$F(U, U_x, U_y, U_{xy}, U_{xx}, U_{yy}, x, y) = 0$

Linear PDE: when coefficients of $U, U_x, U_y; U_{xx}, U_{xy}, U_{yy}$ either depend only on x & y or constant or zero. (not all)

2nd order PDE:

$$A U_{xx} + 2B U_{xy} + C_{yy} + f(U, U_x, U_y, x, y) = 0$$

where A, B, C, are f^ns of x & y or const not all zero.

① Hyperbolic: $B^2 - AC > 0$ $\quad U_{tt} = c^2 U_{xx}$

② Parabolic: $B^2 - AC = 0$

$$U_t = \alpha U_{xx}$$

③ Elliptic: $B^2 - AC < 0$

$$U_{yy} + U_{xx} = 0$$

Parabolic Equation

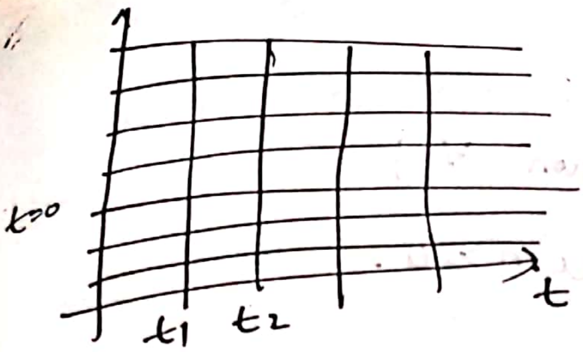
$$\frac{\partial U}{\partial t} = \alpha \frac{\partial^2 U}{\partial x^2}, \quad U(x,t) \quad 0 < x < l, \; t > 0$$

$U(x,t) \longrightarrow$ Temperature

$U(x,0) = f(x)$ Initial condition

B.C.s are described as

$U(0,t) = v_0$, $U(l,t) = v_1$, $\forall t > 0$



Forward Marching in t.

$t_n = n\delta t \qquad x_j = j\delta x$

where $n = 0, 1, 2, \dots N$
$j = 0, 1, 2, \dots M$

$M = \frac{l}{\delta x}$

The domain is subdivided into the number of discrete points. (x_j, t_n)

Let $u_n^j = u(x_j, t_n)$ where $j = 0, 1, \dots m$
$n = 0, 1, 2 \dots$

$\left[\begin{array}{l} t \rightarrow \text{temporal coord.} \\ x \rightarrow \text{spatial coord} \end{array} \right]$

$u_j^0 = f_j \qquad j = 0, 1, \dots M$

$u_0^N = u(0, t_n) = v_0$

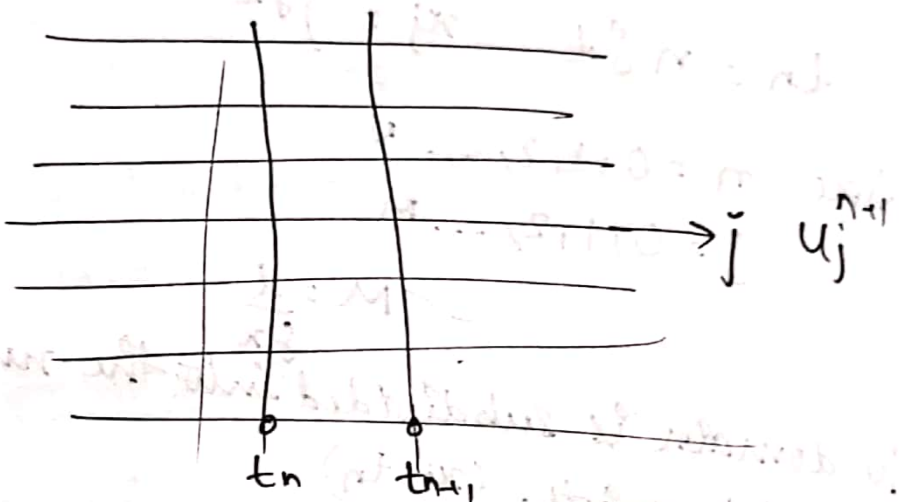
$u_m^N = v_l$

Find u_j^n for $j = 1, 2 \dots M-1$
when $n = 1, 2 \dots$

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

Let u_j^n be known $\forall j$

Find u_j^{n+1} $\forall j$ where $n \geq 0$.



Satisfy the PDE at (x_j, t_n)

$$\left.\frac{\partial u}{\partial t}\right|_j^n = \alpha \left.\frac{\partial^2 u}{\partial x^2}\right|_j^n$$

FORWARD TIME CENTRAL SPACE.

$$\frac{u_j^{n+1} - u_j^n}{\delta t} = \alpha \left(\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\delta x)^2} \right)$$

Let $r = \alpha \frac{\partial t}{(\partial x)^2}$

$u_j^{n+1} = r u_{j+1}^n + (1-2r) u_j^n + r u_{j-1}^n$

unknown is expressed explicitly in terms of known quantities.

This scheme (method) is termed as explicit scheme.

where $n \geq 0$, $j = 1, 2, \ldots M-1$

Q $u_t = u_{xx}$. $u(x,0) = \sin \pi x$. $0 < x < 1$

B.C.s. $u(0,t) = u(1,t) = 0$, $t \geq 0$

$\boxed{\begin{matrix} h = \delta x \\ k = \delta t \end{matrix}}$

$r = 1/4$

$\alpha = 1$ Let $\delta x = 2$ $\delta x = 1/4$

$\frac{1}{4} = \frac{\delta t}{(\delta x)^2}$ $\delta t = 1/64$

$u_0^0 = 0 = u_4^0 = 0$

$u(0,0) = 0 = u(1,0)$

$u(1/4, 0) = 1/\sqrt{2} = u(3/4, 0)$

$u(1/2, 0) = 1$

$u_0^0 = u_4^0 = 0$

$u_1^0 = u_3^0 = 1/\sqrt{2}$

$u_2^0 = 1$

$u_1^1 = 1/4\, u_2^0 + (1 - 1/2) u_1^0 + 1/4\, u_0^0$

$= 1/4 + \frac{1}{2\sqrt{2}}$

$u_1^1 = \frac{\sqrt{2}+1}{4}$

$u_2^1 = 1/4 \times \frac{1}{\sqrt{2}} + (1/2) 1/2 + 1/4\sqrt{2}$

$= \frac{1}{2\sqrt{2}} + 1/4 2$

$u_2^1 = \frac{2+\sqrt{2}}{4}$

$u_3^1 = \frac{1}{4} \times 0 + \frac{1}{4} \times 1 + \frac{1}{2\sqrt{2}}$

$u_3^1 = \frac{1+\sqrt{2}}{4}$

$\therefore n \rightarrow n+1$

$u_1^2 = \frac{1}{4} u_2^1 + (1/2) u_1^1 + 1/4\, u_0^1$

$= \frac{1}{4} \left(\frac{\sqrt{2}+2}{4}\right) + (1/2)\left(\frac{\sqrt{2}+1}{4}\right) + 0$

$u_1^2 = 0.515$

Scanned by CamScanner

$$u_7^2 = \frac{1}{4} \times \frac{\sqrt{2}+1}{4} + \frac{1}{2}\left(\frac{\sqrt{2}+2}{4}\right) + \frac{1}{4}\left(\frac{\sqrt{2}+1}{4}\right)$$

$$= 0.15$$

Home Task.

Stability of the scheme

* Explicit scheme is unstable if $r > 1/2$

$\Rightarrow$ i.e., stable for $r \leq 1/2$

Implicit scheme $\rightarrow$ Is unconditionally stable!

$\Rightarrow$ Stable for any choice of r.

$$r = K\frac{\partial t}{(\partial x)^2} \leq 0.5$$

$$\Rightarrow \partial t \leq (0.5)\frac{(\partial x)^2}{K} \longrightarrow \text{very small}$$

Satisfy the PDE at (x_j, t_{n+1})

$$\left(\frac{\partial u}{\partial t}\right)\Big|_j^{n+1} = \alpha \frac{\partial^2 u}{\partial x^2}\Big|_j^{n+1}$$

to get the solution for $n \rightarrow n+1$

BTCS
Back Time
Cent Space

$$\frac{u_j^{n+1} - u_j^n}{\delta t} = \alpha\left(\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{(\partial x)^2}\right)$$

LAB TASK

Variables with superscript $n+1$ are unknown

$$-ru_{j-1}^{n+1} + (1+2r)u_j^{n+1} - ru_{j+1}^{n+1} = u_j^n$$

$$j = 1, 2, \ldots m-1$$

$r = 1/_{...}$ $\delta x = 1/_{...}$ $\longrightarrow$ Reduce the δx

$u_t = \alpha u_{xx}$, $t > 0$ $0 < x < l$

B.C. : $u(0,t) = U_0$, $u(l,t) = U_l$, $t \geq 0$

I.C. : $u(x,0) = f(x)$, $0 < x < l$.

$r = \alpha \dfrac{\delta t}{(\delta(x))^2}$

Explicit scheme , $r \leq 1/2$ } Stable
Implicit scheme , $\forall r$ }

T.E. ~ $O(\delta t, (\delta x)^2)$

$t >> 1$, $t = 0 \to T \; (>>1)$

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

Integrate both sides wrt t b/w t_n to t_{n+1}.

$$\int_{t_n}^{t_{n+1}} \frac{\partial u}{\partial t}\, dt = \alpha \int_{t_n}^{t_{n+1}} \frac{\partial^2 u}{\partial x^2}\, dt \qquad \delta t$$

$$\Rightarrow u(t_{n+1}, x) - u(t_n, x) = \alpha(t_{n+1} - t_n) \frac{\left(u''(t_{n+1}, x) + u''(t_n, x) \right)}{2}$$

= [illegible]

$\Rightarrow \cancel{u(t_{n+1}, x) - u(t_n, x) =}$

at any point x_j, we can write this as

$$u_j^{n+1} - u_j^n = \alpha \frac{\delta t}{2}\left[\frac{\partial^2 u}{\partial x^2}\Big|_j^{n+1} + \frac{\partial^2 u}{\partial x^2}\Big|_j^n\right]$$

$$u_j^{n+1} - u_j^n = \alpha \frac{\delta t}{2\delta x^2}\Big[u(t_{n+1}, x_{n+1}) - 2u(t_{n+1}, x_n) + u(t_{n+1}, x_{n-1}) + u(t_n, x_{n+1}) - 2u(t_n, x_n) + u(t_n, x_{n-1})\Big]$$

$$u_j^{n+1} - u_j^n = \frac{\alpha \delta t}{\partial x^2}\left[u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1} + u_{j+1}^n - 2u_j^n + u_{j-1}^n\right]$$

$n \geq 0$ and $j = 1, 2, \ldots M-1$

Which can be expressed as

$$a_j u_{j-1}^{n+1} + b_j u_j^{n+1} + c_j u_{j+1}^{n+1} = d_j$$

write $j = 1, 2, \ldots, M-1$

T.E. $\sim O(\delta t^2, \delta x^2)$

Which is an implicit second order accurate scheme

Crank-Nicholson scheme

$$\frac{\partial u}{\partial t}\Big|_j^{t_{n+1/2}} = \alpha \frac{\partial^2 u}{\partial x^2}\Big|_j^{n+1/2}$$

L.T.

$$= \frac{\alpha}{2}\left[\frac{\partial^2 u}{\partial x^2}\Big|_j^{n+1} + \frac{\partial^2 u}{\partial x^2}\right]_j^n$$

$$\frac{u_j^{n+1} - u_j^n}{\cancel{2}.\delta t/\cancel{2}} = \frac{\alpha}{2}\left[\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{(\delta x)^2} + \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\delta x)^2}\right] + O(\delta t^2, \delta x^2)$$

$$u_t = u_{xx}$$

$$u(x,0) = \sin \pi x \quad , 0 < x < 1$$

$$u(0,t) = u(1,t) = 0, t \geq 0$$

$$\delta x = 1/4 \quad , \quad r = 1/6$$

$$\alpha = 1 \qquad r = \frac{\alpha \delta t}{(\delta x)^2} \Rightarrow \delta t = \frac{1}{6 \times 16}$$

$$u_j^{n+1} - u_j^n = \frac{\alpha \delta t}{(\delta x)^2} \left[\begin{array}{l} u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1} \\ + u_{j+1}^n - 2u_j^n + u_{j-1}^n \end{array} \right]$$

$$u_0^0 = 0$$

$$u_0^1 = 0$$

$$\Rightarrow \quad u_1^1 - \cancel{u_1^0} = \frac{1}{6} \left[u_2^1 - 2u_1^1 + \cancel{u_0^1} \right]$$

$$u_1^1 = \frac{1}{6} \left[u_2^1 - 2u_1^1 \right]$$

$$\frac{4}{3} u_1^1 = \frac{1}{6} u_2^1$$

$$\Rightarrow u_1^1 = \frac{u_2^1}{8}$$

$$u_j^{n+1} - u_j^n = \frac{\alpha'}{6}\left[u_{j+1}^{n+1} - 2u_j^{n+1} + u_j^n + u_{j+1}^n - 2u_j^n + u_{j-1}^n\right]$$

$$-\frac{1}{6}\left(u_{j+1}^{n+1} + u_{j+1}^n\right) + \left(\frac{4}{3}u_j^{n+1} - \frac{2}{3}u_j^n\right) - \frac{1}{6}\left(u_{j-1}^{n+1} + u_{j-1}^{n+1}\right) = 0$$

$$-\frac{1}{6}\left(u_2^1 + u_2^0\right) + \left(\frac{4}{3}u_1^1 - \frac{2}{3}u_1^0\right) - \frac{1}{6}\left(u_0^1 + u_0^0\right) = 0$$

$$-\frac{1}{6}\left(u_2^1 + 1\right) + \left(\frac{4}{3}u_1^1 - \frac{\sqrt{2}}{3}\right) = 0$$

$$-\frac{1}{6}(u_2) - \frac{1}{6} + \frac{4}{3}u_1 - \frac{\sqrt{2}}{3} = 0 \quad — (1)$$

$$-\frac{1}{6}\left(u_3^1 + u_3^0\right) + \left(\frac{4}{3}u_2^1 - \frac{2}{3}u_2^0\right) - \frac{1}{6}\left(u_1^1 + u_1^0\right) = 0$$

$$-\frac{1}{6}\left(u_3^1 + \frac{1}{\sqrt{2}}\right) + \left(\frac{4}{3}u_2^1 - \frac{2}{3}\right) - \frac{1}{6}\left(u_1^1 + \frac{1}{\sqrt{2}}\right) = 0$$

$\Rightarrow$

$u_1^1 = 0.6413$
$u_2^1 = 0.907$
$u_3^1 = 0.6413$

$$r = \frac{\delta t}{(\delta x)^2}$$

$\delta t =$

Scanned by CamScanner

$u_j^n \rightarrow U_j^n$ convergence

Lax equivalence Theorem: If the numerical scheme for the linear initial boundary value problem is consistent and stable then the solution of the numerical scheme converges to the exact solution ~~of the PDE~~.

Stability analysis.

$\bar{U}_n^j$ be the exact solution of the finite difference scheme (numerical scheme).

u_j^n is the computed solution.

$$u_j^n = \bar{U}_j^n + \epsilon_j^n \quad -①$$

ϵ_j^n is the error mostly arise due to round off the infinite digits.

Consider the finite scheme, which is the explicit scheme for the initial boundary value problem

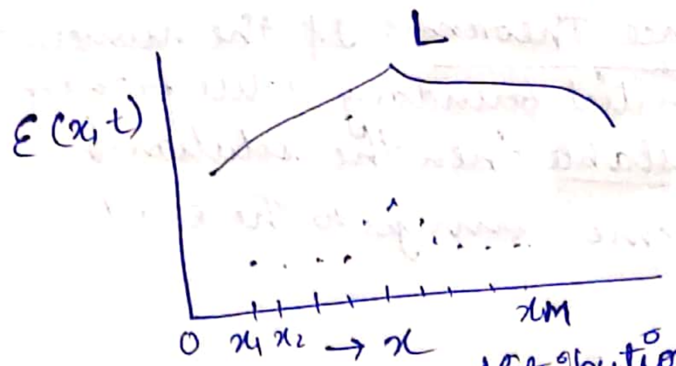
$$u_t = \nu u_{xx} \text{ as}$$

$$\frac{u_j^{n+1} - u_j^n}{\delta t} = \nu \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\delta x)^2} \quad -②$$

Substitute ① in ② and note that ~~since~~ FI is linear and $\bar{u}_j^n$ is the exact solution we get

$$\frac{\epsilon_j^{n+1} - \epsilon_j^n}{\delta t} = \nu \frac{\epsilon_{j+1}^n - 2\epsilon_j^n + \epsilon_{j-1}^n}{(\delta x)^2} \quad -③$$

Thus the error in explicit scheme propagates by ③.



Expand the error distribution $\mathcal{E}(x,t)$ at any time level $t = t_n$ by any finite fourier series.

$$\mathcal{E}(x,t) = \sum_{m=0}^{M} a_m \cos\frac{m\pi x}{L} + b_m \sin\frac{m\pi x}{L}$$

$$0 < x < L$$

$$= \sum_{0}^{M} A_m(t)\, e^{i\frac{m\pi x}{L}}.$$

$A_m(t)$ are arbitrary complex coefficients

at $x = x_j$

$$\mathcal{E}(x_j, t_n) = \sum_{0}^{M} A_m^n \, e^{im\pi x_j / L}$$

This expression for $\mathcal{E}(x,t)$ can be substituted in (3).

Since the difference equation is linear we may examine the behaviour of a single term. Let $\bar{A}^n = \max_{0 \le m \le M} \{|A_m^n|\}$

We examine the behaviour of

$$\mathcal{E}_j^n = \bar{A}^n e^{im\pi x_j / L}$$

$$x_j = j\,\delta x, \quad \theta = \frac{m\pi\,\delta x}{L}$$

Then, $\mathcal{E}_j^n = \bar{A}^n e^{i\theta j}$; θ is a real number

$\epsilon_j^n = \bar{A}^n e^{i\theta j}$, $\theta = \frac{m\pi\,\delta x}{L}$

which represents a wave propagating in the x-direction, with amplitude $\bar{A}^n$ and

$$|\epsilon_j^n| = |\bar{A}^n|$$

We define an amplification factor

$$\xi = \left|\frac{\epsilon_j^{n+1}}{\epsilon_j^n}\right| = \left|\frac{\bar{A}_{n+1}}{\bar{A}_n}\right|$$

Stable if $\xi \le 1$

Unstable if $\xi > 1$.

$\epsilon_j^n = \bar{A}^n e^{i\theta j}$, $\epsilon_{j+1}^n = \bar{A}^n e^{i\theta(j+1)}$

$\epsilon_{j-1}^n = \bar{A}^n e^{i\theta(j-1)}$, $\epsilon_j^{n+1} = \bar{A}^{n+1} e^{i\theta j}$

STABILITY OF THE EXPLICIT SCHEME ②

Substitute the above expressions in ③ we get

$r = \nu \frac{\delta t}{(\delta x)^2}$

$$\bar{A}^{n+1} e^{i\theta j} = r\bar{A}^n e^{i\theta(j-1)} + (1-2r)\bar{A}^n e^{i\theta j} + r\bar{A}^n e^{i\theta(j+1)}$$

$$\Rightarrow \frac{\bar{A}^{n+1}}{\bar{A}^n} = \frac{re^{i\theta(j-1)} + (1-2r)e^{i\theta j} + re^{i\theta(j+1)}}{e^{i\theta j}}$$

$$= \left|re^{-i\theta} + (1-2r) + re^{i\theta}\right| \le 1$$

$1-2r+2r\cos\theta$

$1-2r(1-\cos\theta)$

$$= \left|1-2r+2r\cos\theta\right| \le 1$$

$$= \left|1-2r(1-\cos\theta)\right| \le 1$$

$$1 - 2r(1-\cos\theta) \ge -1 \quad \forall\theta$$

$$2 \ge 2r(1-\cos\theta)$$

$$\boxed{r \le \frac{1}{1-\cos\theta}}$$

$\cos\theta$

$$r \le \frac{1}{\cos\theta} \Rightarrow r \le \frac{1}{2}$$

Home Task if Implicit scheme is unconditionally stable

$$\frac{u_j^{n+1} - u_j^n}{\delta t} = \nu \frac{\left(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}\right)}{(\delta x)^2}$$

$$\frac{\epsilon_j^{n+1} - \epsilon_j^n}{1} = r\left(\epsilon_{j+1}^{n+1} - 2\epsilon_j^{n+1} + \epsilon_{j-1}^{n+1}\right)$$

$$-r\,\epsilon_{j+1}^{n+1} + (2r+1)\,\epsilon_j^{n+1} - r\,\epsilon_{j-1}^{n+1} = \epsilon_j^n$$

$$-r\,A^{n+1} e^{i\theta(j+1)} + (2r-1)\,A^{n+1} e^{i\theta(n)} - r\,A^{n+1} e^{i\theta(n-1)} = A^n e^{i\theta n}$$

$$-re^{i\theta} + (2r-1)e - re^{-i\theta} = \left|\frac{A_n}{A_{n+1}}\right|$$

$$\Rightarrow \left|\frac{A_n}{A_{n+1}}\right| = \left|-re^{i\theta} - re^{-i\theta} + (2r+1)\right| \geq 1$$

if stable

$$= \left|-2r\cos\theta + (2r+1)\right|$$

$$= \left|1 + 2r(1-\cos\theta)\right| \geq 1$$

$$\forall\,\theta \quad \forall\, r$$

Hence implicit scheme is unconditionally stable.

H.T. Crank Nicholson scheme is stable unconditionally.

Scanned by CamScanner

$u_t = \nu u_{xx}$ $\quad u(x,t)$

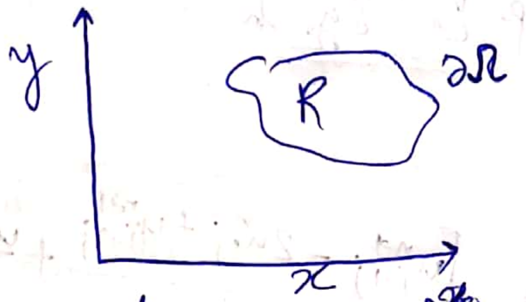
$u(x,y,t)$

$$\frac{\partial u}{\partial t} = \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \nu \nabla^2 u$$

$x, y \in R$, with $\partial\Omega$ as boundary

B.C.s $u(x,y,t)$ is prescribed at $\partial\Omega$

I.C. $u(x,y,0) = f(x,y)$, known $\forall (x,y) \in R$



$$\frac{\partial u}{\partial t} = \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

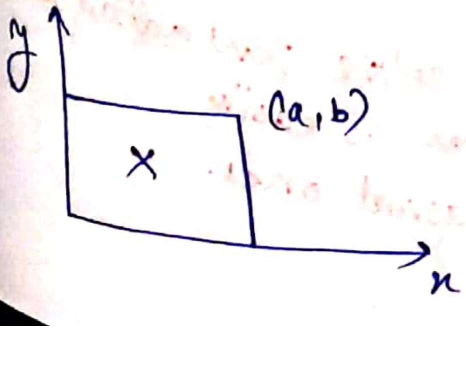
$0 < x < a \qquad t > 0$

$0 < y < b$

B.C $\left.\begin{array}{l} u(0,y,t) = V_0(y) \\ u(a,y,t) = V_a(y) \end{array}\right\} \; 0<y<b$

& $\left.\begin{array}{l} u(x,0,t) = V_0(x) \\ u(x,b,t) = V_b(x) \end{array}\right\} \; 0<x<a$

I.C $u(x,y,0) = f(x,y)$



Applying Crank Nicholson Scheme

$$u_{ij}^n = u(x_i, y_j, t_n)$$

$$\frac{u_{ij}^{n+1} - u_{ij}^n}{\delta t} = \frac{\nu}{2}\left(\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]_{ij}^{n+1} + \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)_{ij}^{n}\right)$$

$$\frac{u_{ij}^{n+1} - u_{ij}^n}{\delta t} = \frac{\nu}{2}\left(\frac{u_{(i+1)j}^{n+1} - 4u_{ij}^{n+1} + u_{ij}^{n+1} + u_{i(j+1)}^{n+1} + u_{i(j-1)}^{n+1}}{h^2} + \frac{u_{(i+1)j}^{n} + u_{(i-1)j}^{n} + u_{i(j+1)}^{n} + u_{i(j-1)}^{n} - 4u_{ij}^{n}}{h^2}\right)$$

if $\delta x = \delta y = h$

o/w

$$\frac{u_{ij}^{n+1} - u_{ij}^n}{\delta t} = \frac{\nu}{2}\left[\frac{u_{(i+1)j}^{n+1} - 2u_{ij}^{n+1} + u_{(i-1)j}^{n+1}}{\delta x^2} + \frac{u_{i(j+1)}^{n+1} - 2u_{ij}^{n+1}}{\delta y^2}\right.$$

& same for superscript n

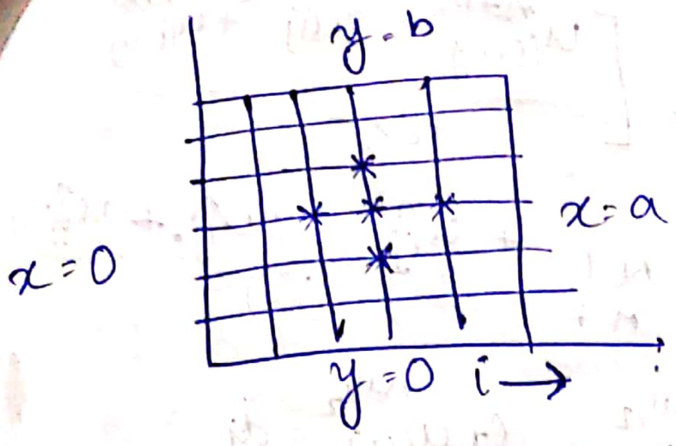
$u_{ij}^{n+1} \longrightarrow (n-1)\times(m-1)$ unknowns

& $(n-1)\times(m-1)$ equations

At any $(i,j)^{th}$ eqn. unknowns are:

$u_{(i-1)j}^{n+1}$, u_{ij}^{n+1}, $u_{(i+1)j}^{n+1}$, u_{ij-1}^{n+1}, u_{ij+1}^{n+1}

We cannot solve pentadiagonal equations as easily as tridiagonal ones.



at t_{n+1}

$a = b = 1$

$\delta x = \delta y = 0.01$

order → 99×99.

Alternating Direction Implicit scheme.
(ADI) Scheme

Solution for $u(x, y, t)$ is determined through the time steps.

Step I : $n \longrightarrow n + 1/2$ and apply implicit (or explicit) difference w.r.t x derivatives and explicit (or implicit) wrt y derivatives.

change the direction (Alternating direction)

$$n \longrightarrow n+1/2 \xrightarrow{\text{Step 1}} n+1$$

Step 2 $n + 1/2 \longrightarrow n+1$, apply explicit or (implicit) diff. wrt. x-derivative, and implicit or (explicit) diff wrt y derivative

The discretization of the PDE.

Step 1: $n \longrightarrow n + 1/2$

$$\frac{u_{ij}^{n+1/2} - u_{ij}^{n}}{\delta t/2} = \nu \left[\frac{\partial^2 u}{\partial x^2}\bigg|_{ij}^{n+1/2} + \frac{\partial^2 u}{\partial y^2}\bigg|_{ij}^{n} \right]$$

$$\frac{u_{ij}^{n+1/2} - u_{ij}^{n}}{\delta t} = \frac{\nu}{2}\left[\frac{u_{(i+1)j}^{n+1/2} - 2u_{ij}^{n+1/2} + u_{(i-1)j}^{n+1/2}}{(\delta x)^2} + \frac{u_{ij+1}^{n} - 2u_{ij}^{n} + u_{ij-1}^{n}}{(\delta y)^2}\right]$$

$i = 1, 2, \dots N-1$

$j = 1, 2, \dots M-1$

At fixed j

$$a_j u_{i-1j}^{n+1/2} + b_j u_{ij}^{n+1/2} + c_j u_{i+1j}^{n+1/2} = d_j$$

$i = 1, 2, \dots N-1$

which forms a tridiagonal system

$$A^j U_j^{n+1/2} = D_j$$

$$U_j^{n+1/2} = \begin{pmatrix} U_{1j}^{n+1/2} \\ U_{2j}^{n+1/2} \\ \vdots \\ U_{n-1j}^{n+1/2} \end{pmatrix}$$ vary for $j = 1, 2, \dots M-1$

Step 2:

$$\frac{u_{ij}^{n+1} - u_{ij}^{n+1/2}}{\delta t} = \frac{\nu}{2}\left[\frac{u_{i-1j}^{n+1/2} - 2u_{ij}^{n+1/2} + u_{i+1j}}{\delta x^2} + \frac{u_{ij+1}^{n+1} - 2u_{ij}^{n+1} + u_{ij-1}^{n+1}}{\delta y^2}\right]$$

$$a_i u_{ij-1}^{n+1} + b_i u_{ij}^{n+1} + c_i u_{ij+1}^{n+1} = d_i$$

$i = 1, 2, \dots N-1$

which again forms a tridiagonal system

H.T. show that on elimination of $n+1/2$ terms this forms a tridiagonal system.

At a fixed $i\ (= 1, 2, \dots N-1)$ we can express the system

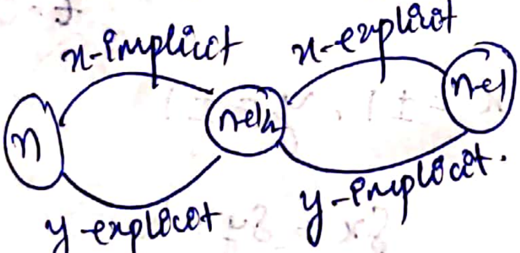
$$a^i u_{ij-1}^{n+1} + b_i u_{ij}^{n+1} + c_i u_{ij+1}^{n+1} = d_i.$$

$$j = 1, 2, \dots M-1$$

Thus for $i = (1, 2, \dots N-1)$

$$A^i U_i^{n+1} = D^i$$

$$U_i^{n+1} = \begin{bmatrix} U_{i1}^{n+1} \\ U_{i2}^{n+1} \\ \vdots \\ U_{iM-1}^{n+1} \end{bmatrix}$$

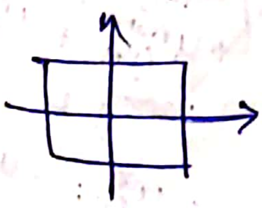


L.T. $\dfrac{\partial u}{\partial t} = \nabla^2 u \quad , -1 < x, y < 1, t > 0$

$$u(x, y, 0) = \cos\frac{\pi x}{2} \cos\frac{\pi y}{2}$$

$u = 0$ on $x = \pm 1\ \ y = \pm 1.$

$\delta x = \delta y = 1/2$



$r = 1/6$

$$r = \nu \frac{\partial t}{(\partial x^2)}$$

ADI scheme is second order accurate in both time & space coordinates. It is unconditionally stable.

$$u_{ij}^{n+1/2} - u_{ij}^{n} = r\left[u_{i+1,j}^{n+1/2} - 2u_{ij}^{n+1/2} + u_{(i-1)j}^{n+1/2} + u_{ij-1}^{n} - 2u_{ij}^{n} + u_{ij+1}^{n}\right]$$

$$u - r u_{(i-1)j}^{n+1/2}$$

$$u_{ij}^{n+1/2}(1+2r) - r u_{i+1,j}^{n+1/2} = u_{ij}^{n} + u_{ij-1}^{n} - 2u_{ij}^{n} + u_{ij+1}^{n}$$

25th March

$$\frac{\delta u}{\delta t} = \nu\left(\frac{\partial^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} + \frac{\delta^2 u}{\delta z^2}\right)$$

$$u_t = \nu \nabla^2 u$$

$$u_{ijk}^{n} \quad n \longrightarrow n + 1/3 \longrightarrow n + 2/3 \longrightarrow n+1$$

$$u_t = u_{xx} + u_{yy}$$

$$u(x,y,0) = \cos\frac{\pi x}{2}\cos\frac{\pi y}{2} \qquad -1 \le x, y \le 1$$

$t = 0$

$u = 0$ on $x = \pm 1$, $y = \pm 1$

$\delta x = \delta y = h$

$\delta t = k$

Applying Crank Nicholson Scheme $\quad r = \frac{k}{h^2}$

$$\cancel{\frac{u_{ij}^{n+1} - u_{ij}^{n}}{k} = \frac{1}{2}\left(\nabla^2 u^{n+1} + \nabla^2 u^{n}\right)}$$

$$\cancel{= \frac{1}{2}\left(\frac{u_{i+1j}^{n+1} - 4u_{ij}^{n+1} + u_{i-1j}^{n+1} + u_{ij+1}^{n+1} + u_{ij-1}^{n+1}}{h^2} + \frac{u_{i+1j}^{n} + u_{i-1j}^{n} - 4u_{ij}^{n} + u_{ij+1}^{n} + u_{ij-1}^{n}}{h^2}\right)}$$

$$\cancel{\frac{u_{ij}^{n+1} - u_{ij}^{n}}{} = \frac{1}{2} r \left(u_{i+1j}^{n+1} + u_{i-1j}^{n+1} - 4u_{ij}^{n+1} + u_{ij+1}^{n+1} + u_{ij-1}^{n+1} + u_{i+1j}^{n} + u_{i-1j}^{n} - 4u_{ij}^{n} + u_{ij+1}^{n} + u_{ij-1}^{n} \right)}$$

Step I : $n \longrightarrow n+1/2$ $\quad x \rightarrow$ implicit

$y \rightarrow$ explicit

For $j = 1, \dots M-1$

$$A^j U_j^{n+1/2} = D^j$$

$$U_j^{n+1/2} = \begin{bmatrix} u_{1j}^{n+1/2} \\ \vdots \\ u_{N-1j}^{n+1/2} \end{bmatrix}$$

$$\frac{u_{ij}^{n+1/2} - u_{ij}^{n}}{k} = \frac{1/2}{h^2} \left[u_{i+1j}^{n+1/2} - 2u_{ij}^{n+1/2} + u_{i-1j}^{n+1/2} + u_{ij+1}^{n} - 2u_{ij}^{n} + u_{ij-1}^{n} \right] r$$

$$a_i^j u_{i-1j}^{n+1/2} + b_i^j u_{ij}^{n+1/2} + c_i^j u_{i+1j}^{n+1/2} = d_i^j$$

$a_i^j = \frac{r}{2}$

$b_i^j = -r-1$

$c_i^j = \frac{r}{2}$

$d_i^j = -u_{ij}^{n} - \frac{r}{2} \left[u_{ij-1}^{n} + u_{ij+1}^{n} - 2u_{ij}^{n} \right]$

$i = 1, 2, \dots M-1$

(Because B.C. are 0)

$n \geq 0.$

$n+1 \rightarrow n+1$

Step 2: $n+1/2$

$$a_i^j u_{ij-1}^{n+1} + b_i^j u_{ij}^{n+1} + c_i^j u_{ij+1}^{n+1} = d_i^j$$

$x \rightarrow$ explicit
$y \rightarrow$ implicit

$$A^i U_i^{n+1} = D_i$$

$$U_i^{n+1} = \begin{bmatrix} u_{i1}^{n+1} \\ u_{i2}^{n+1} \\ \vdots \\ u_{iM-1}^{n+1} \end{bmatrix}$$

$$\nabla^2 u = 0$$

$$0 \le x \le a$$

$$0 \le y \le b$$

u is prescribed on the boundary
Boundary value Problem (BVP)

or $R = \{x, y \mid x^2 + y^2 < a^2\}$

$\partial\Omega \quad x^2 + y^2 = a^2.$

$$\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0 \text{ or } f(x, y)$$

$\downarrow$

Laplace equation

$$\frac{\quad}{\delta x^2} + \frac{u_{ij+1}^{(k)} - 2u_{ij}^{(k+1)} + u_{ij-1}^{(k+1)}}{\delta y^2} = 0$$

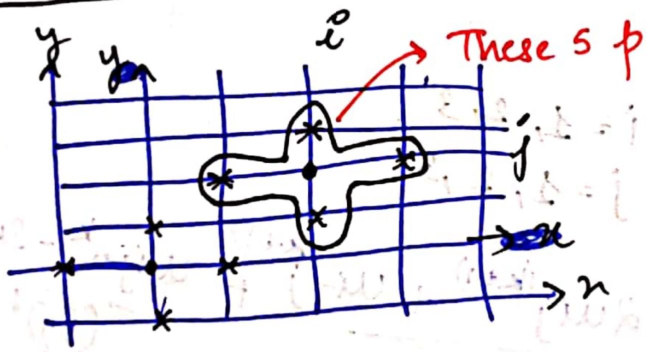
$$0 \le x \le a \;\; \& \;\; 0 \le y \le b$$

$i = 1, 2, \ldots N-1;$

$j = 1, 2, \ldots M-1;$

The eqns are $(N-1) \times (M-1)$ equations in $(N-1) \times (M-1)$ unknowns.

Gauss-Seidel Iterative method



These 5 points are inter-related.

$u_{ij}^{(0)}$ is assumed $\forall \; i = 1, 2, \ldots N-1$
$\qquad\qquad\qquad\quad j = 1, 2, \ldots M-1$

at any $(k+1)$th iteration

$$u_{ij}^{k+1} = \left(\frac{\left(u_{i-1j}^{(k+1)} + u_{i+1j}^{(k)}\right)\delta y^2}{2} + \frac{\delta x^2 \left(u_{ij-1}^{(k+1)} + u_{ij+1}^{(k+1)}\right)}{2} \right) \times \frac{1}{2\left(\delta x^2 + \delta y^2\right)}$$

$$\Rightarrow u_{ij}^{k+1} = \frac{\delta y^2 \left(u_{i-1j}^{(k+1)} + u_{i+1j}^{k}\right) + \delta x^2 \left(u_{ij-1}^{(k+1)} + u_{ij+1}^{(k)}\right)}{2\left(\delta x^2 + \delta y^2\right)}$$

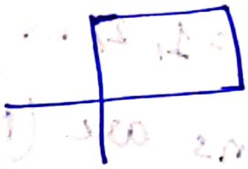
Continue the Iteration

$$\max_{\substack{1 \le i \le N-1 \\ 1 \le j \le M-1}} \left| u_{ij}^{(k+1)} - u_{ij}^{(k)} \right| \le \epsilon$$

Uniform convergence criterion.

26 March

Q) $u_{xx} + u_{yy} = 0$



Lab Task

$\delta x, \delta y < 1$

$0 \le x, y \le 4$ $\quad u = x^2 y^2$ on the boundaries

$\delta x = \delta y = 1$

$u_{ij}^{(0)} = 0$

$i = 1, 2, 3$

$j = 1, 2, 3$

$$\frac{u_{i+1j}^{(k)} - 2u_{ij}^{(k+1)} + u_{i-1j}^{(k+1)}}{(\delta x)^2} + \frac{u_{ij+1}^{k} - 2u_{ij}^{(k+1)} + u_{ij-1}^{(k+1)}}{(\delta y)^2} = 0$$

$$u_{ij}^{(k+1)} = \frac{1}{4}\left(u_{i+1j}^{(k)} + u_{i(j+1)}^{k} + u_{ij-1}^{(k+1)} + u_{ij}^{(k+1)} \right)$$

$k \ge 0$

For $i = 1, 2$

$j = 1, 2$

$\cancel{u_{ij}^{(k+1)}}$

Max

$u_{00}^{0} = 0 = u_{10}^{0} = u_{20}^{0} = u_{30}^{0} = u_{40}^{0}$

u_{01}^{0} ...

$$u_{11}^{1} = \frac{1}{4}\left(u_{21}^{(0)} + u_{12}^{(0)} + u_{10}^{(1)} + u_{01}^{(1)} \right)$$

$$u_{11}^{1} = \cancel{1/2}\ 0$$

$$u_{12}^{1} = \frac{1}{4}\left(u_{22}^{(0)} + u_{13}^{(0)} + u_{11}^{(1)} + u_{02}^{(1)} \right)$$

$$u_{12}^{1} = 0$$

$$u_{13}^{1} = \frac{1}{4}(u$$

$$u_{14}^{1} = 16$$

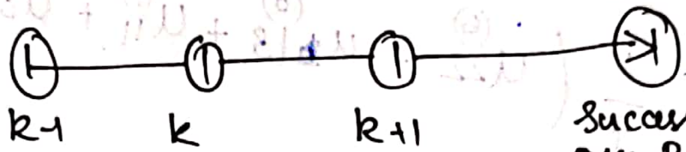
Q $u_{xx} + u_{yy} = x^2 + y^2 \quad 0 < x, y < 1$

$u(x,y) = 0$ on the boundary

Lab Task

$\delta x = \delta y = 0.25$

$\delta x = \delta y < 1$



Successive over Relaxation.

$u_{ij}^{(k)}, u_{ij}^{(k+1)}$

$$\bar{u}_{ij}^{(k+1)} = \bar{u}_{ij}^{(k)} + \omega \left(u_{ij}^{(k+1)} - \bar{u}_{ij}^{(k)}\right)$$

$$1 < \omega < 2$$

where $\bar{u}_{ij}^{k}$ + $\bar{u}_{ij}^{k+1}$ are the modified value using SOR.

$$\bar{u}_{ij}^{(k+1)} = \omega u_{ij}^{k+1} + (1-\omega)\bar{u}_{ij}^{(k)}$$

$$k \geq 0, \; 1 < \omega < 2.$$

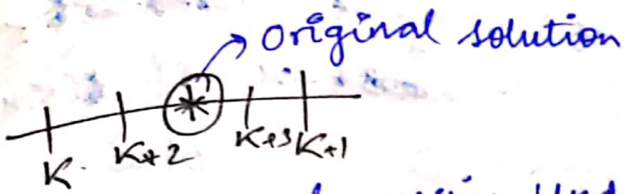
ω is Referred as the Relaxation Param

$u_{ij}^{(k+1)}$ is the one obtained by Gauss Seidel Iteration.

$k \geq K$ apply SOR

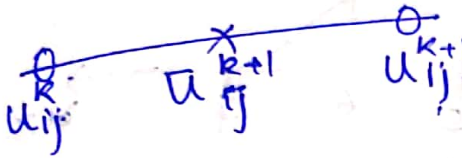
$\bar{u}_{ij}^{(k)} \longrightarrow$ Previous Iterated value

Lab Task Check whether you get rapid convergence via SOR method.



Successive Under Relaxation.

if $0 < \omega < 1$ then it is called SUR.



Transport equation

$$\frac{\delta u}{\delta x} + \frac{c\,\delta u}{\delta y} = \nu \frac{\delta^2 u}{\delta x^2}.$$

till now we have ignored such terms in the equation

If $\nu \simeq 0$,

$$\frac{\delta u}{\delta t} + c\frac{\delta u}{\delta x} = 0 \quad \text{FTCS}$$

- ~~U......~~

$$\frac{dt}{1} = \frac{dx}{c} = \frac{du}{0}$$

$$u = f(x - ct)$$

where f is arbitrary.

$$u(x,t) = f(x-ct)$$

$$u(x,0) = x, \quad 0 \le x \le 1.$$
$$= 2-x; \quad 1 < x \le 2$$
$$= 0, \quad o/w$$

Let $c = 1$. Find $u(x,5) = ?$

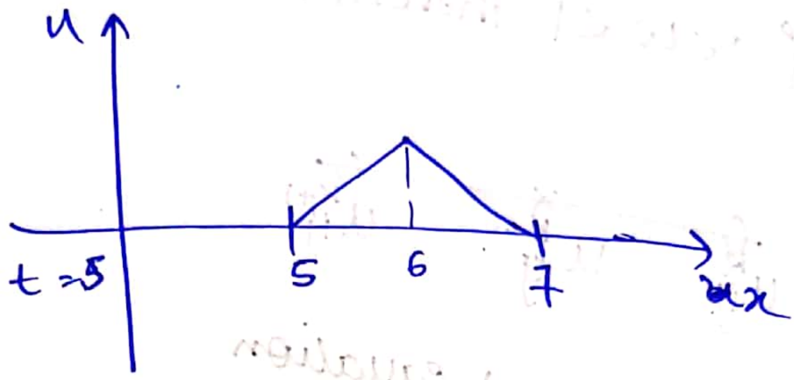
$$f(x) = x \quad 0 \le x < 1$$
$$f(x) = 2-x \quad 1 < x < 2$$

$$u(x,5) = f(x-5t) = x - 5t$$

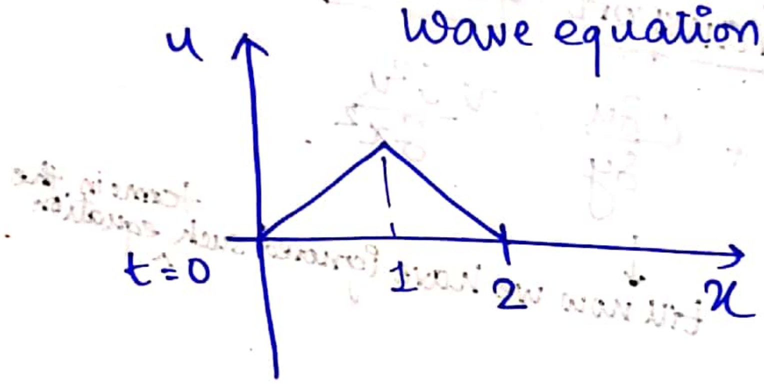
~~$f(x-5t) = x-5t$ if $5 \le x \le 6$~~

~~$= 7-x$ if $6 < x-5t \le 7$~~

~~$= 0$ o/w.~~



Wave equation.



$$f(x-5) = \begin{cases} x-5 & \text{if } 5 < x \le 6 \\ 7-x & \text{if } 6 < x \le 7 \\ 0 & \text{o/w} \end{cases}$$

<u>Check stability</u>

$$\boxed{\frac{\delta u}{\delta t} + \frac{c\,\delta u}{\delta x} = 0}$$ FTCS.

$$\frac{u_i^{n+1} - u_i^n}{\delta t} + c\,\frac{u_{i+1}^n - \cancel{2u} - u_{i-1}^n}{2\delta x} = 0$$

$$\nu = c\frac{\delta t}{\delta x}$$

$$\Rightarrow u_j^{n+1} - u_j^n + \nu\,\frac{(u_{j+1}^n - \cancel{2u_j^n} - u_{j-1})}{2} = 0$$

$\Rightarrow \epsilon_j^n = A^n e^{i\theta j}$

$\theta \to$ phase angle

$A^n \to$ Amplitude.

$$\Rightarrow A^{n+1} e^{i\theta j} - A^n e^{i\theta j} + \nu \left(\frac{A^n e^{i\theta (j+1)} - A^n e^{i\theta (j-1)}}{2} \right) = 0$$

$$\Rightarrow \frac{A^{n+1}}{A^n} = 1 - \nu \left(\frac{e^{i\theta} - e^{-i\theta}}{2} \right)$$

$$|\xi| = \left| \frac{A^{n+1}}{A^n} \right| = |1 - \nu i \cancel{\cos\theta} \sin\theta|$$

$$|1 + \nu^2 \sin^2\theta| \leq 1$$

$$\underline{\underline{|\xi| > 1}}.$$

UNSTABLE

FTCS is unconditionally unstable.

FTBS:

$$\frac{u_j^{n+1} - u_j^n}{\delta t} + c \frac{u_j^n - u_{j-1}^n}{\delta x} = 0$$

$\Rightarrow u_j^{n+1} - u_j^n + \nu (u_j^n - u_{j-1}^n) = 0$

$\Rightarrow A^{n+1} e^{i\theta j} - A^n e^{i\theta j} + \nu (A^n e^{i\theta j} - A^n e^{i\theta (j-1)})$

$\Rightarrow \frac{A^{n+1}}{A^n} = 1 - \nu (1 - e^{-i\theta})$

$\left| \frac{A^{n+1}}{A^n} \right| = |(1 - \nu) + \nu e^{-i\theta}|$

$$|1 - \nu + \nu\cos\theta - \nu i\sin\theta|$$

$$|1 - (\nu(1-\cos\theta) - \nu i\sin\theta)|$$

$$|1 - 2\nu\sin^2\theta/2 - \nu i\sin\theta|$$

$$1 + 4\nu^2\sin^4\theta/2 - 4\nu\sin^2\theta/2 + \nu^2\sin^2\theta$$

$$-1 \le 1 + 4\nu^2\sin^4\theta/2 + 4\nu^2\sin^4\theta/2\cos^4\theta/2 - 4\nu\sin^2\theta/2 \le 1$$

$$-2 \le 4\nu\sin^2\theta/2\,(2\nu\sin^2\theta/2 + \nu\cos^2\theta/2) - 4\nu\sin^2\theta/2 \le 0$$

$$-2 \le 4\nu^2\sin^2\theta/2 - 4\nu\sin^2\theta/2 \le 0$$

$$-2 \le (4\nu^2 - 4\nu)\overset{max}{\sin^2\theta/2} \le 0$$

$$-2 \le (4\nu^2 - 4\nu) \le 0$$

$$-1/2 \le (\nu^2 - \nu) \le 0$$

$1/2 \quad 1/4 - 1/2$

Stable if $\nu \in (0,1)$.

$$\nu = \frac{c\delta t}{\delta x}$$

$\Rightarrow$ if $c>0$ FTBS is stable $\frac{c\delta t}{\delta x} < 1$

if $c<1$ FTBS is unstable

$$\frac{\delta u}{\delta t} + c\frac{\delta u}{\delta x} = 0 \quad \text{FTFS}$$

$$\frac{u_j^{n+1} - u_j^n}{\delta t} + \frac{c}{\delta x}(u_{j+1}^n - u_j^n) = 0$$

$$A^{n+1}e^{i\theta j} - A^n e^{i\theta j} + \nu(A^n e^{i\theta j+1} - A^n e^{i\theta j}) = 0$$

$$\frac{A^{n+1}}{A^n} = 1 - \nu(e^{i\theta} - 1)$$

$$= 1 - \nu(\cos\theta - 1 + i\sin\theta)$$

$$\left|\frac{A^{n+1}}{A^n}\right| = |1 - \nu(\cos\theta - 1) - \nu i \sin\theta|$$

$$= |1 + \nu^2(\cos\theta - 1)^2 - 2\nu(\cos\theta - 1) + \nu^2\sin^2\theta|$$

$$= |1 + \nu^2\cos^2\theta + \nu^2 - 2\nu\cos\theta - 2\nu(\cos\theta - 1) + \nu^2\sin^2\theta|$$

$$= |1 + 2\nu^2 - 2\nu\cos\theta - 2\nu(\cos\theta - 1)|$$

$$= 1 \cancel{\pm 2\nu(\nu-1)\cos\theta - 2\nu\cos\theta - 1) + 2\nu}$$

$$= \cancel{1 + 2\nu\cos\theta(\nu - 2) + 2\nu}$$

$$= 1\cancel{+2\nu}\,\nu -$$

$$-1 \le |1 + 2\nu^2 - 2\nu(2\cos\theta + 1)| \le 1$$

$$-2 \le 2\nu^2 - 2\nu(2\cos\theta + 1) \le 0$$

$$2\nu^2 -$$

$\nu \in (-1, 0)$. stable

$\Rightarrow \quad c < 0$

$\Rightarrow \quad c\,\dfrac{\delta x}{\delta t} < 1$

$$\frac{\delta u}{\delta t} + c\frac{\delta u}{\delta x} = 0$$

$u(x, 0) = f(x)$

$u(a, t) = V_a \quad t \geq 0$

FTCS unstable

If $c > 0$ FTBS stable. $|\nu| \leq 1$

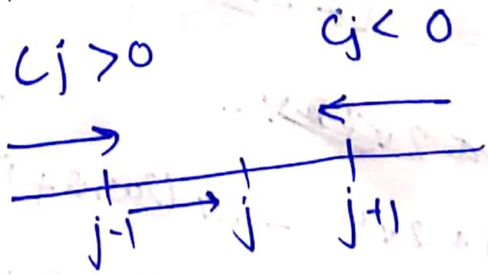
$c < 0$ FTFS stable $|\nu| \leq 1$.

If $c = c(x)$ a known function.

if $c_j > 0$ $$\frac{u_j^{n+1} - u_j^n}{\delta t} + c_j\,\frac{u_j^n - u_{j-1}^n}{\delta x} = 0$$

$c_j < 0$ $$\frac{u_j^{n+1} - u_j^n}{\delta t} + c_j\,\frac{u_{j+1}^n - u_j^n}{\delta x} = 0$$

Upwind scheme



T.E.

Modified equation, T.E. compatibility.

Lax equivalence Theorem :

Lab Task

$$u(x,0) = \begin{cases} 20x, & 0 \le x \le 0.05 \\ 20(0.1-x), & 0.05 \le x \le 0.1 \\ 0, & 0.1 \le x. \end{cases}$$

$c = 1$

201st April

$\nu = c\,\delta t/\delta x \quad , c > 0$

FTBS yields

$$u_j^{n+1} = (1-\nu)u_j^n + \nu u_{j-1}^n$$

Expand by taylor series abt (x_j, t_n)

$$\cancel{u_j^n} + \delta t\, u_t|_j^n + \frac{\delta t^2}{2} u_{tt}|_j^n + \cdots$$

$$= \cancel{(1-\nu)u_j^n} + \cancel{\nu u_j^n} - \nu\,\delta x\, u_x|_j^n + \nu \frac{\delta x^2}{2} u_{xx}|_j^n + \cdots$$

as the pt we get. (x_j, t_n) we get

$$\delta t\, u_t + \nu\,\delta x\, u_x + \frac{\delta t^2}{2} u_{tt} + \frac{\delta t^3}{3} u_{ttt} + \cdots$$

$$= \nu \frac{\delta x^2}{2} u_{xx} - \frac{\nu(\delta x)^3}{3!} u_{xxx} + \cdots$$

We eliminate u_{tt}

Diff ⊛ wrt t on both sides

$$\delta t\, u_{tt} + \nu\,\delta x\, u_{xt} + \frac{\delta t^2}{2} u_{ttt} + \frac{\delta t^3}{3!} u_{tttt} + \cdots$$

$$= \nu \frac{\delta x^2}{2} u_{xxt} - \frac{\nu(\delta x)^3}{3} u_{xxxt}$$

Discretize equation

$$\frac{u_j^{n+1}-u_j^n}{\delta t} + c\frac{u_j^n - u_{j-1}^n}{\delta x} = 0$$

about (x_j, t_n)

$$\frac{1}{\delta t}\left[\cancel{u_j^n} + \delta t\, u_t\big|_j^n + \delta t^2 u_{tt}\big|_j^n + \cdots - \cancel{u_j^n}\right]$$

$$+\frac{c}{\delta x}\left[\cancel{u_j^n} - \cancel{u_j^n} + \delta x\, u_x\big|_j^n - \frac{(\delta x)^2}{2!}u_{xx}\big|_j^n + \cdots\right] = 0$$

at (x_j, t_n)

$$u_t + c u_x = -\frac{\delta t}{2}u_{tt} + \frac{c\,\delta x}{2}u_{xx}$$

$$-\frac{\delta t^2}{3!}u_{ttt} - \frac{c\,\delta x^2}{3!}u_{xxx} \quad -(*)$$

[N.B. If $u(x,t)$ satisfies the PDE then LHS is zero]

Differentiate $(*)$ w.r.t t

(1) $$u_{tt} + c u_{xt} = -\frac{\delta t}{2}u_{ttt} + \frac{c\,\delta x}{2}u_{xxt}$$

$$-\frac{\delta t^2}{3!}u_{tttt} - \frac{c\,\delta x^2}{3!}u_{xxxt}$$

Diff wrt x

$$u_{tx} + c u_{xx} = -\frac{\delta t}{2} u_{ttx} + \frac{c\delta x}{2} u_{xxx} - \frac{\delta t^2}{3!} u_{tttx} - \frac{c\delta x^2}{3!} u_{xxxx}$$

$\text{II} \times (-c) + (\text{I})$

$$u_{tt} - c^2 u_{xx} = \delta t\left[-\frac{u_{ttt}}{2} + \frac{c}{2} u_{ttx} + O(\delta t)\right] + \delta(x)\left[\frac{u_{xxt}}{2} - \frac{c^2 u_{xxx}}{2} + O(\delta x)\right]$$

Substituting this form of u_{tt} is $(*)$ we get

$$u_t + cu_x = -\frac{\delta t}{2}\left[c^2 u_{xx} + O(\delta t)\right] + c\frac{\delta x}{2}\left[u_{xx} + O(\delta x^2)\right]$$

$$= c\frac{\delta x}{2}(1-\nu)u_{xx} + O(\delta t^2) + O(\delta x^2)$$

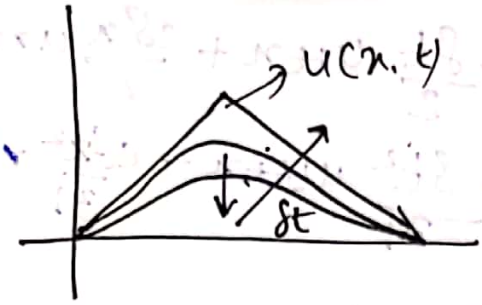
We get the modified eq^n

$$u_t + cu_x = c\frac{\delta x}{2}(1-\nu)u_{xx} + O(\delta t^2, \delta t\delta x, \delta x^2) \quad -③$$

Through the FTBS we are solving the PDE $-①$, which is referred as the modified equation.

If $u(x,t)$ be the exact solution of the PDE. $u_t + cu_x = 0$. Then ① will provide the truncation error.

Scanned by CamScanner



$u_t + c u_x$
$- c \frac{\delta x}{2} (1-\nu) u_{xx}$

$=$

$\nu = 0.1$

Dissipative scheme: If the least order term in the T.E. possesses 2nd or even order derivatives, i.e., u_{xx} or u_{xxxx} then the scheme is referred as dissipative scheme.

Dispersion error ⟶ If the T.E. contains odd order spatial derivative of u i.e. u_{xxx} etc.

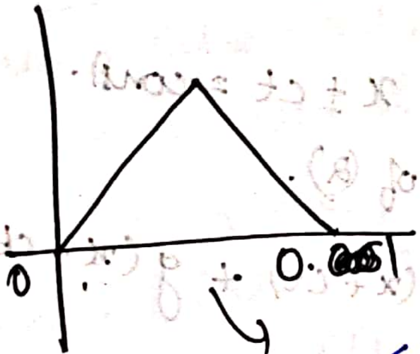
CHECK. FTCS has dispersion error

$u_t + u_x = 0$

$$u(x,0) = \begin{cases} 20x & 0 \le x \le 0.05 \\ 20(0.1-x) & 0.05 \le x \le 0.1 \\ 0, & 0.1 \le x \end{cases}$$

LT $u(0,t) = 0$

$v = \frac{c\,\delta t}{\delta x} = 0.1, 0.3, 0.6, 0.8.$



$t \neq 0$

$\delta x \le 0.05$

Q $u_t + u_x = 0$

$u(0,t) = 0,$

$$u(x,0) = \begin{cases} x^2, & 0 \le x \le 1 \\ 0 & x > 1 \end{cases}$$

$v = \frac{c\,\delta t}{\delta x}$ Increase the choice of v.

$\delta x = 0.125$

fix δt $\longrightarrow$ $v = 0.5, 0.25$

2nd April

$u_t - cu_x = 0$ —①

$u_{tt} - cu_{xt} = 0$

$u_{xt} - cu_{tt} = 0$

$u_{tt} - c^2 u_{xx} = 0$

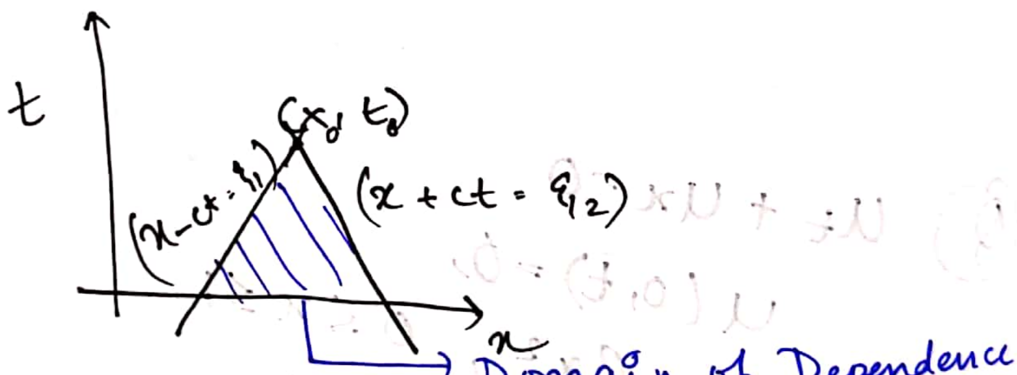
i.e. $u_{tt} = c^2 u_{xx}$

Thus $u(x,t)$ of PDE $u_t \pm cu_x = 0$ also satisfy $\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial^2 x} = 0$

Characteristic curves: $x \pm ct = \text{const}$.

General solution of ②.

$$u(x,t) = f(x+ct) + g(x-ct)$$



$u(x,0) = F(x)$

$F(x) = f(x) + g(x)$.

$x - ct = x_0 - ct_0$

$x + ct = x_0 + ct_0$

$u(x_0, t_0) = f(\xi_1) + g(\xi_2)$

$= f(x_0 - ct_0) + g(x_0 + ct_0)$.

~~Numerical Domain of Dependence should be containing the~~ For stability of the numerical scheme

Domain of dependence of the numerical scheme for solution at (x_j, t_{n+1}), should contain the analytical domain of dependence

$\Rightarrow$ ~~DFE~~ CFL criteria

(Courant - Fredericks - Lewis criteria)

Advection Diffusion Equation

$u_t + c u_x = \nu u_{xx}$ $\quad t > 0$, $0 < x < L$

c ↳ speed of the wind (say) (convection)

$\nu << 1$

I.C. $u(x, 0) = f(x)$, $0 \le x \le L$.

$u(0, t) = V_0$, $u(L, t) = U_L$ for $t > 0$.

B.C

Linear Burgers equation

Non Linear Burgers eqn is

↳ $u_t + u u_x = \nu u_{xx}$

If $\nu << 1$, then

$u_t + c u_x = \nu u_{xx}$

is dominated by the hyperbolic part, a central difference scheme will lead to instability

$$\frac{u_j^{n+1} - u_j^n}{\delta t} + c\left(\frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2\delta x} + \frac{u_{j+1}^n - u_{j-1}^n}{2\delta x}\right)$$

$$= \frac{\nu}{2}\left[\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\delta x^2} + \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\delta x^2}\right]$$

Scanned by CamScanner

FTCS

$$\frac{u_j^{n+1} - u_j^n}{\delta t} + c\left(\frac{u_{j+1}^n - u_{j-1}^n}{2\delta x}\right)$$

$$= \nu\left(\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\delta x^2}\right)$$

Modified eqn

$$u_t + c u_x = \left(\mu - c^2\frac{\delta t}{2}\right) u_{xx}$$

$$+ O(\delta t^2, \delta x \delta t, \delta x^2)$$

T.E. $\longrightarrow O(\delta t, \delta x^2)$

T.E. $\longrightarrow 0$ as $\delta t, \delta x \longrightarrow 0$

$\Rightarrow$ compatible

Stability implies coefficient of u_{xx} must be +ve

$$\mu - c^2\frac{\delta t}{2} \geq 0.$$

$$\Rightarrow \boxed{c^2\frac{\delta t}{2} \leq \mu}$$

For dissipation.

also, $\cancel{c\frac{\delta x}{\mu}}\; r = \mu\frac{\delta t}{(\delta x)^2} \le 1/2$

$$r = \mu\frac{\delta t}{(\delta x)^2} \le 1/2$$

$$\nu = c\frac{\delta t}{\delta x}$$

$$Re_{\sigma x} = \frac{c\,\delta x}{\mu} = \frac{c\,\delta t}{\delta x}\cdot\frac{(\delta x)^2}{\mu\,\delta t}$$

Mesh Reynold's number

$$= \nu / r$$

$$\nu^2 = c^2\left(\frac{\delta t}{\delta x}\right)^2$$

$$\frac{\nu^2}{r} = c^2\frac{\delta t^2}{\delta x^2}\cdot\frac{\delta x^2}{\mu\,\delta t} = \frac{c^2\delta t}{\mu} \le 2$$

also $r \le 1/2$

$\nu^2 \le 2r$ and $r \le 1/2$

$$\frac{\nu}{r} \le \frac{2}{\nu}$$

$$\frac{\nu}{r} \ge 2\nu$$

$$\Rightarrow \boxed{2\nu \le Re_{\sigma x} \le \frac{2}{\nu}}$$

Nonlinear Burgers equation

$$\boxed{u_t + u u_x = \nu u_{xx}} \quad ; \quad 0 < x < L, \quad t > 0$$

$$u(x, 0) = f(x), \quad 0 < x < L$$

$$u(0, t) = U_0, \quad u(L, t) = U_L, \quad t > 0$$

If $\nu \sim O(1)$

Crank - Nicholson scheme is stable.

$$\frac{u_j^{n+1} - u_j^n}{\delta t} + \frac{1}{2}\left[u \frac{\delta u}{\delta x}\bigg|_j^n + u \frac{\delta u}{\delta x}\bigg|_j^{n+1}\right] = \frac{\nu}{2}\left[\frac{\delta^2 u}{\delta x^2}\bigg|_j^{n+1} + \frac{\delta^2 u}{\delta x^2}\bigg|_j^n\right]$$

We solve by iteration.

At any iteration $(K+1)$ level the discretized equation is.

Quasilinearization Technique

$$\frac{u_j^{n+1(K+1)} - u_j^n}{\delta t} + \frac{1}{2}\left[u \frac{\delta u}{\delta x}\bigg|_j^n + u^{(k)} \frac{\delta u}{\delta x}^{(K+1)}\bigg|_j^{n+1}\right] = \frac{\nu}{2}\left[\frac{\delta^2 u}{\delta x^2}^{(K+1)}\bigg|_j^{n+1} + \frac{\delta^2 u}{\delta x^2}\bigg|_j^n\right]$$

At time level $n+1$

$$u_j^{(0)} = u_j^{n}, \quad n \geq 0$$

At every time level $(n+1)$ we solve the discretized eq^n iteratively.

$$\frac{u_j^{n+1\,(k+1)} - u_j^n}{\delta t} + \frac{1}{2}\left[\cancel{\ldots}\right.$$

$$+ \frac{1}{2}\, a\left(u_j^n \left(\frac{u_{j+1}^n - u_{j-1}^n}{\delta x} \right) + u^{n+1\,(k+1)} \left(\frac{u_{j+1}^{n+1\,(k+1)} - u_{j-1}^{n+1\,(k+1)}}{\delta x} \right) \right.$$

$$= \frac{\nu}{2}\left[\frac{u_{j+1}^{n+1\,(k+1)} - 2u_j^{n+1\,(k+1)} + 2u_{j-1}^{n+1}}{\delta x^2} + \frac{u_{j+1}^n - 2u_j^n + 2u_{j-1}^n}{\delta x^2} \right]$$

$$\cancel{\frac{u_j^{n+1\,(k+1)} - u_j^n}{\delta t}}$$

$$A_i u_{j-1}^{n+1\,(k+1)} + B_i u_j^{n+1\,(k+1)} + C_i u_{j+1}^{n+1\,(k+1)} = D_i$$

$$u\frac{\delta u}{\delta x}\bigg|_j^{n+1} = u_j^{(u)}\frac{\delta u}{\delta x}^{(k+1)}\bigg|_j^{n+1}$$

We can also apply Newton's Linearization Technique

at $(k+1)$th iteration

$$\left(u_j^{n+1}\right)^{k+1} = \left(u_j^{n+1}\right)^k + \Delta u_j^n$$

$$\Delta \longrightarrow \text{error}$$

$$\Delta u_j^{n+1} \ \forall j$$

Apply upwind scheme to discretize the 1st order derivative

$$u_t + uu_{xx} = \nu u_{xx}$$

$u > 0 \qquad u_x \longrightarrow BS$

$u < 0 \qquad u_x \longrightarrow FS.$

$$u_{tt} - c^2 u_{xx} = 0 \quad , t > 0 \quad 0 < x < L$$

I.C. $\quad u(x,0) = f(x)$

$\quad \frac{\delta u}{\delta t}(x,0) = g(x)$

B.C. $\quad u(0,t) = ? \quad u(l,t) =$

$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\delta t^2} - c^2\left(\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\delta x^2}\right) = 0$$

$u_j^0 = f_j$

$\frac{\partial u}{\partial t}(x,0) = \frac{u_j^1 - u_j^{-1}}{2\delta t} = g(x)$

$u_j^1 - u_j^{-1} = 2\delta t\, g_j$

$\boxed{u_j^{-1} = u_j^1 - 2\delta t\, g_j} \quad \forall j$

$u(0,t) = u(1,t) = 0$ B.C.

$u(x,0) = \sin \pi x$ I.C.

$\frac{\delta u}{\delta t}(x,0) = 0, \quad 0 \le x \le 1$

$u_{tt} = u_{xx}$ Lab Task

(Q) $u_t + u u_x = 0.5\, u_{xx}$ Lab Task

$u(x,0) = x$

$u(0,t) = 0, \quad u(1,t) = 1$