also, it can be shown that if both the links a and c rotate through full circles, the link b also makes one complete revolution relative to the fixed link d.

The mechanism thus obtained is known as *crank-crank* or *double-crank* or *drag-crank mechanism* or *rotary-rotary converter.* Figure 1.35 shows all the three links a, b and c rotating through one complete revolution.

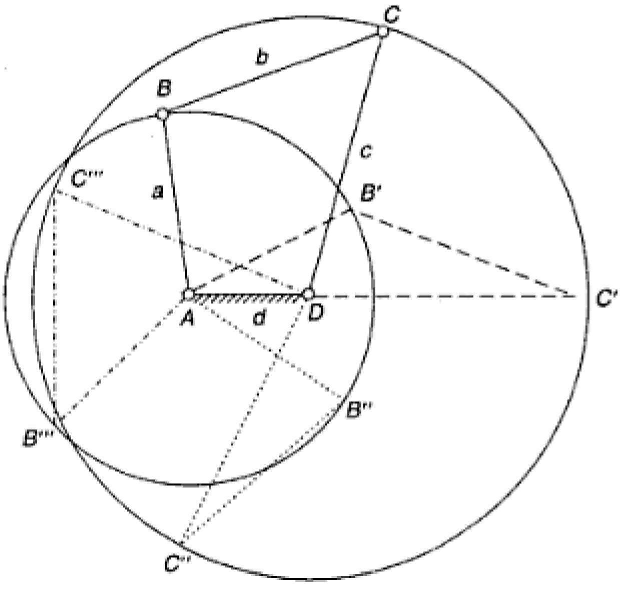


Fig. 1.35

In the above consideration, the rotation of the links is observed relative to the fixed link d. Now, consider the movement of b relative to either a or c. The complete rotation of b relative to a is possible if the angle $\angle ABC$ can be more than 180° and relative to c if the angle $\angle DCB$ more than 180°. From the positions of the links in Fig. 1.35(b) and (c), it is clear that these angles cannot become more than 180° for the above stated conditions.

Now, as the relative motion between two adjacent links remains the same irrespective of which link is fixed to the frame, different mechanisms (known as *inversions*) obtained by fixing different links of this kind of chain will be as follows:

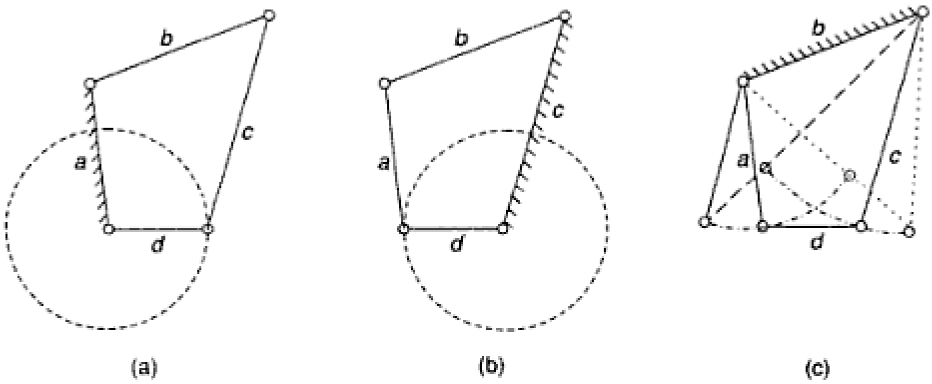


Fig. 1.36

1. If any of the adjacent links of link *d*, i.e., *a* or *c* is fixed, *d* can have a full revolution (crank) and the link opposite to it oscillates (rocks). In Fig. 1.36(a), *a* is fixed, *d* is the crank and *b* oscillates whereas in Fig. 1.36(b), *c* is fixed, *d* is the crank and *b* oscillates. The mechanism is known as *crank-rocker* or *crank-lever mechanism* or *rotary-oscillating converter.*
2. If the link opposite to the shortest link, i.e., link *b* is fixed and the shortest link *d* is made a coupler, the other two links *a* and *c* would oscillate [Fig. 1.36(c)]. The mechanism is known as a *rocker-rocker* or *double-rocker* or *double-lever mechanism* or *oscillating-oscillating converter.*

A linkage in which the sum of the lengths of the longest and the shortest links is less than the sum of the lengths of the other two links, is known as a *class-I,* four-bar linkage.

When the sum of the lengths of the largest and the shorted links is more than the sum of the lengths of the other two links, the linkage is known as a *class-II*, four-bar linkage. In such a linkage, fixing of any of the links always results in a *rocker-rocker* mechanism. In other words, the mechanism and its inversions give the same type of motion (of a *double-rocker* mechanism).

The above observations are summarised in *Grashof's law* which states *that a four-bar mechanism has at least one revolving link if the sum of the lengths of the largest and the shortest links is less than the sum of lengths of the other two links.*

Further, if the *shortest link is fixed,* the chain will act as a double-crank mechanism in which links adjacent to the fixed link will have complete revolutions. If the *link opposite to the shortest link is fixed,* the chain will act as double-rocker mechanism in which links adjacent to the fixed link will oscillate. If the *link adjacent to the shortest link is fixed,* the chain will act as crank-rocker mechanism in which the shortest link will revolve and the link adjacent to the fixed link will oscillate.

If the sum of the lengths of the largest and the shorted links is equal to the sum of the lengths of the other two links, i.e., when equalities exist, the four inversions, in general, result in mechanisms similar to those as given by Grashof's law, except that sometimes the links may become collinear and may have to be guided in the proper direction. Usually, the purpose is served by the inertia of the links. A few special cases may arise when equalities exist. For example, *parallel-crank four-bar linkage* and *deltoid* linkage.

Parallel-Crank Four-Bar Linkage If in a four-bar linkage, two opposite links are parallel and equal in length, then any of the links can be made fixed. The two links adjacent to the fixed link will always act as two cranks. The four links form a parallelogram in all the positions of the cranks, provided the cranks rotate in the same sense as shown in Fig. 1.37.

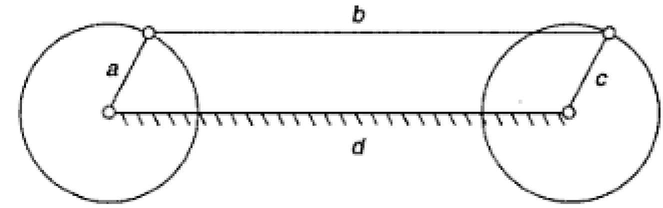


Fig. 1.37

The use of such a mechanism is made in the coupled wheels of a locomotive in which the rotary motion of one wheel is transmitted to the other wheel. For kinematic analysis, link *d* is treated as fixed and the relative motions of the other links are found. However, in fact, *d* has a translatory motion parallel to the rails.

Deltoid Linkage In a deltoid linkage (Fig. 1.38), the equal links are adjacent to each other. When any of the shorter links is fixed, a double-crank mechanism is obtained in which one revolution of the longer link causes two revolutions of the other shorter link. As shown in Fig. 1.38 (a), when the link *c* rotates through half a revolution and assumes the position *DC'*, the link *a* has completed a full revolution.

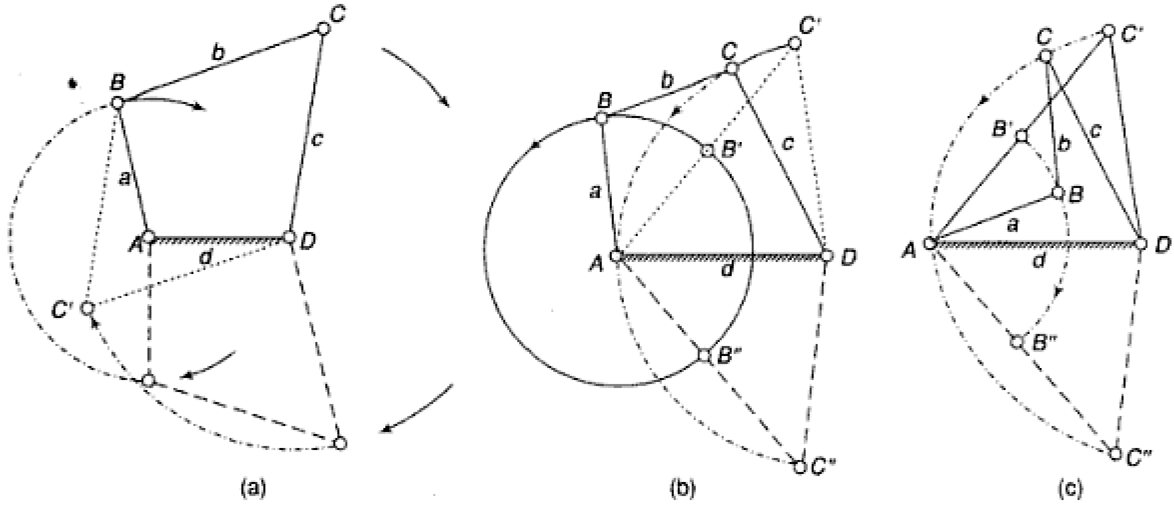


Fig. 1.38

When any of the longer links is fixed, two crank-rocker mechanisms are obtained [Fig. 1.38(b) and (c)]

Example 1.8 *Find all the inversion of the chain given in Fig. 1.39.*

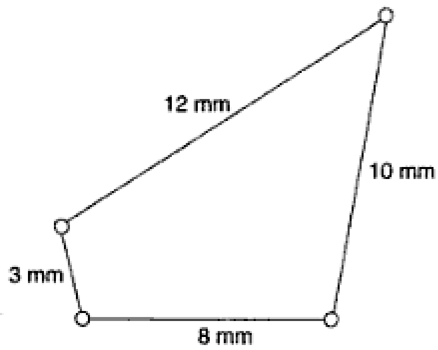


Fig. 1.39

Solution

(a) Length of the longest link = 12 mm
Length of the shortest link = 3 mm
Length of other links = 10 mm and 8 mm
Since $12 + 3 < 10 + 8$, it belongs to the class-I mechanism and according to Grashoff's law, three distinct inversions are possible.

Shortest link fixed, i.e., when the link with 3-mm length is fixed, the chain will act as double-crank mechanism in which links with lengths of 12 mm and 8 mm will have complete revolutions.

Link opposite to the shortest link fixed, i.e., when the link with 10-mm length is fixed, the chain will act as double-rocker mechanism in which links with lengths of 12 mm and 8 mm will oscillate.

Link adjacent to the shortest link fixed, i.e., when any of the links adjacent to the shortest link, i.e., link with a length of 12-mm or 8 mm is fixed, the chain will act as crank-rocker mechanism in which the shortest link of 3-mm length will revolve and the link with 10-mm length will oscillate.

Example 1.9 *Figure 1.40 shows some four-link mechanisms in which the figures indicate the dimensions in standard units of length. Indicate the type of each mechanism whether crank-rocker or double-crank or double-rocker.*

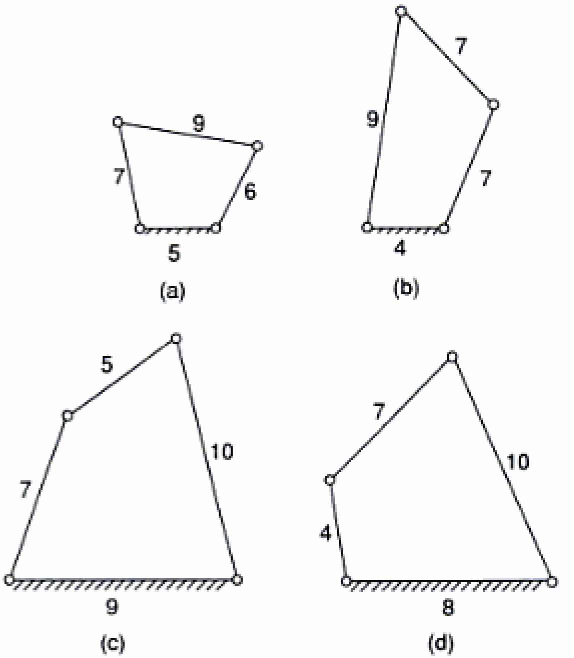


Fig. 1.40

Solution

(a) Length of the longest link = 9
Length of the shortest link = 5
Length of other links = 7 and 6
Since 9 + 5 > 7 + 6, it does not belong to the class-I mechanism. Therefore, it is a double-rocker mechanism.

(b) Length of the longest link = 9
Length of the shortest link = 4
Length of other links = 7 and 7
Since 9 + 4 < 7 + 7, it belongs to the class-I mechanism. In this case as the shortest link is fixed, it is a double-crank mechanism.

(c) Length of the longest link = 10
Length of the shortest link = 5
Length of other links = 9 and 7
Since 10 + 5 < 9 + 7, it belongs to the class-I mechanism. In this case as the link opposite to the shortest link is fixed, it is a double-rocker mechanism.

(d) Length of the longest link = 10
Length of the shortest link = 4
Length of other links = 8 and 7
Since 10 + 4 < 8 + 7, it belongs to the class-I mechanism. In this case as the link adjacent to the shortest link is fixed, it is a crank-rocker mechanism.

Example 1.10 *Figure 1.41 shows a plane mechanism in which the figures indicate the dimensions in standard units of length. The slider C is the driver. Will the link AG revolve or oscillate?*

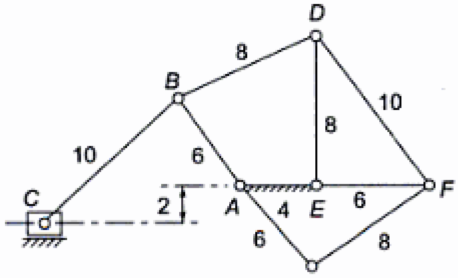


Fig. 1.41

Solution The mechanism has three sub-chains:

(i) *ABC*, a slider-crank chain
(ii) *ABDE*, a four-bar chain
(iii) *AEFG*, a four-bar chain

DEF is a locked chain as it has only three links.

- As the length *BC* is more than the length *AB* plus the offset of 2 units, *AB* acts as a crank and can revolve about *A*.
- In the chain *ABDE*,
  Length of the longest link = 8
  Length of the shortest link = 4
  Length of other links = 8 and 6

Since 8 + 4 < 8 + 6, it belongs to the class-I mechanism. In this case as the shortest link is fixed, it is a double-crank mechanism and thus *AB* and *ED* can revolve fully.

- In the chain *AEFG*,
  Length of the longest link = 8
  Length of the shortest link = 4
  Length of other links = 6 and 6

Since 8 + 4 = 6 + 6, it belongs to the class-I mechanism. As the shortest link is fixed, it is a double-crank mechanism and thus EF and AG can revolve fully.

As DEF is a locked chain with three links, the link EF revolves with the revolving of ED. With the revolving of ED, AG also revolves.

1.14 MECHANICAL ADVANTAGE

The *mechanical advantage* (MA) of a mechanism is the ratio of the output force or torque to the input force or torque at any instant. Thus for the linkage of Fig. 1.42, if friction and inertia forces are ignored and the input torque T_2 is applied to the link 2 to drive the output link 4 with a resisting torque T_4 then

Power input = Power output

$$T_2\,\omega_2 = T_4\,\omega_4$$

or $$\text{MA} = \frac{T_4}{T_2} = \frac{\omega_2}{\omega_4}$$

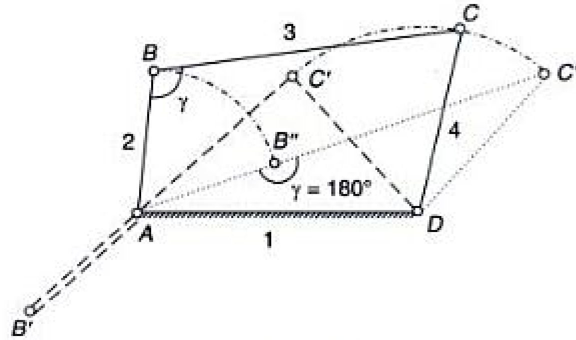


Fig. 1.42

Thus, it is the reciprocal of the velocity ratio. In case of crank-rocker mechanisms, the velocity ω_4 of the output link DC (rocker) becomes zero at the extreme positions ($AB'C'D$ and $AB''C''D$), i.e., when the input link AB is in line with the coupler BC and the angle γ between them is either zero or 180°, it makes the mechanical advantage to be infinite at such positions. Only a small input torque can overcome a large output torque load. The extreme positions of the linkage are known as *toggle* positions.

1.15 TRANSMISSION ANGLE

The angle μ between the output link and the coupler is known as *transmission angle*. In Fig. 1.43, if the link AB is the input link, the force applied to the output link DC is transmitted through the coupler BC. For a particular value of force in the coupler rod, the torque transmitted to the output link (about the point D) is maximum when the transmission angle μ is 90°. If links BC and DC become coincident, the *transmission angle* is zero and the mechanism would lock or jam. If μ deviates significantly from 90°, the torque on the output link decreases. Sometimes, it may not be sufficient to overcome the friction in the system and the mechanism may be locked or jammed. Hence μ is usually kept more than 45°. The best mechanisms, therefore, have a transmission angle that does not deviate much from 90°.

Applying cosine law to triangles ABD and BCD (Fig. 1.43),

$$a^2 + d^2 - 2ad\cos\theta = k^2 \qquad \text{(i)}$$

and $$b^2 + c^2 - 2bc\cos\mu = k^2 \qquad \text{(ii)}$$

From (i) and (ii),

$$a^2 + d^2 - 2ad\cos\theta = b^2 + c^2 - 2bc\cos\mu$$

or $$a^2 + d^2 - b^2 - c^2 - 2ad\cos\theta + 2bc\cos\mu = 0$$

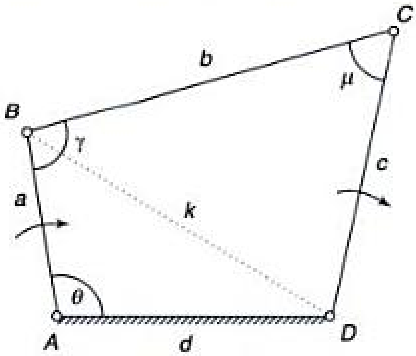


Fig. 1.43

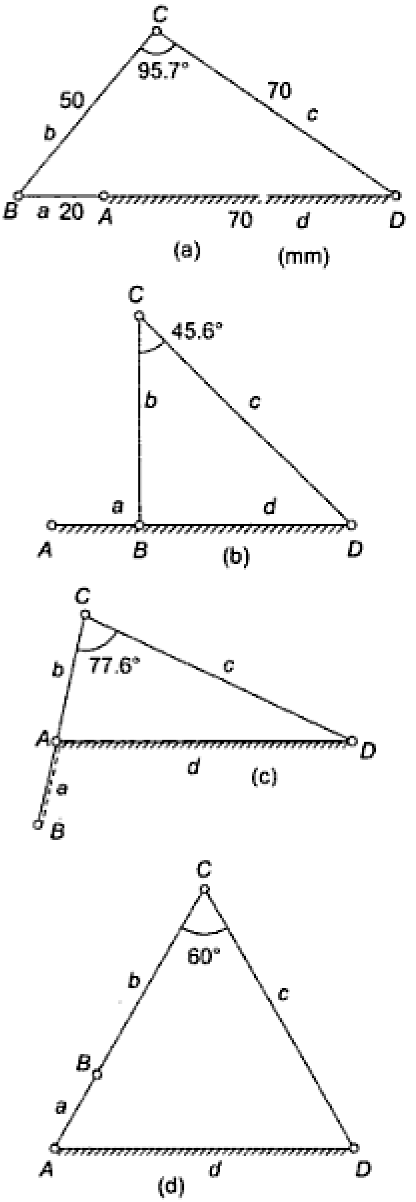


Fig. 1.50

Solution In this mechanism,

Length of the longest link = 70 mm

Length of the shortest link = 20 mm

Length of other links = 70 and 50 mm

Since 70 + 20 < 70 + 50, it belongs to the class-I mechanism. In this case as the link adjacent to the shortest link is fixed, it is a crank-rocker mechanism.

Maximum transmission angle is when θ is 180° [Fig. 1.50(a)],

Thus $(a + d)^2 = b^2 + c^2 - 2bc \cos \mu$

$$(20 + 70)^2 = 50^2 + 70^2 - 2 \times 50 \times 70 \cos \mu$$

$$8100 = 2500 + 4900 - 7000 \cos \mu$$

$$\cos \mu = -0.1$$

$$\mu = 95.7°$$

Minimum transmission angle is when θ is 0° [Fig. 1.50(b)],

Thus $(70 - 20)^2 = 50^2 + 70^2 - 2 \times 50 \times 70 \cos \mu$

$$2500 = 2500 + 4900 - 7000 \cos \mu$$

$$\cos \mu = 0.7$$

$$\mu = 45.6°$$

The two toggle positions are shown in Figs 1.50(c) and (d).

Transmission angle for first position,

$$d^2 = (b - a)^2 + c^2 - 2(b - a)c \cos \mu$$

$$70^2 = 30^2 + 70^2 - 2 \times 30 \times 70 \cos \mu$$

$$4900 = 900 + 4900 - 4200 \cos \mu$$

$$\cos \mu = 0.214$$

$$\mu = 77.6°$$

As c and d are of equal length [Fig. 1.50(c)], it is an isosceles triangle and thus input angle $\theta = (77.6° + 180°) = 257.6°$

Transmission angle for second position Fig. 1.50(d),

$$d^2 = (b + a)^2 + c^2 - 2(b + a)c \cos \mu$$

$$70^2 = 70^2 + 70^2 - 2 \times 70 \times 70 \cos \mu$$

$$4900 = 4900 + 4900 - 9800 \cos \mu$$

$$\cos \mu = 0.5$$

$$\mu = 60°$$

(or as all the sides of the triangle of Fig. 1.50(d) are of equal length, it is an equilateral triangle and thus transmission angle is equal to 60°)

And the input angle, $\theta = 60°$

- The above results can also be obtained graphically by drawing the figures to scale and measuring the angles.

1.16 THE SLIDER-CRANK CHAIN

When one of the turning pairs of a four-bar chain is replaced by a sliding pair, it becomes a *single slider-crank chain* or simply a *slider-crank chain*. It is also possible to replace two sliding pairs of a four-bar chain to get a *double slider-crank chain* (Sec. 1.17). Further, in a slider-crank chain, the straight line path of the slider may be passing through the fixed pivot O or may be displaced. The distance e between the fixed pivot O and the straight line path of the slider is called the *offset* and the chain so formed an *offset slider-crank chain.*

Different mechanisms obtained by fixing different links of a kinematic chain are known as its *inversions*. A slider-crank chain has the following inversions:

First Inversion

This inversion is obtained when link 1 is fixed and links 2 and 4 are made the crank and the slider respectively [Fig. 1.51(a)].

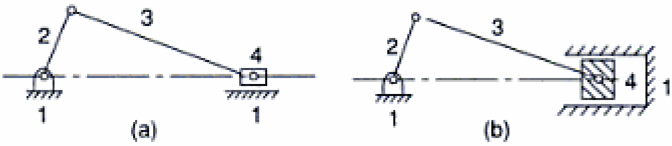


Fig. 1.51

Applications

1. Reciprocating engine
2. Reciprocating compressor

As shown in Fig. 1.51(b), if it is a reciprocating engine, 4 (piston) is the driver and if it is a compressor, 2 (crank) is the driver.

Second Inversion

Fixing of the link 2 of a slider-crank chain results in the second inversion.

The slider-crank mechanism of Fig. 1.51(a) can also be drawn as shown in Fig. 11.52(a). Further, when its link 2 is fixed instead of the link 1, the link 3 along with the slider at its end B becomes a crank. This makes the link 1 to rotate about O along with the slider which also reciprocates on it [Fig. 1.52(b)].

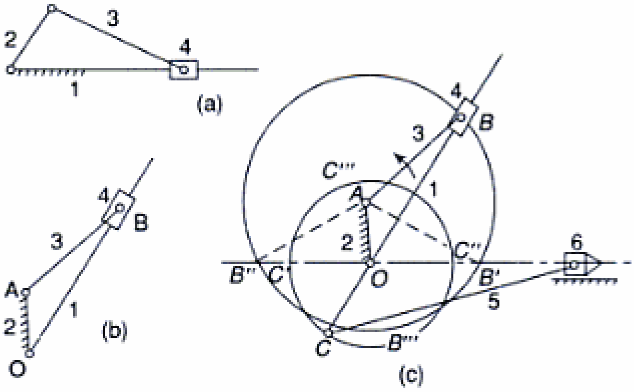


Fig. 1.52

Applications

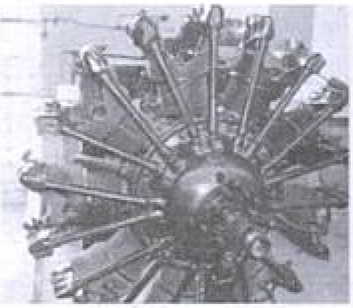
1. Whitworth quick-return mechanism
2. Rotary engine

Whitworth Quick-Return Mechanism It is a mechanism used in workshops to cut metals. The forward stroke takes a little longer and cuts the metal whereas the return stroke is idle and takes a shorter period.

Slider 4 rotates in a circle about A and slides on the link 1 [Fig. 1.52(c)]. C is a point on the link 1 extended backwards where the link 5 is pivoted. The other end of the link 5 is pivoted to the tool, the forward stroke of which cuts the metal. The axis of motion of the slider 6 (tool) passes through O and is perpendicular to OA, the fixed link. The crank 3 rotates in the counter-clockwise direction.

Initially, let the slider 4 be at B' so that C be at C'. Cutting tool 6 will be in the extreme left position. With the movement of the crank, the slider traverses the path $B'BB''$ whereas the point C moves through $C'CC''$. Cutting tool 6 will have the forward stroke. Finally, the slider B assumes the position B'' and the cutting tool 6 is in the extreme right position. The time taken for the forward stroke of the slider 6 is proportional to the obtuse angle $B''AB'$ at A.

Similarly, the slider 4 completes the rest of the circle through the path $B''B'''B'$ and C passes through $C''C'''C'$. There is backward stroke of the tool 6. The time taken in this is proportional to the acute angle $B''AB'$ at A.



A nine-cylinder rotary engine

Let

θ = obtuse angle $B'AB''$ at A

β = acute angle $B'AB''$ at A

Then,

$$\frac{\text{Time of cutting}}{\text{Time of return}} = \frac{\theta}{\beta}$$

Rotary Engine Referring Fig. 1.52(b), it can be observed that with the rotation of the link 3, the link 1 rotates about O and the slider 4 reciprocates on it. This also implies that if the slider is made to reciprocate on the link 1, the crank 3 will rotate about A and the link 1 about O.

In a rotary engine, the slider is replaced by a piston and the link 1 by a cylinder pivoted at O. Moreover, instead of one cylinder, seven or nine cylinders symmetrically placed at regular intervals in the same plane or in parallel planes, are used. All the cylinders rotate about the same fixed centre and form a balanced system. The fixed link 2 is also common to all cylinders (Fig. 1.53).

Thus, in a rotary engine, the crank 2 is fixed and the body 1 rotates whereas in a reciprocating engine (1st inversion), the body 1 is fixed and the crank 2 rotates.

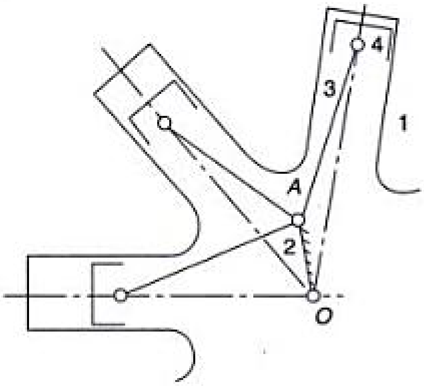


Fig. 1.53

Third Inversion

By fixing the link 3 of the slider-crank mechanism, the third inversion is obtained [Fig. 1.54(a)]. Now the link 2 again acts as a crank and the link 4 oscillates.

Applications

1. Oscillating cylinder engine
2. Crank and slotted-lever mechanism

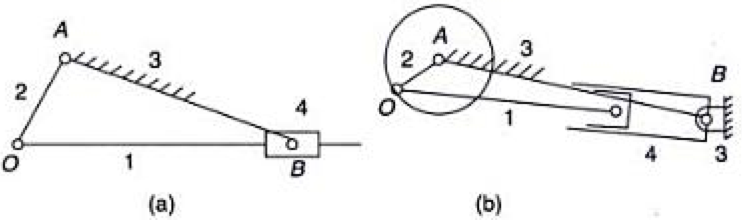


Fig. 1.54

Oscillating Cylinder Engine As shown in Fig. 1.54(b), the link 4 is made in the form of a cylinder and a piston is fixed to the end of the link 1. The piston reciprocates inside the cylinder pivoted to the fixed link 3. The arrangement is known as oscillating cylinder engine, in which as the piston reciprocates in the oscillating cylinder, the crank rotates.

Crank and Slotted-Lever Mechanism If the cylinder of an oscillating cylinder engine is made in the form of a guide and the piston in the form of a slider, the arrangement as shown in Fig. 1.55(a) is obtained. As the crank rotates about A, the guide 4 oscillates about B. At a point C on the guide, the link 5 is pivoted, the other end of which is connected to the cutting tool through a pivoted joint.

Figure 1.55(b) shows the extreme positions of the oscillating guide 4. The time of the forward stroke is proportional to the angle θ whereas for the return stroke, it is proportional to angle β, provided the crank rotates clockwise.

Comparing a crank and slotted-lever quick-return mechanism with a Whitworth quick-return mechanism, the following observations are made:

1. Crank 3 of the Whitworth mechanism is longer than its fixed link 2 whereas the crank 2 of the slotted-lever mechanism is shorter than its fixed link 3.
2. Coupler link 1 of the Whitworth mechanism makes complete rotations about its pivoted joint O with the fixed link. However, the coupler link 4 of the slotted-lever mechanism oscillates about its pivot B.
3. The coupler link holding the tool can be pivoted to the main coupler link at any convenient point C in both cases. However, for the same displacement of the tool, it is more convenient if the point C is taken on the extension of the main coupler link (towards the pivot with the fixed link) in case of the Whitworth mechanism and beyond the extreme position of the slider in the slotted-lever mechanism.

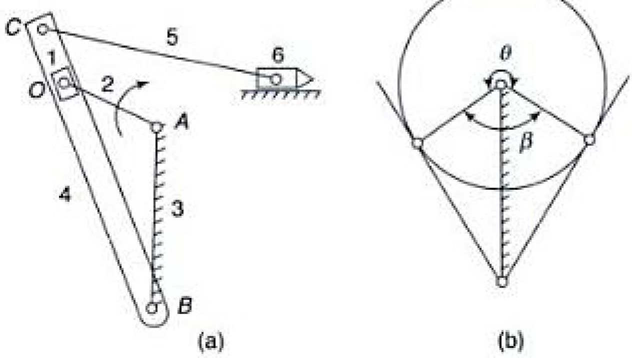


Fig. 1.55

A shaping machine. Shaping machines are fitted with quick-return mechanisms.

Fourth Inversion

If the link 4 of the slider-crank mechanism is fixed, the fourth inversion is obtained [Fig. 1.56(a)]. Link 3 can oscillate about the fixed pivot B on the link 4. This makes the end A of the link 2 to oscillate about B and the end O to reciprocate along the axis of the fixed link 4.

Application Hand-pump

Figure 1.56(b) shows a hand-pump. Link 4 is made in the form of a cylinder and a plunger fixed to the link 1 reciprocates in it.

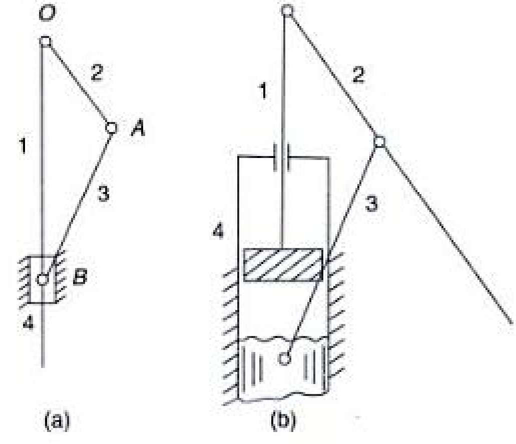


Fig. 1.56

Example 1.13 *The length of the fixed link of a crank and slotted-lever mechanism is 250 mm and that of the crank is 100 mm. Determine the*

(i) inclination of the slotted lever with the vertical in the extreme position,

(ii) ratio of the time of cutting stroke to the time of return stroke, and

(iii) length of the stroke, if the length of the slotted lever is 450 mm and the line of stroke passes through the extreme positions of the free end of the lever.

Solution Refer Fig. 1.57.

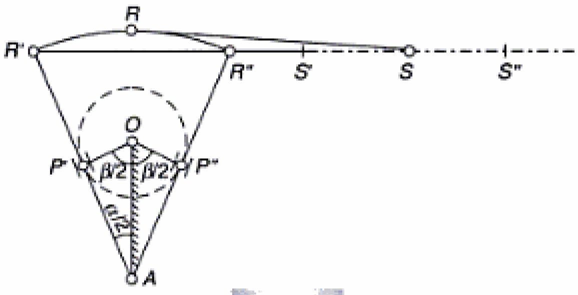


Fig. 1.57

$OA = 250$ mm $\qquad OP' = OP'' = 100$ mm
$AR' = AR'' = AR = 450$ mm

$$\cos\frac{\beta}{2} = \frac{OP'}{OA} = \frac{100}{250} = 0.4$$

or $\quad \dfrac{\beta}{2} = 66.4° \qquad$ or $\quad \beta = 132.8°$

(i) Angle of the slotted lever with the vertical
$\alpha/2 = 90° - 66.4° = 23.6°$

(ii) $\dfrac{\text{Time of cutting stroke}}{\text{Time of return stroke}}$

$$= \frac{360° - \beta}{\beta} = \frac{360° - 132.8°}{132.8°} = 1.71$$

(iii) Length of stroke $= S'S'' = R'R''$

$$= 2\,AR'.\sin(\alpha/2)$$
$$= 2 \times 450 \sin 23.6°$$
$$= 360.3 \text{ mm}$$

1.17 DOUBLE SLIDER-CRANK CHAIN

A four-bar chain having two turning and two sliding pairs such that two pairs of the same kind are adjacent is known as a double-slider-crank chain [Fig. 1.58(a)]. The following are its inversions.

First Inversion

This inversion is obtained when the link 1 is fixed and the two adjacent pairs 23 and 34 are turning pairs and the other two pairs 12 and 41 sliding pairs.

Application Elliptical trammel

Elliptical Trammel Figure 1.58(b) shows an elliptical trammel in which the fixed link 1 is in the form of guides for sliders 2 and 4. With the movement of the sliders, any point C on the link 3, except the midpoint of AB will trace an ellipse on a fixed plate. The midpoint of AB will trace a circle.

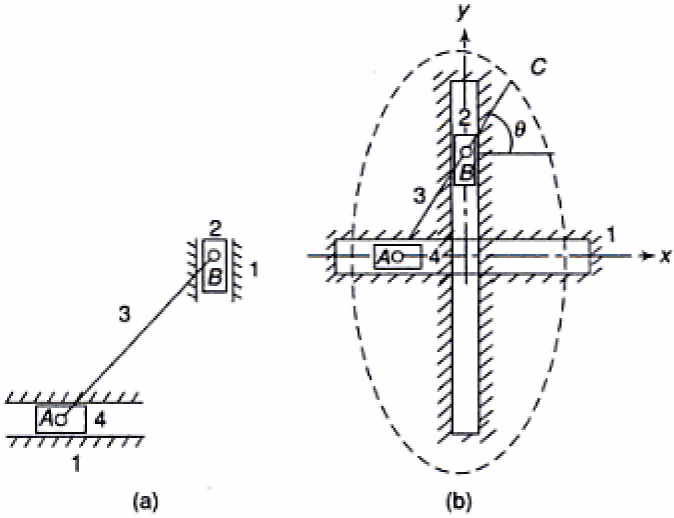


Fig. 1.58

Let at any instant, the link 3 make angle θ with the X-axis. Considering the displacements of the sliders from the centre of the trammel,

$$x = BC\cos\theta \text{ and } y = AC\sin\theta$$

$$\therefore \quad \frac{x}{BC} = \cos\theta \text{ and } \frac{y}{AC} = \sin\theta$$

$$\frac{F}{2\sin\alpha} = \frac{R}{\cos\alpha}$$

or $$2\tan\alpha = \frac{F}{R}$$

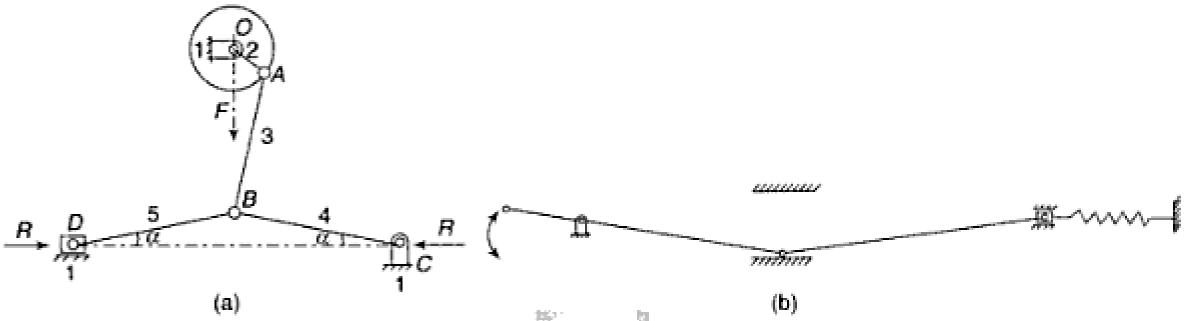


Fig. 1.62

As $\alpha \to 0$, $\tan\alpha \to 0$. Thus for a small value of the force F, R approaches infinity. In a stone crusher, a large resistance at D is overcome with a small force F in this way. Figure 1.62(b) shows another such mechanism.

Indexing Mechanisms

An *indexing mechanism* serves the purpose of dividing the periphery of a circular piece into a number of equal parts. Indexing is generally done on gear cutting or milling machines.

An indexing mechanism consists of an index head in which a spindle is carried in a headstock [Fig. 1.63(a)]. The work to be indexed is held either between centres or in a chuck attached to the spindle. A 40-tooth worm wheel driven by a single-threaded right-hand worm is also fitted to the spindle. At the end of the worm shaft an adjustable index crank with a handle and a plunger pin is also fitted. The plunger pin can be made to fit into any hole in the index plate which has a number of circles of equally spaced holes as shown in Fig. 1.63(b). An index head is usually provided with a number of interchangeable index plates to cover a wide range of work. However, the figure shows only the circle of 17 holes for sake of clarity.

As the worm wheel has 40 teeth, the number of revolutions of the index crank required to make one revolution of the work is also 40. The number of revolutions of the crank, needed for a proper division of the work into the desired number of divisions, can be calculated as follows:

- If a work is to be divided into 40 divisions, the crank should be given one complete revolution; if 20 divisions, two revolutions for each division, and so on.
- If the work is to be divided into 160 divisions, obviously the crank should be rotated through one-fourth of a rotation. For such cases, an index plate with a number of holes divisible by 4 such as with 16 or 20 holes can be chosen.
- If the work is to be divided into 136 parts, the use of the index plate will be essential since the rotation of the crank for each division will be 40/136 or five-seventeenth of a turn. Thus, a plate with 17 holes is selected in this case. To obviate the necessity of counting the holes at each partial turn of the crank, an index sector with two arms which can be set and clamped together at any angle is also available. In this case, this can be set to measure off 5 spaces. Starting with the crankpin in the hole a, a cut would be made in the work. The crank is rotated and the pin is made to enter into the hole b, 5 divisions apart and a second cut is made in the work. In a similar way, a third cut is made by rotating the crank again through five divisions with the help of an index sector, and so on. Usually, index tables are provided to ascertain the number of turns of the crank and the number of holes for the given case.

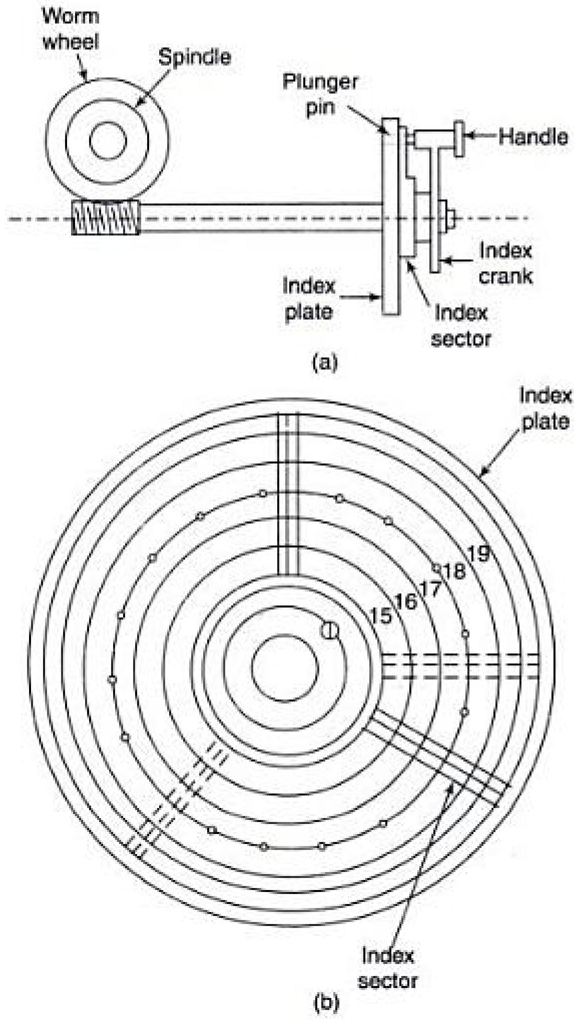
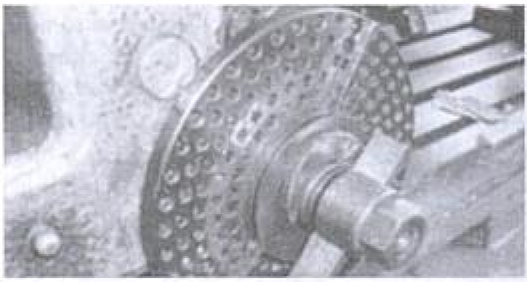


Fig. 1.63



Index plate of an indexing mechanism

Summary

1. *Kinematics* deals with the relative motions of different parts of a mechanism without taking into consideration the forces producing the motions whereas *dynamics* involves the calculation of forces impressed upon different parts of a mechanism.
2. *Mechanism* is a combination of a number of rigid bodies assembled in such a way that the motion of one causes constrained and predictable motion of the others whereas a *machine* is a mechanism or a combination of mechanisms which, apart from imparting definite motions to the parts, also transmits and modifies the available mechanical energy into some kind of useful work.
3. There are three types of constrained motion: *completely constrained, incompletely constrained* and *successfully constrained.*
4. A *link* is a resistant body or a group of resistant bodies with rigid connections preventing their

relative movement. A link may also be defined as a member or a combination of members of a mechanism, connecting other members and having motion relative to them.

5. A *kinematic pair* or simply a pair is a joint of two links having relative motion between them.
6. A pair of links having surface or area contact between the members is known as a *lower pair* and a pair having a point or line contact between the links, a *higher pair*.
7. When the elements of a pair are held together mechanically, it is known as a *closed pair*. The two elements are geometrically identical. When two links of a pair are in contact either due to force of gravity or some spring action, they constitute an *unclosed pair*.
8. Usual types of joints in a chain are binary joint, ternary joint and quaternary joint
9. *Degree of freedom of a pair* is defined as the number of independent relative motions, both translational and rotational, a pair can have.
10. A *kinematic chain* is an assembly of links in which the relative motions of the links is possible and the motion of each relative to the other is definite.
11. A *redundant chain* does not allow any motion of a link relative to the other.
12. A *linkage* or *mechanism* is obtained if one of the links of a kinematic chain is fixed to the ground.
13. *Degree of freedom of a mechanism* indicates how many inputs are needed to have a constrained motion of the other links.
14. *Kutzback's criterion* for the degree of freedom of plane mechanisms is

    $F = 3(N-1) - 2P_1 - 1P_2$
15. *Gruebler's criterion* for degree of freedom of plane mechanisms with single-degree of freedom joints only is

    $F = 3(N-1) - 2P_1$
16. *Author's criterion* for degree of freedom and the number of joints of plane mechanisms with turning pairs is

    $F = N - (2L + 1)$

    $P_1 = N + (L - 1)$
17. In a four-link mechanism, a link that makes a complete revolution is known as a *crank*, the link opposite to the fixed link is called the *coupler* and the fourth link is called a *lever* or *rocker* if it oscillates or another crank, if it rotates.
18. In a Watts six-bar chain, the ternary links are direct connected whereas in a Stephenson's six-bar chain, they are not direct connected.
19. If a system has one or more links which do not introduce any extra constraint, it is known as *redundant link* and is not counted to find the degree of freedom.
20. If a link of a mechanism can be moved without causing any motion to the rest of the links of the mechanism, it is said to have a *redundant degree of freedom*.
21. The *mechanical advantage* (MA) of a mechanism is the ratio of the output force or torque to the input force or torque at any instant.
22. The angle μ between the output link and the coupler is known as *transmission angle*.
23. Different mechanisms obtained by fixing different links of a kinematic chain are known as its *inversions*.
24. The mechanisms used to overcome a large resistance of a member with a small driving force are known as *snap action* or *toggle* mechanisms.
25. An *indexing mechanism* serves the purpose of dividing the periphery of a circular piece into a number of equal parts.

Exercises

1. Distinguish between
   (i) mechanism and machine
   (ii) analysis and synthesis of mechanisms
   (iii) kinematics and dynamics
2. Define: kinematic link, kinematic pair, kinematic chain.
3. What are rigid and resistant bodies? Elaborate.
4. How are the kinematic pairs classified? Explain with examples.
5. Differentiate giving examples:
   (i) lower and higher pairs
   (ii) closed and unclosed pairs
   (iii) turning and rolling pairs
6. What do you mean by degree of freedom of a

kinematic pair? How are pairs classified? Give examples.

7. Discuss various types of constrained motion.
8. What is a redundant link in a mechanism?
9. How do a Watt's six-bar chain and Stephenson's six-bar chain differ?
10. What is redundant degree of freedom of a mechanism?
11. What are usual types of joints in a mechanism?
12. What is the degree of freedom of a mechanism? How is it determined?
13. What is Kutzback's criterion for degree of freedom of plane mechanisms? In what way is Gruebler's criterion different from it?
14. How are the degree of freedom and the number of joints in a linkage can be found when the number of links and the number of loops in a kinematic chain are known?
15. What is meant by equivalent mechanisms?
16. Define Grashof's law. State how is it helpful in classifying the four-link mechanisms into different types.
17. Why are parallel-crank four-bar linkage and deltoid linkage considered special cases of four-link mechanisms?
18. Define mechanical advantage and transmission angle of a mechanism.
19. Describe various inversions of a slider-crank mechanism giving examples.
20. What are quick-return mechanisms? Where are they used? Discuss the functioning of any one of them.
21. How are the Whitworth quick-return mechanism and crank and slotted-lever mechanism different from each other?
22. Enumerate the inversions of a double-slider-crank chain. Give examples.
23. Describe briefly the functions of elliptical trammel and scotch yoke.
24. In what way is Oldham's coupling useful in connecting two parallel shafts when the distance between their axes is small?
25. What are snap-action mechanisms? Give examples.
26. What is an indexing mechanism? Describe how it is used to divide the periphery of a circular piece into a number of equal parts.
27. For the kinematic linkages shown in Fig. 1.64, find the degree of freedom (F).

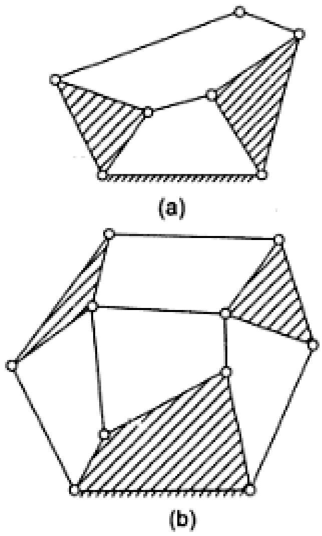


Fig. 1.64

[(a) 1 (b) 0]

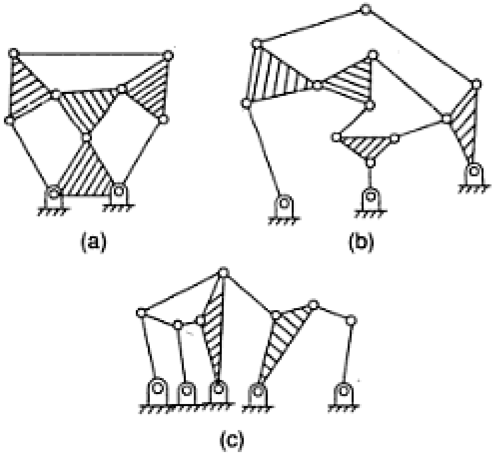


Fig. 1.65

28. For the kinematic linkages shown in Fig 1.65, find the number of binary links (N_b), ternary links (N_t), other links (N_o), total links N, loops L, joints or pairs (P_1), and degree of freedom (F).

[(a) $N_b = 3$; $N_t = 4$; $N_o = 0$; $N = 7$; $L = 3$; $P_1 = 9$; $F = 0$

(b) $N_b = 7$; $N_t = 5$; $N_o = 0$; $N = 12$; $L = 4$; $P_1 = 15$; $F = 3$

(c) $N_b = 8$; $N_t = 2$; $N_o = 1$; $N = 11$; $L = 5$; $P_1 = 15$; $F = 0$]

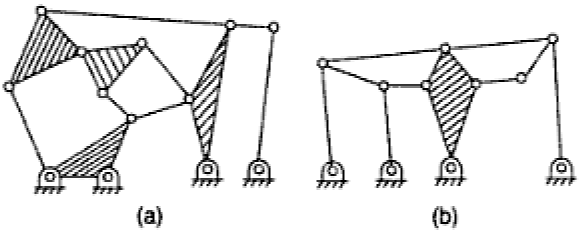


Fig. 1.66

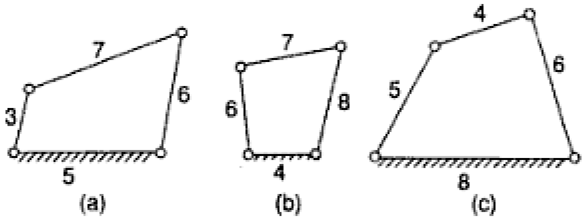


Fig. 1.67

29. Show that the linkages shown in Fig. 1.66 are structures. Suggest some changes to make them mechanisms having one degree of freedom. The number of links should not be changed by more than ±1.
30. A linkage has 14 links and the number of loops is 5. Calculate its
    (i) degrees of freedom
    (ii) number of joints
    (iii) maximum number of ternary links that can be had.

    Assume that all the pairs are turning pairs.

    (3; 18; 8)
31. Figure 1.67 shows some four-link mechanisms in which the figures indicate the dimensions in standard units of length. Indicate the type of each mechanism, whether it is crank-rocker or double-crank or double-rocker.

    [(a) crank-rocker (b) double-crank (c) double-rocker]
32. A crank-rocker mechanism *ABCD* has the dimensions AB = 30 mm, BC = 90 mm, CD = 75 mm and AD (fixed link) = 100 mm. Determine the maximum and the minimum values of the transmission angle. Locate the toggle positions and indicate the corresponding crank angles and the transmission angles.

    (103°, 49°, θ = 228°, μ = 92°, θ = 38.5°, μ = 56°)

3. At the end of the second vector, place the beginning of the third vector, and so on.
4. Joining of the beginning of the first vector and the end of the last vector represents the sum of the vectors. Figure 2.3 shows the addition of four vectors.

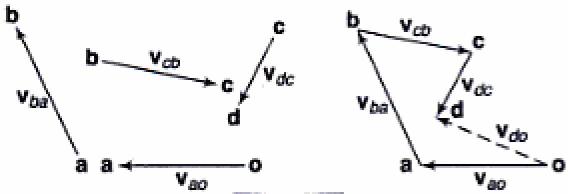


Fig. 2.3

$$\mathbf{v}_{do} = \mathbf{v}_{dc} + \mathbf{v}_{cb} + \mathbf{v}_{ba} + \mathbf{v}_{ao} \tag{2.2}$$
$$= \mathbf{v}_{ao} + \mathbf{v}_{ba} + \mathbf{v}_{cb} + \mathbf{v}_{dc}$$
$$\mathbf{od} = \mathbf{oa} + \mathbf{ab} + \mathbf{bc} + \mathbf{cd}$$

Equation 2.1 may be written as,

Vel. of B rel. to A = Vel. of B rel. to O – Vel. of A rel. to O (2.3)

$$\mathbf{v}_{ba} = \mathbf{v}_{bo} - \mathbf{v}_{ao}$$

or $$\mathbf{ab} = \mathbf{ob} - \mathbf{oa}$$

This shows that in Fig. 2.2, **ab** also represents the subtraction of **oa** from **ob** [Fig. 2.4(a)]

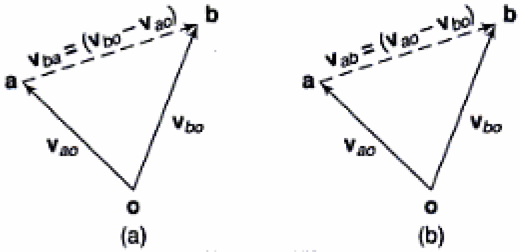


Fig. 2.4

Also $$\mathbf{v}_{ab} = -\mathbf{v}_{ba} = \mathbf{v}_{ao} - \mathbf{v}_{bo}$$

or $$\mathbf{ba} = \mathbf{oa} - \mathbf{ob}$$

This has been shown in Fig. 2.4 (b).

Thus, the difference of two vectors is given by the closing side of a triangle, the other two sides of which are formed by placing the two vectors tail to tail, the sense being towards the vector quantity from which the other is subtracted.

2.4 MOTION OF A LINK

Let a rigid link OA, of length r, rotate about a fixed point O with a uniform angular velocity ω rad/s in the counter-clockwise direction [Fig.2.5 (a)]. OA turns through a small angle $\delta\theta$ in a small interval of time δt. Then A will travel along the arc AA' as shown in [Fig.2.5(b)].

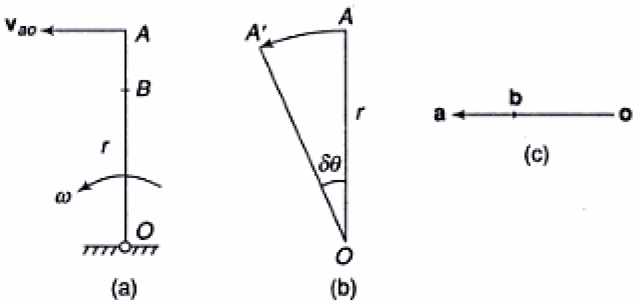


Fig. 2.5

Velocity of A relative to $O = \dfrac{\text{Arc}AA'}{\delta t}$

or

$$\mathbf{v}_{ao} = \frac{r\delta\theta}{\delta t}$$

In the limits, when $\delta t \to 0$

$$\mathbf{v}_{ao} = r\frac{d\theta}{dt} \tag{2.4}$$
$$= r\omega$$

The direction of v_{ao} is along the displacement of A. Also, as δt approaches zero ($\delta t \to 0$), AA' will be perpendicular to OA. Thus, velocity of A is ωr and is perpendicular to OA. This can be represented by a vector **oa** (Fig. 2.5 c). The fact that the direction of the velocity vector is perpendicular to the link also emerges from the fact that A can neither approach nor recede from O and thus, the only possible motion of A relative to O is in a direction perpendicular to OA.

Consider a point B on the link OA.

Velocity of $B = \omega \,.\, OB$ perpendicular to OB.

If **ob** represents the velocity of B, it can be observed that

$$\frac{\mathbf{ob}}{\mathbf{oa}} = \frac{\omega OB}{\omega OA} = \frac{OB}{OA} \tag{2.5}$$

i.e., **b** divides the velocity vector in the same ratio as B divides the link.

Remember, the velocity vector v_{ao} [Fig. 2.5(c)] represents the velocity of A at a particular instant. At other instants, when the link OA assumes another position, the velocity vectors will have their directions changed accordingly.

Also, the magnitude of the instantaneous linear velocity of a point on a rotating body is proportional to its distance from the axis of rotation.

2.5 FOUR-LINK MECHANISM

Figure 2.6(a) shows a four-link mechanism (quadric-cycle mechanism) $ABCD$ in which AD is the fixed link and BC is the coupler. AB is the driver rotating at an angular speed of ω rad/s in the clockwise direction if it is a crank or moving at this angular velocity at this instant if it is a rocker. It is required to find the absolute velocity of C (or velocity of C relative to A).

Writing the velocity vector equation,

Vel. of C rel. to A = Vel. of C rel. to B + vel. of B rel. to A

$$\mathbf{v}_{ca} = \mathbf{v}_{cb} + \mathbf{v}_{ba} \tag{2.6}$$

The velocity of any point relative to any other point on a fixed link is always zero. Thus, all the points on a fixed link are represented by one point in the velocity diagram. In Fig. 2.6(a), the points A and D, both lie on the fixed link AD. Therefore, the velocity of C relative to A is the same as velocity of C relative to D.

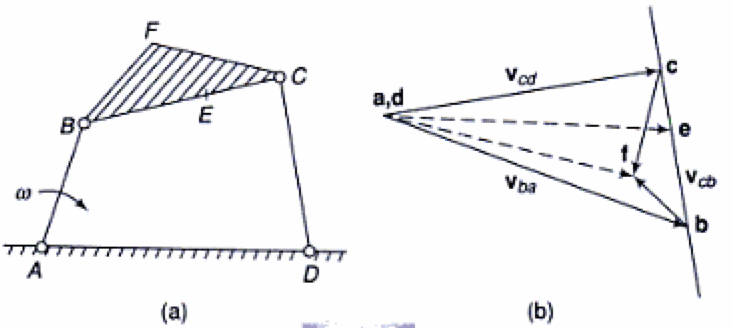


Fig. 2.6

Equation (2.6) may be written as,

$$\mathbf{v}_{cd} = \mathbf{v}_{ba} + \mathbf{v}_{cb}$$

or

$$\mathbf{dc} = \mathbf{ab} + \mathbf{bc}$$

where $\mathbf{v}_{ba}$ or $\mathbf{ab} = \omega AB$; $\perp$ to AB

$\mathbf{v}_{cb}$ or **bc** is unknown in magnitude ; $\perp$ to BC

$\mathbf{v}_{cd}$ or **dc** is unknown in magnitude ; ⊥ to DC

The velocity diagram is constructed as follows:

1. Take the first vector **ab** as it is completely known.
2. To add vector **bc** to **ab**, draw a line ⊥ BC through **b,** of any length. Since the direction-sense of **bc** is unknown, it can lie on either side of **b.** A convenient length of the line can be taken on both sides of **b.**
3. Through **d,** draw a line ⊥ DC to locate the vector **dc.** The intersection of this line with the line of vector **bc** locates the point **c**.
4. Mark arrowheads on the vectors **bc** and **dc** to give the proper sense. Then **dc** is the magnitude and also represents the direction of the velocity of C relative to A (or D). It is also the absolute velocity of the point C (A and D being fixed points).
5. Remember that the arrowheads on vector **bc** can be put in any direction because both ends of the link BC are movable. If the arrowhead is put from **c** to **b**, then the vector is read as **cb.** The above equation is modified as

$$\mathbf{dc} = \mathbf{ab} - \mathbf{cb} \qquad (\mathbf{bc} = -\,\mathbf{cb})$$

or

$$\mathbf{dc} + \mathbf{cb} = \mathbf{ab}$$

Intermediate Point

The velocity of an intermediate point on any of the links can be found easily by dividing the corresponding velocity vector in the same ratio as the point divides the link. For point E on the link BC,

$$\frac{\mathbf{be}}{\mathbf{bc}} = \frac{BE}{BC}$$

ae represents the absolute velocity of E.

Offset Point

Write the vector equation for point F,

$$\mathbf{v}_{fb} + \mathbf{v}_{ba} = \mathbf{v}_{fc} + \mathbf{v}_{cd}$$

or

$$\mathbf{v}_{ba} + \mathbf{v}_{fb} = \mathbf{v}_{cd} + \mathbf{v}_{fc}$$

or

$$\mathbf{ab} + \mathbf{bf} = \mathbf{dc} + \mathbf{cf}$$

The vectors $\mathbf{v}_{ba}$ and $\mathbf{v}_{cd}$ are already there on the velocity diagram.

$\mathbf{v}_{fb}$ is ⊥ BF, draw a line ⊥ BF through **b;**

$\mathbf{v}_{fc}$ is ⊥ CF, draw a line ⊥ CF through **c;**

The intersection of the two lines locates the point **f.**

af or **df** indicates the velocity of F relative to A (or D) or the absolute velocity of F.

2.6 VELOCITY IMAGES

Note that in Fig. 2.6, the triangle **bfc** is similar to the triangle BFC in which all the three sides **bc**, **cf** and **fb** are perpendicular to BC, CF and FB respectively. The triangles such as **bcf** are known as velocity images and are found to be very helpful devices in the velocity analysis of complicated shapes of the linkages. Thus, any offset point on a link in the configuration diagram can easily be located in the velocity diagram by drawing the velocity image. While drawing the velocity images, the following points should be kept in mind:

1. The velocity image of a link is a scaled reproduction of the shape of the link in the velocity diagram from the configuration diagram, rotated bodily through 90° in the direction of the angular velocity.
2. The order of the letters in the velocity image is the same as in the configuration diagram.
3. In general, the ratio of the sizes of different images to the sizes of their respective links is different in the same mechanism.

2.7 ANGULAR VELOCITY OF LINKS

1. Angular Velocity of BC

(a) Velocity of C relative to *B*, v_{cb} = **bc** (Fig. 2.6)

Point *C* relative to *B* moves in the direction-sense given by $\mathbf{v}_{cb}$ (upwards). Thus, *C* moves in the counter-clockwise direction about *B*.

$$\mathbf{v}_{cb} = \omega_{cb} \times BC = \omega_{cb} \times CB$$

$$\omega_{cb} = \frac{v_{cb}}{CB}$$

(b) Velocity of *B* relative to *C*, $\mathbf{v}_{bc}$ = **cb**

B relative to *C* moves in a direction-sense given by v_{bc} (downwards, opposite to **bc**), i.e., *B* moves in the counter-clockwise direction about *C* with magnitude ω_{bc} given by

$$\frac{v_{bc}}{BC}$$

It can be seen that the magnitude of $\omega_{cb} = \omega_{bc}$ as $v_{cb} = v_{bc}$ and the direction of rotation is the same. Therefore, angular velocity of a link about one extremity is the same as the angular velocity about the other. In general, the angular velocity of link *BC* is ω_{bc} ($= \omega_{cb}$) in the counter-clockwise direction.

2. Angular Velocity of CD

Velocity of *C* relative to *D*,

$$v_{cd} = \mathbf{dc}$$

It is seen that *C* relative to *D* moves in a direction-sense given by v_{cd} or *C* moves in the clockwise direction about *D*.

$$\omega_{cd} = \frac{v_{cd}}{CD}$$

2.8 VELOCITY OF RUBBING

Figure 2.7 shows two ends of the two links of a turning pair. A pin is fixed to one of the links whereas a hole is provided in the other to fit the pin. When joined, the surface of the hole of one link will rub on the surface of the pin of the other link. The velocity of rubbing of the two surfaces will depend upon the angular velocity of a link relative to the other.

Fig. 2.7

Pin at A (Fig. 2.6a)

The pin at A joins links AD and AB. AD being fixed, the velocity of rubbing will depend upon the angular velocity of AB only.

Let r_a = radius of the pin at A
Then velocity of rubbing = $r_a \cdot \omega$

Pin at D

Let r_d = radius of the pin at D
Velocity of rubbing = $r_d \cdot \omega_{cd}$

Pin at B

$\omega_{ba} = \omega_{ab} = \omega$ clockwise

$\omega_{bc} = \omega_{cb} = \dfrac{v_{bc}}{BC}$ counter-clockwise

Since the directions of the two angular velocities of links AB and BC are in the opposite directions, the angular velocity of one link relative to the other is the sum of the two velocities.

Let r_b = radius of the pin at B
Velocity of rubbing = $r_b\,(\omega_{ab} + \omega_{bc})$

Pin at C

$\omega_{bc} = \omega_{cb}$ counter-clockwise
$\omega_{dc} = \omega_{cd}$ clockwise

Let r_c = radius of the pin at C
Velocity of rubbing = $r_c\,(\omega_{bc} + \omega_{dc})$

In case it is found that the angular velocities of the two links joined together are in the same direction, the velocity of rubbing will be the difference of the angular velocities multiplied by the radius of the pin.

2.9 SLIDER-CRANK MECHANISM

Figure 2.8(a) shows a slider-crank mechanism in which OA is the crank moving with uniform angular velocity ω rad/s in the clockwise direction. At point B, a slider moves on the fixed guide G. AB is the coupler joining A and B. It is required to find the velocity of the slider at B.

Writing the velocity vector equation,

Vel. of B rel. to O = Vel. of B rel. to A + Vel. of A rel. to O

$$v_{bo} = v_{ba} + v_{ao}$$

$$v_{bg} = v_{ao} + v_{ba}$$

or $$\mathbf{gb} = \mathbf{oa} + \mathbf{ab}$$

$\mathbf{v}_{bo}$ is replaced by v_{bg} as O and G are two points on a fixed link

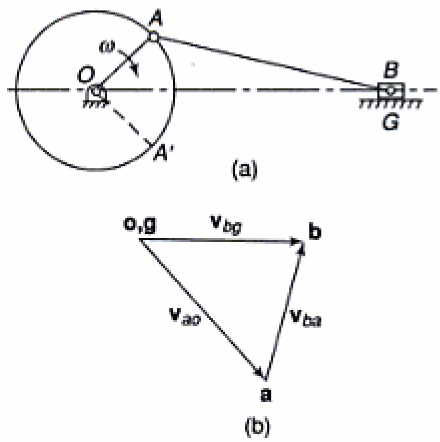


Fig. 2.8

(iii) *position and velocity of a point F on the connecting rod having the least absolute velocity*

(iv) *angular velocity of the connecting rod*

(v) *velocities of rubbing at the pins of the crankshaft, crank and the cross-head having diameters 80, 60 and 100 mm respectively.*

Solution Figure 2.10(a) shows the configuration diagram to a convenient scale.

$$v_{ao} = \omega_{ao} \times OA = 20 \times 0.48 = 9.6 \text{ m/s}$$

The vector equation is $\mathbf{v}_{bo} = \mathbf{v}_{ba} + \mathbf{v}_{ao}$

or

$$\mathbf{v}_{bg} = \mathbf{v}_{ba} + \mathbf{v}_{ao}$$

or

$$\mathbf{v}_{bg} = \mathbf{v}_{ao} + \mathbf{v}_{ba}$$

or

$$\mathbf{gb} = \mathbf{oa} + \mathbf{ab}$$

Take the vector $\mathbf{v}_{ao}$ to a convenient scale in the proper direction and sense [Fig. 2.10 (b)].

$\mathbf{v}_{ba}$ is $\perp AB$, draw a line $\perp AB$ through **a**;

The slider B has a linear motion relative to the guide G. Draw a line parallel to the direction of motion of the slider through **g** (or **o**). Thus, the point **b** is located.

(i) Velocity of the slider, v_b = **ob** = 9.7 m/s

(ii) Locate the point **e** on **ba** extended such that

$$\frac{\mathbf{ae}}{\mathbf{ba}} = \frac{AE}{BA}$$

ba = 5.25 m/s on measuring from the diagram.

$$\therefore\ \mathbf{ae} = 5.25 \times \frac{0.45}{1.60} = 1.48 \text{ m/s}$$

v_e = **oe** = 10.2 m/s

(iii) To locate a point F on the connecting rod which has the least velocity relative to the crankshaft or has the least absolute velocity, draw **of** $\perp$ **ab** through **o**.

Locate the point F on AB such the $\dfrac{AF}{AB} = \dfrac{\mathbf{af}}{\mathbf{ab}}$

or

$$AF = 1.60 \times \frac{2.76}{5.25} = 0.84 \text{ m}$$

v_f = **of** = 9.4 m/s

(iv) $\omega_{ba} = \dfrac{v_{ba}}{AB} = \dfrac{5.25}{1.60} = 3.28$ rad/s clockwise

(v) (a) Velocity of rubbing at the pin of the crankshaft (at O)

$= \omega_{ao}\, r_o = 20 \times 0.04 = 0.8$ m/s

$$\left(r_o\ \frac{80}{2} = 40 \text{ mm}\right)$$

(b) ω_{oa} is counter-clockwise and ω_{ba} is clockwise.

Velocity of rubbing at the crank pin $A = (\omega_{oa} + \omega_{ba})\, r_a$

$= (20 + 3.28) \times 0.03$

$= 0.698$ m/s

(c) At the cross-head, the slider does not rotate and only the connecting rod has the angular motion.

Velocity of rubbing at the cross-head pin at B

$= \omega_{ab}\, r_b = 3.28 \times 0.05 = 0.164$ m/s

Example 2.3 *Figure 2.11a shows a mechanism in which OA = QC = 100 mm, AB = QB = 300 mm and CD = 250 mm. The crank OA rotates at 150 rpm in the clockwise direction. Determine the*

(i) velocity of the slider at D

(ii) angular velocities of links QB and AB

(iii) rubbing velocity at the pin B which is 40 mm in diameter

Solution $\omega_{ao} = \dfrac{2\pi \times 150}{60} = 15.7$ rad/s

$$v_{ao} = \omega_{ao} \times OA = 15.7 \times 0.1 = 1.57 \text{ m/s}$$

The vector equation for the mechanism $OABQ$,

$$\mathbf{v}_{bo} = \mathbf{v}_{ba} + \mathbf{v}_{ao}$$

or $\mathbf{v}_{bq} = \mathbf{v}_{ao} + \mathbf{v}_{ba}$ or **qb = oa + ab**

Take the vector $\mathbf{v}_{ao}$ to a convenient scale in the proper direction and sense [Fig. 2.11 (b)].

$\mathbf{v}_{ba}$ is $\perp AB$, draw a line $\perp AB$ through **a**;

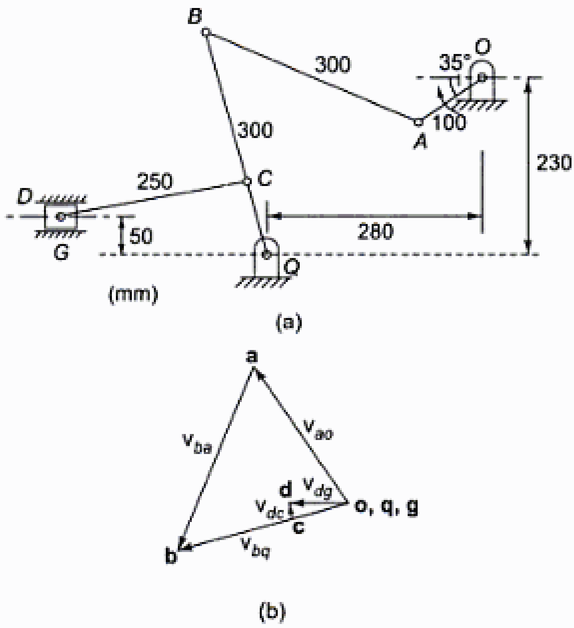


Fig. 2.11

$\mathbf{v}_{bq}$ is $\perp QB$, draw a line $\perp QB$ through **q**;

The intersection of the two lines locates the point **b**.

Locate the point **c** on **qb** such that

$$\frac{\mathbf{qc}}{\mathbf{qb}} = \frac{100}{300} = 0.3$$

The vector equation for the mechanism *QCD*,

$$\mathbf{v}_{dq} = \mathbf{v}_{dc} + \mathbf{v}_{cq} \quad \text{or} \quad \mathbf{v}_{dg} = \mathbf{v}_{cq} + \mathbf{v}_{dc}$$

or $\quad$ **gd** = **qc** + **cd**

$\mathbf{v}_{dc}$ is $\perp DC$, draw a line $\perp DC$ through **c**;

For $\mathbf{v}_{dg}$, draw a line through **g**, parallel to the line of stroke of the slider in the guide *G*.

The intersection of the two lines locates the point **d**.

(i) The velocity of slider at *D*, v_d = **gd** = 0.56 m/s

(vi) $\omega_{bq} = \dfrac{v_{bq}}{QB} = \dfrac{1.69}{0.3} = 5.63$ rad/s counter-clockwise

(vii) $\omega_{ba} = \dfrac{v_{ba}}{AB} = \dfrac{1.89}{0.3} = 6.3$ rad/s counter-clockwise

As both the links connected at *B* have counter-clockwise angular velocities,

velocity of rubbing at the crank pin

$$B = (\omega_{ba} - \omega_{bq})\, r_b$$
$$= (6.3 - 5.63) \times 0.04 = 0.0268 \text{ m/s}$$

Example 2.4 *An engine crankshaft drives a reciprocating pump through a mechanism as shown in Fig. 2.12(a). The crank rotates in the clockwise direction at 160 rpm. The diameter of the pump piston at F is 200 mm. Dimensions of the various links are*

OA = 170 mm (crank) *CD = 170 mm*
AB = 660 mm *DE = 830 mm*
BC = 510 mm

For the position of the crank shown in the diagram, determine the

(i) velocity of the crosshead E

(ii) velocity of rubbing at the pins A, B, C and D, the diameters being 40, 30, 30 and 50 mm respectively

(iii) torque required at the shaft O to overcome a pressure of 300 kN/m² at the pump piston at F

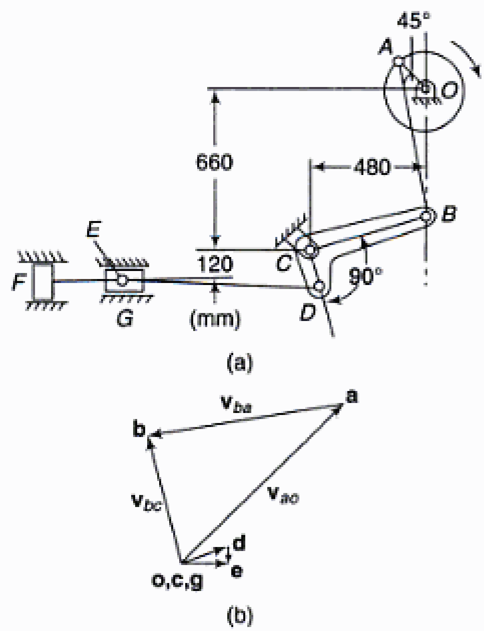


Fig. 2.12

Solution:

$$\omega_{ao} = \frac{2\pi N}{60} = \frac{2\pi \times 160}{60} = 16.76 \text{ rad/s}$$

$$v_{ao} = \omega_{ao} \times OA = 16.76 \times 0.17 = 2.85 \text{ m/s}$$

Writing the vector equation for the mechanism *OABC*,

$$\mathbf{v}_{bo} = \mathbf{v}_{ba} + \mathbf{v}_{ao}$$

or

$$\mathbf{v}_{bc} = \mathbf{v}_{ao} + \mathbf{v}_{ba}$$

or

$$\mathbf{cb} = \mathbf{oa} + \mathbf{ab}$$

Take the vector $\mathbf{v}_{ao}$ to a convenient scale [Fig. 2.12(b)]

$\mathbf{v}_{ba}$ is $\perp AB$, draw a line $\perp AB$ through **a**;

$\mathbf{v}_{bc}$ is $\perp BC$, draw a line $\perp BC$ through **c**.

The intersection of the two lines locates the point ***b***. Velocity of *B* relative to *C* is upwards for the given configuration. Therefore, the link *BCD* moves counter-clockwise about the pivot *C*.

$$\frac{v_{dc}}{v_{bc}} = \frac{DC}{BC}$$

or $v_{dc} = 1.71 \times \dfrac{0.17}{0.51} = 0.57$ m/s $\qquad (\perp DC)$

Writing the vector equation for the mechanism *CDE*,

$$\mathbf{v}_{ec} = \mathbf{v}_{ed} + \mathbf{v}_{dc}$$

or

$$\mathbf{v}_{eg} = \mathbf{v}_{dc} + \mathbf{v}_{ed}$$

or

$$\mathbf{ge} = \mathbf{cd} + \mathbf{de}$$

Take $\mathbf{v}_{dc}$ in the proper direction and sense from **c** assuming *D* in the configuration diagram as an offset point on link *CB*:

$\mathbf{v}_{ed}$ is $\perp DE$, draw a line $\perp DE$ through **d**.

For $\mathbf{v}_{eg}$, draw a line through **g**, parallel to the direction of motion of the slider *E* in the guide *G*.

This way the point **e** is located.

(i) The velocity of the crosshead,

$v_e = \mathbf{oc} = \underline{0.54 \text{ m/s}}$

(ii) (a) ω_{oa} and ω_{ba} both are clockwise.

$$\omega_{ba} = \frac{\mathbf{ab}}{AB} = \frac{2.49}{0.66} = 3.77 \text{ rad/s}$$

Velocity of rubbing at the pin $A = (\omega_{oa} - \omega_{ba})\, r_a$

$$= (16.76 - 3.77) \times \frac{0.04}{2}$$

$= \underline{0.26 \text{ m/s}}$

(b) ω_{ab} is clockwise and ω_{cd} is counter-clockwise.

$$\omega_{cb} = \frac{v_{cb}}{CB} = \frac{1.71}{0.51} = 3.35 \text{ rad/s}$$

Velocity of rubbing at $B = (\omega_{ab} + \omega_{cb})\, r_b$

$= (3.77 + 3.35) \times 0.015 \ldots\ (\omega_{ab} = \omega_{ba})$

$= \underline{0.107 \text{ m/s}}$

(c) Velocity of rubbing at $C = \omega_{bc} . r_c$

$$= 3.35 \times \frac{0.03}{2} = \underline{0.05 \text{ m/s}}$$

(d) ω_{cd} and ω_{ed}, both are counter-clockwise

$\omega_{cd} = \omega_{bc} = 3.35$ rad/s ... (*BCD* is one link)

$$= \omega_{ed} = \frac{v_{ed}}{ED} = \frac{0.15}{0.83} = 0.18 \text{ rad/s}$$

Velocity of rubbing at $D = (\omega_{cd} - \omega_{ed})\, r_d$

$$= (3.35 - 0.18) \times \frac{0.05}{2}$$

$= \underline{0.079 \text{ m/s}}$

(iii) Work input = work output

$T.\omega = F.v$

where T = torque on the crankshaft

ω = angular velocity of the crank

F = force on the piston

v = velocity of the piston $= v_f = v_e$

Thus, neglecting losses,

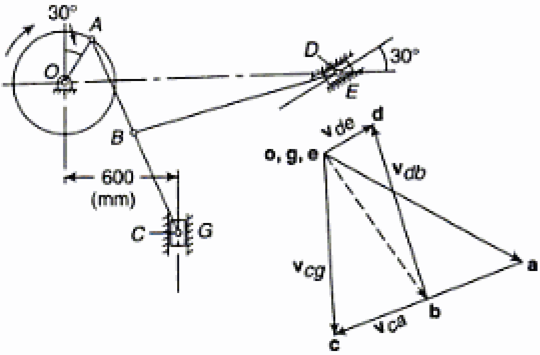
$$T = \frac{F.v}{\omega} = \frac{\pi}{4}(0.02)^2 \times 300 \times 10^3 \times \frac{0.54}{16.76}$$

$= \underline{303.66 \text{ N.m}}$

Example 2.5 *Figure 2.13(a) shows a mechanism in which OA = 300 mm, AB = 600 mm, AC = BD = 1.2 m. OD is horizontal for the given configuration. If OA rotates at 200 rpm in the clockwise direction, find*

(iv) linear velocities of C and D

(v) angular velocities of links AC and BD



(a) (b)

Fig. 2.13

Solution: $\omega_a = \dfrac{2\pi \times 200}{60} = 20.94$ rad/s

$v_a = \omega_a OA = 20.94 \times 0.3 = 6.28$ m/s

Writing the vector equation for the mechanism *OAC*,

$$\mathbf{v}_{co} = \mathbf{v}_{ca} + \mathbf{v}_{ao}$$

or

$$\mathbf{v}_{cg} = \mathbf{v}_{ao} + \mathbf{v}_{ca}$$

or

$$\mathbf{gc} = \mathbf{oa} + \mathbf{ac}$$

Take the vector $\mathbf{v}_{ao}$ to a convenient scale [Fig. 2.13(b)].

$\mathbf{v}_{ca}$ is $\perp AC$, draw a line $\perp AC$ through **a**;

$\mathbf{v}_{cg}$ is vertical, draw a vertical line through **g** (or **o**).

The intersection of the two lines locates the point **c**. Locate the point **b** on **ac** as usual. Join **ob** which gives $\mathbf{v}_{bo}$. Writing the vector equation for the mechanism *OABD*,

$$\mathbf{v}_{do} = \mathbf{v}_{db} + \mathbf{v}_{bo}$$

or

$$\mathbf{v}_{de} = \mathbf{v}_{bo} + \mathbf{v}_{db}$$

or

$$\mathbf{ed} = \mathbf{ob} + \mathbf{bd}$$

$\mathbf{v}_{db}$ is $\perp BD$, draw a line $\perp BD$ through **b**;

For $\mathbf{v}_{de}$, draw a line through **e**, parallel to the line of stroke of the piston in the guide *E*.

The intersection locates the point **d**.

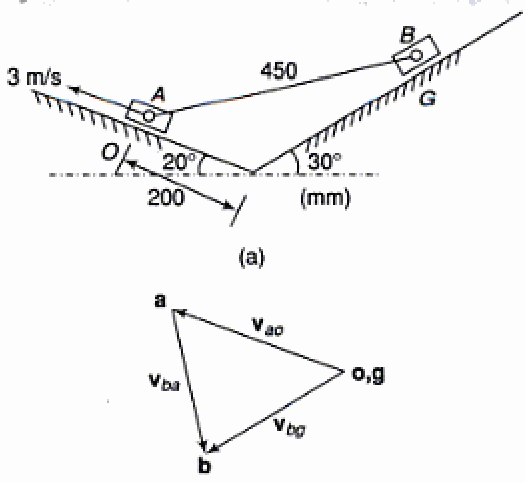
$v_c = \mathbf{oc} = \underline{5.2 \text{ m/s}}$

$v_d = \mathbf{od} = \underline{1.55 \text{ m/s}}$

$$\omega_{ac} = \omega_{ca} = \frac{v_{ca}}{AC} = \frac{5.7}{1.20} = \underline{4.75 \text{ rad/s}} \text{ clockwise}$$

$$\omega_{bd} = \omega_{db} = \frac{v_{db}}{BD} = \frac{5.17}{1.20} = \underline{4.31 \text{ rad/s}} \text{ clockwise}$$

Example 2.6 *For the position of the mechanism shown in Fig. 2.14(a), find the velocity of the slider B for the given configuration if the velocity of the slider A is 3 m/s.*



(a)

(b)

Fig. 2.14

Solution The velocity vector equation is

$$\mathbf{v}_{bo} = \mathbf{v}_{ba} + \mathbf{v}_{ao}$$

or $\mathbf{v}_{bg} = \mathbf{v}_{ao} + \mathbf{v}_{ba}$

or $\mathbf{gb} = \mathbf{oa} + \mathbf{ab}$

Take the vector $\mathbf{v}_{ao}$ (= 3 m/s) to a convenient scale [Fig. 2.14(b)]

$\mathbf{v}_{ba}$ is $\perp AB$, draw a line AB through **a**;

For $\mathbf{v}_{bg}$, draw a line through **g** parallel to the line of stroke of the slider B on the guide G.

The intersection of the two lines locates the point **b**.

Velocity of B = **gb** = 2.67 m/s.

Example 2.7 *In a mechanism shown in Fig. 2.15(a), the angular velocity of the crank OA is 15 rad/s and the slider at E is constrained to move at 2.5 m/s. The motion of both the sliders is vertical and the link BC is horizontal in the position shown. Determine the*

(i) rubbing velocity at B if the pin diameter is 15 mm

(ii) velocity of slider D.

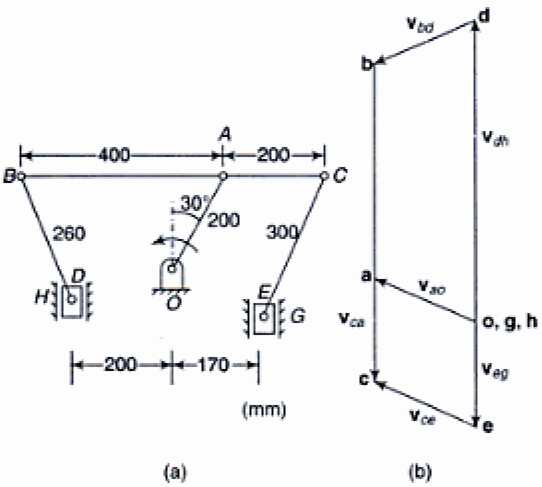


Fig. 2.15

Solution $v_a = \omega_a OA = 15 \times 0.2 = 3$ m/s

Draw the velocity diagram as follows:

- Take vector **oa** to a suitable scale (2.15b).
- Consider two points G and H on the guides of sliders E and F respectively. In the velocity diagram, the points **g** and **h** coincide with **o**. Through **g**, take a vector **ge** parallel to direction of motion of the slider E and equal to 2.5 m/s using some scale.
- Through **e** draw a line $\perp EC$ and through **a**, a line $\perp AC$, the intersection of these two lines locates the point **c**.
- Locate the point **b** on the vector **ca** so that **ca**/**cb** = CA/CB.
- Through **b**, draw a line $\perp BD$ and through **h**, a line parallel to direction of motion of the slider D, the intersection of these two lines locates the point **d**.

(i) Angular velocity of link BD = **bd**/BD = 2.95/0.26 =11.3 rad/s (counter-clockwise)
Angular velocity of link BC = **bc**/BC = 8.4/0.6 =14 rad/s (clockwise)
Thus velocity of rubbing at

$$B = (\omega_{bd} + \omega_{bc})r_b$$
$$= (11.3 + 14) \times 0.015$$
$$= \underline{0.38 \text{ m/s}}$$

(ii) The velocity of the slider D = **hd** = $\underline{8.3 \text{ m/s}}$

Example 2.8 *The lengths of various links of a mechanism shown in Fig. 2.16(a) are as follows:*

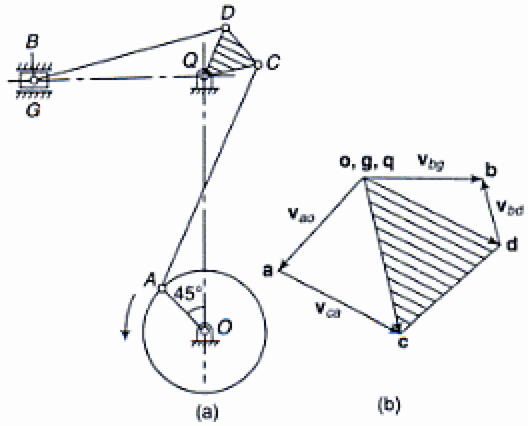


Fig. 2.16

OA = 150 mm	*CD = 125 mm*
AC = 600 mm	*BD = 500 mm*
CQ = QD = 145 mm	*OQ = 625 mm*

Example 2.11 *The mechanism of a stone-crusher is shown in Fig. 2.19a along with various dimensions of links in mm. If crank OA rotates at a uniform velocity of 120 rpm, determine the velocity of the point K (jaw) when the crank OA is inclined at an angle of 30° to the horizontal. What will be the torque required at the crank OA to overcome a horizontal force of 40 kN at K?*

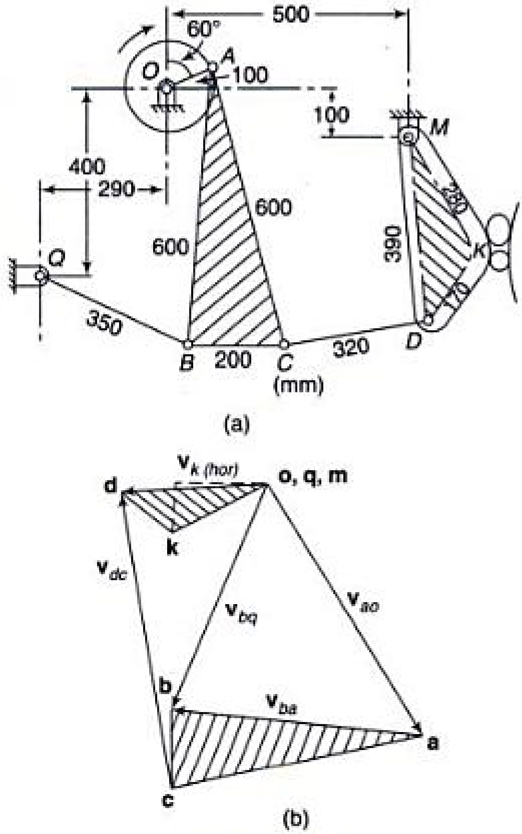


Fig. 2.19

Solution $\omega_{ao} = \dfrac{2\pi \times 120}{60} = 12.6$ rad/s

$$v_{ao} = \omega_{ao} \times OA = 12.6 \times 0.1 = 1.26 \text{ m/s}$$

Write the vector equation for the mechanism *OABQ* and complete the velocity diagram as usual [(Fig. 2.19(b)]. Make Δ**bac** similar to ΔBAC (both are read clockwise).

Write the vector equation for the mechanism *OACDM* and complete the velocity diagram. Make Δ**dmk** similar to ΔDMK (both are read clockwise).

$$v_k = \mathbf{ok} = 0.45 \text{ m/s}$$
$$v_k \text{ (horizontal)} = 0.39 \text{ m/s}$$
$$T.\omega_{ao} = F_k\, v_k \text{ (horizontal)}$$

$$T = \frac{40\,000 \times 0.39}{12.6} = \underline{1242 \text{ N.m}}$$

2.10 CRANK- AND SLOTTED-LEVER MECHANISM

While analysing the motions of various links of a mechanism, sometimes we are faced with the problem of describing the motion of a moveable point on a link which has some angular velocity. For example, the motion of a slider on a rotating link. In such a case, the angular velocity of the rotating link along with the linear velocity of the slider may be known and it may be required to find the absolute velocity of the slider.

A crank and slotted-lever mechanism, which is a form of quick-return mechanism used for slotting and shaping machines, depicts the same form of motion [Fig. 2.20(a)]. *OP* is the crank rotating at an angular velocity of ω rad/s in the

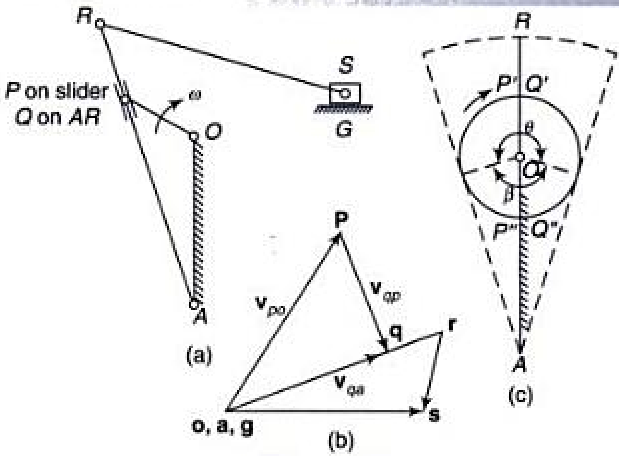


Fig. 2.20

clockwise direction about the centre *O*. At the end of the crank, a slider *P* is pivoted which moves on an oscillating link *AR*.

In such problems, it is convenient if a point *Q* on the link *AR* immediately below *P* is assumed to exist (*P* and *Q* are known as coincident points). As the crank rotates, there is relative movement of the points *P* and *Q* along *AR*.

Writing the vector equation for the mechanism *OPA*,

$$\text{Vel. of } Q \text{ rel. to } O = \text{Vel. of } Q \text{ rel. to } P + \text{Vel. of } P \text{ rel. to } O$$

$$\mathbf{v}_{qo} = \mathbf{v}_{qp} + \mathbf{v}_{po}$$

or

$$\mathbf{v}_{qa} = \mathbf{v}_{po} + \mathbf{v}_{qp}$$

or

$$\mathbf{aq} = \mathbf{op} + \mathbf{pq}$$

In this equation,

$\mathbf{v}_{po}$ or **op** = $\omega.OP$; $\perp$ to OP

$\mathbf{v}_{qp}$ or **pq** is unknown in magnitude ; | | to *AR*

$\mathbf{v}_{qa}$ or **aq** is unknown in magnitude ; $\perp$ to *AR*

Take the vector $\mathbf{v}_{po}$ which is fully known [Fig. 2.20 (b)].

$\mathbf{v}_{qp}$ is | | *AR*, draw a line | | to *AR* through **p;**

$\mathbf{v}_{qa}$ is $\perp AR$, draw a line $\perp AR$ through **a** (or **o**).

The intersection locates the point **q.**

The vector equation for the above could also have been written as

$$\text{Vel. of } P \text{ rel. to } A = \text{Vel. of } P \text{ rel. to } Q + \text{Vel. of } Q \text{ rel. to } A$$

$$\mathbf{v}_{pa} = \mathbf{v}_{pq} + \mathbf{v}_{qa}$$

or

$$\mathbf{v}_{po} = \mathbf{v}_{qa} + \mathbf{v}_{pq}$$

or

$$\mathbf{op} = \mathbf{aq} + \mathbf{qp}$$

Take the vector $\mathbf{v}_{po}$ which is completely known.

$\mathbf{v}_{qa}$ is $\perp AR$, draw a line $\perp AR$ through **a;**

$\mathbf{v}_{pq}$ is | | *AR*, draw a line | | *AR* through **p.**

The intersection locates the point **q.** Observe that the velocity diagrams obtained in the two cases are the same except that the direction of $\mathbf{v}_{pq}$ is the reverse of that of $\mathbf{v}_{qp}$.

As the vectors **oq** and **qp** are perpendicular to each other, the vector $\mathbf{v}_{po}$ may be assumed to have two components, one perpendicular to *AR* and the other parallel to *AR*.

The component of velocity along AR, i.e., **qp** indicates the relative velocity between Q and P or the velocity of sliding of the block on link AR.

Now, the velocity of R is perpendicular to AR. As the velocity of Q perpendicular to AR is known, the point **r** will lie on vector **aq** produced such that **ar/aq** = AR/AQ

To find the velocity of ram S, write the velocity vector equation,

$$\mathbf{v}_{so} = \mathbf{v}_{sr} + \mathbf{v}_{ro}$$

or

$$\mathbf{v}_{sg} = \mathbf{v}_{ro} + \mathbf{v}_{sr}$$

or

$$\mathbf{gs} = \mathbf{or} + \mathbf{rs}$$

$\mathbf{v}_{ro}$ is already there in the diagram. Draw a line through **r** perpendicular to RS for the vector $\mathbf{v}_{sr}$ and a line through **g**, parallel to the line of motion of the slider S on the guide G, for the vector $\mathbf{v}_{sg}$. In this way the point **s** is located.

The velocity of the ram S = **os** (or **gs**) towards right for the given position of the crank.

Also, $\omega_{rs} = \dfrac{v_{rs}}{RS}$ clockwise

Usually, the coupler RS is long and its obliquity is neglected.

Then **or** ≈ **os**

Referring Fig. 2.20 (c),

$$\frac{\text{Time of cutting}}{\text{Time of return}} = \frac{\theta}{\beta}$$

When the crank assumes the position OP' during the cutting stroke, the component of velocity along AR (i.e., **pq**) is zero and **oq** is maximum (= **op**)

Let r = length of crank (= OP)
l = length of slotted lever (= AR)
c = distance between fixed centres (= AO)
ω = angular velocity of the crank

Then, during the cutting stroke,

$$v_{s\max} = \omega \times OP' \times \frac{AR}{AQ} = \omega r \times \frac{l}{c+r}$$

This is by neglecting the obliquity of the link RS, i.e. assuming the velocity of S equal to that of R.

Similarly, during the return stroke,

$$v_{s\max} = \omega \times OP'' \times \frac{AR}{AQ''} = \omega r \times \frac{l}{c-r}$$

$$\frac{v_{s\max}(\text{cutting})}{v_{s\max}(\text{return})} = \frac{\omega r \dfrac{1}{c+r}}{\omega r \dfrac{1}{c-r}} = \frac{c-r}{c+r}$$

Example 2.12 *Figure 2.21(a) shows the link mechanism of a quick return mechanism of the slotted lever type, the various dimensions of which are*

OA = 400 mm, OP = 200 mm, AR = 700 mm, RS = 300 mm

For the configuration shown, determine the velocity of the cutting tool at S and the angular velocity of the link RS. The crank OP rotates at 210 rpm.

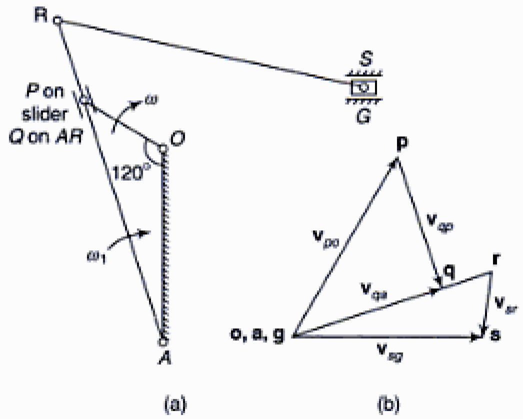


Fig. 2.21

Solution $\omega_{po} = \dfrac{2\pi \times 210}{60} = 22$ rad/s

Draw the configuration to a suitable scale. The vector equation for the mechanism *OPA*,

$$\mathbf{v}_{qa} = \mathbf{v}_{po} + \mathbf{v}_{qp} \quad \text{or} \quad \mathbf{aq} = \mathbf{op} + \mathbf{pq}$$

In this equation,

$$\mathbf{v}_{po} \text{ or } \mathbf{op} = \omega . OP = 22 \times 0.2 = 4.4 \text{ m/s}$$

Take the vector $\mathbf{v}_{po}$ which is fully known [Fig. 2.21(b)].

$\mathbf{v}_{qp}$ is | | *AR*, draw a line | | to *AR* through **p;**

$\mathbf{v}_{qa}$ is ⊥ *AR*, draw a line ⊥ *AR* through **a** (or **o**).

The intersection locates the point **q.** Locate the point **r** on the vector **aq** produced such that **ar/aq** = *AR/AQ*.

Draw a line through **r** perpendicular to *RS* for the vector $\mathbf{v}_{sr}$ and a line through **g**, parallel to the line of motion of the slider *S* on the guide *G*, for the vector $\mathbf{v}_{sg}$. In this way the point **s** is located.

The velocity of the ram *S* = **os** (or **gs**) = 4.5 m/s

It is towards right for the given position of the crank.

Angular velocity of link *RS*,

$$\omega_{rs} = \frac{v_{rs}}{RS} = \frac{1.4}{0.3} = 4.67 \text{ rad/s clockwise}$$

Example 2.13 *For the inverted slider-crank mechanism shown in Fig. 2.22(a), find the angular velocity of the link QR and the sliding velocity of the block on the link QR. The crank OA is 300 mm long and rotates at 20 rad/s in the clockwise direction. OQ is 650 mm and QOA = 40°*

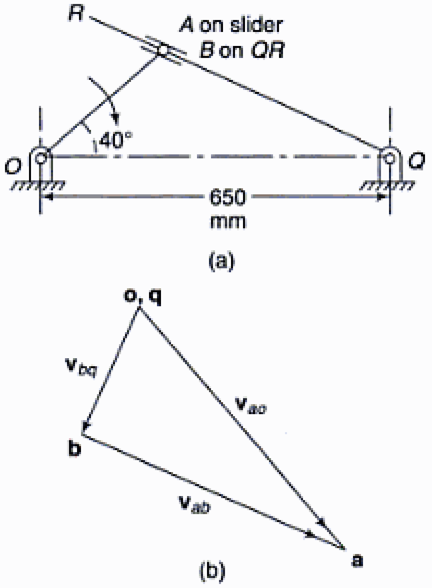


Fig. 2.22

Solution The velocity vector equation can be written as usual.

$$\mathbf{v}_{aq} = \mathbf{v}_{ab} + \mathbf{v}_{bq} \quad \text{or} \quad \mathbf{v}_{bo} = \mathbf{v}_{ba} + \mathbf{v}_{ao}$$

$$\mathbf{v}_{ao} = \mathbf{v}_{bq} + \mathbf{v}_{ab} \qquad \mathbf{v}_{bq} = \mathbf{v}_{ao} + \mathbf{v}_{ba}$$

$$\mathbf{oa} = \mathbf{qb} + \mathbf{ba} \qquad \mathbf{qb} = \mathbf{oa} + \mathbf{ab}$$

$\mathbf{v}_{ao}$ is fully known and after taking this vector, draw lines for $\mathbf{v}_{bq}$ and $\mathbf{v}_{ab}$ (or $\mathbf{v}_{ba}$) and locate the point **b**. obviously, the direction-sense of $\mathbf{v}_{ab}$ is opposite to that of $\mathbf{v}_{ba}$. Figure 2.22 (b) shows the solution of the first equation.

$$\omega_{qr} = \omega_{qb} = \frac{v_{qb} \text{ or } v_{bq}}{BQ}$$

$$= \frac{2.55}{0.46} \qquad (BQ = 0.46 \text{ m on measuring})$$

$$= \underline{5.54 \text{ rad/s}} \text{ counter – clockwise}$$

Sliding velocity of block = $\mathbf{v}_{ba}$ or **ab** = 5.45 m/s

Example 2.14 *For the position of the mechanism shown in Fig. 2.23(a), calculate the angular velocity of the link AR. OA is 300 mm long and rotates at 20 rad/s in the clockwise direction. OQ = 650 mm and QOA = 40°*

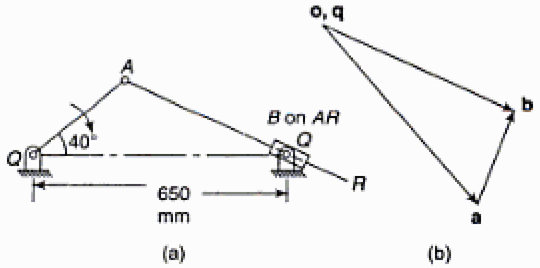


Fig. 2.23

Solution $v_{ao} = 20 \times 0.3 = 6$ m/s

Writing the vector equation,

$$\mathbf{v}_{bo} = \mathbf{v}_{ba} + \mathbf{v}_{ao} \quad \text{or} \quad \mathbf{v}_{aq} = \mathbf{v}_{ab} + \mathbf{v}_{bq}$$

Solving the first one,

$$\mathbf{v}_{bq} = \mathbf{v}_{ao} + \mathbf{v}_{ba}$$

or **qb = oa + ab**

Take $\mathbf{v}_{ao}$ to a convenient scale [Fig. 2.23(b)].

$\mathbf{v}_{ba}$ is $\perp AB$, draw a line $\perp AB$ through **a**;

$\mathbf{v}_{bq}$ is along AB, draw a line || to AB through **q**.

The intersection locates the point **b**.

$$\omega_{ar} = \omega_{ab} = \frac{v_{ab} \text{ or } v_{ba}}{AB} = \frac{2.55}{0.46}$$

= $\underline{5.54 \text{ rad/s}}$ counter–clockwise

Example 2.15 *In the pump mechanism shown in Fig. 2.24(a), OA = 320 mm, AC = 680 mm and OQ = 650 mm. For the given configuration, determine the*

(i) angular velocity of the cylinder

(ii) sliding velocity of the plunger

(iii) absolute velocity of the plunger

The crank OA rotates at 20 rad/s clockwise.

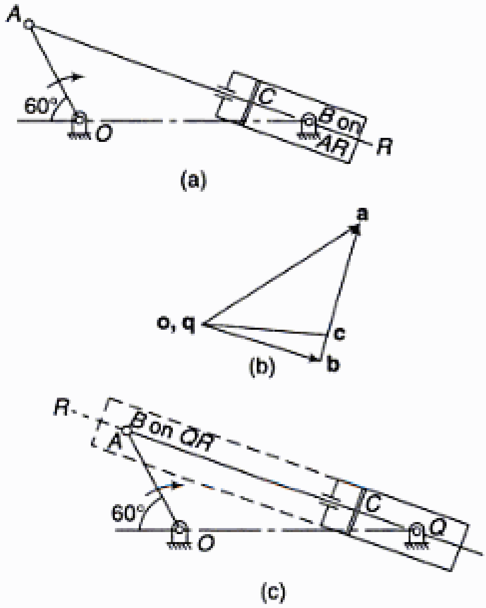


Fig. 2.24

Solution $v_{ao} = 0.32 \times 20 = 6.4$ m/s

Method I Produce AC to R. Line AC passes through the pivot Q. Let B be a point on AR beneath Q.

Writing the vector equation,

$$\mathbf{v}_{bo} = \mathbf{v}_{ba} + \mathbf{v}_{ao} \quad \text{or} \quad \mathbf{v}_{aq} = \mathbf{v}_{ab} + \mathbf{v}_{bq}$$

Solving any of these equations leads to same velocity diagram except for the direction of $\mathbf{v}_{ba}$ and $\mathbf{v}_{ab}$.

Taking the latter equation,

$$\mathbf{v}_{aq} = \mathbf{v}_{ab} + \mathbf{v}_{bq}$$

or

$$\mathbf{v}_{ao} = \mathbf{v}_{bq} + \mathbf{v}_{ab}$$

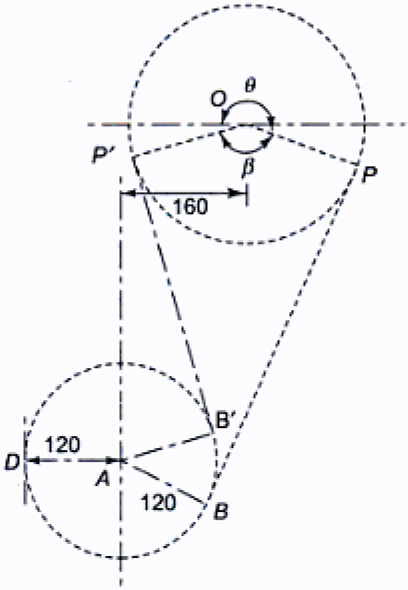


Fig. 2.27

(iii) Stroke of the rack = angular displacement of the quadrant × its radius

= angle $BAB' \times AB$

$$= 44 \times \frac{\pi}{180} \times 120 = 92.2 \text{ mm}$$

($\angle BAB' = 44°$ by measurement)

Example 2.18 *In the swiveling-joint mechanism shown in Fig. 2.28(a), AB is the driving crank rotating at 300 rpm clockwise.*

The lengths of the various links are

$AD = 650$ mm, $AB = 100$ mm, $BC = 800$ mm, $DC = 250$ mm, $BE = CE$, $EF = 400$ mm, $OF = 240$ mm, $FS = 400$ mm

For the given configuration of the mechanism, determine the

(i) velocity of the slider block S

(ii) angular velocity of the link EF

(iii) velocity of the link EF in the swivel block

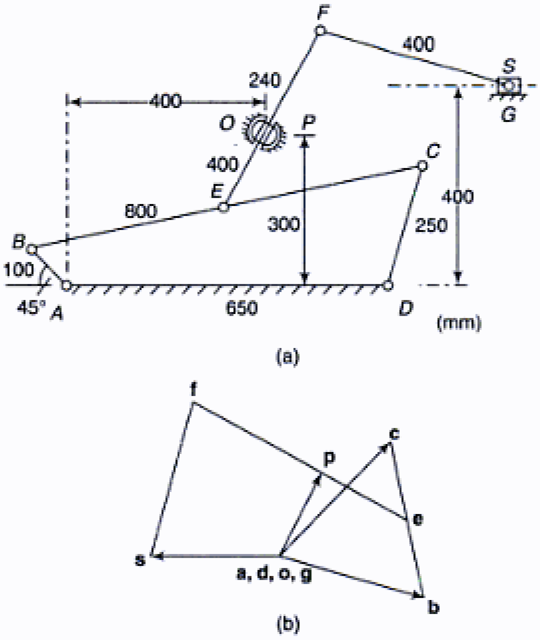


Fig. 2.28

Solution $$\omega_{ba} = \frac{2\pi \times 300}{60} = 31.4 \text{ rad/s}$$

$$v_b = 31.4 \times 0.1 = 3.14 \text{ m/s}$$

The velocity diagram is completed as follows:

- Draw the velocity diagram of the four-link mechanism *ABCD* as usual starting with the vector **ab** as shown in Fig. 2.28(b).
- Locate the point **e** in the velocity diagram at the midpoint of **bc** as the point *E* is the midpoint of *BC*. Let *Q* be a point on the link *EF* at the joint *O*. Draw a line $\perp EQ$ through **e**, a point on which will represent the velocity of *Q* relative to *E*.
- The sliding velocity of link *EF* in the joint at the instant is along the link. Draw a line parallel to *EF* through **o**, the intersection of which with the previous line locates the point **q**.
- Extend the vector **eq** to **f** such that **ef**/**eq** = *EF*/*EQ*.

- Through **f** draw a line $\perp FS$, and through **g** a line parallel to line of stroke of the slider. The intersection of the two lines locates the point **s**.

Thus, the velocity diagram is completed.

(i) The velocity of slider S = **gs** = 2.6 m/s

(ii) The angular velocity of the link EF

$$= \frac{v_{fe}}{EF} = \frac{\mathbf{ef}}{EF} = \frac{4.9}{0.4} = 12.25 \text{ rad/s (ccw)}$$

(iii) The velocity of the link EF in the swivel block = **oq** =1.85 m/s

2.11 ALGEBRAIC METHODS

Vector Approach

In Sec. 2.10, the concept of coincident points was introduced. However, complex algebraic methods provide an alternative formulation for the kinematic problems. This also furnishes an excellent means of obtaining still more insight into the meaning of the term *coincident points.*

Let there be a plane moving body having its motion relative to a fixed coordinate system xyz (Fig. 2.29). Also, let a moving coordinate system $x'y'z'$ be attached to this moving body. Coordinates of the origin A of the moving system are known relative to the absolute reference system. Assume that the moving system has an angular velocity ω also.

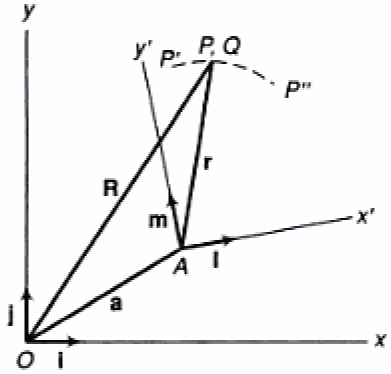


Fig. 2.29

Let

i,j,k	unit vectors for the absolute system
l,m,n	unit vectors for the moving system
ω	angular velocity of rotation of the moving system
R	vector relative to fixed system
r	vector relative to moving system

Let a point P move along path $P'PP''$ relative to the moving coordinate system $x'y'z'$. At any instant, the position of P relative to the fixed system is given by the equation

$$\mathbf{R} = \mathbf{a} + \mathbf{r} \tag{i}$$

where $\mathbf{r} = x'\mathbf{l} + y'\mathbf{m} + z'\mathbf{n}$

Thus, (i) may be written as, $\mathbf{R} = \mathbf{a} + x'\mathbf{l} + y'\mathbf{m} + z'\mathbf{n}$

Taking the derivatives with respect to time to find the velocity,

$$\dot{\mathbf{R}} = \dot{\mathbf{a}} + (\dot{x}'\mathbf{l} + \dot{y}'\mathbf{m} + \dot{z}'\mathbf{n}) + (x'\dot{\mathbf{l}} + y'\dot{\mathbf{m}} + z'\dot{\mathbf{n}}) \tag{ii}$$

The first term in this equation indicates the velocity of the origin of the moving system. The second term refers to the velocity of P relative to the moving system. The third term is due to the fact that the reference system has also rotary motion with angular velocity ω.

Also, $\dot{\mathbf{l}} = \omega \times \mathbf{l}$, $\quad \dot{\mathbf{m}} = \omega \times \mathbf{m}$, $\quad \dot{\mathbf{n}} = \omega \times \mathbf{n}$

of one body with respect to the other body. However, even with this limitation, the instantaneous centre is a useful tool for understanding the kinematics of planar motion. In our further discussions, this centre will be called the *I-centre*.

In case the perpendiculars to $\mathbf{v}_a$ and $\mathbf{v}_b$ at A and B respectively meet outside the body p, the I-centre will lie outside the body p [Fig. 2.32(a)]. If the directions of $\mathbf{v}_a$ and $\mathbf{v}_b$ are parallel and the perpendiculars at A and B meet at infinity, the I-centre of the body lies at infinity. This is the case when the body has a linear motion [Fig. 2.32(b)].

As the body p rotates about the point I at the instant and the velocity of any point on the body is proportional to the distance of the point from I, the velocity of the point I itself would be zero (the distance being zero). This implies that the two bodies p and q are relatively at rest or there is no relative motion between the two at the I-centre.

Now imagine that the body q is also in motion relative to a third body r (Fig. 2.33). Then the motion of the point I relative to the third body would be the same whether this point is considered on the body p or q.

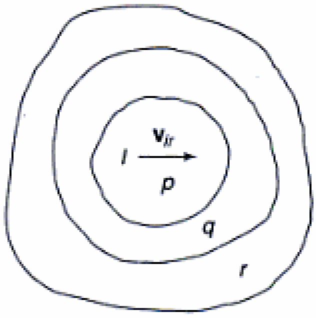


Fig. 2.33

Notation

An I-centre is a centre of rotation of a moving body relative to another body. If a body p is in motion relative to a fixed body q, the centre of rotation (I-centre) may be named as pq. However, in case of relative motions, the body q can also be imagined to rotate relative to body p (i.e., as if the body p is fixed for the moment) about the same centre. Thus, centre of rotation or I-centre can be named qp also.

This shows that the I-centre of the two bodies p and q in relative motion can be named either pq or qp meaning the same thing. In general, the I-centre will be named in the ascending order of the alphabets or digits, i.e., 13, 35, pq, eg, etc.

Number of I-Centres

For two bodies having relative motion between them, there is an I-centre. Thus, in a mechanism, the number of I-centres will be equal to possible pairs of bodies or links.

Let N = Number of I-centres

n = number of bodies or links

Then, $$N = \frac{n(n-1)}{2}$$

2.13 KENNEDY'S THEOREM

Consider three plane bodies p, q and r; r being a fixed body. p and q rotate about centre pr and qr respectively relative to the body r. Thus, pr is the I-centre of bodies p and r whereas qr is the I-centre of bodies q and r. Assume the I-centre of the bodies p and q at the point pq as shown in Fig. 2.34.

Now, p and q both are moving relative to a third fixed body r. Therefore, the motion of their mutual I-centre pq is to be the same whether this point is considered in the body p or q. (Refer to Sec. 2.12).

If the point pq is considered on the body p, its velocity v_p is perpendicular to the line joining pq and pr.

If the point pq is considered on the body q, its velocity v_q is perpendicular to the line joining pq and qr.

It is found that the two velocities of the I-centre pq are in different directions which is impossible. Therefore, the I-centre of the bodies p and q cannot be at the assumed position pq. The velocities v_p and v_q of the I-centre will be same only if this centre lies on the line joining pr and qr.

In words, if three plane bodies have relative motion among themselves, their I-centre must lie on a straight line. This is known as *Kennedy's theorem.*

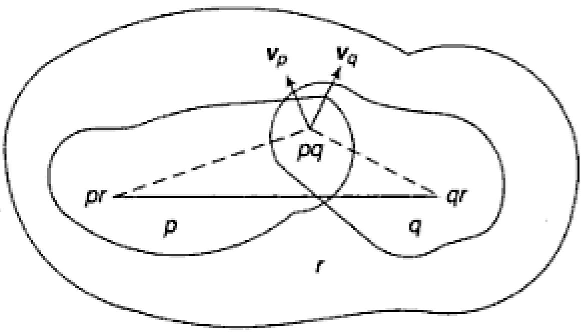


Fig. 2.34

2.14 LOCATING I-CENTRES

The procedure to locate I-centres of a mechanism is being illustrated with the help of the following example of a four-link mechanism.

Figure 2.35 shows a four-link mechanism *ABCD*, the links of which have been named as 1, 2, 3 and 4. The number of I-centres is given by

$$N = \frac{n(n-1)}{2} = \frac{4(4-1)}{2} = 6$$

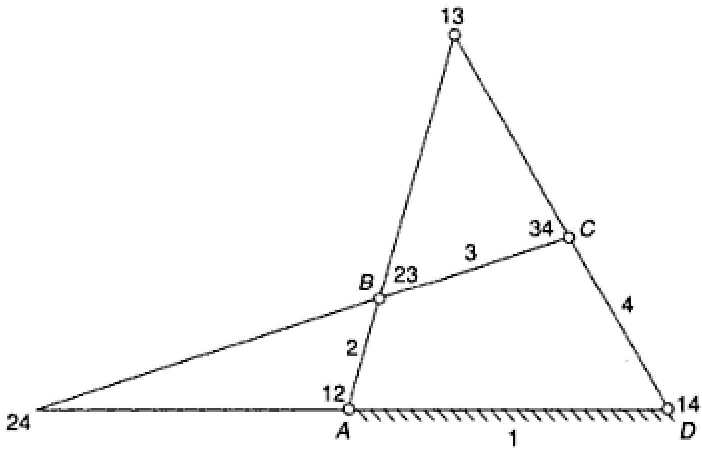


Fig. 2.35

Now, as the centre of rotation of 2 relative to 1 is at A, the I-centre 12 for the links 1 and 2 lies at A. Also, the location of A is not going to change with the rotation of the link 2. Therefore, this I-centre is referred as the *fixed I-centre.* Similarly, 14 is another fixed I-centre for the links 1 and 4 located at D.

Link 3 rotates about B relative to the link 2 and thus the I-centre 23 for links 2 and 3 lies at B. With the movement of the links, the position of the pin-joint B will change and so will the position of the I-centre. However, at all times, the I-centre will be located at the pin joint. Thus, 23 is known as a *permanent* but not a fixed I-centre. Similarly, 34 is another permanent but not a fixed I-centre for the links 3 and 4.

The above I-centres have been located by inspection only. The other two I-centres 13 and 24 which are neither fixed nor permanent can be located easily by applying Kennedy's theorem as explained below.

I-Centre 13

First, consider three links 1, 2 and 3. One more link 2 has been added to links 1 and 3 with the condition that the I-centres 12 and 23 are already known and the third I-centre 13 is to be located.

Now, as the three links 1, 2 and 3 have relative motions among themselves, their I-centres lie on a straight line. Thus, I-centre 13 lies on the line joining 12 and 13 (or line AB).

Similarly, consider the links 1, 4 and 3. Their I-centres are 14, 34 and 13. Out of these, 14 and 34 are already known. Therefore, I-centre 13 lies on the line joining 14 and 34 (or DC). The intersection of the line joining 12 and 23 (or produced) with the line joining 14 and 34 (or produced) locates the I-centre 13.

I-Centre 24

Considering two sets of links 2, 1, 4 and 2, 3, 4; the I-centre would lie on the lines 12–14 and 23–34. The interaction locates the I-centre 24.

There is a convenient way of keeping the track of the I-centres located by inspection and by Kennedy's theorem.

Mark points as the corners of a regular polygon having same number of sides as the number of links in the mechanism.

Name them according to the links of the mechanism. Join the points of which the I-centres have been located by inspection, by firm lines. Then go on joining the points, of which the I-centres are being located by Kennedy's theorem, by dotted lines.

For example, for a four-link mechanism, mark the points 1, 2, 3 and 4 as shown in Fig. 2.36(a). Join 12, 23, 34 and 14 (or 41) by firm lines after locating these I-centres by inspection [Fig. 2.36(b)]. In Fig. 2.29(c) these centres have been encircled for the record.

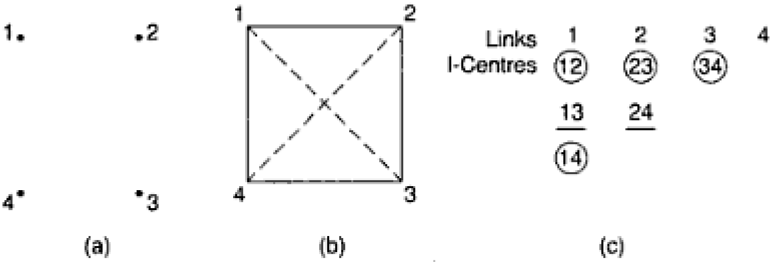


Fig. 2.36

To find the I-centre 13, join 1 to 3 by a dotted line [Fig. 2.36(b)]. The construction shows that the I-centre lies on the line joining I-centres 12 and 23, and the line joining 14 and 34 (or 43). Locate the I-centre actually on the intersection of the two lines in the configuration diagram of the mechanism. In Fig. 2.36(c), 13 is underlined to note that the I-centre has been located by Kennedy's theorem. Similarly, find the I-centre 24 by joining 2 and 4 and locate the point on the intersection of the lines 12–14 and 23–34.

It was mentioned in Section 2.15 that the I-centre is generally not located at the centre of curvature of the apparent path taken by a point of one body with respect to the other body. In the above example of a four-link mechanism, the I-centre of the pivot point B on the coupler relative to the fixed link is at 13, whereas its apparent path is a circular curve about the fixed pivot A which means A is its centre of curvature and the length AB is the radius of curvature. Also, the I-centre of the pivot point C on the coupler relative to the fixed link is again at 13, whereas its apparent path is rotation about the fixed pivot D.

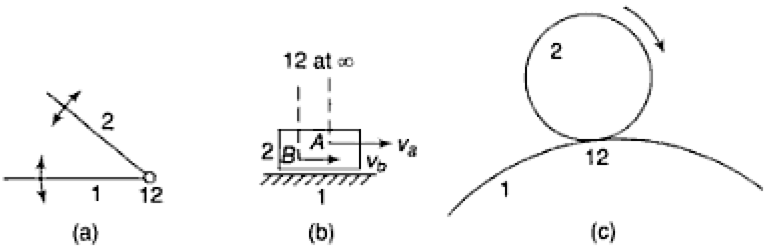


Fig. 2.37

Rules to Locate I-Centres by Inspection

1. In a pivoted joint, the centre of the pivot is the I-centre for the two links of the pivot [Fig. 2.37 (a)].
2. In a sliding motion, the I-centre lies at infinity in a direction perpendicular to the path of motion of the slider. This is because the sliding motion is equivalent to a rotary motion of the links with the radius of curvature as infinity [Fig. 2.37(b)].

3. In a pure rolling contact of the two links, the I-centre lies at the point of contact at the given instant [Fig. 2.37(c)]. It is because the two points of contact on the two bodies have the same linear velocity and thus there is no relative motion of the two at the point of contact which is the I-centre (Refer Sec. 2.12).

2.15 ANGULAR-VELOCITY-RATIO THEOREM

When the angular velocity of a link is known and it is required to find the angular velocity of another link, locate their common I-centre. The velocity of this I-centre relative to a fixed third link is the same whether the I-centre is considered on the first or the second link (Sec. 2.13). First consider the I-centre to be on the first link and obtain the velocity of the I-centre. Then consider the I-centre to be on the second link and find its angular velocity.

For example, if it is required to find the angular velocity of the link 4 when the angular velocity of the link 2 of a four-link mechanism is known, locate the I-centre 24 (Fig. 2.35). Imagine link 2 to be in the form of a flat disc containing point 24 and revolving about 12 or A. Then

$$v_{24} = \omega_2 (24 - 12)$$

Now, imagine the link 4 to be large enough to contain point 24 and revolving about 14 or D. Then

$$v_{24} = \omega_4 (24 - 14)$$

or

$$\omega_4 = \frac{v_{24}}{24-14} = \omega_2 \left(\frac{24-12}{24-14}\right)$$

or

$$\frac{\omega_4}{\omega_2} = \frac{24-12}{24-14}$$

The above equation is known as the *angular-velocity-ratio theorem*. In words, the angular velocity ratio of two links relative to a third link is inversely proportional to the distances of their common I-centre from their respective centres of rotation.

In the above case, the points 12 and 14 lie on the same side of 24 on the line 24–14 and the direction of rotation of the two links (2 and 4) is the same, i.e., clockwise or counter-clockwise. Had they been on the opposite sides of the common I-centre, the direction would have been opposite.

Example 2.19 *In a slider-crank mechanism, the lengths of the crank and the connecting rod are 200 mm and 800 mm respectively. Locate all the I-centres of the mechanism for the position of the crank when it has turned 30° from the inner dead centre. Also, find the velocity of the slider and the angular velocity of the connecting rod if the crank rotates at 40 rad/s.*

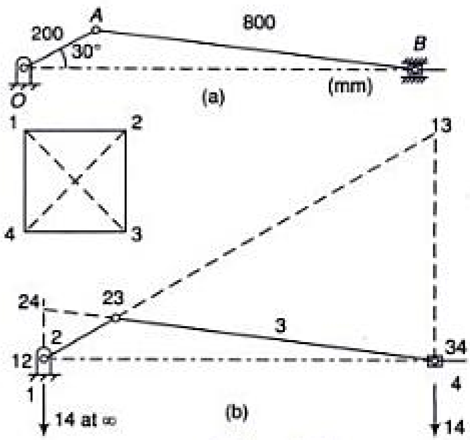


Fig. 2.38

Solution The slider-crank mechanism is shown in Fig. 2.38(a). Name the four links as 1, 2, 3 and 4. Locate the various I-centres as follows:

(i) Locate I-centres 12, 23 and 34 by inspection. They are at the pivots joining the respective links. As the line of stroke of the slider is horizontal, the I-centre 14 lies vertically upwards or downwards at infinity as shown in Fig. 2.38(b).

(ii) Take four points in the form of a square and mark them as 1, 2, 3 and 4. Join 12, 23, 34 and 14 by firm lines as these have been located by inspection.

(iii) I-centre 24 lies at the intersection of lines joining the I-centres 12, 14 and 23, 34 by Kennedy's theorem. Joining of 12 and 14 means a vertical line through 12. This I-centre can be shown in the square by a dotted line to indicate that this has been located by inspection.

(iv) I-centre 13 lies at the intersection of lines joining the I-centres 12, 23 and 14, 34. Joining of 34 and 14 means a vertical line through 34. Show this I-centre in the square by a dotted line.

Thus, all the I-centres are located.

As velocity of the link 2 is known and the velocity of the link 4 is to be found, consider the I-centre 24. The point 24 has the same velocity whether it is assumed to lie in link 2 or 4. First, assume 24 to lie on the link 2 which rotates at angular velocity of 40 rad/s.

Linear velocity of I-centre 24 = 40 × (12–24) = 40 × 0.123

= 4.92 m/s in the horizontal direction

Now, when this point is assumed in the link 4, it will have the same velocity which means the linear velocity of the slider is the same as of the point 24.

Thus, linear velocity of the slider = 4.92 m/s

Example 2.20 *Figure 2.39(a) shows a six-link mechanism. The dimensions of the links are OA = 100 mm, AB = 580 mm, BC = 300 mm, QC = 100 mm and CD = 350 mm. The crank OA rotates clockwise at 150 rpm. For the position when the crank OA makes an angle of 30° with the horizontal, determine the*

(i) linear velocities of the pivot points B, C and D

(ii) angular velocities of the links AB, BC and CD

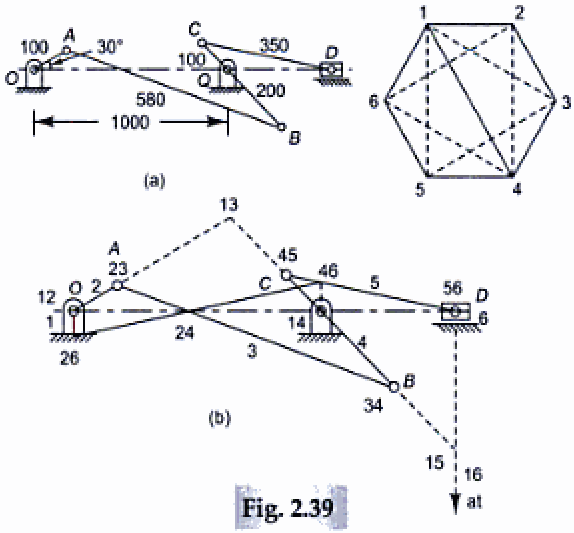


Fig. 2.39

Solution
$$\omega_2 = \frac{2\pi \times 150}{60} = 15.7 \text{ rad/s}$$

$$v_a = \omega_2 .OA = 15.7 \times 0.1 = 1.57 \text{ m/s}$$

Locate I-centres 12, 23, 34, 45, 56, 16 and 14 by inspection.

- Locate 13 which lies on the intersection of 12–23 and 14–34 [Fig. 2.39(b)]
- Locate 15 which lies on the intersection of 14–45 and 56–16 (16 is at ∞)

(i) Now, at the instance, the link 3 rotates about the I-centre 13.

$$\text{Thus, } \frac{v_b}{v_a} = \frac{13-34}{13-23} \text{ or } v_b = \frac{453}{265} \times 1.57 = 2.66 \text{ m/s}$$

$$\text{and } \frac{v_c}{v_b} = \frac{QC}{QB} \text{ or } v_c = \frac{100}{200} \times 2.66 = 1.33 \text{ m/s}$$

At the instance, the link 5 rotates about the I-centre 15.

intersection of lines joining I-centres 12, 16 and 25, 56.

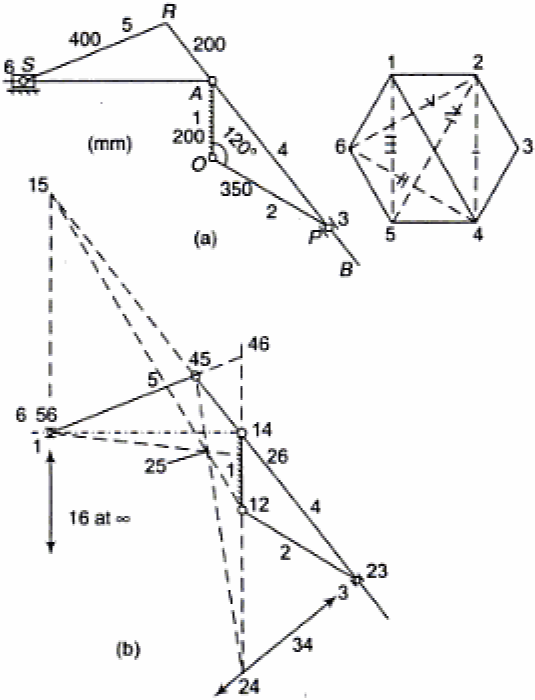


Fig. 2.42

Now, as the velocity of the I-centre 26 is the same whether it is considered to lie on the link 2 or 6,

$v_{26} = \omega_2 .(12\text{–}26) = v_s$

or $\quad v_s = \omega_2 .(12\text{–}26) = 10 \times 0.137 = 1.37$ m/s

Example 2.24 *Solve Example 2.4 by the instantaneous centre method.*

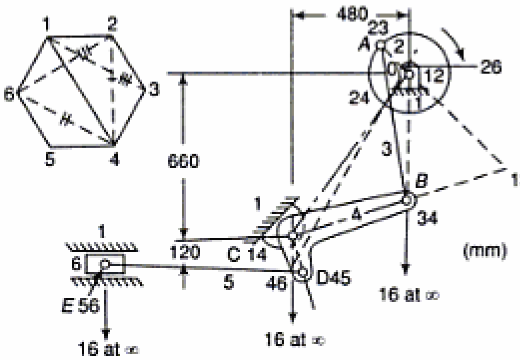


Fig. 2.43

Solution Draw the configuration to a suitable scale as shown in Fig. 2.43.

(a) To find the velocity of E or the link 6, it is required to locate the I-centre 26 as the velocity of a point A on the link 2 is known. After locating I-centres by inspection, locate I-centres 24, 46 and 26 by Kennedy's theorem.
First consider 26 to be on the crank 2.
$v_{26} = \omega(12\text{–}26) = 16.76 \times 0.032 = 0.536$ m/s (horizontal)

When the point 26 is considered on the link 6, all points on it will have the same velocity as the point 26.

Velocity of the crosshead = 0.536 m/s

(b) (i) To find the velocity of rubbing at A (or 23), ω_2 and ω_3 are required.
Locate I-centre 13. Then

$$\omega_3(23\text{–}13) = \omega_2\,(23\text{–}12)$$

$$\therefore \quad \omega_3 = 16.76 \times \frac{0.17}{0.756} = 3.77 \text{ rad/s}$$

ω_3 is clockwise as 13 and 12 lie on the same side of 23.

Velocity of rubbing at $A = (\omega_3 - \omega_2)\, r_a$

$$= (16.76 - 3.77) \times \frac{0.04}{2} = 0.26 \text{ m/s}$$

(ii) For velocity of rubbing at B, ω_2 and ω_4 are required. ω_3 was calculated above.

$$\omega_4(34 - 14) = \omega_3(34 - 13)$$

$$\omega_4 = 3.77 \times \frac{0.45}{0.51} = 3.33 \text{ rad/s}$$

ω_4 is counter-clockwise as 14 and 13 lie on the opposite sides of 34 and ω_3 is clockwise.
Thus, velocity of rubbing at B can be calculated.

(iii) As ω_4 is known, the velocity of rubbing at C can be known.
Similarly, locate the I-centre 15 and obtain ω_5 from the relation,

$$\omega_5 = \omega_4\left(\frac{45-14}{45-15}\right)$$ and determine the velocity of rubbing at D.

Torque is determined in the same way as in Example 2.3.

Example 2.25 *The configuration diagram of a wrapping machine is given in Fig. 2.44(a). Determine the velocity of the point P on the bell-crank lever DCP if the crank OA rotates at 80 rad/s.*

Solution ω_2 is known, ω_6 is required.

Locate the I-centre 26 by first finding the 13 and 16 by Kennedy's theorem. [Fig. 2.44(b)].

Then $\omega_6(26-16) = \omega_2(26-12)$

$$\omega_6 = \omega_2 \times \frac{26-12}{26-16} = 80 \times \frac{47}{383} = 9.82 \text{ rad/s}$$

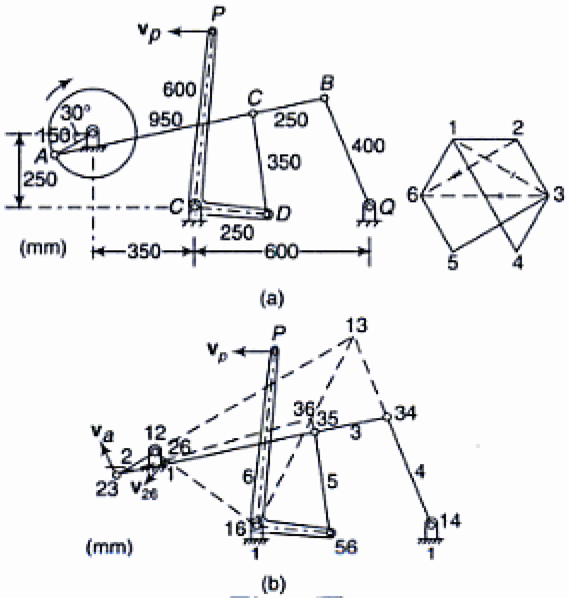


Fig. 2.44

It is counter-clockwise as 16 and 12 lie on the opposite sides of 26 and ω_2 is clockwise.

Thus $v_p = \omega_6 \times (16 - P) = 9.82 \times 600 = \underline{5.89 \text{ m/s}}$

Example 2.26 *Figure 2.45(a) shows the mechanism of a sewing machine needle box. For the given configuration, find the velocity of the needle fixed to the slider D when the crank OA rotates at 40 rad/s.*

Solution Locate the I-centre 26 (Fig. 2.45b).

Consider 26 to lie on the link 2.

$v_{26} = \omega_2 \times (12 - 26) = 40 \times 22.4 = 896$ mm/s vertically downwards.

Consider 26 to lie on the link 6.

Velocity of needle = Velocity of slider = v_{26} = $\underline{896 \text{ mm/s}}$

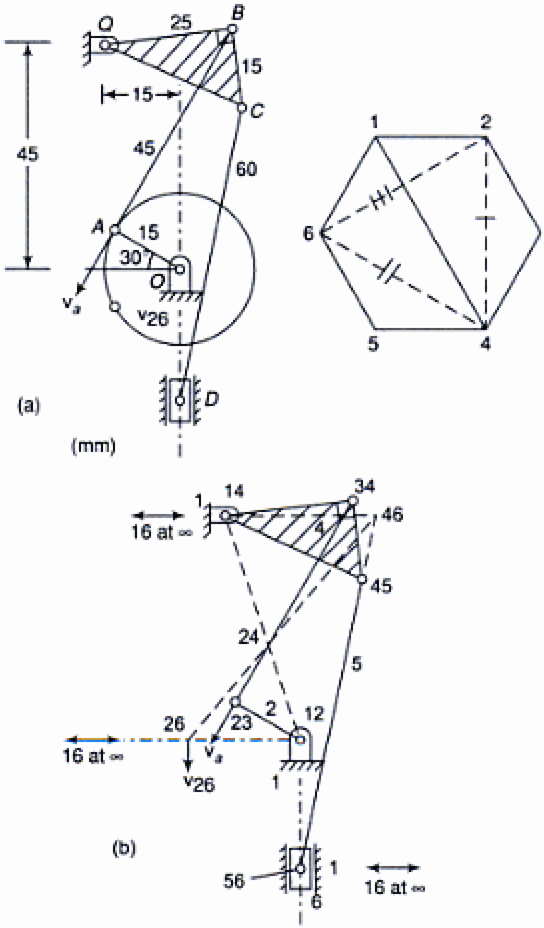


Fig. 2.45

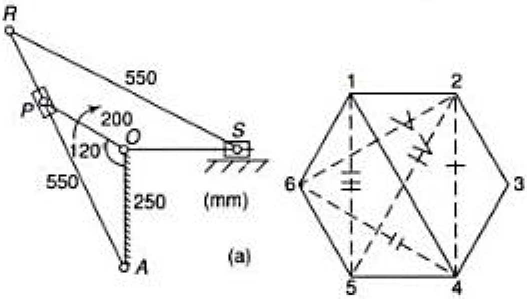
Example 2.27 *Figure 2.46(a) represents a shaper mechanism. For the given configuration, what will be the velocity of the cutting tool at S and the angular velocities of the links AR and RS. Crank OP rotates at 10 rad/s.*

Solution Locate the I-centre 26 [Fig. 2.46(b)]

(i) $v_6 = v_{26} = \omega_2 \times (12 - 26) = 10 \times 0.166 =$ 1.66 m/s

(ii) $\omega_4 = \omega_2\left(\dfrac{24-12}{24-14}\right) = 10 \times \left(\dfrac{183}{430}\right)$

$= 4.25$ rad/s (clockwise)



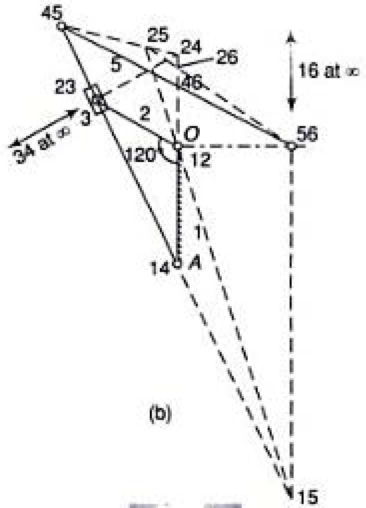


Fig. 2.46

Similarly,

$$\omega_5 = \omega_2\left(\frac{25-12}{25-15}\right) = 10 \times \left(\frac{210}{1060}\right)$$

$=1.98$ rad/s (clockwise)

or

$$\omega_5 = \omega_4\left(\frac{45-14}{45-15}\right) = 4.25\left(\frac{552}{1187}\right) =$$

1.97 rad/s (clockwise)

2.16 CENTRODE

An I-centre is defined only for an instant and changes as the mechanism moves. A *centrode* is the locus of the I-centre of a plane body relative to another plane body for the range of motion specified or during a finite period of time.

There are two types of centrodes:

1. Space Centrode (or Fixed Centrode) of a Moving Body

It is the locus of the I-centre of the moving body relative to the fixed body.

2. Body Centrode (or Moving Centrode) of a Moving Body

It is the locus of the I-centre of the fixed body relative to the movable body, i.e., the locus of the

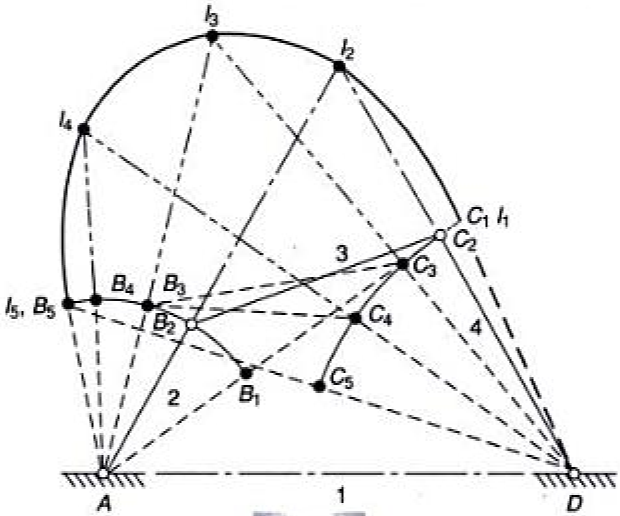


Fig. 2.47

I-centre assuming the movable body to be fixed and the fixed body to be movable.

In a four-link mechanism shown in Fig. 2.47, the link 1 is fixed. The locus of the I-centre of links 1 and 3 over a range of motion of the link 3 is the space centrode. Five positions of the I-centre, i.e., I_1, I_2, I_3, I_4 and I_5 have been obtained and joined with a smooth curve which is the space centrode. If the link 3 is assumed to be fixed and 1 movable, the locus of the I-centre of 1 and 3 is the body centrode. This has been shown in Fig. 2.43 for five positions of the link 1.

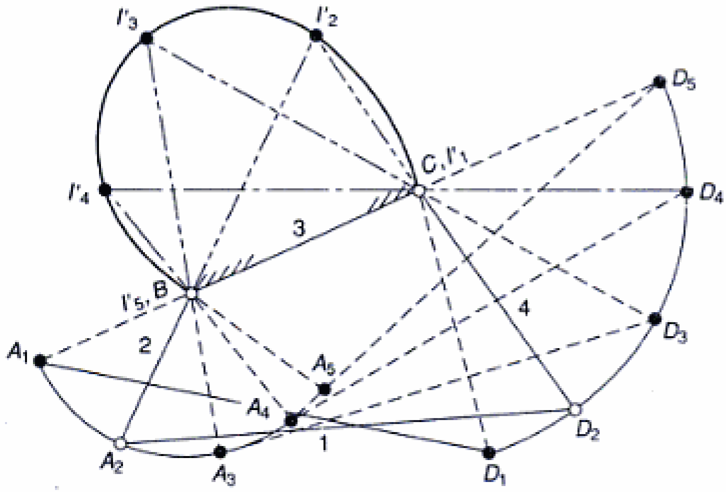


Fig. 2.48

Comparing Figs 2.47 and 2.48, observe that the first position AB_1C_1 (I_1) D of Fig. 2.47 is exactly similar to the first position A_1BC $(I_1')D_1$ of Fig. 2.48. The second positions of the two figures are also similar. Similarly, the third position $AB_3I_3C_3D$ of Fig. 2.47 is exactly similar to the third position $A_3BI_3'CD_3$ of Fig. 2.48, and so on. Thus, Δs $B_2C_2I_2$, $B_3C_3I_3$ and $B_4C_4I_4$ are similar to Δs BCI_2', BCI_3' and BCI_4' respectively. This implies that the positions of the I-centre of Fig. 2.43 can be obtained directly by constructing on B_2C_2 Δs, similar to $B_2C_2I_2$ (already exists), $B_3C_3I_3$ and $B_4C_4I_4$. I_1' lies on C_2 and I_5' on B_2.

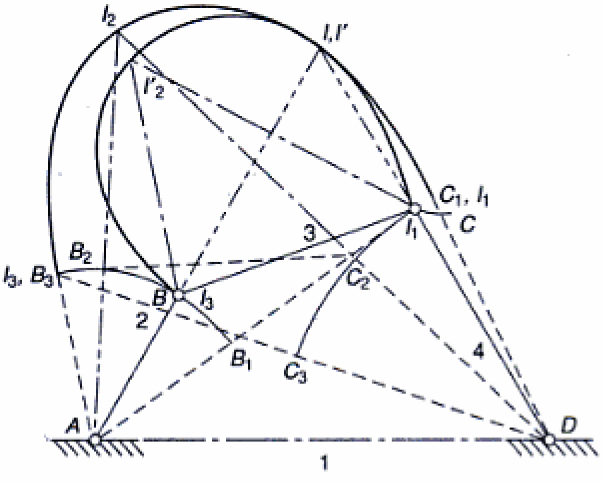


Fig. 2.49

In Fig. 2.49, space and body centrodes of the link 3 relative to 1 have been obtained in the same diagram considering four positions of the link 3.

Figure 2.50 shows the fixed centode (attached to the fixed link 1) and the moving centrode (attached to the moving link 3) with links 2 and 4 removed entirely. Now, if the moving centrode is made to roll on the fixed centrode without slip, the coupler link 3 will exactly traverse the same motion as it had in the original mechanism. This is because a point of rolling contact is always an I-centre in different positions of the link 3.

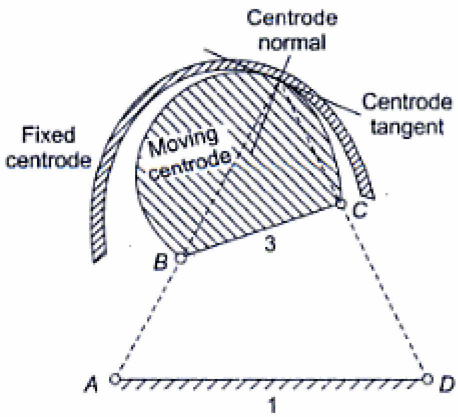


Fig. 2.50

Thus, *the plane motion of a rigid body relative to another rigid body is equivalent to the rolling motion of one centrode on the other.*

The instant point of rolling contact is the instantaneous centre. The common tangent and the common normal to the two centrodes are known as the *centrode tangent* and the *centrode normal* respectively.

Summary

1. A machine or a mechanism, represented by a skeleton or a line diagram, is commonly knows as a *configuration diagram*.
2. Velocity is the derivative of displacement with respect to time and is proportional to the slope of the tangent to the displacement–time curve at any instant.
3. A vector is a line which represents a vector quantity such as force, velocity and acceleration.
4. The magnitude of the instantaneous linear velocity of a point on a rotating body is proportional to its distance from the axis of rotation.
5. The velocity of an intermediate point on any of the links can be found easily by dividing the corresponding velocity vector in the same ratio as the point divides the link.
6. *Velocity images* are found to be very helpful devices in the velocity analysis of complicated linkages. The order of the letters in the velocity image is the same as in the configuration diagram.
7. The angular velocity of a link about one extremity is the same as the angular velocity about the other.
8. The *instantaneous centre of rotation* of a body relative to another body is the centre about which the body rotates at the instant.
9. In a mechanism, the number of I-centres is given by $N = n(n-1)/2$
10. If three plane bodies have relative motion among themselves, their I-centres must lie on a straight line. This is known as *Kennedy's theorem.*
11. When the angular velocity of a link is known and it is required to find the angular velocity of another link, locate their common I-centre. The velocity of this I-centre relative to a fixed third link is the same whether the I-centre is considered on the first or the second link.
12. A *centrode* is the locus of the I-centre of a plane body relative to another plane body for the range of motion specified or during a finite period of time.
13. The plane motion of a rigid body relative to another rigid body is equivalent to the rolling motion of one centrode on the other.

Exercises

1. What is a configuration diagram? What is its use?
2. Describe the procedure to construct the diagram of a four-link mechanism.
3. What is a velocity image? State why it is known as a helpful device in the velocity analysis of complicated linkages.
4. What is velocity of rubbing? How is it found?
5. What do you mean by the term 'coincident points'?
6. What is *instantaneous centre of rotation?* How do you know the number of *instantaneous centres* in a mechanism?
7. State and prove Kennedy's theorem as applicable to instantaneous centres of rotation of three bodies. How is it helpful in locating various instantaneous centres of a mechanism?
8. State and explain *angular-velocity-ratio theorem* as applicable to mechanisms.
9. What do you mean by *centrode* of a body? What are its types?
10. What are fixed centrode and moving centrode? Explain.
11. Show that the plane motion of a rigid body relative to another rigid body is equivalent to the rolling motion of one centrode on the other.
12. In a slider-crank mechanism, the stroke of the slider is one-half the length of the connecting rod. Draw a diagram to give the velocity of the slider at any instant assuming the crankshaft to turn uniformly.
13. In a four-link mechanism, the crank *AB* rotates at 36 rad/s. The lengths of the links are $AB = 200$ mm, $BC = 400$ mm, $CD = 450$ mm and $AD = 600$ mm. *AD* is the fixed link. At the instant when *AB* is at right angles to *AD*, determine the velocity of
   (i) the midpoint of link *BC*
   (ii) a point on the link *CD*, 100 mm from the pin connecting the links *CD* and *AD*.
   (6.55 m/s; 1.45 m/s)
14. For the mechanism shown in Fig. 2.51, determine the velocities of the points *C*, *E* and *F* and the angular velocities of the links *BC*, *CDE* and *EF*.

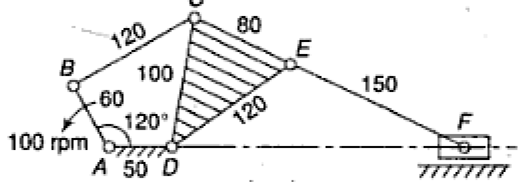


Fig. 2.51

(0.83 m/s; 0.99 m/s; 0.81 m/s; 5.4 rad/s ccw; 8.3 rad/s ccw; 6.33 rad/s ccw)

3 ACCELERATION ANALYSIS

Introduction

Velocity of a moving body is a vector quantity having magnitude and direction. A change in the velocity requires any of the following conditions to be fulfilled:

- A change in the magnitude only
- A change in the direction only
- A change in both magnitude and direction

The rate of change of velocity with respect to time is known as *acceleration* and it acts in the direction of the change in velocity. Thus acceleration is also a vector quantity. To find linear acceleration of a point on a link, its linear velocity is required to be found first. Similarly, to find the angular acceleration of a link, its angular velocity has to be found. Apart from the graphical method, algebraic methods are also discussed in this chapter. After finding the accelerations, it is easy to find inertia forces acting on various parts of a mechanism or machine.

3.1 ACCELERATION

Let a link OA, of length r, rotate in a circular path in the clockwise direction as shown in Fig. 3.1(a). It has an instantaneous angular velocity ω and an angular acceleration α in the same direction, i.e., the angular velocity increases in the clockwise direction.

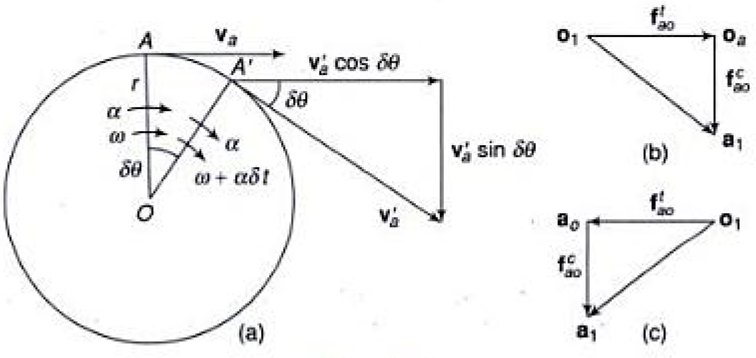


Fig. 3.1

Tangential velocity of A, $v_a = \omega r$

In a short interval of time δt, let OA assume the new position OA' by rotating through a small angle $\delta\theta$.

Angular velocity of OA', $\omega'_a = \omega + \alpha\,\delta t$

Tangential velocity of A', $v'_a = (\omega + \alpha\,\delta t)\,r$

The tangential velocity of A' may be considered to have two components; one perpendicular to OA and the other parallel to OA.

Change of Velocity Perpendicular to OA

Velocity of $A \perp$ to $OA = v_a$

Velocity of $A' \perp$ to $OA = v'_a \cos\delta\theta$

$\therefore$ change of velocity $= v'_a \cos\delta\theta - v_a$

$$\text{Acceleration of } A \perp \text{ to } OA = \frac{(\omega + \alpha.\delta t)r\cos\delta\theta - \omega r}{\delta t}$$

In the limit, as $\delta t \to 0$, $\cos\delta\theta \to 1$

$\therefore$ acceleration of $A \perp$ to $OA = \alpha\, r$

$$= \left(\frac{d\omega}{dt}\right) r \qquad \ldots\left(\alpha = \frac{d\omega}{dt}\right)$$

$$= \frac{dv}{dt} \qquad (3.1)$$

This represents the rate of change of velocity in the tangential direction of the motion of A relative to O, and thus is known as the *tangential acceleration* of A relative to O. It is denoted by f^t_{ao}.

Change of Velocity Parallel to OA

Velocity of A parallel to $OA = 0$

Velocity of A' parallel to $OA = v'_a \sin\delta\theta$

$\therefore$ change of velocity $= v'_a \sin\delta\theta - 0$

$$\text{Acceleration of } A \text{ parallel to } OA = \frac{(\omega + \alpha\,\delta t)r\sin\delta\theta}{\delta t}$$

In the limit, as $\delta t \to 0$, $\sin\delta\theta \to \delta\theta$

$$\text{Acceleration of } A \text{ parallel to } OA = \omega r\frac{d\theta}{dt}$$

$$= \omega r.\omega \qquad \ldots\left(\omega = \frac{d\theta}{dt}\right)$$

$$= \omega^2 r \qquad (3.2)$$

$$= \frac{v^2}{r} \ldots\ldots (v = \omega r) \qquad (3.3)$$

This represents the rate of change of velocity along OA, the direction being from A towards O or towards the centre of rotation. This acceleration is known as the *centripetal* or the *radial acceleration* of A relative to O and is denoted by f^c_{ao}.

Figure 3.1(b) shows the centripetal and the tangential components of the acceleration acting on A. Note the following:

Total acceleration of C relative to $B = \mathbf{b_1\ c_1}$
Total acceleration of $C = \mathbf{d_1\ c_1}$

Angular Acceleration of Links

From the foregoing discussion, it can be observed that the tangential component of acceleration occurs due to the angular acceleration of a link.

Tangential acc. of B rel. to A,

$$\mathbf{f}_{\mathbf{ba}}^{\mathbf{t}} = \alpha.AB = \alpha.BA$$

where α = angular acceleration of the link AB

Thus, angular acceleration of a link can be found if the tangential acceleration is known.

Referring to Fig. 3.2,

Tangential acc. of C rel. to B, $\quad \mathbf{f}_{\mathbf{cb}}^{\mathbf{t}} = \mathbf{c_b\ c_1}$

i.e., acceleration of C relative to B is in a direction $\mathbf{c_b}$ to $\mathbf{c_1}$ or in a counter-clockwise direction about B.

As $\quad f^t_{cb} = \alpha_{cb} CB$

$\therefore \quad \alpha_{cb} = f^t_{cb}/CB$

Tangential acc. of B rel. to C, $f^t_{bc} = \mathbf{c_1\ c_b}$

i.e., acceleration of B relative to C is in a direction $\mathbf{c_1}$ to $\mathbf{c_b}$ or in counter-clockwise direction about C with magnitude, $\alpha_{bc} = f^t_{bc} / BC$ which is the same as α_{cb}.

Thus, angular acceleration of a link about one extremity is the same in magnitude and direction as the angular acceleration about the other.

Tangential acc. of C rel. to D, $\mathbf{f}_{\mathbf{cd}}^{\mathbf{t}} = \mathbf{c_d\ c_1}$

i.e., C relative to D moves in a direction from $\mathbf{c_d}$ to $\mathbf{c_1}$ or C moves in the counter-clockwise direction about D.

$$\alpha_{cd} = \frac{f^t_{cd}}{CD} = \frac{c_d c_1}{CD}$$

3.3 ACCELERATION OF INTERMEDIATE AND OFFSET POINTS

Intermediate Point

The acceleration of intermediate points on the links can be obtained by dividing the acceleration vectors in the same ratio as the points divide the links. For point E on the link BC (Fig. 3.2),

$$\frac{BE}{BC} = \frac{\mathbf{b_1\ e_1}}{\mathbf{b_1\ c_1}}$$

$\mathbf{a_1\ e_1}$ gives the total acceleration of the point E.

Offset Points

The acceleration of an offset point on a link, such as F on BC (Fig. 3.2), can be determined by applying any of the following methods:

1. Writing the vector equation,

$$\mathbf{f}_{fb} + \mathbf{f}_{ba} = \mathbf{f}_{fc} + \mathbf{f}_{cd}$$

or
$$\mathbf{f}_{ba} + \mathbf{f}_{fb} = \mathbf{f}_{cd} + \mathbf{f}_{fc}$$

or
$$\mathbf{f}_{ba} + \mathbf{f}^{c}_{fb} + \mathbf{f}^{t}_{fb} = \mathbf{f}_{cd} + \mathbf{f}^{c}_{fc} + \mathbf{f}^{c}_{fc}$$

or
$$\mathbf{a}_1\mathbf{b}_1 + \mathbf{b}_1\mathbf{f}_b + \mathbf{f}_b\mathbf{f}_1 = \mathbf{d}_1\mathbf{c}_1 + \mathbf{f}_1\mathbf{f}_c + \mathbf{f}_c\mathbf{f}_1$$

The equation can be easily solved graphically as shown in Fig. 3.2(d). $\mathbf{a}_1\mathbf{f}_1$ represents the acceleration of F relative to A or D.

2. Writing the vector equation,

$$\mathbf{f}_{fa} = \mathbf{f}_{fb} + \mathbf{f}_{ba}$$
$$= \mathbf{f}_{ba} + \mathbf{f}_{fb}$$
$$= \mathbf{f}_{ba} + \mathbf{f}^{c}_{fb} + \mathbf{f}^{t}_{fb}$$

or
$$\mathbf{a}_1\mathbf{f}_1 = \mathbf{a}_1\mathbf{b}_1 + \mathbf{b}_1\mathbf{f}_b + \mathbf{f}_b\mathbf{f}_1$$

$\mathbf{f}_{ba}$ already exists on the acceleration diagram.

$$\mathbf{f}^{c}_{fb} = \frac{(bf)^2}{BF}, \parallel FB, \text{ direction towards } B.$$

$$\mathbf{f}^{t}_{fb} = \alpha_{fb} \times FB = \alpha_{cb} \times FB$$

$$= \frac{\mathbf{f}^{t}_{cb}}{CB} \times FB \perp \text{ to } FB; \text{ direction } \mathbf{b} \text{ to } \mathbf{f}$$

$\alpha_{fb} = \alpha_{cb}$, because angular acceleration of all the points on the link BCF about the point B is the same (counter-clockwise).

$\mathbf{f}_{fa}$ can be found in this way.

3. ***By acceleration image method*** In the previous chapter, it was mentioned that velocity images are useful in finding the velocities of offset points of links. In the same way, *acceleration images* are also helpful to find the accelerations of offset points of the links. The acceleration image of a link is obtained in the same manner as a velocity image. It can be proved that the triangle $\mathbf{b}_1\mathbf{c}_1\mathbf{f}_1$ is similar to the triangle BCF in Figs 3.2(d) and (a).

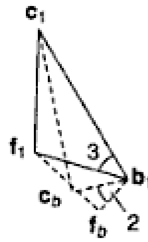


Fig. 3.3

Let ω' = angular velocity of the link BCF

α = angular acceleration of the link BCF

Referring to Figs 3.2(a) and 3.3,

$$\frac{\mathbf{b}_1\mathbf{f}_b}{\mathbf{b}_1\mathbf{c}_b} = \frac{\omega'^2 BF}{\omega'^2 BC} = \frac{BF}{BC} = \frac{\alpha BF}{\alpha BC} = \frac{\mathbf{f}_b\mathbf{f}_1}{\mathbf{c}_b\mathbf{c}_1}$$

$\mathbf{b}_1\mathbf{f}_b\mathbf{f}_1$ and $\mathbf{b}_1\mathbf{c}_b\mathbf{c}_1$ are two right-angled triangles in which the ratio of the two corresponding sides is the same as proved above. Therefore, the two triangles are similar.

$$\frac{\mathbf{b}_1\mathbf{f}_1}{\mathbf{b}_1\mathbf{c}_1} = \frac{BF}{BC} = k$$

Also, $\angle \mathbf{f}_b\mathbf{b}_1\mathbf{f}_1 = \angle \mathbf{c}_b\mathbf{b}_1\mathbf{c}_1$

or
$$\angle \mathbf{f}_b\mathbf{b}_1\mathbf{f}_1 = \angle \mathbf{c}_b\mathbf{b}_1\mathbf{f}_1 = \angle \mathbf{c}_b\mathbf{b}_1\mathbf{c}_1 - \angle \mathbf{c}_b\mathbf{b}_1\mathbf{f}_1$$

or
$$\angle 3 = \angle 2 = \angle 1 \quad (\because \mathbf{b}_1\mathbf{f}_b \parallel BF, \mathbf{b}_1\mathbf{c}_b \parallel BC)$$

Now, in Δs $\mathbf{b_1f_1c_1}$ and BFC,

$$\angle 3 = \angle 1$$

and $$\frac{\mathbf{b_1 f_1}}{\mathbf{b_1 c_1}} = \frac{BF}{BC} = k$$

Therefore, the two triangles are similar.

Thus, to find the acceleration of an offset point on a link, a triangle similar to the one formed in the configuration diagram can be made on the acceleration image of the link in such a way that the sequence of letters is the same, i.e., $\mathbf{b_1f_1c_1}$ is clockwise, so should be BFC.

An easier method of making the triangle $\mathbf{b_1f_1}$ similar to BFC is by marking BC' on BC equal to $\mathbf{b_1c_1}$ and drawing a line parallel to CF, meeting BF in F'. $BC'F'$ is the exact size of the triangle to be made on $\mathbf{b_1c_1}$. Take $\mathbf{b_1f_1} = BF'$ and $\mathbf{c_1f_1} = C'F'$.

Thus, the point $\mathbf{f_1}$ is obtained.

3.4 SLIDER-CRANK MECHANISM

The configuration and the velocity diagrams of a slider-crank mechanism discussed in Sec. 2.8 have been reproduced in Figs. 3.4(a) and (b).

Writing the acceleration equation,

Acc. of B rel. to O = Acc. of B rel. to A + Acc. of A rel. to O

$$\mathbf{f}_{bo} = \mathbf{f}_{ba} + \mathbf{f}_{ao}$$

$$\mathbf{f}_{bg} = \mathbf{f}_{ao} + \mathbf{f}_{ba} = \mathbf{f}_{ao} + \mathbf{f}^c_{ba} + \mathbf{f}^t_{ba}$$

$$\mathbf{g_1\,b_1} = \mathbf{o_1\,a_1} + \mathbf{a_1\,b_a} + \mathbf{b_a\,b_1}$$

The crank OA rotates at a uniform velocity, therefore, the acceleration of A relative to O has only the centripetal component. Similarly, the slider moves in a linear direction and thus has no centripetal component.

Setting the vector table:

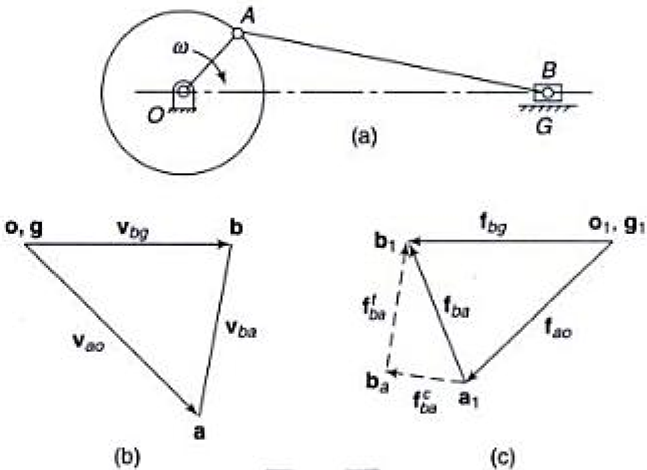


Fig. 3.4

SN	Vector	Magnitude	Direction	Sense
1.	$\mathbf{f}_{ao}$ or $\mathbf{o_1a_1}$	$\frac{(\mathbf{oa})^2}{OA}$	$\parallel OA$	$\rightarrow$ O
2.	$\mathbf{f}^c_{ba}$ or $\mathbf{a_1\,b_a}$	$\frac{(\mathbf{ab})^2}{AB}$	$\parallel AB$	$\rightarrow$ A
3.	$\mathbf{f}^t_{ba}$ or $\mathbf{b_a\,b_1}$	-	$\perp AB$	-
4.	$\mathbf{f}_{bg}$ or $\mathbf{g_1b_1}$	-	$\parallel$ to line of motion of B	-

Construct the acceleration diagram as follows:

1. Take the first vector $\mathbf{f}_{ao}$.
2. Add the second vector to the first.
3. For the third vector, draw a line ⊥ to AB through the head $\mathbf{b}_a$ of the second vector.
4. For the fourth vector, draw a line through $\mathbf{g}_1$ parallel to the line of motion of the slider.

   This completes the velocity diagram.

   Acceleration of the slider $B = \mathbf{o}_1\, \mathbf{b}_1$ (or $\mathbf{g}_1\, \mathbf{b}_1$)

   Total acceleration of B relative to $A = \mathbf{a}_1\, \mathbf{b}_1$

Note that for the given configuration of the mechanism, the direction of the acceleration of the slider is opposite to that of the velocity. Therefore, the acceleration is negative or the slider decelerates while moving to the right.

Example 3.1 *Figure 3.5(a) shows the configuration diagram of a four-link mechanism along with the lengths of the links in mm. The link AB has an instantaneous angular velocity of 10.5 rad/s and a retardation of 26 rad/s² in the counter-clockwise direction. Find*

(i) the angular accelerations of the links BC and CD

(ii) the linear accelerations of the points E, F and G

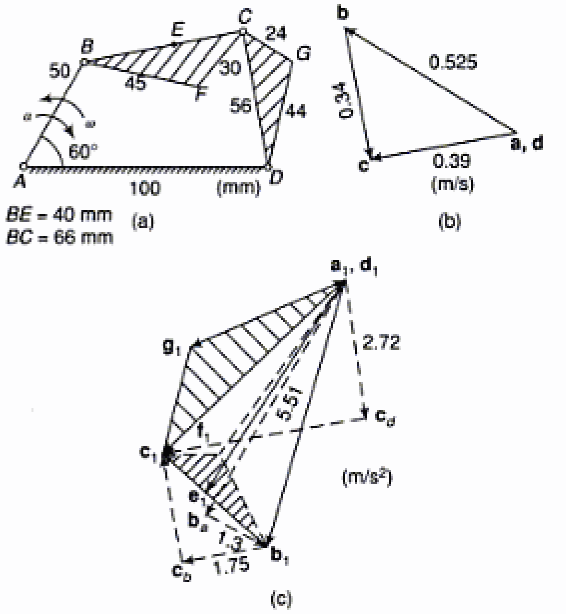


Fig. 3.5

Solution $v_b = 10.5 \times 0.05 = 0.525$ m/s

Complete the velocity diagram [Fig. 3.5(b)] as explained in Example 2.1.

Writing the vector equation for acceleration,

Acc. of C rel. to A = Acc. of C rel. to B + Acc. of B rel. to A

$$\mathbf{f}_{ca} = \mathbf{f}_{cb} + \mathbf{f}_{ba}$$

or $$\mathbf{f}_{cd} = \mathbf{f}_{ba} + \mathbf{f}_{cb}$$

or $$\mathbf{d}_1\, \mathbf{c}_1 = \mathbf{a}_1\, \mathbf{b}_1 + \mathbf{b}_1\, \mathbf{c}_1$$

Each vector has a centripetal and a tangential component,

$$\therefore \quad \mathbf{f}^c_{cd} + \mathbf{f}^t_{cd} = \mathbf{f}^c_{ba} + \mathbf{f}^t_{ba} + \mathbf{f}^c_{cb} + \mathbf{f}^t_{cb}$$

or $$\mathbf{d}_1\, \mathbf{c}_d + \mathbf{c}_d\, \mathbf{c}_1 = \mathbf{b}_a + \mathbf{b}_a\, \mathbf{b}_1 + \mathbf{b}_1\, \mathbf{c}_b + \mathbf{c}_b\, \mathbf{c}_1$$

Set the vector table (Table 1) on the next page.

Draw the acceleration diagram as follows:

(i) Take the pole point $\mathbf{a}_1$ or $\mathbf{d}_1$ [Fig. 3.5(c)].

(ii) Starting from $\mathbf{a}_1$, take the first vector $\mathbf{a}_1\, \mathbf{b}_a$.

(iii) To the first vector, add the second vector and to the second vector, add the third.

(iv) The vector 4 is known in direction only. Therefore, through the head $\mathbf{c}_b$ of the third vector, draw a line, ⊥ to BC. The point $\mathbf{c}_1$ of the fourth vector is to lie on this line.

(v) Start with $\mathbf{d}_1$ and take the fifth vector $\mathbf{d}_1\, \mathbf{c}_d$.

Table 1

SN	Vector	Magnitude (m/s²)	Direction	Sense
1.	f^c_{ba} or $a_1 b_a$	$\frac{(ab)^2}{AB} = \frac{(0.525)^2}{0.05} = 5.51$	$\parallel AB$	$\rightarrow A$
2.	f^t_{ba} or $b_a b_1$	$\alpha \times AB = 26 \times 0.05 = 1.3$	$\perp AB$ or $\parallel$ **ab**	$\rightarrow$ a
3.	f^t_{cb} or $b_1 c_b$	$\frac{(bc)^2}{BC} = \frac{(0.34)^2}{0.066} = 1.75$	$\parallel BC$	$\rightarrow B$
4.	f^t_{cb} or $b_b c_1$	-	$\perp B$	-
5.	f^c_{cd} or $d_1 c_d$	$\frac{(dc)^2}{DC} = \frac{(0.39)^2}{0.56} = 2.72$	$\parallel DC$	$\rightarrow D$
6.	f^t_{cd} or $c_d c_1$	-	$\perp B$	-

(vi) The sixth vector is known in direction only. Draw a line $\perp$ to DC through head $\mathbf{c_d}$ of the fifth vector, the intersection of which with the line in the step (d) locates the point $\mathbf{c_1}$.

(vii) Join $\mathbf{a_1b_1}$, $\mathbf{b_1c_1}$ and $\mathbf{d_1c_1}$.

Now, $\mathbf{a_1b_1}$ represents the total accelerations of the point B relative to the point A.

Similarly, $\mathbf{b_1c_1}$ is the total acceleration of C relative to B and $\mathbf{d_1c_1}$ is the total acceleration of C relative to D.

[Note In the acceleration diagram shown in Fig. 2.5c, the arrowhead has been put on the line joining points $\mathbf{b_1}$ and $\mathbf{c_1}$ in such a way that it represents the vector for acceleration of C relative to B. This satisfies the above equation. As the same equation

$$\mathbf{f}_{cd} = \mathbf{f}_{ba} + \mathbf{f}_{cb}$$

can also be put as

$$\mathbf{f}_{cd} + \mathbf{f}_{bc} = \mathbf{f}_{ba}$$

$$\mathbf{d_1 c_1} + \mathbf{c_1 b_1} = \mathbf{a_1 b_1}$$

This shows that on the same line joining $\mathbf{b_1}$ and $\mathbf{c_1}$, the arrowhead should be marked in the other direction so that the vector represents the acceleration of B relative to C to satisfy the latter equation.

This implies that in case both the ends of a link are in motion, the arrowhead may be put in either direction or no arrowhead is put at all. This is because every time it is not necessary to write the acceleration equation. The acceleration equation is helpful only at the initial stage for better comprehension.]

(i) *Angular accelerations*

$$\alpha_{bc} = \frac{f^t_{cb} \text{ or } \mathbf{c_b\, c_1}}{BC} = \frac{2.25}{0.066} = \underline{34.09 \text{ rad/s}^2}$$

counter-clockwise

$$\alpha_{cd} = \frac{f^t_{cd} \text{ or } \mathbf{c_d\, c_1}}{CD} = \frac{4.43}{0.056}$$

$= \underline{79.11 \text{ rad/s}^2}$ counter-clockwise

(ii) *Linear accelerations*

(a) Locate point $\mathbf{e_1}$ on $\mathbf{b_1\, c_1}$ such that

$$\frac{\mathbf{b_1e_1}}{\mathbf{b_1c_1}} = \frac{BE}{BC}$$

$f_e = \mathbf{a_1e_1} = \underline{5.15 \text{ m/s}^2}$

(b) Draw $\Delta\, \mathbf{b_1c_1f_1}$ similar to $\Delta\, BCF$ keeping in mind that BCF as well as $\mathbf{b_1c_1f_1}$ are read in the same order (clockwise in this case).

$f_f = \mathbf{a_1f_1} = \underline{4.42 \text{ m/s}^2}$

(c) Linear acceleration of the point G can also be found by drawing the acceleration image of the triangle DCG on $\mathbf{d_1c_1}$ in the acceleration diagram such that the order of the letters remains the same.

$f_g = \mathbf{d_1g_1} = \underline{3.9 \text{ m/s}^2}$

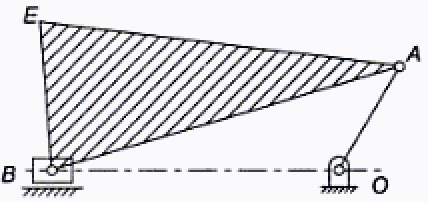


Fig. 3.8

To find the point on the connecting rod which has zero acceleration at this instant, draw triangle ABE on the configuration diagram similar to $\mathbf{a_1b_1o_1}$ such that the letters are in the same order, i.e., clockwise (Fig. 3.8). Then E is a point on the connecting rod with zero acceleration as it corresponds to zero acceleration of point O.

Example 3.4 *In the mechanism shown in Fig. 3.9(a), the crank OA rotates at 210 rpm clockwise. For the given configuration, determine the acceleration of the slider D and angular acceleration of the link CD.*

Solution $v_a = \dfrac{2\pi \times 210}{60} \times 0.1 = 2.2$ m/s

Complete the velocity diagram as follows:

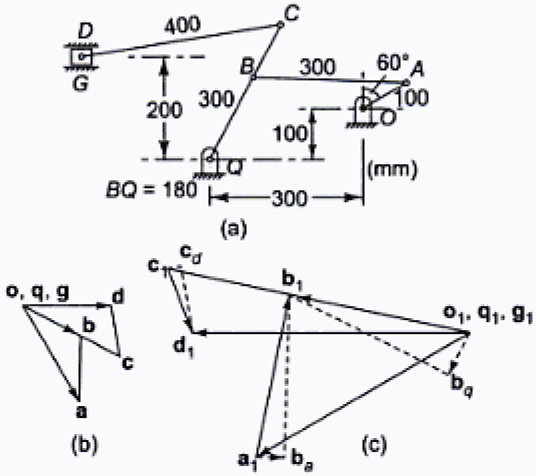


Fig. 3.9

- For the four-link mechanism $OABQ$, complete the velocity diagram as usual [Fig. 3.9(b)].
- Locate point **c** on vector **ob** extended so that $\dfrac{cq}{bq} = \dfrac{CQ}{BQ} = \dfrac{300}{180} = 1.667$
- Draw a horizontal line through **g** for the vector $\mathbf{v}_{dg}$ and a line $\perp CD$ for the vector $\mathbf{v}_{dc}$, the intersection of the two locates the point **d**.

  Thus the velocity diagram is completed.

Set the vector table (Table 4).

Table 4

S.N.	Vector	Magnitude (m/s²)	Direction	Sense
1.	$\mathbf{f}^c_{ao}$ or $\mathbf{o_1\ a_1}$	$\dfrac{(\mathbf{oa})^2}{OA} = \dfrac{(2.2)^2}{0.1} = 48.4$	$\parallel OA$	$\to O$
2.	$\mathbf{f}^c_{ba}$ or $\mathbf{a_1\ b_a}$	$\dfrac{(\mathbf{ab})^2}{AB} = \dfrac{(1.29)^2}{0.3} = 5.55$	$\parallel AB$	$\to A$
3.	$\mathbf{f}^t_{ba}$ or $\mathbf{b_a\ b_1}$	-	$\perp AB$	-
4.	$\mathbf{f}^c_{bq}$ or $\mathbf{q_1\ b_q}$	$\dfrac{(\mathbf{bq})^2}{BQ} = \dfrac{(1.29)^2}{0.18} = 9.25$	$\parallel BQ$	-
5.	$\mathbf{f}^t_{bq}$ or $\mathbf{b_q\ b_1}$	-	$\perp BQ$	-
6.	$\mathbf{f}^c_{dc}$ or $\mathbf{c_1\ c_d}$	$\dfrac{(\mathbf{cd})^2}{CD} = \dfrac{(1.01)^2}{0.4} = 2.55$	$\parallel CD$	$\to C$
7.	$\mathbf{f}^t_{dc}$ or $\mathbf{c_d\ d_1}$	-	$\perp CD$	-
8.	$\mathbf{f}_{dg}$ or $\mathbf{g_1\ d_1}$	-	$\parallel$ to slider motion	-

Links AC and CQ each can have centripetal and tangential components.

$$\mathbf{f}^{t}_{cq} + \mathbf{f}^{t}_{cq} = \mathbf{f}^{t}_{ao} + \mathbf{f}^{t}_{ca} + \mathbf{f}^{t}_{ca}$$

or $$\mathbf{q_1\, c_q + c_q\, c_1 = o_1\, a_1 + a_1\, c_a + c_a\, a_1}$$

Set the following vector table (Table 5).

Complete the acceleration vector diagram $\mathbf{o_1a_1c_1q_1}$ as usual [Fig. 3.10(c)].

Draw $\Delta\mathbf{c_1q_1d_1}$ similar to ΔCQD such that both are read in the same sense, i.e., clockwise.

Write the vector equation for the slider-crank mechanism QDB,

$$\mathbf{f}_{bq} = \mathbf{f}_{bd} + \mathbf{f}_{dq}$$

or $$\mathbf{f}_{bg} = \mathbf{f}_{dq} + \mathbf{f}_{bd}$$

or $$\mathbf{g_1b_1 = q_1d_1 + d_1b_1}$$

From this equation $\mathbf{q_1d_1}$ is already drawn in the diagram and $\mathbf{g_1b_1}$ is a linear acceleration component.

$$\mathbf{f}_{bg} = \mathbf{f}_{dq} + \mathbf{f}^{c}_{bd} + \mathbf{f}^{t}_{bd}$$

or $$\mathbf{g_1b_1 = q_1d_1 + d_1b_d + b_db_1}$$

Set the following vector table (Table 6).

Complete the acceleration vector diagram $\mathbf{q_1\, d_1\, b_1\, g_1}$.

(i) $f_g = \mathbf{g_1\, b_1} = 7$ m/s^2 towards left

As the acceleration $\mathbf{f}_b$ is opposite to $\mathbf{v}_b$, the slider is decelerating.

(ii) $$\alpha_{ac} = \frac{\mathbf{f}^{t}_{ca} \text{ or } \mathbf{c_ac_1}}{AC} = \frac{13.8}{0.6} = \underline{23 \text{ rad/s}^2}$$

counter-clockwise

Table 5

SN	Vector	Magnitude (m/s^2)	Direction	Sense
1.	$\mathbf{f}_{ao}$ or $\mathbf{o_1\, a_1}$	$\frac{(\mathbf{oa})^2}{OA} = \frac{(0.94)^2}{0.15} = 5.92$	$\parallel OA$	$\rightarrow O$
2.	$\mathbf{f}^{c}_{ca}$ or $\mathbf{a_1c_a}$	$\frac{(\mathbf{ac})^2}{AC} = \frac{(1.035)^2}{0.60} = 1.79$	$\parallel AC$	$\rightarrow A$
3.	$\mathbf{f}^{t}_{ca}$ or $\mathbf{c_ac_1}$	-	$\perp AC$	-
4.	$\mathbf{f}^{c}_{cq}$ or $\mathbf{q_1c_q}$	$\frac{(\mathbf{qc})^2}{QC} = \frac{(1.14)^2}{0.145} = 8.96$	$\parallel QC$	$\rightarrow Q$
5.	$\mathbf{f}^{t}_{cq}$ or $\mathbf{c_qc_1}$	-	$\perp QC$	-

Table 6

SN	Vector	Magnitude (m/s^2)	Direction	Sense
1.	$\mathbf{f}_{dq}$ or $\mathbf{q_1\, d_1}$	Already drawn	-	-
2.	$\mathbf{f}^{c}_{bd}$ or $\mathbf{d_1b_d}$	$\frac{(\mathbf{db})^2}{DB} = \frac{(0.495)^2}{0.50} = 0.49$	$\parallel DB$	$\rightarrow D$
3.	$\mathbf{f}^{t}_{bd}$ or $\mathbf{b_d\, b_1}$	-	$\perp DB$	-
4.	$\mathbf{f}_{bg}$ or $\mathbf{g_1\, b_1}$	-	$\parallel$ to slider motion	-

$$\alpha_{cqd} = \frac{\mathbf{f}_{cq}^{t} \text{ or } \mathbf{c}_q\mathbf{c}_1}{QC} = \frac{2.0}{0.145} = \underline{13.8 \text{ rad/s}^2}$$

counter-clockwise

$$\alpha_{bd} = \frac{\mathbf{f}_{bd}^{t} \text{ or } \mathbf{b}_d\mathbf{b}_1}{BD} = \frac{7.2}{0.5} = \underline{14.4 \text{ rad/s}^2}$$

clockwise

Example 3.6 *In the mechanism shown in Fig. 3.11(a), the crank OA rotates at 210 rpm clockwise. For the given configuration, determine the velocities and accelerations of the sliders B, D and F.*

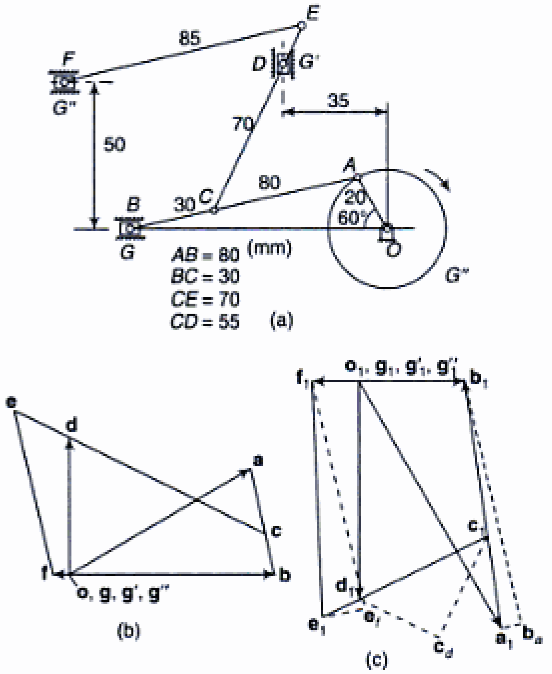


Fig. 3.11

Solution $v_a = \dfrac{2\pi \times 210}{60} \times 0.2 = 4.4$ m/s

Complete the velocity diagram as follows [Fig. 3.11(b)]:

- For the slider-crank mechanism *OAB*, complete the velocity diagram as usual.
- Locate the point **c** on the vector **ab**.
- Draw a vertical line through **g′** for the vector $\mathbf{v}_{dg}'$ and a line $\perp CD$ for the vector $\mathbf{v}_{dc}$, the intersection of the two locates the point **d**.
- Extend the vector **cd** to **e** such that **ce**/**cd** = *CE*/*CD*.
- Draw a horizontal line through **g″** for the vector $\mathbf{v}_{fg}''$ and a line $\perp EF$ for the vector $\mathbf{v}_{fe}$, the intersection of the two locates the point **f**.

Thus, the velocity diagram is completed.

Velocity of slider B = **gb** = <u>4.65 m/s</u>

Velocity of slider D = **g′d** = <u>2.85 m/s</u>

Velocity of slider F = **g″f** = <u>0.35 m/s</u>

Set the vector table (Table 7) as shown in the following page:

The acceleration diagram is drawn as follows:

(i) From the pole point $\mathbf{o}_1$, take the first vector $\mathbf{o}_1\mathbf{a}_1$ [Fig. 3.11(c)].

(ii) Add the second vector by placing its tail at $\mathbf{b}_1$.

(iii) For the third vector $\mathbf{f}_{ba}^{t}$, draw a line $\perp AB$ through $\mathbf{b}_1$ and for the fourth vector a horizontal line through **g**, the intersection of the two lines locates point $\mathbf{b}_1$.

(iv) Locate point $\mathbf{c}_1$ on the vector $\mathbf{a}_1\mathbf{b}_1$.

(v) Add the vector for centripetal acceleration $\mathbf{f}_{dc}^{c}$ of link *CD* and for its tangential component, draw a perpendicular line to it.

(vi) For the vector 7, draw a vertical line through **g′**, the intersection of this line to the previous line locates the point $\mathbf{d}_1$.

(vii) Join $\mathbf{c}_1\mathbf{d}_1$ and locate point $\mathbf{e}_1$ on its extension.

(viii) Take the vector 8 and draw line $\mathbf{e}_1\mathbf{e}_f$ parallel to *EF* and draw a line for the tangential component.

(ix) For the vector 10, take a horizontal line through $\mathbf{g}_1''$ and the intersection of this with the previous line locates the point $\mathbf{f}_1$.

This completes the acceleration diagram.

Acceleration of slider B = $\mathbf{g}_1\mathbf{b}_1$ = 36 <u>m/s²</u>

Acceleration of slider D = $\mathbf{g}'_1\mathbf{d}_1$ = <u>74 m/s</u>

Acceleration of slider F = $\mathbf{g}''_1\mathbf{f}_1$ = <u>16 m/s</u>

Example 3.8 *An Andrew variable-stroke engine mechanism is shown in Fig. 3.13(a). The crank OA rotates at 100 rpm. Find the*

(i) linear acceleration of the slider at D

(ii) angular acceleration of the links AC, BC and CD.

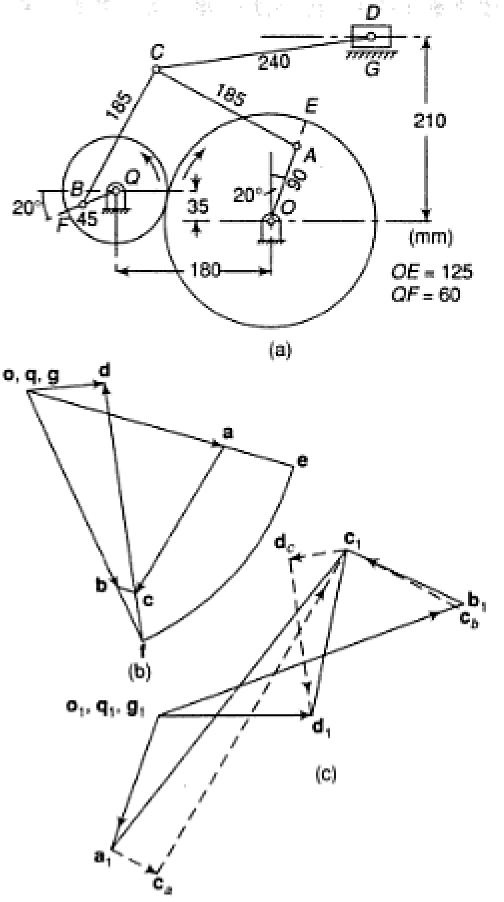


Fig. 3.13

Solution $v_a = \dfrac{2\pi \times 100}{60} \times 0.09 = 0.94$ m/s

Complete the velocity diagram as shown in Fig. 3.13(b). The procedure is explained in Example 2.10.

Write the acceleration vector equation noting that the cranks OA and QB rotate at different uniform speeds.

For the linkage $OACBQ$,

$$\mathbf{f}_{ca} + \mathbf{f}_{ao} = \mathbf{f}_{cb} + \mathbf{f}_{bq} \quad \text{or} \quad \mathbf{f}_{ao} + \mathbf{f}_{ca} = \mathbf{f}_{bq} + \mathbf{f}_{cb}$$

$$\mathbf{o}_1\,\mathbf{a}_1 + \mathbf{a}_1\,\mathbf{c}_1 = \mathbf{q}_1\,\mathbf{b}_1 + \mathbf{b}_1\,\mathbf{c}_1$$

Links AC and BC each have two components,

$$\mathbf{f}_{ao} + \mathbf{f}^{c}_{ca} + \mathbf{f}^{t}_{ca} = \mathbf{f}_{bq} + \mathbf{f}^{c}_{cb} + \mathbf{f}^{t}_{cb}$$

or $\quad \mathbf{o}_1\,\mathbf{a}_1 + \mathbf{a}_1\,\mathbf{c}_a + \mathbf{c}_a\,\mathbf{c}_1 = \mathbf{q}_1\,\mathbf{b}_1 + \mathbf{b}_1\,\mathbf{c}_b + \mathbf{c}_b\,\mathbf{c}_1$

Set the following vector table (Table 9).

Draw the acceleration diagram as follows:

(a) From the pole point $\mathbf{o}_1$, take the first vector and add the second vector to it as shown in Fig. 3.13 (c).

(b) Through the head $\mathbf{c}_a$ of the second vector, draw a line ⊥ to AC for the third vector.

(c) From $\mathbf{q}_1$ (or $\mathbf{o}_1$), take the fourth vector and add the fifth vector to it.

(d) Through the head $\mathbf{c}_b$ of the fifth vector, draw a line ⊥ to BC for the sixth vector.

The intersection of the lines drawn in steps (b) and (d) locates the point $\mathbf{c}_1$.

Now,

$$\mathbf{f}_{do} = \mathbf{f}_{dc} + \mathbf{f}_{co} \quad \text{or} \quad \mathbf{f}_{dg} = \mathbf{f}_{co} + \mathbf{f}_{dc}$$

Since $\mathbf{f}_{dc}$ has two components,

$$\mathbf{f}_{dg} = \mathbf{f}_{co} + \mathbf{f}^{c}_{dc} + \mathbf{f}^{t}_{dc}$$

or $\quad \mathbf{g}_1\,\mathbf{d}_1 = \mathbf{o}_1\,\mathbf{c}_1 + \mathbf{c}_1\,\mathbf{d}_c + \mathbf{d}_c\,\mathbf{d}_1$

Set the following vector table (Table 10).

From $\mathbf{c}_1$ draw the second vector and draw a line ⊥ to CD through the head of the second vector. Draw a line parallel to the line of motion of the slider through $\mathbf{g}_1$. Thus, the point $\mathbf{d}_1$ is located.

(i) $f_d = \mathbf{o}_1\mathbf{d}_1 = \underline{10.65 \text{ m/s}^2}$

(ii) $\alpha_{ac} = \dfrac{\mathbf{f}^{t}_{ca} \text{ or } \mathbf{c}_a\mathbf{c}_1}{AC} = \dfrac{26.4}{0.185} = \underline{142.7 \text{ rad/s}^2}$

clockwise

$$\alpha_{bc} = \frac{\mathbf{f}^{t}_{cb} \text{ or } \mathbf{c}_b\,\mathbf{c}_1}{BC} = \frac{8.85}{0.185} = \underline{47.8 \text{ rad/s}^2}$$

counter-clockwise

$$\alpha_{cd} = \frac{\mathbf{f}^{t}_{dc} \text{ or } \mathbf{d}_c\,\mathbf{d}_1}{CD} = \frac{11.2}{0.24} = \underline{46.7 \text{ rad/s}^2}$$

clockwise

Table 9

SN	Vector	Magnitude (m/s²)	Direction	Sense
1.	$\mathbf{f_{ao}}$ or $\mathbf{o_1 a_1}$	$\frac{(\mathbf{oa})^2}{OA} = \frac{(0.94)^2}{0.09} = 9.87$	$\parallel OA$	$\rightarrow O$
2.	$\mathbf{f_{ca}^c}$ or $\mathbf{a_1 c_a}$	$\frac{(\mathbf{ac})^2}{AC} = \frac{(0.81)^2}{0.185} = 3.55$	$\parallel AC$	$\rightarrow A$
3.	$\mathbf{f_{ca}^t}$ or $\mathbf{c_a c_1}$	-	$\perp AC$	-
4.	$\mathbf{f_{bq}}$ or $\mathbf{q_1 b_1}$	$\frac{(\mathbf{qb})^2}{QB} = \frac{(1.0)^2}{0.045} = 22.2$	$\parallel QB$	$\rightarrow Q$
5.	$\mathbf{f_{cb}^c}$ or $\mathbf{b_1 c_1}$	$\frac{(\mathbf{bc})^2}{BC} = \frac{(0.12)^2}{0.185} = 0.078$	$\parallel BC$	$\rightarrow B$
6.	$\mathbf{f_{cb}^t}$ or $\mathbf{c_b c_a}$	-	$\perp BC$	-

Table 10

SN	Vector	Magnitude	Direction	Sense
1.	$\mathbf{f_{co}}$ or $\mathbf{o_1 c_1}$	Already drawn	-	-
2.	$\mathbf{f_{dc}^c}$ or $\mathbf{c_1 d_c}$	$\frac{(\mathbf{cd})^2}{CD} = \frac{(1.0)^2}{0.24} = 4.17$	$\parallel CD$	$\rightarrow C$
3.	$\mathbf{f_{dc}^t}$ or $\mathbf{d_c d_1}$	-	$\perp CD$	-
4.	$\mathbf{f_{dg}}$ or $\mathbf{g_1 d_1}$	-	$\parallel$ to motion of D	-

3.5 CORIOLIS ACCELERATION COMPONENT

It is seen that the acceleration of a moving point relative to a fixed body (fixed coordinate system) may have two components of acceleration; the centripetal and the tangential. However, in some cases, the point may have its motion relative to a moving body (moving coordinate) system, for example, motion of a slider on a rotating link. The following analysis is made to investigate the acceleration at that point.

Let a link AR rotate about a fixed point A on it (Fig. 3.14). P is a point on a slider on the link.

At any given instant,

Let ω = angular velocity of the link

α = angular acceleration of the link

v = linear velocity of the slider on the link

f = linear acceleration of the slider on the link

r = radial distance of point P on the slider

In a short interval of time δt, let $\delta\theta$ be the angular displacement of the link and δr, the radial displacement of the slider in the outward direction.

After the short interval of time δt, let

$\omega' = \omega + \alpha\delta t$ = angular velocity of the link
$v' = v + f.\delta t$ = linear velocity of the slider on the link
$r' = r + \delta r$ = radial distance of the slider

Acceleration of P Parallel to AR

Initial velocity of P along $AR = v = v_{pq}$
Final velocity of P along $AR = v' \cos\delta\theta - \omega' r' \sin\delta\theta$
Change of velocity along $AR = (v' \cos\delta\theta - \omega' r' \sin\delta\theta) - v$
Acceleration of P along AR

$$= \frac{(v + f\delta t)\cos\delta\theta - (\omega + \alpha\delta t)(r + \delta r)\sin\delta\theta - v}{\delta t}$$

In the limit, as $\delta t \to 0$
$\cos\delta\theta \to 1$ and $\sin\delta\theta \to \delta\theta$

$$\text{Acceleration of } P \text{ along } AR = f - \omega r \frac{d\theta}{dt}$$
$$= f - \omega r w = f - \omega^2 r$$
$$= \text{Acc. of slider–centripetal. acc.}$$

This is the acceleration of P along AR in the radially outward direction. f will be negative if the slider has deceleration while moving in the outward direction or has acceleration while moving in the inward direction.

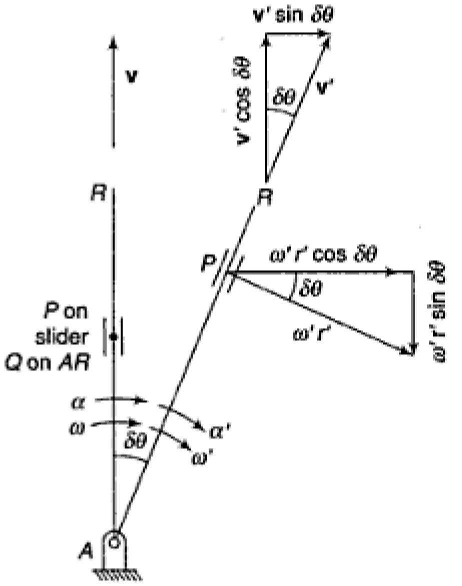


Fig. 3.14

Acceleration of P Perpendicular to AR

Initial velocity of $P \perp$ to $AR = \omega r$
Final velocity of $P \perp$ to $AR = v' \sin\delta\theta + \omega' r' \cos\delta\theta$
Change of velocity $\perp$ to $AR = (v' \sin\delta\theta + \omega' r' \cos\delta\theta) - \omega r$
Acceleration of $P \perp$ to AR

$$= \frac{(v + f\delta t)\cos\delta\theta - (\omega + \alpha\delta t)(r + \delta r)\cos\delta\theta - \omega r}{}$$

In the limit, as $\delta t \to 0$
$\cos\delta\theta \to 1$ and $\sin\delta\theta \to \delta\theta$

$$\text{Acceleration of } P \perp \text{ to } AR = v\frac{d\theta}{dt} + \omega\frac{dr}{dt} + r\alpha$$
$$= v\omega + \omega v + r\alpha = 2\omega v + r\alpha$$
$$= 2\omega v + \text{tangential acc.}$$

This is the acceleration of P perpendicular to AR. The component $2\omega v$ is known as the *Coriolis acceleration component*. It is positive if both ω and v are either positive or negative.

Thus, the coriolis component is positive if the

- link AR rotates clockwise and the slider moves radially outwards
- link rotates counter-clockwise and the slider moves radially inwards.

Otherwise, the Coriolis component will be negative.

These observations can be summarised into the following rule:

The direction of the Coriolis acceleration component is obtained by rotating the radial velocity vector **v** through 90° in the direction of rotation of the link.

Let Q be a point on the link AR immediately beneath the point P at the instant. Then

Note that in the present case, the sliding acceleration $\mathbf{a_1q_a}$ is in the opposite direction to the sliding velocity **qp**. This signifies that the slider is decelerating.

Also,

$$\mathbf{f}_{sa} = \mathbf{f}_{sr} + \mathbf{f}_{ra}$$
$$\mathbf{f}_{sg} = \mathbf{f}_{ra} + \mathbf{f}_{sr}$$
$$= \mathbf{f}_{ra} + \mathbf{f}^{c}_{sr} + \mathbf{f}^{t}_{sr}$$
$$\mathbf{g_1\, s_1 = a_1\, r_1 + r_1\, s_r + s_r\, s_1}$$

This equation can be solved as usual.

Total acc. of S relative to R, $\mathbf{f}_{sr} = \mathbf{r_1\, s_1}$

Acceleration of $S = \mathbf{g_1\, s_1}$ or $\mathbf{a_1\, s_1}$ or $\mathbf{o_1\, s_1}$

The direction of $\mathbf{g_1s_1}$ is opposite to the direction of motion of the slider S indicating that the slider is decelerating.

Example 3.9 *Figure 3.16(a) shows a slider moving outwards on a rod with a velocity of 4 m/s when its distance from the point O is 1.5 m. At this instant, the velocity of the slider is increasing at a rate of 10 m/s². The rod has an angular velocity of 6 rad/s counter-clockwise about O and an angular acceleration of 20 rad/s² clockwise. Determine the absolute acceleration of the slider.*

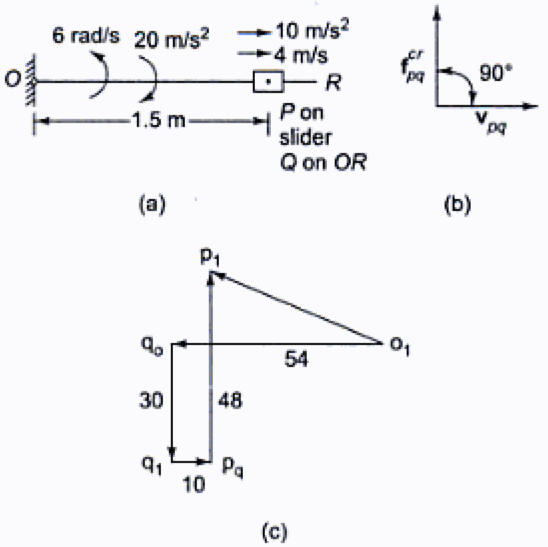


Fig. 3.16

Solution

Writing the acceleration vector equation,

$$\mathbf{f}_{po} = \mathbf{f}_{pq} + \mathbf{f}_{qo} = \mathbf{f}_{qo} + \mathbf{f}_{pq} = \mathbf{f}^{c}_{qo} + \mathbf{f}^{t}_{qo} + \mathbf{f}^{s}_{pq} + \mathbf{f}^{cr}_{pq}$$

or $$\mathbf{o_1\, p_1 = o_1q_o + q_oq_1 + q_1p_q + p_q\, p_1}$$

Set the following vector table (Table 12):

Figure 3.16(b) shows how to obtain the direction of the coriolis component. The velocity vector of the slider is rotated through 90° in the angular direction of the rod.

Draw the acceleration diagram as follows [Fig. 3.16(c)]:

1. From the pole point $\mathbf{o}_1$, take the first vector $\mathbf{o_1q_o}$.
2. Add to it the second vector $\mathbf{q_oq_1}$.

Table 12

SN	Vector	Magnitude (m/s²)	Direction	Sense
1.	$\mathbf{f}^{c}_{qo}$ or $\mathbf{o_1\, q_o}$	$\omega^2.r = 6^2 \times 1.5 = 54$	$\parallel OR$	←
2.	$\mathbf{f}^{t}_{qo}$ or $\mathbf{q_o\, q_1}$	$\alpha_{or} \times OQ = 20 \times 1.5 = 30$	$\perp OR$	↓
3.	$\mathbf{f}^{s}_{pq}$ or $\mathbf{q_1\, p_q}$	10	$\parallel OR$	→
4.	$\mathbf{f}^{cr}_{pq}$ or $\mathbf{p_q\, p_1}$	$2\omega.\, v_{pq} = 2 \times 6 \times 4 = 48$	$\perp OR$	

3. Add the third vector to the second vector.
4. To add the fourth vector, place the tail of the fourth vector at the head of the third vector and the final point $\mathbf{p_1}$ is located.
5. Join $\mathbf{o_1\,p_1}$.

   On measurement, $\mathbf{o_1\,p_1} = 47.5$ m/s^2

   or by calculation from the acceleration diagram

   $= \sqrt{(54-10)^2 + (48-30)^2} = 47.5$ m/s^2

Example 3.10 *Fig. 3.17(a) shows the link mechanism of a quick-return mechanism of the slotted-lever type, the various dimensions of which are*

OA = 400 mm, OP = 200 mm, AR = 700 mm, RS = 300 mm.

For the configuration shown, determine the acceleration of the cutting tool at S and the angular acceleration of the link RS. The crank OP rotates at 210 rpm.

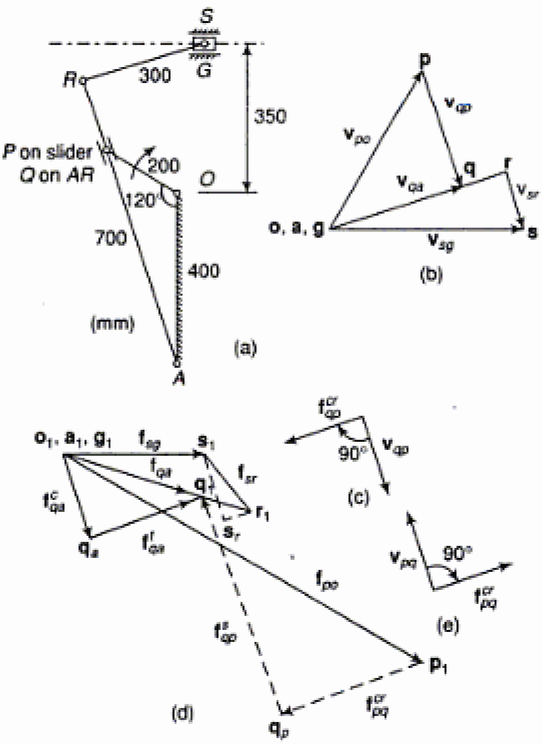


Fig. 3.17

Solution The velocity diagram has been reproduced in Fig. 3.17(b) from Fig. 2.21.

$$\mathbf{v_{po}} \text{ or } \mathbf{op} = \omega . OP = \frac{2\pi \times 210}{60} \times 0.2 = 22 \times 0.2 = 4.4 \text{ m/s}$$

Writing the acceleration vector equation,

$$\mathbf{f}_{qo} = \mathbf{f}_{qp} + \mathbf{f}_{po}$$
$$\mathbf{f}_{qa} = \mathbf{f}_{po} + \mathbf{f}_{qp}$$

or $\quad \mathbf{a_1\,q_1} = \mathbf{o_1\,p_1} + \mathbf{p_1\,q_1}$

or $\quad \mathbf{f^c_{qa}} + \mathbf{f^t_{qa}} = \mathbf{f^c_{po}} + \mathbf{f^{cr}_{qp}} + \mathbf{f^s_{qp}}$

or $\quad \mathbf{a_1\,q_a} + \mathbf{q_a\,q_1} = \mathbf{o_1\,p_1} + \mathbf{p_1\,q_p} + \mathbf{q_p\,q_1}$

Set the following vector table (Table 13) shown in the following page.

Direction of f^{cr}_{qp} is obtained by rotating $\mathbf{v_{qp}}$ through 90° in the direction of angular movement of link QA as shown in Fig. 3.17(c) (clockwise in this case). Draw the acceleration diagram as follows [Fig. 3.17(d)]:

1. From the pole point $\mathbf{o_1}$, take the first vector $\mathbf{o_1\,p_1}$.
2. Add to it the second vector $\mathbf{p_1q_p}$.
3. Add the third vector to the second vector. For the fourth vector, draw a line parallel to AQ, through the head $\mathbf{q_p}$ of the third vector.
4. From the pole point $\mathbf{a_1}$ or $\mathbf{o_1}$, take the fifth vector and for the sixth vector, draw a line perpendicular to AQ through the head $\mathbf{q_a}$ of the fifth vector.

   This way the point $\mathbf{q_1}$ is located.
5. Join $\mathbf{q_1}$ and $\mathbf{a_1}$ and extend to $\mathbf{r_1}$ such that

   $$\frac{\mathbf{a_1r_1}}{\mathbf{a_1q_1}} = \frac{AR}{AQ}$$

   Writing the vector equation,

   $$\mathbf{f}_{so} = \mathbf{f}_{sr} + \mathbf{f}_{ro}$$

   or $\quad \mathbf{f}_{sg} = \mathbf{f}_{ro} + \mathbf{f}_{sr}$

   $\quad\quad = \mathbf{f}_{ro} + \mathbf{f}^c_{sr} + \mathbf{f}^t_{sr}$

   or $\quad \mathbf{g_1\,s_1} = \mathbf{o_1\,r_1} + \mathbf{r_1\,s_r} + \mathbf{s_r\,s_1}$

$\mathbf{f}_{ro}$ is already available on the acceleration diagram. $\mathbf{f}_{sg}$ is horizontal.

$$\mathbf{f^c_{sr}} = \frac{(\mathbf{rs})^2}{RS} = \frac{(1.41)^2}{0.3} = 6.63 \text{ m/s}^2$$

Complete the vector diagram as usual.

Acceleration of the cutting tool, $\mathbf{f_s} = \mathbf{o_1\,s_1}$ = 32.8 m/s^2

$$\alpha_{rs} = \frac{\mathbf{f^t_{rs}} \text{ or } \mathbf{s_1s_r}}{RS} = \frac{15.7}{0.3} = \underline{52.3 \text{ rad/s}} \text{ clockwise}$$

Table 13

SN	Vector	Magnitude (m/s²)	Direction	Sense
1.	$\mathbf{f}^{c}_{po}$ or $\mathbf{o_1 p_1}$	$\dfrac{(\mathbf{op})^2}{OP} = \dfrac{(4.4)^2}{0.2} = 96.8$	$\parallel OP$	$\to O$
2.	$\mathbf{f}^{cr}_{qp}$ or $\mathbf{p_1 q_p}$	$2\,\omega_{ra}\,\mathbf{v}_{qp} = 35.5^*$	$\perp AQ$	Refer *
3.	$\mathbf{f}^{s}_{qp}$ or $\mathbf{q_p q_1}$	–	$\parallel AQ$	–
4.	$\mathbf{f}^{c}_{qa}$ or $\mathbf{a_1 q_a}$	$\dfrac{(\mathbf{aq})^2}{AQ} = \dfrac{(3.26)^2}{0.52} = 20.4$	$\parallel AQ$	$\to A$
5.	$\mathbf{f}^{t}_{qa}$ or $\mathbf{q_a q_1}$	–	$\perp AQ$	–

$${}^*\mathbf{f}^{cr}_{qp} = 2\omega_{ra}v_{qp} = 2\frac{v_{ra}}{RA}v_{qp} = 2\times\frac{4.36}{0.7}\times 2.85 = 35.5 \text{ m/s}^2$$

Note: In case the problem is to be worked out without writing the vector equation and if the Coriolis acceleration component $\mathbf{f}^{cr}_{pq}$ is considered instead of $\mathbf{f}^{cr}_{qp}$, then note that

- the magnitude of the Coriolis component remains the same.
- in order to find the direction, the velocity vector $\mathbf{v}_{qp}$ is to be rotated through 90° as shown in Fig. 3.17e. The direction of $\mathbf{f}^{cr}_{qp}$ is found to be opposite to $\mathbf{f}^{cr}_{pq}$. Now, one will be tempted to place this vector towards right of $\mathbf{p_1}$ in the acceleration diagram. However, if that is done, the vector would be read as $\mathbf{p_1 q_p}$ which means $\mathbf{f}^{cr}_{qp}$ and not $\mathbf{f}^{cr}_{pq}$. Thus, again the vector $\mathbf{f}^{cr}_{qp}$ has to be placed at the same place, i.e., on the left of $\mathbf{p_1}$ which means the acceleration diagram obtained will be the same.

Example 3.11 *Figure 3.18(a) shows the Scotch yoke mechanism. At the instant shown in the figure, the crank OP has an angular velocity of 10 rad/s and an angular acceleration of 30 rad/s². Determine the acceleration of the slider P in the guide and the horizontal acceleration of the guide.*

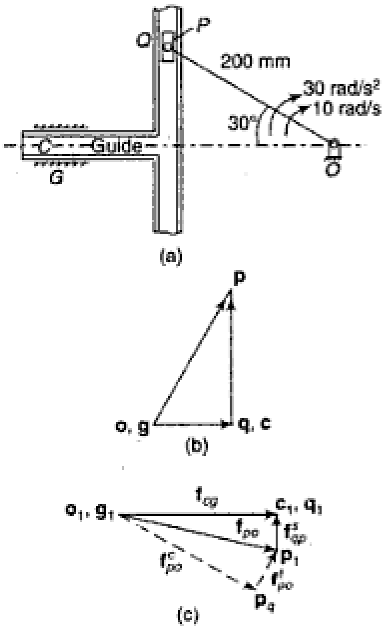


Fig. 3.18

Solution $v_{po} = 10 \times 0.2 = 2$ m/s

To draw the velocity diagram, take a coincident point Q just beneath P on the guide link. Take another point C on the guide link. Now, proceed as follows [Fig. 3.18(b)]:

1. Take the vector **op** equal to 2 m/s to some suitable scale.
2. The velocity of *Q* relative to *P* is along the guide path. Therefore, draw a line parallel to this path (vertical) through **p** to locate the point **q**.
3. The velocity of *C* relative to *G* is along the guide path at *G* or is horizontal. Thus, draw a horizontal line through **g** to locate point **c**.
4. Now, *Q* and *C* are two fixed points on the same link and the distance between them does not vary. Therefore, the points **q** and **c** in the velocity diagram coincide. Thus, the intersection of lines drawn in steps 2 and 3 locates points **q** or **c**.

Now, $\mathbf{f}_{po}^{c}$ or $\mathbf{o}_1\,\mathbf{p}_o = \dfrac{(\mathbf{op})^2}{OP} = \dfrac{2^2}{0.2} = 20\ \text{m/s}^2$

$\mathbf{f}_{po}^{t}$ or $\mathbf{p}_o\,\mathbf{p}_1 = 30 \times 0.2 = 6\ \text{m/s}^2$

Draw acceleration diagram as follows [Fig. 3.18(c)]:

1. First take the centripetal acceleration component $\mathbf{f}_{po}^{c}$ or $\mathbf{o}_1\mathbf{p}_o$ and add the tangential component $\mathbf{f}_{po}^{t}$ or $\mathbf{p}_o\mathbf{p}_1$ to it.
2. Now, the linear acceleration of sliding of *Q* relative to *P* is vertical. Thus, draw a line to locate point $\mathbf{q}_1$ on that.
3. Draw a horizontal line through $\mathbf{g}_1$ to locate the point $\mathbf{c}_1$ on that.
4. As there is zero velocity between *Q* and *P*, they are to be the coinciding points in the acceleration diagram also. Thus, the intersection of lines drawn in steps 2 and 3 locates the point $\mathbf{q}_1$ or $\mathbf{c}_1$.

Acceleration of slider *P* = $\mathbf{f}_{pq}$ or $\mathbf{q}_1\mathbf{p}_1$ = 4.75 m/s²

and horizontal acceleration of guide = $\mathbf{f}_{cg}$ or $\mathbf{g}_1\,\mathbf{c}_1 = 20.5\ \text{m/s}^2$

It is to be noted that in this example, *Q* and *P* are two coincident points, but still there is no Coriolis component. This is because the link (guide) on which the slider is moving does not have any angular motion and thus ω for that is zero.

Example 3.12 *A Whitworth quick-return mechanism has been shown in Fig. 3.19(a). The dimensions of the links are OP (crank) = 240 mm, OA = 150 mm, AR = 165 mm and RS = 430 mm. The crank OP has an angular velocity of 2.5 rad/s and an angular deceleration of 20 rad/s² at the instant. Determine the*

(i) acceleration of the slider S

(ii) angular acceleration of links AR and RS

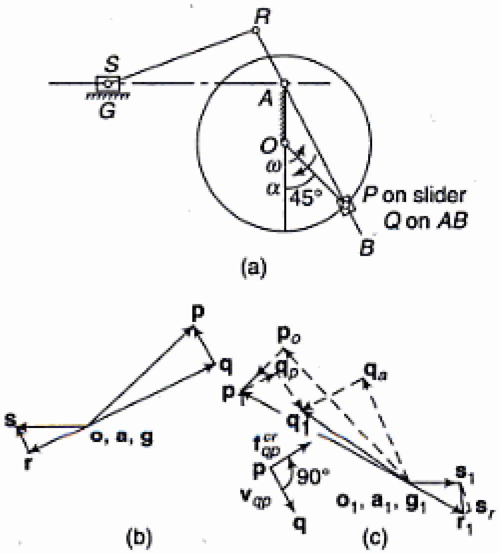


Fig. 3.19

Solution The velocity diagram has been reproduced in Figs 3.19(b) from Fig. 2.25(b).

Writing the acceleration vector equation,

$$\mathbf{f}_{qo} = \mathbf{f}_{qp} + \mathbf{f}_{po} \qquad \text{or} \qquad \mathbf{f}_{pa} = \mathbf{f}_{pq} + \mathbf{f}_{qa}$$

Both the equations lead to the same acceleration diagram except that the direction sense of the acceleration vectors between *P* and *Q* are reversed.

Taking the first one,

$$\mathbf{f}_{qo} = \mathbf{f}_{qp} + \mathbf{f}_{po}$$

$$\mathbf{f}_{qa} = \mathbf{f}_{po} + \mathbf{f}_{qp}$$

or $\quad \mathbf{a}_1\,\mathbf{q}_1 = \mathbf{o}_1\,\mathbf{p}_1 + \mathbf{p}_1\,\mathbf{q}_1$

Each has two components,

$$\mathbf{f}_{qa}^{c} + \mathbf{f}_{qa}^{t} = \mathbf{f}_{po}^{c} + \mathbf{f}_{po}^{t} + \mathbf{f}_{qp}^{cr} + \mathbf{f}_{qp}^{s}$$

or $\quad \mathbf{a_1 q_a + q_a q_1 = o_1 p_o + p_o p_1 + p_1 q_p + q_p q_1}$

Set the following vector table (Table 14):

The direction of f_{qp}^{cr} is obtained by rotating $\mathbf{v}_{qp}$ through 90° in the direction of angular movement of the link *QA* or *BA* (counter-clockwise in this case) Draw the acceleration diagram as follows [Fig. 3.19(c)]:

1. From the pole point $\mathbf{o_1}$, take the first vector and add to it the second vector.
2. Add the third vector to the second vector. For the fourth vector, draw a line parallel to *AQ*, through the head $\mathbf{q_p}$ of the third vector.
3. From the pole point $\mathbf{a_1}$ or $\mathbf{o_1}$, take the fifth vector and for the sixth vector, draw a line perpendicular to *AQ* through the head $\mathbf{q_a}$ of the fifth vector.

   This way the point $\mathbf{q_1}$ is located.
4. Join $\mathbf{q_1}$ and $\mathbf{a_1}$ and extend to $\mathbf{r_1}$ such that

$$\frac{\mathbf{a_1 r_1}}{\mathbf{a_1 q_1}} = \frac{AR}{AQ}$$

Writing the vector equation,

$$\mathbf{f}_{so} = \mathbf{f}_{sr} + \mathbf{f}_{ro}$$

or
$$\mathbf{f}_{sg} = \mathbf{f}_{ro} + \mathbf{f}_{sr}$$
$$= \mathbf{f}_{ro} + \mathbf{f}_{sr}^{c} + \mathbf{f}_{sr}^{t}$$

or
$$\mathbf{g_1 s_1 = o_1 r_1 + r_1 s_r + s_r s_1}$$

$\mathbf{f}_{ro}$ is already available on the acceleration diagram. $\mathbf{f}_{sg}$ is horizontal.

$$\mathbf{f}_{sr}^{c} = \frac{(\mathbf{rs})^2}{RS} = \frac{(0.12)^2}{0.43} = 0.033 \text{ m/s}^2$$

Complete the vector diagram as usual.

$$\mathbf{f}_s = \mathbf{o_1 s_1} = 0.39 \text{ m/s}^2$$

$$\alpha_{ar} = \alpha_{qa} = \frac{\mathbf{f}_{qa}^{t} \text{ or } \mathbf{q}_a \, \mathbf{q}_1}{QA} = \frac{0.57}{0.365}$$

$= \underline{1.56 \text{ rad/s}^2}$ clockwise

$$\alpha_{rs} = \frac{\mathbf{f}_{rs}^{t} \text{ or } \mathbf{s}_1 \, \mathbf{s}_r}{RS} = \frac{0.24}{0.43}$$

$= \underline{0.558 \text{ rad/s}^2}$ clockwise

Example 3.13 *One cylinder of a rotary engine is shown in the configuration diagram shown in Fig. 3.20(a). OA is the fixed crank, 200 mm long. OP is the connecting rod and is 520 mm long. The line of stroke is along AR and at the instant is inclined at 30° to the vertical. The body of the engine consisting of cylinders rotates at a uniform speed of 400 rpm about the fixed centre A. Determine the*

(i) acceleration of piston (slider) inside the cylinder

(ii) angular acceleration of the connecting rod

Table 14

SN	Vector	Magnitude (m/s²)	Direction	Sense
1.	$\mathbf{f}_{po}^{c}$ or $\mathbf{o_1 \, p_o}$	$\frac{(\mathbf{op})^2}{OP} = \frac{(0.6)^2}{0.24} = 1.5$	‖ *OP*	→ *O*
2.	$\mathbf{f}_{po}^{t}$ or $\mathbf{p_o \, p_1}$	$\alpha_{op} \times OP = 20 \times 0.24 = 0.48$	⊥ *OP* or ‖ **op**	→ **o**
3.	$\mathbf{f}_{qp}^{cr}$ or $\mathbf{p_1 \, q_p}$	$2\,\omega_{ba}\,\mathbf{v}_{qp} = 0.38$*	⊥ *AQ*	Refer *
4.	$\mathbf{f}_{qp}^{s}$ or $\mathbf{q_p \, q_1}$	-	‖ *AQ*	-
5.	$\mathbf{f}_{qa}^{c}$ or $\mathbf{a_1 \, q_a}$	$\frac{(\mathbf{aq})^2}{\mathbf{AQ}} = \frac{(0.585)^2}{0.365} = 0.93$	‖ *AQ*	→ *A*
6.	$\mathbf{f}_{qa}^{t}$ or $\mathbf{q_a \, q_1}$	-	⊥ *AQ*	-

$$^{*}\mathbf{f}_{qp}^{cr} = 2\omega_{ba} v_{qp} = 2\frac{v_{qa}}{\mathbf{QA}} v_{qp} \quad (\omega_{ba} = \omega_{qa}) = 2 \times \frac{0.585}{0.365} \times 0.118 \ (v_{qa} = \mathbf{aq} = 0.585) = 0.38 \text{ m/s}^2$$

$$= 2 \times \frac{1.95}{0.16} \times 1.85 = 45.1 \text{ m/s}^2$$

ω_{qe} is found to be counter-clockwise.

The direction for Coriolis component is taken by rotating v_{qe} through 90° in the direction of angular movement of the link *QE* (counter-clockwise in this case). The acceleration diagram is completed as usual.

Acceleration of sliding of link *EF* in the trunnion $= \mathbf{q_o q_1} = \underline{4.86 \text{ m/s}^2}$

This shows that it is downwards or opposite to the velocity. Thus, it is deceleration.

Example 3.15 *In the pump mechanism shown in Fig. 3.22(a), OA = 320 mm, AC = 680 mm and OQ = 650 mm. For the given configuration, determine*

(i) linear (sliding) acceleration of slider C relative to cylinder walls

(ii) angular acceleration of the piston rod

Solution The velocity diagram has been reproduced in Fig. 3.22(b) from Example 2.15.

The problem can be solved by either of the two methods discussed for velocity diagram in Example 2.15.

Writing the acceleration vector equation for the latter configuration,

$$\mathbf{f}_{aq} = \mathbf{f}_{ab} + \mathbf{f}_{bq}$$

or

$$\mathbf{f}_{ao} = \mathbf{f}_{bq} + \mathbf{f}_{ab}$$

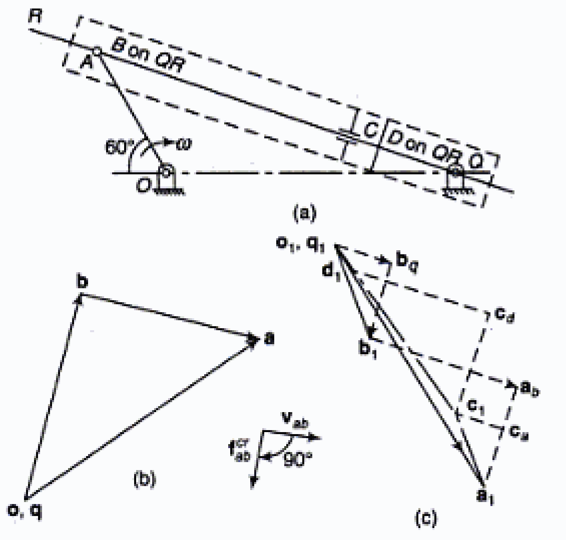


Fig. 3.22

$$= \mathbf{f}^c_{bq} + \mathbf{f}^t_{bq} + \mathbf{f}^s_{ab} + \mathbf{f}^{cr}_{ab}$$

or $\quad \mathbf{o_1 a_1} = \mathbf{q_1 b_q} + \mathbf{b_q b_1} + \mathbf{b_1 a_b} + \mathbf{a_b a_1}$

Set the vector table (Table 17)

The direction of $\mathbf{f}^{cr}_{ab}$ is obtained by rotating $\mathbf{v}_{ba}$ through 90° in the direction of ω_{bq} (clockwise).

Draw the acceleration diagram as given below:

1. Take the first vector [Fig. 3.22(c)].
2. From point $\mathbf{q_1}$ (pole point), take the second vector and through the head of it, draw a line perpendicular to *QB* for the third vector.

Table 17

SN	Vector	Magnitude (m/s²)	Direction	Sense
1.	$\mathbf{f}_{ao}$ or $\mathbf{o_1 a_1}$	$\frac{(\mathbf{oa})^2}{OA} = \frac{(6.4)^2}{0.32} = 128$	$\parallel OA$	$\rightarrow O$
2.	$\mathbf{f}^c_{bq}$ or $\mathbf{q_1 b_q}$	$\frac{(\mathbf{bq})^2}{BQ} = \frac{(4.77)^2}{0.85} = 26.8$	$\parallel QB$	$\rightarrow Q$
3.	$\mathbf{f}^t_{bq}$ or $\mathbf{b_q b_1}$	-	$\perp QB$	-
4.	$\mathbf{f}^s_{ab}$ or $\mathbf{b_1 f_b}$	-	$\parallel QB$	-
5.	$\mathbf{f}^{cr}_{ab}$ or $\mathbf{a_b a_1}$	47.1*	$\perp QB$	-

$$*\mathbf{f}^{cr}_{ab} = 2\omega_{rq} v_{ab} = 2\frac{v_{bq}}{BQ}\mathbf{ba} \quad (\omega_{rq} = \omega_{bq}) = 2 \times \frac{4.77}{0.85} \times 4.2 = 47.1 \text{ m/s}^2$$

3. Take the fifth vector from the point $\mathbf{a}_1$ such that the vector is in the proper direction and sense.
4. For the fourth vector, draw a line parallel to QB through the tail $\mathbf{a}_b$ of the fifth vector.

The intersection of the lines drawn in steps (2) and (4) locates point $\mathbf{b}_1$.

Let D be a point on QB beneath the point C.

Acc. of C rel. to cylinder walls

$= f_{cd}^s = f_{ab}^s = \underline{72 \text{ m/s}^2}$

Writing the vector equation,

$$\mathbf{f}_{cq} = \mathbf{f}_{cd} + \mathbf{f}_{dq}$$
$$= \mathbf{f}_{dq} + \mathbf{f}_{cd}$$
$$= \mathbf{f}_{dq} + \mathbf{f}_{cd}^s + \mathbf{f}_{cd}^{cr}$$

or $$\mathbf{q}_1\mathbf{c}_1 = \mathbf{q}_1\mathbf{d}_1 + \mathbf{d}_1\mathbf{c}_d + \mathbf{c}_d\mathbf{d}_1$$

$$\mathbf{f}_{cd}^s = \mathbf{f}_{ab}^s = \underline{72 \text{ m/s}^2} \text{ parallel to } QB.$$
$$\mathbf{f}_{cd}^{cr} = 2\omega_{rq} v_{cd} = 2\omega_{rq} v_{ab} = f_{ab}^{cr}$$

Locate point $\mathbf{d}_1$ on $\mathbf{q}_1\mathbf{b}_1$ such that $\dfrac{\mathbf{q}_1\mathbf{d}_1}{\mathbf{q}_1\mathbf{b}_1} = \dfrac{QD}{QB}$

To the vector $\mathbf{f}_{dq}$, add the vector $\mathbf{f}_{cd}^s$ and then $\mathbf{f}_{cd}^{cr}$.

Thus, the point $\mathbf{c}_1$ is located.

Join $\mathbf{a}_1$ to $\mathbf{c}_1$.

$\mathbf{a}_1\mathbf{c}_1$ represents total acceleration of C relative to A. This has two components.

$$\mathbf{f}_{cd}^t \perp \text{ to } CA = \mathbf{a}_1\mathbf{c}_a$$

and $$\mathbf{f}_{ca}^c \parallel \text{ to } CA\ (= \mathbf{c}_a\mathbf{c}_1)$$

$$\alpha_{ca} = \frac{\mathbf{f}_{ca}^t}{CA} = \frac{14}{0.68} = \underline{20.59 \text{ rad/s}^2}$$

counter-clockwise

Example 3.16 *The dimensions of a four-link mechanism are as under [Fig. 3.23(a)]:*
AB = 35 mm, BC = 40 mm, CD = 45 mm and AD = 70 mm. At the instant when ∠DAB= 75°, the link AB rotates with an angular velocity of 10 rad/s in the counter-clockwise direction. Given the coincident points P attached to the link CD and Q attached to the link BC such that BQ = 30 mm and CQ = 20 mm. Determine the acceleration of P relative to Q (or the link BC).

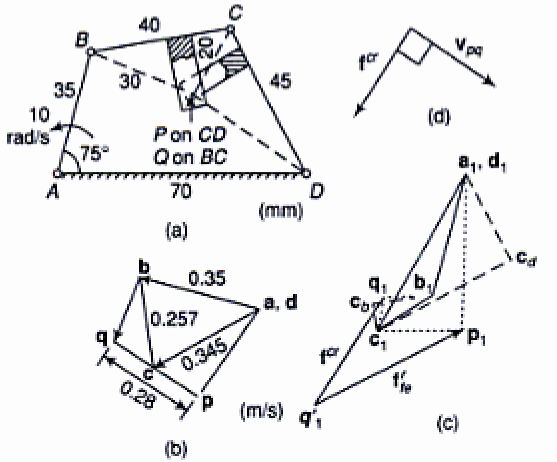


Fig. 3.23

Solution $v_b = 10 \times 0.035 = 0.35$ m/s

Complete the velocity diagram as shown in Fig. 3.23(b). Point Q can be located by drawing the velocity image of the triangle BCQ and P by drawing the velocity image of the triangle DCP.

For acceleration diagram, we have,

Acc. of C rel. to A = Acc. of C rel. to B + Acc. of B rel. to A

$$\mathbf{f}_{ca} = \mathbf{f}_{cb} + \mathbf{f}_{ba}$$

or $$\mathbf{f}_{cd} = \mathbf{f}_{ba} + \mathbf{f}_{cb}$$

or $$\mathbf{d}_1\mathbf{c}_1 = \mathbf{a}_1\mathbf{b}_1 + \mathbf{b}_1\mathbf{c}_1$$

Writing in terms of components,

$\therefore$ $$\mathbf{f}_{cd}^c + \mathbf{f}_{cd}^t = \mathbf{f}_{ba}^c + \mathbf{f}_{cb}^c + \mathbf{f}_{cb}^t$$

or $$\mathbf{d}_1\mathbf{c}_d + \mathbf{c}_d\mathbf{c}_1 = \mathbf{a}_1\mathbf{b}_1 + \mathbf{b}_1\mathbf{c}_b + \mathbf{c}_b\mathbf{c}_1$$

Set the following vector table (Table 18):

Draw the acceleration diagram $\mathbf{a}_1\mathbf{b}_1\mathbf{c}_1\mathbf{d}_1$ following the steps of Example 3.1. Locate the point $\mathbf{p}_1$ by drawing the acceleration image of the triangle DCP on the vector $\mathbf{c}_1\mathbf{d}_1$ and $\mathbf{q}_1$ by drawing the acceleration image of the triangle BCQ on the vector $\mathbf{b}_1\mathbf{c}_1$ [Fig.3.23(c)]

Now,

$\mathbf{f}_{pa}$ = Acc. of P rel. to Q + Acc. of Q rel. to A + Coriolis acceleration component

$$= \mathbf{f}_{pq} + \mathbf{f}_{qa} + \mathbf{f}^{cr}$$

In the above equation $\mathbf{f}_{pa}\ (\mathbf{a}_1\mathbf{p}_1)$ and $\mathbf{f}_{qa}\ (\mathbf{a}_1\mathbf{q}_1)$ are known and it is required to find $\mathbf{f}_{pq}$. Therefore, first we have to find $\mathbf{f}^{cr}$.

Table 18

SN	Vector	Magnitude (m/s²)	Direction	Sense
1.	f^c_{ba} or $a_1\ b_1$	$\frac{(ab)^2}{AB} = \frac{(0.35)^2}{0.035} = 3.5$	$\parallel AB$	$\rightarrow A$
2.	f^c_{cb} or $b_1\ c_b$	$\frac{(bc)^2}{BC} = \frac{(0.257)^2}{0.04} = 1.65$	$\parallel BC$	$\rightarrow B$
3.	f^t_{cb} or $c_b\ c_1$	-	$\perp BC$	-
4.	f^c_{cd} or $d_1\ c_d$	$\frac{(dc)^2}{DC} = \frac{(0.345)^2}{0.045} = 2.65$	$\parallel DC$	$\rightarrow D$
5.	f^t_{cd} or $c_d\ c_1$	-	$\perp DC$	-

$$*f^{cr} = 2\omega_{bc}\, v_{pq}$$

$$= 2\frac{v_{bc}}{BC}\mathbf{qp}$$

$$= 2\times\frac{0.257}{0.04}\times 0.28 = 3.6 \text{ m/s}^2$$

Its direction is given by as shown in Fig. 3.23(d) by rotating the vector $\mathbf{v}_{pq}$ in the direction of angular velocity of BC which is clockwise in this case.

f_{pq} is represented by vector $\mathbf{q_1\ p_1}$ = 4.38 m/s²

This is the acceleration of P relative to Q (or the link BC), i.e, the acceleration which an observer stationed on link BC would report as the acceleration of the point P.

3.7 ALGEBRAIC METHODS

Let us consider the same system of a plane moving body having its motion relative to a fixed coordinate system xyz as was taken in Section 2.11. A moving coordinate system $x'y'z'$ is attached to this moving body as before (Fig. 3.24). Coordinates of the origin A of the moving system are known relative to the absolute reference system and the moving system has an angular velocity ω also.

To find the acceleration of P, a procedure similar to the one adopted for velocity is used here also.

Vector Approach

Equation 2.7 is

$$\mathbf{v}_p = \mathbf{v}_b + \mathbf{v}^R + \omega \times \mathbf{r}$$

Differentiating it to obtain the acceleration of P,

$$\dot{\mathbf{v}}_p = \dot{\mathbf{v}}_b + \dot{\mathbf{v}}^R + \dot{\omega}\times\mathbf{r} + \omega\times\dot{\mathbf{r}}$$

where $\dot{\omega}$ is the angular acceleration of rotation of the moving system.

and $\dot{\mathbf{v}}^R$ is obtained by differentiating $(\dot{x}'\mathbf{l} + \dot{y}'\mathbf{m} + \dot{z}'\mathbf{n})$,

i.e.,

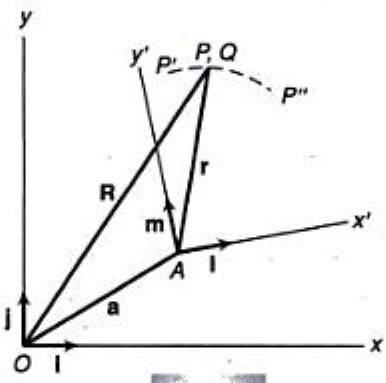


Fig. 3.24

$$\dot{v}^R = (\ddot{x}'l + \ddot{y}'m + \ddot{z}'n) + (\dot{x}'\dot{l} + \dot{y}'\dot{m} + \dot{z}'\dot{n})$$
$$= (\ddot{x}'l + \ddot{y}'m + \ddot{z}'n) + \omega(\dot{x}'l + \dot{y}'m + \dot{z}'n)$$
$$= f^R + \omega X v^R$$

$$\omega \times \dot{r} = \omega \times \frac{d}{dt}(x'l + y'm + z'n)$$
$$= \omega \times (\dot{x}'l + \dot{y}'m + \dot{z}'n) + \acute{E}(x'\dot{l} + y'\dot{m} + z'\dot{n})$$
$$= \omega \times v^R + \omega \times (\omega \times r)$$

But $\dot{v}_p = f_p$ and $\dot{v}_b = f_b$

Therefore,

$$\mathbf{f_p} = \mathbf{f_b} + (\mathbf{f^R} + \omega \times \mathbf{v^R}) + \dot{\omega} \times \mathbf{r} + [\omega \times \mathbf{v^R} + \omega \times (\omega \times \mathbf{r})]$$
$$= \mathbf{f_b} + \mathbf{f^R} + 2\omega \times \mathbf{v^R} + \dot{\omega} \times \mathbf{r} + \omega \times (\omega \times \mathbf{r}) \qquad \text{(i)}$$

Now absolute acceleration of *Q*, the coincident point may be written as

$$\mathbf{f}_{qa} = \mathbf{f}_{qb} + \mathbf{f}_{ba}$$
$$= \mathbf{f}_{ba} + \mathbf{f}_{qb}$$
$$= \mathbf{f_b} + \frac{\mathbf{d}}{\mathbf{dt}}(\omega \times \mathbf{r})$$
$$= \mathbf{f_b} + \dot{\omega} \times \mathbf{r} + \omega \times (\omega \times \mathbf{r})$$

Thus equation (i) reduces to

$$\mathbf{f_p} = \mathbf{f_{qa}} + \mathbf{f^R} + 2\omega \mathbf{X} \mathbf{v^R}$$
$$= \mathbf{f_{qa}} + \mathbf{f^R} + \mathbf{f^C}$$

where $\mathbf{f}_{qa}$ is the absolute acceleration of *Q*, $\mathbf{f^R}$ is the acceleration of *P* relative to the moving system or relative to *Q*, and $\mathbf{f^C}$ is known as the Coriolis component of acceleration.

The above equation may be written as

$$\mathbf{f}_{pa} = \mathbf{f}_{qa} + \mathbf{f}_{pq} + \mathbf{f}^c$$
$$\mathbf{f}_{pa} = \mathbf{f}_{pq} + \mathbf{f}_{qa} + \mathbf{f}^c \qquad (3.6)$$

Acc. of *P* rel. to *A* = Acc. of *P* rel. to *Q* + Acc. of *Q* rel. to *A* + Coriolis Acc.

Use of Complex Numbers

Equation 2.8 is

$$v = \dot{r}e^{i\theta} + ir\dot{\theta}\, e^{i\theta}$$

Differentiating it with respect to time,

$$\mathbf{f} = (\ddot{r}e^{i\theta} + \dot{r}i\,\dot{\theta}e^{i\theta}) + (ir\ddot{\theta}\,e^{i\theta} + i\dot{r}\dot{\theta}\,e^{i\theta} + i^2 r\dot{\theta}\,e^{i\theta})$$
$$= (\ddot{r}e^{i\theta} - r\dot{\theta}^2)e^{i\theta} + i(r\ddot{\theta} + 2\dot{r}\dot{\theta})e^{i\theta} \qquad (3.7)$$

The first part of this equation indicates the radial or *centripetal acceleration* and the second part, the *transverse acceleration* in polar coordinates.

$$\mathbf{f} = (f - \omega^2 r) + (r\alpha + 2\omega v)$$
$$= f + (r\alpha - \omega^2 r) + 2\omega v \qquad (3.8)$$

= Acc. of *P* rel. to *Q* + Acc. of *Q* rel. to *A* + Coriolis acceleration component i.e., the same equation as before.

3.8 KLEIN'S CONSTRUCTION

In Klein's construction, the velocity and the acceleration diagrams are made on the configuration diagram itself. The line that represents the crank in the configuration diagram also represents the velocity and the acceleration of its moving end in the velocity and the acceleration diagrams respectively. For a slider-crank mechanism, the procedure to make the Klein's construction is described below.

Slider-Crank Mechanism

In Fig. 3.25, *OAB* represents the configuration of a slider-crank mechanism. Its velocity and acceleration diagrams are as shown in Figs. 3.4(b) and (c). Let r be the length of the crank *OA*.

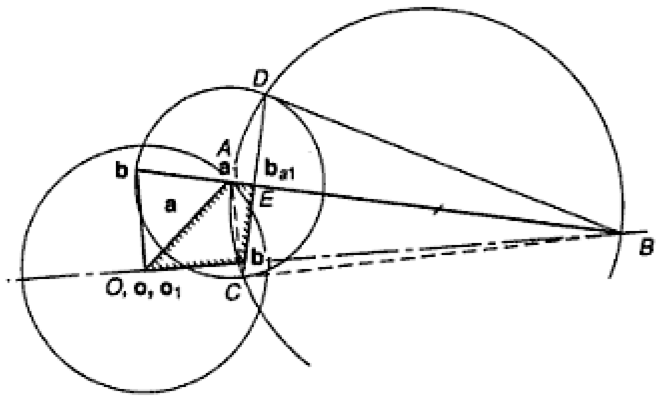


Fig. 3.25

Velocity Diagram For velocity diagram, let r represent v_{ao}, to some scale. Then for the velocity diagram, length $oa = \omega r = OA$.

From this, the scale for the velocity diagram is known.

Produce *BA* and draw a line perpendicular to *OB* through *O*. The intersection of the two lines locates the point **b**. the figure, **oab** is the velocity diagram which is similar to the velocity diagram of Fig. 3.4(b) rotated through 90° in a direction opposite to that of the crank.

Acceleration Diagram For acceleration diagram, let r represent f_{ao},

$$\therefore \quad \mathbf{o_1\, a_1} = \omega^2 r = OA$$

This provides the scale for the acceleration diagram.

Make the following construction:

1. Draw a circle with **ab** as the radius and **a** as the centre.
2. Draw another circle with *AB* as diameter.
3. Join the points of intersections *C* and *D* of the two circles. Let it meet *OB* at $\mathbf{b_1}$ and *AB* at *E*.

Then $\mathbf{o_1a_1b_ab_1}$ is the required acceleration diagram which is similar to the acceleration diagram of Fig. 3.4(c) rotated through 180°.

The proof is as follows:

Join *AC* and *BC*.

AEC and *ABC* are two right-angled triangles in which the angle *CAB* is common. Therefore, the triangles are similar.

$$\frac{AE}{AC} = \frac{AC}{AB} \quad \text{or} \quad AE = \frac{(AC)^2}{AB} \quad \text{or} \quad \mathbf{a_1\, b_a} = \frac{(\mathbf{ab})^2}{AB} = f_{ba}^c$$

Thus, this acceleration diagram has all the sides parallel to that of acceleration diagram of Fig. 3.4(c) and also has two sides $\mathbf{o_1a_1}$ and $\mathbf{a_1b_a}$ representing the corresponding magnitudes of the acceleration. Thus, the two diagrams are similar.

of the other. In the four-bar linkage of Fig. 3.2(a), A on the fixed link 1 is the centre of curvature of B on the moving coupler 3. Then considering the inversion, i.e., assuming link 3 to be fixed and releasing the fixed link 1, B on the link 3 is also the centre of curvature of the point A on the link 1. The two points are known as the *conjugates* of each other. The distance between them is called the radius of curvature of either locus.

3.11 HARTMANN CONSTRUCTION

The Hartmann construction is a graphical method to find the location of the centre of curvature of the locus of a point on a moving body. Let there be two bodies having a relative planar motion between them. Consider two curvatures of the two actual centrodes (Section 2.16) in the region near the point of contact at the instant. Let a circle with centre O' represent the circle corresponding to the curvature of fixed centrode and O, the centre of circle corresponding to the curvature of the moving centrode (Fig. 3.27). For the sake of convenience, the two circles may be called the fixed and the moving centrodes. Let I be the point of contact of the two centrodes which is also the instantaneous centre. The centrode tangent and the centrode normal are also shown in the figure.

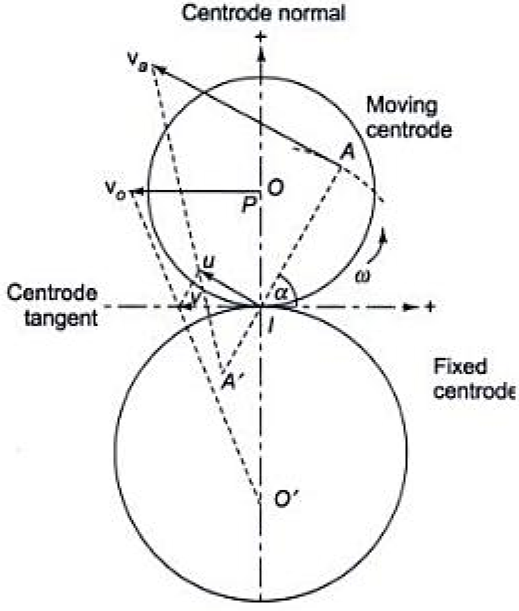


Fig. 3.27

Let the moving centrode roll on the fixed centrode with angular velocity ω. Then as I is also the instantaneous centre, the velocity of the point O is

$$v_o = \omega . OI$$

As the motion of moving centrode advances, the point of contact P moves along with some velocity v. Since at any instant, the line joining O with O' must pass through P, the velocity of P must be given by

$$v = \frac{IO'}{OO'} v_o$$

The velocity of any arbitrary point A on the moving centrode, i.e., a point on the coupler whose conjugate point is to be found is given by, $v_a = \omega . AI$.

To find the conjugate point of the point A, the Hartmann construction is as follows:

1. Take a vector representing the velocity $\mathbf{v}_0$ of the point O by drawing a line perpendicular to OI to a suitable scale. Also, take a vector representing $\mathbf{v}_a$, the velocity of A by taking a line perpendicular to AI and drawn to the same scale.
2. Draw the velocity vector $\mathbf{v}$ to indicate the velocity of the point I by drawing a line parallel to $\mathbf{v}_0$ and intersecting with the line joining O' with the end point of the vector $\mathbf{v}_0$.
3. Take a component of $\mathbf{v}$ parallel to $\mathbf{v}_a$. Let this vector be called $\mathbf{u}$.
4. Join end points of the vectors $\mathbf{v}_a$ and $\mathbf{u}$. Then the intersection of this line with AI provides the requisite conjugate point A', giving the radius of curvature of the locus of A as AA'.

3.12 EULER–SAVARY EQUATION

An analytical expression known as the *Euler–Savary equation* for the location of the conjugate point of A is derived as follows:

If α is the angle between the centrode tangent and the line AP (Fig. 3.27), then as $v = \dfrac{IO'}{OO'} v_o$,

$$u = v\sin\alpha = \frac{IO'}{OO'} v_o . \sin\alpha = \frac{IO'}{OO'} .(\omega . OP)\sin\alpha = \frac{IO'.OI}{OO'} .\omega . \sin\alpha \quad \text{(i)}$$

Also,

$$u = \frac{IA'}{AA'} v_a = \frac{IA'}{AA'} .(\omega . AI) = \frac{IA'.AI}{AA'} .\omega \quad \text{(ii)}$$

From (i) and (ii), $\dfrac{IO'.OI}{OO'} .\omega . \sin\alpha = \dfrac{IA'.AI}{AA'} .\omega$

or

$$\frac{AA'}{AI.IA'} \sin\alpha = \frac{OO'}{OI.IO'}$$

or

$$\left(\frac{AI}{AI.IA'} + \frac{IA'}{AI.IA'}\right)\sin\alpha = \frac{OI}{OI.IO'} + \frac{IO'}{OI.IO'}$$

or

$$\left(\frac{1}{IA'} + \frac{1}{AI}\right)\sin\alpha = \frac{1}{IO'} + \frac{1}{OI}$$

or

$$\left(\frac{1}{AI} - \frac{1}{A'I}\right)\sin\alpha = \frac{1}{OI} - \frac{1}{O'I} \quad \text{(iii)}$$

This is known as one form of the Euler–Savary equation. This is useful to locate the conjugate point A' of the point A when the radii of curvature of the two centrodes are known.

For any other point B at an angle β with the centrode tangent whose conjugate point is B' (Fig. 3.28), the above equation may be written as

$$\left(\frac{1}{BI} - \frac{1}{B'I}\right)\sin\beta = \frac{1}{OI} - \frac{1}{O'I}$$

Let this point be a particular point in the moving centrode such that it satisfies the equation

$$\frac{\sin\beta}{BI} = \frac{1}{OI} - \frac{1}{O'I}$$

This means that the term $1/B'I$ is zero which indicates that the point B is such that its conjugate point lies at infinity on the line joining BI.

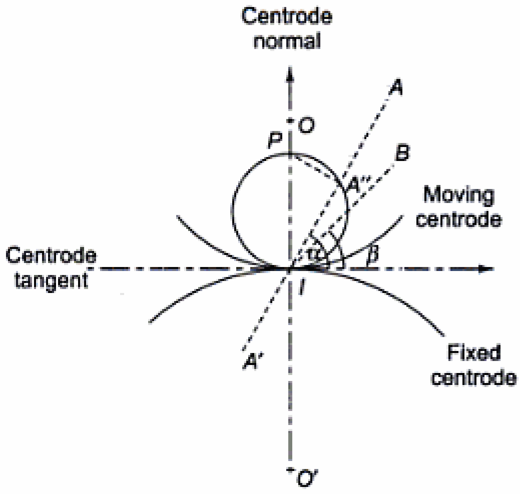


Fig. 3.28

Similarly, for a point P on the centrode normal whose conjugate point is at infinity on the line IO, $\dfrac{1}{PI} = \dfrac{1}{OI} - \dfrac{1}{O'I}$ as angle β is 90° and $\sin\beta$ is 1.

This also indicates that $PI = BI/\sin\beta$. The point P is known as the *inflection pole*.

Thus, to find more points whose conjugate points are at infinity, the equation $PI = BI/\sin\beta$ or $BI = PI/\sin\beta$ must be satisfied. This equation defines a circle whose diameter is IP as shown in Fig. 3.28. The circle is known as the *inflection circle*. Each point on this circle has an infinite radius of curvature at the instant and its conjugate point lies at infinity.

Thus, on the line AI, the point A'' intersecting the circle indicates that its conjugate point is at infinity and thus,

$$\frac{\sin\alpha}{A''I} = \frac{1}{OI} - \frac{1}{O'I} \tag{iv}$$

From (iii) and (iv), $\dfrac{1}{A''I} = \dfrac{1}{AI} - \dfrac{1}{A'I}$ or $\dfrac{1}{A''I} = \dfrac{A'I - AI}{AI.A'I}$ or $\dfrac{1}{A''I} = \dfrac{A'I + IA}{AI.A'I}$

or $\quad AI.A'I = A''I.A'A$

$\quad AI(A'A - IA) = (AI - AA'')A'A$

or $\quad AI.A'A - AI.IA = AI.A'A - AA''.A'A$

or $\quad AI.AI = AA''.AA'$

or $$AI^2 = AA''.AA' \tag{3.9}$$

This is the second form of the Euler–Savary equation and is more useful than the first form as this does not require knowing the curvatures of the two centrodes. However, it requires drawing the inflection circle which can easily be drawn.

In applying the above equation, AA' and AA'' are to lie on the same side of A.

Example 3.17 *A slider-crank mechanism is shown in Fig. 3.29(a). The dimensions are: OA = 20 mm, AB = 25 mm, AD = 10 mm and DC = 10 mm.*

Draw the inflection circle for the motion of the coupler and find the instantaneous radius of curvature of the path of the coupler point C.

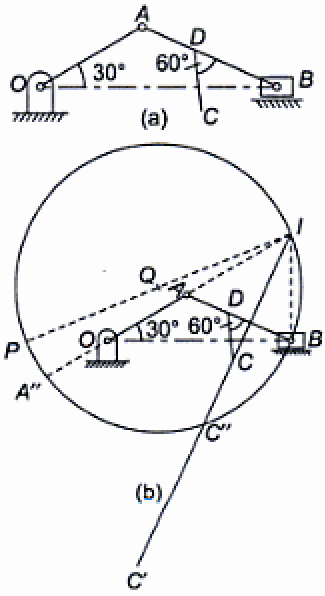


Fig. 3.29

Solution Locate the instantaneous centre of I at the intersection of OA and a line perpendicular to the direction of motion of the slider [Fig. 3.29(b)]. Apart from I, the point B also lies on the inflection circle as its centre of curvature is at infinity. One more point is needed to draw the inflection circle which can be obtained as follows:

As O is the centre of curvature of the point A, extend AO to A'' such that

$AA'' = \dfrac{AI^2}{AO} = \dfrac{26.7^2}{20} = 35.6$ mm (on measurement $AI = 26.7$ mm) (Eq. 3.9)

Locate A'' as shown in the figure on the same side of A as A'. Thus, A'' is a point whose centre of curvature is at infinity.

Now draw a circle passing through points I, B and A'' by taking right bisectors of IB and BA'' (not shown in the figure) intersecting at the centre Q of the circle.

Diameter of the inflection circle, $IP = 62.5$ mm

To find the centre of curvature of the point C on the coupler, join IC intersecting the inflection circle at C''. Then C'' is a point having centre of curvature at infinity as this point lies on the inflection circle. Locate a point on IC or its extension such that

$$CC' = \frac{CI^2}{CC''} = \frac{29.9^2}{15.9} = 52.9 \text{ mm}$$

Locate C' as shown in the figure on the same side of C as C''. Then C' is the requisite centre of curvature of the point C.

3.13 BOBILLIER CONSTRUCTION

This is another graphical method by which inflection circle can be drawn without requiring the curvatures of the centrodes.

Let A and B be two points on the moving body which are not collinear with I (Fig. 3.30). Let A' and B' be their conjugate points respectively at the instant. Join AB and $A'B'$ and let their intersection be at Q. Then the line passing through I and Q is known as the *collineation axis*. This axis is specific for the two rays AA' and BB' and for another set of points A and B. Even on these rays, Q will have a different location and thus a different collineation axis.

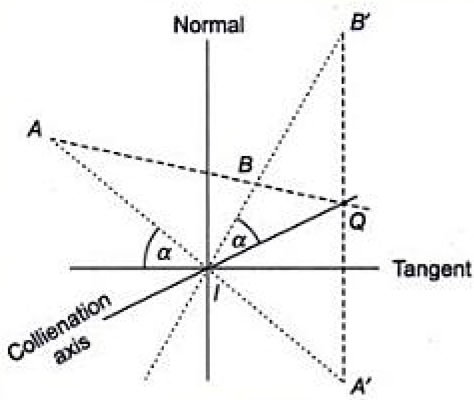


Fig. 3.30

Bobillier theorem It states that *the angle subtended by one of the rays (AA′ or BB′) with the centrode tangent is equal to negative of the angle subtended by the other ray with the collineation axis.*

In Fig. 3.30, the ray AA' subtends angle α with the centrode tangent and the ray BB' subtends the same negative angle with the collineation axis.

Proof

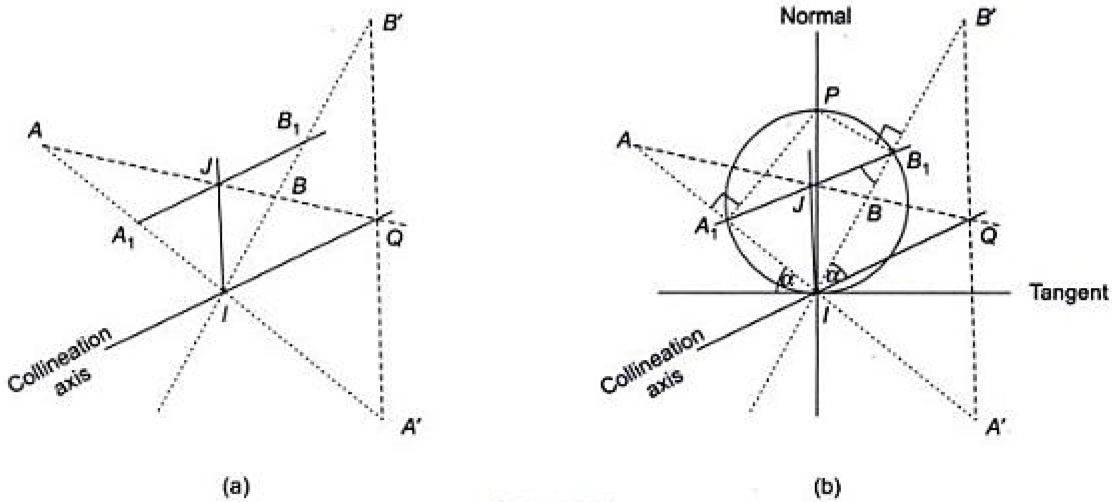


Fig. 3.31

Let A and A', and B and B' be the known pairs of conjugate points [Fig. 3.31(a)].

Make the following construction:

1. Locate the point I, the instantaneous centre of velocity at the intersection of two rays AA' and BB'.
2. Locate the point Q at the intersection of rays AB and $A'B'$.
3. Join IQ to obtain the collineation axis.
4. Draw a line parallel to $A'B'$ intersecting AB at J.
5. Draw a line parallel to IQ through J intersecting AA' and BB' at A_1 and B_1 respectively.
6. Draw a circle passing through I, A_1 and B_1 (Fig. 3.31b). A convenient way of drawing the circle is by drawing $A_1P \perp AA'$ and $B_1P \perp BB'$ intersecting two perpendicular lines at P. Now IP is the diameter of the inflection circle as it subtends a 90° angle at points A_1 and B_1 indicating that A_1 and B_1 are the points in the semicircles with diameter IP. Thus, P is the inflection pole. Draw the circle with IP as the diameter.
7. As IP is also the centrode normal, draw the centrode tangent as shown in the figure.

Let α be the angle which IA_1 subtends with the centrode tangent. Now, arc IA_1 is inscribed by the chord IA_1 which is at an angle α with the centrode tangent and subtends the angle IB_1A_1 at the circumference of the inflection circle. Therefore, the angle IB_1A_1 is also equal to α. As A_1B_1 is parallel to PQ and is intersected by IB', the angle IB_1A_1 is also equal to the angle QIB_1 i.e., equal to α. Thus, the angle subtended by one of the rays with the centrode tangent is equal to the negative of the angle subtended by the other ray with the collineation axis. Thus the construction satisfies the Bobillier theorem.

Method to find a conjugate point of another arbitrary point

Let the inflection circle be drawn and the centrode tangent and normal be known and it is required to find the conjugate point of C (Fig. 3.32). The point P is the inflection pole, i.e., its conjugate point P' lies at infinity and thus the ray PP' is perpendicular to the tangent to the centrode tangent. This suggestes that according to the Bobillier theorem, the other ray CC' will be perpendicular to the collineation axis. But as the point C' must lie on IC, the collineation axis can be drawn by drawing a line perpendicular to IC at I. Since Q is a point of intersection of two rays PC and $P'C'$, it can be located at the intersection of PC and the collineation axis. Now as Q also lies on $P'C'$, joining of $P'Q$ means a line parallel to IP, the intersection of this line with IC locates the point C'.

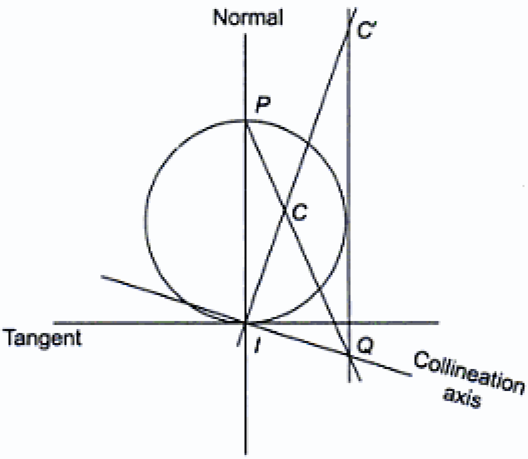


Fig. 3.32

Thus, the procedure to find the conjugate point C' of any arbitrary point C is as follows:

- Draw the collineation axis by drawing a line perpendicular to IC through I. Locate Q at the intersection of PC with the collineation axis.
- Draw a line parallel to IP through Q intersecting the line IC at C', the requisite conjugate point of C.

Example 3.18 *Use the Bobillier theorem to determine the centre of curvature of the coupler curve of the point E of the four-bar mechanism shown in Fig. 3.33(a). The dimensions are AD = AB = 60 mm, BC = CD = 25 mm. AD is the fixed link and E is the midpoint of BC.*

3. The rate of change of velocity along the radial direction is known as the *centripetal* or *radial acceleration*, the direction being towards the centre of rotation.
4. The angular acceleration of a link about one extremity is the same in magnitude and direction as the angular acceleration about the other and is found by dividing the tangential acceleration with the length of the link.
5. *Acceleration images* are helpful to find the accelerations of offset points of the links. The acceleration image of a link is obtained in the same manner as a velocity image.
6. Acceleration of a point on a link relative to a coincident point on a moving link is the sum of absolute acceleration of the coincident point, acceleration of the point relative to coincident point and the *Coriolis* acceleration.
7. The *Hartmann construction* is a graphical method to find the location of the centre of curvature of the locus of a point on the moving body.
8. The *Euler–Savary equation* is expressed as $AI^2 = AA''.AA'$
9. The Bobillier construction is another graphical method by which an inflection circle can be drawn without requiring the curvatures of the centrodes.
10. The *Bobillier theorem* states that *the angle subtended by one of the rays (AA' or BB') with the centrode tangent is equal to the negative of the angle subtended by the other ray with the collineation axis.*
11. The locus of all such points on the coupler which have stationary curvature at the instant is known as the *cubic of the stationary curvature* or the *circling-point curve*.

Exercises

1. What are centripetal and tangential components of acceleration? When do they occur? How are they determined?
2. Describe the procedure to draw velocity and acceleration diagrams of a four-link mechanism. In what way are the angular accelerations of the output link and the coupler found?
3. What is an acceleration image? How is it helpful in determining the accelerations of offset points on a link?
4. What is the Coriolis acceleration component? In which cases does it occur? How is it determined?
5. Explain the procedure to construct Klein's construction to determine the velocity and acceleration of a slider-crank mechanism.
6. Explain the term conjugates in relation to two points on two plain bodies.
7. Explain the Hartmann construction to find the location of the centre of curvature of the locus of a point on a moving body.
8. What is Euler–Savary equation? What are its two forms? Explain how these are used to find the location of conjugate points.
9. Use the Bobillier theorem to show that the inflection circle can be drawn without requiring the curvatures of the centrodes.
10. Define the term *cubic of the stationary curvature*. Explain one graphical method to draw it.
11. A crank and rocker mechanism *ABCD* has the following dimensions:

    AB = 0.75 m, BC = 1.25 m, CD = 1 m, AD = 1.5 m.
    BE = 437.5 mm, CE = 87.5 mm and CF = 500 mm
    E and F are two points on the coupler link BC. AD is the fixed link. BEC is read clockwise and F lies on BC produced. Crank AB has an angular velocity of 20.94 rad/s counter-clockwise and a deceleration of 280 rad/s² at the instant $\angle DAB = 60°$. Find the
    (i) instantaneous linear acceleration of C, E and F
    (ii) instantaneous angular velocities and accelerations of links BC and CD

    [(i) 166 m/s², 330 m/s², 161 m/s² (ii) ω_{bc} = 5.92 rad/s cw, ω_{cd} = 11.5 rad/s ccw, ω_{bc} = 229 rad/s² ccw, α_{cd} = 100 rad/s² ccw]

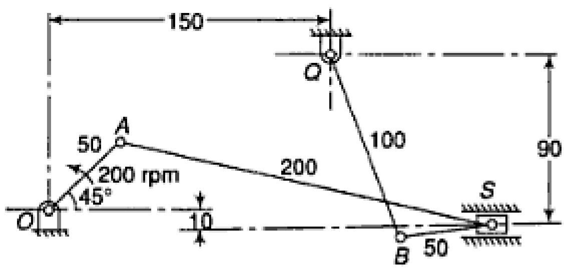


Fig. 3.35

12. Figure 3.35 shows a mechanism in which O and Q are the fixed centres. Determine the acceleration of the slider S and the angular acceleration of the link BQ for the given configuration.

    (14.5 m/s² towards left; 114 rad/s² cw)

13. In a simple steam engine, the lengths of the crank and the connecting rod are 100 mm and 400 mm respectively. The weight of the connecting rod is 50 kg and its centre of mass is 220 mm from the cross-head centre. The radius of gyration about the centre of mass is 120 mm. If the engine speed is 300 rpm, determine for the position when the crank has turned 45° from the inner-dead centre,
   (i) the velocity and acceleration of the centre of mass of the connecting rod
   (ii) the angular velocity and acceleration of the rod
   (iii) the kinetic energy of the rod
   [(i) 2.7 m/s, 80 m/s² (ii) 5.7 rad/s, 173 rad/s² (iii) 194 N.m]
14. From the data of a reciprocating pump given in Example 2.4, find the linear acceleration of the cross-head *E* and the angular accelerations of the links *BCD* and *DE*.
   [9.25 m/s²; 60.8 rad/s²; 5.12 rad/s²]
15. Figure 3.36 shows a toggle mechanism in which the crank OA rotates at 120 rpm. Find the velocity and the acceleration of the slider at D.
   (0.17 m/s; 0.83 m/s²)

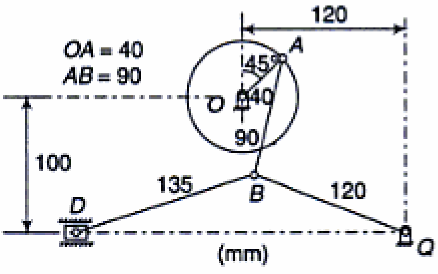


Fig. 3.36

16. In a crank and slotted-lever quick-return mechanism (Fig. 3.15a), the distance between the fixed centres *O* and *A* is 250 mm. Other lengths are: *OP* = 100 mm, *AR* = 400 mm, *RS* = 150 mm and ∠*AOP* = 120°. Uniform speed of the crank is 60 rpm clockwise. Line of stroke of the ram is perpendicular to *OA* and is 450 mm above *A*. Calculate the velocity and the acceleration of the ram *S*. **(0.64 m/s; 1.55 m/s²)**
17. For the inverted slider-crank mechanism of Example 2.13, determine the angular acceleration of the link *QR*. **(358 rad/s)**
18. In the pump mechanism shown in Fig. 3.22(a), the crank *OA* is 50 mm long and the piston rod *AC* is 150 mm long. The lengths *OQ* and *CQ* are 250 mm and 80 mm respectively. The crank rotates at 300 rpm in the clockwise direction. Determine the
   (a) velocity of the piston relative to walls
   (b) angular velocities of rod *AC* and the cylinder
   (c) sliding acceleration of the piston relative to cylinder
   (d) velocity of piston (absolute)
   (e) angular acceleration of the piston rod *BC*
   [(a) 1.51 m/s (b) 2.06 rad/s ccw of both, rod *AC* and cylinder (c) 16 m/s² (d) 1.5 m/s (e) 239 rad/s² ccw]

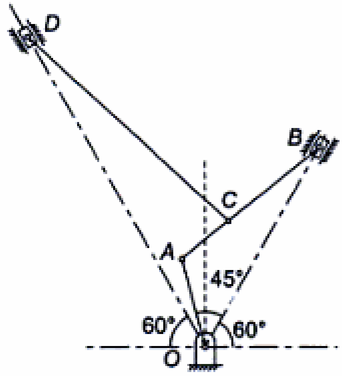


Fig. 3.37

19. In the mechanism shown in Fig. 3.37, the crank *OA* drives the sliders *B* and *D* in straight paths through connecting links *AB* and *CD*. The lengths of the links are *OA* = 150 mm, *AB* = 300 mm, *AC* = 100 mm, *CD* = 450 mm. *OA* rotates at 60 rpm clockwise and at the instant has angular retardation of 16 rad/s². Determine (i) the velocity and acceleration of sliders *B* and *D*, and (ii) the angular velocity and angular acceleration of link *CD*.
   (0.92 m/s, 0.31 m/s, 5.55 m/s², 5.49 m/s²; 2.07 rad/s, 6.53 rad/s²)

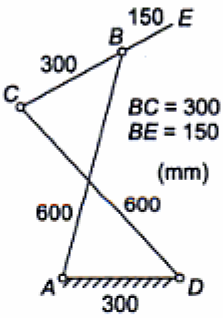


Fig. 3.38

20. For the motion of the coupler relative to the fixed link of the four-link mechanism as shown in Fig. 3.38, locate the position of the centre of curvature of the point *E* using the Bobillier theorem.

4 COMPUTER-AIDED ANALYSIS OF MECHANISMS

Introduction

The analyses of the velocity and the acceleration, given in chapters 2 and 3, depend upon the graphical approach and are suitable for finding out the velocity and the acceleration of the links of a mechanism in one or two positions of the crank. However, if it is required to find these values at various configurations of the mechanism or to find the maximum values of maximum velocity or acceleration, it is not convenient to draw velocity and acceleration diagrams again and again. In that case, analytical expressions for the displacement, velocity and acceleration in terms of the general parameters are derived. A desk-calculator or digital computer facilitates the calculation work.

4.1 FOUR-LINK MECHANISM

Displacement Analysis

A four-link mechanism shown in Fig. 4.1 is in equilibrium. a, b, c and d represent the magnitudes of the links AB, BC, CD and DA respectively. θ, β and φ are the angles of AB, BC and DC respectively with the x-axis (taken along AD). AD is the fixed link. AB is taken as the input link whereas DC as the output link.

As in any configuration of the mechanism, the figure must enclose, the links of the mechanism can be considered as vectors. Thus, vector displacement relationships can be derived as follows.

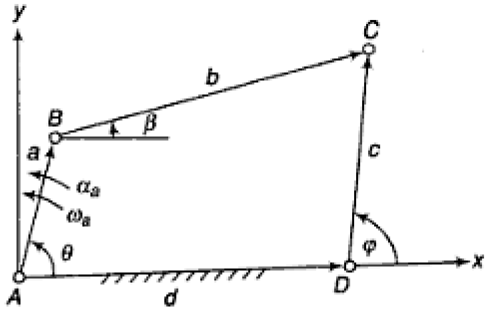


Fig. 4.1

Displacement along x-axis

$$a\cos\theta + b\cos\beta = d + c\cos\phi \tag{4.1}$$

(The equation is valid for $\angle\phi$ more than 90° also.)

or

$$b\cos\beta = c\cos\varphi - a\cos\theta + d$$

or

$$(b\cos\beta)^2 = (c\cos\varphi - a\cos\theta + d)^2$$

$$= c^2\cos^2\varphi + a^2\cos^2\theta + d^2 - 2ac\cos\theta\cos\varphi - 2ad\cos\theta + 2cd\cos\varphi \tag{4.2}$$

Displacement along y-axis

$$a \sin\theta + b \sin\beta = c \sin\phi \tag{4.3}$$

or
$$b \sin\beta = c \sin\varphi - a \sin\theta$$

or
$$(b \sin\beta)^2 = (c \sin\varphi - a \sin\theta)^2$$
$$= c^2 \sin^2\varphi + a^2 \sin^2\theta - 2ac \sin\theta \sin\varphi \tag{4.4}$$

Adding equations (4.2) and (4.4),

$$b^2 = c^2 + a^2 + d^2 - 2ac \cos\theta \cos\varphi - 2ad \cos\theta + 2cd \cos\varphi - 2ac \sin\theta \sin\varphi \tag{4.5}$$

Put

$$a^2 - b^2 + c^2 + d^2 = 2k$$

Then,

$$2cd \cos\varphi - 2ac \cos\theta \cos\varphi - 2ac \sin\theta \sin\varphi - 2ad \cos\theta + 2k = 0$$

or

$$cd \cos\varphi - ac \cos\theta \cos\varphi - ac \sin\theta \sin\varphi - ac \cos\theta + k = 0 \tag{4.6}$$

From trigonometric identities,

$$\sin\varphi = \frac{2\tan\left(\dfrac{\varphi}{2}\right)}{1+\tan^2\left(\dfrac{\varphi}{2}\right)}$$

$$\cos\varphi = \frac{1-\tan^2\left(\dfrac{\varphi}{2}\right)}{1+\tan^2\left(\dfrac{\varphi}{2}\right)}$$

$$cd\left[\frac{1-\tan^2(\varphi/2)}{1+\tan^2(\varphi/2)}\right] - ac\cos\theta\left[\frac{1-\tan^2(\varphi/2)}{1+\tan^2(\varphi/2)}\right] - ac\sin\theta\left[\frac{2\tan(\varphi/2)}{1+\tan^2(\varphi/2)}\right] - ad\cos\theta + k = 0$$

Multiplying throughout by $\left[1+\tan^2\left(\dfrac{\varphi}{2}\right)\right]$

$$cd - cd\tan^2\left(\frac{\varphi}{2}\right) - ac\cos\theta + ac\cos\theta\tan^2\left(\frac{\varphi}{2}\right) - 2ac\sin\theta\tan\left(\frac{\varphi}{2}\right)$$
$$- ad\cos\theta - ad\cos\theta\tan^2\left(\frac{\varphi}{2}\right) + k + k\tan^2\left(\frac{\varphi}{2}\right) = 0$$

$$[k - a(d-c)\cos\theta - cd]\tan^2\left(\frac{\varphi}{2}\right) + [-2ac\sin\theta]\tan\left(\frac{\varphi}{2}\right) + [k - a(d+c)\cos\theta + cd] = 0$$

or

$$A \tan^2\left(\frac{\varphi}{2}\right) + B \tan\left(\frac{\varphi}{2}\right) + C = 0$$

where

$$A = k - a\,(d - c)\cos\theta - cd$$
$$B = -\,2ac\sin\theta$$
$$C = k - a\,(d + c)\cos\theta + cd$$

Equation (4.6) is a quadratic in $\tan\left(\frac{\varphi}{2}\right)$. Its two roots are

$$\tan\left(\frac{\varphi}{2}\right) = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

or

$$\varphi = 2\tan^{-1}\left[\frac{-B \pm \sqrt{B^2 - 4AC}}{2A}\right] \tag{4.7}$$

Thus, the position of the output link, given by angle φ, can be calculated if the magnitude of the links and the position of the input link are known, i.e., a, b, c, d and θ are known.

A relation between the coupler link position β and the input link position θ can also be found as below:

Equations (4.1) and (4.3) can be written as,

$$c\cos\varphi = a\cos\theta + b\cos\beta - d \tag{4.8}$$

$$c\sin\varphi = a\sin\theta + b\sin\beta \tag{4.9}$$

Squaring and adding the two equations,

$$c^2 = a^2 + b^2 + d^2 + 2ab\cos\theta\cos\beta - 2bd\cos\beta - 2ad\cos\theta + 2ab\sin\theta\sin\beta$$

Put $\quad a^2 + b^2 - c^2 + d^2 = 2k'$

$$-\,2bd\cos\beta + 2ab\cos\theta\cos\beta + 2ab\sin\theta\sin\beta - 2ad\cos\theta + 2k' = 0$$

$$-\,bd\cos\beta + ab\cos\theta\cos\beta + ab\sin\theta\sin\beta - ad\cos\theta + k' = 0 \tag{4.10}$$

Equation (4.10) is identical to Eq. 4.6 and can be obtained from the same by substituting β for φ, $-b$ for c and k' for k.

Thus, the solution of Eq. (4.10) will be,

$$\beta = 2\tan^{-1}\left[\frac{-E \pm \sqrt{E^2 - 4DF}}{2D}\right] \tag{4.11}$$

where $\quad D = k' - a(d + b)\cos\theta + bd$

$$E = 2ab\sin\theta$$

$$F = k' - a(d - b)\cos\theta - bd$$

β can also be found directly from relation (4.3) after calculating φ.

```
#include<stdio.h>
  #include<conio.h>
  #include<math.h>
  void main()
  {
    int i,j,iht,th,theta,limit,ins;
    float a,b,c,d,vela,acca,thet,aa,bb,cc,bet1,bet2,betd1,
    betd2,num1,num2,phi1,ph1,unroot,undroot,pi,k,phh,phi2,
    ph2,vel2,dthet;
    float num[2],phi[2],ph[2],bet[2],betd[2],b1[2],b2[2],
    b3[2],b4[2],c1[2],c2[2],c3[2],c4[2],accc[2],accb[2],
    velb[2],velc[2];
    clrscr();

    printf("enter values a,b,c,d,vela,acca,theta,limit\n");
    scanf("%f%f%f%f%f%f%d%d",&a,&b,&c,&d,&vela,&acca,
    &theta,&limit);
    printf( "   thet  vela  acca     phi     beta ");
    printf( "   velc   velb   accc   accb \n");
    ins=1;
    if(vela==0 && acca>0)ins=0;

    pi=4*atan(1);
    iht=360/theta;
    if(vela>0 && acca==0){ins=0;iht=360/theta; }
    if(ins==1)iht=theta;
    dthet=pi*2/iht;
    if(vela==0 && acca>0)iht=iht+limit/theta;
      for(j=0;j<iht+1;j++)
      {
        if(j>(iht-360/theta-1) && ins==0)acca=0;
        thet=j*dthet;
        if(ins==1){j=iht; thet=theta*pi/180;}
        th=theta*j;
        if(ins==1)th=theta;
        k=(a*a-b*b+c*c+d*d)/2;
        aa=k-a*(d-c)*cos(thet)-c*d;
        bb=-2*a*c*sin(thet);
        cc=k-a*(d+c)*cos(thet)+c*d;
        unroot=bb*bb-4*aa*cc;
          if(unroot>0)
```

```
        {
          undroot=sqrt(unroot);
          num[0]=-bb+undroot;
          num[1]=-bb-undroot;
            for(i=0;i<2;i++)
            {
              phi[i]=atan(num[i]*.5/aa)*2;
              ph[i]=phi[i]*180/pi;
              bet[i]=asin((c*sin(phi[i])-a*sin(thet))/b);
              betd[i]=bet[i]*180/pi;
              velc[i]=(a*vela*sin(bet[i]-thet))/(c*sin(bet[i]
              -phi[i]));
              velb[i]=(a*vela*sin(phi[i]-thet))/(b*sin(bet[i]
              -phi[i]));
              c1[i]=a*acca*sin(bet[i]-thet);
              c2[i]=a*pow(vela,2)*cos(bet[i]-thet)+
              b*pow(velb[i],2);
              c3[i]=c*pow(velc[i],2)*cos(phi[i]-bet[i]);
              c4[i]=c*sin(bet[i]-phi[i]);
              accc[i]=(c1[i]-c2[i]+c3[i])/c4[i];
              b1[i]=a*acca*sin(phi[i]-thet);
              b2[i]=a*pow(vela,2)*cos(phi[i]-thet);
              b3[i]=b*pow(velb[i],2)*cos(phi[i]-bet[i])
              -c*pow(velc[i],2);
              b4[i]=b*sin(bet[i]-phi[i]);
              accb[i]=(b1[i]-b2[i]-b3[i])/b4[i];
              printf( "%6.2d %6.2f%8.2f %8.2f %8.2f %6.2f
              %6.2f %6.2f %6.2f\n",th,vela,acca,ph[i],betd[i],
              velc[i],velb[i],accc[i],accb[i]);
            }
        }
        vela=sqrt(vela*vela+2*acca*dthet);
      }
      getch();
  }
```

Fig. 4.2

Figure 4.2 shows a program in *C* for solving such a problem. The program can be used to find the angular velocities and accelerations of the output and coupler links for the following cases:

1. Link *AB* is a crank and rotates at uniform angular velocity. In this case, the acceleration of the input link will be zero. If the link *AB* is not a crank but a rocker, the program will make the calculations only for feasible cases.
2. Link *AB* is a crank and starts from the stationary position. In this case, the initial velocity is zero and a value of the acceleration has to be provided along with the limit of the angle up to which the acceleration continues. At that angle when the maximum velocity is attained, the acceleration automatically reduces to zero and the onward the crank starts rotating at constant angular velocity. Further, calculations are made for one complete revolution.
3. For instant values of input velocity and acceleration, only one calculation is made for that specified position.

Various input variables are

`a, b, c, d`	Magnitudes of links *AB, BC, CD* and *DA* respectively (mm)
`vela`	Angular velocity of the input link *AB* (m/s)
`acca`	Angular acceleration of the input link (m/s^2)(acceleration is taken positive, deceleration negative)
`theta`	The interval of the input angle, i.e., the results are to be taken with a difference of 10°, 20° or 30°, etc., starting from zero
`Limit`	Angle up to which acceleration continues (for the case 2; in the other cases any value may be given)

The output variables are

`thet`	Angular displacement of the input link *AB* (degrees)
`phi`	Angular displacement of the output link *DC* (degrees)
`beta`	Angular displacement of the coupler link *BC* (degrees)
`velc`	Angular velocity of the output link *DC* (rad/s)
`velb`	Angular velocity of the coupler link *BC* (rad/s)
`accc`	Angular acceleration of the output link (rad/s^2)
`accb`	Angular acceleration of the coupler link (rad/s^2)

The results are obtained in sets of two possible solutions for each position of the input link. In case the input *AB* is not a crank, the results are obtained for the possible positions only. The counter-clockwise direction is considered as positive and the clockwise as negative.

4.2 USE OF COMPLEX ALGEBRA

For a four-link mechanism, we can write

$$\mathbf{a} + \mathbf{b} - \mathbf{c} - \mathbf{d} = \mathbf{0} \tag{4.24}$$

Transforming it into complex polar form,

$$a\, e^{i\theta} + b\, e^{i\beta} - c\, e^{i\phi} - d = 0 \tag{4.25}$$

Now, we know, $e^{i\,\theta} = \cos\theta + i\sin\theta$

Thus, transforming this equation into complex rectangular form and separating the real and imaginary terms,

$$a\cos\theta + b\cos\beta = d + c\cos\phi \tag{4.26}$$

and

$$a\sin\theta + b\sin\beta = c\sin\phi \tag{4.27}$$

which are the same equations as 4.1 and 4.3 and thus can be solved to find β and θ.

Differentiating Eq. (4.25) with respect to t,

$$ia\dot{\theta}\, e^{i\theta} + ib\dot{\beta}\, e^{i\beta} - ic\dot{\varphi}\, e^{i\varphi} = 0 \tag{4.28}$$

or
$$ia\, \omega_a\, e^{i\,\theta} + ib\, \omega_b\, e^{i\,\beta} - ic\, \omega_c\, e^{i\varphi} = 0 \tag{4.29}$$

Again, transforming this equation into complex rectangular form and separating the real and imaginary terms,

$$a\, \omega_a \cos\theta + b\omega_b \cos\beta - c\, \omega_c \cos\varphi = 0 \tag{4.30}$$

$$-a\, \omega_a \sin\theta - b\omega_b \sin\beta + c\, \omega_c \sin\varphi = 0 \tag{4.31}$$

which are the same equations as 4.13 and 4.15 and thus can be solved to find ω_b and ω_c.

Differentiating Eq. (4.28) with respect to t,

$$ia(\ddot{\theta}e^{i\theta} + i\dot{\theta}^2 e^{i\theta}) + ib(\ddot{\beta}\; e^{i\beta} + i\dot{\beta}^2 e^{i\beta}) - ic(\ddot{\varphi}e^{i\varphi} + i\dot{\varphi}^2 e^{i\varphi}) = 0 \tag{4.32}$$

or
$$ia(\alpha_a e^{i\theta} + i\omega_a{}^2 e^{i\theta}) + ib(\alpha_b \ddot{\beta}\; e^{i\beta} + i\omega_b{}^2 e^{i\beta}) - ic(\alpha_c e^{i\varphi} + i\omega_c{}^2 e^{i\varphi}) = 0 \tag{4.33}$$

Transforming this equation into complex rectangular form and separating the real and imaginary terms,

$$-a\alpha_a \sin\theta - a\omega_a^2 \cos\theta - b\alpha_b \sin\beta - b\omega_b^2 \cos\beta + c\alpha_c \sin\varphi + c\omega_c^2 \cos\varphi = 0 \tag{4.34}$$

$$a\alpha_a \cos\theta - a\omega_a^2 \sin\theta + b\alpha_b \cos\beta - b\omega_b^2 \sin\beta - c\alpha_c \cos\varphi + c\omega_c^2 \sin\varphi = 0 \tag{4.35}$$

which are the same equations as 4.20 and 4.21 and can be solved as before.

4.3 THE VECTOR METHOD

We have

$$\mathbf{a} + \mathbf{b} - \mathbf{c} - \mathbf{d} = \mathbf{0}$$

Assuming that the angles β and ϕ have been determined by any of the above methods, differentiate the above equation with respect to time,

$$\omega_a \times \mathbf{a} + \omega_b \times \mathbf{b} - \omega_c \times \mathbf{c} = \mathbf{0} \qquad (a,\ b,\ c \text{ and } d \text{ are constants}) \tag{4.36}$$

Let $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$ and $\hat{\mathbf{c}}$ be the unit vectors along **a**, **b** and **c** vectors. In plane–motion mechanisms, all the angular velocities are in the **k** direction. Therefore,

$$a\omega_a\, (\mathbf{k} \times \hat{\mathbf{a}}) + b\, \omega_b\, (\mathbf{k} \times \hat{\mathbf{b}}) - c\, \omega_c\, (\mathbf{k} \times \hat{\mathbf{c}}) = \mathbf{0} \tag{4.37}$$

Take the dot product with $\hat{\mathbf{b}}$,

$$a\, \omega_a\, (\mathbf{k} \times \hat{\mathbf{a}}).\, \hat{\mathbf{b}} + b\omega_b\, (\mathbf{k} \times \hat{\mathbf{b}})\, \hat{\mathbf{b}} - c\, \omega_c\, (\mathbf{k} \times \hat{\mathbf{c}})\, \hat{\mathbf{b}} = 0$$

$$a\, \omega_a\, (\mathbf{k} \times \hat{\mathbf{a}}).\, \hat{\mathbf{b}} + 0 - c\omega_c\, (\mathbf{k} \times \hat{\mathbf{c}})\, \hat{\mathbf{b}} = 0$$

or
$$\omega_c = -\frac{a\omega_a\, (\mathbf{k} \times \hat{\mathbf{a}}).\, \hat{\mathbf{b}}}{c(\mathbf{k} \times \hat{\mathbf{b}}).\, \hat{\mathbf{b}}} \tag{4.38}$$

Taking the dot product with $\hat{\mathbf{c}}$,

$$a\, \omega_a\, (\mathbf{k} \times \hat{\mathbf{a}}).\, \hat{\mathbf{c}} + b\, \omega_b\, (\mathbf{k} \times \hat{\mathbf{b}}).\, \hat{\mathbf{c}} - c\, \omega_c\, (\mathbf{k} \times \hat{\mathbf{c}})\, \hat{\mathbf{c}} = 0$$

$$a\,\omega_a\,(\mathbf{k}\times\hat{\mathbf{a}}).\hat{\mathbf{c}} + b\,\omega_b\,(\mathbf{k}\times\hat{\mathbf{b}}).\hat{\mathbf{c}} - 0 = 0$$

or
$$\omega_b = -\frac{a\,\omega_a\,(\mathbf{k}\times\hat{\mathbf{a}}).\hat{\mathbf{c}}}{b(\mathbf{k}\times\hat{\mathbf{b}}).\hat{\mathbf{c}}} \tag{4.39}$$

It can be shown that Eqs 4.38 and 4.39 are the same as Eqs 4.16 and 4.17 as follows:

$$(\mathbf{k}\times\hat{\mathbf{a}}).\hat{\mathbf{b}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ \cos\theta & \sin\theta & 0 \end{vmatrix}.\hat{\mathbf{b}} = (-\sin\theta.\mathbf{i} + \cos\theta.\mathbf{j}).\hat{\mathbf{b}}$$

$$= (-\sin\theta.\mathbf{i} + \cos\theta.\mathbf{j}).(\cos\beta\,\mathbf{i} + \sin\beta\,\mathbf{j})$$

$$= -\sin\theta\cos\beta + \cos\theta\sin\beta$$

$$= \sin(\beta - \theta)$$

Similarly,
$$(\mathbf{k}\times\hat{\mathbf{c}}).\hat{\mathbf{b}} = \sin(\beta - \phi)$$

Therefore,
$$\omega_c = -\frac{a\omega_a\,(\mathbf{k}\times\hat{\mathbf{a}}).\hat{\mathbf{b}}}{c(\mathbf{k}\times\hat{\mathbf{c}}).\hat{\mathbf{b}}} = \frac{a\omega_\alpha\sin(\beta-\theta)}{c\sin(\beta-\varphi)} \tag{4.40}$$

In the same way,
$$w_b = -\frac{a\,\omega_a\,(\mathbf{k}\times\hat{\mathbf{a}}).\hat{\mathbf{c}}}{b(\mathbf{k}\times\hat{\mathbf{c}}).\hat{\mathbf{c}}} = -\frac{a\omega_\alpha\sin(\varphi-\theta)}{b\sin(\varphi-\beta)} \tag{4.41}$$

which are the same equations as equations 4.16 and 4.17.

Differentiating Eq. 4.36 with respect to time to get the accelerations,

$$\dot{\omega}_a\times\mathbf{a} + \omega_a\times(\omega_a\times\mathbf{a}) + \dot{\omega}_b\times\mathbf{b} + \omega_b\times(\omega_b\times\mathbf{b}) - \dot{\omega}_c\times\mathbf{c} - \omega_c\times(\omega_c\times\mathbf{c}) = \mathbf{0}$$

or

$$\alpha_a\times\mathbf{a} + \omega_a\times(\omega_a\times\mathbf{a}) + \alpha_b\times\mathbf{b} + \omega_b\times(\omega_b\times\mathbf{b}) - \alpha_c\times\mathbf{c} - \omega_c\times(\omega_c\times\mathbf{c}) = \mathbf{0}$$

or

$$a\,\alpha_a(\mathbf{k}\times\hat{\mathbf{a}}) - a\,\omega_a^2\,\hat{\mathbf{a}} + b\,\alpha_b(\mathbf{k}\times\hat{\mathbf{b}}) - b\,\omega_b^2\,\hat{\mathbf{b}} - c\alpha_c(\mathbf{k}\times\hat{\mathbf{c}}) + c\,\omega_c^2\,\hat{\mathbf{c}} = 0 \tag{4.42}$$

Take the dot product of this equation with $\hat{\mathbf{b}}$,

$$a\,\alpha_a(\mathbf{k}\times\hat{\mathbf{a}}).\hat{\mathbf{b}} - a\,\omega_a^2\,\hat{\mathbf{a}}.\hat{\mathbf{b}} + 0 - b\,\omega_b^2\,\hat{\mathbf{b}}.\hat{\mathbf{b}} - c\alpha_c(\mathbf{k}\times\hat{\mathbf{c}}).\hat{\mathbf{b}} + c\,\omega_c^2\,\hat{\mathbf{c}}.\hat{\mathbf{b}} = 0$$

$$\alpha_c = \frac{a\,\alpha_a(\mathbf{k}\times\hat{\mathbf{a}}).\hat{\mathbf{b}} - a\,\omega_a^2\,\hat{\mathbf{a}}.\hat{\mathbf{b}} - b\omega_b^2 + c\omega_c^2\,\hat{\mathbf{c}}.\hat{\mathbf{b}}}{c(\mathbf{k}\times\hat{\mathbf{c}}).\hat{\mathbf{b}}} \tag{4.43}$$

120	10.5	0.0	-145.28	-48.22	-0.77	3.20	26.64	4.54
120	10.5	0.0	123.49	26.44	3.77	-0.20	-0.66	21.44
160	10.5	0.0	-144.69	-36.44	1.12	2.75	29.94	-16.40
160	10.5	0.0	136.77	28.52	2.96	1.32	-22.41	23.99
200	10.5	0.0	-136.77	-28.52	2.96	1.32	22.41	-23.99
200	10.5	0.0	144.69	36.44	1.12	2.75	-29.94	16.40
240	10.5	0.0	-123.49	-26.44	3.77	-0.20	0.66	-21.44
240	10.5	0.0	145.28	48.22	-0.77	3.20	-26.64	-4.54
280	10.5	0.0	-110.16	-29.87	2.92	-1.62	-27.30	22.42
280	10.5	0.0	139.02	58.74	-2.51	2.03	-26.17	-31.04
320	10.5	0.0	-103.82	-38.99	0.07	-3.15	-56.46	-20.96
320	10.5	0.0	126.30	61.47	-4.06	-0.83	-15.50	-50.99

Fig. 4.4

4.4 SLIDER-CRANK MECHANISM

Figure 4.5 shows a slider-crank mechanism in which the strokeline of the slider does not pass through the axis of rotation of the crank. Angle β in clockwise direction from the x-axis is taken as negative.

Let e = eccentricity (distance CD).

Displacement along x-axis,

$$a \cos\theta + b \cos(-\beta) = d \tag{4.46}$$

or

$$b \cos\beta = d - a\cos\theta \tag{4.46a}$$

Displacement along y-axis,

$$a \sin\theta + b \sin(-\beta) + e \tag{4.47}$$

or

$$b \sin\beta = e - a\sin\theta \tag{4.47a}$$

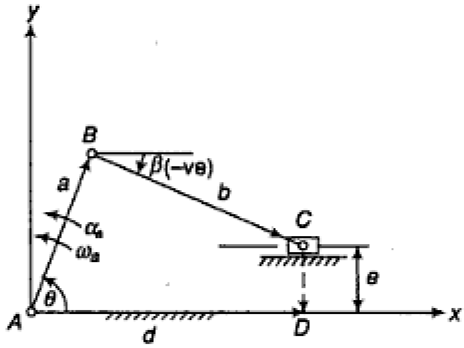


Fig. 4.5

Squaring Eqs (4.46a) and (4.47a) and adding,

$$b^2 = a^2\cos^2\theta + d^2 - 2ad\cos\theta + a^2\sin^2\theta + e^2 - 2ae\sin\theta$$
$$= a^2 + e^2 + d^2 - 2ae\sin\theta - 2ad\cos\theta$$

or

$$d^2 - (2a\cos\theta)d + a^2 - b^2 + e^2 - 2ae\sin\theta = 0$$

or

$$d^2 + C_1 d + C_2 = 0 \tag{4.48}$$

where $C_1 = -2a\cos\theta$

$C_2 = a^2 - b^2 + e^2 - 2ae\sin\theta$

Equation (4.48) is a quadric in d. Its two roots are,

$$d = \frac{-C_1 \pm \sqrt{C_1^2 - 4C_2}}{2} \tag{4.49}$$

Thus, if the parameters a, b, e and θ of the mechanism are known, the output displacement can be computed.

Also, from Eq. (4.47a),

$$\beta = \sin^{-1} \frac{e - a \operatorname{in} \theta}{b} \tag{4.50}$$

Velocity Analysis

Differentiating Eqs. (4.46) and (4.47) with respect to time,

$$-a\omega_a \sin\theta - b\,\omega_b \sin\beta - \dot{d} = 0 \tag{4.51}$$

$$a\omega_a \cos\theta + b\,\omega_b \cos\beta = 0 \tag{4.52}$$

Multiply Eq. (4.51) by $\cos\beta$ and Eq. (4.52) by $\sin\beta$ and add,

$$a\omega_a (\sin\beta \cos\theta - \cos\beta \sin\theta) - \dot{d}\cos\beta = 0$$

$$\dot{d} = \frac{a\omega_a \sin(\beta - \theta)}{\cos\beta} = \tag{4.53}$$

From Eq. (4.52),

$$\omega_b = -\frac{a\omega_a \cos\theta}{b\cos\beta} \tag{4.54}$$

ω_b provides the angular velocity of the coupler-link whereas $\dot{d}$ gives the linear velocity of the slider.

Acceleration Analysis

Differentiating Eqs (4.51) and (4.52) with respect to time,

$$-\left[a\alpha_a \sin\theta + a\omega_a^2 \cos\theta\right] - \left[b\alpha_b \sin\beta + b\omega_b^2 \cos\beta\right] - \ddot{d} = 0 \tag{4.55}$$

$$\left[a\alpha_a \cos\theta + a\omega_a^2 \sin\theta\right] - \left[b\alpha_b \cos\beta + b\omega_b^2 \sin\beta\right] = 0 \tag{4.56}$$

Multiply Eq. (4.55) by $\cos\beta$ and Eq. (4.56) by $\sin\beta$ and add,

or $$a\alpha_a \sin(\beta - \theta) - a\omega_a^2 \cos(\beta - \theta) - b\omega_b^2 - \ddot{d}\cos\beta = 0$$

$$\ddot{d} = \frac{a\alpha_a \sin(\beta - \theta) - a\omega_a^2 \cos(\beta - \theta) - b\omega_b^2}{\cos\beta} \tag{4.57}$$

From Eq. (4.56)

$$\alpha_b = \frac{a\alpha_a \cos\theta - a\omega_a^2 \sin\theta - b\omega_b^2 \sin\beta}{b\cos\beta} \tag{4.58}$$

α_b provides the angular acceleration of the coupler-link whereas $\ddot{d}$ gives the linear acceleration of the slider.

Figure 4.6 shows a program to solve this type of problem. It can be used for the same type of three cases as for the four-link mechanism.

```
#include<stdio.h>
#include<conio.h>
#inculde<math.h>
void main()
{
    int j,iht,th,theta,limit,ins;
    float a,b,e,c1,c2,c3,c4,vela,acca,thet,pi,dthet,bet,
    velb,vels,accs,accb;
    clrscr();

    printf("enter values a,b,e,vela,acca,theta,limit\n");
    scanf("%f%f%f%f%f%d%d%",&a,&b,&e,&vela,&acca,&theta,
    &li   mit);
    printf( " thet vela acca beta ");
    printf( " velc velb accc accb \n");
    ins=1;
    if(vela==0 && acca>0)ins=0;
    pi=4*atan(1);
    iht=360/theta;
    if(vela>0 && acca==00) {ins=0;iht=360/theta; }
    if(ins==1)iht=theta;
    dthet=pi*2/iht;
    if(vela==0 && acca>0)iht=iht+limit/theta;
          for(j=0;j<iht+1;j++)
          {
            if(j>(iht-360/theta-1) && ins==0)acca=0;
            thet=j*dthet;
            if(ins==1) {j=iht; thet=theta*pi/180;}
            th=theta*j;
            if(ins==1)th=theta;
            bet=asin((e-a*sin(thet))/b);
            vels=-a*vela*sin(thet-bet)/(cos(bet)*1000);
            velb=-a*vela*cos(thet)/b*cos(bet);
            c1=a*acca*sin(bet-thet)-b*pow(velb,2);
            c2=a*pow(vela,2)*cos(bet-thet);
            accs=(c1-c2)/(cos(bet)*1000);
            c3=a*acca*cos(thet)-a*pow(vela,2)*sin(thet);
            c4=b*pow(velb,2)*sin(bet);
            accb=-(c3-c4)/(b*cos(bet));
            printf( "%6.2d %6.2f %6.2f %6.2f %6.2f%8.2f
            %8.2f %8.2f\n",th,vela,acca,bet*180/pi,vels,
            velb,accs,accb);
            vela=sqrt(vela*vela+2*acca*dthet);

          }
    getch();
}
```

Fig. 4.6

The input variables are

`a,b,e`	The magnitudes *a*, *b* and *e* respectively (mm)
`vela`	Angular velocity of the input link *AB* (m/s)
`acca`	Angular acceleration of the input link (m/s^2)
`theta`	The interval of the input angle (degrees)
`limit`	Angle up to which acceleration continues, in case the crank starts from stationary position (in other cases any value may be given)

The output variables are

`thet`	Angular displacement of the input link *AB* (degrees)
`bet`	Angular displacement of link *AB* (rad/s)
`vels`	Linear velocity of the slider (m/s)
`velb`	Angular velocity of link *BC* (rad/s)
`accs`	Linear acceleration of the slider (m/s^2)
`accb`	Angular acceleration of link *BC* (rad/s^2)

Example 4.3 *In a slider-crank mechanism, the lengths of the crank and the connecting rod are 480 mm and 1.6 m respectively. It has an eccentricity of 100 mm. Assuming a velocity of 20 rad/s of the crank OA, calculate the following at an interval of 30°:*

(i) Velocity and the acceleration of the slider
(ii) Angular velocity and angular acceleration of the connecting rod

Solution The input and the output have been shown in Fig. 4.7. The results have been obtained at an interval of 30° of the input link (crank).

```
Enter values of a, b, e, vela, acca, theta, limit
480  1600  100  20  0  30  0
```

thet	vela	acca	beta	vels	velb	accs	accb
00	20.0	0.0	3.58	0.60	-5.99	-249.49	2.25
30	20.0	0.0	-5.02	-5.53	-5.18	-200.88	57.88
60	20.0	0.0	-11.38	-9.28	-2.94	-76.65	104.27
90	20.0	0.0	-13.74	-9.60	0.00	46.94	123.53
120	20.0	0.0	-11.38	-7.35	2.94	115.35	104.27
150	20.0	0.0	-5.02	-4.07	5.18	131.67	57.88
180	20.0	0.0	3.58	-0.60	5.99	134.51	2.25
210	20.0	0.0	12.27	2.99	5.08	144.94	-55.80
240	20.0	0.0	18.80	6.68	2.84	138.98	-107.04
270	20.0	0.0	21.25	9.60	-0.00	74.68	-128.76
300	20.0	0.0	18.80	9.95	-2.84	-53.02	-107.04
330	20.0	0.0	12.27	6.61	-5.08	-187.61	-55.80

Fig. 4.7

4.5 COUPLER CURVES

A coupler curve is the locus of a point on the coupler link. A four-link mechanism *ABCD* with a coupler point *E* (offset) is shown in Fig. 4.8. Let the x-axis be along the fixed link *AD*.

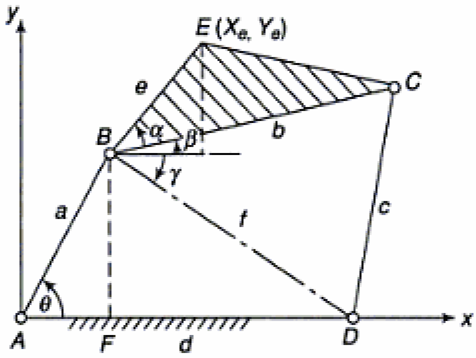


Fig. 4.8

Let $BE = e$ and $\angle CBE = \alpha$

Angles β and γ are defined as shown in the diagram.

Let X_e and Y_e be the coordinates of the point *E*.

Then,

$$X_e = a\cos\theta + e\cos(\alpha+\beta) \tag{4.59}$$

$$Y_e = a\sin\theta + e\sin(\alpha+\beta) \tag{4.60}$$

In these equations a, e, θ and α are known. To know the coordinates X_e and Y_e, it is necessary to express β in terms of known parameters, i.e., a, b, c, d, e, θ and α.

In ΔBDC, applying cosine law,

$$\cos(\beta+\gamma) = \frac{b^2+f^2-c^2}{2bf}$$

or

$$\beta+\gamma = \cos^{-1}\left[\frac{b^2+f^2-c^2}{2bf}\right]$$

$$\beta = \cos^{-1}\left[\frac{b^2+f^2-c^2}{2bf}\right]-\gamma \tag{4.61}$$

where

$$\tan\gamma = \frac{BF}{FD} = \frac{BF}{AD-AF} = \frac{a\sin\theta}{d-a\cos\theta}$$

or

$$\gamma = \tan^{-1}\left[\frac{a\sin\theta}{d-a\cos\theta}\right] \tag{4.62}$$

f^2 can be found by applying the cosine law to ΔABD,

i.e.,

$$f^2 = a^2+d^2-2ad\cos\theta$$

Having found the value of the angle β, the coordinates of the point *E* can be known for different values of θ from Eqs (4.59) and (4.60).

A coupler curve can also be obtained in case of a slider-crank mechanism (Fig. 4.9). The angle *CBE* is α and the eccentricity is e.

Draw $BL \perp AD$ and $CF \perp BL$

$$X_e = a\cos\theta + e\cos(\alpha-\beta) \tag{4.63}$$

$$Y_e = a\sin\theta + e\sin(\alpha-\beta) \tag{4.64}$$

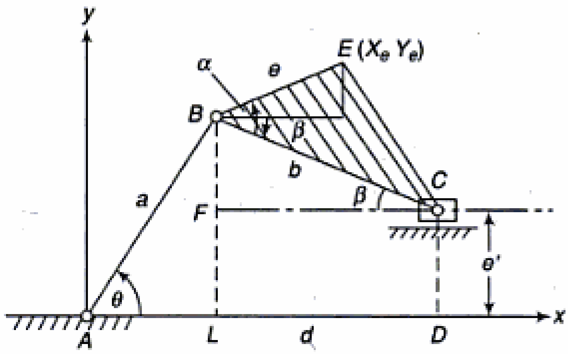


Fig. 4.9

where $C_1 = -2a\cos\theta$
$C_2 = a^2 - b^2 + e^2 - 2ae\sin\theta$

(ii) The angle of the coupler, $\beta = \sin^{-1}\dfrac{e - a\sin\theta}{b}$

(iii) The velocities of the slider and the angular velocity of the coupler are given by

$$\dot{d} = \frac{a\omega_a \sin(\beta-\theta)}{\cos\beta} \quad \text{and} \quad \omega_b = \frac{a\omega_a \cos\theta}{b\cos\beta}$$

(iv) The accelerations of the slider and the angular velocity of the coupler are given by

$$\ddot{d} = \frac{a\alpha_a \sin(\beta-\theta) - a\omega_a^2 \cos(\beta-\theta) - b\omega_b^2}{\cos\beta} \quad \text{and}$$

$$\alpha_b = \frac{a\alpha_a \cos\theta - a\omega_a^2 \sin\theta - b\omega_b^2 \sin\beta}{b\cos\beta}$$

Exercises

1. Find expressions to determine the angles of the output link and coupler of a four-link mechanism. Deduce relations for the angular velocity and accelerations of the same links.
2. Deduce expressions to find the linear velocity and acceleration and angular velocity and angular acceleration of the coupler of a slider-crank mechanism.
3. What are coupler curves? Deduce expressions to draw the same in case of a four-link mechanism and slider-crank mechanism.
4. Derive expressions for the displacement, velocity and acceleration analyses of an inverted slider-crank mechanism.
5. In a four-link mechanism (Fig. 4.1), the dimensions of the links are AB = 30 mm, BC = 80 mm, CD = 40 mm and AD = 75 mm. If OA rotates at a constant angular velocity of 30 rad/s in the clockwise direction, calculate the angular velocities and the angular accelerations of links BC and CD for values of θ at an interval of 30°.
6. In a slider-crank mechanism (Fig. 4.5), the crank AB = 50 mm, BC = 160 mm and eccentricity e = 15 mm. For the angle θ = 45°, angular velocity of AB = 8 rad/s with an angular acceleration of 12 rad/s² (both clockwise), find the linear velocity and the acceleration of the slider and the angular velocity and the angular acceleration of the connecting rod analytically.
   (0.32 m/s, 1.98 m/s², 1.78 rad/s, 16.53 rad/s²)
7. Derive expressions to find the angular displacement, angular velocity and the angular acceleration of the link EF of a six-link mechanism shown in Fig. 4.13. AB is the input link having an angular velocity of ω rad/s in the counter-clockwise direction.

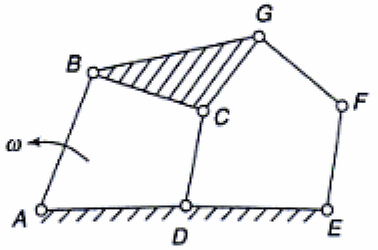


Fig. 4.13

8. Derive expressions for the coupler curves of an inverted slider-crank mechanism.
9. For the data of Example 4.3, take some more coupler points by taking different values of BE and $\angle\alpha$ and draw coupler curves for the same. Make a cardboard model of the mechanism and obtain the coupler curve by rotating the crank through 360°.

5 GRAPHICAL AND COMPUTER-AIDED SYNTHESIS OF MECHANISMS

Introduction

Dimensional synthesis of a pre-conceived type mechanism necessitates determining the principal dimensions of various links that satisfy the requirements of motion of the mechanism. A mechanism of preconceived type may be a four-link or a slider-crank mechanism. Principal dimensions involve link lengths, angular positions, position of pivots, eccentricities, angle between bell-crank levers and linear distance of sliders, etc. Synthesis of mechanisms may be done by graphical methods or by analytical means that involves the use of calculators and computers. In general, the types of synthesis may be classified as under:

1. *Function generation* It requires correlating the rotary or the sliding motion of the input and the output links. The motion of the output and the input links may be prescribed by an arbitrary function $y = f(x)$. This means if the input link moves by x, the output link moves by $y = f(x)$ for the range $x_o \leq x \leq x_{n+1}$ There lies n values of independent parameters $(x_1, x_2, ...x_n)$ in the range between x_o and x_{n-1}. In case of rotary motions of the input and the output links, when the input link rotates through an angle θ, the output link moves through an angle ϕ corresponding to the value of the dependent variable $y = f(x)$. In case of slider-crank mechanism, the output is in the form of displacement s of the slider. It is to be noted that a four-link mechanism can match the function at only a limited number of prescribed points. However, it is a widely used mechanism in the industry since a four-bar is easy to construct and maintain and in most of the cases exact precision at many points is not required.
2. *Path generation* When a point on the coupler or the floating link of a mechanism is to be guided along a prescribed path, it is said to be a path generation problem. This guidance of the path of the point may or may not be coordinated with the movement of the input link and is generally called *with prescribed timing* or *without prescribed timing*.
3. *Motion generation* In this type, a mechanism is designed to guide a rigid body in a prescribed path. This rigid body is considered to be the coupler or the floating link of a mechanism.

If the above tasks are to be accomplished at fewer positions, it is simple to design a mechanism. However, when it is required to synthesize a mechanism to satisfy the input and the output links at larger number of positions, only an approximated solution can be obtained giving least deviation from the specified positions. In this chapter, both graphical as well as analytical methods to design a four-link mechanism and a slider-crank mechanism are being discussed.

PART A: GRAPHICAL METHODS

5.1 POLE

If it is desired to guide a body or link in a mechanism from one position to another, the task can easily be accomplished by simple rotation of the body about a point known as the *pole*. In Fig. 5.1, a link B_1C_1 is shown to move to another position B_2C_2 by rotating it about the pole P_{12}. This pole is easily found graphically by joining the midnormals of any two corresponding points on the link such as B_1B_2 and C_1C_2. If the pole point happens to fall off the frame of the machine, two fixed pivots, one each anywhere along the two midnormals will serve the purpose. In the figure, A and B are taken to be the fixed pivots. The configuration also happens to be a four-link mechanism $ABCD$ in two positions AB_1C_1D and AB_2C_2D in which the coupler link BC has moved from the position B_1C_1 to B_2C_2. The input link AB and the output link DC have moved through angles θ_{12} and ϕ_{12} respectively in the clockwise direction (Fig. 5.1).

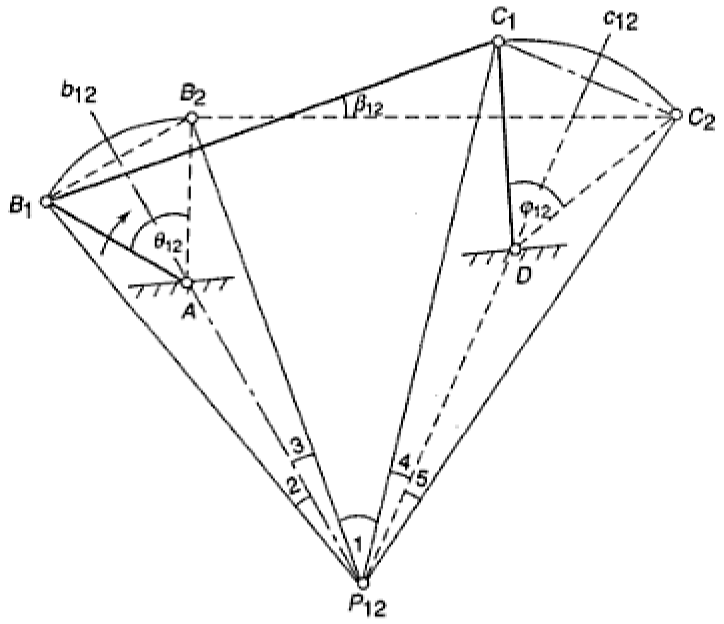


Fig. 5.1

Thus, a pole P_{12} of the coupler link BC is its centre of rotation with respect to the fixed link for the motion of the coupler from B_1C_1 to B_2C_2. Each point on the link BC describes a circular arc with centre at the pole P_{12}. Thus, a line joining the two positions of a point on the link is a chord of the circle and the midnormal (perpendicular bisector) of the chord passes through the centre of rotation P_{12}. B and C are also two points on the link BC. B moves from B_1 to B_2 while C from C_1 to C_2. Therefore, B_1B_2 and C_1C_2 are the chords of the two circles and their midnormals b_{12} and c_{12} also pass through or intersect at the centre of their rotation, i.e., at P_{12}.

Properties of Pole Point

1. As $AB_1 = AB_2$, the midnormal b_{12} of B_1B_2 passes through the fixed pivot A. Similarly, the midnormal of C_1C_2 passes through pivot D.
2. The coupler link BC is rotated about P_{12} from the position B_1C_1 to B_2C_2,

   $\therefore \quad \Delta B_1P_{12}C_1 \equiv \Delta B_2P_{12}C_2$

   $\therefore \quad \angle 2 + \angle 3 + \angle 1 = \angle 1 + \angle 4 + \angle 5$

   i.e., angle subtended by B_1C_1 at P_{12} = angle subtended by B_2C_2 at P_{12}

   or the angle subtended by BC at P_{12} in two positions is the same.
3. From (2), $\angle 2 + \angle 3 + \angle 1 = \angle 1 + \angle 4 + \angle 5$

   or $\quad \angle 2 + \angle 3 = \angle 4 + \angle 5$

   i.e., B_1B_2 and C_1C_2 subtend equal angles at P_{12}.
4. P_{12} lies on the midnormal of B_1B_2,

   $\therefore \quad \angle 2 = \angle 3$

   Similarly, $\angle 4 = \angle 5$
5. $\quad \angle 2 + \angle 3 = \angle 4 + \angle 5$

   But $\angle 2 = \angle 3$ and $\angle 4 = \angle 5$,

$\therefore \quad \angle 2 = \angle 4$

and $\quad \angle 3 = \angle 5$

i.e., the input and the output links subtend equal angles at P_{12} in their corresponding positions.

6. $\angle 2 + \angle 3 + \angle 1 = \angle 1 + \angle 4 + \angle 5$

$= \angle 1 + \angle 4 + \angle 3 \qquad (\angle 3 = \angle 5)$

i.e., the angle subtended by the coupler link is equal to that subtended by the fixed pivots A and D.

7. The triangle $B_1P_{12}C_1$ moves as one link about P_{12} to the position $B_2P_{12}C_2$.
   Angular displacement of coupler B_1C_1 = Angular displacement of $P_{12}C_1$ = Angular displacement of $P_{12}B_1$
   i.e., $\beta_{12} = \angle 4 + \angle 5 = \angle 2 + \angle 3$

5.2 RELATIVE POLE

A *pole* of a moving link is the centre of its rotation with respect to a fixed link. However, if the rotation of the link is considered relative to another moving link, the pole is known as the *relative pole*. The relative pole can be found by fixing the link of reference and observing the motion of the other link in the reverse direction.

For the four-link mechanism of Fig. 5.2, the pole of BC relative to AB is at the pivot B. The pole of DC relative to AB can be found as follows:

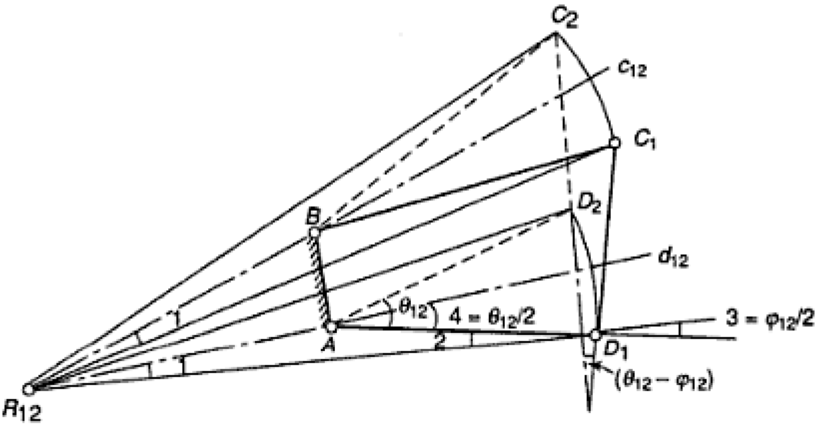


Fig. 5.2

Let θ_{12} = angle of rotation of AB (clockwise)
φ_{12} = angle of rotation of DC (clockwise)

Make the following constructions:

1. Assume A and B as the fixed pivots and rotate AD_1 about A through angle θ_{12} in the counter-clockwise direction (opposite to the direction of rotation of AB). Let D_2 be the new position after the rotation of AD (AB fixed).
2. Locate the point C_2 by drawing arcs with centres B and D_2 and radii equal to BC_1 and D_1C_1 respectively. Then ABC_2D_2 is known as the inversion of ABC_1D_1.
3. Draw midnormals of D_1D_2 and C_1C_2 which pass through A and B and intersect at R_{12} which is the required relative pole.

Now $(\varphi_{12} - \theta_{12})$ = Angle of rotation of the output link DC relative to the input link AB.

This angle is negative if $DC > AB$ and is positive if $DC < AB$.

Angular displacement of $R_{12}D_1$ = angular displacement of D_1C_1

[Refer Sec. 5.2 (7)]

$\angle D_1R_{12}D_2 = -(\angle\varphi_{12} - \angle\theta_{12}) \qquad$ (assuming $DC > AB$)

or $\quad 2\angle 1 = -(\angle\varphi_{12} - \angle\theta_{12})$

5.3 FUNCTION GENERATION BY RELATIVE POLE METHOD

The problems of function generation for two and three accuracy positions are easily solved by the relative pole method as discussed below:

(a) Four-link Mechanisms

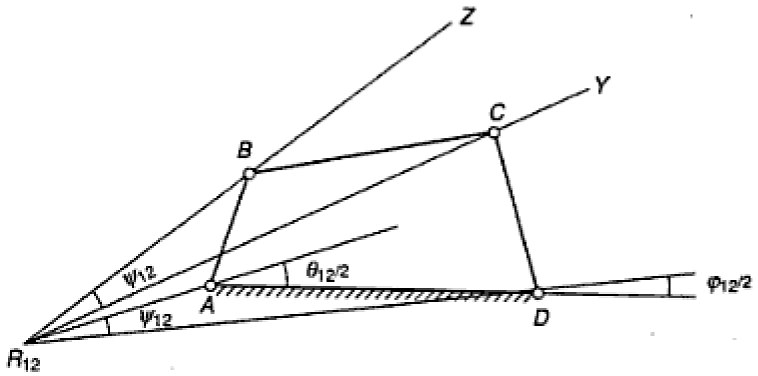


Fig. 5.4

Two-position synthesis Let for a four-link mechanism, the positions of the pivots A and D along with the angular displacements θ_{12} (angle between θ_1 and θ_2) and φ_{12} (angle between φ_1 and φ_2) of the driver and the driven links respectively be known.

To design the mechanism (Fig. 5.4), first locate the relative pole R_{12} by the procedure given in Sec. 5.2.

Now, angle subtended by the coupler BC at R_{12}

= angle subtended by the fixed pivots A and D at R_{12}

$$= \frac{1}{2}\angle\theta_{12} - \frac{1}{2}\angle\varphi_{12} \qquad \text{(assuming } DC > AB\text{)}$$

$$= \angle\psi_{12}$$

Adopt any of the following alternatives to design the required mechanism:

1. At point R_{12}, construct an angle ψ_{12} at an arbitrary position. Join any two points on the two arms of the angle to obtain the coupler link BC of the mechanism. Join AB and DC to have the driver and the driven links respectively.
2. Locate the point C arbitrary so that DC is the output link. Construct an angle $CR_{12}Z = \psi_{12}$. Take any point B on $R_{12}Z$. Join AB and BC.
3. Instead of locating the point C as above, locate the point B arbitrary so that AB is the input link. Construct an angle $BR_{12}Y = \psi_{12}$. Take any point C on $R_{12}Y$. Join BC and DC.

Then $ABCD$ is the required four-link mechanism.

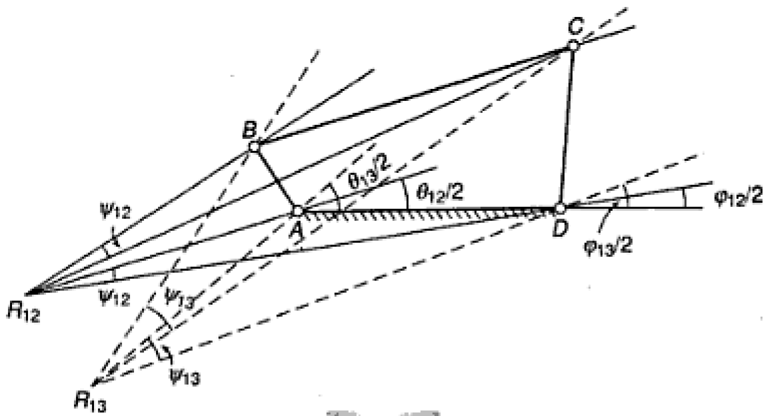


Fig. 5.5

Three-position synthesis If instead of one angular displacement of the input and of the output link, two displacements of the input (θ_{12} and θ_{13}) and two of the output (φ_{12} and φ_{13}) are known, find R_{12} and R_{13} as shown in Fig. 5.5.

Let ψ_{12} and ψ_{13} = angles made by the fixed link at R_{12} and R_{13} respectively.

Construct the angles ψ_{12} and ψ_{13} at the points R_{12} and R_{13} respectively in arbitrary positions such that the arms of the angles intersect at B and C in convenient positions.

[Fig. 5.8(a)] and about D through an angle 15° (= $\varphi_{12}/2$) taking both counter-clockwise. The point of intersection of the two positions of AD after rotation about A and D is the relative pole R_{12}. Similarly, locate R_{13}.

3. At point R_{12}, construct an angle of 15° (= $\theta_{12}/2 - \varphi_{12}/2$) at an arbitrary suitable position. At the point R_{13}, construct an angle of 20° (= $\theta_{13}/2 - \varphi_{13}/2$) in such a way that the intersection of its two arms with that of the arms of the previous angle locates points B and C at suitable positions.
4. Join AB, BC and CD.

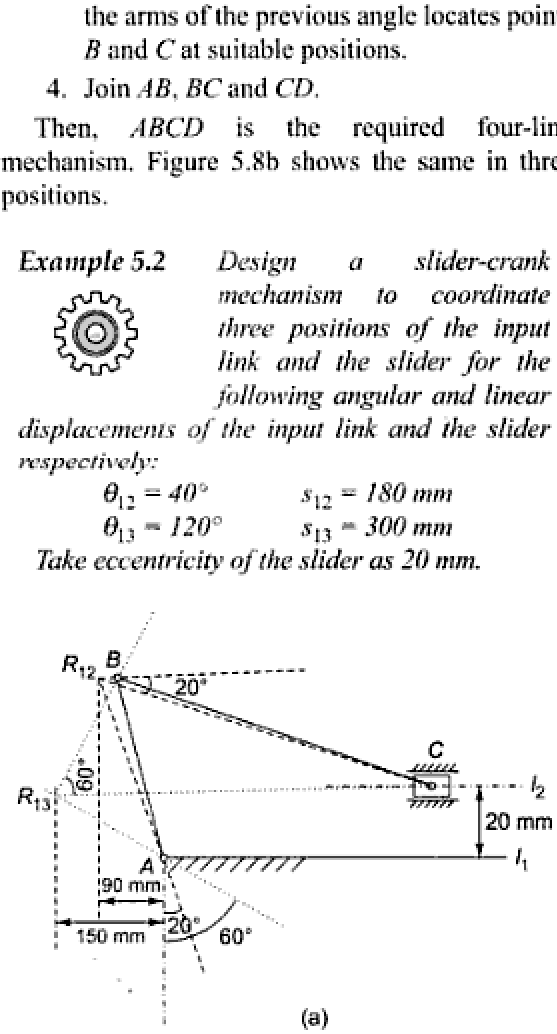
Then, $ABCD$ is the required four-link mechanism. Figure 5.8b shows the same in three positions.

Example 5.2 *Design a slider-crank mechanism to coordinate three positions of the input link and the slider for the following angular and linear displacements of the input link and the slider respectively:*

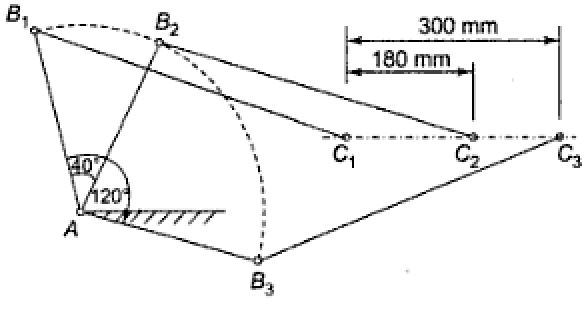
$\theta_{12} = 40°$ $\quad s_{12} = 180$ mm

$\theta_{13} = 120°$ $\quad s_{13} = 300$ mm

Take eccentricity of the slider as 20 mm.



(a)



(b)

Fig. 5.9

Solution The required slider-crank mechanism can be designed as follows:

1. Draw two parallel lines l_1 and l_2 20 mm apart from each other [Fig. 5.9(a)].
2. Take an arbitrary point A on the line l_1 for the fixed ground pivot.
3. Locate the relative pole R_{12} by rotating a vertical line through A about A through an angle of 20° (=$\theta_{12}/2$) counter-clockwise and drawing a vertical line at 90 mm (= $s_{12}/2$) to the left of A. Similarly, locate the relative pole R_{13} by rotating vertical line through A about A through an angle 60° (=$\theta_{13}/2$) counter-clockwise and drawing a vertical line at 150 mm (= $s_{13}/2$) to the left of A.
4. At point R_{12}, construct an angle of 20° (= $\theta_{12}/2$) and at point R_{13}, construct an angle of 60° (= $\theta_{13}/2$) in such a way that the intersection of their arms locate the points B and C (on l_2) at suitable positions.
5. Join AB and BC.

Then, ABC is the required slider-crank mechanism. Figure 5.9(b) shows the same in three positions.

5.4 INVERSION METHOD

Basically, the relative pole method is derived from the kinematic inversion principle. But there is no visible inversion of the planes during the solution of the problems. In the inversion method, there is direct use of the concept of inversion.

A four-link mechanism *ABCD* is shown in two positions AB_1C_1D and AB_2C_2D in Fig. 5.10. The input and the output links *AB* and *DC* are moved through angles θ_{12} and φ_{12} respectively in the clockwise direction.

Rotate *AD* through θ_{12} in a direction opposite to the rotation of *AB* and get the inversion $AB_1C_2'D'$. It can be observed that the configuration AB_2C_2D has been rotated about *A* through an angle θ_{12} in the counter-clockwise direction to obtain the figure $AB_1C_2'D'$. Make the following observations:

1. Point C_2 is rotated through an angle θ_{12} in the counter-clockwise direction with the centre at *A*.
2. $C_1C'_2$ lies on a curve with the centre of rotation at B_1. Therefore, B_1 lies on the midnormal of $C_1C'_2$.

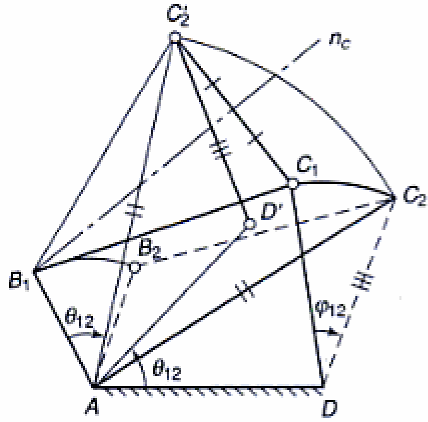


Fig. 5.10

5.5 FUNCTION GENERATION BY INVERSION METHOD

The problems of function generation for two, three and four accuracy positions can be solved by the inversion method as follows:

(a) Four-Link Mechanism

Two-position synthesis Let the distance between the fixed pivots *A* and *D*, and the angles θ_{12} and φ_{12} be known. To design the mechanism, proceed as follows:

1. Draw a line segment *AD* of length equal to the distance between the fixed pivots (Fig. 5.11).
2. At point *D*, construct an angle $C_1DC_2 = \varphi_{12}$ (clockwise) at an arbitrary position, selecting a suitable output crank length $DC_1 = DC_2$.
3. Rotate point C_2 in the counter-clockwise direction through an angle θ_{12} with *A* as centre and obtain the point C_2'.
4. Join C_1C_2' and draw its midnormal. Select a suitable point B_1 on it.

   AB_1C_1D is the required four-link mechanism in which B_1C_1 is the coupler.

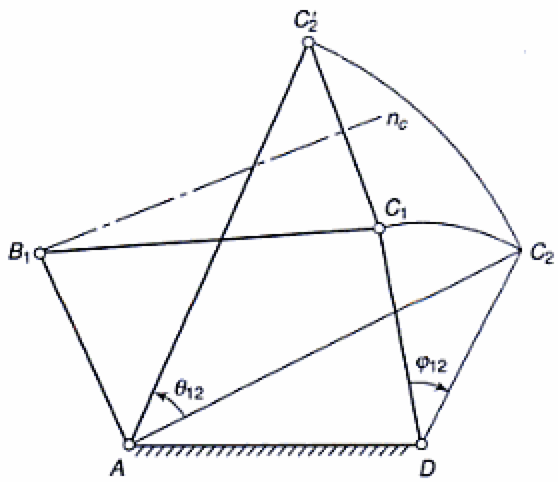


Fig. 5.11

Three-position synthesis If two angular displacements of the input link (θ_{12} and θ_{13}) and two of the output link (φ_{12} and φ_{13}) are known, proceed as below:

1. Draw a line segment AD of length equal to the distance between the fixed pivots (Fig. 5.12).
2. Choose some suitable length of the output link DC. Draw it at some suitable angle with the fixed link AD and locate its three positions DC_1, DC_2 and DC_3 as its angular displacements are known.
3. Find the points C'_2 and C'_3 after rotating AC_2 and AC_3 about A through angles θ_{12} and θ_{13} respectively in the counter-clockwise direction.
4. Intersection of the midnormals of $C_1 C'_2$ and $C_1 C'_3$ locates the point B_1. Then, AB_1C_1D is the required four-link mechanism.

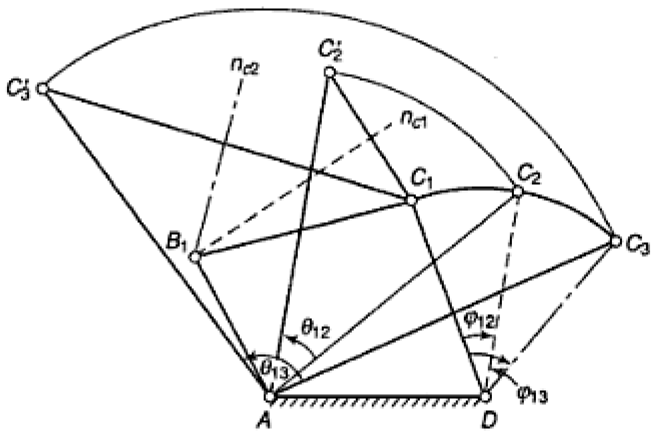


Fig. 5.12

The mechanism could also have been obtained by drawing the input link AB in three positions and rotating DB_2 and DB_3 through angles φ_{12} and φ_{13} respectively in the counter-clockwise direction with D as centre.

Four-position synthesis If a four-link mechanism is to be designed for four precision positions of the input and four positions of the output link, it can be designed by *point-position reduction* method. In this method, the point B_1 is chosen at the relative pole R_{12} with an assumed position of fixed link AD. The corresponding positions of B_2, B_3 and B_4 are easily located establishing the input link in four positions. Then by using the inversion method, the mechanism can be designed. The method is given below in brief:

1. Draw a line segment AD of suitable length to be the distance between the fixed pivots (Fig. 5.13).
2. Locate the position of the relative pole by rotating AD about A through angle $\theta_{12}/2$ and about D through an angle $\phi_{12}/2$ both in counter-clockwise direction. Take this as the point B_1.
3. Draw the input link AB in four positions AB_1, AB_2, AB_3 and AB_4 as its angular displacements are known.
4. Find the points B'_2, B'_3 and B'_4 after rotating DB_2, DB_3 and DB_4 about D through angles ϕ_{12}, ϕ_{13} and ϕ_{14} respectively in the counter-clockwise direction. It may be noted that the location of B'_2 is situated at B_1.
5. Intersection of the midnormals of $B'_2B'_3$ and $B'_3B'_4$ locates the point C.

Then AB_1CD is the required mechanism.

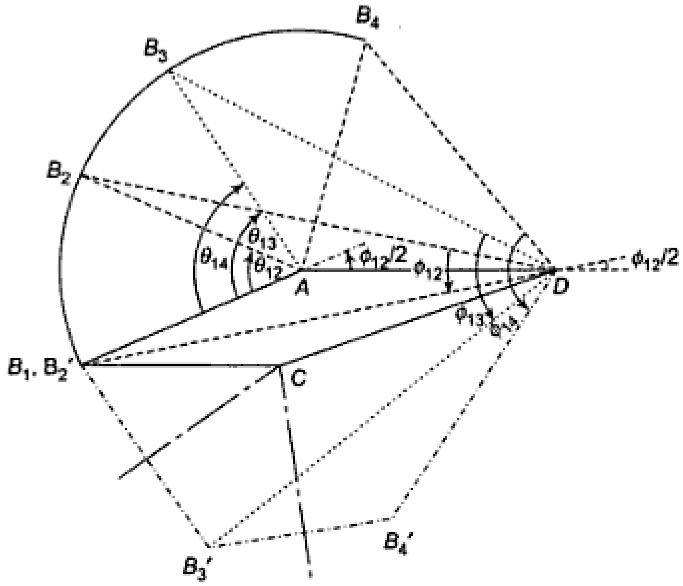


Fig. 5.13

(b) Slider-Crank Mechanism

Two-position synthesis If the angular displacement of the input link θ_{12} and the linear displacement of the slider s_{12} along with the eccentricity e are known, the required slider-crank mechanism is obtained as follows:

1. Draw two parallel lines l_1 and l_2 at a distance e apart (Fig. 5.14).
2. Take an arbitrary point A on the line l_1 for the fixed pivot and two points C_1 and C_2 on the line l_2, a distance s_{12} apart for the initial and the final positions of the slider.
3. Rotate the point C_2 about A through an angle θ_{12} in the counter-clockwise direction to obtain the point C'_2.
4. Join $C_1C'_2$ and draw its midnormal n_c. Take an arbitrary but convenient point B on it.

ABC_1 is the required slider-crank mechanism.

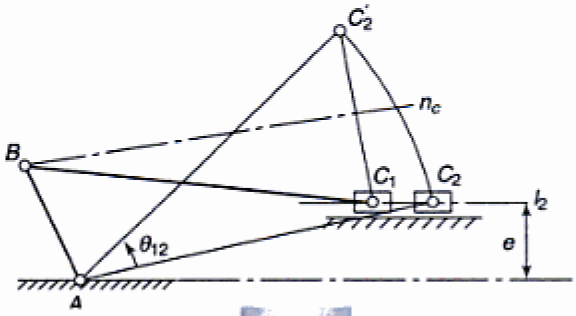


Fig. 5.14

Three-position synthesis For three positions of the input link and three positions of the slider, find C'_2 and C'_3 as usual. Then midnormal of $C_1C'_2$ and $C_1C'_3$ intersect at the point B (Fig.5.15).

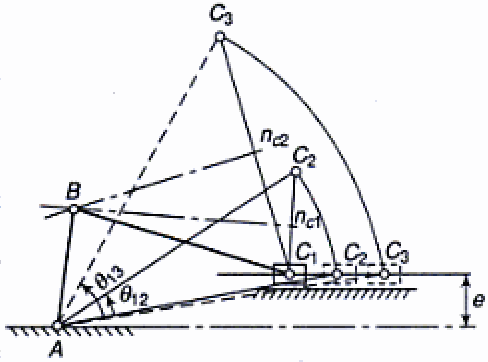


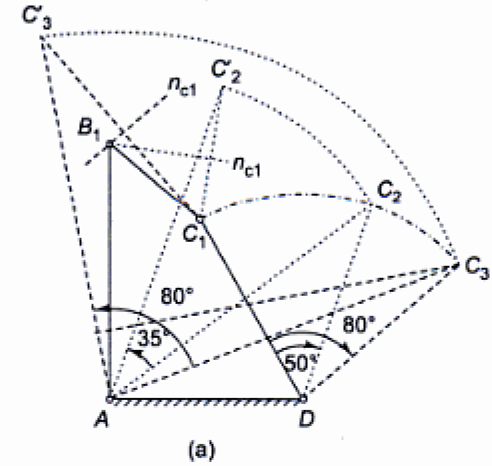
Fig. 5.15

Four-position synthesis A four-position synthesis can be done in the same way as in case of a four-link mechanism.

Example 5.3 *Design a four-link mechanism to coordinate three positions of the input and of the output links for the following angular displacements by inversion method:*

$\theta_{12} = 35°$ $\quad \varphi_{12} = 50°$

$\theta_{13} = 80°$ $\quad \varphi_{13} = 80°$



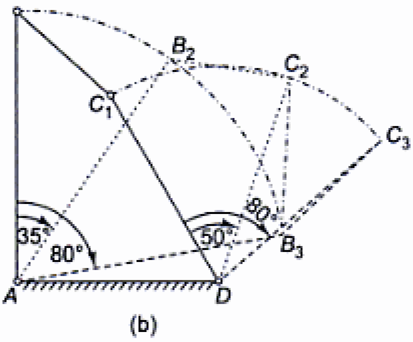
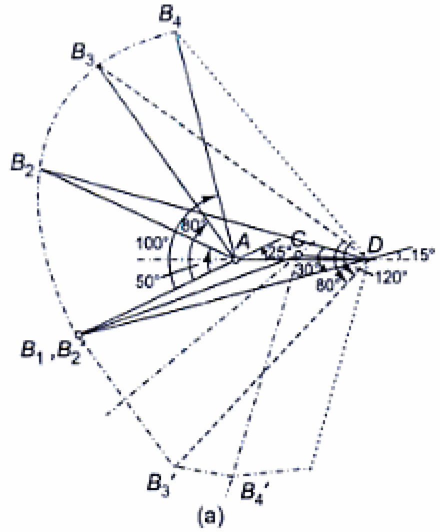


Fig. 5.16

Solution For the given two angular displacements of the input and the output links, proceed as given below:

1. Draw a line segment AD of suitable length to represent the fixed link [Fig. 5.16(a)].
2. Choose a suitable length of the output link DC and a suitable location of C_1. Then locate the positions of C_2 and C_3 by drawing the output link DC in three positions DC_1,

3. Draw the input link AB in four positions AB_1, AB_2, AB_3 and AB_4 as its angular displacements are known.



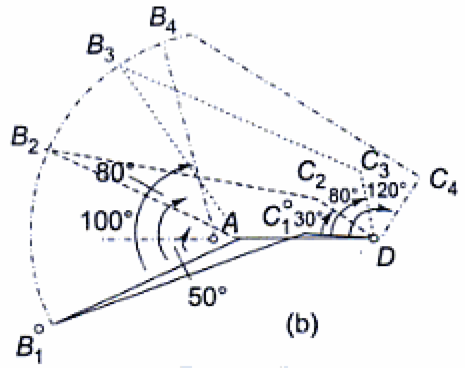


Fig. 5.18

4. Locate the points B'_2, B'_3 and B'_4 by rotating DB_2, DB_3 and DB_4 about D through angles φ_{12}, φ_{13} and φ_{12} respectively in the counter-clockwise direction such that the location of B'_2 is at B_1.
5. Intersection of the midnormals of $B'_2B'_3$ and $B'_3B'_4$ locates the point C.

Then AB_1CD is the required mechanism. Figure 5.18(b) shows the mechanism in the required four positions.

5.6 PATH GENERATION

The problem may be of designing the mechanism without or with prescribed timing, i.e., the guidance of the point on the coupler may or may not be coordinated with the movement of the input link. To design such a mechanism, the method of inversion of mechanisms is used by fixing the coupler and releasing the fixed link. To understand the inversion method, consider a four-link mechanism as shown in Fig. 5.19(a) in two positions $A_1B_1C_1D_1$ and $A_1B_2C_2D_1$. E is an offset point on the coupler which assumes the location E_2 in the second position. In Fig. 5.19(b), the inversion of the mechanism is shown by fixing the coupler B_1C_1 and releasing the fixed link so that the quadrilateral $A_1B_2C_2D$ of figure (a) in exactly the same as the quadrilateral $A_2B_1C_1D_2$ of figure (b). It can be done by taking $\angle A_2\ B_1C_1 = \angle A_1B_2C_2$. Now if triangles $B_2E_2D_1$ and $B_1E_1D_2$ are marked in the two figures, they must be congruent. It can be observed that the point C_1 is the centre of curvature of the arc passing through D_1 and D_2 and thus lies on the right bisector of D_1D_2.

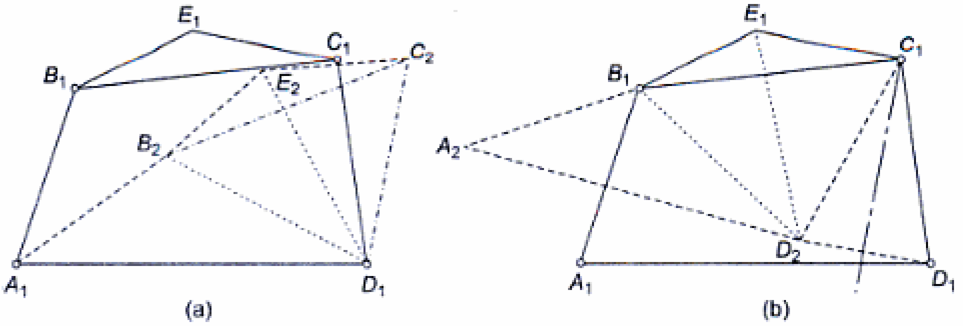
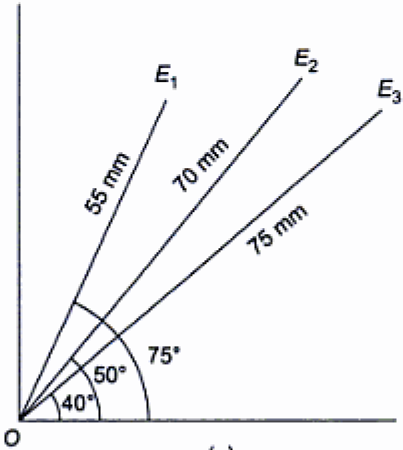


Fig. 5.19

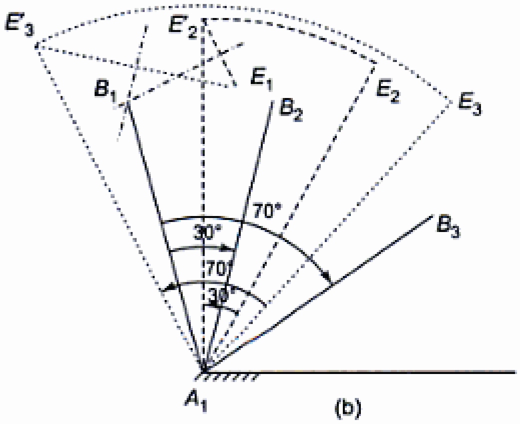
Example 5.6 *Design a four-link mechanism to coordinate the following three positions of the coupler point. The positions are given with respect to coordinate axes:*

$r_1 = 55$ mm	$\alpha_1 = 75°$
$r_2 = 70$ mm	$\alpha_2 = 50°$
$r_3 = 75$ mm	$\alpha_3 = 40°$

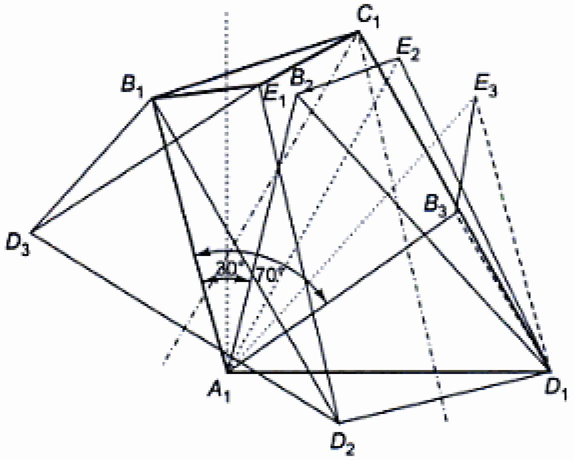
The angular displacements of the input link are to be $\theta_{12} = 30°$ and $\theta_{13} = 70°$.



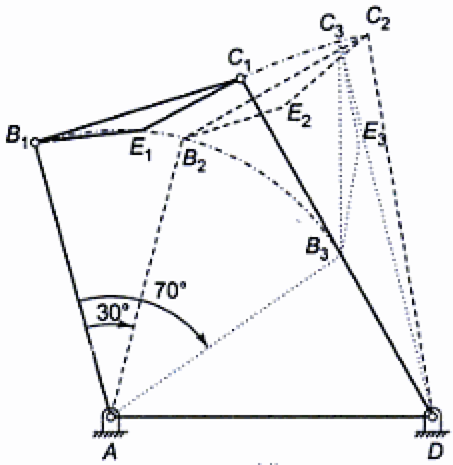
(a)



(b)



(c)



(d)

Fig. 5.22

Solution It is the case of path generation with prescribed timing. The procedure is given below:

1. Locate the three coupler points E_1, E_2 and E_3 as shown in Fig. 5.22(a).
2. Select a suitable location of the pivot A_1 of the fixed link with respect to the positions of the coupler points [Fig. 5.22(b)].
3. Rotate A_1E_2 through an angle 30° ($=\theta_{12}$) in the counter-clockwise direction with A_1 as centre and obtain the point E'_2. Similarly, rotate AE_3 through an angle 70° ($=\theta_{13}$) in the counter-clockwise direction with centre A and obtain the point E'_3.

4. Draw midnormals of $E_1E'_2$ and $E_1E'_3$, the intersection locates the point B_1.
5. Draw the input link in three positions AB_1, AB_2 and AB_3.
6. Select suitable location of the pivot D_1 of the fixed link. Construct $\Delta E_2B_2D_1 \equiv \Delta E_1B_1D_2$ and $\Delta E_3B_3D_1 \equiv \Delta E_1B_1D_3$.
7. Draw midnormals of D_1D_2 and D_2D_3. The intersection of the two locates the point C_1.

Thus, $A_1B_1C_1D_1$ is the required four-link mechanism with the coupler point E_1. Figure 5.22(d) shows the required mechanism in three positions.

5.7 MOTION GENERATION (RIGID-BODY GUIDANCE)

Let a rigid body be guided through three prescribed positions. It is required to design a four-link mechanism of which this rigid body will be a coupler. The rigid body is shown in Fig. 5.23 in three given positions. To find the lengths of the four links of the mechanism, proceed as follows:

1. Take any two arbitrary suitable points B and C on the rigid body and locate these on the body in three positions. It is assumed that the point B_1, B_2, B_3 and E_1, E_2 and E_3 are non-collinear.
2. Find the centre A of the circle passing through B_1, B_2, and B_3. Similarly, let the centre of the circle passing through C_1, C_2 and C_3 be D.
3. Join AB_1, B_1C_1 and C_1D.

Then, AB_1C_1D is the required mechanism which takes the coupler B_1C_1 through B_2C_2 and B_3C_3.

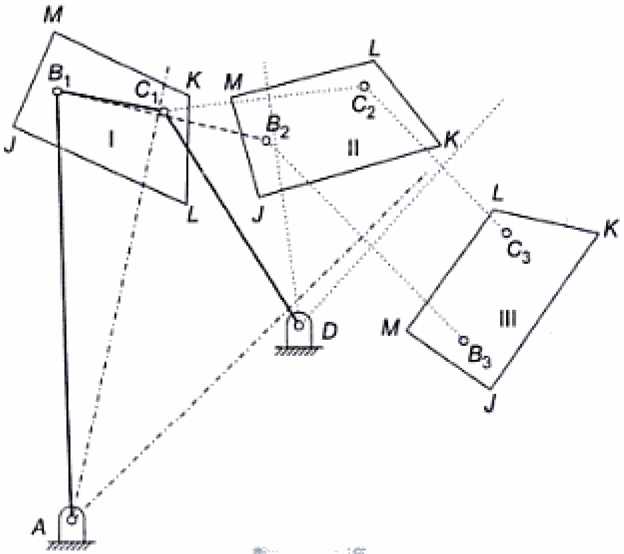


Fig. 5.23

In the above case of motion generation, the choice of the ground pivots is not with the designer. Many times, it becomes necessary to fix the locations of these pivots beforehand due to constraint of space. Such type of problem can also be solved by the inversion method as discussed in Section 5.6. In such cases, proceed as follows:

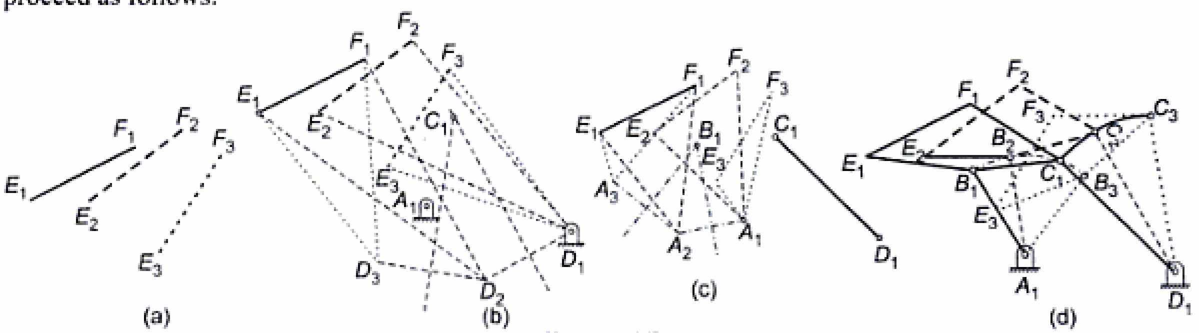


Fig. 5.24

1. Take any two arbitrary points E and F on the rigid body and locate these on the body in three positions [Fig. 5.24(a)].
2. Let A_1 and D_1 be the locations of the ground pivots [Fig. 5.24(b)].
3. Construct $\Delta E_2F_2D_1 \equiv \Delta E_1F_1D_2$ and $\Delta E_3F_3D_1 \equiv \Delta E_1F_1D_3$.
4. The centre of the arc through D_1, D_2 and D_3 is the crank pin C_1. To locate it, draw midnormals of D_1D_2 and D_2D_3. The intersection of the two is the pivot point C_1 on the rigid body or the coupler.
5. Construct $\Delta E_2F_2A_1 \equiv \Delta E_1F_1A_2$ and $\Delta E_3F_3A_1 \equiv \Delta E_1F_1A_3$ [Fig. 5.24(c)].
6. The centre of the arc through A_1, A_2 and A_3 is the crank pin B_1. Draw midnormals of A_1A_2 and A_2A_3. The intersection of the two locates the pivot point A_1 on the rigid body or coupler.

Then $A_1B_1C_1D_1$ is the required mechanism which takes the coupler $B_1E_1F_1C_1$ through $B_2E_2F_2C_2$ and $B_3E_3F_3C_3$ [Fig. 5.24(d)].

PART B: COMPUTER-AIDED SYNTHESIS OF MECHANISMS

5.8 FUNCTION GENERATION

A four-link mechanism shown in Fig. 5.25 is in equilibrium. Let a, b, c and d be the magnitudes of the links AB, BC, CD and DA respectively. θ, β and φ are the angles of AB, BC and DC respectively with the X-axis (taken along AD). AD is the fixed link. AB and DC are the input and output links respectively of the mechanism.

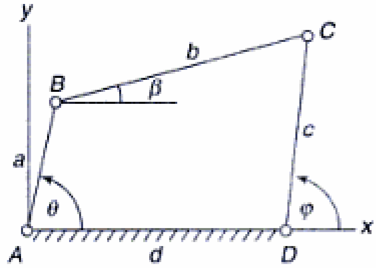


Fig. 5.25

Considering the links to be vectors, displacement along the X-axis

$a \cos \theta + b \cos \beta = d + c \cos \phi$ (The equation is valid for $< \phi$ more than 90° also.)

or $\quad b \cos \beta = c \cos \varphi - a \cos \theta + d$

or $\quad (b \cos \beta)^2 = (c \cos \varphi - a \cos \theta + d)^2$

$$= c^2 \cos^2 \varphi + a^2 \cos^2 \theta + d^2 - 2ac \cos \theta \cos \varphi - 2ad \cos \theta + 2cd \cos \varphi \quad \text{(i)}$$

Displacement along Y-axis

$$a \sin \theta + b \sin \beta = c \sin \phi$$

or $\quad b \sin \beta = c \sin \varphi - a \sin \theta$

or $\quad (b \sin \beta)^2 = (c \sin \varphi - a \sin \theta)^2$

$$= c^2 \sin^2 \varphi + a^2 \sin^2 \theta - 2ac \sin \theta \sin \varphi \quad \text{(ii)}$$

Adding (i) and (ii),

$$b^2 = c^2 + a^2 + d^2 - 2ac \cos \theta \cos \varphi - 2ad \cos \theta + 2cd \cos \varphi - 2ac \sin \theta \sin \varphi$$

or $\quad 2cd \cos\varphi - 2ad \cos\theta + a^2 - b^2 + c^2 + d^2 = 2ac(\cos\theta \cos \varphi + \sin\theta \sin \varphi)$

Dividing throughout by $2ac$,

$$\frac{d}{a} \cos \varphi - \frac{d}{c} \cos \theta + \frac{a^2 - b^2 + c^2 + d^2}{2ac} = \cos(\theta - \varphi) = \cos(\varphi - \theta)$$

This is known as *Freudenstein's* equation and can be written as,

$$k_1 \cos\varphi + k_2 \cos\theta + k_3 = \cos(\theta - \varphi) \quad (5.1)$$

where

$$k_1 = \frac{d}{a}; k_2 = -\frac{d}{c}; \quad \text{and} \quad k_3 = \frac{a^2 - b^2 + c^2 + d^2}{2ac}$$

Let the input and the output are related by some function such as $y = f(x)$ and for the specified positions

$\theta_1, \theta_2, \theta_3$ = three positions of input link (given)

and $\varphi_1, \varphi_2, \varphi_3$ = three positions of output link (given)

It is required to find the values of a, b, c and d to form a four-link mechanism giving the prescribed motions of the input and the output links.

Equation (5.1) can be written as,

$$k_1 \cos\varphi_1 + k_2 \cos\theta_1 + k_3 = \cos(\theta_1 - \varphi_1)$$
$$k_1 \cos\varphi_2 + k_2 \cos\theta_2 + k_3 = \cos(\theta_2 - \varphi_2)$$
$$k_1 \cos\varphi_3 + k_2 \cos\theta_3 + k_3 = \cos(\theta_3 - \varphi_3)$$

k_1, k_2, and k_3 can be evaluated by Gaussian elimination method or by the Cramer's rule.

$$\Delta = \begin{vmatrix} \cos\varphi_1 & \cos\theta_1 & 1 \\ \cos\varphi_2 & \cos\theta_2 & 1 \\ \cos\varphi_3 & \cos\theta_3 & 1 \end{vmatrix}$$

$$\Delta_1 = \begin{vmatrix} \cos(\theta_1 - \varphi_1) & \cos\theta_1 & 1 \\ \cos(\theta_2 - \varphi_2) & \cos\theta_2 & 1 \\ \cos(\theta_3 - \varphi_3) & \cos\theta_3 & 1 \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} \cos\varphi_1 & \cos(\theta_1 - \varphi_1) & 1 \\ \cos\varphi_2 & \cos(\theta_2 - \varphi_2) & 1 \\ \cos\varphi_3 & \cos(\theta_3 - \varphi_3) & 1 \end{vmatrix}$$

$$\Delta_3 = \begin{vmatrix} \cos\varphi_1 & \cos\theta_1 & \cos(\theta_1 - \varphi_1) \\ \cos\varphi_2 & \cos\theta_2 & \cos(\theta_2 - \varphi_2) \\ \cos\varphi_3 & \cos\theta_3 & \cos(\theta_3 - \varphi_3) \end{vmatrix}$$

k_1, k_2 and k_3 are given by,

$$k_1 = \frac{\Delta_1}{\Delta}; \quad k_2 = \frac{\Delta_2}{\Delta}; \quad k_3 = \frac{\Delta_3}{\Delta}$$

Knowing k_1, k_2 and k_3, the values of a, b, c and d can be computed from the relations

$$k_1 = \frac{d}{a}; \quad k_2 = -\frac{d}{c}; \quad k_3 = \frac{a^2 - b^2 + c^2 + d^2}{2ac}$$

Value of either a or d can be assumed to be unity to get the proportionate values of other parameters.

```
#include<stdio.h>
#include<conio.h>
#include<math.h>
void main()
{
    float a,b,c,p1,p2,p3,t1,t2,t3,th2,th3,a1,a2,a3,del,
    rad,ph1,ph2,ph3,del1,del2,del3;
    clrscr();
    printf("enter values of th1,th2,th3,ph1,ph2,ph3;\n");
    scanf("%f%f%f%f%f%f",&th1,&th2,&th3,&ph1,&ph2,&ph3);
```

Conditions for this to be minimum are,

$$\frac{\partial S}{\partial k_1} = 0, \frac{\partial S}{\partial k_2} = 0 \text{ and } \frac{\partial S}{\partial k_3} = 0$$

i.e.
$$\sum_{i=1}^{n} 2[k_1 \cos\varphi_i + k_2 \cos\theta_i + k_3 - \cos(\theta_i - \varphi_i)]\cos\varphi_i = 0$$

or

$$k_1 \sum k_1 \cos^2 \varphi_i + k_2 \sum \cos\theta_i \cos\varphi_i + k_3 \sum \cos\varphi_i = \sum \cos(\theta_i - \varphi_i)\cos\varphi_i \tag{5.2}$$

Similarly,

$$k_1 \sum \cos\varphi_i \cos\theta_i + k_2 \sum \cos^2 \theta_i + k_3 \sum \cos\theta_i = \sum \cos(\theta_i - \varphi_i)\cos\theta_i \tag{5.3}$$

and

$$k_1 \sum \cos\varphi_i + k_2 \sum \cos\theta_i + k_3 \sum 1 = \sum \cos(\theta_i - \varphi_i) \tag{5.4}$$

These are three simultaneous linear, homogenous equations in three unknowns k_1, k_2, and k_3. These can be solved by using Cramer's rule or other means.

Figure 5.27 shows a program to find the ratio of different links using the least-square technique. The input variables are

```
#include<stdio.h>
#include<conio.h>
#include<math.h>
void main()
{
     int i,k;
     float a,b1,b2,b3,tt,b,c,d,p1,p2,p3,t1,t2,t3,th1,th2,
     th3,a1,a2,a3,del,rad,ph1,ph2,ph3,del1,del2,del3;
     float th[10],ph[10];
     clrscr();

     printf("enter i the number of positions\n");
     scanf("%d",&i);
     printf("enter i values of th[i] and ph[i]\n");
     for(k=0;k<i;k++)     scanf("%f",&th[k]);
     for(k=0;k<i;k++)     scanf("%f",&ph[k]);
     rad=4*atan(1)/180;
       for(k=0;k<i;k++)
       {
             p1=p1+pow(cos(ph[k]*rad),2);
             p2=p2+(cos(th[k]*rad))*(cos(ph[k)*rad));
             p3=p3+cos(ph[k]*rad);
             t1=p2;
             t2=t2+(cos(th[k]*rad)*(cos(th[k]*rad);
             t3=t3+cos(th[k)*rad);
             b1=p3;
             b2=t3;
             b3=i;
             tt=cos((th[k)-ph[k))*rad);
             a1=a1+tt*cos(ph[k]*rad);
```

a	b	c	d
1.56	0.66	1.33	1.00

Fig. 5.28

The mechanism is shown in Fig. 5.29.

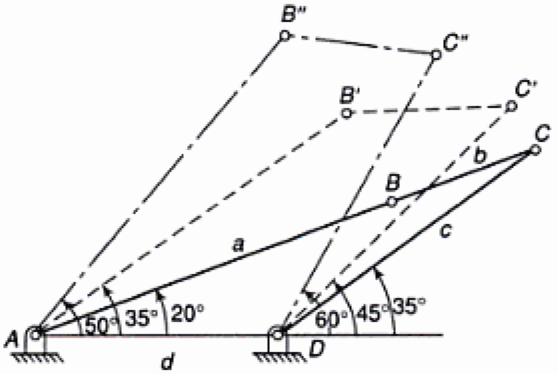


Fig. 5.29

Example 5.8 *Design a four-link mechanism when the motions of the input and the output links are governed by a function $y = x^2$ and x varies from 0 to 2 with an interval of 1. Assume θ to vary from 50° to 150° and ϕ from 80° to 160°.*

Solution The angular displacement of the input link is governed by x whereas that of the output link, by y.

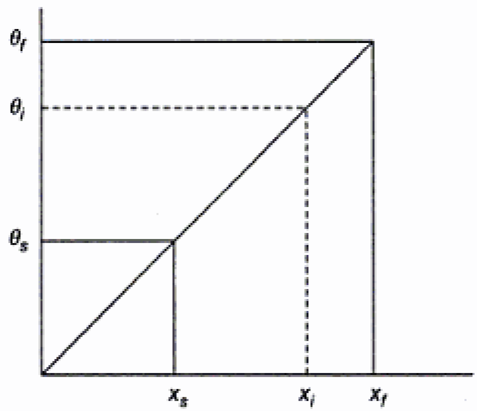


Fig. 5.30

Let subscripts s, f and i indicate the start, final and any value in the range.

Range of $x = x_f - x_s = 2 - 0 = 2$ and thus

$x_1 = 0; \quad x_2 = 1; \quad x_3 = 2$

The corresponding values of y are according to function, $y = x^2$

Range of $y = y_f - y_s = 4 - 0 = 4$ and ;

$y_1 = 0; \quad y_2 = 1; \quad y_3 = 4$

Range of $\theta = \theta_f - \theta_s = 150° - 50° = 100°$

Range of $\phi = \phi_f - \phi_s = 160° - 80° = 80°$

Refer Fig. 5.30 which indicates a linear relationship between x and θ. Thus

$$\frac{\theta_i - \theta_s}{\theta_f - \theta_s} = \frac{x_i - x_s}{x_f - x_s}$$

or

$$\theta_i = \theta_s + \frac{\theta_f - \theta_s}{x_f - x_s}(x_i - x_s) = \theta_s + \frac{\Delta\theta}{\Delta x}(x_i - x_s);$$

Thus, $\theta_1 = 50° + \dfrac{100°}{2} \times 0 = 50°$;

$$\theta_2 = 50° + \frac{100°}{2} \times 1 = 100°;$$

$$\theta_3 = 50° + \frac{100°}{2} \times 2 = 150°$$

Similarly,

$$\varphi_i = \varphi_s + \frac{\varphi_f - \varphi_s}{y_f - y_s}(y_i - y_s) = \varphi_s + \frac{\Delta\varphi}{\Delta y}(y_i - y_s);$$

or $\varphi_1 = 80° + \dfrac{80°}{4} \times 0 = 80°$;

$$\varphi_2 = 80° + \frac{80°}{4} \times 1 = 100°;$$

$$\varphi_3 = 80° + \frac{80°}{4} \times 4 = 160°$$

This can be written in a tabular form:

Position	x	y	θ	φ
1	0	0	50°	80°
2	1	1	100°	100°
3	2	4	150°	160°

Thus, we have the following equations,

$$k_1 \cos 80° + k_2 \cos 50° + k_3 = \cos 30°$$
$$k_1 \cos 100° + k_2 \cos 100° + k_3 = \cos 0°$$

$k_1 \cos 160° + k_2 \cos 150° + k_3 = \cos 10°$

Using Cramer's rule,

$\Delta = -0.3850$

$\Delta_1 = -0.1052 \quad k_1 = 0.273 = \dfrac{d}{a}$

$\Delta_2 = 0.1079 \quad k_2 = -0.280 = -\dfrac{d}{c}$

$\Delta_3 = 0.3844 \quad k_3 = 0.988 = \dfrac{a^2 - b^2 + c^2 + d^2}{2ac}$

which gives

$a = 3.66$ units
$b = 1.02$ units
$c = 3.57$ units
and $d = 1$ unit

Figure 5.31 shows the required mechanism.

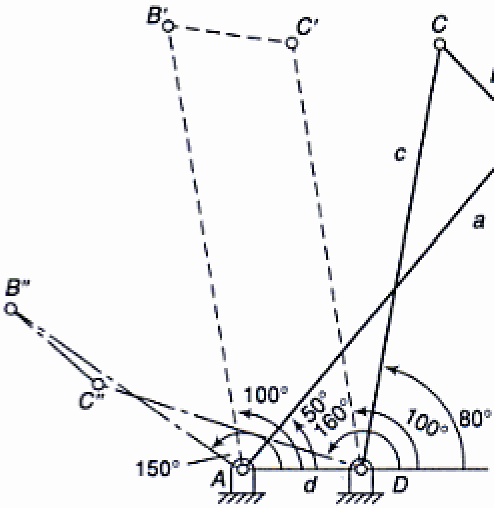


Fig. 5.31

Example 5.9 *Design a four-link mechanism when the motions of the input and the output links are governed by a function $y = 2\log_{10} x$ and x varies from 2 to 4 with an interval of 1. Assume θ to vary from 30° to 70° and ϕ from 40° to 100°.*

Solution Let subscripts s, f and i indicate the start, final and any value in the range.

Range of $x = x_f - x_s = 4 - 2 = 2$ and thus

$x_1 = 2; \quad x_2 = 3; \quad x_3 = 4$

The corresponding values of y are according to function, $y = 2\log_{10} x$

Range of $y = y_f - y_s = (2\log_{10}4) - (2\log_{10}2) = 1.204 - 0.602 = 0.602$

and $y_1 = 0.602; \quad y_2 = 2\log_{10} 3 = 0.954; \quad y_3 = 1.204;$

Range of $\theta = \theta_f - \theta_s = 70° - 30° = 40°$

Range of $\phi = \phi_f - \phi_s = 100° - 40° = 60°$

As $\theta_1 = \theta_s = 30°$ and $\theta_3 = \theta_f = 70°$, there is no need of finding them.

$$\theta_i = \theta_s + \frac{\Delta\theta}{\Delta x}(x_i - x_s) \text{ and thus}$$

$$\theta_2 = 30° + \frac{40°}{2} \times 1 = 50°$$

Similarly, As $\varphi_1 = \varphi_s = 40°$ and $\varphi_3 = \varphi_f = 100°$, there is no need of finding them.

$$\varphi_2 = 40° + \frac{60°}{0.602}(0.954 - 0.602) = 75°$$

This can be written in a tabular form also.

Position	x	y	θ	φ
1	2	0.602	30°	40°
2	3	0.954	50°	75°
3	4	1.204	70°	100°

Thus, we have the following equations,

$k_1 \cos 40° + k_2 \cos 30° + k_3 = \cos(-10°)$
$k_2 \cos 75° + k_2 \cos 50° + k_3 = \cos(-25°)$
$k_3 \cos 100° + k_2 \cos 70° + k_3 = \cos(-30°)$

Using Cramer's rule,

$\Delta = 0.0560$

$\Delta_1 = 0.0146 \quad k_1 = 0.2607 = \dfrac{d}{a}$

$\Delta_2 = -0.0135 \quad k_2 = -0.241 = -\dfrac{d}{c}$

$\Delta_3 = 0.0557$

$$k_3 = 0.995 = \frac{a^2 - b^2 + c^2 + d^2}{2ac}$$

which gives

$a = 3.83$ units
$b = 1.14$ units
$c = 4.14$ units
and $d = 1$ unit

Figure 5.32 shows the required mechanism.

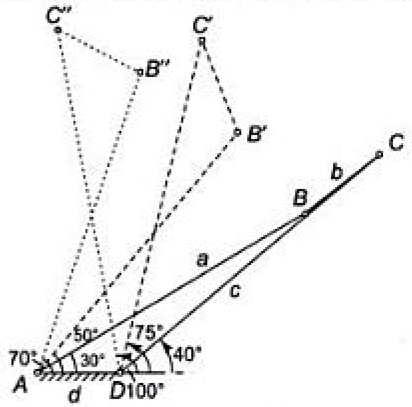


Fig. 5.32

Example 5.10 *Design a four-link mechanism to coordinate the motion of the input and the output links governed by a function $y = \log x$ for $0 < x \leq 8$. Take $\delta x = 1$. The range for θ is from 15° to 120° whereas for, φ it is from 20° to 150°.*

Solution The angular positions of the input and the output links are tabulated below:

X	y	θ	φ
1	0	15°	20°
2	0.69	30°	*63°
3	1.10	45°	89°
4	1.39	60°	107°
5	1.61	75°	121°
6	1.79	90°	132°
7	1.95	105°	142°
8	2.08	120°	150°

$$* \ 20° + (150° - 20°) \times \frac{0.69}{2.08} = 63°$$

It is required to design the mechanism so that the input and the output links pass through eight specified positions. It is not possible to design such a mechanism. However, using the least-square technique, a mechanism may be devised which gives the least deviation from the specified positions.

The dimensions of various links are shown in Fig. 5.33 using the program given in Fig. 5.27.

```
Enter i the number of specified positions 8
Enter i values of th[i] and ph[i]
  15  30  45  60  75  90  105  120
  20  63  89  107  121  132  142  150
   a       b        c    d
  2.42   0.91     2.37 1.00
```

Fig. 5.33

Figure 5.34 shows the required mechanism which will give the least deviation from the specified positions.

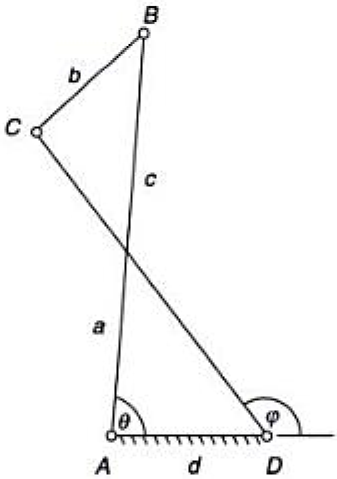


Fig. 5.34

5.9 CHEBYCHEV SPACING

In function-generation problems, the output is related to the input through a function $y = f(x)$ and it is required to obtain the dimensions of a linkage to satisfy this relationship. In general, a linkage synthesis problem does not have exact solution over its entire range of travel. However, it is usually possible to design a linkage which exactly satisfies the desired function at a few chosen positions known as *precision* or *accuracy*

5.10 PATH GENERATION

A four-link mechanism $ABCD$ with a coupler point E is shown in Fig. 5.38. Three positions of the input link (θ_1, θ_2, θ_3) and three positions of the coupler point E given by three values of r and α, i.e., r_1, r_2, r_3 and α_1, α_2, α_3 are known. It is required to find the dimensions of a, c, e and f along with the location of the pivots A and D given by g, γ and h, ψ respectively so that the coupler point E generates the specified path with the motion of the input link AB.

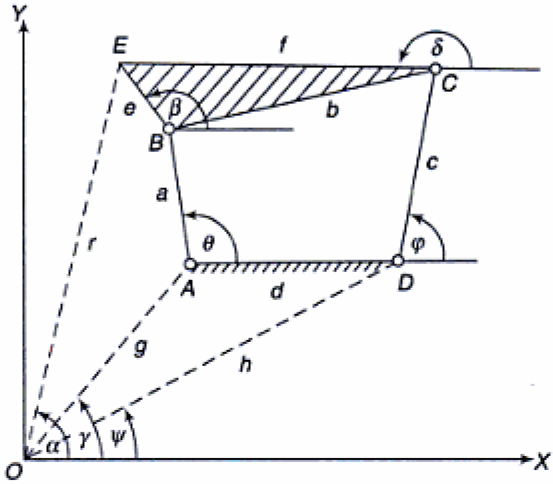


Fig. 5.38

For the loop $OABE$, considering the links to be vectors

$$g \cos \gamma + a \cos \theta + e \cos \beta - r \cos \alpha = 0 \quad (5.5)$$

and
$$g \sin \gamma + a \sin \theta + e \sin \beta - r \sin \alpha = 0 \quad (5.6)$$

or
$$e \cos \beta = r \cos \alpha + g \cos \gamma - a \cos \theta$$

and
$$e \sin \beta = r \sin \alpha + g \sin \gamma - a \sin \theta$$

Squaring and adding,

$$e^2 = r^2 + g^2 + a^2 - 2gr (\cos \alpha \cos \gamma + \sin \alpha \sin \gamma)$$
$$+ 2ag (\cos \theta \cos \gamma + \sin \theta \sin \gamma) - 2ar (\cos \theta \cos \alpha + \sin \theta \sin \alpha)$$

or
$$2ar \cos (\theta - \alpha) + 2gr \cos (\alpha - \gamma) + (e^2 - a^2 - g^2) = r^2 + 2ag \cos (\theta - \gamma)$$

or
$$2ar \cos (\theta - \alpha) + 2gr \cos (\alpha - \gamma) + k = r^2 + 2ag \cos (\theta - \gamma) \quad (5.7)$$

where
$$k = e^2 - a^2 - g^2 \quad (5.8)$$

Inserting the values of r_1, r_2, r_3; $\alpha_1, \alpha_2, \alpha_3$ and $\theta_1, \theta_2, \theta_3$, we obtain three equations. The unknowns are a, g, e and γ. Thus, for three equations, there are four unknowns and therefore, equations cannot be solved. However, the value of one of the unknown can be assumed. Assuming the value of γ, we are left with three unknowns a, g, e and there are three equations to solve them.

Even now, the equations cannot be solved as such, as these are non-linear equations. However, by making the following substitutions, these can be solved easily.

Let

$$\left.\begin{aligned} a &= l_a + \lambda m_a \\ g &= l_g + \lambda m_g \\ k &= l_k + \lambda m_k \end{aligned}\right\} \quad (5.9)$$

where
$$\begin{aligned}\lambda &= ag \\ &= (l_a + \lambda m_a)(l_g + \lambda m_g) \\ &= l_a l_g + \lambda l_a m_g + \lambda l_g m_a + \lambda^2 m_a m_g\end{aligned}$$

or
$$m_a m_g \lambda^2 + (l_a m_g + l_g m_a - 1)\lambda + l_a l_g = 0$$

or
$$A\lambda^2 + B\lambda + C = 0$$

or
$$\lambda = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \quad (5.10)$$

where
$$A = m_a m_g,$$

$$B = l_a m_g + l_g m_a - 1$$
$$C = l_a l_g$$

Thus, Eq. (5.7) becomes,

$$2(l_a + \lambda m_a) r \cos(\theta - \alpha) + 2(l_g + \lambda m_g) r \cos(\alpha - \gamma) + l_k + \lambda m_k = r^2 + 2\lambda \cos(\theta - \gamma)$$

Separating the components into two groups; one with and the other without λ,

$$l_a[2r\cos(\theta - \alpha)] + l_g[2r\cos(\alpha - \gamma)] + l_k = r^2 \tag{5.11}$$

$$m_a[2r\cos(\theta - \alpha)] + m_g[2r\cos(\alpha - \gamma)] + m_k = 2\cos(\theta - \gamma) \tag{5.12}$$

From Eq. (5.11), three equations can be written as,

$$l_a[2r_1\cos(\theta_1 - \alpha_1)] + l_g[2r_1\cos(\alpha_1 - \gamma)] + l_k = r_1^2 \tag{5.13}$$

$$l_a[2r_2\cos(\theta_2 - \alpha_2)] + l_g[2r_2\cos(\alpha_2 - \gamma)] + l_k = r_2^2 \tag{5.14}$$

$$l_a[2r_3\cos(\theta_3 - \alpha_3)] + l_g[2r_3\cos(\alpha_3 - \gamma)] + l_k = r_3^2 \tag{5.15}$$

These are three linear equations l_a, l_g and l_k and can be solved by applying Cramer's rule or by other means.

Similarly, m_a, m_g, and m_k can also be found.

As l_a, l_g, l_k and m_a, m_g, m_k have been found, a, g and k can be calculated from the relations of Eq. (5.9).

Also, $e = \sqrt{k + a^2 + g^2}$ [from Eq. (5.8)]

From Eq. (5.5), three values of β can be found,

$$e\cos\beta = r\cos\alpha - g\cos\gamma - a\cos\theta$$

$$\beta_1 = \cos^{-1}\left[\frac{r_1\cos\alpha_1 - g\cos\gamma - a\cos\theta_1}{e}\right] \tag{5.16}$$

Similarly, β_2 and β_3 can be found.

Thus, we have obtained the values of a, e, g, γ and β. The whole procedure can be repeated for the loop *ODCE*. The following equations are formed,

$$h\cos\psi + c\cos\varphi + f\cos\delta - r\cos\alpha = 0 \tag{5.17}$$

$$h\sin\psi + c\sin\varphi + f\sin\delta - r\sin\alpha = 0 \tag{5.18}$$

These equations are similar to Eqs (5.5) and (5.6).

Assuming

$$f = l_f + \lambda' m_f$$
$$h = l_h + \lambda' m_h$$
$$p = l_p + \lambda' m_p$$

Two sets of equations similar to Eqs (5.11) and (5.12) are obtained by eliminating ϕ as given below:

$$l_f[2r\cos(\delta - \alpha)] + l_k[2r\cos(\alpha - \psi)] + l_p = r^2 \tag{5.19}$$

$$m_f[2r\cos(\delta - \alpha)] + m_h[2r\cos(\alpha - \psi)] + m_p = 2\cos(\delta - \psi) \tag{5.20}$$

In these equations, α and r are known. ψ can be assumed. Also, assuming δ_1, the values of δ_2 and δ_3 can be found as follows:

The angular displacements of the coupler link *BCE* is the same at the points *B* and *C*,

$$\delta_2 - \delta_1 = \beta_2 - \beta_1$$

or

$$\delta_2 = \delta_1 + (\beta_2 - \beta_1) \tag{5.21}$$

Similarly,

$$\delta_3 = \delta_1 + (\beta_3 - \beta_1) \tag{5.22}$$

Solving the Eqs (5.19) and (5.20), the values of f, h and c can be known.

As the points *A, B, E, C* and *D* are located, the dimensions *a, b, c, d, e* and *f* can be obtained. Figure 5.39 shows a program for the solution of such a problem.

```
#include<stdio.h>
#include<conio.h>
#include<math.h>
void main()
{
     FILE*fp;
     int k,j;
     float a1,a2,a3,a11,a22,a33,a12,a21,g12,g21,e12,e21,ak1,
     ak2,ak3,a11,a12,a13,aa,g1,g2,g3,t11,t22,t33,tb1,tb2,tb3,
     gg,gamm,ss,si,d11,d22,d33,r1,r2,r3,p1,p2,p3,t1,t2,t3,
     c1,c2,c3,alg,ala,alk,ama,amg,amk,bb,cc,a11,e1,e2,squ,
     bet1,bet2,bet3,p11,p22,p33,e3,gs,del,del1,del2,del3,rad;
     clrscr();
     printf("Enter values of tb1,tb2,tb3,r1,r2,r3,a11,a11,
     a12,a13,");
     printf("gamm,si,del1\n");
     scanf("%f%f%f%f%f%f%f%f%f%f%f%f",&tb1,&tb2,&tb3,&r1,
     &r2,&r3,&a11,&a12,&a13,&gamm,&si,&del1);
     rad=4*atan(1)/180;
     t11=tb1*rad;
     t22=tb2*rad;
     t33=tb3*rad;
     a11=a11*rad;
     a22=a12*rad;
     a33=a13*rad;
     gg=gamm*rad;
     ss=si*rad;
     d11=del1*rad;
      for(j=0;j<3;j++)
      {
        p1=2*r1*cos(t11-a11);
        p2=2*r2*cos(t22-a22);
        p3=2*r3*cos(t33-a33);
        t1=2*r1*cos(a11-gg);
        t2=2*r2*cos(a22-gg);
        t3=2*r3*cos(a33-gg);
        c1=r1*r1;
        c2=r2*r2;
        c3=r3*r3;
           for(k=0;k<2;k++)
           {
           del=p1*(t2-t3)+t1*(p3-p2)+(p2*t3-p3*t2);
           del1=c1*(t2-t3)+t1*(c3-c2)+(c2*t3-c3*t2);
           del2=p1*(c2-c3)+c1*(p3-p2)+(p2*c3-p3*c2);
           del3=p1*(t2*c3-t3*c2)+t1*(c2*p3-c3*p2);
           +c1*(p2*t3-p3*t2);
           ak1=del1/del;
```

```
 ak2=del2/del;
 ak3=del3/del;
 if(k==0)
 {
        ala=ak1;
        alg=ak2;
        alk=ak3;
        c1=2*cos(t11-gg);
        c2=2*cos(t22-gg);
        c3=2*cos(t33-gg);
        }
 }
ama=ak1;
amg=ak2;
amk=ak3;
aa=ama*amg;
bb=ala*amg+alg*ama-1;
cc=ala*alg;
squ=bb*bb-4*aa*cc;
if(squ>0)
 {
 all=sqrt(squ);
 a11=(-bb-all)/(2*aa);
 a12=(-bb+all)/(2*aa);
 a1=ala+a11*ama;
 g1=alg+a11*amg;
 a2=ala+a12*ama;
 g2=alg+a12*amg;
 e1=sqrt(alk+a11*amk+a1*a1+g1*g1);
 e2=sqrt(alk+a12*amk+a2*a2+g2*g2);
 if(j==0){printf("   g      a      e");
 printf("     h      c      f\n"); }
 if(j==1){printf("%8.2f %8.2f %8.2f %8.2f %8.2f %8.2f
 \n",g12,a12,e12,g1,e1,a1);
 printf("%8.2f %8.2f %8.2f %8.2f %8.2f %8.2f\n",
 g12,a12,e12,g2,e2,a2); }
 if (j==2) {printf("%8.2f %8.2f %8.2f %8.2f %8.2f
 %8.2f \n",g21,a21,e21,g1,e1,a1);
 printf("%8.2f %8.2f %8.2f %8.2f %8.2f %8.2f \n",
 g21,a12,e21,g2,e2,a2); }
  if(j==0)
  {
        g12=g1;
        a12=a1;
        e12=e1;
        g21=g2;
        a21=a2;
        e21=e2;
        gs=gg;
        p11=t11;
```

```
                p22=t22;
                p33=t33;
            }
            if(j==1)
            {
                g1=g21;
                a1=a21;
                e1=e21;
                t11=d11;
                t22=d22;
                t33=d33;
                gg=gs;
                t11=p11;
                t22=p22;
                t33=p33;
                }
            bet1=acos((r1*cos(a11)-g1*cos(gg)-a1*cos(t11))/e1);
            bet2=acos((r2*cos(a22)-g1*cos(gg)-a1*cos(t22»/e1);
            bet3=acos((r3*cos(a33)-g1*cos(gg)-a1*cos(t33»)/e1);
            d22=d11+bet2-bet1;
            d33=d11+bet3-bet1;
            a3=a2;
            g3=g2;
            e3=e2;
            p11=t11;
            p22=t22;
            p33=t33;
            t11=d11;
            t22=d22;
            t33=d33;
            gs=gg;
            gg=ss;
    }
 }
 getch();
}
```

Fig. 5.39

The input variables are:

`tb1, tb2, tb3`	angular displacement of the input link *AB* (degrees)
`r1, r2, r3`	radial distances of the coupler point from origin (mm)
`al1, al2, al3`	angular position of the coupler point (degrees)
`gamm`	assumed value of the angle γ (degrees)
`si`	assumed value of the angle ψ (degrees)
`del1`	assumed value of the angle δ_1 (degrees)

The output variables are

`g, a, e, h, c, f`	distances or lengths of the links in mm

If more than three positions of the input link along with the same number of positions of the coupler point from Eq. (5.5) and (5.6) instead of β. The equations formed are exactly the same if θ is replaced by β in Eqs (5.11) and (5.12). Also, as β is directly known, there is no need of using Eq. 5.16.

The program given in Fig. 5.43 solves this type of problem. The input variables are

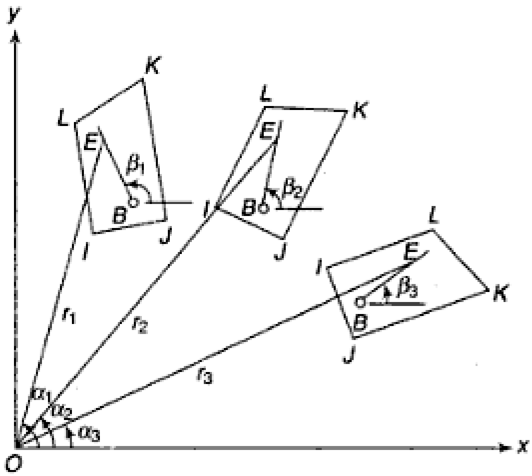


Fig. 5.42

```
#include<stdio.h>
#include<conio.h>
#include<math.h>
void main ( )
 {
  FILE *fp;
  int k,j;
  float a1,a2,a3,a11,a22,a33,a12,a21,g12
  ,g21,e12,e21,ak1,
  ak2,ak3,al1,al2,al3,aa,g1,g2,g3,t11,t
  22,t33,tb1,tb2,
  tb3,gg,gamm,ss,si,d11,d22,d33,r1,r2,r
  3, p1,p2,p3,t1,t2,
  t3,c1,c2,c3,alg,ala,alk,ama,amg,amk,bb,cc,all,e1,e2,
  squ,bet1,bet2,bet3,p11,p22,p33,e3,gs,del,del1,del2,
  del3,rad;
  clrscr( ) ;
  printf("Enter values of tb1,tb2,tb3,r1,r2,r3,al1,al1,");
  printf("al2,al3,gamm,si,del1\n");
  scanf("%f%f%f%f%f%f%f%f%f%f%f%f",&tb1,&tb2,&tb3,&r1,&r2,
  &r3,&al1,&al2,&al3,&gamm,&si,&del1);
  rad=4*atan(1)/180;
  t11=tb1*rad;
  t22=tb2*rad;
  t33=tb3*rad;
  a11=al1*rad;
  a22=al2*rad;
  a33=al3*rad;
  gg=gamm*rad;
  ss=si*rad;
  d11=del1*rad;
  for (j=0; j<3; j++)
  {
   p1=2*r1*cos(t11-a11);
   p2=2*r2*cos(t22-a22);
   p3=2*r3*cos(t33-a33);
   t1=2*r1*cos(a11-gg);
   t2=2*r2*cos(a22-gg);
   t3=2*r3*cos(a33-gg);
   c1=r1*r1;
   c2=r2*r2 ;
   c3=r3*r3;
    for(k=0;k<2;k++)
```

```
{
 del=p1*(t2-t3)+t1*(p3-p2)+(p2*t3-p3*t2);
 del1=c1*(t2-t3)+t1*(c3-c2)+(c2*t3-c3*t2);
 del2=p1*(c2-c3)+c1*(p3-p2)+(p2*c3-p3*c2);
 del3=p1*(t2*c3-t3*c2)+t1*(c2*p3-c3*p2)
 +c1*(p2*t3-p3*t2);
 ak1=del1/del;
 ak2=del2/del;
 ak3=del3/del;
  if (k==0)
  {
   a1a=ak1;
   a1g=ak2;
   a1k=ak3;
   c1=2*cos(t11-gg);
   c2=2*cos(t22-gg):
   c3=2*cos(t33-gg):
  }
 }
ama=ak1;
amg=ak2:
amk=ak3:
aa=ama*amg:
bb=a1a*amg+a1g*ama-1:
cc=a1a*a1g;
squ=bb*bb-4*aa*cc:
 if (squ>0)
 {
  a11=sqrt(squ):
  a11=(-bb-a11)/(2*aa):
  a12=(-bb+a11)/(2*aa):
  a1=a1a+a11*ama:
  g1=a1g+a11*amg:
  a2=a1a+a12*ama;
  g2=a1g+a12*amg:
  e1=sqrt(a1k+a11*amk+a1*a1+g1*g1):
  e2=sqrt(a1k+a12*amk+a2*a2+g2*g2):
  if(j==0) ( printf("       g       e      a")
        printf("   h        c    f\n"):   }
  if(j==1) (printf("%8.2f %8.2f %8.2f %8.2f %8.2f
  %8.2f \n",g12,a12,e12,g1,e1,a1):
  printf("%8.2f %8.2f %8.2f %8.2f %8.2f %8.2f \n",
  g12,a12,e12,g2,e2,a2): }
  if(j==2){printf("%8.2f %8.2f %8.2f %8.2f %8.2f
  %8.2f \n",g21,a21,e21,g1,e1,a1):
  printf("%8.2f %8.2f %8.2f %8.2f %8.2f %8.2f \n",
  g21,a21,e21,g2,e2,a2);}
  if (j==0)
  {
        g12=g1:
```

```
            a12=a1;
            e12=e1:
            g21=g2:
            a21=a2:
            e21=e2:
            gs=gg:
            p11=t11:
            p22=t22;
            p33=t33:
        }
        if(j==1)
        {
            g1=g21:
            a1=a21:
            e1=e21;
            t11=d11:
            t22=d22:
            t33=d33:
            gg=gs:
            t11=p11;
            t22=p22;
            t33=p33;
      }
    d22=d11+t22-t11;
    d33=d11 +t33-t11;
    a3=a2;
    g3=g2;
    e3=e2;
    p11=t11;
    p22=t22;
    p33=t33;
    t11=d11;
    t22=d22;
    t33=d33;
    gs=gg;
    gg=ss;
  }
        }
 getch( );
}
```

Fig. 5.43

The input variables are

`tb1, tb2, tb3`	angles β_1, β_2 and β_3 respectively (degrees)
`r1, r2, r3`	radial distances of point *E* from the origin (mm)
`al1, al2, al3`	angular position of point *E* (degrees)
`gamm`	assumed value of the angle γ (degrees)

`si`	assumed value of the angle ψ (degrees)
`dell`	assumed value of the angle δ_1 (degrees

The output variables are

`g, a, e, h, c, f`	distances or lengths of the links in mm

Example 5.13 *Design a four-link mechanism to guide a rigid body through three finitely separated positions given by*

$\beta_1 = 105°$	$r_1 = 80$ mm	$\alpha_1 = 65°$
$\beta_2 = 95°$	$r_2 = 90$ mm	$\alpha_2 = 56°$
$\beta_3 = 85°$	$r_3 = 96$ mm	$\alpha_3 = 48°$

Assume the values of γ, ψ and δ_1 as 20°, 10° and 150° respectively.

Solution: Figure 5.44 shows the input and the four sets of values of the output obtained by using the program of Fig. 5.43.

```
Enter values of tb1, tb2, tb3, r1, r2, r3, al1, al2, al3, gamm, si, dell
105  95  85  80  90  96  65  56  48  20 10  150
      g             e             a             h             c             f
   27.04        102.24         49.76         32.33        240.67       -201.61
   27.04        102.24         49.76         81.82         27.25         66.79
   53.75         40.60         16.14         32.33        240.67       -201.61
   53.75         40.60         16.14         81.82         27.25         66.79
```

Fig. 5.44

Summary

1. Dimensional synthesis of a pre-conceived type mechanism seeks to determine the principal dimensions of various links that satisfy the requirements of motion of the mechanism.
2. *Function generation* involves correlating the rotary or the sliding motion of the input and the output links. The motion of the output and the input links may be prescribed by an arbitrary function $y = f(x)$.
3. When a point on the coupler or the floating link of a mechanism is to be guided along a prescribed path, it is said to be a *path-generation* problem. This guidance of the path of the point may or may not be coordinated with the movement of the input link.
4. *In motion generation*, a mechanism is designed to guide a rigid body in a prescribed path.
5. A *pole* of a moving link is the centre of its rotation with respect to a fixed link.
6. If the rotation of the link is considered relative to another moving link, the pole is known as the *relative pole.*
7. The problems of function generation for two and three accuracy positions are easily solved by the relative pole method.
8. In the inversion method, there is direct use of the concept of inversion.
9. *Freudenstein's* equation is
$$\frac{d}{a}\cos\varphi - \frac{d}{c}\cos\theta + \frac{a^2 - b^2 + c^2 + d^2}{2ac} = \cos(\theta - \varphi) = \cos(\varphi - \theta)$$
and is used to coordinate positions of the input and output links of the four-link mechanism.
10. For n accuracy positions in the range $x_0 \le x \le x_{n+1}$, the Chebychev spacing is given by
$$x_i = \frac{x_{n+1} + x_0}{2} - \frac{x_{n+1} - x_0}{2}\cos\frac{(2i-1)\pi}{2n}$$
where i = 1, 2, 3 ....n

Exercises

1. What do you mean by dimensional synthesis of a pre-conceived type mechanism?
2. Explain the terms: function generation, path generation and motion generation.

3. What is the pole of a coupler link of four-link mechanism? Enumerate its properties. What is a relative pole?
4. Describe the procedure to design a four-link mechanism by relative pole method when three positions of the input $(\theta_1, \theta_2, \theta_3)$ and the output link $(\varphi_1, \varphi_2, \varphi_3)$ are known.
5. Describe the procedure to design a slider-crank mechanism by relative pole method when three positions of the input link $(\theta_1, \theta_2, \theta_3)$ and the slider (s_1, s_2, s_3) are known.
6. Discuss the procedure to design the mechanisms by inversion method.
7. What is *Freudenstein's* equation? How is it helpful in designing a four-link mechanism when three positions of the input $(\theta_1, \theta_2, \theta_3)$ and the output link $(\varphi_1, \varphi_2, \varphi_3)$ are known?
8. What is least-square technique? When is it useful in designing a four-link mechanism?
9. What do you mean by *precision* or *accuracy* points in the design of mechanisms? What is *structural error*?
10. What is *Chebychev spacing*? What is its significance?
11. Design a four-link mechanism to coordinate three positions of the input and the output links for the following angular displacements using relative pole method:
    $\theta_{12} = 50°$ $\quad\varphi_{12} = 40°$
    $\theta_{13} = 70°$ $\quad\varphi_{13} = 75°$
12. Design a slider-crank mechanism to coordinate three positions of the input link and the slider for the following angular and linear displacements of the input link and the slider respectively:
    $\theta_{12} = 30°$ $\quad s_{12} = 100$ mm
    $\theta_{13} = 90°$ $\quad s_{13} = 200$ mm
    Take eccentricity of the slider as 10 mm. Use the relative pole method.
13. In a four-link mechanism, the angular displacements of the input link are 30° and 75° and of the output link, 40° and 65° respectively. Design the mechanism using the inversion method.
14. Design a slider-crank mechanism to coordinate three positions of the input and of the slider when the angular displacements of the input link are 40° and 75° and linear displacements of the slider are 55 mm and 90 mm respectively with an eccentricity of 20 mm. Use the inversion method.
15. For the following angular displacements of the input and the output links, design a four-link mechanism:
    $\theta_{12} = 40°$ $\quad\varphi_{12} = 45°$
    $\theta_{13} = 85°$ $\quad\varphi_{13} = 75°$
    $\theta_{13} = 120°$ $\quad\varphi_{13} = 110°$
16. Design a four-link mechanism that coordinates the following three positions of the coupler point if the positions are indicated with respect to coordinate axes:
    $r_1 = 60$ mm $\quad\alpha_1 = 75°$
    $r_2 = 75$ mm $\quad\alpha_2 = 60°$
    $r_3 = 85$ mm $\quad\alpha_3 = 50°$
    The angular displacements of the input link are $\theta_{12} = 40°$ and $\theta_{13} = 75°$.
17. Design a four-link mechanism to coordinate three positions of the input and the output links given by
    $\theta_1 = 25°$ $\quad\varphi_1 = 30°$
    $\theta_2 = 35°$ $\quad\varphi_2 = 40°$
    $\theta_3 = 50°$ $\quad\varphi_3 = 60°$ (5.6, 0.17, 4.88, 1)
18. Design a four-link mechanism when the motions of the input and the output links are governed by the function $y = 2x^2$ and x varies from 2 to 4 with an interval of 1. Assume θ to vary from 40° to 120° and ϕ from 60° to 132°. (1.73, 0.70, 1.78, 1.00)
19. Design a four-link mechanism to coordinate the motions of the input and the output links governed by a function $y = 2 \log x$ for $2 < x < 12$. Take $\Delta x = 1$. Assume suitable ranges for θ and φ.
20. Design a four-link mechanism if the motions of the input and the output links are governed by a function $y = x^{1.5}$ and x varies from 1 to 4. Assume θ to vary from 30° to 120° and φ from 60° to 130°. The length of the fixed link is 30 mm. Use Chebychev spacing of accuracy points.
21. Design a four-link mechanism to guide a rigid body through three positions of the input link with three positions of the coupler point, the data for which is given below:

$\theta_1 = 40°$	$r_1 = 90$ mm	$\alpha_1 = 78°$
$\theta_2 = 55°$	$r_2 = 40$ mm	$\alpha_2 = 90°$
$\theta_3 = 70°$	$r_3 = 75$ mm	$\alpha_3 = 95°$

22. Design a four-link mechanism, the coupler point of which traces a coupler curve that is approximated by ten positions given by the following data

$\theta_1 = 160°$	$r_1 = 57$ mm	$\alpha_1 = 70°$
$\theta_2 = 130°$	$r_2 = 76$ mm	$\alpha_2 = 65°$
$\theta_3 = 98°$	$r_3 = 88$ mm	$\alpha_3 = 55°$
$\theta_4 = 73°$	$r_4 = 98$ mm	$\alpha_4 = 45°$
$\theta_5 = 32°$	$r_5 = 92$ mm	$\alpha_5 = 30°$
$\theta_6 = -15°$	$r_6 = 89$ mm	$\alpha_6 = 20°$
$\theta_7 = -25°$	$r_7 = 82$ mm	$\alpha_7 = 19°$
$\theta_8 = -70°$	$r_8 = 53$ mm	$\alpha_8 = 25°$
$\theta_9 = -125°$	$r_9 = 38$ mm	$\alpha_9 = 50°$
$\theta_{10} = -165°$	$r_{10} = 42$ mm	$\alpha_{10} = 70°$

$$OA = OQ; \qquad AB = AC$$

and $$BP = PC = CQ = QB$$

OA is the fixed link and OQ is a rotating link. It can be proved that as the link OQ moves around O, P moves in a straight line perpendicular to OA. All the joints are pin-jointed.

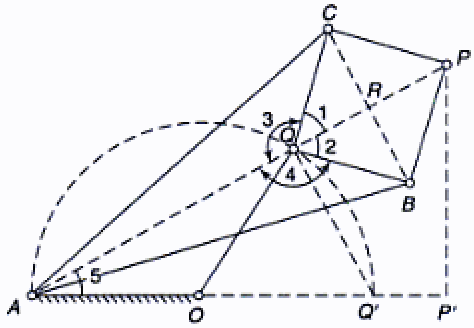


Fig. 6.2

Since $BPCQ$ is a rhombus,

QP always bisects the angle BQC,

i.e.,

$$\angle 1 = \angle 2 \qquad \text{(i)}$$

in all the positions

Also, in Δs AQC and AQB,

AQ is common,

$AC = AB$

$QC = QB$

$\therefore$ Δs are congruent in all positions.

or $$\angle 3 = \angle 4 \qquad \text{(ii)}$$

Adding (i) and (ii),

$\angle 1 + \angle 3 = \angle 2 + \angle 4$

But $\angle 1 + \angle 2 + \angle 3 + \angle 4 = 360°$

$\therefore$ $\angle 1 + \angle 3 = \angle 2 + \angle 4 = 180°$

or A, Q and P lie on a straight line.

Let PP' be the perpendicular on AO produced.

Δs AQQ' and APP' are similar because $\angle 5$ is common and $\angle AQQ' = \angle AP'P = 90°$

$$\therefore \quad \frac{AQ}{AP'} = \frac{AQ'}{AP}$$

or
$$\begin{aligned} AQ'.AP' &= (AQ)(AP) \\ &= (AR - RQ)(AR + RP) \\ &= (AR - RQ)(AR + RQ) \qquad \ldots (RP = RQ) \\ &= (AR)^2 - (RQ)^2 \\ &= [(AC)^2 - (CR)^2) - [(CQ)^2 - (CR)^2] \end{aligned}$$

or
$$AP' = \frac{(AC)^2 - (CQ)^2}{AQ'}$$

= constant, as AC, CQ and AQ' are always fixed

This means that the projection of P and AQ produced is constant for all the configurations.

Thus, PP' is always a normal to AO produced or P moves in a straight line perpendicular to AO.

Example 6.1 *Figure 6.3(a) shows the link MAC which oscillates on a fixed centre A. Another link OQ oscillates on the centre O. The links AB and AC are equal. Also BP = PC = CQ = QB. Locate the position of O such that,*

(i) P moves in a straight line

(ii) P moves in a circle with centre A

(iii) P moves in a circle with centre at OA produced

(iv) P moves in a circle with centre O and Q moves in a straight line

(modify the lengths of links if necessary)

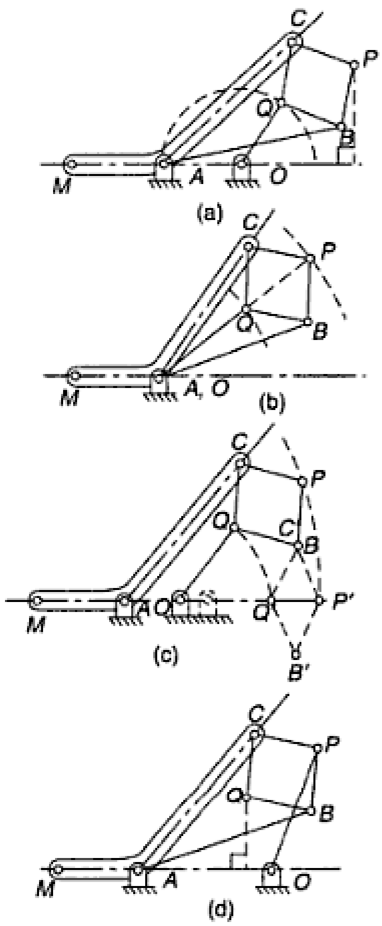


Fig. 6.3

Solution

(i) As in the Paucellier mechanism, O is located by drawing a straight line through A and perpendicular to the motion of P such that $AO = OQ$ [Fig. 6.3(a)].

(ii) If O is made to coincide with A, AQ would be equal to OQ. Thus, Q and P will be fixed on AP. Q will rotate about A and thus P will also rotate in a circle about A with AP as the radius [Fig. 6.3(b)].

(iii) From the above two cases, it can be observed that in (i) P moves in a circle with the centre at infinity on OA produced and in (ii) P moves in circle with the centre at A. Thus, if P is to move in a circle with the centre in-between A and infinity on OA produced, O must lie in-between O and A or in other words OQ should be greater than OA [Fig. 6.3(c)].

(iv) The mechanism will be similar to the Paucellier mechanism. P is to be joined with O by a link so that P moves in a circle about O and $OA = OP$. The lengths can be modified in two ways [Fig. 6.3(d)].

(a) OA is increased and OA and OP are made equal.

(b) Lengths AB and AC are reduced in such a way that $OA = OP$.

2. Hart Mechanism

A Hart mechanism consists of six links as shown in Fig. 6.4 such that

$$AB = CD; \quad AD = BC \quad \text{and } OE = OQ$$

OE is the fixed link and OQ, the rotating link. The links are arranged in such a way that $ABDC$ is a trapezium (AC parallel to BD). Pins E and Q on the links AB and AD respectively, and the point P on the link CB are located in such a way that

$$\frac{AE}{AB} = \frac{AQ}{AD} = \frac{CP}{CB} \qquad \text{(i)}$$

It can be shown that as OQ rotates about O, P moves in a line perpendicular to EO produced.

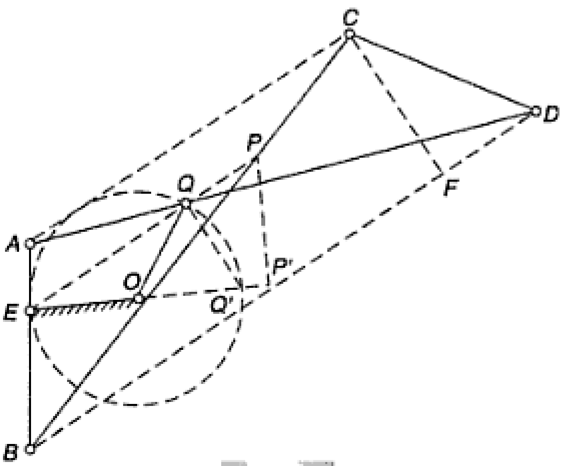


Fig. 6.4

In ΔABD $\dfrac{AE}{AB} = \dfrac{AQ}{AD}$ (Given)

Therefore, EQ is parallel to BD and thus parallel to AC.

In ΔABC $\dfrac{AE}{AB} = \dfrac{CP}{CB}$ (Given)

Therefore, EP is parallel to AC and thus parallel to BD.

Now, EQ and EP are both parallel to AC and BD and have a point E in common; therefore, EQP is a straight line.

Δs AEQ and ABD are similar ($\because EQ \parallel BD$).

$$\therefore \qquad \frac{EQ}{BD} = \frac{AE}{AB} \text{ or } EQ = BD \times \frac{AE}{AB} \qquad \text{(ii)}$$

Δs BEP and BAC are similar ($\because EP \parallel AC$).

$$\therefore \qquad \frac{EP}{AC} = \frac{BE}{BA} \text{ or } EP = AC \times \frac{BE}{AB} \qquad \text{(iii)}$$

Δs EQQ' and $EP'P$ are similar, because $\angle QEQ'$ or $\angle PEP'$ is common and $\angle EQQ' = \angle QP'P = 90°$.

$$\therefore \qquad \frac{EQ}{EP'} = \frac{EQ'}{EP}$$

or
$$EQ' \times EP' = EQ \times EP$$
$$= \left(BD \times \frac{AE}{AB}\right)\left(AC \times \frac{BE}{AB}\right) \qquad \text{[from (ii) and (iii)]}$$

or
$$EP' = \frac{AE \times BE}{(EQ')(AB)^2}[(BD)(AC)]$$
$$= \frac{AE \times BE}{(EQ')(AB)^2}\left[(BF + FD)(BF - FD)\right]$$
$$= \frac{AE \times BE}{(EQ')(AB)^2}\left[(BF)^2 - (FD)^2\right]$$
$$= \frac{AE \times BE}{(EQ')(AB)^2}\left[\{(BC)^2 - (CF)^2\} - \{(CD)^2 - (CF)^2\}\right]$$
$$= \frac{AE \times BE}{(EQ')(AB)^2}\left[(BC)^2 - (CD)^2\right]$$

= constant, as all the parameters are fixed.

Thus, EP' is always constant. Therefore, the projection of P on EO produced is always the same point or P moves in a straight line perpendicular to EO.

Example 6.2 *A circle with EQ' as diameter has a point Q on its circumference. P is a point on EQ produced such that if Q turns about E, $EQ.EP$ is constant. Prove that the point P moves in a straight line perpendicular to EQ'.*

Solution Let PP' be perpendicular to EQ' produced (Fig. 6.5).

For any position of Q on the circumference of the circle with diameter EQ', Δs EQQ' and $EP'P$ are similar ($\angle QEQ'$ is common and $\angle EQQ' = \angle EP'P = 90°$).

$$\therefore \qquad \frac{EQ}{EQ'} = \frac{EP'}{EP}$$

or $\qquad EQ'.\,EP' = EQ.\,EP$

or $$EP' = \frac{EQ.EP}{EQ'}$$

= constant, as EQ' is fixed and EQ. EP = constant

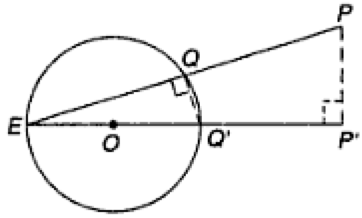


Fig. 6.5

Thus, EP' will be constant for all positions of Q. Therefore, the location of P' is fixed which means that P moves in a straight line perpendicular to EQ'.

3. Scott–Russel Mechanism

A Scott–Russel mechanism consists of three movable links; OQ, PS and slider S which moves along OS. OQ is the crank (Fig. 6.6). The links are connected in such a way that

$$QO = QP = QS$$

It can be proved that P moves in a straight line perpendicular to OS as the slider S moves along OS.

As $QO = QP = QS$, a circle can be drawn passing through O, P and S with PS as the diameter and Q as the centre.

Now, O lies on the circumference of the circle and PS is the diameter. Therefore, $\angle POS$ is a right angle. This is true for all the positions of S and is possible only if P moves in a straight line perpendicular to OS at O.

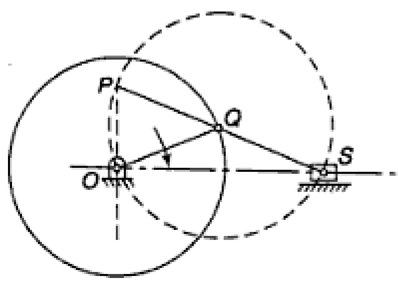


Fig. 6.6

Note that in such a mechanism, the path of P is through the joint O which is not desirable. This can be avoided if the links are proportioned in a way that QS is the mean proportional between OQ and QP, i.e.,

$$\frac{OQ}{QS} = \frac{QS}{QP}$$

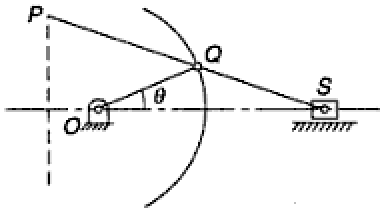


Fig. 6.7

However, in this case P will approximately traverse a straight line perpendicular to OS and that also for small movements of S or for small values of the angle θ (Fig. 6.7). A mathematical proof of this, being not simple, is omitted here. However, by drawing the mechanism in a number of positions, the fact can be verified.

Usually, this is known as the *modified Scott–Russel mechanism.*

4. Grass–Hopper Mechanism

This mechanism is a derivation of the modified Scott–Russel mechanism in which the sliding pair at S is replaced by a turning pair. This is achieved by replacing the slider with a link AS perpendicular to OS in the mean position. AS is pin-jointed at A (Fig. 6.8).

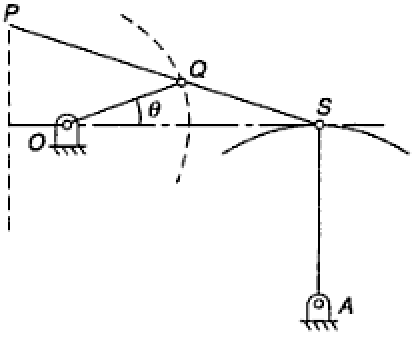


Fig. 6.8

If the length AS is large enough, S moves in an approximated straight line perpendicular to AS (or in line with OS) for small angular movements. P again will move in an approximate straight line if QS is the mean proportional between OQ and QP, i.e.,

$$\frac{OQ}{QS} = \frac{QS}{QP}$$

Example 6.3 *In a Grass–Hopper mechanism shown in Fig. 6.9, OQ = 80 mm, SQ = 120 mm and SP = 300 mm. Find the magnitude of the vertical force at P necessary to resist a torque of 100 N.m applied to the link OQ when it makes angles of 0°, 10° and 20° with the horizontal.*

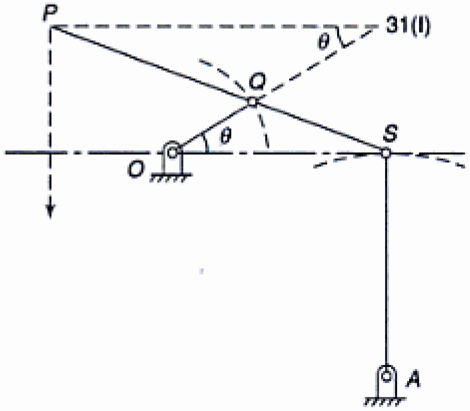


Fig. 6.9

Solution

$OQ = 80$ mm

$QS = 120$ mm

$QP = 300 - 120 = 180$ mm

$$\frac{OQ}{QS} = \frac{QS}{QP}, \text{ i.e., } \frac{80}{120} = \frac{120}{180}$$

As the condition for the dimensions of the Grass–Hopper mechanism is satisfied, P moves in an approximate straight line for small angles of OQ with the horizontal.

Now

$$F_p \times v_p = T_q \times \omega_q$$

$$F_p = \frac{T_p \omega_q}{v_p} = \frac{T_q}{v_p} \cdot \frac{v_q}{OQ} \qquad \text{(i)}$$

Locate the I-centre (instantaneous centre) of the link SP. It is at 31 as the directions of motions of points P and Q on it are known.

$$\frac{v_q}{v_p} = \frac{IQ}{IP} = \frac{OQ}{OS}$$

($\because$ Δs IQP and OQS are similar)

(i) becomes $F_p = \dfrac{T_q}{OS}$

When $\theta = 0°$, $OS = 80 + 120 = 200$ mm

$$F_p = \frac{100}{0.2} = 500 \text{ N}$$

When $\theta = 10°$,

$$OS = 80 \cos 10° + \sqrt{(120)^2 - (80 \sin 10°)^2}$$
$$= 198 \text{ mm}$$

$$F_p = \frac{100}{0.198} = \underline{505.05 \text{ N}}$$

When $\theta = 20°$,

$$OS = 80 \cos 20° + \sqrt{(120)^2 - (80 \sin 20°)^2}$$
$$= 192 \text{ mm}$$

$$F_p = \frac{100}{0.192} = \underline{520.8 \text{ N}}$$

As the angle θ increases, P moves in only approximate straight line and thus the calculations for F_p are not exact.

5. Watt Mechanism

It is a very simple mechanism. It has four links OQ, OA, QB and AB. OQ is the fixed link. Links OA and QB can oscillate about centres O and Q respectively. It is seen that if P is a point on the link AB such that $PA/PB = QB/OA$, then for small oscillations of OA and QB, P will trace an approximately straight line. This has been shown in Fig. 6.10 for three positions.

In earlier times, the mechanism was used by Watt to guide the piston, as it was difficult to machine plane surfaces.

The lengths of the opposite links of each parallelogram should also be equal, i.e.,

$$AC = BD,\ CE = DF,\ EG = FH \text{ and } GI = HJ$$

It can be seen that any number of parallelograms can be used to form this ruler. The dimensions of the mechanism ensure that IJ moves parallel to AB.

Lazy Tongs In this mechanism (Fig. 6.14), O is pin-jointed and is a fixed point. The point A slides in the vertical guides while all other points are pin-jointed. All the links are of equal length. As A moves vertically, P will move in an approximate horizontal line. The use of such a mechanism can be made in supporting a bulb (of a table lamp) or telephone, etc.

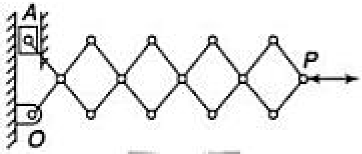


Fig. 6.14

Universal Drafting Machine In such a mechanism (Fig. 6.15), two parallelograms of the links are formed.

$$AB = CD \quad \text{and} \quad AC = BD$$

The link AB is fixed.

As $ABDC$ is a parallelogram, CD always remains parallel to AB. C and D are pin-jointed to a disc D_1. Thus, the disc D_1 can have translatory motion in a plane but not angular motions.

EF is another link on the disc D_1 pin-jointed at the ends E and F. As the orientation of the disc D_1 is fixed, the direction of EF is also fixed.

Also,

$$EG = FH \quad \text{and} \quad EF = GH$$

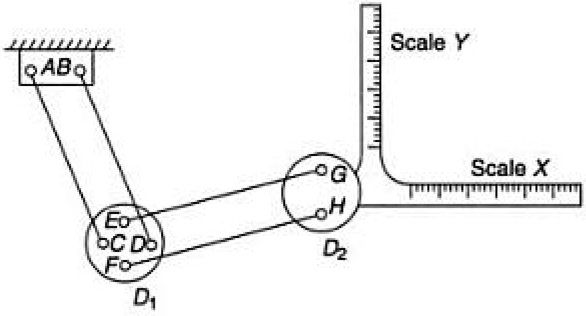


Fig. 6.15

Thus, the direction of GH is always parallel to EF or there is no angular movement of the disc D_2. Therefore, scales X and Y will always be along the horizontal and the vertical directions.

A universal drafting machine is extensively used as a substitute for T-square and set-square.

6.3 ENGINE INDICATORS

An *indicator* of a reciprocating engine is an instrument that keeps the graphical record of pressure inside the cylinder during the piston stroke.

An indicator consists of an *indicator cylinder* with a *piston*. The indicator cylinder is connected to the engine cylinder. Thus, varying pressure of the gas or steam is communicated to the indicator piston, the displacement of which is constrained by a spring to get a direct measure of the gas or the steam pressure. The displacement is recorded by a pencil on paper, wrapped on a drum, to a suitable scale with the help of a straight-line mechanism.

The following are the usual types of indicators:

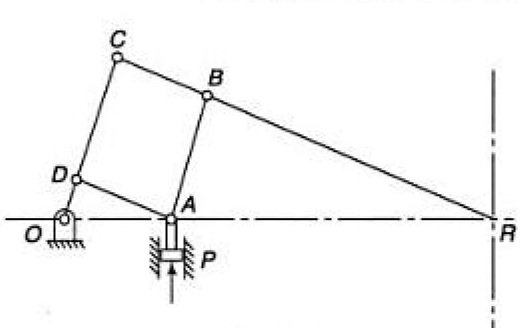


Fig. 6.16

1. Simplex Indicator

This indicator employs the mechanism of a pantograph. As shown in Fig. 6.16, O is the fixed pivot whereas $ABCD$ is a parallelogram formed by the four links. R is a point on the link CB produced to trace the path of A or P (movement of piston). Also, refer to Fig. 6.1.

P moves in a vertical straight line within the guides or the indicator cylinder. Its movement is controlled by the steam or the gas pressure to be measured. Thus, R also moves in a vertical straight line recording the variation of pressure with the help of a pencil recorder.

Example 6.4 *Design a pantograph for an indicator to be used to obtain the indicator diagram of an engine. The distance between the fixed point and the tracing point is 180 mm. The indicator diagram should be three times the gas pressure inside the cylinder of the engine.*

Solution

Refer Fig. 6.16,

$$OR = 180 \text{ mm and } \frac{OR}{OA} = 3 \text{ (given)}$$

or $\dfrac{180}{OA} = 3$

or $OA = 60$ mm

The relationship of the different arms of a simplex indicator is as follows:

$$\frac{OR}{OA} = \frac{OC}{OD} = \frac{CR}{CB} = 3$$

Choose convenient dimensions of OD and DA. Let these be 30 mm and 50 mm respectively.

Thus, as $ABCD$ is to be a parallelogram and the above relation is to be fulfilled, the other dimensions will be

$$OC = 30 \times 3 = 90 \text{ mm}$$
$$CR = 50 \times 3 = 150 \text{ mm}$$

Construct the diagram as follows:

1. Locate D by making arcs of radii 30 mm and 50 mm with centres O and A respectively.
2. Produce OD to C such that $OC = 90$ mm.
3. Join CR.
4. Draw AB parallel to OC.

Thus, the required pantograph is obtained.

2. Crosby Indicator

This indicator employs a modified form of the pantograph. The mechanism has been shown in Fig. 6.17.

To have a vertical straight line motion of R, it must remain in line with O and P, and also the links OC and PB must remain approximately parallel.

As P lies on the link 3 and R on 5, locate the *I*-centres 31 and 51. If the directions of velocities of any two points on a link are known, the I-centre can be located easily which is the intersection of the perpendiculars to the directions of velocities at the two points.

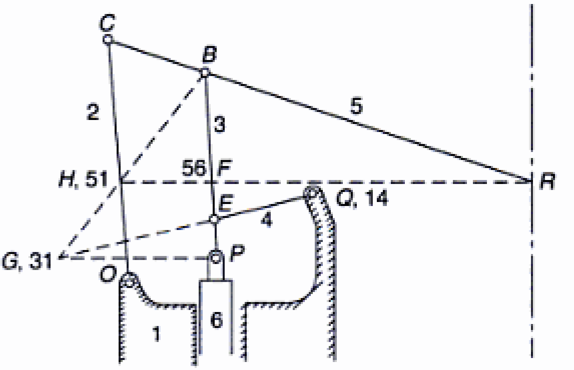


Fig. 6.17

First, locate 31 as the directions of velocities of P and E on the link 3 are known.

- The direction of velocity of P is vertical. Therefore, 31 lies on a horizontal line through P.
- The direction of velocity of E is perpendicular to QE. Therefore, 31 lies on QE (or QE produced).

The intersection of QE produced with the horizontal line through P locates the point 31.

Thus, the link 3 has its centre of rotation at 31 (link 1 is fixed) and the velocity of any point on the link is proportional to its distance from 31, the direction being perpendicular to a line joining the point with the I-centre.

To locate 51, the directions of velocities of B and C are known.

- The direction of velocity of B is $\perp$ to $31 - B$. Therefore, 51 lies on $31 - B$.
- The direction of velocity of C is $\perp$ to OC. Therefore, 51 lies on OC.

Thus, 51 can be located.

Now, the link 5 has its centre of rotation at 51. The direction of velocity of the point R on this link will be perpendicular to 51-R. To have a vertical motion of R, it must lie on a horizontal line through 51.

The ratio of the velocities of R and P is given by,

$$\frac{v_r}{v_p} = \frac{v_r}{v_b} \cdot \frac{v_b}{v_p} \qquad (B \text{ is common to 3 and 5})$$

$$= \frac{51-R}{51-B} \cdot \frac{31-B}{31-P}$$

$$= \frac{51-R}{51-B} \cdot \frac{51-B}{51-F} \qquad (\because \Delta\text{s } BPG \text{ and } BFH \text{ are similar})$$

$$= \frac{51-R}{51-F}$$

$$= \frac{CR}{CB} \qquad (\because \Delta\text{s } CRH \text{ and } BRF \text{ are similar})$$

$$= \text{constant}$$

This shows that the velocity or the displacement of R will be proportional to that of P.

Alternatively, locate the I-centre 56 by using Kennedy's theorem. It will be at the point F (the intersection of lines joining I-centres 16, 15 and 35, 36, not shown in the figure).

First, consider this point 56 to lie on the link 6. Its absolute velocity is the velocity of 6 in the vertical direction (1 being fixed).

Now, consider the point 56 to lie on the link 5. The motion of 5 is that of rotation about 51 (1 being fixed). Thus, velocity of R on the link 5 can be found as the velocity of 56, another point on the same link is known.

$$\frac{v_r}{v_f} = \frac{51-R}{51-F}$$

or

$$= \frac{51-R}{51-F} \qquad (v_f = v_p)$$

$$= \frac{CR}{CB}$$

3. Thomson Indicator

A Thomson indicator employs a Grass–Hopper mechanism $OCEQ$. R is the tracing point which lies on CE produced as shown in Fig. 6.18.

The best position of the tracing point R is obtained as discussed below:

Locate the I-centres 31 and 51 as in case of a Crosby indicator. The directions of velocities of two points C and E on the link 5 are known; therefore, first locate the I-centre 51.