MATH-GA 2791 Derivative Securities Final Project

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1 Scheme 1: lognormal & Hull-White

Assumptions:
We used the historical data from the year of 2019 to estimate our parameters.

According to historical data on Jan 3, 2019 (our t_0), we set the parameters: $r_f = -0.25\%$ (1-year Euro rate from European Central Bank), $q = 0.0261$ (dividend yield from Yahoo Finance)

Parameters assumed by us: $\sigma_r = 0.8$, $a = 0.3$ (the mean reversion parameter), $k_1 = 0.53$, $k_2 = 0.07$.

We set yield-to-maturity ytm1 = 2.5 (data from U.S. treasury department) to calculate $p(0, T)$. Then we use US Treasury yield curve rate (see Appendix, Figure 6) to calculate $p(0, T - \Delta)$, using numerical linear interpolation. Based on this, we find yield-to-maturity ytm2. Then find $p(0, T)$ and $p(0, T - \Delta)$ based on the following:

$$p(0,T) = \frac{1}{1 + \text{ytm1}/100}$$

$$p(0,T-\Delta) = \frac{1}{(1 + \text{ytm2}/100)^{0.75}}$$

Then, we start two-factor simulation with time-steps:

$$\frac{dS}{S} = (r_f - q)dt + \sigma_s dW^{Q^f}$$

Quantoed into USD with quanto adjustment: σ_x:x = USD/EURO-vol (historical volatility of X from market), ρ_{xs}:historical correlation between log-returns of the stock and the exchange rate

$$\frac{dS}{S} = (r_f - q - \rho_{xs}\sigma_s\sigma_x)dt + \sigma_s dW^{Q^d}$$

Hull-White model for short rate: mean-aversion term $a(t)$ (known=0.03): $\theta(t)$ is closed form of $p(0, t)$, can make assumptions about this discount factor (known from the market); σ_r: Hull-White interest rate vol (assumed 0.8 here);

$$dr = (\theta(t) - ar)dt + \sigma_r dZ^{Q^d}$$

Assume a correlation between stock and rate $\rho_{S,r} = 0.65$ (between dZ^{Q^d} and dW^{Q^d}).
Need to simulate the two-factor model between stock and rate: choose small timesteps of $\Delta t = \frac{1}{252}$ to discretize Hull-White short rate model:

$$r_{k+1} - r_k = (\theta(t_k) - ar_k)\Delta t + \sigma_r\sqrt{\Delta t}\tilde{\varepsilon} \text{ ,where } \theta(t) = e^{-at}$$

$$S_T = S_0 \exp\left((r_f - q - \rho_{x,S}\sigma_x\sigma_S)T + \sigma_S\sqrt{T}\varepsilon\right)$$

For path j, have $S^j(t_k)$ and $r^j(t_k)$ with respective terms $\varepsilon, \tilde{\varepsilon} \sim \mathcal{N}(0,1)$ correlated with $\rho_{S,r}$ **(instantaneous correlation between t and $t+\Delta t$)**.
$L(0, T-\Delta, T)$ is known forward rate, and $L(T-\Delta, T-\Delta, t)$ is unknown, given rate $r(T-\Delta)$.

$$L(T-\Delta, T-\Delta, T) = \frac{1-p(T-\Delta, T)}{\Delta \cdot p(T-\Delta, T)}$$

with $p(T-\Delta, T)$ using the term structure

$$p(T-\Delta, T) = \frac{p(0,T)}{p(0,T-\Delta)} \times \exp\left(B(T-\Delta,T)f(0,T-\Delta) - \frac{\sigma^2}{4a}B^2(T-\Delta,T)(1-e^{-2at}) - B(T-\Delta,T)r(T-\Delta)\right)$$

with

$$B(T-\Delta, T) = \frac{1}{a}\left(1-e^{-a\Delta}\right)$$

$$p(t,T) = \mathbb{E}^{Q^d}\left[\exp\left(-\int_t^T r_u du\right)\right]$$

$$f(t,T) = \mathbb{E}^{Q^d}[r(T)] = -\frac{\partial \ln p(t,T)}{\partial T}$$

$$f(0,t) = -\frac{\partial \ln p(0,t)}{\partial T}, \text{ where } p(0,T) = \mathbb{E}_0^{Q^d}\left[\exp\left(-\int_0^t r_u du\right)\right]$$

All are known from the Hull-White model except $r(T-\Delta)$ from our simulated path. Determine, by Monte-Carlo,

$$r^j(T-\Delta) \to P^j(T-\Delta) \to L^j(T-\Delta, T-\Delta, T)$$

$$S^j(T)$$

Get payoff $\Pi_T^j(L,S) = \max\left[0, \left(\frac{S(T)}{S(0)} - k\right) \cdot \left(\frac{L^j(T-\Delta,T-\Delta,T)}{L(0,T-\Delta,T)} - k'\right)\right]$ under Q^d. Then, discount path by path based on

$$\exp\left(-\int_0^T r_u du\right) \sim \exp\left(-\sum_{k=1}^{N} r(t_k)\Delta t\right)$$

Last, find out

$$\mathbb{E}^Q\left[\exp\left(-\int_0^T r_u du\right) \times \Pi_T\right]$$

Remark: The bond prices we used in the term structure, $p(0,T-\Delta)$, $p(0,T)$, and $f(0,T-\Delta)$are calculated by interpolation with historical data since there doesn't exist bonds with maturity of 9 months.

2 Scheme 2: lognormal & lognormal LIBOR

Assumptions:
S_t and LIBOR are both log-normal distributed. Parameters from historical data are used here as same as those in the Scheme 1, including σ_x, σ_s, and ρ_{sx}. We assume the volatility of bonds and LIBOR:$\sigma_p = 0.05$ and $\sigma_L = 0.1$

$L(t, T-\Delta, T)$ is martingale under Q^T.

$$\frac{dL}{L} = \sigma_L dZ^{Q^T}$$

$$\frac{dS}{S} = (r_f - q - \rho_{x,S}\sigma_S\sigma_x + \sigma_s\sigma_p\rho_{S,p})dt + \sigma_S dW^{Q^T}$$

Q^T is under USD, with numeraire $p(t,T)$. $\rho_{S,p}$ is the correlation between stock and bond (used to adjust the drift changes), and $\rho_{S,L}$ is the correlation between stock and LIBOR **(overall correlation between 0 and T)**, and also between $\varepsilon, \tilde{\varepsilon} \sim \mathcal{N}(0,1)$. We assume $\rho_{S,L} = 0.65$. Approximately,

$$\rho_{S,p} = -\rho_{S,L}$$

We do not take small time steps in this case.

$$S_T = S_0 \exp\Big((r_f - q - \rho_{x,S}\sigma_x\sigma_S + \rho_{p,S}\sigma_S\sigma_p)T + \sigma_S\sqrt{T}\varepsilon\Big)$$

$$L(0, T-\Delta, T) = L_0 \exp(\sigma_L\sqrt{T-\Delta}\tilde{\varepsilon})$$

We set

$$L_0 = f(0,T)$$

where $f(0,T)$ is found by interpolating $\ln p(0,T)$ first and then differentiate numerically
Remark: In our model, we assumed that $p(0,T)$ is observable but $p(0,T-\Delta)$ is not, so we use the LIBOR ratio we get from simulation to calculate the "implied" $p(0,T-\Delta)$

$$L(0,T-\Delta,T) = -\frac{p(0,T) - p(0,T-\Delta)}{\Delta \cdot p(0,T)} \Leftrightarrow p(0,T-\Delta) = p(0,T) + L(0,T-\Delta,T)\cdot\Delta\cdot p(0,T)$$

By no arbitrage condition,

$$p(0,T-\Delta)\cdot p(T-\Delta,T) = p(0,T)$$

Hence,

$$p(T-\Delta,T) = \frac{p(0,T)}{p(0,T-\Delta)}$$

where both are observed from market, which is different from Scheme 1 where we calculated from simulation.
Find L_ratio in the payoff:

$$\frac{(1 - p(T-\Delta,T)) \times (p(0,T))}{p(T-\Delta,T)\times(p(0,T-\Delta) - p(0,T)))}$$

Find payoff $\Pi_T = \max\left[0, \left(\frac{S(T)}{S(0)} - k\right)\cdot\left(\frac{L^j(T-\Delta,T-\Delta,T)}{L(0,T-\Delta,T)} - k'\right)\right]$, with $r(T-\Delta)$ known, then discount back using constant US risk-free rate (2.5%).

3 Results & Explanations

The resulting price from Scheme 1(Hull-White) is \$8.5649, and the price from Scheme 2(Lognormal LIBOR) is \$0.2163. Noticing the significant inconsistency, we examine the stock and LIBOR parts in the final payoff separately, as shown in figure 1. S_part denotes $\frac{S(T)}{S(0)}$ and L_part denotes $\frac{L(T-\Delta,T-\Delta,T)}{L(0,T-\Delta,T)}$.

S_part in Scheme 1 is around \$0.97298, and S_part in Scheme 2 is around \$0.97840. This slight difference does make sense because the only difference between the simulation of stock price is that in scheme 2, there's an extra drift term of $\sigma_s \sigma_p \rho_{S,p}$. Since all of them have relatively low values, like 0.2 or so, we expect that the additional drift term won't affect the final stock price that much. Therefore, it is reasonable that the two schemes don't differ a lot in the simulated price of the stock.

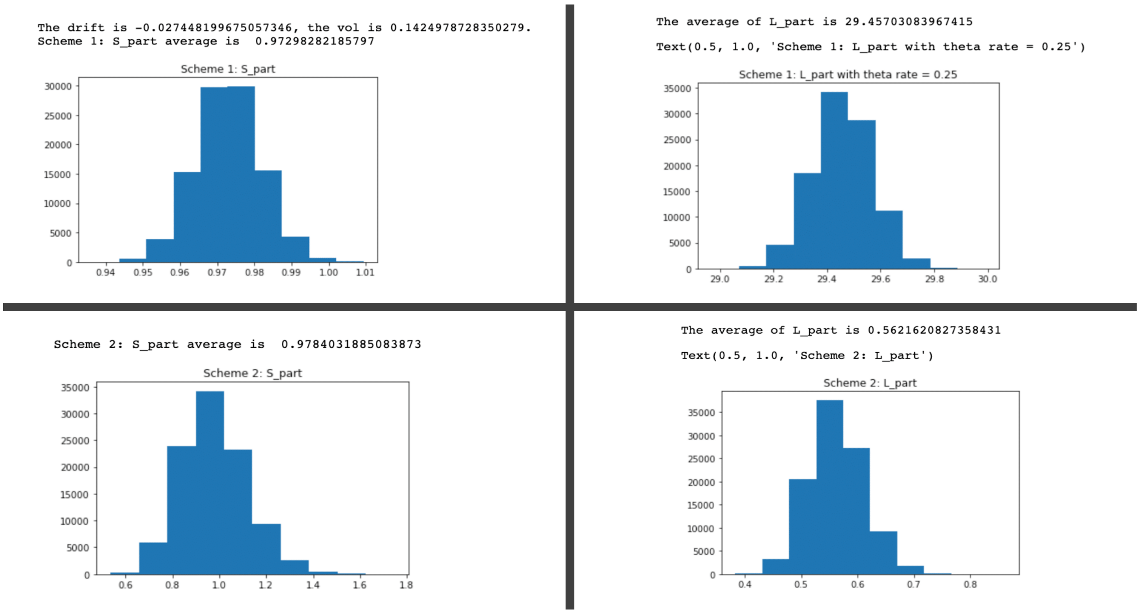


Figure 1: Comparison of Stock and LIBOR parts in the product payoff with theta rate = 0.25

We first look at the discount factor of scheme 1 and scheme 2. The discount factor in scheme 1 is around 0.65, while the discount factor in scheme 2 is around 0.99, as shown in figure 2. This is a big difference but it is not enough to explain the difference in L_part.

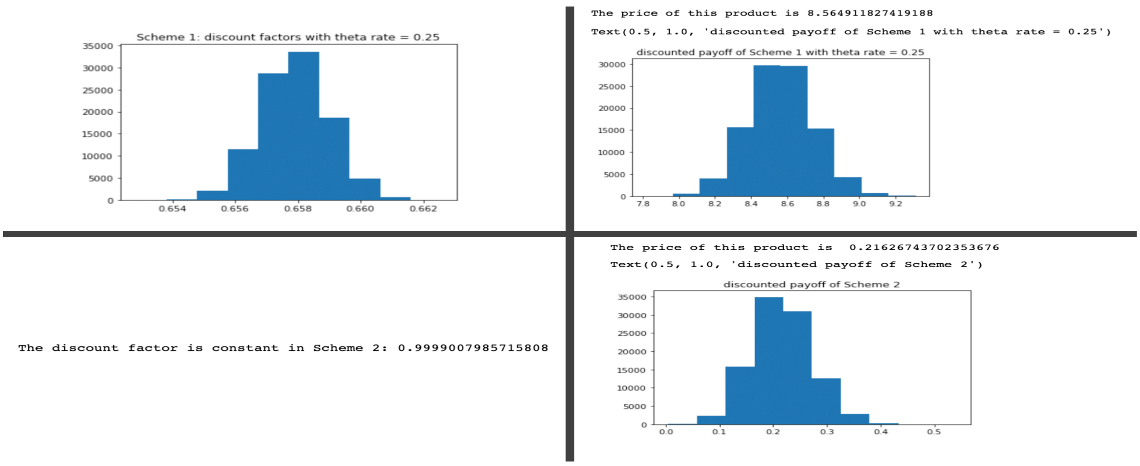


Figure 2: Discount Factors and Discounted Payoff Distributions with theta rate = 0.25

The main difference, however, is caused by the LIBOR part in the payoff. The average L_part in scheme 1 is \$29.46 while the average L_part in scheme 2 is \$0.56. This difference is quite significant, and as we analyzed each step of our simulation, we found that the problem is that our simulated short-rate at $T-\Delta$ is too high. As shown in the first picture of Figure 3, which is the short rate path of one of our simulation, the curve is approximately an upward sloping straight line with no randomness. Also, the short rate ends up being 0.7 at the T, which is far from short rates we see in reality. After adjusting the different parameters in the simulation, we realized that the problem was that our $\theta(t)$ decays too slow since we set it to be $e^{-0.25t}$, and its value is relatively large compare to the stochastic term, so the stochastic term adds little randomness in the path of the short rate. In order to solve this, we tried to multiply the decaying rate by 10, so $\theta(t)$ is now $e^{-2.5t}$. This is the result shown in the second graph of the Figure 3. There's an obvious curvature in the graph, and that it does show certain fluctuation at the right tail of the graph, but the effect is not strong enough and the simulated short rate is still too high. Therefore, we multiplied it by 10 again, resulted a new $\theta(t)$ of e^{-25t}. The short rate path is shown in the third graph, which has an obvious fluctuation at the right tail, and a reasonable short rate of around 0.03.We then chose this to be the decaying parameter in $\theta(t)$ in scheme 1.

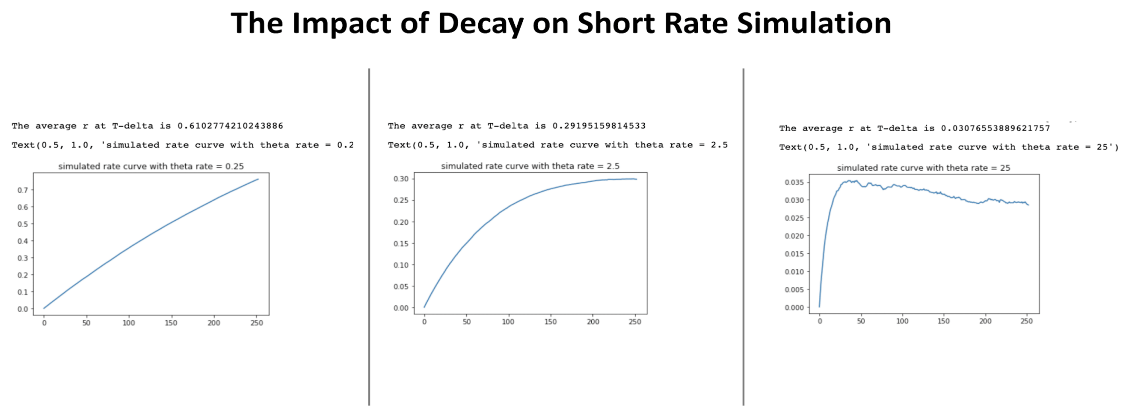


Figure 3: Short Rate Simulations with increasing theta rate

Now take a look at the other results under new decaying speed. The discount factor under $\theta(t) = e^{2.5t}$ becomes around 0.8, the average L_part is 15.5, and the average price is 5.48. The discount factor under $\theta(t) = e^{25t}$ becomes around 0.96, the average L_part is 4.95, and the average price is 2.09. There's still a difference between the two simulated prices, which is caused by the difference between the simulated L_part. We conclude that this difference is caused by the different interest rate models.

One more thing that matters is our strike prices, as we choose 0.53 and 0.07 as our strike prices. As both of them have a S_part average of around 0.97, the $\frac{S(T)}{S(0)} - k$ part in the payoff is like a deep in the money option. Scheme 1 has an average L_part of 4.95 and scheme 2 has an average L_part of 0.56, so the $\frac{L(T-\Delta,T-\Delta,T)}{L(0,T-\Delta,T)} - k'$ part is also like a deep in the money option. If we set the Strike price for the Stock part to be 0.96, which is sort of in the money, the price of this product under scheme 1 will be 0.0629 and the price of this product under scheme 2 will be 0.0293. Based on that, if we also set the strike price for the LIBOR part to be 4.6, which is close to the average L_part of scheme 1, but still much higher than the L_part of scheme 2, the two prices will be 0.049 and 0.186, so the price given by scheme 2 is now higher than the price given by scheme 2. Another thing that matters is the co-movements of Stock part and LIBOR part. If both of them are lower than the strike prices or higher than the strike price, the payoff will be positive; if one of them is higher than its strike but the other one is lower, the payoff will be 0. Since in each of the models, the Brownian motion for the stock and the Brownian motion for rates are indeed correlated (in our code we used Cholesky decomposition to generate correlated ϵ's), this will also affect the payoff of the product.

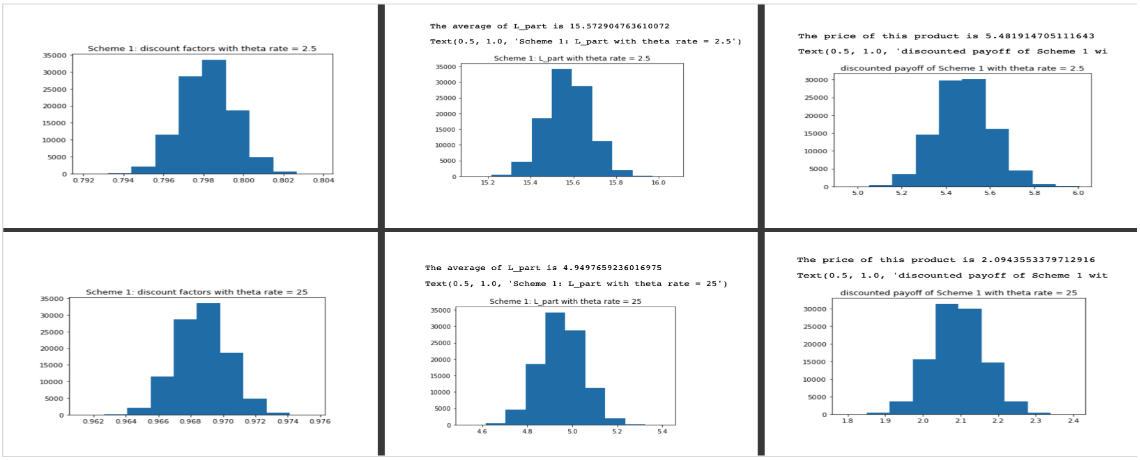


Figure 4: Comparison of discount factors, LIBOR parts, and average price with theta rates of 2.5 and 25

4 Appendix

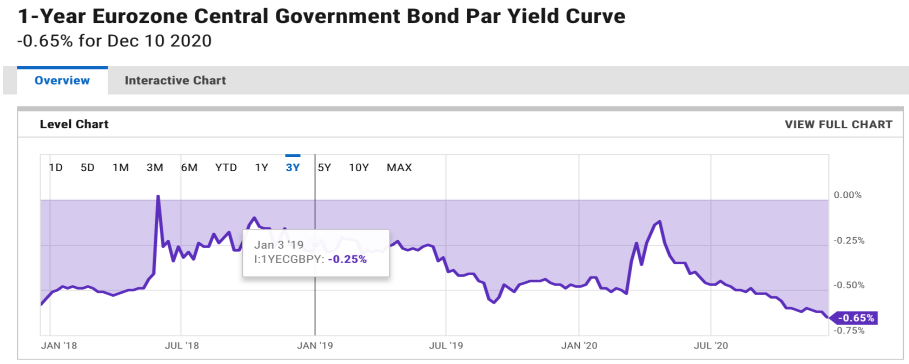
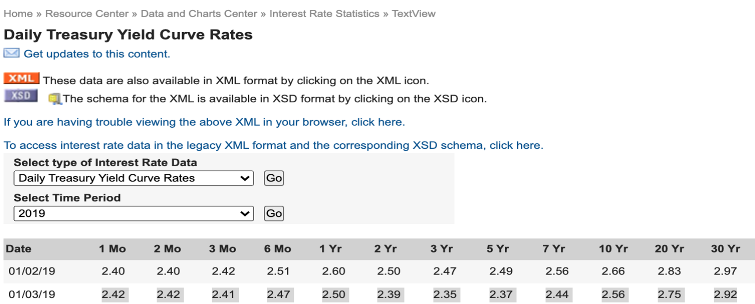


Figure 5: 1 year Euro rate on Jan. 3, 2019, data from European Central Bank. Source: `https://ycharts.com/indicators/1year_eurozone_central_government_bond_par_yield_curve`.



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01/02/19	2.40	2.40	2.42	2.51	2.60	2.50	2.47	2.49	2.56	2.66	2.83	2.97
01/03/19	2.42	2.42	2.41	2.47	2.50	2.39	2.35	2.37	2.44	2.56	2.75	2.92

Figure 6: Daily US Treasury Yield Curve Rate on Jan. 3, 2019. We apply these rates for interpolation to calculate $p(0, T-\Delta)$. Source: `https://www.treasury.gov/resource-center/data-chart-center/interest-rates/Pages/TextView.aspx?data=yieldYear&year=2019`.