Teacher's Sign / Remarks

why do we study vector space?

$2x+3y=0$ —— (1)

$3x+2y=0$ —— (2)

we use the properties of vector spaces to solve these.

we have set / system of equations, $(V, F, +, \cdot)$

V — x, y
F — coeffs

$V \neq \varphi$ $F \neq 0$
↓ set   ↓ Field

define binary operation $+ : V \times V \to V$
$\cdot : F \times V \to V$

Then, $(V, F, +, \cdot)$ is called vector space if the following conditions are satisfied.

1. $x+(y+z) = (x+y)+z \quad \forall x, y, z \in V$
2. $(x+y) = (y+x) \quad \forall x, y \in V$
3. $\exists 0 \in V$, s.t. $0+x = x \quad \forall x \in V$
4. For each $x \in V$ $\exists y \in V$ s.t. $x+y=0$
5. $\alpha(\beta x) = \alpha\beta x \quad \forall \alpha, \beta \in F, \forall x \in V$ 6. $\alpha(x+y) = \alpha x + \alpha y$
7. $(\alpha+\beta).x = \alpha x + \beta x \quad \forall \alpha \in F, x \in V$
8. $1x = x \quad \forall x \in V$

Ex: $(R, R, +, \cdot)$

$(R, Q, +, \cdot)$ → set of rational numbers

$(R, \mathbb{C}, +, \cdot)$ → Not vector space as complex multiplication is not closed.

$(\mathbb{Q}, \mathbb{R}, +, \cdot)$ — Not a vector space

Ex: $V = \{ f \mid f : \mathbb{R} \to \mathbb{R},\ f(1) = 0 \}$

$f : V \times V \longrightarrow V$ defined by

$f(x) + g(x) = (f+g)(x) \quad \forall\ f, g \in V.$

$\therefore F \times V \longrightarrow V$

But if $f(1) = 1 \Rightarrow$ Not a vector space.

Ex: $V = \{$set of all $m \times n$ matrices$\}$

$(V, \mathbb{R}, \pm, \cdot)$ — vector space

$\bar{V} = \{$set of all non-singular matrices$\}$

$0 \notin \bar{V}$ — Not a vector space

$V = \{$set of all singular matrices$\}$

V — Not a vector space.

Ex: $V = \{ A \in M_{n \times n} \mid tr(A) = 0 \}$

$-(V, \mathbb{R}, +, \cdot)$ — Vector space.

$c[a,b]$ = Set of all continuous functions from $\underline{a}$ to $\underline{b}$

$(c[a,b], \mathbb{R}, +, \cdot)$ — This is a vector space.

Ex: $(\mathbb{R}^n, \mathbb{R}, +, \cdot)$ — vector space.

Ex: $(\mathbb{C}, \mathbb{C}, +, \cdot)$ —

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Subspace:

Let $(V, F, +, \cdot)$ be a VS

Then $W \in V$. Then W is called subspace of V if $(W, F, +, \cdot)$ is a vector space.

$(R, R, +, \cdot)$ – vector space / Sub space

$(\{0\}, R, +, \cdot)$ – Sub space of $(R, R, +, \cdot)$

$(N, R, +, \cdot)$ – Not vector space.

Ex: There are only two subspaces of R.

$(Q, Q, +, \cdot) \subset (R, Q, +, \cdot)$

$M_{n\times n}$ = set of all $n\times n$ matrices

W_1 {set of all symmetric matrices}

W_2 {set of all skew-symmetric matrices}

Th. Let $(V, F, +, \cdot)$. Let $W \subseteq V$. Then W is a subspace of V iff $\alpha x + \beta y \in W$ $\forall \alpha, \beta \in F$, $\forall x, y \in W$.

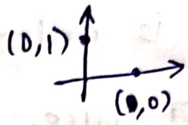
proof: First assume that W is subspace that means $(W, F, +, \cdot)$ is a vectorspace. So,

$\alpha x + \beta y \in W \Leftrightarrow \alpha x \in W$

$\alpha(x+y) \in W$

Let V be a VS and W_1 and W_2 are subspaces.

Q1. Will $W_1 \cup W_2$ be subspace of V?



$(1,1) \notin (x,y)$ axes

If $\alpha_1 \in W_1$

$\alpha_2 \in W_2$

$\not\Rightarrow \alpha_1 + \alpha_2 \in W_1 \cup W_2$

No compulsion for $\alpha_1+\alpha_2$ to belong to $W_1 \cup W_2$.

Q. Will $W_1 \cap W_2$ be subspace of V?

$\alpha \in W_1 \cap W_2 \Rightarrow \alpha \in W_1 \Rightarrow \cancel{a\alpha \in W_1 \cap W_2}$

$\beta \in W_1 \cap W_2 \qquad \alpha \in W_2 \qquad \cancel{b\beta \in W_1 \cap W_2}$

$\beta \in W_1 \qquad \cancel{a\alpha + b\beta \in W_1 \cap W_2}$

$\beta \in W_2$

(or) $a\alpha + b\beta \in W_1$

$a\alpha + b\beta \in W_2 \Rightarrow a\alpha + b\beta \in W_1 \cap W_2$

If W_i is subspace of V then,

$W_1 \cap W_2 \ldots\ldots \cap W_n$ is subspace of V.

Even when $n=\infty$ this works!

Arbitary intersection of any subspaces is also a subspace.

Q. Is $W_1 + W_2$ subspace of V.

$W_1 + W_2 = \{x+y \mid x \in W_1, y \in W_2\}$

$\alpha \in W_1 + W_2 \Rightarrow \exists x_1, y_1 \quad x_1 + y_1 = \alpha$

$\beta \in W_1 + W_2 \Rightarrow \exists x_2\, y_2 \quad x_2 + y_2 = \beta$

$\cancel{x_2 = ax_1 \quad y_2 = ay_1 \quad \exists x_2, y_2}$

$\Rightarrow \cancel{\exists x_2, y_2 \quad x_2 + y_2 = a\alpha}$

$\cancel{\exists x_3, y_3 \quad x_3 + y_3 = b\beta}$

$$a\alpha + b\beta = z_1 + z_2$$

$$a\alpha + b\beta = a(x_1+y_1) + b(x_2+y_2)$$

$$= \underbrace{ax_1 + bx_2}_{\subset W_1} + \underbrace{ay_1 + by_2}_{\subset W_2}$$

$\therefore \; W_1 + W_2 \subseteq V$

$\rightarrow W_1 \subseteq W_1 + W_2$

$\rightarrow W_2 \subseteq W_1 + W_2$

$W_1 + W_2$ is the smallest subspace countaining W_1 and W_2 both.

Let $W_1 \subset S$ $\qquad \alpha, \beta$ – Field element

$W_2 \subset S$ $\qquad x \in W_1 \quad y \in W_2$

$\Rightarrow \alpha x + \beta y \subset S \quad \forall \alpha, \beta \quad \forall x, y$

$\Rightarrow W_1 + W_2 \subset S$?

Need to prove every element of $W_1 + W_2$ is of form $\alpha x + \beta y$.

Let $z \in W_1 + W_2$

$\Rightarrow z = x + y \quad \exists x \in W_1 \; y \in W_2$

with, $\alpha = 1 \quad \beta = 1$

$\therefore \boxed{W_1 + W_2 \subseteq S}$

Let V be a VS and let W_1 and W_2 be two subspaces of V such that $W_1 \cap W_2 = \{0\}$

$$W_1 + W_2 = \{x+y \mid x \in W_1, y \in W_2\}$$

If $W_1 \cap W_2 = \{0\}$ then $W_1 + W_2$ is unique.

That means,

$\nexists\, x_1\ x_2$, $y_1\ y_2$ such that $x_1 + y_1 = z$ and $x_2 + y_2 = z$

$\Rightarrow x_1 + y_1 = x_2 + y_2$

$\Rightarrow x_1 - x_2 = y_2 - y_1$

$x_1 - x_2 \in W_1$, $y_2 - y_1 \in W_2$ $\Rightarrow$ $x_1 = x_2$, $y_1 = y_2$

since $\{0\}$ is only common.

Here we can write as $W_1 \oplus W_2$ (internal direct sum)

W_1 : x-axis

W_2 : y-axis

$W_1 \cap W_2$: $(0,0)$

$W_1 \oplus W_2$: $\mathbb{R}^2$

Let V be a VS

Let $x_1, x_2, \ldots, x_n \in V$

and let $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$

we call $\sum_{i=1}^{n} \alpha_i x_i$ as linear combination of $x_1, x_2, x_3, \ldots, x_n$.

$S = \{ \sum \alpha_i x_i \mid \alpha_i \in F, i = 1, 2, \ldots, n \}$

S- set of linear combinations

S- vector space.

$W \subseteq V$

$LS(W) = ?$

All finite linear combinations of elements of W.

Linearly dependent:

If $\sum_{i=1}^{n} c_i x_i = 0$ where atleast one $c_i \neq 0$ Then they are called linearly independent.

Else they are linearly dependent.

$\{0\}$ is linearly dependent.

Infact any set countaining $\{0\}$ is linearly dependent.

$1 \cdot 0 + \sum 0 \cdot x_i = 0$ ✓ $c_0 \neq 0$

Every non-zero vector is LI

* $(R, Q, +, \cdot)$

$1, \sqrt{2}$ – Linearly Independent

$\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}, \ldots$ – Linearly Independent

$\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ are LI

doesn't mean $\{\alpha + \alpha_1, \ldots, \alpha + \alpha_n\}$ is LI

They are linearly dependent only when α belongs to the linear span of $\{\alpha_1, \dots, \alpha_n\}$

proof:

$$c_1(\alpha+\alpha_1) + c_2(\alpha+\alpha_2) + \dots + c_n(\alpha+\alpha_n) = 0$$

$$\Rightarrow \frac{c_1}{\Sigma c_i} \cdot \alpha_1 + \frac{c_2}{\Sigma c_i} \cdot \alpha_2 + \dots = -\alpha$$

$\Sigma c_i \neq 0$. Because they are not all linearly dependent.

$\therefore$ If they are linearly dependent then $\alpha \in LS\{\alpha_1, \alpha_2, \dots, \alpha_n\}$

A sets S is linearly dependent if it has finite set of vectors which are LD.

A set S is linearly independent if it is not independent

α

te set of

independent

Basis:

A maximal L.I. subset of V.

A LI set S which spans V.

$(\mathbb{R}, \mathbb{R}, +, \cdot)$

$S = \{1\}$

$LS(\{1\}) = \mathbb{R}$

$(\mathbb{R}^2, \mathbb{R}, +, \cdot)$

$B = \{(0,1), (1,0)\}$

$LS(B) = \mathbb{R}^2$

↳ prove by $LS(B) \subseteq \mathbb{R}^2$ / Trivial

$\mathbb{R}^2 \subseteq LS(B)$ / Showing $\forall a \in \mathbb{R}^2 \Rightarrow a \in LS(B)$

$(x, y) = x(1,0) + y(0,1)$

$V = M_{2\times 2}$

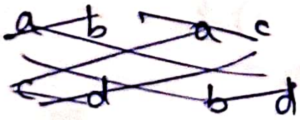
$$B = \left\{ \begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix} \right\}$$

$S(B) = V$?

$$\begin{pmatrix}a&b\\c&d\end{pmatrix} = a\begin{pmatrix}1&0\\0&0\end{pmatrix} + b\begin{pmatrix}0&1\\0&0\end{pmatrix} + c\begin{pmatrix}0&0\\1&0\end{pmatrix} + d\begin{pmatrix}0&0\\0&1\end{pmatrix}$$

$\therefore M_{2\times 2} = V \subseteq S(B)$

$\therefore S(B) = V$



• V = set of 2×2 symmetric matrices

$B = \{ \ldots, \begin{pmatrix}0&0\\0&1\end{pmatrix}\}$

V = set of all skew symmetric matrices

$$B = \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$$

$$V = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \middle| \; a_{11} = 0 \right\}$$

$$B = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$$

$$V = \{ A_{2\times 2} \mid tr(A) = 0 \}$$

$$B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$$

1. Guess the Basis
2. check linear dependency
3. $LS(B) = V$ proof $\rightarrow$ $LS(B) \subseteq V$ & $V \subseteq LS(B)$

$V = \{$ set of all continuous $\}$

$S = \{ \sin x, \cos x \} \rightarrow$ Independent

$LS(S) = \alpha \sin x + \beta \cos x = f(x) \quad \forall x,$ given f

$V = \{ (x_n)_{n \in N} \}$

V = set of real sequences

$B = \{ (1, 0, 0, \dots), (0, 1, 0, 0, \dots), \dots\dots, (0, 0, \dots, 1) \}$

$(1, 1, \dots, 1)$ cannot be expressed as linear combination of finite elements of B.

To prove infinite set to be linearly dependent proving for any finite subset could be enough.

Ex: $\mathbb{R}^2 = V$

$B = \{(0,1), (1,0)\}$

$LS(B) = V$

$LS(B) \subseteq V$ (By def)

we have to show that $V \subseteq LS(B)$

$x \in V,\ x = (x_1, y_1)$

$x = (x_1, y_1) = x_1(1,0) + y_1(0,1)$

x is LC of $(0,1)$ and $(1,0)$

$\therefore\ V \subseteq LS(B)$

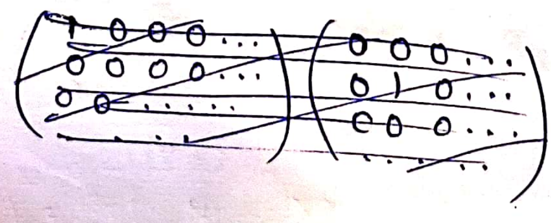
Hence, $V = LS(B)$

1. Each vector space has Basis.
2. Each VS has infinitely many basis.

$\mathbb{R}^?$:

$B = \{(1,0,\dots\dots), (0,1,0\dots\dots), (0,0,1,\dots\dots)\dots\dots, (0,\dots\dots,1)\}$

$V =$ set of all matrices s.t. trace$(A) = 0$.



$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \\ a_{n1} & a_{n2} & \cdots \end{pmatrix}$$

$$\sum_{i=1}^{n} a_{ii} = 0$$

$$B = \left\{ \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & \cdots & \\ \cdots & & & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ \cdots & & & \\ 0 & 0 & 0 & \cdots -1 \end{pmatrix}, \cdots \right.$$

$$\left. \begin{pmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \cdots & & & \end{pmatrix}, \cdots \right\}$$

\# Let V be a vector space

Let B be the basis

Let $x \in V$

$\Rightarrow x \in LS(B)$

For this x, $\exists x_1, x_2, \dots, x_n \in B$

s.t. $x = \sum_{i=1}^{n} c_i x_i$, $c_i \in F$

and also, $x = \sum_{i=1}^{n} b_i x_i \Rightarrow b_i = c_i$

\# $V \longrightarrow VS$

B for V

$\dim(V) = |B|$

\# $V \to B_1$, $V \to B_2$ $\Rightarrow |B_1| = |B_2|$

$\dim(\mathbb{R}^2) = 2$

$\dim(\mathbb{R}, \mathbb{Q}, +, \cdot) =$ infinite dimensional vector space.

$\dim$(set of all 2×2 matrices) $= 4$

$\dim$(set of 2×2 symmetric matrices) $= 3$

$\dim$(set of 2×2 skew-symmetric matrices) $= 1$

– Let V be a VS

B be basis

$\dim(V) = n$

$B = \{x_1, x_2, \dots, x_n\}$

$x \in V (\neq 0)$

$x = \sum_{i=1}^{n} c_i x_i$ where $c_i \in F$ $\forall i$

since $x \neq 0$ atleast one $c_i \neq 0$ let say $i = K$; $c_K \neq 0$.

$B_1 = \{x_1, x_2 \dots, x_{K-1}, x, x_{K+1} \dots\}$

swapped x_K with x

(1). LI ✓

(2). $LS(B_1) = V$ ✓

$LS(B_1) = \sum_{\substack{i=1 \\ (i\neq K)}}^{n} b_i' x_i + b_K \sum_{i=1}^{n} c_i x_i$

$= \sum_{\substack{i=1 \\ (i\neq K)}}^{n} (b_i' + b_K c_i) x_i + x_K c_K = LS(B)$ $\quad c(\neq 0)$

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$\mathbb{R}^2$

$B = \{(0,1), (1,0)\}$

$(1,1) \in \mathbb{R}^2 \qquad (1,1) = (1,0) + (0,1)$

$\qquad\qquad\qquad\qquad = x_1 + x_2$

$B_1 = \{(0,1), (1,1)\}$

$B_2 = \{(1,0), (1,1)\}$

1. Take any element in VS
2. We can replace it by exactly element in Basis given that coeff in linear combination is non-zero.

Let $B = \{(0,1), (1,0), (1,1)\}$

$LS(B) = \mathbb{R}^2$

But B is not linearly independent.

B is not a Basis.

V- finite

$S \subseteq V$ s.t. $LS(S) = V$

Can we construct a basis from S.

S is not LI $\Rightarrow$ Not Basis.

$\exists x \in S$ s.t.

$x = \sum_{i=1}^{n} c_i x_i$ atleast one $x_i \neq 0$ say $i = n$

$S_1 = S - \{x_i\}$

$LS(S_1) = V$

If S_1 is LI, then S_1 is self basis.

V- finite dimensional VS

$S \subseteq VS$ s.t. $LS(S) = VS$

Then we can construct a basis of V from S.

If (i). S is LI; S itself is the basis

(ii). S is not LI; there exists $\exists x \in S$ such that $x_1, x_2, \dots, x_n \in S$ such that $\sum c_i x_i = x$ where $c_i \in F$.

given

(iii). $S - \{x\} = S_1$

Again come to step (i).

continue untill we will not get basis

(iv). $LS(S_1) = V$ (true)

* If $0 \in S$, then delete 0 from S.

$\underline{V = \mathbb{R}^3}$

$S = \{(1,1,1)\}$ i). Not Basis

ii). LI

$LS(S) \neq V$

$(1,0,0) \in V$ & $(1,0,0) \notin LS(S)$

$S_1 = \{(1,1,1), (1,0,0)\} \longrightarrow$ LI

$LS(S_1) \neq V$

$(0,1,0) \in V$ & $(0,1,0) \notin LS(S_1)$

$S_2 = \{(1,1,1), (1,0,0), (0,1,0)\} \longrightarrow$ LI

$LS(S_2) = V$

$\therefore$ S_2 is the basis.

$V = M_{2\times 2}$

$S = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$

Extend S to a basis for V.

$LS(S) \neq V$

$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \notin LS(S) \in V$

$S_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$

$LS(S_2) \neq V$

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \notin LS(S_2) \in V$

$S_3 = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$

$LS(S_3) \neq V$

$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \notin LS(S_3) \in V$

$S_4 = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$

$LS(S_4) = V \Rightarrow V = LS(S_4)$

Let V be FDV and $\dim(V) = n$.

Let $S \subseteq V$ be a LI.

If $|S| = n$, then S is a basis of V.

$\Rightarrow$ (i). S is LI

To show, $LS(S) = V$.

If $LS(S) \neq V$ $\exists x \in V$ s.t. $x \notin LS(S)$.

$S_1 = S \cup \{x\} \to LI$

$|S_1| = n+1$

since, $\dim(V) = n$, you never have a LI subset which countains n+1 elements.

$\# \; V = \mathbb{R}^5$

$W_1 = LS\{(1,0,0,0,0), (0,1,0,0,0)\}$

$W_2 = LS\{(0,1,0,0,0), (0,0,1,0,0)\}$

$W_1 + W_2$ is also subspace of V.

$\downarrow$

Finite dimensional

$\dim(V_1 + V_2) = 3$

$\forall z \in W_1 + W_2 \;; \; z = c_1(1,0,0,0,0) + c_2(0,1,0,0,0) + c_3(0,1,0,0,0) + c_4(0,0,1,0,0)$

$= c_1(1,0,0,0,0) + (c_2+c_3)(0,1,0,0,0) + c_4(0,0,1,0,0)$

$\exists c_i \in F$

$\therefore W_1 + W_2 \subseteq B_1 \cup B_2$

$B_1 + B_2 \subseteq W_1 + W_2$

$\therefore \underline{\dim(W_1 + W_2) = 3}$

$\dim(W_1 \cap W_2) = 1$

$\dim(V) = 5$

$\dim(W_1) = 2$

$\dim(W_2) = 2$

$\dim(W_1 + W_2) = 3$

$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$

Th: Let V be finite dimensional VS.
Let W_1 and W_2 be supspaces of V.
Then $\dim(W_1+W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$

proof: $W_1 \cap W_2$ is a subspace of V.
Take a basis for $W_1 \cap W_2$ let
$S = \{\alpha_1, \alpha_2 \ldots, \alpha_k\}$ be basis of
S in LI in $W_1 \cap W_2$.

S is LI in both W_1 and W_2.
We can extend S to be basis for W_1 and W_2.

let $S_1 = S \cup \{\beta_1, \beta_2 \ldots, \beta_m\}$ be basis of W_1
$S_2 = S \cup \{\gamma_1, \gamma_2 \ldots \gamma_p\}$ be basis of W_2.

$\alpha \in W_1 + W_2$
$\alpha = \alpha_1 + \alpha_2$ where, $\alpha_1 \in W_1$ & $\alpha_2 \in W_2$

$\alpha_1 = a_1\alpha_1 + \ldots + a_k\alpha_k + b_1\beta_1 + b_2\beta_2 + \ldots + b_m\beta_m$
$\alpha_2 = c_1\alpha_1 + \ldots + c_k\alpha_k + d_1\gamma_1 + d_2\gamma_2 + \ldots + d_p\gamma_p$

$\alpha = \sum(a_i+c_i)\alpha_i + \sum b_i\beta_i + \sum d_i\gamma_i$

$\dim(W_1+W_2) = k+m+p$

$W_1+W_2 \subseteq LS(\alpha_i, \beta_i, \gamma_i)$
$LS(\alpha_i, \beta_i, \gamma_i) \subseteq W_1+W_2$
$\alpha_i, \beta_i, \gamma_i$ are LI

$\dim(W_1) = k+m$
$\dim(W_2) = k+p$
$\dim(W_1 \cap W_2) = k$

$\dim(W_1+W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$

If they LD;

$$\Sigma a_i\alpha_i + \Sigma b_i\beta_i = -\Sigma c_i\gamma_i \qquad \therefore \quad \Sigma c_i\gamma_i \in W_1 \cap W_2$$

$\therefore$ contradiction at $\underline{S_2}$.

$V = M_{2\times 2}$

$W_1 =$ symmetric

$W_2 =$ skew-symmetric

$\dim(W_1+W_2)$

$$W_1 = \begin{pmatrix} a_1 & b_1 \\ b_1 & c_1 \end{pmatrix} \qquad W_2 = \begin{pmatrix} a_2 & 0 \\ 0 & c_2 \end{pmatrix} \qquad \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}$$

$$W_1+W_2 = \begin{pmatrix} a_1 & b_1-a \\ b_1+a & c_1+c_2 \end{pmatrix} \Rightarrow \dim(W_1+W_2) = 4$$

$\dim(W_1) = 3 \qquad \dim(W_2) = 1 \qquad \dim(W_1 \cap W_2) = 0$

$W_3 =$ set of all matrices with trace $(A) = 0$

$\dim(W_2) = 1$

$\dim(W_3) = 3 \qquad\qquad \dim(W_2+W_3) = 3$

$\dim(W_1 \cap W_3) = 1$

$$W_3 = \begin{pmatrix} -a & d_1 \\ +b & a \end{pmatrix} \qquad W_2 = \begin{pmatrix} 0 & -d \\ d & 0 \end{pmatrix}$$

$$W_2+W_3 = \begin{pmatrix} -a & d_1-d \\ (b+d) & a \end{pmatrix} \cancel{= a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \overset{a}{(b-d)}\begin{pmatrix} 0 & 0 \\ 0 & - \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}$$

$$= a\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + (d_1-d)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + (b+d)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Q2.

$\dim(W_1) = 3$
$\dim(W_2) = 3$

$\dim(W_1 \cap W_2)$: $a = x$
$a = -y$ $\boxed{x = -y}$
$b = -x$
$z = c$

$\therefore W_1 \cap W_2 = \left\{ \begin{pmatrix} x & -x \\ -x & z \end{pmatrix} \middle| \, x, z \in \mathbb{R} \right\}$ $\dim(W_1 \cap W_2) = 2$

$\dim(W_1 + W_2) = 3 + 3 - 2 = 4$

$\dim(W_1 + W_2)$: $\begin{pmatrix} a+x & y-a \\ b-x & z+c \end{pmatrix}$: 4

$V = \{ f : \mathbb{R} \to \mathbb{R} \mid f(-x) = f(x) \}$

$(f+g)(x) = f(x) + g(x)$

~~f+g~~ $f(-x) = f(x) \Rightarrow -f(-x) = -f(x)$; $-f(x) \in V$

$0 \in V$

$f(x) = f(-x) \Rightarrow \lambda f(x) = +\lambda f(-x) \in V$

V is a vector space.

$V = \{ f : \mathbb{R} \to \mathbb{R} \mid f(-x) = -f(x) \}$

V is a vector space.

$\mathbb{R}^+ = \{$ set of non-negative real numbers $\}$

$x, y \in \mathbb{R}^+$

$x \oplus y = xy$

check whether $\mathbb{R}^+$ is vector space or not.

$(\mathbb{R}^+, \mathbb{R}^+, \oplus, \cdot)$

1. $x \oplus y = y \oplus x$
2. $x \in \mathbb{R}^+,\ \forall \lambda \in \mathbb{R}^+ \Rightarrow x\lambda \in \mathbb{R}^+$
3. $x \in \mathbb{R}^+,\ y \in \mathbb{R}^+ \Rightarrow xy \in \mathbb{R}^+$

Note: $\mathbb{R}^+$ cannot be field.

$1 \oplus y = y$ $\{$ 1 is identity element $\}$

$0 \in \mathbb{R}^+$, But doesn't have inverse

Not a vector space

$\mathbb{R}^+ = \{x \mid x > 0;\ x \in \mathbb{R}\}$ $\quad F = \mathbb{R}$

$x \oplus y = xy$

$V = M_{2\times 2}$

$$W_1 = \left\{ \begin{pmatrix} a & -a \\ b & c \end{pmatrix} \middle|\ a, b, c \in \mathbb{R} \right\}$$

$$W_2 = \left\{ \begin{pmatrix} x & y \\ -x & z \end{pmatrix} \middle|\ x, y, z \in \mathbb{R} \right\}$$

Q1. W1 and W2 both are subspaces of V or not?

$\alpha \in F,\ x \in W_1 \Rightarrow \alpha x \in W_1$

$x \in W_1,\ y \in W_1 \Rightarrow \alpha x + \beta y \in W_1$ ✓

$0 \in W_1$ (sure)

$\mathbb{R}^n \to VS$

$x, y \in \mathbb{R}^n$, $x = (x_1, x_2 \ldots, x_n)$
$y = (y_1, y_2, \ldots, y_n)$

$$x \cdot y = x_1 \cdot y_1 + x_2 y_2 + \ldots + x_n y_n$$

$$\langle x, y \rangle = x \cdot y = \sum_{i=1}^{n} x_i y_i$$

1. $\langle x, x \rangle \geq 0 \quad \forall x \in \mathbb{R}^n$

If $\langle x, x \rangle = 0 \Rightarrow x = 0$

$\Rightarrow x_1^2 + x_2^2 + \ldots + x_n^2 = 0 \Rightarrow x_i = 0 \;\; \forall i \text{ e} \Rightarrow x = 0$

2. $\langle x, y \rangle = \langle y, x \rangle$
3. $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$
4. $\langle x+z, y \rangle = \langle x, y \rangle + \langle z, y \rangle \quad \forall x, y, z \in \mathbb{R}^n$

Let $(V, \mathbb{R}, +, \cdot)$ be a VS be a arbitary vector space
A mapping $\langle , \rangle : V \times V \longrightarrow \mathbb{R}$ s.t. $\langle , \rangle$ satisfies the following conditions:

1. $\langle x, x \rangle \geq 0 \quad \forall x \in V$ / $\langle x, x \rangle = 0$ iff $x = 0$
2. $\langle x, y \rangle = \langle y, x \rangle \quad \forall x, y \in \cancel{\mathbb{R}} V$
3. $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle \quad \forall x, y \in \cancel{\mathbb{R}} V, \; \alpha \in \mathbb{R}$ (field)
4. $\langle x+z, y \rangle = \langle x, y \rangle + \langle z, x \rangle \quad \forall x, y, z \in \cancel{\mathbb{R}} V$

* $V \times V = \{ (x, y) \mid x \in V, y \in V \}$

$\langle , \rangle : V \times V \longrightarrow \mathbb{R}$

A mapping respects/satisfies these conditions, it is called inner product.

R^2 $\quad x, y \in R^2$

$x = (x_1, x_2)$

$y = (y_1, y_2)$

define: $\langle x, y \rangle = x_1 y_1$

1. $\langle x, x \rangle = x_1^2 \geq 0$ ✓
2. $\langle x, y \rangle = \langle y, x \rangle = y_1 x_1$ ✓
3. $\langle \alpha x, y \rangle = \alpha x_1 y_1$

   $\alpha \langle x, y \rangle = \alpha x_1 y_1$ ✓
4. $\langle x+z, y \rangle = (x_1 + z_1) y_1$

   $\langle x, y \rangle = x_1 y_1$

   $\langle z, y \rangle = z_1 y_1$

   ✓

But, $\langle (0,1), (0,1) \rangle = 0$ But, $(0,1) \neq (0,0)$

Hence, not an inner product. Property (1).

$\langle x, y \rangle = -(x_1^2 + y_1^2)$

property (1) doesn't hold true.

$M_2(\mathbb{R})$; $\langle A, B \rangle = \text{trace}(AB^T)$

1. $\langle A, A \rangle = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix}^T = \begin{pmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{pmatrix}$

   (2) ✓

2. $Tr\left(\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} a_5 & a_7 \\ a_6 & a_8 \end{pmatrix}\right) = a_1 a_5 + a_2 a_6 \; a_3 a_7 + a_4 a_8$

   $Tr\left(\begin{pmatrix} a_5 & a_6 \\ a_7 & a_8 \end{pmatrix} \begin{pmatrix} a_1 & a_3 \\ a_2 & a_4 \end{pmatrix}\right) = a_5 a_1 + a_6 a_2 + a_7 a_3 + a_8 a_4$ ✓

4. ✓

$\langle A,B\rangle = \text{trace}(AB)$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad A^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\text{trace}(A^2) = a^2 + bc + bc + d^2$

$= a^2 + d^2 + 2bc$

(Not necessarily ≥ 0)

ex: $\begin{pmatrix} 1 & 500 \\ -1 & 1 \end{pmatrix}$

Not an inner product.

$A \in M_2(\mathbb{R}) \quad x, y \in \mathbb{R}^2 \quad \langle x,y\rangle = x^t A y$

$x = (x_1, x_2)$

$y = (y_1, y_2)$

$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \quad \langle x,y\rangle = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{2\times 1} \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}_{2\times 2} (y_1 \ \ y_2)_{1\times 2}$

$= \cancel{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} \cancel{(x_1 y_1}$

$= \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$= (x_1 a_1 + x_2 a_3 \quad x_1 a_2 + x_2 a_4) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$= x_1 a_1 y_1 + x_2 a_3 y_1 + x_1 a_2 y_2 + x_2 y_2 a_4$

1. $\langle x,x\rangle = x_1^2 a_1 + x_2 x_1 a_3 + x_1 x_2 a_2 + x_2^2 a_4$

(Not compulsorily True, depends on A)

If $A = I$; $\langle x,y\rangle = x.y$

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~~Let $(V,\mathbb{R})$ be a VS~~

~~An inner product of V is a mapping $\langle , \rangle : V\times V \to \mathbb{R}$~~

vector space $= \mathbb{C}$ $\quad \langle , \rangle : \mathbb{C}\times\mathbb{C} \to \mathbb{C}$

Field $= \mathbb{C}$

$z_1 = x_1 + iy_1$

$z_2 = x_2 + iy_2$

$\langle z_1, z_2\rangle = z_1 . z_2$

$= (x_1x_2 - y_1y_2) + i\,(x_1y_2 + x_2y_1)$

1. $\langle z_1, z_1\rangle = (x_1^2 - y_1^2) + 2i\,x_1y_1$

if, $\langle z_1, z_1\rangle = 0 \Rightarrow z_1 = 0$

2. $\langle \alpha z_1, z_2\rangle =$ we want define complement.

$\langle z_1, z_2\rangle = z_1\bar{z}_2$

1. $\langle z_1 ; z_1\rangle = |z_1|^2 \geq 0$

8. $\langle z_1, z_1\rangle = 0 \Rightarrow |z_1| = 0 \Rightarrow z_1 = 0$

2. $\langle z_1, z_2\rangle = z_1\bar{z}_2 = (x_1x_2 + y_1y_2) + i\,(x_1y_2 - x_2y_1)$

$\langle z_2, z_1\rangle = z_2\bar{z}_1 = (x_1x_2 + y_1y_2) + i\,(x_2y_1 - x_1y_2)$

condition (2) is not satisfied.

So we define $\langle x, y\rangle = \overline{\langle y, x\rangle}$

Now, its a inner product.

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Let $(V, \mathbb{C})$ be a VS An inner product on V is a mapping $\langle , \rangle : V \times V \to \mathbb{C}$ such that:

1. $\langle x, x \rangle \geq 0 \quad \forall x \in V$
   $\langle x, x \rangle = 0 \Rightarrow x = 0$
2. $\langle x, y \rangle = \overline{\langle y, x \rangle} \quad \forall x \; \forall y$
3. $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle ; \alpha \in \mathbb{C} ; \; \forall x, y \in V$
4. $\langle x + z, y \rangle = \langle x, y \rangle + \langle z, y \rangle \quad \forall x, y, z \in V$

– $M_2(\mathbb{C}) = V$

$\text{trace}(AB^t) = \langle A, B \rangle$

$$A = \begin{vmatrix} a_1 & a_2 \\ a_3 & a_4 \end{vmatrix} \quad B = \begin{vmatrix} b_1 & b_2 \\ b_3 & b_4 \end{vmatrix}$$

$$AB^t = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} b_1 & b_3 \\ b_2 & b_4 \end{pmatrix} = \begin{pmatrix} a_1b_1 + a_2b_2 & a_1b_3 + a_2b_4 \\ a_3b_1 + a_4b_2 & a_3b_3 + a_4b_4 \end{pmatrix}$$

$\text{trace}(AB^t) = a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4$

1. $\langle z_1, z_1 \rangle = \langle A, A \rangle = a_1^2 + b_2^2 + a_3^2 + a_4^2 \not\geq 0$

$\langle A, B \rangle = \text{trace}(AB^*)$

$B^* = (\overline{B})^T$

$$AB^* = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} \overline{b_1} & \overline{b_3} \\ \overline{b_2} & \overline{b_4} \end{pmatrix} = \_\_\_$$

$\text{trace}(AB^*) = a_1\overline{b_1} + a_2\overline{b_2} + a_3\overline{b_3} + a_4\overline{b_4}$

1. $\langle A, A \rangle = |A|^2 \geq 0$ & $\langle A, A \rangle = 0 \Rightarrow |A| = 0$
2. $\langle A, B \rangle = a_1\overline{b_1} + a_2\overline{b_2} + a_3\overline{b_3} + a_4\overline{b_4}$
   $\langle B, A \rangle = \overline{a_1}b_1 + \overline{a_2}b_2 + \overline{a_3}b_3 + \overline{a_4}b_4 \qquad \langle A, B \rangle = \overline{\langle B, A \rangle}$

3. $\langle \alpha A, B \rangle = (\alpha a_1)\overline{b_1} + (\alpha a_2) b_2 + (\alpha a_3) b_3 + (\alpha a_4) b_4$

$= \alpha \langle A, B \rangle$

4. $\langle A+C, B \rangle = \Sigma (a_1 + c_1)\overline{b_1} = \Sigma a_1 \overline{b_1} + \Sigma c_1 \overline{b_1}$

$= \langle A, B \rangle + \langle C, B \rangle$

$\therefore$ $\langle A, B \rangle$ is inner product

$C(\mathbb{C}, \mathbb{C})$

$\langle z_1, z_2 \rangle = |z_1|^2$

1. $\langle z_1, z_2 \rangle \geq 0 \quad \forall z_1$

$\langle z_1, z_1 \rangle = 0 \Rightarrow z_1 = 0$ ✓

2. $\langle z_1, z_2 \rangle \neq \langle z_2, z_1 \rangle$

Not an inner product

$C(\mathbb{C}, \mathbb{C})$

$\langle z_1, z_2 \rangle = |z_1|^2 + |z_2|^2$

property (3), $\langle \alpha z_1, z_2 \rangle = |\alpha|^2 |z_1|^2 + |z_2|^2$

Not an inner product.

Let V be an inner product space

Let $x, y \in V$: We say x and y are orthogonal

iff $\langle x, y \rangle = 0$.

$(\mathbb{R}^2, \langle , \rangle)$ $\quad u = (1,1)$, $v = (1,-1)$

u, v are orthogonal

$M_2(\mathbb{R})$ $\quad \langle A, B \rangle = \text{trace}(AB^T)$

$= a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4$

$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ $\quad B = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \Rightarrow tr(AB^T) = 0$

A, B are orthogonal in $(M_2(\mathbb{R}), \langle , \rangle)$

Linearly dependent vectors cannot be orthogonal

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ (non-zero) any

Let $u, v \in V$ $\quad u \neq 0$, $v \neq 0$

u, v are orthogonal

$c_1u + c_2v = 0$

$\langle c_1u + c_2v, u \rangle = \langle 0, u \rangle$

$\Rightarrow c_1\langle u, u \rangle + c_2\langle v, u \rangle = 0$

$\Rightarrow c_1\langle u, u \rangle = 0$

$\therefore \boxed{c_1 = 0}$ Thus $c_2 = 0$

So, u and v are LI

$u = (1,0)$
$v = (1,1)$

u and v are not orthogonal.

Let V be an inner product space
Let $x, y \in V$. We say x and y are orthogonal
if $\langle x, y \rangle = 0$.

$(\mathbb{R}^2, \langle , \rangle)$

$u = (1,1)$
$v = (1,-1)$

u, v are orthogonal

$M_2(\mathbb{R})$

$\langle A, B \rangle = \operatorname{trace}(AB^T)$
$= a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4$

$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \Rightarrow tr(AB^T) = 0$

A, B are orthogonal in $(M_2(\mathbb{R}), \langle , \rangle)$

Linearly dependent vectors cannot be orthogonal

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ (non-zero) only

let $u, v \in V$ $\quad u \neq 0$, $v \neq 0$

u, v are orthogonal

$c_1u + c_2v = 0$

$\langle c_1u + c_2v, u \rangle = \langle 0, u \rangle$

$\Rightarrow c_1\langle u, u \rangle + c_2\langle v, u \rangle = 0$

$\Rightarrow c_1\langle u, u \rangle = 0$

$\therefore \boxed{c_1 = 0}$ Thus, $c_2 = 0$

So, u and v are LI

$u = (1,0)$
$v = (1,1)$

u and v are not orthogonal.
They are LI.

Linear Independency $\Rightarrow$ Orthogonality

Let $S \subseteq V$ and $S \neq \langle \phi \rangle$

Then, S is said to be orthogonal set if for all x, y $\langle x, y\rangle = 0$ and $x \neq y$.

$S^{\perp} = \{ x \in V \mid \langle x, y\rangle = 0 \ \forall y \in S \}$

Then $S^{\perp}$ is said to be orthogonal complement of S.

$\underline{\mathbb{R}^3}$

$S = (0,1,0)$

$S^{\perp} = \{ (x, 0, y) \mid x, y \in \mathbb{R} \}$

$S = (1,1,1)$

$S^{\perp} = \{ (x, y, -x-y) \mid x, y \in \mathbb{R} \}$

gonal
ero) only

$S^{\perp}$ is always a vector space of V.

$\alpha_1, \alpha_2, \dots, \alpha_K \in V$ and $u \in V$

$\langle u, \alpha_i \rangle = 0 \quad i = 1, 2, \dots, n$

$\Rightarrow \langle u, c_1\alpha_1 + c_2\alpha_2 + \dots + c_i\alpha_i \rangle = 0$

u is orthogonal to every vector of $LS\{\alpha_1, \alpha_2, \dots, \alpha_K\}$

Let V be a ~~ortho~~ inner product space

Let $S \subseteq V$ and $u \in V$. u is said to orthogonal to S if $\langle u, v\rangle = 0 \ \forall v \in S$.

$M_2(\mathbb{R})$

$S = \langle$ set of 2×2 symmetric matrices $\}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} ; M_2 = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

$$tr\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d \end{pmatrix}\begin{pmatrix} a & b \\ b & c \end{pmatrix}\right) = 0 \Rightarrow a_1a + bb_1 + bb_1 + cc_1 = 0$$

$$\Rightarrow aa_1 + bb_1 + cd_1 = 0 + c_1b$$

(a_1, b_1, c_1, d_1)

$(0, 1, -1, 0)$

$$S^T = LS\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}$$

<u>Sol:</u> $S = LS\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$

S^T must be orthogonal to basis so that it is orthogonal to all element

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S^T \Rightarrow$ $a = 0$, $d = 0$, $b + c = 0$

$$\therefore S^T = LS\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}$$

$C(\mathbb{R})$: set of all continuos functions from $\mathbb{R} \to \mathbb{R}$

check whether $C(\mathbb{R})$ is finite dimensional or not.

Suppose $C(\mathbb{R})$ is finite dimensional and

let $\dim(VS) = n$.

$V \longrightarrow VS$, $\dim(V) = n$ } we cannot countain or get more than n LI vectors.

$S = \langle 1, x, \ldots, x^n \}$

such that $f(1) = f(3)$.

1. $0(1) = 0(3)$ ✓
2. $-f(1) = -f(3)$ ✓
3. $f(1) + g(1) = (f+g)(1)$ ✓

- Not finite dimensional.

$S = \langle 1, (x-2)^2, \ldots \}$ even powers of $(x-2)$

$\langle f, g\rangle = \int_0^1 f(x) g(x)\, dx.$ $\mathbb{R} \to \mathbb{R}$

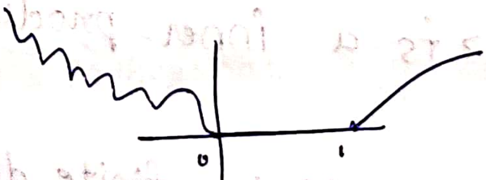
(1). $\langle f, f\rangle \geq 0$ ✓
$\langle f, f\rangle = 0 \Rightarrow f = 0$ ✓

(2). $\langle f, g\rangle = \langle g, f\rangle$ ✓

(3). $\langle \alpha f, g\rangle = \alpha \langle f, g\rangle$ ✓

(4). $\langle f+h, g\rangle = \langle f, g\rangle + \langle h, g\rangle$ ✓

Not an inner product



$S = \langle x \}$; $S^\perp$:

$\langle x, g\rangle = \int_0^1 x\, g(x)\, dx = 0$ $\cancel{= \frac{x^2}{2} g(x) - \int \frac{x^2}{2} g'(x) dx} = 0$

Note: $\int_a^b f^2(x)\,dx = 0 \Rightarrow f(x) = 0$ in $[a,b]$

V – finite dimensional vector space

can you construct inner product of V?

V has finite Basis B

Let $B = \{\alpha_1, \dots, \alpha_n\}$

Let $x, y \in V$

$x = c_1\alpha_1 + c_2\alpha_2 + \dots + c_n\alpha_n$

$y = b_1\alpha_1 + b_2\alpha_2 + \dots + b_n\alpha_n$

$\exists c_i$ and $\exists b_i$ such that $c_i, b_i \in F(\mathbb{C})$

$\langle x, y\rangle = \sum_{i=1}^{n} c_i \bar{b}_i$ check whether inner product or not.

$\langle x, x\rangle = \sum_{i=1}^{n} c_i \bar{c}_i \geq 0 \Rightarrow \langle x, x\rangle = 0 \Rightarrow x = 0$

$\langle \alpha x, y\rangle = \alpha \langle x, y\rangle$ ✓

$\langle x, y\rangle = \overline{\langle y, x\rangle}$? $\sum_{i=1}^{n} c_i \bar{b}_i = \sum_{i=0}^{n} \overline{\bar{c}_i b_i}$ ✓

$\langle x+z, y\rangle = \langle x, y\rangle + \langle z, y\rangle$ ✓

$\langle x, y\rangle$ is a inner product.

Every non-trivial finite dimensional vector space has an inner product.

V – FDVS

$S \subseteq V$ $S = \{\alpha_1, \ldots \alpha_n\}$, $\alpha_i \neq 0$ for $i = 1, 2, \ldots, n$

If S is ~~original~~ orthogonal, then S is linearly independent.

converse is not true.

$(\mathbb{R}^3, \langle , \rangle)$

$\alpha_1 = (1,0,0)$
$\alpha_2 = (1,1,0)$
$\alpha_3 = (0,0,1)$

$S = \{\alpha_1, \alpha_2, \alpha_3\}$ is LI

S is not orthogonal as $\langle \alpha_1, \alpha_2 \rangle = 1 \neq 0$

$\{\alpha_1, \alpha_3, \alpha_2 - \alpha_1\}$ is orthogonal.

in $(\mathbb{R}^2, \langle , \rangle)$

let u and v are not orthogonal

But are LI

$\langle u, v \rangle \neq 0$

$\langle u, v - \alpha u \rangle = 0$ (say)

$\Rightarrow \langle u, v \rangle = \alpha \langle u, u \rangle$

$\therefore$ u and $v - \dfrac{\langle u, v \rangle}{\langle u, u \rangle} . u$ are orthogonal.

$(\mathbb{R}^2, \langle , \rangle)$ $u = (1,1)$
$v = (2,3)$

$(2,3) - \left(\dfrac{2+3}{1+1}\right)(1,1) = (2,3) - (2.5, 2.5) = (-0.5, 0.5)$

and u are orthogonal

$(\mathbb{R}^3, \langle , \rangle)$

$S = \{u, v, w\}$ are linearly independent.

$c_1 \langle u,u \rangle + c_2 \langle v,u \rangle + c_3 \langle w,u \rangle = 0$

$u_1, \ldots, u_k$ LI

$\alpha_1 = u_1$

$\alpha_2 = c_1 u_1 + c_2 u_2$

$\alpha_3 = b_1 \alpha_1 + \ldots + b_3 \alpha_3$

$\alpha_4 = d_1 \alpha_1 + d_2 \alpha_2 + d_3 \alpha_3 + d_4 \alpha_4$

$\langle \alpha_1, \alpha_2 \rangle = 0$

$\langle \alpha_3, \alpha_1 \rangle = 0$

$\langle \alpha_2, \alpha_3 \rangle = 0$

(construct this way)

V - an inner product space

$\{\alpha, \beta, \gamma\}$ is LI

But not orthogonal

$\{\alpha_1, \beta_1, \gamma_1\}$ - orthogonal

$\alpha_1 = a_1 \alpha + a_2 \beta + a_3 \gamma$

$\alpha_2 = b_1 \alpha + b_2 \beta + b_3 \gamma$

$\alpha_3 = c_1 \alpha + c_2 \beta + c_3 \gamma$ (say)

suppose $\alpha_1 = \alpha$

let $\beta_1 = c\alpha + \beta$

$\langle \alpha_1, \beta_1 \rangle = 0 \Rightarrow c \langle \alpha_1, \alpha \rangle + \langle \alpha, \beta \rangle = 0$

$\Rightarrow c = -\dfrac{\langle \alpha_1, \beta \rangle}{\langle \alpha_1, \alpha \rangle}$

$\Rightarrow \beta_1 = \beta - \dfrac{\alpha_1 \langle \alpha_1, \beta \rangle}{\langle \alpha_1, \alpha \rangle} = \beta - \dfrac{\langle \beta, \alpha_1 \rangle}{\langle \alpha, \alpha_1 \rangle} \alpha_1$

~~let $\gamma_1 = a\alpha + b\beta + \gamma$~~

~~$\langle \alpha_1, \gamma_1 \rangle = 0 \Rightarrow$~~

$\gamma_1 = b_1\alpha_1 + b_2\beta_1 + \gamma$

$\langle \alpha_1, \gamma_1 \rangle = 0 \Rightarrow \quad \langle \gamma_1, \alpha_1 \rangle = 0$

$\Rightarrow b_1 \langle \alpha_1, \alpha_1 \rangle + \quad \langle \gamma, \alpha_1 \rangle = 0$

$$b_1 = -\frac{\langle \gamma, \alpha_1 \rangle}{\langle \alpha_1, \alpha_1 \rangle}$$

$\langle \beta_1, \gamma_1 \rangle = 0 \Rightarrow \quad \langle \gamma_1, \beta_1 \rangle = 0$

$$\Rightarrow b_2 = -\frac{\langle \gamma, \beta_1 \rangle}{\langle \beta_1, \beta_1 \rangle}$$

$$\gamma_1 = \gamma - \frac{\langle \gamma, \alpha_1 \rangle}{\langle \alpha_1, \alpha_1 \rangle}\alpha_1 - \frac{\langle \gamma, \beta_1 \rangle}{\langle \beta_1, \beta_1 \rangle}\beta_1$$

$$\beta_1 = \beta - \frac{\langle \beta, \alpha_1 \rangle}{\langle \alpha_1, \alpha_1 \rangle}\alpha_1$$

$\alpha_1 = \alpha$

V – an inner product space

$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ – LI but not orthogonal

$\{\beta_1, \beta_2, \beta_3, \beta_4\}$ – orthogonal

$\beta_1 = \alpha_1$

$$\beta_4 = \alpha_4 - \frac{\langle \alpha_4, \alpha_1 \rangle}{\langle \alpha_1, \alpha_1 \rangle}\alpha_1 - \frac{\langle \alpha_4, \beta_2 \rangle}{\langle \beta_2, \beta_2 \rangle}\beta_2 - \frac{\langle \alpha_4, \beta_3 \rangle}{\langle \beta_3, \beta_3 \rangle}\beta_3$$

$\{\alpha_1, \alpha_2 \ldots, \alpha_n\}$ LI but not orthogonal

$\{\beta_1, \ldots, \beta_n\}$ are orthogonal

$$\Rightarrow \beta_n = \alpha_n - \frac{\langle \alpha_n, \beta_1 \rangle}{\langle \beta_1, \beta_1 \rangle}\beta_1 - \cdots\cdots - \frac{\langle \alpha_n, \beta_{n-1} \rangle}{\langle \beta_{n-1}, \beta_{n-1} \rangle}\beta_{n-1}$$

V– an inner product space

$\underline{V = \mathbb{R}^3}$

$\alpha_1 = (1,0,0)$; $\alpha_2 = (0,1,0)$; $\alpha_3 = (1,1,0)$

$\alpha_1, \alpha_2, \alpha_3$ are LD.

$\beta_1 = \alpha_1 = (1,0,0)$

$\beta_2 = \alpha_2 - \frac{0}{1} = (0,1,0)$

$\beta_3 = \alpha_3 - \frac{\langle \beta_2, \alpha_3 \rangle}{\langle \beta_2, \beta_2 \rangle} \beta_2 - \frac{\langle \beta_1, \alpha_3 \rangle}{\langle \beta_1, \beta_1 \rangle} \beta_1$

$= \alpha_3 - \left(\frac{1}{1}\right) \beta_2 - \left(\frac{1}{1}\right) \beta_1 = (1,1,0) - (1,1,0)$

$= (0,0,0)$

Can we construct an orthogonal from a LD set?
(Exercise)

$\underline{\mathbb{R}^3}$

$\alpha_1 = (1,0,0)$ $\alpha_2 = (1,1,0)$ LI but not orthogonal

$\left.\begin{array}{l} \beta_1 = \alpha_1 \\ \beta_2 = \alpha_2 - \alpha_1 = (0,1,0) \end{array}\right\}$ orthogonal

$LS\{\alpha_1, \alpha_2\} = LS\{\alpha_1, \beta_2\}$

$\underline{\mathbb{R}^4}$

$\alpha_1 = (1,0,0,0)$ $\alpha_2 = (0,1,0,0)$ $\alpha_3 = (0,0,1,0)$

$\alpha_1, \alpha_2, \alpha_3$ are LI

$\beta_1 = \alpha_1$ $LS\{\alpha_1, \alpha_2, \alpha_3\} = LS\{\beta_1, \beta_2, \beta_3\}$

$\beta_2 = \alpha_2 - \alpha_1$

$\beta_3 = \alpha_3$

V - Finite dimensional inner product space

$\langle \alpha_1, \dots, \alpha_n \rangle$

$\exists$ an orthogonal set $\langle \beta_1, \dots, \beta_n \rangle$

such that $LS\langle \alpha_1, \dots, \alpha_n \rangle = LS\langle \beta_1, \dots, \beta_n \rangle$

Gram Schmidt orthogonalisation Theorem.

V - Finite dimensional IPS β: Basis.

$\beta = \langle \alpha_1, \alpha_2 \dots, \alpha_n \rangle$

$\beta' = \langle \beta_1, \beta_2 \dots, \beta_n \rangle \rightarrow$ orthogonal

$LS\langle \beta \rangle = LS\langle \beta' \rangle$

β' will also be basis for V.

β' - orthogonal basis for V.

$\underline{\beta' \text{ is LI}}$

If it is non-zero orthogonal set then β' is LI.

\# V - finite dimensional IPS

$\beta = \langle \alpha_1, \alpha_2 \dots, \alpha_n \rangle$

$x \in V$

$x = c_1\alpha_1 + \dots + c_n\alpha_n$ // Tough to find c_i

But if $\beta' = \langle \beta_1, \dots, \beta_n \rangle$ // orthogonal Basis

$x = b_1\beta_1 + \dots + b_n\beta_n$

$\langle x, \beta_1 \rangle = b_1 \langle \beta_1, \beta_1 \rangle \Rightarrow b_1 = \dfrac{\langle x, \beta_1 \rangle}{\langle \beta_1, \beta_1 \rangle} \; \dots \; b_n = \dfrac{\langle x, \beta_n \rangle}{\langle \beta_n, \beta_n \rangle}$

"Advantage of using orthogonal Basis"

$V \longrightarrow$ FDIPS

~~W subspace of V~~ $(W \subseteq V)$

$W^{\perp} = \{x \in V \mid \langle x, y\rangle = 0 \ \forall y \in W\}$

$\longrightarrow$ orthogonal complement of W.

check whether subspace or not ?

$x \in W^{\perp}$ & $z \in W^{\perp} \Rightarrow x+z \in W^{\perp}$

$x \in W^{\perp} \Rightarrow \alpha x \in W^{\perp}$

$0 \in W^{\perp} \Rightarrow W^{\perp}$ is a subspace

$V = \mathbb{R}^3$

$W = \langle (1,0,0), (1,1,0)\rangle$

Find out $W^{\perp}$.

Sol: $X \in V \quad \langle X, y\rangle = 0 \quad \forall y \in W \qquad X = (x,y,z)$

$\Rightarrow \quad x = 0 \quad \& \quad x+y=0 \Rightarrow y = 0$

$\therefore \quad X = \{(0,0,z) \mid z \in \mathbb{R}\}$

Find out $W \cap W^{\perp} = \varphi$

$W = LS\{(1,0,0), (1,1,0)\}$

$W^{\perp} = LS\{(0,0,1)\}$

$W \cap W^{\perp} = 0 = (0,0,0)$

$W \oplus W^{\perp} = \mathbb{R}^3$

$V = M_{2\times 2}(\mathbb{R})$

$\langle A, B \rangle = \text{trace}(AB^t)$

$W = \{\text{set of all } 2\times 2 \text{ matrices}\}$ symmetric

$W^\perp = ?$

$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \quad B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \quad A = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_4 \end{pmatrix}$

$\langle A, B \rangle = a_1 b_1 + a_2 b_3 + a_2 b_2 + a_4 b_4 = 0 \quad \forall a_1, a_2, a_3, a_4$

$= a_1 b_1 + a_2(b_2 + b_3) + a_4 b_4 = 0$

$\Rightarrow b_1 = b_4 = b_2 + b_3 = 0$

$W^\perp = LS\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\} \quad W \cap W^\perp = \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right\}$

$W \oplus W^\perp = M_{2\times 2}(\mathbb{R})$

Theorem:

Let V be a finite dimensional inner product space. Let W be a subspace of V. Then, $V = W \oplus W^\perp$.

We can show, $W \cap W^\perp = \{0\}$

V be FDIPS

let $v \in V$ and $v \neq 0$.

Then show that $W = \{ w \in V \mid \langle w, v \rangle = 0 \}$

is a subspace of V and $\dim(W) = \dim(V) - 1$.

<u>Sol:</u> $w_1 \in W \quad \langle w_1, v \rangle = 0$

$w_2 \in W \quad \langle w_2, v \rangle = 0$

$$\Rightarrow \langle w_1 + w_2, v \rangle = 0$$

$$\therefore w_1 + w_2 \in W$$

$\langle \lambda w_1, v \rangle = 0$

$\langle 0, v \rangle = 0$

From gram schmidt theorem, we can generate OB with an element as $\underline{v}$.

$$V = LS\{ \underbrace{v, \alpha_1, \ldots, \alpha_{n-1}}_{\text{orthogonal Basis for } V} \}$$

where,

[redacted] $\langle \alpha_i, v \rangle = 0$

$\Rightarrow W = LS\{\alpha_1, \ldots, \alpha_{n-1}\}$

$\therefore \dim(W) = \dim(V) - 1$

$\gamma \in W \Rightarrow \gamma \in V$

$\gamma = c_0 v + c_1 \alpha_1 + \ldots + c_{n-1} \alpha_{n-1}$

$\forall\, c_i \in F$

$$\Rightarrow \langle \textstyle\sum c_i \alpha_i , v \rangle = 0 \quad \Longrightarrow \quad LS\{\alpha_1, \dots, \alpha_{n-1}\} \subseteq W$$

$$\langle \gamma, v \rangle = c_0 \langle v, v \rangle \Rightarrow c_0 = 0$$

$$\therefore \forall \gamma \in W \quad \gamma = c_1\alpha_1 + \dots + c_{n-1}\alpha_{n-1} \Rightarrow W \subseteq LS\{\alpha_1, \dots, \alpha_{n-1}\}$$

$$W = LS\{\alpha_1, \dots, \alpha_{n-1}\}$$

<u>Alternate:</u> $\dim(V) = \dim(W_1 \oplus W_2)$

$$W_1 \cap W_2 = \varphi$$

$$\Rightarrow \dim(V) = \dim(W_1) + \dim(W_2)$$

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AI TRIPLE CAMERA

$V = P_3[x]$ = { set of real numbers co.efficients polynomials of degree ≤ 3 }

$$\langle p_1(x), p_2(x) \rangle = \int_{-1}^{1} p_1(x) p_2(x)\, dx$$

$$W = \{ p(x) \in P_3[x] \mid p(-x) = -p(x) \}$$

find out $W^\perp$

$$\langle p_1, p_2 \rangle = \int_0^1 p_1 p_2\, dx + \int_{-1}^{0} p_1 p_2\, dx$$

$$\Rightarrow \int_0^1 \left(p_1(x) p_2(x) + p_1(-x) p_2(-x) \right) dx = 0$$

$$\Rightarrow \int_0^1 p_1(x) \left(p_2(x) - p_2(-x) \right) dx = 0 \qquad \forall\, p_1(x) \in V$$

$\forall$ even polynomials $\in W^\perp \Rightarrow LS\{1, x^2\} \subseteq W^\perp$

let $q(x) \in W^\perp \Rightarrow \int_0^1 p_1(x)(q(x) - q(-x)) = 0 \quad \forall\, p_1(x) \in V$

$\therefore$ ~~$W^\perp \subseteq LS\{1, x^2\}$~~

$$\int_0^1 (ax^3 + bx)(q(x) - q(-x)) = 0 \qquad \forall\, a, b \in \mathbb{R}$$

$$\Rightarrow q(x) = q(-x)$$

V – finite dimensional IPS

$W \subseteq V$ – subspace

$V = W \oplus W^{\perp}$

$\underline{x \in V}$ $\quad \exists!\, x_1 \in W$ and $\quad \exists!\, x_2 \in W^{\perp}$ $\quad \boxed{! \to \text{unique}}$

such that $\quad x = \underbrace{x_1}_{\text{orthogonal projection of } x \text{ on } W} + \underbrace{x_2}_{\text{orthogonal projection of } x \text{ on } W^{\perp}}$

orthogonal projection of x on W

orthogonal projection of x on $W^{\perp}$

\# $V = \mathbb{R}^3$

$W = LS\{(1,1,0), (1,0,0)\} = LS\{(1,0,0), (0,1,0)\}$

$x = (1,1,1) \qquad W^{\perp} = LS\{(0,0,1)\}$

$x = x_1 + x_2$

$x_1 = (1,1,0)$

$x_2 = (0,0,1)$

\# $B = \{\alpha_1, \dots, \alpha_n\}$ is a basis of W

Extend B' to the basis of W.

$B' = \{\alpha_1, \dots, \alpha_K, \beta_1, \beta_2, \dots, \beta_p\}$

$B' = \{u_1, \dots \dots u_{K+p}\}$ orthogonal basis of V.

$x \in V$

$x = c_1 u_1 + \dots + c_{K+p} u_{K+p}$

$= \underbrace{\sum_{i=1}^{K} c_i u_i}_{\in W} + \underbrace{\sum_{i=K+1}^{K+p} c_i u_i}_{\in W^{\perp}} = x_1 + x_2$

where, $c_i = \dfrac{\langle x, u_i \rangle}{\langle u_i, u_i \rangle}$

$\mathbb{R}^2$ $\quad W = LS\langle(1,0)\rangle$

$x = (1,2)$

$W^{\perp} = LS\langle(0,1)\rangle$

$x_1 = (1,0)$

$x_2 = (0,2)$

$V = M_{2\times2}(\mathbb{R})$ $\quad \langle A,B\rangle = tr(AB^t)$

$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ $\quad W =$ set of all symmetric matrices

$$tr\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} a_1 & a_3 \\ a_2 & a_4 \end{pmatrix} = 0$$

$$\Rightarrow aa_1 + ba_2 + ba_3 + ca_4 = 0$$

$x_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$; $x_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \Rightarrow$

$W^{\perp} = LS\left\{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right\}$

$A = x_1 + x_2$

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = c_1\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c_2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c_3\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + c_4\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$\Rightarrow c_1 = 1$ $\quad c_2 + c_4 = 1 \Rightarrow c_2 = c_4 = 1/2$

$c_2 - c_4 = 0$

$c_3 = 1$

$$x_1 = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix} \qquad x_2 = \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix}$$

\# $\underline{V = \mathbb{R}^3}$

$W = LS\{(1,1,0),(0,0,1)\}$

$\alpha = (1,2,3)$ $\qquad (x+y=0) \quad (z=0)$

$W^{\perp} = LS\{(1,-1,0)\}$

$(1,2,3) = a(1,1,0) + b(0,0,1) + c(1,-1,0)$

$\Rightarrow a + c = 1 \qquad a = 1.5$

$a - c = 2 \qquad c = -0.5$

$b = 3 \qquad b = 3$

$x_1 = (1.5, 1.5, 3) = \left(\frac{3}{2}, \frac{3}{2}, 3\right)$

$x_2 = (-0.5, 0.5, 0) = \left(\frac{-1}{2}, \frac{1}{2}, 0\right)$

V-IPS

$x \in V \quad \|x\| = \langle x, x \rangle^{1/2}$

$x, y \in V \quad |\langle x, y \rangle| \leq \|x\| \|y\|$

$\langle x, x \rangle^{1/2} \langle y, y \rangle^{1/2}$

- cauchy-schwarz inequality

$u = x - ty \qquad t \in \mathbb{R}$

$\langle u, u \rangle = \langle x - ty, x - ty \rangle = \langle x, x \rangle + \langle x, -ty \rangle + \langle -ty, x \rangle + \langle y, y \rangle$

$= \|x\|^2 - t\langle x, y \rangle - t\overline{\langle x, y \rangle} + \|y\|^2 t^2$

$= \|x\|^2 + t^2\|y\|^2 - 2t\, Re(\langle x, y \rangle) \geq 0$

$at^2 + bt + c \geq 0 \quad \forall t \in \mathbb{R}$

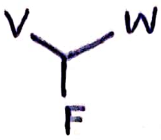
$b^2 - 4ac \leq 0 \quad$ as $\quad a \geq 0$

$\Rightarrow \quad 4(Re(\langle x, y \rangle))^2 - 4\|x\|^2\|y\|^2 \leq 0$

$\Rightarrow \quad |Re[\langle x, y \rangle]|^2 \leq \|x\|^2\|y\|^2$

Linear Transformation:

Let V, W be two vector spaces over the same field F



$T: V \to W$ is called as linear Transformation if,

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y) \qquad \forall \alpha, \beta \in F$$

$V = W = F = \mathbb{R}$

$T(x) = 0 \quad \forall x \in \mathbb{R}$

Yes, A LT

$T(x) = x \quad \forall x \in \mathbb{R}$

$T(x) = \alpha x \quad \forall x \in \mathbb{R}$

Give me another LT on $\mathbb{R} \to \mathbb{R}$ other than $T(x) = \alpha x$.

T is a linear Transformation from $\mathbb{R}$ to $\mathbb{R}$

No, $\{T(x) = \alpha x \mid \alpha \in \mathbb{R}\}$ <u>$\mathbb{R}$ to $\mathbb{R}$</u>

$T(x, y) = x + y$ Yes.

$T(x, y) = x - y$ Yes.

$T(x, y) = \alpha x + \beta y$ Yes.

Proof that only $\alpha x + \beta y$ is LT for $\mathbb{R}^2 \to \mathbb{R}$

$$T(x(1,0) + y(0,1)) = x\,T(1,0) + y\,T(0,1)$$
$$= \alpha x + \beta y$$

$T(x,y) = x+y+1$ No.

$V = \mathbb{R}^n$; $W = \mathbb{R}$

$T(x_1, \dots, x_n) = \alpha_1 x_1 + \dots + \alpha_n x_n$

$V = \mathbb{R}^n$ $W = \mathbb{R}^m$

$A \in M_{m \times n}$

$T(X) = AX, \ X \in \mathbb{R}^n$

$$T(\alpha X + \beta Y) = A(\alpha X + \beta Y)$$
$$= \alpha AX + \beta AY$$
$$= \alpha T(X) + \beta T(Y)$$

$V = C[0,1]$; $W = \mathbb{R}$

$$T(f) = \int_0^1 f(x)\,dx$$

$$T(\alpha f + \beta g) = \alpha T(f) + \beta T(g)$$

$V = \mathbb{R}^3$; $W = \mathbb{R}^3$

1). $T(x,y,z) = (x,y,0)$ ✓

2). $T(x,y,z) = (x-y,z,y)$ ✓

3). $T(x,y,z) = (x-y+1,z,y)$ ×

$\mathbb{R}^n \longrightarrow \mathbb{R}^m$

$T(x_1, \dots, x_n) = (T_1, \dots, T_m)$

$T_i : \mathbb{R}^n \to \mathbb{R}$

$T(x,y,z) = (x,y^2,0)$ ×

$T: V \to W$

$T(0_V) = 0_W$

$\alpha = \beta = 0_F \in \mathbb{F}$

$T(0_F \cdot x + 0_F \cdot y) = 0_F \cdot T(x) + 0_F \cdot T(y)$

$= 0_W$

class test 30th Aug : 6–7pm

V, W
$\vee$
F

$\dim(V) = n$

$\{u_1, \ldots, u_n\}$ be a basis of V.

can you construct a linear transformation using $\{u_1, \ldots, u_n\}$?

$T: V \to W$

$T(u_i) = w_i \qquad i = 1, 2, \ldots, n$

$w_1, w_2, \ldots, w_n \in W$

$F(u_i) = w_i \qquad$ for $\quad i = 1, 2, \ldots, n$

can you tell me the linear Transformation.

$x \in V, \quad x = \sum_{i=1}^{n} \alpha_i u_i$

$T(x) = \sum_{i=1}^{n} \alpha_i T(u_i), \quad \alpha_i \in F, \; i = 1, 2, \ldots, n$

T is well defined.

$T(x) = \Sigma \alpha_i \omega_i$

$T'(x) = T'(\alpha_1 u_1 + \dots + \alpha_n u_n)$

$= \Sigma \alpha_i T'(u_i)$

$= \Sigma \alpha_i \omega_i$

$\Rightarrow T(x) = T'(x) \quad \forall x \in V$

$\therefore$ If we fix linear transformation for basis, then it will be unique for that basis mapped transformation.

$T: V \to W$

T is one-one if $T(x) = T(y) \Rightarrow x = y$ $\quad \forall x, y \in V$

$T: \mathbb{R}^3 \to \mathbb{R}^2$

$$T(x_1, x_2, x_3) = (x_1, 0)$$

$$L\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$$

$$T\begin{pmatrix}1\\0\\0\end{pmatrix} = (1,0) \quad T\begin{pmatrix}0\\1\\0\end{pmatrix} = (0,0) \quad T\begin{pmatrix}0\\0\\1\end{pmatrix} = (0,0)$$

$\{T(1,0,0), T(0,1,0), T(0,0,1)\}$ is dependent.

S is LI $\not\Rightarrow$ $T(S)$ is LI in W

$T: \mathbb{R}^3 \to \mathbb{R}^2$

$$T(x_1, x_2, x_3) = (x_1 - x_2, x_2 - x_3)$$

$$\left.\begin{array}{l} T(1,0,0) = (1,0) \\ T(0,1,0) = (-1,1) \\ T(0,0,1) = (0,-1) \end{array}\right\} \text{LD}$$

$T: \mathbb{R}^3 \to \mathbb{R}^3$

$T(x_1, x_2, x_3) = (x_1 + x_2, x_2, x_3)$

T is one-one

$$\left.\begin{array}{l} T(1,0,0) = (1,0,0) \\ T(0,1,0) = (1,1,0) \\ T(0,0,1) = (0,0,1) \end{array}\right\} \text{LI set}$$

n it will

on.

$T: V \to W$ be a linear Transformation and one-one

Let $S = \{u_1, \ldots, u_k\}$ be a L.I. Then $T(S)$ is LI.

$T(S) = \{T(u_1), \ldots, T(u_k)\}$ is LI

let us prove, $\quad \exists \alpha_i$

$\sum \alpha_i u_i = 0$

$\sum \alpha_i T(u_i) = 0_W \Rightarrow \sum T(\alpha_i u_i) = 0_W$

$\sum \alpha_i u_i = 0_V \Rightarrow \alpha_i = 0_F$

$\Rightarrow$ [illegible]

$\Rightarrow$ [illegible] [illegible]

$\therefore$ $T(S)$ is LI

$T: \mathbb{R}^3 \to \mathbb{R}^2$

$T(x_1, x_2, x_3) = (x_1, 0)$

$T^{-1}\begin{pmatrix}1\\1\end{pmatrix} = \{\varphi\}$

$T^{-1}\begin{pmatrix}1\\0\end{pmatrix} = \{(1,a,b) \mid a, b \in \mathbb{R}\}$

$T: V \to W \quad$ LI

$w_1, \ldots, w_k \in W \quad$ st $\quad T(v_i) = w_i \quad i = 1, 2, \ldots, n$

If $\{w_1, \ldots, w_n\}$ is LI

$\{v_1, \ldots, v_n\}$

$\sum \alpha_i v_i = 0_v$

$T(\sum \alpha_i v_i) = T(0_v)$

$\Rightarrow \sum \alpha_i \omega_i \qquad = 0_w$

$\Rightarrow \alpha_1 = \alpha_2 = \dots\dots = \alpha_n = 0$

Let $T: V \to W$ be a linear transformation,

$S = \{ x \in V \mid T(x) = 0_w \}$

↳ This set is called kernel of T

it is denoted by $\ker(T)$

$\ker(T)$ is a subspace of V or not.

$x \in V \Rightarrow \alpha x \in V$

$T(\alpha x) = \alpha T(x) = 0_w \Rightarrow \alpha x \in \ker(T)$

$x \in V,\ y \in V \Rightarrow x + y \in V$

$T(x+y) = T(x) + T(y)$

$= 0_w$

$\therefore x + y \in \ker(T)$

$T: \mathbb{R}^3 \to \mathbb{R}^2$

$T(x_1, x_2, x_3) = (x_1, 0, \cancel{0})$ ← Not ϕ

Find out $\ker(T)$.

<u>Sol:</u> $T(x_1, x_2, x_3) = 0_w = (0, 0, \cancel{0})$

$\Rightarrow x_1 = 0$

$\therefore \ker(T) = LS\{(0,1,0), (0,0,1)\}$

T is not one-one.

$T: \mathbb{R}^3 \to \mathbb{R}^3$

$T(x_1, x_2, x_3) = (x_1 - x_2, x_2, x_3)$

$Ker(T)$:

$T(x_1, x_2, x_3) = (0,0,0)$

$x_1 = x_2 \; ; \; x_2 = 0 \; ; \; x_3 = 0$

$Ker(T) = \{(0,0,0)\}$

T is one-one

If T is 1-1, then $ker(T) = \{0_v\}$

$T(x) = 0_w \quad ; \; T(0_v) = 0_w$

$\Rightarrow x = 0_v$ since, T is 1-1

$\therefore \; ker(T) = \{0_v\}$

$T: V \to W$ and $Ker(T) = \{0_v\}$

can you say anything about T?

$x_1, x_2 \in V$

$T(x_1) = T(x_2) \Rightarrow T(x_1 - x_2) = 0_W$

$\Rightarrow x_1 - x_2 \in ker(T)$

$\Rightarrow x_1 - x_2 = 0_v$

$\Rightarrow x_1 = x_2 \quad / \; T$ is one-one

Let $T: V \to W$ be a linear transformation,

$S = \{ T(x) \mid x \in V \} \subseteq W$

The set is called range of T and denoted by $R(T)$.

Is R a subspace of W?

$$\underbrace{T(x_1) + T(x_2)}_{\in S} = T(x_1 + x_2)$$

$x_1 + x_2 \in V \Rightarrow T(x_1 + x_2) \in S$

$$\alpha T(x_1) = T(\alpha x_1)$$

$\alpha x_1 \in V \Rightarrow T(\alpha x_1) \in S$

$\therefore R$ is a subspace of W.

$T: \mathbb{R}^3 \to \mathbb{R}^3$

$T(x_1, x_2, x_3) = (x_1 - x_2, x_2, x_3)$

$R(T) = \mathbb{R}^3$

$T: \mathbb{R}^3 \to \mathbb{R}^2$

$T(x_1, x_2, x_3) = (x_1, 0)$

Find out $R(T)$.

$R(T) = LS\{(x, 0)\}$

$T: V \to W$

$Ker(T)$ is subspace of V

$R(T)$ is subspace of W

$\dim(Ker(T)) = nullity(T)$

$\dim(R(T)) = rank(T)$

If V is finite dimensional $T: V \to W$

$\dim(V) = \text{nullity}(T) + \text{rank}(T)$

Rank–Nullity Theorem:

$$\dim(V) = \text{nullity}(T) + \text{rank}(T)$$

$T: M_n(\mathbb{R}) \to \mathbb{R}$

$T(A) = \text{trace}(A)$

1. Check T is linear or not.

$T(A+B) = T(A) + T(B)$ ✓

$T(\alpha A) = \alpha T(A)$ ✓

Yes!

2. Find $R(T)$ and $\ker(T)$

$R(T) = \{x_{11} + \dots + x_{nn} \mid x_{ii} \in \mathbb{R}\}$

$= \mathbb{R}$

$\ker(T) \Rightarrow x_{11} + \dots + x_{nn} = 0$

$x_{nn} = -(x_{11} + \dots + x_{(n-1)(n-1)})$

$$M = \begin{pmatrix} x_{11} & x_{12} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \\ \cdots & \cdots & & -(x_{11} + \dots + x_{(n-1)(n-1)}) \end{pmatrix} \in \ker(T)$$

$$\ker(T) = Ls\left\{ \begin{pmatrix} 1 & 0 & 0 \\ \cdots & \cdots & \cdots \\ \cdots & & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \\ \cdots & \cdots & \cdots & \end{pmatrix}, \dots \right\}$$

3. $\text{rank}(T) = 1$

$\ker(T) = \underline{n^2 - 1}$

$\underline{\dim(V)}$

$\{u_1, \ldots, u_n\}$ is basis of V. $y = \sum \alpha_i u_i$

$T(y) = \sum \alpha_i T(u_i)$

$R(T) \subseteq LS\langle T(u_i) \ldots \}$

$y \in LS\langle T(u_1) \ldots, T(u_n)\}$

$\Rightarrow y = \beta_1 T(u_1) + \ldots + \beta_n T(u_n)$

$= T(\sum \beta_i u_i) \subseteq R(T)$

$\therefore \quad R(T) = LS\langle T(u_1) \ldots T(u_n)\}$ $\rightarrow$ Basis of W

$\dim(R(T)) \leq n$

Let $S = \langle u_1, \ldots, u_k\}$ be basis of $\ker(T)$

we extend S to a basis for V which is

$\langle u_1 \ldots, u_k, v_1, \ldots v_{n-k}\}$ —(1)

$T(u_i) = 0_\omega$ for $i = 1, \ldots, k$

$x \in V$, $x = \alpha_1 u_1 + \ldots + \beta_1 v_1 + \ldots + \beta_{n-k} v_{n-k}$

$T(x) = \beta_1 T(v_1) + \ldots + \beta_{n-k} v_{n-k}$

$R(T) \subseteq LS\langle T(v_1) \ldots, T(v_{n-k})\rangle$

~~$\dim(R(T)) \geq n-k$~~

$y \in LS\langle T(v_1) \ldots, T(v_{n-k})\}$

$y = \beta_1 T(v_1) + \ldots + \beta_{n-k} T(v_{n-k}) + 0.0 + \ldots$

$y \in R(T) \Rightarrow R(T) = LS\langle T(v_1) \ldots T(v_{n-k})\}$

Need to check whether LI/LD

$c_1 T(v_1) + \ldots + c_n T(v_n) = 0$

$\Rightarrow T(c_1 v_1 + \ldots + c_n v_n) = 0_W$

$\hookrightarrow \in \ker(T)$ (Not true from (1))

$$\Rightarrow \quad \dim(R(T)) = n-k$$
$$\dim(N(T)) = \frac{k}{\dim(V)}$$

\# $T: M_n(\mathbb{R}) \longrightarrow \mathbb{R}$, $x \in \mathbb{R}^n$ $\quad x \neq 0$

$T(A) = x^t A x$

1. $Ker(T)$, $R(T)$

$$T(A) = O_m \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix}^T_{1\times n} \begin{pmatrix} a_{11} & a_{12} & \cdots \\ \cdots & \cdots & \\ \cdots & \cdots & a_{nn} \end{pmatrix}_{n\times n} (x_1 \ x_2 \ \cdots \ x_n)^T_{1\times n}$$

$$= (x_1 a_{11} + \cdots x_n a_{n1} \quad x_2 a_{12} + \cdots + x_2 a_{n2} \quad \cdots) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$= \sum\sum x_i x_j a_{ij} = 0$$

$R(T) = \{0\}$ or $R(T) = \mathbb{R}$

$R(T) = \mathbb{R}$

$rank(T) = 1 \qquad Ker(T) = n^2 - 1$

$R(T) \subseteq \mathbb{R}$

$\Rightarrow R(T) = \{0\}$ (or) $R(T) = \mathbb{R}$

\# $T(A) = \det(A)$

$T(A+B) \neq T(A) + T(B)$.

$A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & 3 \end{pmatrix}$

$\langle , \rangle : \mathbb{R}^2 \to \mathbb{R}$ defined by $\langle x,y \rangle = y^t A x$

1. check whether IP on $\mathbb{R}^2$ or not.

$$\langle x,y \rangle = (y_1 \ \ y_2 \ \ y_3) \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= (2y_1 + y_2 - y_3 \quad y_1 + y_2 \quad 3y_3 - y_1) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= x_1(2y_1 + y_2 - y_3) + x_2(y_1 + y_2) + x_3(3y_3 - y_1)$$

$$\langle x,x \rangle = 2x_1^2 + x_1x_2 - x_1x_3 + x_1x_2 + x_2^2 + 3x_3^2 - x_1x_3$$

$$= 2x_1^2 + x_2^2 + 3x_3^2 + 2x_1x_2 - 2x_1x_3$$

$$= (x_1 + x_2)^2 + (x_1 - x_3)^2 + 2x_3^2$$

1). $\langle x,x \rangle = 0 \Rightarrow$ $x_1 = -x_2$, $x_1 = x_3$, $x_3 = 0$ ✓ $\Rightarrow x = 0_{\mathbb{R}^2}$

$\langle x,x \rangle \geq 0 \ \forall x.$

2). $\langle x,y \rangle = \langle y,x \rangle$

$$\langle y,x \rangle = y_1(2x_1 + x_2 - x_3) + y_2(x_1 + x_2) + y_3(3x_3 - x_1)$$

$$= x_1(2y_1 + y_2 - y_3) + x_2(y_1 + y_2) + x_3(3y_3 - y_1) ✓$$

3). $\langle \alpha x,y \rangle = \alpha \langle x,y \rangle$ ✓

4) $\langle x+z,y \rangle = \langle x,y \rangle + \langle z,y \rangle$ ✓

$\underline{2.}$ $S = \langle (x_1, x_2, -(x_1+x_2)) \mid \in \mathbb{R}^3 \}$

let $y \in S^{\perp}$,

$$y^t A \begin{pmatrix} x_1 \\ x_2 \\ -(x_1+x_2) \end{pmatrix} = 0$$

$\Rightarrow x_1(2y_1+y_2-y_3) + x_2(y_1+y_2) - (x_1+x_2)(3y_3-y_1) = 0$

$\Rightarrow x_1(3y_1-4y_3+y_2) + x_2(2y_1+y_2-3y_3) = 0$

$\Rightarrow$ $3y_1+y_2 = 4y_3$
$2y_1+y_2 = 3y_3$

$\underline{y_1 = y_2 = y_3}$

$S^{\perp} = LS\langle (1,1,1) \}$

$\langle (1,1,1),(1,1,1) \rangle = 6$

$\Rightarrow$ orthonormal basis: $\left(\frac{1}{\sqrt6}, \frac{1}{\sqrt6}, \frac{1}{\sqrt6}\right)$ (B) say

$\underline{3.}$ $(1,0,-1) \perp B \notin B$.

$\|(1,0,-1)\| = 1+4+2 = 7$

$EB_1 = \left\{ \left(\frac{1}{\sqrt7}, 0, \frac{-1}{\sqrt7}\right), \left(\frac{1}{\sqrt6}, \frac{1}{\sqrt6}, \frac{1}{\sqrt6}\right) \right\}$ $LS(EB_1) \neq \mathbb{R}^2$

$\left(\frac{-1}{\sqrt7}, \frac{2}{\sqrt7}, \frac{-1}{\sqrt7}\right) \perp EB_1 \notin EB_1$ $(-1,2,1) = 1+4+2 = 7$

$EB_2 = \left\{ \left(\frac{1}{\sqrt7}, 0, \frac{-1}{\sqrt7}\right), \left(\frac{-1}{\sqrt7}, \frac{2}{\sqrt7}, \frac{1}{\sqrt7}\right), \left(\frac{1}{\sqrt6}, \frac{1}{\sqrt6}, \frac{1}{\sqrt6}\right) \right\}$

$LS(EB_2) = \mathbb{R}^3$

$(x_1, x_2, x_3) = LS\{(1,0,-1),(0,1,-1)\}$

$u_1 = (1,0,-1) = \left(\frac{1}{\sqrt{7}}, 0, \frac{-1}{\sqrt{7}}\right)$

$u_2 = (0,1,-1) - \alpha u_1$

$\langle u_2 - \alpha u_1, u_1 \rangle = 0 \Rightarrow \alpha = \frac{\langle u_2, u_1 \rangle}{\langle u_1, u_1 \rangle}$

$\langle u_2, u_1 \rangle = y_1 + y_2 + y_1 - 3y_3$

$= 2 + 0 + 3 = 5$

$\langle u_1, u_1 \rangle = 1 + 4 + 2 = 7$

$v_2 = (0,1,-1) - \frac{5}{7}(1,0,-1)$

$= \left(\frac{-5}{7}, 1, \frac{-2}{7}\right)$

$\langle v_2, v_2 \rangle = \left(\frac{2}{7}\right)^2 + \left(\frac{3}{7}\right)^2 + \frac{8}{7} = \frac{8 + 13}{49} = \frac{3}{7}$

$\hat{v}_2 = \left(\frac{-5}{\sqrt{21}}, \frac{1}{\sqrt{21}}, \frac{-2}{\sqrt{21}}\right)$

$B = \left\{\left(\frac{1}{\sqrt{7}}, 0, \frac{-1}{\sqrt{7}}\right), \left(\frac{-5}{\sqrt{21}}, \frac{7}{\sqrt{21}}, \frac{-2}{\sqrt{21}}\right)\right\}$

3. $(1,1,1) \in B^{\perp} \quad \notin B \quad \& \quad LS\{B\} \neq \mathbb{R}^3$

$\langle (1,1,1), (1,1,1) \rangle = 6$

$B = \left\{\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right), \left(\frac{1}{\sqrt{7}}, 0, \frac{-1}{\sqrt{7}}\right), \left(\frac{-5}{\sqrt{21}}, \frac{7}{\sqrt{21}}, \frac{-2}{\sqrt{21}}\right)\right\}$

Isomorphism:

Let V and W be two vector spaces over the same field F.

A linear transformation $T: V \to W$ is said to be isomorphism if T is a bijection.

\# $T: R^3 \to R^2$

$T(x_1, x_2, x_3) = (x_1 - x_2, x_3)$

check whether it is an isomorphism or not.

T is one-one (not) ×

\# $T: R^3 \to R^3$

$T(x_1, x_2, x_3) = (x_1, x_2, x_3)$

T is isomorphism.

\# $V = C[0,1]$ $W = R$

$T: V \to W$

$T(f(x)) = \int_0^1 f(x)dx$

Not one-one.

$f, g = \frac{1}{2}, x$ $T(f) = T(g) = \frac{1}{2}$

\# $T: \mathbb{R}^3 \to \mathbb{R}^3$ $A \in M_3(\mathbb{R})$

let $T(x) = Ax$

check whether isomorphism or not.

ame

norphism

if $\det(A) \neq 0 \Rightarrow$ one-one & onto $\Rightarrow$ Isomorphism.

if $\det(A) = 0 \Rightarrow$ not one-one & not onto $\Rightarrow$ not isomorphism

Let V and W be two vector spaces over the same field F.

Then V is said to be isomorphic to W if there is isomorphism from V to W.

Let V and W be two finite dimensional vector spaces over the same field F. Assume that V is isomorphic to W. Then $\dim(V) = \dim(W)$. (iff)

$V = R^m \quad W = R^n \quad ; m \neq n$

They are not isomorphic.

$V = R^6 \quad W = M_{3\times 2}(R)$

$\dim(V) = 6$

$\dim(W) = 6$

$$T(x_1, x_2, x_3, x_4, x_5, x_6) = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 + x_1 \end{pmatrix}^T$$

They are isomorphic.

V is isomorphic iff $\dim(V) = \dim(W)$.

1. T is well defined
2. T is linear
3. T is bijective. T is isomorphism from V to W.

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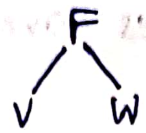
$V = \mathbb{R}^3$ $W =$ symmetric 3×3

Not isomorphic as $\dim(V) = 3$
$\dim(W) = 6$



$\mathcal{L}(V, W) = \{$ set of all linear Transformation from V to W $\}$

$\mathcal{L}(V, W) \longrightarrow$ vector space over the field F.

* $\dim(\mathcal{L}(V, W))$ depends on both V and W.

If V and W are finite, then $\mathcal{L}(V, W)$ are finite dimensional.

$V = \mathbb{R}^3$ W = symmetric 3×3

Not isomorphic as $\dim(V) = 3$
$\dim(W) = 6$

$\mathcal{L}(V,W) = \{$ set of all linear Transformation from V to W $\}$

$\mathcal{L}(V,W) \longrightarrow$ vector space over the field F

* $\dim(\mathcal{L}(V,W))$ depends on both V and W.

If V and W are finite, then $\mathcal{L}(V,W)$ are finite dimensional.