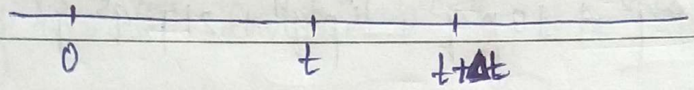
Poisson Distribution

$\boxed{\lambda > 0}$

$\lambda \leftarrow$ rate of arrival



X_t : no. of arrivals that occur in the time interval $[0, t]$
$R_{X_t} = \{0, 1, 2, \dots\}$.

Assumptions based on observations:

(a) The prob that exactly one arrival occurs in the interval of length Δt is approximately $(\lambda \cdot \Delta t)$
(This prob. is actually equal to $\lambda \Delta t + O_1(\Delta t)$
where $\frac{O_1(\Delta t)}{\Delta t} \to 0$ as $\Delta t \to 0$.)

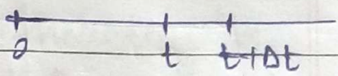
(a) The prob. of exactly zero arrivals is approx. $1 - \lambda \cdot \Delta t$

(b) The prob of two or more arrivals is approx. equal to $O_3(\Delta t)$
where $\frac{O_3(\Delta t)}{\Delta t} \to 0$ as $\Delta t \to 0$.

$p(x) = P(X_t = x) = p_x(t)$ $\qquad x = 0, 1, 2, \dots$

Consider, (after fixing t)

$$p_0(t + \Delta t) = [1 - \lambda(\Delta t)]\, p_0(t)$$

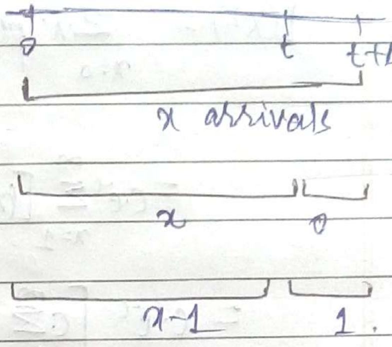


Prob. of No arrival ↑

$$\Rightarrow \frac{p_0(t+\Delta t) - p_0(t)}{\Delta t} = -\lambda p_0(t)$$

$$\boxed{\lim_{\Delta t \to 0} \left[\frac{p_0(t+\Delta t) - p_0(t)}{\Delta t}\right] = p_0'(t) = -\lambda p_0(t)} \quad -(1)$$

For any $x > 0$ (i.e. no. of arrivals is greater than or equal to 1)



$$p_x(t+\Delta t) = (1-\lambda\cdot\Delta t)\, p_x(t) + \lambda\cdot\Delta t\, p_{x-1}(t)$$

$$\frac{p_x(t+\Delta t) - p_x(t)}{\Delta t} = \lambda\, p_{x-1}(t) - \lambda\, p_x(t)$$

$$\boxed{\lim_{\Delta t \to 0}\left[\frac{p_x(t+\Delta t) - p_x(t)}{\Delta t}\right] = p_x'(t) = \lambda\, p_{x-1}(t) - \lambda\, p_x(t)} \quad \text{--- (2)}$$

The solution to this system of differential eq^{n}s (1) and (2) is

$$\boxed{p_x(t) = \frac{(\lambda t)^x e^{-\lambda t}}{x!}} \qquad x = 0, 1, 2, \ldots$$

Prob (x arrivals upto time t)

Put $c = \lambda t$ for const. t.

$$\boxed{\begin{aligned} p_x = p(X=x) &= \frac{c^x e^{-c}}{x!} \qquad x = 0, 1, 2, \ldots \\ &= 0 \qquad \text{otherwise} \end{aligned}}$$

$X \sim$ Poisson (λt) (t = fixed time)

$$E(X) = \mu = \sum_{x=0}^{\infty} x\, p(x) = \sum_{x=0}^{\infty} \frac{x\, e^{-c} c^x}{x!} = c\, e^{-c} \sum_{x=1}^{\infty} \frac{c^{x-1}}{(x-1)!} \quad = e^{c}$$

$$\boxed{\mu = c}$$

Ex. Prove $\boxed{Var(X) = c}$

$$E(X^2) = \sum_{x=0}^{\infty} x^2 p(x) = \sum_{x=0}^{\infty} x^2 \frac{e^{-c} c^x}{x!}$$

$$= c e^{-c} \sum_{x=1}^{\infty} [(x-1)+1] \frac{c^{x-1}}{(x-1)!}$$

$$= c \cdot e^{-c} \left[c \sum_{x=2}^{\infty} \frac{c^{x-2}}{(x-2)!} + \sum_{x=1}^{\infty} \frac{c^{x-1}}{(x-1)!} \right]$$

$$= c \cdot e^{-c} \left[c \cdot e^{c} + e^{c} \right] = c^2 + c$$

$$Var(X) = E(X^2) - [E(X)]^2 = (c^2+c) - (c)^2 = c^2 + c - c^2$$

$$\boxed{Var(X) = c}$$

$$M_X(t) = e^{c(e^t - 1)}$$

$$mgf = M_X(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} p(x) = \sum_{x=0}^{\infty} \frac{e^{tx} e^{-c} c^x}{x!}$$

$$= e^{-c} \sum_{x=0}^{\infty} \frac{(e^t c)^x}{x!} = e^{-c} e^{(e^t c)}$$

$$\boxed{M_X(t) = e^{c(e^t - 1)}}$$

Poisson as an approximation of Binomial

$X \sim$ Binomial (n, p)

$R_X = \{0, 1, 2, \ldots n\}$

$X \sim$ Poisson (c)

$R_X = \{0, 1, 2 \ldots\}$

$$p(x) = \binom{n}{x} p^x (1-p)^{n-x} \quad x = 0, 1, 2, \ldots n$$

$$= 0 \quad \text{Otherwise}$$

$c = np$, $p = \frac{c}{n}$, $1-p = 1-\frac{c}{n} = \frac{n-c}{n}$

$$p(x) = \frac{n!}{x!\,(n-x)!}\, p^x (1-p)^{n-x}$$

$$= \frac{n(n-1)\ldots(n-x+1)}{x!} \left(\frac{c}{n}\right)^x \left(\frac{n-c}{n}\right)^{n-x}$$

$$= \frac{c^x}{x!}\; 1\cdot\left(1-\frac{1}{n}\right)\ldots\left(1-\frac{x-1}{n}\right)\left[1-\frac{c}{n}\right]^n\left[1-\frac{c}{n}\right]^{-x}$$

take limit as $n\to\infty$ and $p\to 0$, but $np = c$.

$$\boxed{p(x) = \frac{c^x e^{-c}}{x!}} \qquad x = 0, 1, 2, \ldots$$

$\lim_{n\to\infty}\left(1-\frac{c}{n}\right)^n$

$= e^{-\frac{c}{n}\times n} = e^{-c}$

CONTINUOUS TIME DISTRIBUTIONS

28/02/17

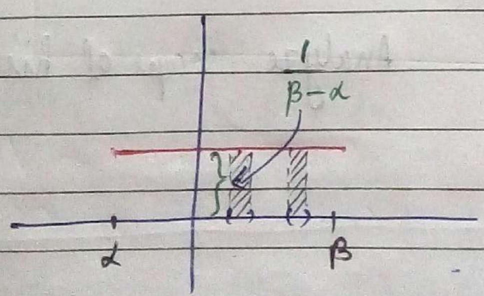
Continuous Uniform:-

The uniform density function

$$f(x) = \frac{1}{\beta-\alpha} \qquad \alpha \le x \le \beta \qquad (\alpha<\beta)$$

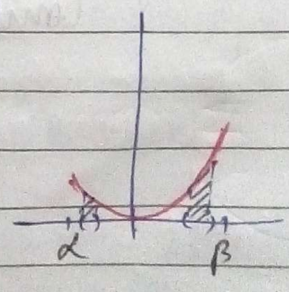
$$= 0 \qquad \text{otherwise}$$

$X \sim \text{Uniform}(\alpha, \beta)$



$f(x) = x^2 \qquad -\delta \le x \le 2\delta$

$= 0 \qquad$ otherwise



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$$E(X) = \int_{\alpha}^{\beta} \frac{x\,dx}{\beta-\alpha} = \frac{\beta+\alpha}{2}$$

$$Var(X) = \int_{\alpha}^{\beta} \frac{x^2\,dx}{\beta-\alpha} - \left(\frac{\beta+\alpha}{2}\right)^2 = \frac{(\beta-\alpha)^2}{12}$$

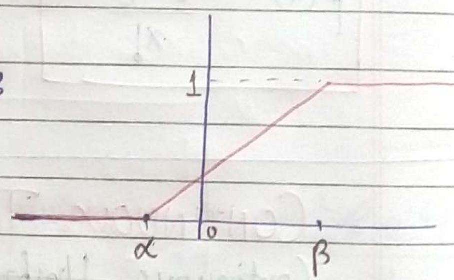
$$M_X(t) = \int_{\alpha}^{\beta} e^{tx} \frac{1}{\beta-\alpha}\,dx = \frac{e^{t\beta}-e^{t\alpha}}{t(\beta-\alpha)}$$

$$\int_{\alpha}^{x} \frac{1}{\beta-\alpha}\,dx = \frac{x-\alpha}{\beta-\alpha}$$

$F(x) = 0 \qquad -\infty < x < \alpha$

$= \dfrac{x-\alpha}{\beta-\alpha} \qquad \alpha < x < \beta$

$= 1 \qquad x > \beta$



Example.

$a.b_1b_2b_3\ldots$

(1) a (2) $a+1$

$.b_1b_2b_3\ldots < 0.5$ (1)

≥ 0.5 (2)

Take 1000 noise plot histogram of error occuring on averaging the no's.

100, 1000, 10k...

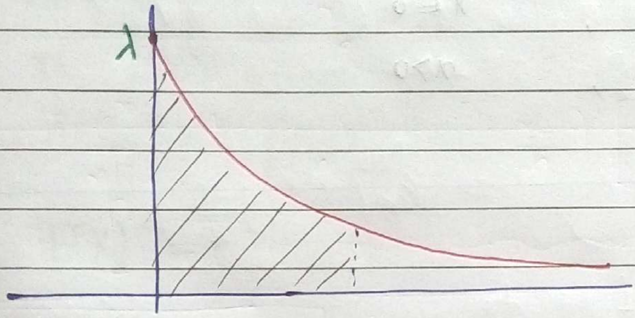
Analyze shape of histogram.

Computer program.

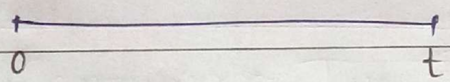
EXPONENTIAL DISTRIBUTION

A continuous random variable X is said to follow exponential distribution with parameter $\lambda > 0$ if it has following pdf :-

$$f_X(x) = \lambda e^{-\lambda x} \qquad x \geq 0$$
$$= 0 \qquad \text{otherwise}$$



POISSON DISTRIBUTION (discrete)



X_t

$p_x(t)$ = prob. of x arrivals in $(0,t)$

$p_0(t)$ = prob. of 0 arrivals in $(0,t)$.
$= e^{-\lambda t}$

X = waiting time until first arrival

$$p(X \leq t) = 1 - e^{-\lambda t}$$

$\lambda x = c$

$X \sim \exp(\lambda)$

$$E(X) = \int_0^\infty x f_X(x)\,dx = \int_0^\infty x\lambda e^{-\lambda x}\,dx = \frac{1}{\lambda}\int_{\mathrm{I}}^{\infty} \underset{\mathrm{I}}{c}\, \underset{\mathrm{II}}{e^{-c}}\,dc$$

$$= \frac{1}{\lambda}\left[c(-e^{-c}) - \int_0^\infty 1(-e^{-c})\,dc\right]_0^\infty$$

$$= \frac{1}{\lambda}\left[-c\cdot e^{-c}\right]_0^\infty + \left(-c\cdot e^{-c}\right)_0^\infty = -\frac{2}{\lambda}\left[e^{-c}\right]_0^\infty$$

$$E(X) = 1/\lambda$$

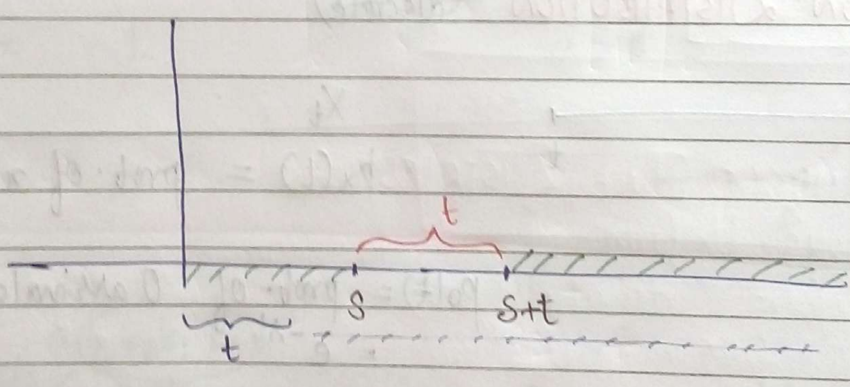
$$Var(X) = \int_0^\infty x^2 f_X(x)\,dx - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

$$M_X(t) = \left(1 - \frac{t}{\lambda}\right)^{-1} \qquad t<\lambda$$

$$F(x) = \begin{cases} 0, & x \le 0 \\ 1 - e^{-\lambda x}, & x>0 \end{cases}$$

Memoryless property

$$P(X > t+s \mid X > s) = P(X>t)$$



GAMMA DISTRIBUTION

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\,dx$$

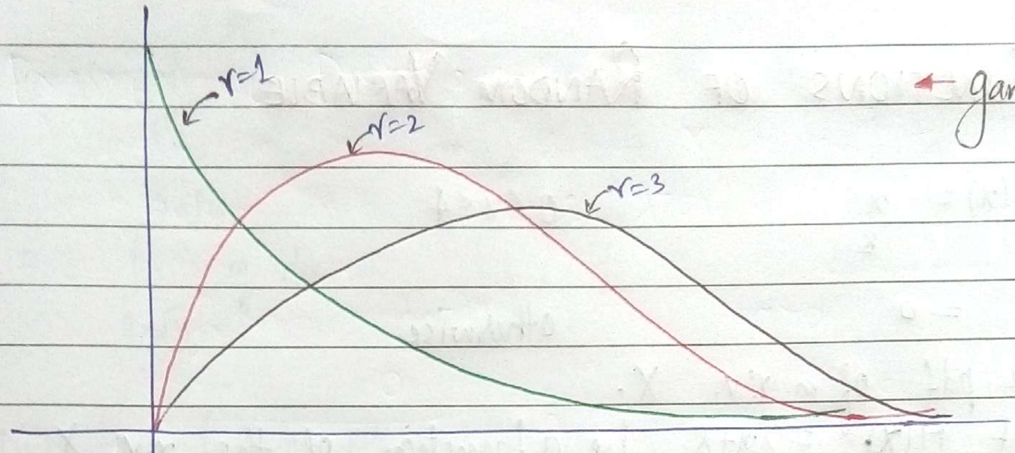
$$*\ \Gamma(n) = (n-1)\,\Gamma(n-1)$$

A random variable X is said to follow gamma distribution if it has pdf

$$\boxed{f_X(x) = \begin{cases} \dfrac{\lambda}{\Gamma(r)} (\lambda x)^{r-1} e^{-\lambda x}, & x>0 \\ = 0, & \text{otherwise} \end{cases}}$$

$r \to$ shape parameter
$\lambda \to$ scale parameter

gamma dist → clustering prob.



← gamma(r, 1) pdf

r = shape parameter
λ = scale parameter

$$E(X) = \frac{r}{\lambda} \qquad Var(X) = \frac{r}{\lambda^2}$$

$$M_X(t) = \left(1 - \frac{t}{\lambda}\right)^{-r}$$

WEIBULL DISTRIBUTION

Density function

$$f_X(x) = \begin{cases} \frac{\beta}{\delta}\left(\frac{x-\gamma}{\delta}\right)^{\beta-1} \exp\left[-\left(\frac{x-\gamma}{\delta}\right)^{\beta}\right] & \text{for } x \geq \gamma \\ 0 & \text{otherwise} \end{cases}$$

$$E(X) = \gamma + \delta\,\Gamma\left(1 + \frac{1}{\beta}\right)$$

$$Var(X) = \delta^2 \left\{\Gamma\left(1 + \frac{2}{\beta}\right) - \left[\Gamma\left(1 + \frac{1}{\beta}\right)\right]^2\right\}$$

CDF of X

$$F_X(x) = 1 - \exp\left[-\left(\frac{x-\gamma}{\delta}\right)^{\beta}\right] \qquad x \geq \gamma$$

ANUBHAV JAIN NOTES

FUNCTIONS OF RANDOM VARIABLE r.v.

Ex:

$$f_X(x) = \frac{x}{8} \qquad 0 \le x \le 4$$
$$= 0 \qquad \text{otherwise}$$

is the pdf of a r.v. X.

Let $H(X) = 2x+8$ be a function of the r.v. X

let $Y = H(X) = 2X+8$

$$\underbrace{\text{Prob}(Y \le y)}_{F_Y(y)} = \text{Prob}(2X+8 \le y) = \text{Prob}_X\left(X \le \frac{y-8}{2}\right)$$

$$= \int_0^{\frac{y-8}{2}} f_X(x)\,dx \qquad 8 \le y \le 16$$

$$= \int_0^{\frac{y-8}{2}} \frac{x}{8}\,dx = \left[\frac{x^2}{16}\right]_0^{\frac{y-8}{2}} = \frac{(y-8)^2}{64}$$

$$F_Y(y) = \text{Prob}_X\left(X \le \frac{y-8}{2}\right) = \frac{1}{64}(y^2 - 16y + 64)$$

$$\boxed{f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{y}{32} - \frac{1}{4} \qquad 8 \le y \le 16. \qquad = 0 \qquad \text{Otherwise}}$$

Ex:

$$f_X(x) = \frac{x}{8} \qquad 0 \le x \le 4$$
$$= 0 \qquad \text{Otherwise}$$

is the pdf of a r.v. X

Let $Y = H(X) = (X-2)^2$ be a function of X.

Compute pdf of Y.

$0 \le y \le 4$

$$F_Y(y) = \text{Prob}(Y \le y) = \text{Prob}_Y((X-2)^2 \le y) = \cancel{\text{Prob}_X(X \le 2 - \sqrt{|y|})}$$

$$\text{Prob}_X(2 - \sqrt{|y|} \le X \le 2 + \sqrt{|y|})$$

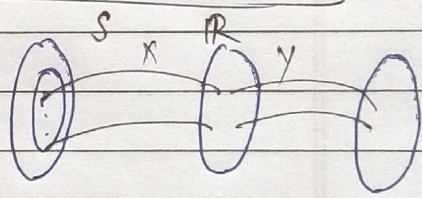
$$= \int_0^{2-\sqrt{|y|}} \frac{x}{8}\,dx = \left[\frac{x^2}{16}\right]_0^{2-\sqrt{|y|}} = \frac{(2-\sqrt{|y|})^2}{16}$$

$$= \int_{2-\sqrt{y}}^{2+\sqrt{y}} \frac{x}{8}\,dx = \left[\frac{x^2}{16}\right]_{2-\sqrt{y}}^{2+\sqrt{y}} = \frac{(2+\sqrt{y})^2 - (2-\sqrt{y})^2}{16}$$

$$F_Y(y) = \frac{(4+y+4\sqrt{y}) - (4+y-4\sqrt{y})}{16} = \frac{\sqrt{y}}{2}$$

$$f_Y(y) = \frac{1}{2}\,\frac{1}{2}\,y^{-1/2} = \frac{1}{4\sqrt{y}}$$

$$\Rightarrow \boxed{\begin{aligned} f_Y(y) &= \frac{1}{4\sqrt{y}} && 0 \le y \le 4 \\ &= 0 && \text{otherwise.} \end{aligned}}$$



Theorem:-

X is a continuous random variable with pdf f_X which satisfies $f_X(x) > 0$ for $a < x < b$ and $y = H(x)$ is a continuous strictly monotonic (increasing/decreasing) function of X, then the random variable $Y = H(X)$ has pdf given by

$$f_Y(y) = f_X(x) \cdot \left| \left(\frac{dx}{dy}\right) \right|$$

where $x = H^{-1}(y)$. If H is increasing, $f_Y(y) > 0$ if $H(a) < y < H(b)$

Proof: Assume H is increasing function

$$F_Y(y) = P_Y(Y \le y) = P_X(H(X) \le y)$$
$$= P_X(X \le H^{-1}(y))$$
$$= F_X(H^{-1}(y))$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X(H^{-1}(y))$$ chain rule

$$= \frac{dF_X}{dx} \cdot \frac{dx}{dy}$$

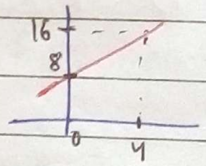
$$\boxed{f_Y(y) = f_X(x) \cdot \frac{dx}{dy}}$$

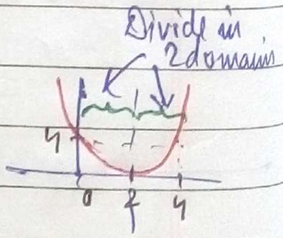
Ex. $f_X(x) = \frac{x}{8}$, $0 \le x \le 4$

$= 0$, otherwise

$Y_1 = H_1(x) = 2x+8 \rightarrow x = \frac{1}{2}y - 4$

$Y_2 = H_2(x) = (x-2)^2$

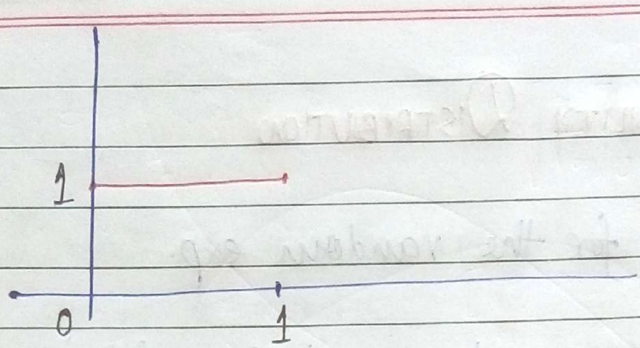




$$f_Y(y) = \frac{x}{8} \cdot \frac{1}{2} = \frac{1}{16}\left(\frac{y}{2} - 4\right)$$
$$= \frac{y}{32} - \frac{1}{4} \quad (8 \le y \le 16)$$

Theorem:

Suppose that X is a random variable with CDF $F_X(x)$. Then the random variable $Y = F_X(x)$ has a uniform distribution on $[0,1]$.



$$f_X(x) = \lambda e^{-\lambda x} \quad x>0$$
$$= 0 \quad \text{otherwise}$$

$X \sim \exp(\lambda)$

$F_X(x) = 1 - e^{-\lambda x} = y$

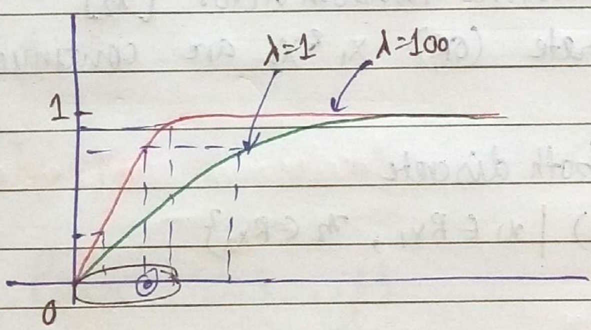
$e^{-\lambda x} = 1 - y$

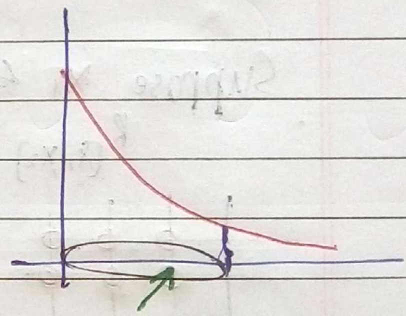
$x = -\frac{1}{\lambda}\ln(1-y)$

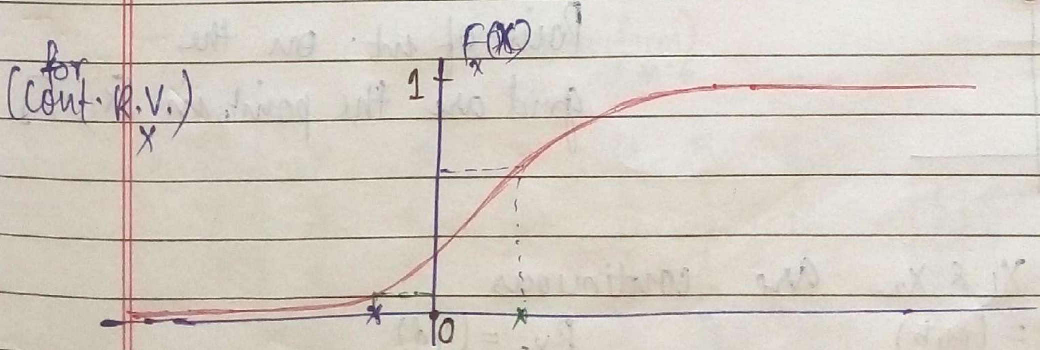
$$f_Y(y) = f_X(x)\cdot\frac{dx}{dy}$$
$$= \lambda e^{-\lambda x}\cdot\frac{1}{\lambda e^{-\lambda x}} = 1. \quad (0 \le x \le 1)$$

(uniform dist.)

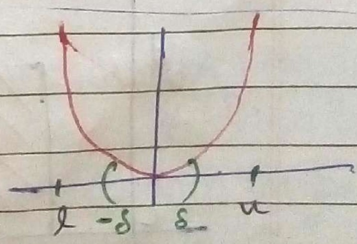
$$\frac{dx}{dy} = -\frac{1}{\lambda}\frac{(-1)}{(1-y)} = \frac{1}{\lambda(e^{-\lambda x})}$$







$$f(x) = x^2 \quad \underset{l}{-2\cdot 3^{-1/3}} \le x \le \underset{u}{3^{-1/3}}$$
$$= 0 \quad \text{otherwise}$$



$\frac{2\delta^3}{3} = P(-\delta \le x \le \delta) \approx 0$

Joint Probability Distribution

S: Sample space for the random exp.

$X_1, X_2, \ldots, X_k$ are random variables defined on S.
Then vector $[X_1, X_2, \ldots X_k]$ is called a random vector

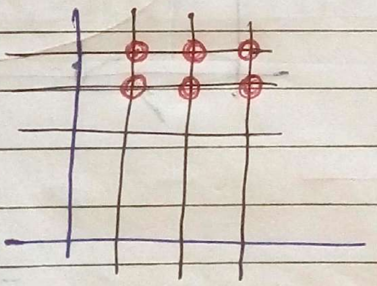
$$R_{X_1, X_2, X_3, \ldots, X_k} = \{ (x_1, x_2, \ldots, x_k) \mid x_1 \in R_{X_1}, x_2 \in R_{X_2}, \ldots, x_k \in R_{X_k} \}$$

Range of $\{x_1, x_2, \ldots, x_n\}$ which means the set of all possible values $[X_1, X_2, \ldots, X_k]$ can take.

Our Interest: Two dimensional random vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
Both X_1 & X_2 are discrete (OR) X_1 & X_2 are continuous.

Suppose X_1 & X_2 are both discrete

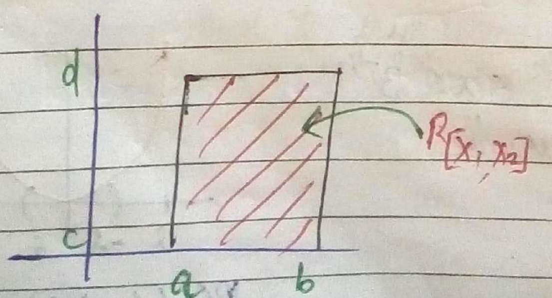
$$R_{[X_1, X_2]} = \{ (x_1, x_2) \mid x_1 \in R_{X_1}, x_2 \in R_{X_2} \}$$



Point of int. on the grid are the points in $[X_1, X_2]$

Suppose X_1 & X_2 are continuous

$R_{X_1} = (a, b)$ $R_{X_2} = (c, d)$



Definition – Bivariate probability functions.

1. Discrete case : To every outcome $(x_1, x_2) \in R_{[X_1, X_2]}$ we assign a number

$$p(x_1, x_2) = P(X_1 = x_1 ; X_2 = x_2)$$

where

* $p(x_1, x_2) \geq 0 \quad \forall (x_1, x_2) \in R_{[X_1, X_2]}$

and

* $\sum_{x_2} \sum_{x_1} p(x_1, x_2) = 1$.

$X_1 = x_1$ \ $X_2 = x_2$	★	★	★ x_2	★
★				
★ x_1			$p(x_1, x_2)$	
★				
★				

Tabular form of pmf

1

Joint PMF of $[X_1, X_2]$

$R_{X_1} = \{1, 2, 3, \ldots 6\}$
Fair
(Discrete uniform) $n=6$

$R_{X_2} = \{0, 1\}$.
Fair
(Bernoulli)

$X_2 = x_2$ \ $X_1 = x_1$	1	2	3	4	5	6	
0	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{2}$
1	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{2}$
	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$\boxed{1}$

marginal distribution of X_2

$p(1,0) = P(X_1 = 1, X_2 = 0)$

marginal distribution of X_1

2. Continuous case :

If $[X_1, X_2]$ is a continuous random vector with range space R. Then the function f, the joint density function, has the following properties :

* $f(x_1, x_2) \geq 0 \quad \forall (x_1, x_2) \in R$

* $\iint_R f(x_1, x_2)\, dx_1\, dx_2 = 1$

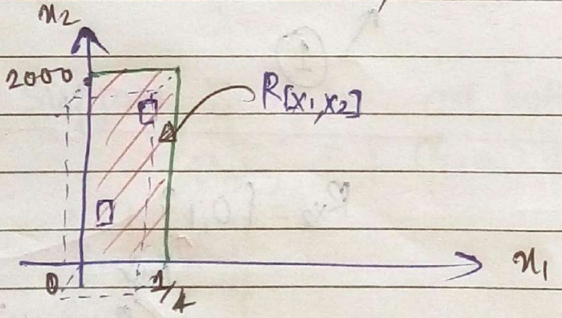
→ Note: $f(x_1, x_2)$ in this case does NOT represent probability

Typical probability statement

$$P(a \leq X_1 \leq b,\ c \leq X_2 \leq d) = \int_a^b \int_c^d f(x_1, x_2)\, dx_2\, dx_1$$

Ex: $f(x_1, x_2) = \frac{1}{500}$, $0 \leq x_1 \leq 1/4$, $0 \leq x_2 \leq 2000$

$= 0$ otherwise



Continuous uniform density

(i) $f(x_1, x_2) \geq 0 \quad \forall (x_1, x_2) \in R_{[X_1, X_2]}$

(ii) $\iint_R f(x_1, x_2)\, dx_2\, dx_1 = \int_0^{1/4} \int_0^{2000} \frac{1}{500}\, dx_2\, dx_1 = 1$.

Joint pdf.

Marginal distributions

Let (X_1, X_2) be a discrete random vector with joint pmf $p(x_1, x_2)$. Then the marginal distribution of X_1 is given by

Marginal distribution of X_1: * $p_1(x_1) = \sum_{x_2} p(x_1, x_2)$ for all x_1 (Column sum distr)

Marginal distribution of X_2 is given by

* $p_2(x_2) = \sum_{x_1} p(x_1, x_2)$ for all x_2 (row sum distr.)

Let (X_1, X_2) be a continuous random vector with joint pdf $f(x_1, x_2)$. Then the marginal density of X_1 is given by

* $$f(x_1) = \int_{x_2} f(x_1, x_2)\, dx_2 \quad \forall\ x_1$$

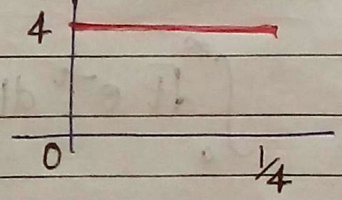
Similarly, the marginal density of X_2 is given by

* $$f(x_2) = \int_{x_1} f(x_1, x_2)\, dx_1 \quad \forall\ x_2$$

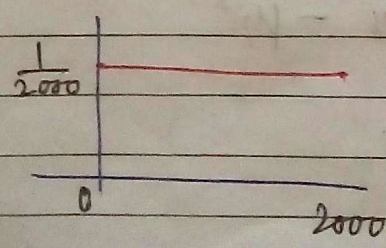
Ex. $f(x_1, x_2) = \frac{1}{500}$ $\quad 0 \le x_1 \le 1/4\ ; \quad 0 \le x_2 \le \cancel{20}\ 2000$

$= 0$ otherwise

$$f_1(x_1) = \int_0^{2000} \frac{1}{500}\, dx_2 = 4 \qquad 0 \le x_1 \le \frac{1}{4}$$



$$f_2(x_2) = \int_0^{1/4} \frac{1}{500}\, dx_1 = \frac{1}{2000} \qquad 0 \le x_2 \le 2000$$



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Definitions:-

$E(X_1) = \mu_1 = \sum_{x_i} x_i \, p_1(x_i)$ for discrete case

$= \int_x x \, f_1(x)$ for continuous case

Similarly

$Var(X_1) = \sum_{x_i} x_i^2 p(x_i) - \mu_1^2$ for discrete case

$= \int_x x^2 f_1(x) - \mu_1^2$ for cont. case

Ex.

$$f(x_1, x_2) = 4x_1x_2 \, e^{-(x_1^2+x_2^2)} \quad x_1>0, \; x_2>0$$
$$= 0 \quad \text{otherwise}$$

$$f_1(x_1) = \int_{x_2=0}^{\infty} 4x_1x_2 \, e^{-(x_1^2+x_2^2)} \, dx_2$$

$$= 4x_1 e^{-x_1^2} \int_0^{\infty} x_2 \, e^{-x_2^2} \, dx_2$$

$$= 2x_1 e^{-x_1^2} \int_0^{\infty} e^{-t} \, dt = 2x_1 e^{-x_1^2} \left[\frac{e^{-t}}{-1}\right]_0^{\infty}$$

$$\Rightarrow f_1(x_1) = 2x_1 e^{-x_1^2}, \quad x_1 > 0$$
$$= 0, \quad \text{otherwise}$$

$$E(X_1) = \int_0^{\infty} x_1 \, 2x_1 e^{-x_1^2} \, dx_1 = \int_0^{\infty} \sqrt{t} \, e^{-t} \, dt$$

$$E(X_1) = \iint_R x_1 \, f(x_1,x_2) \, dx_1 \, dx_2$$

$$Var(X_2) = \iint_R x_2^2 \, f(x_1,x_2) \, dx_1 \, dx_2 - \mu_2^2$$

Conditional distributions

Let $[X_1, X_2]$ be a discrete random vector with joint pmf $p(x_1, x_2)$ and marginal pmfs of X_1 & X_2 as $p_1(x_1)$ & $p_2(x_2)$ respectively. Then conditional probability distribution of X_2 given $X_1 = x_1$ is

$$p_{X_2}\Big|_{X_1=x_1} = \frac{p(x_1, x_2)}{p_1(x_1)} \quad \forall\, x_1, x_2$$

for $p_1(x_1) > 0$

Similarly, the conditional distribution of X_1 given $X_2 = x_2$ is

$$p_{X_1}\Big|_{X_2=x_2} = \frac{p(x_1, x_2)}{p_2(x_2)} \quad \forall\, x_1, x_2$$

for $p_2(x_2) > 0$

$$p_{X_2}\Big|_{X_1=0} = \boxed{\begin{aligned} p_{X_2}\big|_0 &= \frac{1}{6} & x_2 &= 1, 2, \ldots, 6 \\ &= 0 & &\text{otherwise} \end{aligned}}$$

$$\frac{p(0,0)}{p_1(0)} \qquad \frac{p(0,1)}{p_1(0)} \qquad \frac{p(0,6)}{p_1(0)}$$

$$p_{X_2}\big|_0 = \boxed{p_2(x_2) = \frac{p(\overset{0}{x_1}, x_2)}{p_1(\underset{0}{x_1})}}$$

very crucial role in definition of independence of r.v.'s

Definition: Let $[X_1, X_2]$ be a continuous random vector with joint pdf $f(x_1, x_2)$ and marginal densities of X_1, X_2 being $f_1(x_1)$ & $f_2(x_2)$ resp. Then the conditional density of X_1 given $X_2 = x_2$

$$f_{X_1}\Big|_{X_2=x_2} = \frac{f(x_1, x_2)}{f_2(x_2)} \quad \forall\, x_1, x_2$$

$f_2(x_2) \neq 0$

Similarly, the conditional density of X_2 given $X_1 = x_1$

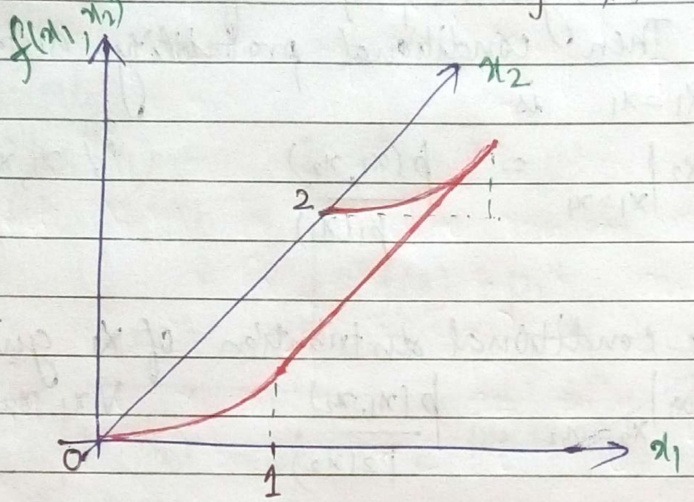
$$f_{X_2}\Big|_{X_1=x_1} = \frac{f(x_1, x_2)}{f_1(x_1)} \quad \forall\, x_1, x_2$$

$f_1(x_1) \neq 0$

Ex: Consider the joint pdf of $[X_1, X_2]$ given by $f(x_1,x_2) = x_1^2 + \frac{x_1 x_2}{3}$, $0 \le x_1 \le 1$, $0 \le x_2 \le 2$

$f(x_1,x_2) = 0$, otherwise

Find $f_{X_2|x_1}$ and $f_{X_1|x_2}$



$$f_{X_2|x_1} = \frac{f(x_1,x_2)}{f_1(x_1)} \qquad 0 \le x_1 \le 1, \quad 0 \le x_2 \le 1.$$

$$= \frac{x_1^2 + \frac{x_1 x_2}{3}}{f_1(x_1)}$$

$*$ $$f_1(x_1) = \int_{x_2=0}^{2} f(x_1,x_2)\,dx_2 = \int_{x_2=0}^{2} \left(x_1^2 + \frac{x_1 x_2}{3}\right) dx_2$$

$$= x_1^2 \left[x_2\right]_0^2 + \frac{x_1}{3}\left[\frac{x_2^2}{2}\right]_0^2$$

$$= 2x_1^2 + \frac{2x_1}{3}$$

$$\Rightarrow f_{X_2|x_1} = \frac{x_1^2 + \frac{x_1 x_2}{3}}{2x_1^2 + \frac{2x_1}{3}} = \frac{x_1 + \frac{x_2}{3}}{2x_1 + 2/3} = \frac{3x_1 + x_2}{6x_1 + 2} = \boxed{\frac{3x_1 + x_2}{2(3x_1+1)}}$$

$*$ $$f_2(x_2) = \int_{x_1=0}^{1} \left(x_1^2 + \frac{x_1 x_2}{3}\right) dx_1 = \left[\frac{x_1^3}{3} + \frac{x_1^2 x_2}{6}\right]_{x_1=0}^{1} = \frac{1}{3} + \frac{x_2}{6}$$

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$$\Rightarrow f_{X_1|x_2} = \frac{\frac{x_1^2 + x_1x_2}{3}}{\frac{1}{3}+\frac{x_2}{6}} = \frac{2(3x_1^2 + x_1x_2)}{(x_2+2)} = \boxed{\frac{2x_1(x_2+3x_1)}{(x_2+2)}}$$

$$\text{Prob}(0 \le X_1 \le 0.5 \mid X_2 = 1) \longleftrightarrow \int_0^{0.5} f_{X_1|_1}\, dx_1 \quad (f_{X_1|x_2=1})$$

$$= \int_0^{0.5} \frac{2x_1(3x_1+1)}{3}\, dx_1$$

$$= \left[\frac{2x_1^3}{3} + \frac{\cancel{2}}{3}\frac{x_1^2}{\cancel{2}}\right]_0^{0.5}$$

$$= \frac{2}{3}\cdot\frac{1}{8} + \frac{1}{3}\cdot\frac{1}{4} = \frac{2+2}{24} = \boxed{\frac{1}{6}}$$

$f_{X_1|x_2} \rightarrow$ No. of conditional densities = No. of possible values of x_2 = (∞ in this case) $0 \le x_2 \le 2$

$$E(X_1 \mid x_2) = \sum_{x_1} x_1\, p_{X_1|x_2} \quad \text{for discrete case}$$

$$= \int_{x_1} x_1\, f_{X_1|x_2} \quad \text{for continuous case}$$

Independence of random variables

Defn:

Let $[X_1, X_2]$ be a discrete random vector with joint pmf $p(x_1, x_2)$ and marginal pmf of X_1 & X_2 being $p_1(x_1)$ & $p_2(x_2)$ resp. Then the random variables X_1 & X_2 are independent if and only if

$$p(x_1,x_2) = p_1(x_1)\cdot p_2(x_2) \quad \forall\, x_1 \,\&\, x_2$$

ANUBHAV JAIN NOTES

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Defn: Let (X_1, X_2) be a continuous random vector with joint pdf $f(x_1, x_2)$ and marginal pdfs of x_1, x_2 being $f_1(x_1)$ & $f_2(x_2)$ respectively. Then r.v. x_1, x_2 are independent if and only if

$$f(x_1, x_2) = f_1(x_1) \cdot f_2(x_2) \qquad \forall\, x_1, x_2.$$

Theorem:- Let (X_1, X_2) be a discrete random vector. Then

$$p_{X_2|x_1}(x_2) = p_2(x_2) \quad \text{and}$$
$$p_{X_1|x_2}(x_1) = p_1(x_1)$$

for all x_1 and x_2 if and only if X_1 & X_2 are independent.

- Let (X_1, X_2) be a continuous random vector. Then

$$f_{X_2|x_1}(x_2) = f_2(x_2) \quad \text{and}$$
$$f_{X_1|x_2}(x_1) = f_1(x_1)$$

for all x_1 and x_2 if and only if X_1 & X_2 are independent.

Ex.

$$f(x_1, x_2) = \frac{1}{500} \qquad \left\{ \begin{matrix} 0 < x_1 < 1/4 \\ \text{and} \\ 0 < x_2 < 2000 \end{matrix} \right\}$$
$$= 0 \qquad \text{otherwise}$$

$$f_1(x_1) = 4 \qquad 0 < x_1 < 1/4$$
$$= 0 \qquad \text{otherwise}$$

$$f(x_2) = \frac{1}{2000} \qquad 0 < x_2 < \frac{1}{2000}$$
$$= 0 \qquad \text{otherwise}$$

Check: $f_{x_2|x_1} = f_2(x_2)$

$f_{x_1|x_2} = f_1(x_1)$

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Covariance / Correlation

Def: Let (X_1, X_2) be a random vector. Then the covariance between X_1 & X_2, denoted as $cov(X_1, X_2)$ or σ_{12} is defined as

$$Cov(X_1, X_2) = \sigma_{12} = E\left[(X_1 - E(X_1))(X_2 - E(X_2))\right]$$
$$= E\left[(X_1 - \mu_1)(X_2 - \mu_2)\right]$$

Further, correlation coefficient, e is defined as

$$e = \frac{Cov(X_1, X_2)}{\sqrt{Var(X_1) \cdot Var(X_2)}} = \frac{\sigma_{12}}{\sigma_1 \cdot \sigma_2}$$

$$E(X_1 X_2) = \iint x_1 x_2 f(x_1, x_2)\, dx_1\, dx_2$$
$$E(\phi(x_1, x_2)) = \iint \phi(x_1, x_2) f(x_1, x_2)\, dx_1\, dx_2$$

Consequence:

$$\boxed{Cov(X_1, X_2) = E(X_1 X_2) - E(X_1)E(X_2)}$$

$Var(X_1) = E(X_1^2) - [E(X_1)]^2$

Theorem:- If X_1 & X_2 are independent, then
$$E(X_1 X_2) = E(X_1)\, E(X_2)$$

Proof:-

$$E(X_1 X_2) = \iint x_1 x_2 f(x_1, x_2)\, dx_1\, dx_2$$
$$= \iint x_1 x_2 f_1(x_1) f_2(x_2)\, dx_1\, dx_2 \quad (\because X_1 \& X_2 \text{ are independent})$$
$$= \int x_1 f(x_1)\, dx_1 \int x_2 f(x_2)\, dx_2$$
$$= E(X_1) \cdot E(X_2).$$

ANUBHAV JAIN NOTES

Read Ch-4 (complete).

Corollary: If X_1 & X_2 are independent, then

(i) $Cov(X_1, X_2) = 0$

(ii) $\rho_{X_1, X_2} = 0$

Theorem: The value of ρ, the correlation coefficient, lies in the interval $[-1, 1]$. $-1 \le \rho \le 1$

Proof:
$$Q(t) = E\left[(X_1 - E(X_1)) + t(X_2 - E(X_2))\right]^2$$

$$= E(X_1 - E(X_1))^2 + 2tE(X_1 - E(X_1))E(X_2 - E(X_2)) + t^2 E(X_2 - E(X_2))^2$$

Note that $Q(t) \ge 0$, so the discremenant of $Q(t) \le 0$

$$4\{E(X_1 - E(X_1)) \cdot E(X_2 - E(X_2))\}^2 \le 4E(X_1 - E(X_1))^2 E(X_2 - E(X_2))^2$$

$$[Cov(X_1, X_2)]^2 \le Var(X_1) \cdot Var(X_2)$$

$$\Rightarrow \frac{Cov(X_1, X_2)^2}{Var(X_1)\,Var(X_2)} \le 1.$$

$$\Rightarrow -1 \le \rho \le 1$$

Ex.
$$f(x_1, x_2) = \begin{cases} \frac{1}{8}(6 - x_1 - x_2), & 0 \le x_1 \le 2,\ 2 \le x_2 \le 4. \\ = 0, & \text{otherwise} \end{cases}$$

(i) Check whether X_1 & X_2 are independent.

(ii) ρ_{X_1, X_2}

$$\rightarrow f_1(x_1) = \int_{x_2=2}^{4} \frac{1}{8}(6 - x_1 - x_2)\, dx_2 = \left[\frac{1}{8}\left((6 - x_1)x_2 - \frac{x_2^2}{2}\right)\right]_{x_2=2}^{4}$$

$$= \frac{1}{8}\left(12-2x_1-6\right) = \frac{(3-x_1)}{4}.$$

$$f_2(x_2) = \int_{x_1=0}^{2} \frac{1}{8}(6-x_1-x_2)\,dx_1 = \frac{1}{8}\left[(6-x_2)x_1 - \frac{x_1^2}{2}\right]_0^2$$

$$= \frac{1}{4}\left(5-x_2\right).$$

$$f_1(x_1)\cdot f_2(x_2) = \frac{(3-x_1)(5-x_2)}{4\times 4} = \frac{15+x_1x_2-5x_1-3x_2}{16} \neq f(x_1,x_2)$$

$\Rightarrow$ X_1 & X_2 are not independent.

$$E(X_1X_2) = \iint x_1 x_2\, f(x_1x_2)\,dx_1\,dx_2.$$

$$= \frac{1}{8}\int_{x_1=0}^{x_1=2} x_1\left[\int_{x_2=2}^{x_2=4} x_2(6-x_1-x_2)\,dx_2\right]dx_1$$

$$= \frac{1}{8}\int_{x_1=0}^{x_1=2} x_1 \left[(6-x_1)\frac{x_2^2}{2} - \frac{x_2^3}{3}\right]_2^4 dx_1$$

$$= \frac{1}{8}\int_0^2 x_1\left[(6-x_1)6 - \frac{56}{3}\right]dx_1$$

$$= \frac{1}{8}\int_0^2\left(\frac{52}{3}x_1 - 6x_1^2\right)dx_1 = \frac{1}{8}\left[\frac{52}{3}\cdot\frac{x_1^2}{2} - 2x_1^3\right]_0^2$$

$$= \frac{1}{8}\left(\frac{26}{3}(4) - 16\right) = \frac{104-48}{24} = \frac{\cancel{56}^{14}}{\cancel{24}_6} = \frac{7}{3}$$

$$E(X_1) = \int_0^2 x_1\left(\frac{3-x_1}{4}\right)dx_1 = \frac{1}{4}\left[\frac{3x_1^2}{2} - \frac{x_1^3}{3}\right]_0^2 = \frac{1}{4}\left(6-\frac{8}{3}\right)$$

$$= \frac{5}{6}$$

$$E(X_2) = \int_2^4 x_2\left(\frac{5-x_2}{4}\right)dx_2 = \frac{1}{4}\left[\frac{5x_2^2}{2} - \frac{x_2^3}{3}\right]_2^4 = \frac{1}{4}\left[\frac{5}{2}\times\cancel{12}^{6} - \frac{56}{3}\right] = \frac{17}{6}$$

$$\rho = \frac{\sigma_{12}}{\sigma_1 \cdot \sigma_2} = ?$$

$$Var(X_1) = \int_0^2 x_1^2 \cdot \left(\frac{3-x_1}{4}\right) dx_1 - \mu_1^2$$

$$= \frac{1}{4}\left[\frac{3x_1^3}{3} - \frac{x_1^4}{4}\right]_0^2 - \left(\frac{5}{6}\right)^2 = \frac{1}{4}(8-4) - \left(\frac{5}{6}\right)^2 = \frac{11}{36}.$$

$$Var(X_2) = \int_2^4 x_2^2 \left(\frac{5-x_2}{4}\right) dx_2 - \mu_2^2$$

$$= \frac{1}{4}\left[\frac{5x_2^3}{3} - \frac{x_2^4}{4}\right]_2^4 - \left(\frac{17}{6}\right)^2 = \frac{1}{4}\left[\frac{5(56)}{3} - \frac{1}{4}(\overset{60}{\cancel{240}})\right] - \frac{17^2}{36}.$$

$$= \frac{11}{36}$$

$$\sigma_{12} = E(X_1 X_2) - E(X_1)E(X_2)$$

$$= \frac{7}{3} - \frac{5}{6} \cdot \frac{17}{6} = -\frac{1}{36}$$

$$\rho = \frac{-1/36}{\sqrt{\frac{11}{36} \cdot \frac{11}{36}}} = \frac{-1/36}{11/36} = \boxed{\frac{-1}{11}}$$

Distribution function of 2-d random vector.

(X_1, X_2) is a random vector.

$$\boxed{F(x_1, x_2) = Prob\,(X_1 \le x_1,\ X_2 \le x_2).}$$

→ if (X_1, X_2) is a discrete random vector,

$$\boxed{F(x_1, x_2) = \sum_{t_1 \le x_1} \sum_{t_2 \le x_2} p(t_1, t_2)}$$

→ if (X_1, X_2) is continuous, then

$$\boxed{F(x_1, x_2) = \int_{-\infty}^{x_1}\int_{-\infty}^{x_2} f(t_1, t_2)\, dt_1\, dt_2}$$

Generalizing 2-d to n-d.
(Read from Ch 4 of the book).

Linear combinations:-
Let $X_1, X_2, \ldots, X_n$ be random variables.

$$H(X_1, X_2, \ldots, X_n) = a_0 + a_1 x_1 + \ldots + a_n x_n$$
is called as linear combination of the random variables $X_1, \ldots, X_n$

- $$E(H(X_1, X_2, \ldots, X_n)) = E(a_0 + a_1 x_1 + \ldots + a_n x_n)$$

$$= \int\int \ldots \int H(x_1, x_2, \ldots x_n) f(x_1, x_2, \ldots x_n)\, dx_1\, dx_2 \ldots dx_n$$

$$= a_0 + a_1 E(X_1) + a_2 E(X_2) + \ldots + a_n E(X_n).$$

In particular,
$$E(X_1 + X_2 + \ldots + X_n) = \sum_{i=1}^{n} E(X_i)$$

$$\boxed{E(\bar{X}) = E\left(\frac{X_1 + X_2 + \ldots + X_n}{n}\right) = \frac{1}{n}\sum_{i=1}^{n} E(X_i)}$$

$$Var(X_1 + X_2) = E(X_1 + X_2)^2 - [E(X_1 + X_2)]^2$$

$$\boxed{Var(X_1 + X_2) = var(X_1) + Var(X_2) + 2\,Cov(X_1, X_2)}$$

If X_1 & X_2 are independent,
$$var(X_1 + X_2) = Var(X_1) + var(X_2).$$

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In general, if $X_1, X_2, \ldots X_n$ are independent, then

$$\boxed{\operatorname{var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \operatorname{var}(X_i)}$$

Moment generating function

For a random variable X,

$$M_X(t) = E(e^{tX})$$

1. Let $Y = aX$

$$M_Y(t) = E(e^{tY}) = E(e^{taX}) = E(e^{(ta)X}) = M_X(at)$$

$$\boxed{M_Y(t) = M_X(at)}$$

2. Let $X_1, X_2, \ldots X_n$ be independent random variables

Let $Y = X_1 + X_2 + \ldots + X_n$.

Show: $M_Y(t) = M_{X_1}(t) \cdot M_{X_2}(t) \ldots M_{X_n}(t)$.

$$M_Y(t) = E(e^{tY}) = E(e^{t(X_1+X_2+\ldots+X_n)})$$

$$= E(e^{tX_1} \cdot e^{tX_2} \ldots e^{tX_n})$$

$$M_Y(t) = E(e^{tX_1}) \cdot E(e^{tX_2}) \ldots E(e^{tX_n})$$

$X_1, X_2 \ldots X_n$ independent

$$\boxed{M_Y(t) = M_{X_1}(t) \cdot M_{X_2}(t) \ldots M_{X_n}(t)}$$

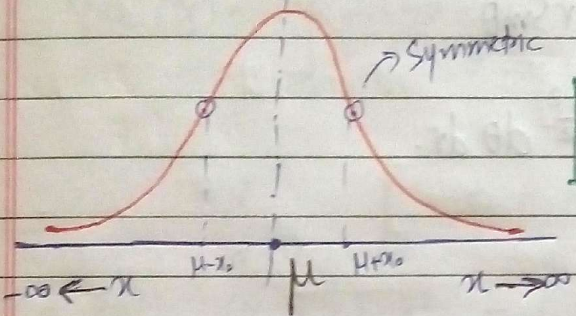
This result easily shows the proof for the expression of mgf of $\operatorname{Bin}(n,p)$ from mgf of Bernoulli(p).

The NORMAL distribution / Gaussian distribution

A continuous random variable X is said to follow normal distribution if it has following density function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$$

$-\infty < x < \infty$
$-\infty < \mu < \infty$
$\sigma > 0$



$$\boxed{X \sim N(\mu, \sigma^2)}$$

Properties of Normal disf^n :-

1. $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$

2. $f_X(x) \geq 0$

3. $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0$

4. $f(x+\mu) = f(\mu - x)$ (This implies f is symmetric about μ)

5. The maximum value of f occurs at μ.

6. The points of inflection are at $x = \mu \pm \sigma$

Prove: $\int_{-\infty}^{\infty} f(x)\, dx = 1$

Let $I = \int_{-\infty}^{\infty} f(x)\, dx$, show $I^2 = 1$.

Let $y = \frac{x-\mu}{\sigma}$

$\left(dy = \frac{dx}{\sigma} \Rightarrow dx = \sigma\, dy\right)$

$$\Rightarrow I = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2}\, dy$$

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$$I^2 = \left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2}\,dy\right]\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2}\,dz\right]$$

$$I^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\left(\frac{y^2+z^2}{2}\right)}\,dy\,dz.$$

$$y = r\cos\theta \qquad z = r\sin\theta$$

$$I^2 = \frac{1}{2\pi}\int_0^{\infty}\int_0^{2\pi} r e^{-\frac{r^2}{2}}\,d\theta\,dr.$$

$$I^2 = \int_0^{\infty} r e^{-r^2/2}\,dr$$

$$t = r^2/2 \Rightarrow dt = r\,dr.$$

$$I^2 = \int_0^{\infty} e^{-t}\,dt = \left[\frac{e^{-t}}{-1}\right]_0^{\infty} = \left(\frac{0-1}{-1}\right) = 1$$

$$\Rightarrow \boxed{I^2 = 1}$$

$$\Rightarrow I = 1 \Rightarrow \boxed{\int_{-\infty}^{\infty} f(x)\,dx = 1}$$

(positivity of f)

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\,dx.$$

Substitute $z = \frac{x-\mu}{\sigma} \Rightarrow x = (\mu + z\sigma)$

$$E(X) = \mu \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\,dz}_{=1} + \sigma \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} z e^{-z^2/2}\,dz}_{=0}$$

$$\boxed{E(X) = \mu}$$

- $\int z \cdot e^{-z^2/2} dz = -e^{-z^2/2}$
- $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$

$$Var(X) = E(X-\mu)^2$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Substitute $z = \frac{x-\mu}{\sigma} \Rightarrow x = \sigma z + \mu$.

$$Var(X) = \int_{-\infty}^{\infty} \sigma^2 z^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}z^2} (\sigma dz)$$

$$= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-\frac{1}{2}z^2} dz = \frac{2\sigma^2}{\sqrt{2\pi}} \int_0^{\infty} \underset{I}{z} \cdot \underset{II}{\left(z e^{-z^2/2}\right)} dz$$

$$= \frac{2\sigma^2}{\sqrt{2\pi}} \left[z \int \left(z \cdot e^{-\frac{z^2}{2}}\right) dz \Big|_0^{\infty} - \int_0^{\infty} \left[\int z \cdot e^{-z^2/2} dz\right] dz \right]$$

$$= \frac{\cancel{2}\sigma^2}{\sqrt{2\pi}} \underbrace{\cancel{\left[-z e^{-z^2/2}\right]_{-\infty}^{\infty}}}_{\to 0} + \frac{\cancel{2}\sigma^2}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} e^{-z^2/2} dz}_{\to 1} = \sigma^2$$

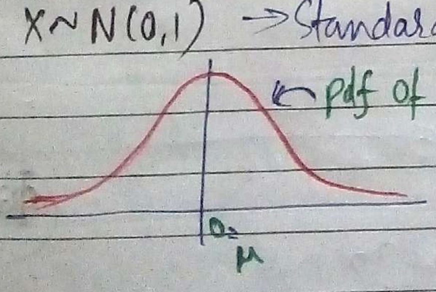
$$\boxed{Var(X) = \sigma^2}$$

The Standard Normal Distribution

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = 1.$$

Density of Normal r.v. with $\mu = 0$, $\sigma^2 = 1$.

$X \sim N(0,1) \rightarrow$ Standard normal r.v.



CDF of the standard normal r.v

$$Z \sim N(0,1)$$

$$\boxed{F_Z(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du}$$

$\Phi(-x) = 1 - \Phi(x)$

$\Phi(x) = F_Z(z)$

Prob $(Z \le z)$ = look from the table.

$X \sim N(\mu, \sigma^2)$ $\qquad\qquad Z \sim N(0,1)$

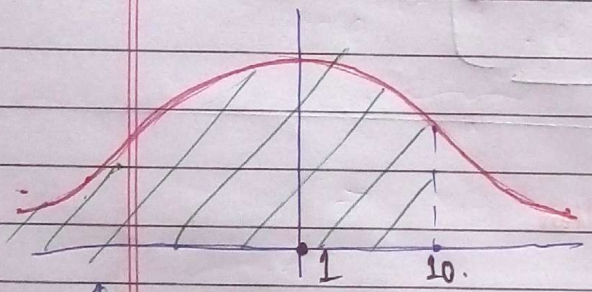
Prob $(X \le x) = F_X(x)$

Prob $(X \le x)$ = Prob $\left(\underbrace{\frac{X-\mu}{\sigma}}_{N(0,1)} \le \frac{x-\mu}{\sigma}\right)$ = $\underbrace{\text{Prob}\left(Z \le \frac{x-\mu}{\sigma}\right)}_{\text{Can be calculated from the table}}$

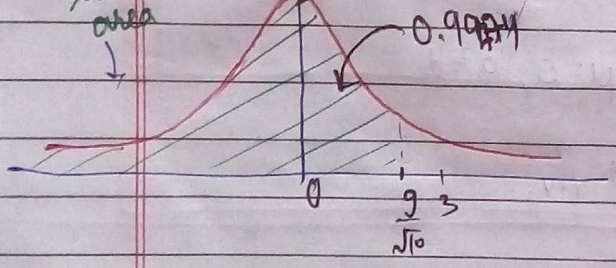
$E\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma}\left[E(X) - E(\mu)\right] = \frac{1}{\sigma}(\mu - \mu) = 0.$

$\text{Var}\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma^2}\left[\text{Var}(X) - \text{Var}(\mu)\right] = \frac{1}{\sigma^2}(\sigma^2 - 0) = 1.$

Problem: $X \sim N(\underset{\mu}{1}, \underset{\sigma^2}{10})$



same area



Prob $(X \le 10)$ = Prob $\left(\frac{X-1}{\sqrt{10}} \le \frac{10-1}{\sqrt{10}}\right)$

= Prob $\left(Z \le \frac{9}{\sqrt{10}}\right)$

= available from table

= Prob $(Z \le 2.846)$

$= \Phi(2.84)$

$= 0.9974$

Problem: $X \sim N(\mu, \sigma^2)$ $\qquad \boxed{\Phi(z) = F_Z(z)}$

Prob $(\mu - 3\sigma \le X \le \mu + 3\sigma)$ = ??

= Prob $\left(\frac{\mu - 3\sigma - \mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{\mu + 3\sigma - \mu}{\sigma}\right)$

= Prob $\left(-3 \le \frac{X-\mu}{\sigma} \le 3\right)$ = Prob $(-3 \le Z \le 3) = \Phi(3) - \Phi(-3)$

$= 2\Phi(3) - 1 = 0.9974$

* Cumulative Standard Normal Distribution Table.

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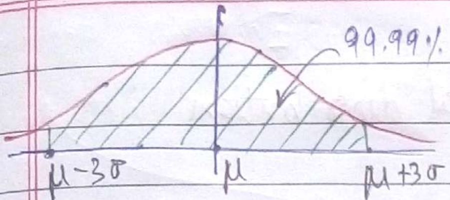
* Six Sigma Rule.



$X \sim N(\mu, \sigma^2)$, $P(\mu-3\sigma < X < \mu+3\sigma) = 0.9974.$

$P(\mu-\sigma < X < \mu+\sigma) = ?$

$P(\mu-2\sigma < X < \mu+2\sigma) = ?$

$$P(\mu-\sigma < X < \mu+\sigma) = P\left(\frac{\mu-\sigma-\mu}{\sigma} \leq \frac{X-\mu}{\sigma} \leq \frac{\mu+\sigma-\mu}{\sigma}\right)$$

$$= P\left(-1 \leq \frac{X-\mu}{\sigma} < 1\right) = P(-1 < Z < 1)$$

$$= \Phi(1) - \Phi(-1) = 2\Phi(1) - 1 = 0.68268$$

$$P(\mu-2\sigma < X < \mu+2\sigma) = 2\Phi(2) - 1 = 0.9545$$

Mgf of $N(\mu, \sigma^2)$

$$\boxed{M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}}$$

$X \sim N(\mu, \sigma^2)$

$$E(e^{tx}) = M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x)\, dx$$

Ex. $Z = \frac{X-\mu}{\sigma}$ for $X \sim N(\mu, \sigma^2)$ $Z \sim N(0,1)$

$$M_Z(t) = M_{\frac{X-\mu}{\sigma}}(t)$$

$$M_{X-\mu}(t) = e^{-\mu t} M_X(t)$$

$$M_{\frac{X-\mu}{\sigma}}(t) = e^{-\frac{\mu t}{\sigma}} M_X(t/\sigma)$$

$$\Rightarrow M_Z(t) = e^{-\frac{\mu t}{\sigma}} e^{\left(\frac{\mu t}{\sigma} + \frac{1}{2}\sigma^2 \frac{t^2}{\sigma^2}\right)} = \boxed{e^{\frac{t^2}{2}}}$$

$M_{X+a}(t) = e^{at} M_X(t)$

$M_{aX}(t) = M_X(at)$

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Reproductive property of normal distribution

Suppose $X_1, X_2, \ldots, X_n$ are independent random variables with mean $\mu_1, \mu_2, \ldots, \mu_n$ and variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$ resp. Then

$$X_1 + X_2 + \cdots + X_n \sim N\left(\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2\right)$$

Pf

$X_i \sim N(\mu_i, \sigma_i^2)$

$M_{X_i}(t) = e^{\mu_i t + \frac{1}{2}\sigma_i^2 t^2}$ $\quad i = 1, 2, \ldots n.$

Since $X_1, X_2, \ldots, X_n$ are independent,

$$M_{X_1+X_2+\cdots+X_n}(t) = M_{X_1}(t) \cdot M_{X_2}(t) \cdots M_{X_n}(t)$$

$$= e^{\mu_1 t + \frac{1}{2}\sigma_1^2 t^2} \cdot e^{\mu_2 t + \frac{1}{2}\sigma_2^2 t^2} \cdots e^{\mu_n t + \frac{1}{2}\sigma_n^2 t^2}$$

$$= e^{\underbrace{(\mu_1+\mu_2+\cdots+\mu_n)}_{\mu} t + \frac{1}{2}\underbrace{(\sigma_1^2+\sigma_2^2+\cdots+\sigma_n^2)}_{\sigma^2} t^2}$$

$$\Rightarrow X_1 + X_2 + \cdots + X_n \sim N\left(\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2\right)$$

Ex: Let independent normal r.v.s X_1, X_2 & X_3 be such that

$X_1 \sim N(12, 0.02)$ $\quad X_3 \sim N(18, 0.04)$

$X_2 \sim N(24, 0.03)$

Then ① compute Prob $(53.8 \le X_1 + X_2 + X_3 \le 54.2)$

② Compute Prob $(11.8 \le X_1 - X_2 + X_3 \le 12.2)$

① $X = X_1 + X_2 + X_3 \sim N(54, 0.09)$

$\mu = 54 \quad \sigma = 0.3$

$$P(53.8 \le \underbrace{(X_1 + X_2 + X_3)}_{X} \le 54.2)$$

$$= P\left(\frac{53.8 - 54}{0.3} \le \frac{X-\mu}{\sigma} \le \frac{54.2-54}{0.3}\right) = P\left(-\frac{2}{3} \le \underbrace{\frac{X-\mu}{\sigma}}_{Z} \le \frac{2}{3}\right)$$

$$= 2\,\Phi\left(\tfrac{2}{3}\right) - 1 = 0.49394.$$

0.74697

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② **Result:-** Let $X_1, X_2, \ldots, X_n$ be independent normal random variables with $X_i \sim N(\mu_i, \sigma_i^2)$ $i=1,2,\ldots n$.

Consider $Y = a_0 + a_1 x_1 + \ldots + a_n x_n$ for some $a_0, a_1, \ldots a_n \in \mathbb{R}$

$$Y \sim N\left(a_0 + \sum_{i=1}^{n} a_i \mu_i, \sum_{i=1}^{n} a_i^2 \sigma_i^2\right)$$

$\Rightarrow$ $Y = X_1 + X_2 + X_3$

$\mu = 12 - 24 + 18 = 6$

$\sigma^2 = 0.02 + 0.03 + 0.04 = 0.09 \Rightarrow \sigma = 0.3$

$$P(11.8 \le Y \le 12.2) = P\left(\frac{11.8-6}{0.3} \le \frac{Y-\mu}{\sigma} \le \frac{12.2-6}{0.3}\right)$$

$$= P\left(\frac{58}{3} \le Z \le \frac{62}{3}\right) = \Phi\left(\frac{62}{3}\right) - \Phi\left(\frac{58}{3}\right)$$

Corollary:

Let $X_1, X_2, \ldots, X_n$ be independent normal random variables with $X_i \sim N(\mu_i, \sigma_i^2)$ $i=1,2,\ldots n$.

Then, $$\bar{X} \sim N\left(\frac{\sum_{i=1}^{n} \mu_i}{n}, \frac{\sum_{i=1}^{n} \sigma_i^2}{n^2}\right)$$

Central Limit Theorem

Let $X_1, X_2, \ldots, X_n$ be a collection of n independent random variables with mean $E(X_i) = \mu_i$ and variances $Var(X_i) = \sigma_i^2$ respectively. Construct $Y = X_1 + X_2 + \ldots + X_n$.

Then $E(Y) = \sum_{i=1}^{n} \mu_i$, $Var(Y) = \sum_{i=1}^{n} \sigma_i^2$

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Then $$Z = \frac{Y - \sum_{i=1}^{n} \mu_i}{\sqrt{\sum_{i=1}^{n} \sigma_i^2}} \sim N(0,1)$$ "approximately".

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$Z \overset{\text{follows}}{\sim} N(0,1)$ "approximately"

Let $F_n(z)$ be the CDF of Z. Then.

$$\lim_{n\to\infty} \frac{F_n(z)}{\Phi(z)} = 1.$$

Computer Exercise:-

(1) Consider

$$X_i \sim \exp(\lambda) \qquad i = 1, 2, \ldots \begin{matrix} 10 \\ 50 \\ 100 \end{matrix}$$

$$X_1 + X_2 + \cdots + X_{10} = Y \qquad (\underset{E(Y)}{\mu_Y}, \underset{Var(Y)}{\sigma_Y^2})$$

$$Z = \frac{Y - \mu_Y}{\sigma_Y} \sim N(0,1)$$

→ Verify if Z follows $N(0,1)$?

Prob 1). $X_1, X_2, \ldots, X_{5000}$ independent random variables with mean $\mu_i = 0.50$ and $\sigma_i^2 = 0.01$.

Then compute $Prob(X_1 + X_2 + \cdots + X_{5000} \geq 2510)$

$$E(Y) = \mu_Y = \sum_{i=1}^{5000} \mu_i = 2500 \qquad \sigma^2 = \sum_{i=1}^{5000} \sigma_i^2 = 5000 \times 0.01 = 50$$

$$\sigma_Y = \sqrt{50} = 5\sqrt{2}$$

Let $Y = X_1 + X_2 + \cdots + X_{5000}$

$$\Rightarrow Prob(Y \geq 2510) = Prob\left(\frac{Y-\mu_Y}{\sigma_Y} \geq \frac{2510 - 2500}{5\sqrt{2}}\right)$$

$$= Prob\left(\left(\frac{Y-\mu_Y}{\sigma_Y}\right) \geq \sqrt{2}\right) = 1 - F_{5000}\left(\frac{Y-\mu}{\sigma}\right)$$

Using Central limit thm,

$$1 - F\left(\frac{Y-\mu}{\sigma}\right) = 1 - \Phi(\sqrt{2}) = 1 - \Phi(1.414)$$

$$= 1 - 0.921465$$

$$= \boxed{0.078535}$$

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CLT

Let $x_1, x_2, \ldots x_n$ be a collection of n independent random variables with mean $E(X_i) = \mu_i$ and variances $Var(x_i) = \sigma_i^2$ respectively. Then:

$$Z_n = \frac{Y - \sum_{i=1}^{n} \mu_i}{\sqrt{\sum_{i=1}^{n} \sigma_i^2}} \qquad \left(Y = X_1 + X_2 + X_3 + \ldots + X_n\right)$$

has approximate $N(0,1)$ distribution, in the sense that $\lim_{n\to\infty} \frac{F_n(z)}{\Phi(z)} = 1$ where $F_n(Z) =$ CDF of (Z).

Corollary:

If $X_1, X_2, \ldots X_n$ are n iid (independent & identically distributed random var.) with mean $E(X_i) = \mu$ and variance $Var(X_i) = \sigma^2$, then

$$Z_n = \frac{Y - n\mu}{\sigma\sqrt{n}} \quad \text{where } (Y = X_1 + X_2 + \ldots + X_n)$$

has approximate $N(0,1)$ distribution.

* Particular case:

Normal approximation to Binomial distribution. In the above result, take $X_i \sim Bernoulli(p)$

$$Y = X_1 + X_2 + \ldots + X_n \sim Bin(n,p)$$

$$Z_n = \frac{Y - np}{\sqrt{npq}} \sim N(0,1) \text{ approximately (using CLT).}$$

$$\Rightarrow \quad Y \sim N(np, npq) \text{ approximately (using CLT).}$$

However, we already know $Y \sim Bin(n,p)$.

$Prob(Y = 5)$ when $n = 100$, $p = 0.7$

$= \binom{100}{5} (0.7)^5 (0.3)^{95}$

exact probability calculation (accurate but very tedious)

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$$\text{Prob}(X \le x) = \text{Prob}\left(Z \le \frac{x-\mu}{\sigma}\right)$$

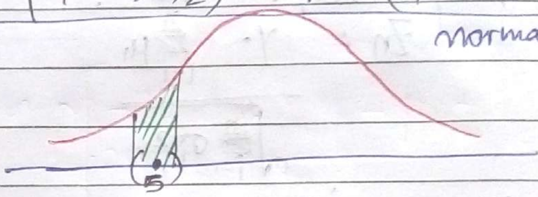
Y is approximated as $N(0,1) \rightarrow$ continuous random variable

$\rightarrow P(Y=5) = 0.$ ← Not true

$$\Rightarrow \underbrace{\text{Prob}(Y=5)}_{\text{exact prob}} = \text{Prob}(Y \le 5+1/2) - \text{Prob}(Y \le 5-1/2)$$

normal approximation

2.68×10^{-43}



$Y \sim \text{Bin}(10, 0.5)$ ($1/2$)

$$\text{Prob}(Y=5) = \text{Prob}(Y \le 5+1/2) - \text{Prob}(Y \le 5-1/2)$$

$\binom{10}{5}(0.5)^5(0.5)^5 = \boxed{0.2461}$

$n=10 \quad p=0.5$

$\mu = np = 5$

$\sigma^2 = npq = 2.5 \Rightarrow \sigma = \sqrt{2.5} = 1.58$

$$\text{Prob}(Y \le 5.5) = \text{Prob}\left(Z \le \frac{5.5-5}{1.58}\right) = \text{Prob}\left(Z \le \overset{0.3162}{\frac{1}{\sqrt{10}}}\right) = \Phi\left(\frac{1}{\sqrt{10}}\right)$$

$$\text{Prob}(Y \le 4.5) = \text{Prob}\left(Z \le \frac{4.5-5}{1.58}\right) = \text{Prob}\left(Z \le \frac{-1}{\sqrt{10}}\right) = \Phi\left(\frac{-1}{\sqrt{10}}\right)$$

$$\text{Prob}(Y=5) = \Phi\left(\frac{1}{\sqrt{10}}\right) - \Phi\left(\frac{-1}{\sqrt{10}}\right) = 2\Phi\left(\frac{1}{\sqrt{10}}\right) - 1$$

$$= 2(0.6217) - 1 = \boxed{0.2434.}$$

Another important random variable

$Y \sim \text{Bin}(n,p)$

$\hat{p} = \frac{Y}{n}$

proportion r.v. $\hat{p} \sim N\left(p, \frac{pq}{n}\right)$ by CLT

Syllabus: Post Midsem $\boxed{CT}$ 9 April (Sun) → 9:30 am – 1 pm 11:30 am – 11 am

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- Joint Dist.
- Sp. types of Cont. Dist
- ~~Normal Dist~~
- Bivariate Normal Dist.
- . . .

10 Q – 2 Mrks each. 20 Marks.

Bivariate normal distribution

A bivariate random vector $[X_1, X_2]$ is said to follow bivariate normal random density if its pdf is given by

$$f(x_1,x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{\frac{-1}{2(1-\rho^2)}\left[\frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2} - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right)\right]\right\}$$

$-\infty < x_1 < \infty$, $-\infty < x_2 < \infty$

$-\infty < \mu_1, \mu_2 < \infty$

$\sigma_1 > 0$, $\sigma_2 > 0$

$-1 \leq \rho \leq 1$

The marginal densities

$$f_1(x_1) = \int_{-\infty}^{\infty} f(x_1,x_2)\,dx_2 = \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2}$$

$X_1 \sim N(\mu_1, \sigma_1^2)$ $\quad -\infty < x_1 < \infty$

$$f_2(x_2) = \int_{-\infty}^{\infty} f(x_1,x_2)\,dx_1 = \frac{1}{\sigma_2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2}$$

$X_2 \sim N(\mu_2, \sigma_2^2)$ $\quad -\infty < x_2 < \infty$

$$\rho = \frac{\sigma_{12}}{\sigma_1\sigma_2}$$

$$\sigma_{12} = Cov(X_1,X_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x_1-\mu_1)(x_2-\mu_2) f(x_1,x_2)\,dx_1\,dx_2$$

$$= E(X_1-\mu_1)(X_2-\mu_2)$$

$(X_1, X_2) \sim BN(\mu_1, \sigma_1^2, \mu_2, \sigma_2^2, \rho)$

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Conditional distributions.

$\Rightarrow$ $$f_{X_1|x_2}(x_1) = \frac{f(x_1, x_2)}{f_2(x_2)}$$ for every x_1 & x_2.

$$\Rightarrow f_{X_1|x_2}(x_1) = \frac{1}{\sigma_1\sqrt{2\pi}\sqrt{1-\rho^2}} \exp\left[\frac{-1}{2\sigma_1^2(1-\rho^2)}\left\{x_1 - \left(\mu_1 + \rho\left(\frac{\sigma_1}{\sigma_2}\right)(x_2-\mu_2)\right)\right\}^2\right]$$

* $$X_1|x_2 \sim N\left(\mu_1 + \rho\left(\frac{\sigma_1}{\sigma_2}\right)(x_2-\mu_2),\ \sigma_1^2(1-\rho^2)\right)$$

- Marginals and conditionals are also normal densities.

$\rightarrow$ $$f_{X_2|x_1}(x_2) = \frac{f(x_1, x_2)}{f_1(x_1)}$$

$$\Rightarrow f_{X_2|x_1}(x_2) = \frac{1}{\sigma_2\sqrt{2\pi}\sqrt{1-\rho^2}} \exp\left[\frac{-1}{2\sigma_2^2(1-\rho^2)}\left\{x_2 - \left[\mu_2+\rho\left(\frac{\sigma_2}{\sigma_1}\right)(x_1-\mu_1)\right]\right\}^2\right]$$

* $$X_2|x_1 \sim N\left(\mu_2 + \rho\left(\frac{\sigma_2}{\sigma_1}\right)(x_1-\mu_1),\ \underbrace{\sigma_2^2(1-\rho^2)}_{\text{independent of } x_1}\right)$$

$(X_1, X_2) \sim BN$

$\mu_1 = 0.2$, $\mu_2 = 1100$, $\sigma_1^2 = 0.02$, $\sigma_2^2 = 525$, $\rho = 0.9$

(i) $E(X_2 | x_1 = 1)$

(ii) $\text{Prob}(X_2 \geq 1080)$

(iii) $\text{Prob}(X_2 \geq 1080 | x_1 = 1) = \text{Prob}(Y \geq 1080) = 1 - \Phi(\ )$

(i) $E(X_2 | x_1 = 1)$ = Mean of Random Variable $X_2 | x_1 = 1$

$\hookrightarrow \mu_2 + \rho\left(\frac{\sigma_2}{\sigma_1}\right)(x_1 - \mu_1)$

$$X_2|x_1 \approx N\left(\mu_2 + \rho\left(\frac{\sigma_2}{\sigma_1}\right)(x_1-\mu_1),\ \sigma_2^2(1-\rho^2)\right)$$

* $E(X_2 | x_1 = 1) = 1100 + 0.9\left(\frac{525}{0.02}\right)^{1/2}(1 - 0.2) = 1216.653.$

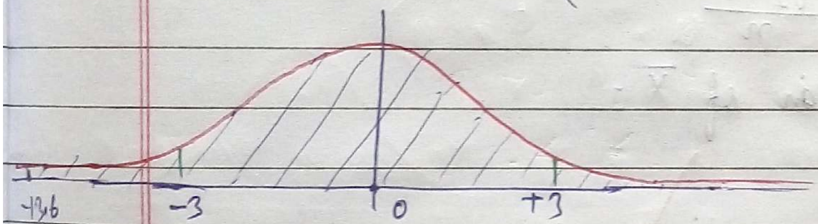
* $\sigma_2^2(1-\rho^2) = 525(0.19) = 99.75.$

(iii) $Prob(X_2 \geq 1080 | x_1 = 1) = Prob(Y \geq 1080)$

$$Y \equiv X_2 | x_1 = 1 \sim N(\underset{\mu}{1216.653}, \underset{\sigma^2}{99.75})$$

$= 1 - Prob(Y < 1080)$ or $Prob\left(\frac{Y-\mu}{\sigma} \geq \frac{1080-\mu}{\sigma}\right)$

$= Prob(Z \geq -13.68) \approx \boxed{1}.$



Log Normal Distribution

Consider a random variable X with range $(0, \infty)$, where $Y = \ln X$ is normally distributed with mean μ_Y and variance σ_Y^2.

The density of X is

$$f_X(x) = \frac{1}{x\sigma_Y} e^{-\frac{1}{2}\left[\frac{\ln x - \mu_Y}{\sigma_Y}\right]^2} \quad x > 0$$

$$= 0. \quad \text{Otherwise}$$

$X \sim$ Log Normal

$E(X) = \mu_X = e^{\mu_Y + \frac{\sigma_Y^2}{2}}$

$Var(X) = \sigma_X^2 = \mu_X^2(e^{\sigma_Y^2} - 1)$