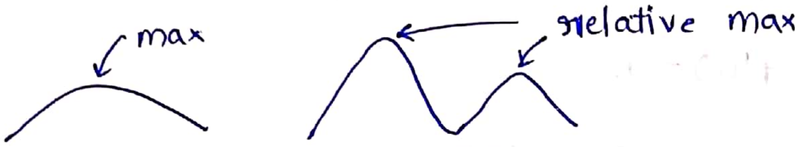
Differentiation:

$f: A \to R$, $c \in A$, c is a cluster point of A. The $\exists$ a number L for given $\epsilon > 0$, there is $\delta > 0$ with

$$0 < |x - c| < \delta \Rightarrow \left| \frac{f(x) - f(c)}{x - c} - L \right| < \epsilon$$

Then we say f' is differentiable and $f'(c) = L$



Theorem: Let $f: (a,b) \to R$ be differentiable at $c \in (a,b)$.

Let $f'(c) > 0$ or $f'(c) = \infty$. The $\exists$ a neibourhood $(c-\delta, c+\delta)$ of c such that

$c + \delta > x > c \Rightarrow f(x) > f(c)$

$c > x > c - \delta \Rightarrow f(x) < f(c)$

If $f'(c) < 0$ or $f'(c) = -\infty$, then $\exists$ a nbhd $(c-\delta, c+\delta)$ such that

$c + \delta > x > c \Rightarrow f(x) < f(c)$

$c > x > c - \delta \Rightarrow f(x) > f(c)$

Relative maximum (local maximum)

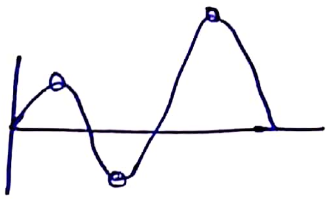
$f: [a,b] \to R$ is relative maximum at c if $\exists$ a nbhd $(c-\delta, c+\delta)$ st $x \in (c-\delta, c+\delta) \Rightarrow f(x) \leq f(c)$

Relative minimum

$f: [a,b] \to R$ is relative minimum at c if $\exists$ a nbhd $(c-\delta, c+\delta)$ st $x \in (c-\delta, c+\delta) \Rightarrow f(x) \geq f(c)$

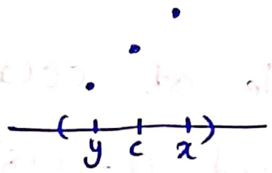
Relative Extremum:

f is relative extremum at c If f is either relative maximum or relative minimum.



<u>Theorem</u>: If $f:[a,b]\to R$ is diff at $c\in(a,b)$ and f is relative extremum at c. Then $f'(c)=0$.

proof: $f(x)>f(c)$



follows from previous theorem

If c is boundary points of $[a,b]$ i.e., $c=a$ or $c=b$ then the theorem is not true.

$f(x)=x$ on $[0,1]$

$f'(0)=1$ $f'(1)=1$

f is relative min at 0, relative max at 1.

$f(x)=|x|$ f is not differentiable at $x=0$, but has relative minimum at $x=0$.

<u>Ex</u>: $f(x)=x^2-3x+5$. Find relative extremum of f.

<u>Sol</u>: $f'(x)=2x-3$

$f'(x)=0 \Rightarrow x=3/2$

$\Rightarrow c=3/2$ is relative extremum for f.

Cor: If $f:[a,b]\to R$ has relative extremum at c. $c\in(a,b)$. Then either f is not differentiable at c (or) $f'(c)=0$.

$$f'(x)=2\left(x-\tfrac{3}{2}\right)$$

$\Rightarrow f'(x)>0$ if $x>3/2$

$f'(x)<0$ if $x<3/2$

First derivative test for extremum:

Let f be continuos on $I=[a,b]$ and $c\in(a,b)$

Assume f is diff at (a,c) and (c,b) Then,

(a). If there is a nbhd $(c-\delta,c+\delta)\in I$ such that $f'(x)\geq 0$ for $x\in(c-\delta,c)$ and $f'(x)\leq 0$ for $x\in(c,c+\delta)$.

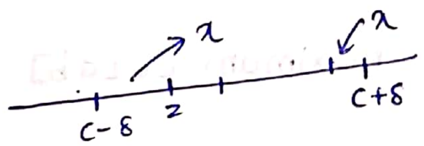
(b). If $\exists$ a nbhd $(c-\delta,c+\delta)\in I$ such that $f'(x)\leq 0$ for $x\in(c-\delta,c+\delta)$ and $f'(x)\geq 0$ for $(c,c+\delta)$. Then f is relative minimum at c.

proof: $f(c)-f(x)\geq 0$ in some nbhd $(c-\delta,c+\delta)$

Take $x\in(c-\delta,c)$ Then $f'(x)\geq 0$

If we can write,

$$f(c)-f(x)=f'(z)(c-x)$$
$$\geq 0 \quad \forall z\in(x,c)$$



Take $x\in(c,c+\delta)$

If we can write, $f(x)-f(c)=f'(y)(x-c)$
$y\in(c,x)$

Then $f'(y)\leq 0$ $x-c\geq 0$ $\Rightarrow f(x)-f(c)\leq 0$

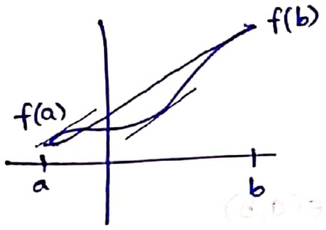
$\Rightarrow f(c)\geq f(x)$

Mean value Theorem:

$f:[a,b] \to \mathbb{R}$ continuos, f is differentiable on (a,b)

Then, $\exists\ c \in (a,b)$ such that

$$f(b) - f(a) = f'(c)(b-a)$$



$$f(b)-f(a) = f'(c)(b-a)$$

$$\Rightarrow \frac{f(b)-f(a)}{b-a} = f'(c)$$

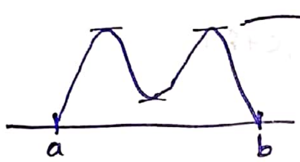
At c, the tangent line is parallel to the line segment.

First take f such that $f(a)=f(b)$

$$\frac{f(b)-f(a)}{b-a} = 0 \Rightarrow f'(c) = 0.$$

Even take f such that $f(a)=f(b)=0$

We have to take $c \in (a,b)$ such that $f'(c)=0$



c is relative extrema

Rolle's Theorem:

$f:[a,b] \to \mathbb{R}$ be continuos, f be differentiable on (a,b), and $f(a) = f(b) =$

Then $\exists$ a point c such that $f'(c)=0$

<u>proof:</u> f is continuos on $[a,b]$. Then f has maximum $c \in [a,b]$

$\Rightarrow$ f has relative maximum at c.

<u>case-1:</u> $c \in (a,b)$

Then $f'(c) = 0$

[$f:[a,b] \to \mathbb{R}$ continuos, differentiable on (a,b) then f has a relative maximum at $c \in [a,b]$ Then $f'(c)=0$]

Case-2: $c=a$

Then $f(c)=f(a)=f(b)=0$

Also assume f is non-constant.

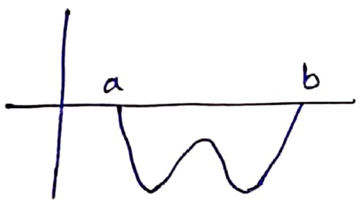
So, f has a minimum $d\in(a,b)$

$\therefore f'(d)=0$

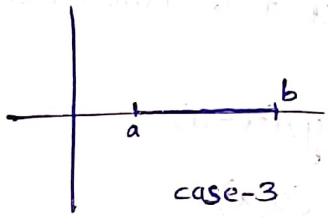
Case-3: $c=a$ f is constant

Then $d=\frac{a+b}{2}$

Mean value Theorem:



case-2



case-3

Generalised Rolle's Theorem:

$f:[a,b]\to R$ continuoss and diff on (a,b)

then if $f(a)=f(b)$ then $\exists c$ st $f'(c)=0$.

Ex: $f:[0,2]\to R$

f is continuos and diff on $[0,2)$

$f(0)=0$ $f(1)=f(2)=1$. show that $\exists c_1\in(0,1)$ such that $f'(c_1)=1$

$\exists c_2\in(1,2)$ such that $f'(c_2)=0$, $\exists c_3\in(0,2)$ such that $f'(c_3)=1/3$

$\to$ part-1: Rolle's

$\to$ part-2: Mean value

$\to$ part-3: darboux.

Mean value Theorem:

$f(b)-f(a)=f'(c)(b-a)$

Theorem: suppose f is continuos on $[a,b]$ diff on (a,b) $f'(x)=0$ $\forall x \in (a,b)$

define, $g(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a)$

$g(a) = 0 = g(b)$ g is continuos on $[a,b]$

Then apply rolles theorem

Then $\exists\, c \in (a,b)$ st $g'(c) = 0 \Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}$

Applications of mean value Theorem:

(1). Theorem: $f:[a,b] \to \mathbb{R}$ is continuos on $[a,b]$ is differentiable on (a,b). If $f'(x) = 0$ $\forall\, x \in (a,b)$ Then f is constant.

proof: $x > a$ $\quad f(x) - f(a) = f'(y)(x-a)$

for some $y \in (a,x)$

$\Rightarrow f(x) = f(a)$

(2). $f,g:[a,b] \longrightarrow \mathbb{R}$ are continuos on $[a,b]$, diff on (a,b)

$f'(x) = g'(x)$ $\forall x \in (a,b)$

Then $f = g + k$ for some constant k.

proof: Take $h : f - g$

$h'(x) = 0$

$\Rightarrow h(x)$ is constant k

$\Rightarrow f(x) = g(x) + k$

(3). Inequality: $-x \leq \sin x \leq x$

proof: $x > 0$ $\quad f(x) - f(0) = \cos c\,(x-0)$

for some $c \in (0,x)$

$-1 \leq \cos x \leq 1$

$\Rightarrow -x \leq f(x) - f(0) \leq x \Rightarrow -x \leq \sin x \leq x$

Mean value Theorem:

$\exists c$ st $f'(c) = \frac{f(b)-f(a)}{b-a}$

Rolle's Theorem:

$f(a) = f(b) = 0$

$\exists c$ st $f'(c) = 0$

(4). Bernoulli's Inequality:

If $\alpha > 1$ then $(1+x)^\alpha \geq 1+\alpha x$ $\forall x \geq -1$ $(x > 0)$

$f(x): (1+x)^\alpha$

$f'(x) = \alpha(1+x)^{\alpha-1}$

Note: $f:[a,b] \to R$ is strictly monotone and continuos Let $J = f[a,b]$

Then $f^{-1}: J \to R$ is strictly monotone and continuos. If f is diff on $[a,b]$ and $f'(c) \neq 0$ $\forall c \in [a,b]$ Then f^{-1} is also differentiable and $(f^{-1})'(f(c)) = \frac{1}{f'(c)}$

$\alpha = \frac{m}{n}$

$(x^{m/n}) = (x^{1/n})^m$ $\qquad x^{1/n} = (x^n)^{-1}$ $\qquad x^{1/n}$ is inverse of x^n

$f(x) - f(0) = f'(c)(x-0)$ for some $c \in (0, x)$ or $(x, 0)$

$\Rightarrow f(x) - 1 = f'(c)x$

$\Rightarrow (1+x)^\alpha - 1 = \alpha(1+c)^{\alpha-1}(x-0)$ for some $c \in (0,x)$ or $(x,0)$

$c > 0$ $\quad (1+c)^{\alpha-1} > 1$

$\Rightarrow (1+x)^\alpha - 1 \geq \alpha x$

$\Rightarrow (1+x)^\alpha \geq 1+\alpha x$

$\boxed{-1 < x < 0} \longrightarrow$ next class

(3). Theorem: $f: J \to R$ is diff on (a,b) and $I = [a,b]$

Then

(i). $f'(x) \geq 0 \quad \forall x \in (a,b)$ if and only if $f(x)$ is increasing function.

(ii). $f'(x) \leq 0 \quad \forall x \in (a,b)$ if and only if $f(x)$ is decreasing function.

proof: For any two points $x < y$

$\exists c \in (x, y)$ such that $f(y) - f(x) = f'(c)(y - x) \geq 0$

$\Rightarrow f(y) - f(x) \geq 0 \quad \forall \ y > x$

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

$x > c \Rightarrow f(x) \geq f(c)$

$\Rightarrow f(x) \geq f(c)$

$\Rightarrow \dfrac{f(x) - f(c)}{x - c} \geq 0$

$x > c \Rightarrow f(x) \leq f(c)$

$\Rightarrow \dfrac{f(x) - f(c)}{x - c} \leq 0$

strictly increasing: $x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$ $\quad f'(z) > 0 \quad \forall z \in I$

$f'(x) > 0 \quad \forall x(a,b) \Rightarrow f(x)$ is strictly increasing

$f(x) = x^3 \quad f(x) = x^{2n+1}$

$f'(0) = 0$ but, f is strictly increasing

lemma: If $f: I \to R$ be diff on I, $c \in I$

(i). $f'(c) > 0 \Rightarrow \exists$ a $\delta > 0$ st $x \in (c, c+\delta)$

$\Rightarrow f(x) > f(c)$

$x \in (c, c-\delta) \Rightarrow f(x) < f(c)$

(ii). $f'(c) < 0 \Rightarrow \exists$ a $\delta > 0$ st $x \in (c, c+\delta)$

$\Rightarrow f(x) < f(c)$

$x \in (c, c-\delta) \Rightarrow f(x) > f(c)$

Pf: $f'(c) > 0$

$\Rightarrow \frac{f(x)-f(c)}{x-c} > 0$ for some $(c-\delta, c+\delta) - \{c\}$

Then $x \in (c, c+\delta) \Rightarrow f(x) > f(c)$

$x \in (c-\delta, c) \Rightarrow f(x) < f(c)$

Exm: $a > b > 0, n \geq 2$ then show that $a^{1/n} - b^{1/n} < (a-b)^{1/n}$

Sol: $a^{1/n} - b^{1/n} < (a-b)^{1/n}$

$\Leftrightarrow 1 - \left(\frac{b}{a}\right)^{1/n} < \left(1 - \frac{b}{a}\right)^{1/n}$

$f(x) := (1-x)^{1/n} + x^{1/n}$

$f'(x) = \frac{1}{n} x^{-1+1/n} + \frac{1}{n}(1-x)^{-1+1/n}$

$x < 1 \Rightarrow f'(x) > 0$

$f(1) - f\left(\frac{b}{a}\right) = f'(c)\left(1 - \frac{b}{a}\right)$ for some $c \in \left(\frac{b}{a}, 1\right)$

> 0

$\Rightarrow f(1) > f\left(\frac{b}{a}\right)$

$\Rightarrow 1^{1/n} + (1-1)^{1/n} > \left(1 - \frac{b}{a}\right)^{\frac{1}{n}} + \left(\frac{b}{a}\right)^{1/n}$

$\Rightarrow 1 - \left(\frac{b}{a}\right)^{1/n} > -\left(1 - \frac{b}{a}\right)^{1/n}$

$f(x) = (x-b)^{1/n} - (x^{1/n} - b^{1/n})$

$f'(x) = \frac{1}{n}(x-b)^{-1+1/n} - \frac{1}{n} x^{-1+1/n}$

$x > b \Rightarrow f'(x) > 0$

$f(a) - f(b) = (a-b) f'(c) \quad c \in (b, a)$

$f(a) - f(b) > 0$

$\Rightarrow a^{1/n} - b^{1/n} < (a-b)^{1/n}$

Exe: $f:[0,2] \rightarrow \mathbb{R}$ diff on $(0,2)$, $f(0)=0$

$f(1)=$ [redacted] $f(2)=$ [redacted]. Then,

(i). $\exists\, c_1 \in (1,2)$ st $f'(c_1)=0$

(ii). $\exists\, c_2 \in (0,2)$ st $f'(c_2)=1/3$

(iii). $\exists\, c_3 \in (0,1)$ st $f'(c_3)=1$

Darboux Theorem:

If f is diff on $[a,b]$ and k is a no between $f'(a)$ and $f'(b)$

Then $\exists$ at least one $c \in (a,b)$ st $f'(c)=k$.

Assume $f'(a) < k < f'(b)$

let. $g(x) = kx - f(x)$ g is continuos and attains a absolute max, say d.

~~$g(a) \neq$~~

$g'(d)=0 \Rightarrow f'(d)=k$ $d \in [a,b]$

$g'(a) = k - f'(a) > 0 \rightarrow$ By lemma (i) g is not max at a

$g'(b) = k - f'(b) < 0 \rightarrow$ By lemma (ii) g is not max at b

$\therefore d \in (a,b)$

Application of First derivative Theorem: for Relative Extremum

a). Find points of relative extremum of $f(x) = x|x^2-12|$, $-2 \le x \le 3$

$f(x) = -x(x^2-12) = 12x - x^3$

$f'(c)=0 \Rightarrow 12 - 3x^2 = 0$

$\Rightarrow x^2 = 4$

$\boxed{x = \pm 2}$

differentiable if $x^2-12 \neq 0$

so, on $[-2,3]$ it is differentiable.

possible points of relative extremum are $-2, 2, 3$.

b). Find the points of relative extrema, intervals on which f is increasing or decreasing of the function.

$$f(x) = \sqrt{x} - 2\sqrt{x+2}, \quad x>0$$

Sol: $$f'(x) = \frac{1}{2\sqrt{x}} - \frac{2}{2\sqrt{x+2}}$$

$$f'(x) = 0 \Rightarrow x = 2/3$$

$x = 2/3$ is the only relative extremum point.

$$x < \frac{2}{3} \Rightarrow f'(x) > 0$$

$$x > \frac{2}{3} \Rightarrow f'(x) < 0$$

By first derivative test, f has rel max at $2/3$.

$x < \frac{2}{3} \Rightarrow f'(x) > 0$

$\Rightarrow f$ is strictly increasing on $(0, 2/3)$

$x > 2/3 \Rightarrow f'(x) < 0$

$\Rightarrow f$ is strictly decreasing on $(2/3, \infty)$

Higher order derivative test for relative extremum

Let I be an interval and x_0 be an interior point of I

$I = [a,b]$ be an interval.

c is interior point if $c \in (a,b)$

Let $n \geq 2$. Let $f, f^{(1)}, f^{(2)}, \ldots, f^{(n)}$ exist and are continuos in a nbhd $(x_0-\delta, x_0+\delta)$ of x_0. Let $f'(x_0) = \ldots = f^{(n-1)}(x_0) = 0$ and $f^{(n)}(x_0) \neq 0$

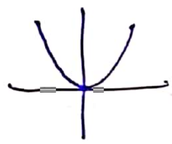
Then,

(i). n is ~~every~~ even $f^{(n)}(x_0) > 0$, then, f has rel minimum at x_0.

$f(x) = x^2$ $\quad n = 2$

$f'(0) = 0$

$f''(0) > 0$



(ii). n is even, $f^{(n)}(x_0) < 0$, then f has rel maximum at x_0.

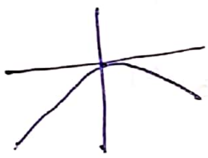
$f(x) = -x^4$

$f'(0) = 0$

$f''(0) = 0$

$f'''(0) = 0$

$f^{(4)}(0) = -12 < 0$



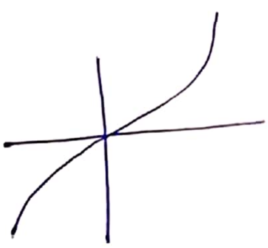
(iii). n is odd, $f^{(n)}(x_0) \neq 0$ has neither relative max, relative min at x_0.

$f(x) = x^3$

$f'(0) = 0$

$f''(0) = 0$

$f'''(0) = 6 > 0$



Taylor's Theorem:

Let $n \in \mathbb{N}$ $I = [a,b]$, let $f: I \to R$ be a function, Let $f^{(1)} \dots f^{(n)}$ exist and continuos on $[a,b]$ and let $f^{(n+1)}$ exist on (a,b). If $x_0 \in I$, then for any $x \in I$, there exists a point between x and x_0 (i.e., $c \in (x, x_0)$ or (x_0, x)) such that

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)(x-x_0)^2}{2!} + \dots + \frac{f^{(n)}(x_0-x_0)^n}{n!} + \frac{f^{(n+1)}(c)(x-x_0)^{n+1}}{(n+1)!}$$

proof of Higher order derivative Test:

$$f(x) = f(x_0) + \frac{(x-x_0)^n}{n!} f^{(n)}(c)$$

case 1: n is even, $f^{(n)}(x_0) < 0$

There exists a neigbourhood δ such that $y \in (x_0-\delta, x_0+\delta) \Rightarrow f^{(n)}(y) < 0$.

Now choose x from $(x_0-\delta, x_0+\delta)$. So, $f^{(n)}(c) < 0$.

$f(x) \leq f(x_0)$ $\forall$ $x \in (x_0-\delta, x_0+\delta)$

$\Rightarrow$ Relative maximum at x_0.

case 3: n is odd, $f^{(n)}(x_0) \neq 0$

Assume $f^{(n)}(x_0) < 0$, there exists a nbhd $(x_0-\delta, x_0+\delta)$ such that $y \in (c_0-\delta, c_0+\delta)$ $\Rightarrow f^{(n)}(y) < 0$. Choose x from $(x_0-\delta, x_0+\delta)$. Then $c \in (x_0-\delta, x_0+\delta)$. Then $f^{(n)}(c) < 0$,

so, $f(x) = f(x_0) + \frac{f^{(n)}(c)(x-x_0)^n}{n!}$

$x > x_0 \Rightarrow f(x) \geq f(x_0)$

$x < x_0 \Rightarrow f(x) \leq f(x_0)$

Exm: determine whether or not $x=0$ is rel maximum or minimum if:

(a). $f(x) = \sin x + \frac{1}{6}x^3$

(b). $f(x) = \cos x - 1 + \frac{1}{2}x^2$

Sol:

$\cancel{f'(x) = \sin x + \frac{x^2}{2}}$

$\cancel{f'(0) = 0}$

$f''\cancel{(x) = \cos x +} x$

(a). $f'(x) = \cos x + \frac{x^2}{2}$

$f'(0) = 1 \neq 0$

f has no relative maximum at $x=0$.

(b). $f(x) = \cos x - 1 + \frac{1}{2}x^2$

$f'(x) = -\sin x + x \qquad f'(0) = 0$

$f''(x) = 1 - \cos x \qquad f''(0) = 0$

$f'''(x) = \sin x \qquad f^{(3)}(0) = 0$

$f^{(4)}(x) = \cos x \qquad f^{(4)}(0) = 1 > 0$.

f has relative ~~maximum~~ minimum at $x=0$.

proof of Taylor's Theorem:

Let $x_0 < x$, consider the interval $[x_0, x]$

Let $F(t) = f(x) - f(t) - (x-t)f'(t) - \frac{(x-t)^2}{2!}f''(t) - \dots - \frac{(x-t)^n}{n!}f^{(n)}(t)$

Let $G(t) := F(t) - \frac{(x-t)^{n+1}}{(x-x_0)^{n+1}} F(x_0)$

$G(x_0) = 0 \qquad G(x) = 0$

using rolles theorem $\exists\ c \in [x_0, x]$ $G'(c) = 0$

$G'(t) = F'(t) + \frac{(n+1)(x-t)^n}{(x-x_0)^{n+1}} F(x_0)$

$G'(c) = 0 \Rightarrow F'(c) + \frac{(n+1)(x-c)^n}{(x-x_0)^{n+1}} F(x_0) = 0$

$F'(t) = -f'(t) + f'(t) - (x-t)f''(t) + \frac{(x-t)^2}{2!} f'''(t) - \dots$

$$G'(t) = -\frac{(x-t)^n}{n!} f^{(n)}(t) + (n+1)\frac{(x-t)^n}{(x-x_0)^{n+1}} F(x_0)$$

$\exists c \in (x, x_0)$ st $\quad G'(c) = 0$

$$\Rightarrow \quad -\frac{(x-c)^n}{n!} f^{(n+1)}(c) + (n+1)\frac{(x-c)^n}{(x-x_0)^{n+1}} F(x_0) = 0$$

$\exists c$ st $\quad \frac{f^{(n+1)}(c)}{(n+1)!} (x-x_0)^{n+1} = F(x_0)$

$$f(x) - f(x_0) - (x-x_0)f'(x_0) - \dots - \frac{(x-x_0)^n}{n!} f^{(n)}(x_0) = \frac{f^{(n+1)}(c)\,(x-x_0)^{n+1}}{(n+1)!}$$

$\Rightarrow \exists c \in (x, x_0)$ such that,

$$f(x) = f(x_0) + (x-x_0)f'(x_0) + \dots + \frac{(x-x_0)^n}{n!} f^{(n)}(x_0) + \frac{(x-x_0)^{n+1} f^{(n+1)}(c)}{(n+1)!}$$

$$\Rightarrow P_n(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f^{(2)}(x_0)(x-x_0)^2}{2!} + \dots + \frac{f^{(n)}(x_0)(x-x_0)^n}{n!}$$

Remainder $= \frac{f^{(n+1)}(x_0)(x-x_0)^{n+1}}{(n+1)!}$

Called as Taylor Polynomial.

$$P_n'(x) = f'(x_0) + f^{(2)}(x_0)(x-x_0) + \frac{f^{(3)}(x_0)(x-x_0)^3}{2!} + \dots$$

$P_n'(x_0) = f'(x)$

$P_n^{(k)}(x_0) = f^{(k)}(x_0) \qquad k = 1, 2, \dots, n$

Applications of Taylor's theorem:

1) approximation e with error $< 10^{-5}$

Sol: $f(x) = e^x \qquad x = 1 \quad x_0 = 0$

$$P_n(x) = f(x_0) + (x-x_0)f'(x_0) + \frac{(x-x_0)^2}{2!}f''(x_0) + \cdots\cdots$$

$$P_n(1) = f(0) + f'(0) + \frac{1}{2}f''(0) + \cdots\cdots + \frac{1}{n!}f^{(n)}(0) + \frac{1}{(n+1)!}f^{(n+1)}(c)$$

$$\left|\frac{f^{(n+1)}(c)}{(n+1)!}\right| < 10^{-5} \Rightarrow \left|\frac{e^c}{(n+1)!}\right| < 10^{-5}$$

$$\Rightarrow (n+1)! > 3,00,000 \Rightarrow \left|\frac{e^c}{(n+1)!}\right| < 10^{-5}$$

$\forall\ c \in (0,1)$

$\Rightarrow n+1 \geq 9$

$\Rightarrow n \geq 8$

$$e \approx P_8(1) = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{8!} = 2.71828$$

2). Show that if $x > 0$, then

$$1 + \frac{1}{2}x - \frac{1}{8}x^2 \leq \sqrt{1+x} \leq 1 + \frac{1}{2}x$$

pf:

$f(x) = \sqrt{1+x} = (1+x)^{1/2} \qquad f(0) = 1$

$f'(x) = \frac{1}{2}(1+x)^{-1/2} \qquad f'(0) = 1/2$

$f''(x) = \frac{-1}{4}(1+x)^{-3/2} \qquad f''(0) = -1/4$

$f^{(3)}(x) = \frac{3}{8}(1+x)^{-5/2} \qquad f^{(3)}(0) = 3/8$

$$f(x) = P_2(x) + R_2(x)$$

$$R_2(x) = \cancel{f(0)}\ \frac{f^{(3)}(c)(x-x_0)^3}{3!} = \frac{x^3 f^{(3)}(c)}{3!} \geq 0$$

$$f(x) = 1 + \frac{x}{2} - \frac{x^2}{8} + R_2(x) \Rightarrow f(x) \geq 1 + \frac{x}{2} - \frac{x^2}{8}$$

$$f(x) = P_1(x) + R_1(x)$$

$$P_1(x) = 1 + \frac{x}{2}$$

$$R_1(x) = \frac{+x^2 f^{(2)}(c)}{2!} \leq 0$$

$$\therefore \quad f(x) \leq 1 + \frac{x}{2}$$

$$\therefore \quad 1 + \frac{x}{2} - \frac{x^2}{8} \leq \sqrt{1+x} \leq 1 + \frac{x}{2}$$

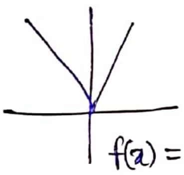
Evaluate $\sqrt{1.2}$.

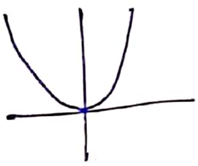
$$1 + \frac{0.2}{2} - \frac{(0.04)}{8} \leq \sqrt{1.2} \leq 1 + \frac{0.2}{2}$$

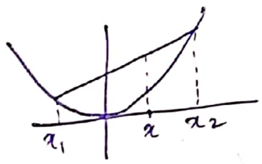
$$\Rightarrow 1.095 \leq \sqrt{1.2} \leq 1.10$$

3). $\left| \sin x - \left(x - \frac{x^3}{6} + \frac{x^5}{120} \right) \right| < \frac{1}{5040}$ for $|x| < 1$.

Convex Functions:







$(1-t) f(x_1) + t f(x_2) \geq f(x)$ where, $x = (1-t)x_1 + t x_2$

$0 \leq t \leq 1 \quad \forall \, x_1, x_2 \in I$

Thm: Let I be an interval and $f: I \to \mathbb{R}$ have a 2nd derivative on I. Then f is convex if and only if $f''(x) \geq 0$.

proof: Assume $f''(x) \geq 0$

$$f(x_1) = f(x_0) + f'(x_0)(x_1 - x_0) + \frac{f''(c)(x_1 - x_0)^2}{2!} \qquad c \in (x_0, x_1)$$

$$\Rightarrow f(x_1) \geq f(x_0) + f'(x_0)(x_1 - x_0)$$

$$f(x_2) \geq f(x_0) + f'(x_0)(x_2 - x_0)$$

$(1-t)f(x_1)+tf(x_2) \geq f(x_0) + f'(x_0)((1-t)x_1+tx_2-x_0)$

$\geq f(x_0)$

Assume $f''(c)<0$ for some $c \in [x_1,x_2]$

$\exists$ a nbhd δ such that $f''(x)<0 \quad \forall x \in (c-\delta, c+\delta)$

$$f(c+\delta) = f(c) + \delta f'(c) + \frac{\delta^2}{2!} f''(c_0) \qquad c_0 \in (c-\delta, c)$$

$$\Rightarrow f(c+\delta) < f(c) + \delta f'(c)$$

$$f(c-\delta) < f(c) - \delta f'(c)$$

$$f(c+\delta) + f(c-\delta) < 2f(c)$$

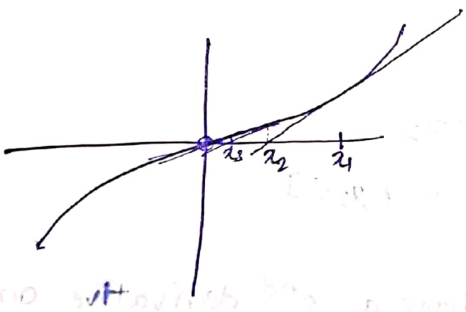
contradiction.

method: $f''(x) = \lim_{h \to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2}$

$$x = \frac{x+h}{2} + \frac{x-h}{2}$$

$$f(x) \leq \frac{1}{2}f(x+h) + \frac{1}{2}f(x-h)$$

Newton's Method:



Depends on how we choose the x_1.

Tangent at $(x_1, f(x_1))$ is

$$\frac{y-f(x_1)}{x-x_1} = f'(x_1)$$

$$y=0 \Rightarrow x = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Let r be zero of $f(x)$

Let $f(a)f(b)<0$

using intermediate value theorem $\exists r \in (a,b)$ st $f(r)=0$

Take $x_1 \in I$

Take Taylor expansion of r wrt to x_1.

$$f(r) = f(x_1) + f'(x_1)(r-x_1) + \frac{f''(c_1)(r-x_1)^2}{2!}$$

c_1 between r and x_1.

$$f(r)=0 \Rightarrow -f(x_1) = f'(x_1)(r-x_1) + \frac{f''(c_1)(r-x_1)^2}{2!}$$

choose $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$$|x_2 - r| = \left|x_1 - \frac{f(x_1)}{f'(x_1)} - r\right| = \left|x_1 - r + r - x_1 + \frac{f''(c_1)(r-x_1)^2}{2f'(x_1)}\right|$$

$$= \frac{f''(c_1)(r-x_1)^2}{2f'(x_1)}$$

Take $f'(x_1) \geq m > 0$ $\quad f''(c_1) \leq M$

$m = \inf\{f'(x) \mid x \in I\}$ $\quad M = \sup\{f''(x)\}$

$$x_2 - r \leq \frac{M}{2m}(x_1 - r)^2 = K(x_1 - r)^2$$

choose δ st $\delta < \frac{1}{K}$, $\delta K < 1$

choose x_1 from $(r-\delta, r+\delta)$ $\quad x_2 - r \leq K\delta^2 = \delta\delta K < \delta$

$$|x_{n+1} - r| \leq K|x_n - r|^2 \leq K\delta^2 \leq \delta$$

$\Rightarrow x_n \longrightarrow r$

Thm: Let $f: I \to \mathbb{R}$ $I = [a,b]$ be twice differentiable. $f(a)f(b) < 0$ and there are constants m and M such that

$|f'(x)| \geq m > 0$ $|f''(x)| \leq M$ $\forall x \in I$ Let $K = \frac{M}{2m}$. Then,

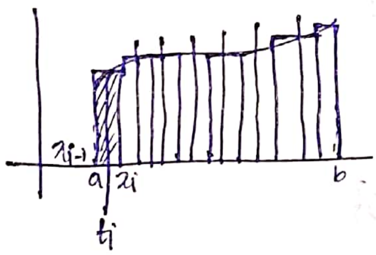
$\exists$ a subinterval $I^* (= [r-\delta, r+\delta])$ countaining a zero r of f st for any $x_1 \in I^*$ the sequence defined by $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ belongs to I^* and (x_n) converges to r. Moreover,

$$|x_{n+1} - r| \leq K |x_n - r|^2$$

Integration

Riemann Integration:

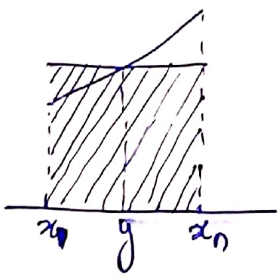
Riemann sum = $\sum f(t_i)(x_i - x_{i-1})$



sum of areas of all rectangles $\sim$ area under the graph

Mathematical analysis by Apostol.

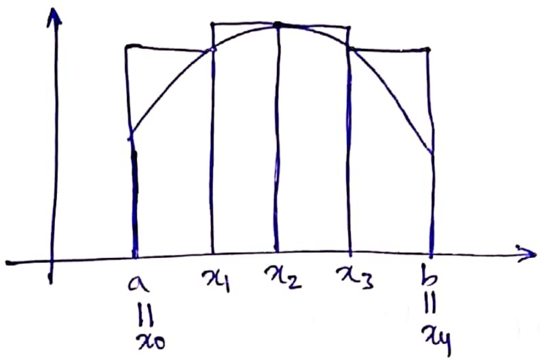
partition: Let $I = [a,b]$ be a closed and bounded interval, let P be a partition on $[a,b]$ which is a finite ordered set $(a = x_0 < x_1 < \dots x_n = b)$ of points of $[a,b]$. P is used to divide $[a,b]$ into non-overlapping.



we to set the $\max\{x_1 - x_0, x_2 - x_1, \ldots, x_n - x_{n-1}\}$ to getter better approximation.

Norm of $P = \|P\| = \max\{x_1 - x_0, x_2 - x_1, \ldots, x_n - x_{n-1}\}$

if $\|P\| \to 0$ we say, sum as the integration.



Let $P = \{a = x_0 < x_1 < \ldots < x_n = b\}$ be a partition of $[a,b]$

$M_k(f) = \max\{f(x) : x \in [x_{k-1}, x_k]\}$

$\sum M_k(f)(x_k - x_{k-1}) \longrightarrow$ upper riemann sum. for the partition P $\quad U(P,f)$

$m_k(f) = \min\{f(x) : x \in [x_{k-1}, x_k]\} \rightarrow$

$\sum m_k(f)(x_k - x_{k-1}) \longrightarrow$ lower riemann sum for the partition P. $\quad L(P,f)$

$L(P,f) \leq$ Actual area $\leq U(P,f)$

$\underline{\int_a^b} f(x)\,dx := \sup\{L(P,f) : P \text{ is partition of } [a,b]\}$

$\overline{\int_a^b} f(x)\,dx = \inf\{U(P,f) : P \text{ is partition of } [a,b]\}$

$\underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx$

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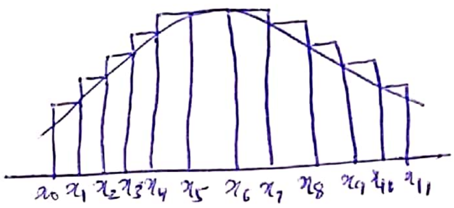
If $\int_{\underline{a}}^{b} f(x)\,dx = \int_{a}^{\overline{b}} f(x)\,dx$ then we say that f is riemann integrable and integration is $\int_{\underline{a}}^{b} f(x)\,dx = \int_{a}^{\overline{b}} f(x)\,dx$ denoted by $\int_a^b f(x)\,dx$.

upper riemann sum:

Trying to integrate f on $[a,b]$ Partition of $[a,b]$ which is a finite ordered set $a = x_0 < x_1 < \dots < x_n = b$



Base is $[x_0, x_1]$ and Height is $\sup f$ on $[x_0, x_1]$

$$U(P,f) = \sum_{k=1}^{n} M_k(f)(x_k - x_{k-1})$$

$$L(P,f) = \sum_{k=1}^{n} m_k(f)(x_k - x_{k-1})$$

$$m_k(f) = \inf\{ f \mid x \in [x_{k-1}, x_k] \}$$

$$M_k(f) = \sup\{ f \mid x \in [x_{k-1}, x_k] \}$$

$L(P,f) \leq$ area under the graph $\leq U(P,f)$

Definition:

$$\overline{I} := \int_a^{\overline{b}} f(x)\,dx = \inf\{ U(P,f) \mid P \text{ is a partition} \}$$

upper integral on f

$$\underline{I} := \int_{\underline{a}}^{b} f(x)\,dx = \sup\{ L(P,f) \mid P \text{ is a partition} \}$$

Lower integral on f

To prove, $\underline{I} \le \bar{I}$

For this we have to prove, $L(P_1,f) \le U(P_2,f)$
for any two partitions P_1 & P_2.

$L(P_1,f) \le L(P_3,f) \le U(P_3,f) \le U(P_2,f)$
for some cleverly chosen partition P_3.

Take $P_3 = P_1 \cup P_2$

Refinement of a partition:

P is a partition on $[a,b]$. If $P' \supseteq P$, these P' is a finer ~~than~~ of P or a refinement of P.

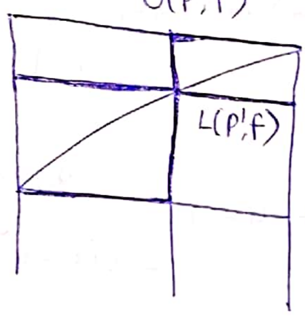
Theorem:

If P' is finer than P then,

(i). $U(P',f) \le U(P,f)$

(ii). $L(P',f) \ge L(P,f)$

(iii)* $L(P_1,f) \le U(P_2,f)$



(iii)* Take $P_3 = P_1 \cup P_2$

$L(P_1,f) \le L(P_3,f) \le U(P_3,f) \le U(P_2,f)$

(i). It's enough to prove for $P' = P \cup \{y\}$
(ii)

$y \in (x_{k-1}, x_k)$

$$U(P,f) = \sum_{k=1}^{n} M_k(x_k - x_{k-1})$$

$$= \sum_{\substack{k=1 \\ k \ne i}}^{n} M_k(f)(x_k - x_{k-1}) + M_i(f)(x_i - x_{i-1})$$

$$\ge \sum_{\substack{k=1 \\ k \ne i}}^{n} M_k(f)(x_k - x_{k-1}) + M_i'(f)(y - x_{i-1}) + M_i''(f)(x_i - y) = U(P',f)$$

where, $M_i'(f) = \sup_{[x_{i-1}, y]} f$

$M_i''(f) = \sup_{[y, x_i]} f$

$$\underline{I} \le \overline{I} = \inf\{U(P,f)\}$$

$\underline{I} = \sup\{L(P,f)\}$

We say f is riemann integrable if $\underline{I} = \overline{I}$ and riemann integral of f on $[a,b]$ is $I = \underline{I} = \overline{I}$. This is denoted by $\int_a^b f(x)dx$.

Example:

(1). A constant function $f(x) = c \quad \forall x \in [a,b]$ is Riemann Integrable.

proof: Take any partition P of $[a,b]$

$$a = x_0 \le a < x_1 < x_2 \dots < x_n = b.$$

$$U(P,f) = \sum_{k=1}^{n} M_k(f)(x_k - x_{k-1})$$

$$M_k(f) = \sup\{f(x) \mid x \in [x_{k-1}, x_k]\}$$

$$U(P,f) = \sum_{k=1}^{n} c(x_k - x_{k-1}) = c(x_n - x_0) = c(b-a)$$

$$L(P,f) = c(b-a)$$

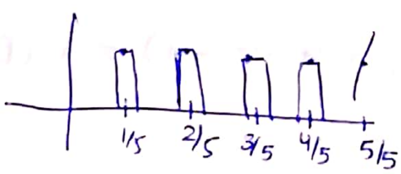
$\overline{I} = c(b-a) = \underline{I}$ $\therefore f$ is riemann integrable $I = c(b-a)$

(2). $f : [0,1] \to R$

$$f(x) := \begin{cases} 0; & \text{otherwise} \\ 1, & x = 1/5, 2/5, 3/5, 4/5, \cancel{5/5} \end{cases}$$

$L(P,f) = 0$ for any partition P

$U(P,f) \ge 0$



choose sub intervals around $1/5, 2/5, 3/5, 4/5$ of length ϵ.

$$0 = x_0 < x_1 = \left(\tfrac{1}{5} - \epsilon\right) < x_2\left(= \tfrac{1}{5} + \epsilon\right) < x_3\left(= \tfrac{2}{5} - \epsilon\right) < x_4\left(\tfrac{2}{5} + \epsilon\right) < x_5\left(= \tfrac{3}{5} - \epsilon\right)$$
$$< x_8\left(\tfrac{4}{5} + \epsilon\right) < x_9 = 1.$$

$$U(P,f) = 0(x_1 - 0) + \tfrac{1}{\cancel{5}}(2\epsilon) + \tfrac{2}{\cancel{5}} 2\epsilon + \tfrac{3}{\cancel{5}} 2\epsilon + \tfrac{4}{\cancel{5}} 2\epsilon + 0(1 - x_8)$$
$$= 8\epsilon$$

$\overline{I} = \inf(U(P,f))$ P is a partition
$= 0.$

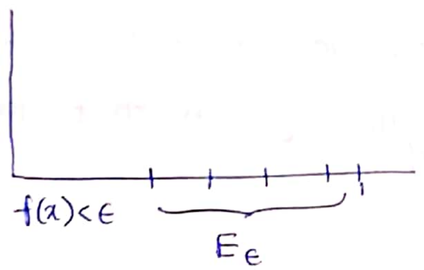
$\Rightarrow f$ is Riemann integrable and $\int_0^1 f(x)dx = 0.$

(iii). $f(x) = \frac{1}{n}$ if $x = \frac{1}{n}$
$= 0$ if $x \neq \frac{1}{n}$ $\quad f:[0,1] \to \mathbb{R}.$

$L(P,f) = 0$ for any partition P
$U(P,f) \geq 0$ for any partition P.

Let $E_\epsilon = \{x \in [0,1] \mid f(x) \geq \epsilon\}$
E_ϵ contains finite number of elements.

Let $n_\epsilon \# E_\epsilon$
let $\delta_\epsilon = \frac{\epsilon}{2n_\epsilon}$ let $N \in \mathbb{N}$ st $\frac{1}{N} < \delta_\epsilon$



Take partition $P = \left\{0, \frac{1}{N}, \frac{2}{N}, \ldots \frac{N-1}{N}, 1\right\}$

$\|P\| = \frac{1}{N} < \delta_\epsilon$

$U(P,f)$ Let $P = P_1 \cup P_2$ where, $P_1 \subset A$ and $P_2 \cap A = \varphi$

$$= \sum_{x_k \in P_1} M_k(f)(x_k - x_{k-1}) + \sum_{x_k' \in P_2} M_{k'}(f)(x_k' - x_{k'-1}) < \epsilon \sum(x_k - x_{k-1}) + \sum_{n_\epsilon}(x_k' - x_{k'-1})$$
$$= \epsilon + \frac{n_\epsilon}{N} < \epsilon + n_\epsilon \delta_\epsilon$$
$$= \epsilon + n_\epsilon \frac{\epsilon}{2\epsilon} = \frac{3\epsilon}{2}$$

$\overline{I} = \underline{I} = 0$

and $I = 0$.

$\therefore f$ is Riemann integrable ~~at x=0~~

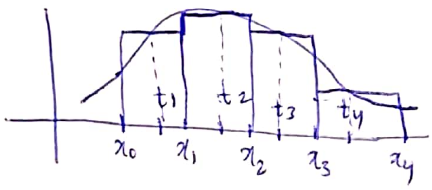
(4). $f(x) = \begin{cases} 1; & x \text{ is rational} \\ 0; & x \text{ is irrational} \end{cases}$

$x \in [0,1]$

$$U(P,f) = \sum_{k=1}^{n} M_k(f)(x_k - x_{k-1}) = 1$$

$$L(P,f) = \sum_{k=1}^{n} m_k(f)(x_k - x_{k-1}) = 0$$

Riemann Sum:



Let P be a partition of $[a,b]$

choose $t_k \in [x_{k-1}, x_k]$

$\hookrightarrow$ Tag of $[x_{k-1}, x_k]$

$\dot{P} = \{ [x_{k-1}, x_k], t_k \} \longrightarrow$ a set of ordered pairs of subintervals and tags.

Riemann sum of a function $[a,b]$ corresponding to a tagged partition $\dot{P}$ is $\sum_{k=1}^{n} f(t_k)(x_k - x_{k-1}) := S(\dot{P}, f)$

A function f on $[a,b]$ is Riemann integrable if $\exists$ a no L and for given $\epsilon > 0$ there exists a partition P_ϵ such that for any refinement P' of P_ϵ, and any tag of P' $|S(\dot{P}, f) - L| < \epsilon$

L is called Riemann integral.

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Theorem:

If f is riemann integrable on $[a,b]$ then the riemann integral of f is unique.

proof: Let L and L' be two riemann integrals. For any given $\epsilon>0$, $\exists P_\epsilon$, such that for any $\epsilon>0$, $\exists P_\epsilon$, such that for any tag on P

$$|S(\dot{P},f)-L| < \epsilon/2$$

For given $\epsilon>0$, $\exists P_\epsilon'$ st for any refinement P' of P_ϵ' and any tag on P', $|S(\dot{P}',f)-L'| < \epsilon/2$

Let $P_\epsilon'' = P_\epsilon \cup P_\epsilon'$. Here, P_ϵ'' is a refinement of both P_ϵ and P_ϵ'.

so, for any partition P'', refinement of P_ϵ'' and any tag on P'';

$|S(P'',f)-L| < \epsilon/2$

$|S(P'',f)-L'| < \epsilon/2$

so, $|L-L'| < \epsilon/2$ sufficient condition:

If for a ~~set~~ no L and for given $\epsilon>0$, $\exists\delta$ such that for any partition P with $\|P\|<\delta$ and any tag on P, $|S(\dot{P},f)-L|<\epsilon$, then f is riemann integrable on $[a,b]$ and L is riemann integrable.

Ex:

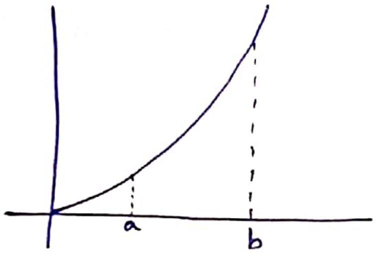
1). Find riemann integral of $f(x) = x^2,\ x \in [a,b]$

Sol:

Enough to prove

~~∃L~~ f on $[a,b]$ is riemann integrable

if $\exists L$ for given $\epsilon > 0, \exists \delta > 0$ such that for any partition such that $\|P\| < \delta$ & any tag on P $(\dot{P})$ $|S(\dot{P}, f) - L| < \epsilon$.



Take any partition with, norm $< \delta$

call as P_ϵ.

$$S(f, \dot{P}) = \sum_{i=1}^{n} t_i^2 (x_i - x_{i-1})$$

$$t_i = \frac{x_i x_{i-1} + x_i^2 + x_{i-1}^2}{3}$$

$$= \sum \left(\frac{x_i^2 + x_i x_{i-1} + x_{i-1}^2}{3} \right) (x_i - x_{i-1})$$

$$= \frac{1}{3} \sum_{i=1}^{n} x_i^3 - x_{i-1}^3 = \frac{1}{3}(b^3 - a^3)$$

show, $\dfrac{x_i^2 + x_i x_{i-1} + x_{i-1}^2}{3} > 0$

$$t_i^2 = \frac{1}{3}(x_i^2 + x_i x_{i-1} + x_{i-1}^2)$$

$$\Rightarrow x_{i-1}^2 < t_i^2 < x_i^2 \Rightarrow x_{i-1} < t_i < x_i$$

Let Q be the same partition P with arbitary tag (Q).

$$|S(f, \dot{P}) - S(f, \dot{Q})| = \left| \sum_{i=1}^{n} (q_i^2 - t_i^2)(x_i - x_{i-1}) \right|$$

$$\to (q_i - t_i)(q_i + t_i)(x_i - x_{i-1})$$

$$\leq \sum_{i=1}^{n} 2b(\delta)(x_i - x_{i-1})$$

Take $\delta = \dfrac{\epsilon}{2b(b-a)}$ $\quad \|P\| < \delta \Rightarrow |x_i - x_{i-1}| < \delta$

$$\Rightarrow |S(f,\dot{P}) - S(f,\dot{Q})| < \frac{2b\epsilon}{2b(b-a)} \sum_{i=1}^{n} (x_i - x_{i-1})$$

$$< \underline{\underline{\epsilon}}$$

$$\Rightarrow \left|S(f,\dot{Q}) - \frac{1}{3}(b^3 - a^3)\right| < \epsilon$$

<u>Riemann's condition:</u>

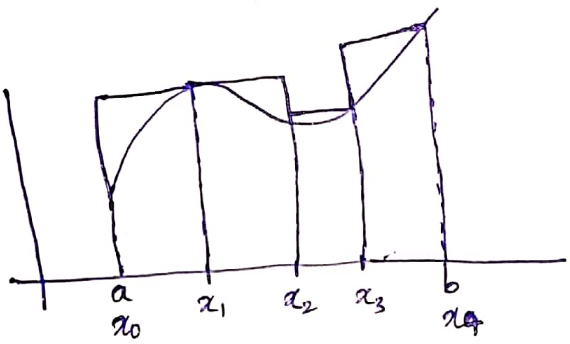
Take P be a partition,

$$M_K(f) = \sup_{x \in [x_{K-1}, x_K]} f(x)$$

$$m_K(f) = \inf_{x \in [x_{K-1}, x_K]} f(x)$$

$$U(P,f) = \sum_{K=1}^{n} M_K(f)(x_K - x_{K-1})$$ upper sum

$$L(P,f) = \sum_{K=1}^{n} m_K(f)(x_K - x_{K-1})$$ lower sum



$$\bar{I} := \inf_P U(P,f) \qquad \bar{\int_a^b} f$$ upper integral

$$\underline{I} := \sup_P L(P,f) = \underline{\int_a^b} f$$ lower integral

It can be proved $\underline{I} \leq \bar{I}$.

f is riemann integrable if $\underline{I} = \bar{I}$.

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f is said to satisfying rieman's condition if for given $\epsilon > 0$ $\exists$ a partition P_ϵ such that for any partition finer than P_ϵ,

$$0 \leq U(P,f) - L(P,f) < \epsilon$$

The following are equivalent:

(1). f is integrable means riemann's sums converge.

(2). $\overline{I} = \underline{I}$

(iii). f satisfies riemann's condition.

Proof:

(i) $\Rightarrow$ (iii) for any partition and tag of P.

$|S(\dot{P},f) - L| < \epsilon/2$

If we take partition Q finer than P_ϵ and any tag Q,

$|S(\dot{Q},f) - L| < \epsilon/2$

$\Rightarrow |S(\dot{P},f) - S(\dot{Q},f)| < \epsilon$

$\Rightarrow \left|\sum_{i=1}^{n} |f(t_i) - f(q_i)|(x_i - x_{i-1})\right| < \epsilon$

$U(P,f) - L(P,f) < \epsilon$

$\Leftrightarrow \sum_{i=1}^{n} (M_i(f) - m_i(f))(x_i - x_{i-1}) < \epsilon$

$\sup_{t_i, q_i \in [x_{i-1}, x_i]} |f(t_i) - f(q_i)| = M_i(f) - m_i(f)$

$\Rightarrow$ For given h, $\exists t_i', q_i'$ st

$|f(t_i') - f(q_i')| > M_i(f) - m_i(f) - h$

$$U(P,f) - L(P,f) = \sum_{i=1}^{n} (M_i(f) - m_i(f))(x_i - x_{i-1})$$

$$< \sum_{i=1}^{n} |f(t_i') - f(q_i')|(x_i - x_{i-1}) + h \sum_{i=1}^{n} (x_i - x_{i-1})$$

$< \epsilon + h(b-a)$ choose, $h = \frac{\epsilon}{b-a}$

$< 2\epsilon$

(iii) $\Rightarrow$ (ii)

$\overline{I}(f) - \varepsilon < U(P_1, f)$ for some partition P_1

$\Rightarrow \overline{I}(f) < U(P_1, f) + \varepsilon$

$U(P_2, f) < L(P_2, f) + \varepsilon$, for a partition P_2 finer than P_ε

(by riemann condition)

$L(P_3, f) \leq \underline{I}(f) + \varepsilon$

(as $\underline{I}(f) = \sup\{L(P,f)\}$)

$P_4 = P_1 \cup P_2 \cup P_3$

$U(P_4, f) \leq U(P_1, f)$ (Because, refinement)

$\overline{I}(f) \leq U(P_1, f) < L(P_1, f) + \epsilon \leq \underline{I}(f) + \epsilon$

$\Rightarrow \overline{I}(f) \leq \underline{I}(f)$

$\Rightarrow \overline{I}(f) = \underline{I}(f)$

(ii) $\Rightarrow$ (i) $\quad L = \overline{I} = \underline{I}$

$\overline{I} = \inf\{U(P,f)\}$

for $\epsilon > 0$, $\exists$ a partition P_1 st,

$U(P_1, f) < \overline{I} + \epsilon$

$\underline{I} = \sup\{L(P,f)\}$

For $\epsilon > 0$, $\exists$ a partition P_2 st

$\underline{I} - \epsilon < L(P_2, f)$

$P_\varepsilon = P_1 \cup P_2$

$U(P_\epsilon, f) \leq U(P_1, f) < \overline{I} + \epsilon$

$L(P_\epsilon, f) \geq L(P_2, f) > \underline{I} - \epsilon$

P finer than P_ϵ, then, $U(P, f) \leq U(P_\epsilon, f) < \overline{I} + \varepsilon$

$L(P, f) \geq L(P_\varepsilon, f) > \underline{I} - \varepsilon$

$\underline{I} - \varepsilon < L(P, f) \leq \underbrace{J}_{S(P,f)} \leq U(P, f) \leq \overline{I} + \varepsilon$

for any tag on $S(P,f)$

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$\Rightarrow I-\varepsilon < S(\dot{P},f) < I+\varepsilon$

$\Rightarrow |S(\dot{P},f)-I| < \varepsilon$

$\Rightarrow f$ is riemann integrable

Ex: $f:[0,1]\to\mathbb{R}$

$f(x) = 0$ if x is rational

$= \frac{1}{x}$ if x is irrational

$\exists$ a sequence $\dot{P}_n$ of tagged partitions with $\|P_n\|\to 0$ and $S(\dot{P}_n,f)$ converges.

Sol: $P_n = \{0,\frac{1}{n},\ldots,\frac{n-1}{n},1\}$

$\|P_n\| = \frac{1}{n} \to 0.$

choose $t_i \in \left(\frac{i-1}{n},\frac{i}{n}\right)$ to be any rational.

$S(\dot{P}_n,f) = \sum f(t_i)\left(\frac{i}{n}-\frac{i-1}{n}\right) = 0$

~~Thm~~ $f\in R[a,b] \Rightarrow f:[a,b]\to\mathbb{R}$ is riemann integrable

~~proof~~: Thm: $f\in R[a,b] \Rightarrow f$ is bounded.

proof: choose $\varepsilon = 1$

Let $\int_a^b f = L$. Then $\exists$ a partition P_ε such that,

$|S(f;\dot{P}_\varepsilon)-L| < 1 \Rightarrow |S(f,\dot{P}_\varepsilon)| < |L|+1$

$P_\varepsilon = \{x_0,x_1,\ldots,x_n\}$

subintervals: $[x_0,x_1],[x_1,x_2],\ldots\ldots[x_{n-1},x_n]$

~~we want to show that, $|S(f;\dot{P}_\varepsilon)-L|>1$~~

~~assuming f is unbounded.~~

~~or, $|S(f;\dot{P}_\varepsilon)| \geq |L|+1$~~

For that we want,

$$|f(t_k)| > \frac{|L|+1 - \sum_{i\neq k} f(t_i)(x_i-x_{i-1})}{(x_i-x_{i-1})}$$

$$|S(f;\dot{P}_\varepsilon)| = |f(t_k)(x_k-x_{k-1})| + \left|\sum_{i\neq k} f(t_i)(x_i-x_{i-1})\right| \geq 1+|L|$$

f is unbounded $\Rightarrow$ at one subinterval is unbdd.

fix $t_i, i\neq k$ Then $\sum_{i\neq k} f(t_i)(x_i-x_{i-1})$ is fixed.

<u>Thm:</u> suppose $f,g \in R[a,b]$ and $k\in\mathbb{R}$

(i). $f+g \in R[a,b]$

(ii). $kf \in R[a,b]$

(iii). $f(x)\leq g(x)\ \forall x\in[a,b] \Rightarrow \int_a^b f \leq \int_a^b g$

<u>proof:</u>

(3). For any given $\varepsilon>0$, $\exists P_\varepsilon$ such that any partition P finer than $P_\varepsilon \Rightarrow$

$$|S(f,\dot{P}) - \int_a^b f| < \varepsilon$$

$$\Rightarrow \int_a^b f - \varepsilon < S(f,\dot{P}) < \varepsilon + \int_a^b f$$

$\exists P_\varepsilon'$ such that,

$$\int_a^b g - \varepsilon < S(g,\dot{P}) < \int_a^b g + \varepsilon$$

choose $P_\varepsilon'' = P_\varepsilon \cup P_\varepsilon'$

P_ε'' satisfies A & B.

$$\int_a^b f - \varepsilon < S(f;P_\varepsilon'') \leq S(g;P_\varepsilon'') < \int_a^b g + \varepsilon$$

True for any ε, $\Rightarrow \int_a^b f \leq \int_a^b g$

Squeeze theorem:

$f \in R[a,b] \Leftrightarrow$ For every $\varepsilon>0$, $\exists$ functions $\alpha_\varepsilon, \omega_\varepsilon \in R[a,b]$ such that $\alpha_\varepsilon \le f \le \omega_\varepsilon$ and $\int_a^b (\omega_\varepsilon - \alpha_\varepsilon) < \varepsilon$

proof ($\Rightarrow$) $f \in R[a,b]$ Then

$$\alpha_\varepsilon = f = \omega_\varepsilon$$

($\Leftarrow$) $\alpha_\varepsilon \in R[a,b]$

for given $\varepsilon>0$, $\exists P_\varepsilon'$ st P is finer than P_ε' and any tag of P

$$\Rightarrow \left|S(\alpha_\varepsilon; \dot{P}) - \int_a^b \alpha_\varepsilon\right| < \varepsilon$$

$\exists P_\varepsilon''$ st P is finer than P_ε'' and any tag on P,

$$\Rightarrow \left|S(\omega_\varepsilon; \dot{P}) - \int_a^b \omega_\varepsilon\right| < \varepsilon$$

Taking $P_\varepsilon = P_\varepsilon' \cup P_\varepsilon''$

if P is finer than P_ε and any tag on P

$$\Rightarrow \left|S(\alpha_\varepsilon, \dot{P}) - \int_a^b \alpha_\varepsilon\right| < \varepsilon \quad \& \quad \left|S(\omega_\varepsilon; \dot{P}) - \int_a^b \omega_\varepsilon\right| < \varepsilon$$

$$\int_a^b \alpha_\varepsilon - \varepsilon < S(\alpha_\varepsilon; \dot{P}) \le S(f; \dot{P}) \le S(\omega_\varepsilon; \dot{P}) < \int_a^b \omega_\varepsilon + \varepsilon$$

If Q is another partition finer than P_ε and any tag on P_ε,

$$-\varepsilon + \int_a^b \alpha_\varepsilon < S(f; \dot{Q}) < \int_a^b \omega_\varepsilon + \varepsilon$$

$$|S(f, \dot{P}) - S(f; \dot{Q})| \le \int_a^b \omega_\varepsilon + \varepsilon - \left(\int_a^b \alpha_\varepsilon - \varepsilon\right)$$

$$\le \int_a^b (\omega_\varepsilon - \alpha_\varepsilon) + 2\varepsilon < 3\varepsilon$$

Cauchy criterion: (C1)

$f \in R[a,b]$ if and only if for given $\varepsilon>0, \exists$ a partition P_ε s.t. for any two partitions P and Q finer than P_ε and any tag P and Q

$$|S(f;\dot{P}) - S(f,\dot{Q})| < \varepsilon$$

another cauchy criterion: (C2)

For given $\varepsilon > 0$, $\exists\, \eta_\varepsilon > 0$ for a partition P, Q with $\|P\| < \eta_\varepsilon$, $\|Q\| < \eta_\varepsilon$ and any tag on P and Q,

$$|S(f,\dot{P}) - S(f,\dot{Q})| < \epsilon \qquad (C2) \Rightarrow (C1)$$

Theorem:

$f \in R[a,b]$ if and only if f satisfies cauchy criterion (C1)

Ex: $f: [0,3] \to \mathbb{R}$

$f(x) = 1$, if $x \in [0,1]$

$f(x) = 2$, if $x \in [1,3]$

Sol: Take a tagged partition P.

Let $\dot{P}_1$ be union of all sub intervals for which corresponding tag $\in [0,1]$

let $\|P\| < \delta$

$$[0, 1-\delta] \subset \dot{P}_1 \subset [0, 1+\delta]$$

If $u \in [0, 1-\delta]$ and $u = [x_{i-1}, x_i] \in P$

$$x_{i-1} \le 1-\delta \Rightarrow x_i - x_{i-1} < \delta$$
$$\Rightarrow x_i < 1$$

$$1(1-\delta) \le S(f, \dot{P}_1) \le 1(1+\delta)$$

Let $\dot{P}_2$ be union of those subintervals such that tags are in $(1,3)$.

$$[1+\delta, 3) \subset P_2 \subset [1-\delta, 3) \qquad P = \dot{P}_1 \cup \dot{P}_2$$

$$2(2-\delta) \le S(f, \dot{P}_2) \le 2(2+\delta)$$

$$5 - 3\delta \le S(f, \dot{P}) \le 5 + 3\delta$$

for any tagged partition $\dot{Q}$ with $\|Q\| < \delta$.

similarly, $5 - 3\delta \le S(f; \dot{Q}) \le 5 + 3\delta$

$\Rightarrow \quad |S(f;\dot{P}) - S(f;\dot{Q})| \le 6\delta = \epsilon$

$\Rightarrow$ f satisfies (C2)

$\Rightarrow$ f satisfies (C1)

$\Rightarrow f \in R[0,3]$

proof: ($\Rightarrow$) $f \in R[a,b] \Rightarrow$ For given $\epsilon > 0$, $\exists$ partition P_ϵ such that for any partition P finer than P_ϵ, and any tag on P,

$$|S(f;\dot{P}) - L| < \epsilon/2$$

Q finer than $P_\epsilon \Rightarrow |S(f;\dot{Q}) - L| < \epsilon/2$

$$\therefore |S(f;\dot{P}) - S(f;\dot{Q})| < \epsilon$$

($\Leftarrow$) choose $\frac{1}{n}$, then $\exists P_n$ and for any two partitions P and Q, finer than P_n and any tag P & Q,

$$|S(f;\dot{P}) - S(f;\dot{Q})| < 1/n$$

we choose P_n $\forall n$ st P_m is finer than P_n whenever $m > n$.

$\Rightarrow |S(f;\dot{P}_n) - S(f;\dot{P}_m)| < \frac{1}{n}$ when $m > n$.

$\Rightarrow (S(f,\dot{P}_n))$ is a cauchy sequence

$\Rightarrow (S(f;\dot{P}_n))$ is convergent

and let the limit be L

$\Rightarrow$ for given $\epsilon > 0$, $\exists N$ st

$$|S(f,\dot{P}_n) - L| < \epsilon \quad \forall n \ge N$$

Take any P finer than $\dot{P}_N$, then

$$|S(f;\dot{P}) - L| \le |S(f;\dot{P}_N) - L| + |S(f;\dot{P}_N) - S(f;\dot{P})|$$

$$< \epsilon + \frac{1}{N} \quad < \epsilon + \epsilon = 2\epsilon$$

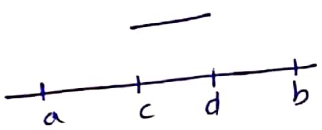
$\frac{1}{N_2} < \epsilon$ Take $\max\{N_1, N_2\} = N$.

step function:

$\varphi : [a,b] \to \mathbb{R}$ is a step function if φ has only many values and each valued being assumed in ones or more subintervals of $[a,b]$.

Elementary step function:

$\exists c,d$ s.t. $a \le c < d \le b$ and $J := [c,d]$ $\quad \varphi_J(x) = 1$ on J
$= 0$ elsewhere



A step function $\varphi = \sum_{j=1}^{n} k_j \varphi_j$

φ_j is an elementary step function.

$k_j \in \mathbb{R}$

φ_j's are riemann integrable

$\Rightarrow \varphi$ is riemann integrable

and $\int_a^b \varphi = \sum_{j=1}^{n} k_j \int_a^b \varphi_j = \sum_{j=1}^{n} k_j (d_j - c_j)$

Lemma: an element step function $\in R(a,b]$

and $\int_a^b \varphi_j = d - c$

$\varphi_j(x) = 1$ if $x \in [c,d]$
$= 0$, elsewhere.

Step functions are riemann integrable:

Step function:

Has finitely many points in the image and each value is being assumed in one or more subintervals of $[a,b]$.

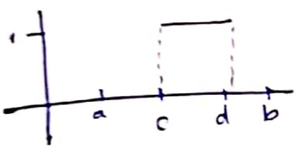
Elementary step function:

$\exists$ a subinterval $[c,d] \subset [a,b]$ st

$$f(x) = \begin{cases} 1, & x \in [c,d] \\ 0, & \text{elsewhere} \end{cases}$$

A step function $f = \sum_{j=1}^{n} k_j \varphi_j$, φ_j is elementary step function.

$$\int_a^b f = \sum_{j=1}^{n} k_j \int \varphi_j = \sum_{j=1}^{n} k_j (d_j - c_j)$$



proof: Let P be a partition with $\|P\| < \delta$ and let $\dot{P}$ be a tagged partition. Let $\dot{P}_1$ be the ~~subset~~ union of subintervals where the corresponding tag lies in $[c,d]$

$$[c+\delta, d-\delta] \subset \dot{P}_1 \subset [c-\delta, d+\delta]$$

$u \in \dot{P}_1 \ \exists \ [x_{i-1}, x_i]$ where $u \in [x_{i-1}, x_i]$ and the corresponding tag $t_i \in J$. $x_{i-1} \leq t_i \leq x_i$

$$t_i \in J \Rightarrow x_{i-1} \in J = [c,d]$$

$$\Rightarrow x_i < x_{i-1} + \delta$$
$$< d + \delta$$

$\cancel{x_{i-1} < x_i - \delta}$ $\quad x_{i-1} > x_i - \delta$

$$\geq c - \delta$$

$\dot{P}_2$ is the complement of $\dot{P}_1$

$$S(\dot{P},f) = S(\dot{P}_1,f) + \underbrace{S(\dot{P}_2,f)}_{\overset{\shortparallel}{0}}$$

$$= S(\dot{P}_1,f)$$

$$(d-\delta)-(c+\delta) \le S(\dot{P}_1,f) \le (d+\delta)-(c-\delta)$$

$$d-c-2\delta \le S(\dot{P}_1,f) \le d-c+2\delta$$

$$|S(\dot{P},f)-(d-c)| < 2\delta$$

<u>Thm:</u>

$f:[a,b]\to R$ continuos then $f\in R[a,b]$

<u>proof:</u>

Let, P_ε a partition of norm $\le \delta$.

we have to choose, $\alpha_\varepsilon \le f \le \omega_\varepsilon$ $\int_a^b (\omega_\varepsilon-\alpha_\varepsilon) < \varepsilon$

Take u_i to be max of f on $[x_{i-1},x_i]$

v_i to be min of f on $[x_{i-1},x_i]$

$$\alpha_\varepsilon(x) = u_i,\ x\in[x_{i-1},x_i)$$
$$= u_n,\ x\in[x_{n-1},x_n]$$

$$\omega_\varepsilon(x) = v_i,\ x\in[x_{i-1},x_i)$$
$$= v_n,\ x\in[x_{n-1},x_n]$$

$$\alpha_\varepsilon \le f \le \omega_\varepsilon$$

$$\int_a^b \omega_\varepsilon - \int_a^b \alpha_\varepsilon = \int_a^b (\omega_\varepsilon-\alpha_\varepsilon) = \sum_{i=1}^{n} (\overset{u_i-v_i}{\cancel{v_i-u_i}})(x_i-x_{i-1})$$

<u>Lemma:</u> $J=[c,d]$ or $(c,d]$ or $[c,d)$ or (c,d) Then $\varphi_j = 1$ if $x\in J$, 0 o/w

$$\int_a^b \varphi_j = d-c$$

f is continuos on $[a,b] \Rightarrow f$ is uniformly continuos.

$\Rightarrow$ For given $\epsilon>0$, $\exists \delta$ st $|f(x)-f(y)|<\epsilon$ when $|x-y|<\delta$.

now, let P_ϵ to be partition of norm $<\delta$.

$u_i =$ max on $[x_{i-1}, x_i]$
$v_i =$ min on $[x_{i-1}, x_i)$

$u_i - v_i < \epsilon$

$$\int_a^b (u_i - v_i)(x_i - x_{i-1}) < \sum_{i=1}^{n} \epsilon(x_i - x_{i-1}) = \epsilon(b-a)$$

Thm: $f:[a,b] \to R$ is monotone. Then $f \in R[a,b]$

Additivity Theorem:

$f:[a,b] \in R$ $c \in (a,b)$ Then $f \in R[a,b]$ If and only if $R[a,c]$ & $R[c,b]$. In both cases, $\int_a^b f = \int_a^c f + \int_c^b f$

Cor: $f \in R[a,b]$ if $[c,d] \subset [a,b]$. Then the restriction of form $[c,d]$ is $\in R[c,d]$

proof: $a<c<b$

$f \in R[a,b] \Rightarrow f \in R[c,b]$

$c>d>b$

$f \in R[c,b] \Rightarrow f \in R[c,d]$

Def:

$f \in R[a,b]$ $a<\alpha<\beta \leq b$

Then $\int_\beta^\alpha f := -\int_\alpha^\beta f$; $\int_\alpha^\alpha f := 0$

Thm: $f \in R[a,b]$ $\alpha, \beta, \gamma \in [a,b]$

Then, $\int_\alpha^\beta f = \int_\alpha^\gamma f + \int_\gamma^\beta f$

proof : $(\Rightarrow)$

$f \in R[a,b]$ For any given $\epsilon>0, \exists$ a partition P_ϵ st for any partition P, Q finer than and any tag on P, Q,

$$|S(P,f) - S(Q,f)| < \epsilon \quad (*)$$

$P_\epsilon' = \{P_\epsilon \cap [a,c]\} \cup \{c\}$

P_ϵ' is a partition of $[a,c]$

Take any two partitions P and Q_1 both finer than P_ϵ',

such that $|S(\dot{P}_1,f) - S(\dot{Q}_1,f)| < \epsilon$

Extend P_1 to a partition P_2 of $[a,b]$

$$P_2 = P_1 \cup \{P_\epsilon \cap [b,c]\}$$

similarly, $Q_2 = Q_1 \cup \{P_\epsilon \cap [c,b]\}$

$$|S(\dot{P}_1,f) - S(\dot{Q}_1,f)|$$

By $(*)$ $|S(\dot{P}_2,f) - S(\dot{Q}_2,f)| < \epsilon$

we can choose same tags for $P_2 \cap [c,b]$ & $Q_2 \cap [c,b]$

$$S(\dot{P}_2,f) - S(\dot{Q}_2,f) = \sum_{P_2} f(t_i)(x_i - x_{i-1}) - \sum_{Q_2'} f(t_i')(x_i' - x_{i-1}')$$

$$= \sum_{P_2 \cap [a,c]} f(t_i)(x_i - x_{i-1}) + \sum_{P_2 \cap [c,b]} f(t_i)(x_i - x_{i-1})$$

$$= - \left(\sum_{Q_2 \cap [a,c]} f(t_i')(x_i' - x_{i-1}') + \sum_{Q_2 \cap [c,b]} f(t_i')(x_i' - x_{i-1}') \right)$$

$R \in R[a,c] \Rightarrow \exists$ a partition P_ϵ' of $[a,c]$ st for a partition P, finer than P_ϵ' and any tag on P

$$|S(\dot{P},f) - L_1| < \epsilon$$

$f \in R[c,b] \Rightarrow \exists$ a partition P_ϵ'' of $[c,b]$ st for a partion P, finer than P_ϵ'' and any tag on P,

$$|S(\dot{P},f) - L_2| < \epsilon.$$

Take $P_\varepsilon = P'_\varepsilon \cup P''_\varepsilon$ $\quad P_\varepsilon$ countains c

Take P be finer than P_ε, P countains c.

Take $P_1 = P \cap [a,c]$ $\quad P_2 = P \cap [c,b]$

For any tag on P

$$|S(\dot{P},f) - (L_1+L_2)| < |S(\dot{P}_1,f) - L_1| + |S(\dot{P}_2,f) - L_2|$$
$$< 2\epsilon$$

Def:

f is given on $[a,b]$.

If F is another function $\quad F'(x) = f(x)$

then F is called ~~an~~ anti-derivative of f.

Fundamental theorem of integral calculus:

suppose that E is a finite set of $[a,b]$ and f, F are defined on $[a,b]$ st,

(i). $F'(x) = f(x) \quad \forall x \in [a,b] - E$

(ii). F is continuos on $[a,b]$

(iii). $f \in R[a,b]$

then $\int_a^b f = F(b) - F(a)$.

Examples:

$F, f : [-a,a] \to \mathbb{R}$

1). $F(x) = |x| \qquad f(x) = sgn(x)$

$F'(x) = f(x) \qquad x \in [-a,a]$

$$\begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

Here $\quad E = \{0\}$

$$\int_{-a}^{a} sgn(x) = F(a) - F(-a) = |a| - |-a|$$
$$= |a| - |a| = 0.$$

2). $F(x) = 2\sqrt{x}$ $\quad$ $F, f : [0,1] \to R$

$f(x) = \frac{1}{\sqrt{x}}, x \neq 0$

we can take $f(0)$ as then $E = \{0\}$

$f \notin R[0,1]$ as f is not bounded.

proof: Assume $E = \{a, b\}$

If $E = \{a, c, b\}$

then break $[a,b]$ on $[a,c]$

$$\int_a^b f = \int_a^c f + \int_c^b f$$

$f \in R[a,b] \Rightarrow \exists$ a no. L & for given $\epsilon > 0$ $\exists$ a partition P_ϵ s.t for any P finer than P_ϵ and any tag on P, $|S(f, \dot{P}) - L| < \epsilon$

$$F(b) - F(a) = \sum_{i=1}^{n} (F(x_i) - F(x_{i-1}))$$

$$F(x_i) - F(x_{i-1}) = F'(u_i)(\overset{x_i - x_{i-1}}{\cancel{u_i - u_{i-1}}}) \quad \text{for some} \quad u_i \in (x_i, x_{i-1})$$

$$= \cancel{f(u_i)(u_i - u_{i-1})} \; f(u_i)(x_i - x_{i-1})$$

$|F(b) - F(a) - L| < \epsilon \Rightarrow F(b) - F(a) = L$

Fundamental theorem of integral calculus(second form)

indefinite integral:

$f \in R[a,b]$ Then $F(x) = \int_a^x f$ is called indefinite integral of f.

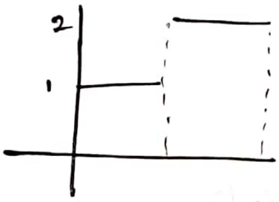
$x \in [a,b]$.

<u>Theorem:</u>

Let $f \in R[a,b]$ also let f be continuos at c. Then the definite integral F is differentiable at c and $F'(c) = f(c)$.

<u>Thm:</u> The indefinite integral F is cont. on $[a,b]$. In fact,

$|F(z) - F(w)| \leq |M||z-w|$ if $f(x) \leq M$; $z, w \in R[a,b]$.



$F(x) = x, \; x < 1$

$F(1) = 1$

$1 < x < 2, \; F(x) = 1 + 2(x-1)$

Though the function is not continuos, derivative is continuos.

$F(z) = \int_a^z f \qquad F(w) = \int_a^w f$

$F(z) - F(w) = \int_z^w f$ (By additivity theorem)

$-M \leq f \leq M \qquad \int_z^w (-M) \leq \int_z^w f \leq \int_z^w (M)$

$\Rightarrow \quad -M(w-z) \leq \int_z^w f \leq M(w-z) \Rightarrow \left| \int_z^w f \right| = |M(w-z)|$

$\Rightarrow \quad |F(z) - F(w)| \leq |M| \, |z - w|$

continuos by cauchy criterion

Let $0 < h < \delta$

$$F(c+h) - F(c) = \int_a^{c+h} f - \int_a^{c} f$$

$$= \int_c^{c+h} f \qquad \text{(By additivity theorem).}$$

$f(x) \leq f(c) + \epsilon$ for $x \in [c, c+\delta)$

$$\Rightarrow \int_c^{c+h} f \leq \int_c^{c+h} f(c) + \epsilon < (f(c)+\epsilon)h$$

$$(f(c) - \epsilon)h \leq \int_c^{c+h} f$$

$$(f(c) - \epsilon)h \leq F(c+h) - F(c) \leq (f(c) + \epsilon)h$$

$$\Rightarrow \left| \frac{F(c+h) - F(c)}{h} - f(c) \right| < \epsilon$$

$$\Rightarrow \boxed{F'(c) = f(c)}$$

Cor: If f is continuos, then the indefinite integral ~~integral~~ F is differentiable on $[a,b]$ and $F'(c) = f(c)$. F is anti-derivative of f.

First form:

(i). $f \in R[a,b]$

(ii). F is continuos

(iii). $F'(c) = f(c)$ on $[a,b] - E$

Then $\int_a^b f = F(b) - F(a)$.

Substitution theorem:

Let $J := [a,b]$ and $\varphi : J \to R$ have a continuos derivative on J.

If f is continuos on an interval countaining $\varphi(J)$ then,

$$\int_a^b f(\varphi(t))\,\varphi'(t)\,dt = \int_{\varphi(a)}^{\varphi(b)} f(x)\,dx$$

Lebesgue ~~Theorem~~ integrability:

Ex: step function $\longrightarrow$ these are cont except at finitely many points.

2). cont functions $\longrightarrow$ Everywhere continuos

3) monotone functions $\longrightarrow$

Thm: $f : I \to R$ be monotone function where I is an interval.

$$x \le y \longrightarrow f(x) \le f(y) \quad ||$$
$$x \le y \longrightarrow f(x) \ge f(y)$$

4). Thomae's function:

$f : [0,1] \longrightarrow R$ by $f(x) = 0$, $x \in Q^c$

$= \frac{1}{n}$, $x = \frac{m}{n}$, m & n are coprime

To show, $f \in R[0,1]$

For given $\epsilon > 0$, $E_\epsilon = \left\{ x \in [0,1] \;\middle|\; f(x) \ge \frac{\epsilon}{2} \right\}$

$x \in \mathbb{Q}^c \Rightarrow x \notin E_\epsilon$

$x \in \mathbb{Q} \Rightarrow \frac{1}{n} \geq \frac{\epsilon}{2}$

$n \leq \frac{2}{\epsilon}$

$\hookrightarrow n$ finite

$\Downarrow$

m finite $\Rightarrow$ m,n pairs finite.

Take $n_\epsilon = \# E_\epsilon$ $\quad \delta_\epsilon = \frac{\epsilon}{4n_\epsilon}$.

Take any tagged partition $\dot{P}$ with norm $< \delta_\epsilon$. Let $\dot{P}_1$ be the subintervals of P with tags in E_ϵ.

$$S(f; \dot{P}_1) = \sum_{\substack{2n_\epsilon \\ \text{subintervals}}} f(t_i)(x_i - x_{i-1})$$

$$\leq \sum_{2n_\epsilon} 1.\delta_\epsilon = \delta_\epsilon (2n_\epsilon) = \frac{\epsilon}{2}$$

$$S(f; \dot{P}_1^c) = \sum_{\dot{P}_1^c} f(t_i)(x_i - x_{i-1}) < \sum \frac{\epsilon}{2}(x_i - x_{i-1}) < \frac{\epsilon}{2}$$

union of subintervals tag is not in E_ϵ

$\therefore S(f; \dot{P}_1 \cup \dot{P}_1^c) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

$\therefore \int_0^1 f = 0$ & $f \in R[0,1]$

it has discontinuity at countably infinite points.

(5). dirichlet functions:

$f(x) = 1$ if x is rational

$\quad\quad 0$ if x is irrational

$f(x) \notin R[0,1]$

Uncountable discontinuity.

Def: A set A is a null set if for given $\epsilon > 0$, $\exists$ a countable collection $\{(a_K, b_K)\}_{K=1}^{\infty}$ of open interval st

$$A \subseteq \bigcup_{K=1}^{\infty} (a_K, b_K) \ \& \ \sum_{K=1}^{\infty} (b_K - a_K) < \epsilon.$$

Ex:

1). A countable set is null set $\langle c_K \rangle$ Take $\epsilon > 0$

$$(a_K, b_K) = \left(c_K - \frac{\epsilon}{2^{K+1}}, c_K + \frac{\epsilon}{2^{K+1}} \right)$$

$$\langle c_K \rangle_K \subset \bigcup_{K=1}^{\infty} (a_K, b_K)$$

$$\& \quad \sum_{K=1}^{\infty} (b_K - a_K) = \sum_{K=1}^{\infty} \frac{\epsilon}{2^K} = \epsilon$$

2). Cantor's set is uncountable, but is null set.

Def^n:

1). A statement Q holds almost everywhere if, $\exists$ a null set z st $Q(x)$ is true for $A - z$.

Integrability criterion (Lebsgue):

A bounded function on A is Riemann integrable If and only if f is almost continuous on $[a,b]$.

composition Theorem:

$f \in R[a,b]$ and

$f([a,b]) \subset [c,d]$ and φ is continuous on $[c,d]$

Then $\varphi \circ f \in R[a,b]$

proof: f is almost everywhere continuos on $[a,b]$ i.e, $\exists$ null set z st. f is continuos on $[a,b]-z$.

$\therefore \varphi o f$ is continuos on $[a,b]-z$

so $\varphi o f \in R[a,b]$

cor: $f \in R[a,b] \Rightarrow |f| \in R[a,b]$ and

$$\left|\int_a^b f\right| \leq \int_a^b |f| \leq M(b-a) \quad \text{if } |f| \leq M.$$

proof:

$\varphi o f$ take $\varphi(x) = |x|$

By composition theorem, $\varphi o f \in R[a,b] \Rightarrow |f| \in R[a,b]$

$$-|f| \leq f \leq |f|$$

$$\Rightarrow -\int_a^b |f| \leq \int_a^b f \leq \int_a^b |f|$$

$$\Rightarrow \left|\int_a^b f\right| \leq \int_a^b |f| \leq M(b-a)$$

Product Theorem:

$f,g \in R[a,b] \Rightarrow f.g \in R[a,b]$

Assignment-7

1. show that the formula in mean-value theorem can be stated as follows:

$$\frac{f(x+h)-f(x)}{h} = f'(x+\theta h)$$
$$0<\theta<1.$$

determine θ as a function of x and h for,

(i). $f(x) = x^2$ (ii). $f(x) = e^x$

Sol: (i). $\frac{(x+h)^2 - x^2}{h} = 2(x+\theta h)$

$\Rightarrow h+2x = 2x+2\theta h$

$\Rightarrow \boxed{\theta = \frac{1}{2}}$

(ii). $\frac{e^{x+h} - e^x}{h} = e^x e^{\theta h}$

$\Rightarrow e^{\theta h} = \frac{e^h - 1}{h}$

$\theta h = \ln\left(\frac{e^h-1}{h}\right) \Rightarrow \theta = \frac{1}{h}\ln\left(\frac{e^h-1}{h}\right)$

3. f has finite derivative f' in $0<x\le 1$ and $|f'(x)|<1$ define $a_n = f\left(\frac{1}{n}\right)$. show that $\lim_{n\to\infty} a_n$ exists.

Sol: $|a_m - a_n| < \left|f\left(\frac{1}{n}\right) - f\left(\frac{1}{m}\right)\right| \le \left|f'(c)\left(\frac{1}{n} - \frac{1}{m}\right)\right|$

$c \in \left(\frac{1}{n} - \frac{1}{m}\right)$

$< \left|\frac{1}{n} - \frac{1}{m}\right|$

$\forall n, m > \frac{2}{\epsilon}$

$|a_m - a_n| < \epsilon$ $\rightarrow$ cauchy criterion.

2. f has finite derivative at (a,b) and $\lim\limits_{x\to b^-} f'(x) = +\infty$.

prove that $\lim\limits_{x\to b^-} f'$ either fails to exist or ~~fails~~ is infinite

Sol: If possible assume $\lim\limits_{x\to b^-} f'(x)$ exists and is finite

let $\lim\limits_{x\to b^-} f(x)=L$

For $\epsilon = 1/2$, $\exists\, \delta>0$ st $|f'(x)-L|<\frac{1}{2}$ $\forall x\in(b-\delta,b)$

$$|f'(x)| \le L+\frac{1}{2}$$

let $(x_n)\longrightarrow b$

Then (x_n) is a cauchy sequence.

$$f(x_n)-f(x_m) = f'(c)(x_n-x_m)$$
$$c\in(x_n-x_m)$$

also assume $x_n\in(b-\delta,b)$

$$f(x_n)-f(x_m) = f'(c)(x_n-x_m)$$
$$c\in(x_n,x_m)$$

$$|f(x_n)-f(x_m)| < (|L|+\tfrac{1}{2})|x_n-x_m|$$

$\therefore f(x_n)$ is cauchy sequence & is convergent.

$\therefore \lim\limits_{x\to b^-} f$ exists. contradiction.

4. f be continuous on $[0,1]$, $f(0)=0$, $f'(x)$ is finite for each $x>0$, f' is an increasing function, show that $g(x)=\frac{f(x)}{x}$ is increasing. (f'' exists)

Sol: $g'(x) = \frac{f'(x)}{x} - \frac{f(x)}{x^2} = \frac{xf'(x)-f(x)}{x^2}$

$h(x) = f'(x)\cdot x - f(x)$

$h(0)=0$

$h'(x) = f'(x)+xf''(x)-f'(x) = xf''(x)\ge 0$ as f' is increasing

$h(x)$ is increasing $\Rightarrow h(x)\ge 0$ $\forall x\ge 0$ $\Rightarrow g'(x)\ge 0$ $\forall x\ge 0$

$\Rightarrow g$ is increasing function.

Mean value theorem:

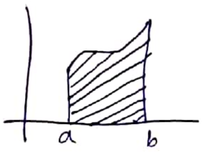
(for Riemann integration):

First mean value Theorem:

If $f \in R[a,b]$ let $m \le f(x) \le M$ $\forall x \in [a,b]$ Then,

$\int_a^b f = c(b-a)$ for some $c \in [m,M]$. If is cont, then,

$\exists z \in (a,b)$ such that $\int_a^b f = f(z)(b-a)$



proof: $m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)$

$$m(b-a) \le \underline{\int_a^b} f \le \overline{\int_a^b} f \le M(b-a)$$

$$f \in R[a,b] \Rightarrow \underline{\int_a^b} f = \overline{\int_a^b} f = \int_a^b f$$

so, $\quad m(b-a) \le \int_a^b f \le M(b-a)$

$$\Rightarrow m \le \frac{\int_a^b f}{b-a} \le M$$

second mean value theorem:

Assume that g is continuos and f is increasing on $[a,b]$ assume,

$A \le f(a+)$, $B \ge f(b-)$. Then $\exists x_0 \in (a,b)$

st

(i). $\int_a^b f(x)g(x)\,dx = A\int_a^{x_0} g(x)dx + B\int_{x_0}^b f(x)dx$

(ii). $f(x) \ge 0$ $\forall x \in [a,b]$ then

$\int_a^b f(x)g(x)dx = B\int_{x_0}^b g(x)dx \quad x_0 \in [a,b]$

sequences and series of functions:

Ex:

(1). Weierstrass approximation theorem:

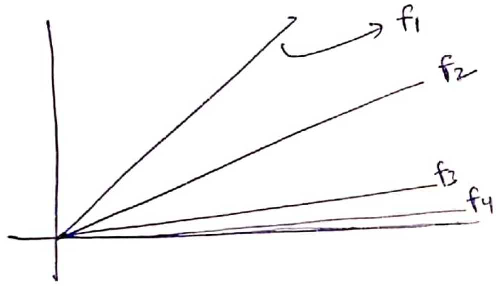
If f is continuos on $[a,b]$. For given $\epsilon > 0$, there exists a polynomial P_ϵ such that $|P_\epsilon(x) - f(x)| < \epsilon \quad \forall x \in [a,b]$.

(2). Fourier series

(3). $f_n(x) = \frac{x}{n} \quad \forall x \in \mathbb{R}, \forall n \in \mathbb{N}$

Fix x_0 we get a sequence, $f_n(x_0) = \left(\frac{x_0}{n}\right)$

$$\lim_{n\to\infty} f_n(x_0) = 0$$



suppose $(f_n(x_0))$ is convergent, then we get a new function $f(x) = \lim_{n\to\infty} f_n(x)$.

pointwise convergence:

Let $f_n : A \to R$ be a sequence of functions. Let $A_0 \subseteq A$ and $f : A_0 \to R$.

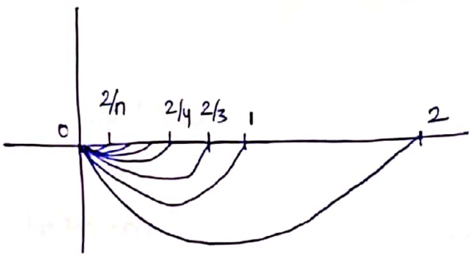
We say (f_n) is pointwise convergent on A_0 if $f(x) = \lim_{n\to\infty}(f_n(x)) \quad \forall x \in A_0$.

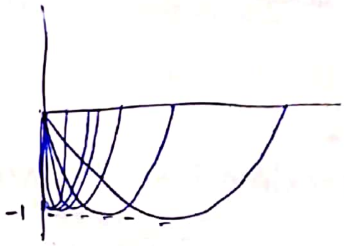
suppose $f_n : A \to R$ and c is a limit point of A, $c \in A$.

$$\lim_{n\to\infty} \lim_{x\to c} f_n(x) = \lim_{x\to c} \lim_{n\to\infty} f_n(x)$$

4). $f_n : [0,2] \longrightarrow R$ be defined by

$$f_n(x) = \begin{cases} 0, & \frac{2}{n} \le x \le 2 \\ n^2x^2 - 2xn, & 0 \le x \le \frac{2}{n}. \end{cases}$$





choose $x, 0 \leq x \leq 2$

choose $N \in \mathbb{N}$ st, $\frac{2}{N} < x$. So, for all

$n \geq N$, $\frac{2}{n} < x \Rightarrow f_n(x) = 0 \quad \forall n \geq N$

$$\Rightarrow \lim_{n\to\infty} f_n(x) = 0$$

$f_n(0) = 0 \quad \forall n \Rightarrow \lim_{x\to\infty} f_n(0) = 0$

$\therefore (f_n(x))$ converges to f where $f \equiv 0$.

$f_n\left(\frac{1}{n}\right) = n^2 \frac{1}{n^2} - 2n\frac{1}{n} = -1$

$\lim_{n\to\infty} f_n\left(\frac{1}{n}\right) = -1 \qquad f_n\left(\lim_{n\to\infty}\left(\frac{1}{n}\right)\right) = \underline{\underline{f_n(0) = 0}}$

$x_n \longrightarrow c$

$\not\Rightarrow \lim_{n\to\infty} f_n(x_n) \longrightarrow f(c)$

<u>(5).</u> $f_n(x) = x^n, x \in \mathbb{R}$

$x = 0, 1 \quad f_n(x) = 0$ and $1 \quad \forall n$

$\lim_{n\to\infty} f_n(x) = 0$ and 1

$0 < x < 1$, $f_n(x) = x^n \to 0$ as $n \to \infty$.

$(f_n(x))$ is convergent.

$-1 < x < 0$, $f_n(x) = x^n \to 0$ as $n \to \infty$

$(f_n(x))$ is convergent.

$x > 1$, $f_n(x) = x^n \longrightarrow$ divergent

$x < -1$, $f_n(x) = x^n \longrightarrow$ divergent

$x = -1$, $f_n(x) \longrightarrow$ divergent

$f_n(x)$ converges on $x \in (-1,1]$

each of the function is continuos, but however the limit function is not continuos.

(6). $f_n(x) = \frac{1}{n}\sin(nx+n)$

$(f_n(x)) \longrightarrow 0$ So, $(f_n) \to f \equiv 0$ pointwise

$|f_n(x) - f| = \left|\frac{1}{n}\sin(nx+n) - 0\right| = \frac{1}{n}|\sin(nx+n)| \leq \frac{1}{n}.$

choose $\epsilon > 0$, $\exists N \in \mathbb{N}$ st $\frac{1}{n} < \epsilon$ $\forall n \geq N$.

$\Rightarrow |f_n - f| < \epsilon$ $\forall n \geq N$ $\forall x$

$f_n(x) = x^n$

$f_n(x) \longleftrightarrow f(x)$ where $f(x) = 0$ if $x \in (-1,1)$
$= 1$ if $x = 1$

$x \neq 1, x > 0,$

$|f_n(x) - f(x)| = |x^n - 0| = |x|^n = x^n$

$|f_n(x) - f(x)| < \epsilon$ if $x^n < \epsilon \Rightarrow n \ln x < \ln \varepsilon$
$\Leftrightarrow n > \frac{\ln \varepsilon}{\ln x}$

choose $N \in \mathbb{N}$ st $N > \left(\frac{\ln \varepsilon}{\ln x}\right)$

depends on x. Then $n \geq N \Rightarrow |f_n(x) - f(x)| < \epsilon$

$f_n, f : A \longrightarrow R$

def: (f_n) converges to f a ~~natural numbers~~ uniformly if for given $\epsilon > 0$, $\exists$ a natural number N st $\forall n \geq N$,

$|f_n(x) - f(x)| < \epsilon$ $\forall x \in A$.

uniform convergence $\Rightarrow$ pointwise convergence.

Lemma: A sequence of functions $f_n : A \to R$ doesn't converge uniformly to f on $A_0 \in A$ if and only if for each $\epsilon > 0$, $\exists$ a sequence $\{n_K\}_K$ of natural numbers and a sequence $\{x_K\}$ of points of A_0 st,

$$|f_{n_K}(x_K) - f(x_K)| > \epsilon \quad \forall K \in N.$$

proof:

suppose (f_n) is not convergent to finite on A_0.

$\Rightarrow$ For some ϵ, we can't find $N \in \mathbb{N}$ st

$$|f_n(x) - f(x)| < \epsilon \quad \forall n \geq N \ \& \ \forall x \in A_0.$$

$\Rightarrow$ for $N_1 \in \mathbb{N}$, $\exists n_1 > N_1$ st $|f_{n_1}(x) - f(x_1)| \geq \epsilon$

choose $n_2 > n_1$ st $|f_{n_2}(x) - f(x_2)| > \epsilon$

for some $x_2 \in A_0$.

Ex 4). $f_n : [0,2] \to R$

$$f_n(x) = \begin{cases} 0, & \frac{2}{n} \leq x \leq 2 \\ n^2x^2 - 2xn ; & 0 \leq x \leq 2/n \end{cases}$$

$f_n(x) \to f(x)$ pointwise convergence

But $f_n \to f$ not uniform.

$$f_n\left(\frac{1}{n}\right) = -1$$

$$\left|f_n\left(\frac{1}{n}\right) - f\left(\frac{1}{n}\right)\right| = |-1-0| = \frac{-1}{}$$

uniform metric:

$f, g : A \to \mathbb{R}$, f and g are bounded.

$$|f(x)-g(x)| \leq |f(x)| + |g(x)| \leq M_1 + M_2$$

$$\sup_{x \in A} |f(x) - g(x)| = d_A(f,g)$$

$$\|f-g\|_A = \sup_{x \in A} |f(x) - g(x)|$$

uniform norm

uniform norm:

$\varphi : A \to R$ is a bounded functions then uniform norm,

$$\|\varphi\|_A := \sup_{x \in A} \{ |\varphi(x)| \}$$

Lemma:

A seq (f_n) of bounded functions on $A \subseteq R$ converges uniformly on A if and only if $\|f_n - f\|_A \to 0$

cauchy criterion for uniform convergence:

Let (f_n) be a sequence of bounded functions on A; converges uniformly to a bounded function f if and only if for each $\epsilon > 0$, $\exists K \in \mathbb{N}$ st, $\|f_m - f_n\| < \epsilon$ $\forall m, n > K$.

Pf:

$$\|f_m - f_n\| < \epsilon \quad \forall m, n \geq K \qquad \begin{matrix} \epsilon > 0 \\ K \in \mathbb{N} \text{ st} \end{matrix}$$

$$\Rightarrow |f_m(x) - f_n(x)| < \epsilon \quad \forall m, n \geq K, \ x \in X.$$

For each $x \in A$ we get a cauchy sequence $(f_n(x))$ which is convergent.

define $\lim f(x) = \lim_{n \to \infty} f_n(x)$

$|f_m(x) - f_n(x)| < \epsilon/2$ take $n \to \infty$ $|f_m(x) - f(x)| < \epsilon$.

$\therefore (f_n) \to f$ uniformly.

<u>Ex:</u> $G_n(x) = x^n(1-x), x \in [0,1]$

Fix x, $G_n(x) \to 0$ as $n \to \infty$.

$$\|G_n - G\|_A = \sup_{x\in A}\{G_n(x) - G(x)\}$$

$$= \sup_{x \in A}\{x^n(1-x)\}$$

$(x^n(1-x))' = x^{n-1}(n-(n+1)x)$

max attained $x_0 \Rightarrow x_0^{n-1}(n-(n+1)x_0) = 0$

$\Rightarrow x_0 = 0$ (or) $x_0 = \frac{n}{n+1}$

$(x^n(1-x))'' = x^{n-2}(n^2 - n - n(n+1)x)$

put $x_0 = \frac{n}{n+1}$, $(x^n(1-x))'' < 0$

$\sup\{x^n(1-x)\} = \left(\frac{n}{n+1}\right)^n\left(1 - \frac{n}{n+1}\right) = \left(1+\frac{1}{n}\right)^{-1}\left(\frac{1}{n+1}\right) \to e^{-1}0 = 0$

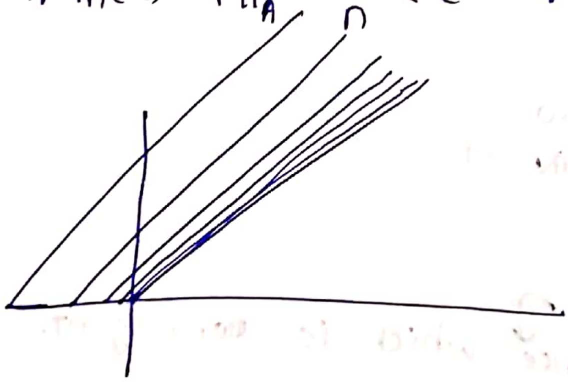
By lemma (before cauchy criteria):

$G_n \to G$ uniformly.

<u>Ex:</u> $f_n(x) = x + \frac{1}{n}$ $\forall x \in \mathbb{R}$

$f_n(x) \to x$

$\| f_n(x) - f\|_A = \frac{1}{n} < \epsilon$ $\forall n > \frac{1}{\epsilon}$



$\therefore \|f_n - f\| \to 0$

Ex: $f_n^2(x) = x^2 + \frac{1}{n^2} + \frac{2x}{n}$

$$\lim_{n\to\infty} f_n^2(x) = x^2$$

$$\| f_n^2 - x^2 \| = \left\| \frac{1}{n^2} + \frac{2x}{n} \right\|$$

proof for non-uniformity.

$$\| f_n^2 - x^2 \| > 1.$$

$$x_n = n^2$$

$$|f_n(x_n) - x_n^2| = \left\| 2n + \frac{1}{n^2} \right\| > 1$$

Ex: $f_n(x) = x^n \quad \forall x \in \mathbb{R}$

$(f_n(x))$ converges $x \in [-1,1]$

to $f(x) \equiv 0$ if $x(-1,1)$

$\equiv 1$ if $x = 1$

property of continuity may not be preserved by pointwise convergence.

Theorem:

If $(f)_n$ a sequence of function converges to f uniformly and each f_n is continuos, then f is also continuos on A.

proof: For given $\epsilon > 0$, $\exists N \in \mathbb{N}$ st

$$|f_n(x) - f(x)| < \epsilon \quad \forall n \geq N, \forall x \in A$$

Let $c \in A$,

$$|f(x) - f(c)| \leq |f(x) - f_n(x) + f_n(x) - f_n(c) + f_n(c) - f(c)|$$

$$\leq |f(x) - f_n(x)| + |f_n(x) - f_n(c)| + |f_n(c) - f(c)|$$

$$\leq 2\epsilon + |f_n(x) - f_n(c)|$$

$$\forall n \geq N.$$

Take $n = N$,

$$|f(x) - f(c)| \leq 2\epsilon + |f_N(x) - f_N(c)|$$

using continuity of f_N, $|f_N(x) - f_N(c)| < \epsilon$
$\forall\ |x-c|<\delta$

$\therefore\ |f(x) - f(c)| < \epsilon\ \ \forall\ |x-c|<\delta.$

Ex: $f_n(x) = x^n \longrightarrow f(x) = 1$ if $x=1$
$= 0$ if $x \in (-1,1)$

pointwise convergence

The limit function is not continuos, and not differentiable.

(b). $f_n(x) = \sum_{K=1}^{n} 2^{-K} \cos(3^K x)$

each function is differentiable

$(f_n) \to f = \sum_{K=1}^{\infty} 2^{-K} \cos(3^K x)$

uniform

f is not diff

Thm: Let $J \in \mathbb{R}$ be a bounded interval and let (f_n) be a sequence of functions on J to $\mathbb{R}$. Suppose that $\exists\, x_0 \in J$ st $(f_n(x_0))$ converges, each f_n is differentiable, the sequence (f_n') converges to a function g on J.

Then (f_n) converges uniformly to some function f, f is differentiable and $f' = g$.

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Weake's Theorem for differentiability:

Let $I = [a,b]$, let (f_n) be a sequence of functions on $I \to \mathbb{R}$ that converges on I to f. Suppose each derivative f_n' is cont. and (f_n') to g uniformly. Then f is diff. and $f'(x) = g(x)$.

Ex: $f_n(x) = \frac{x^n}{n}$; $x \in [0,1]$

$f_n \to f \equiv 0$

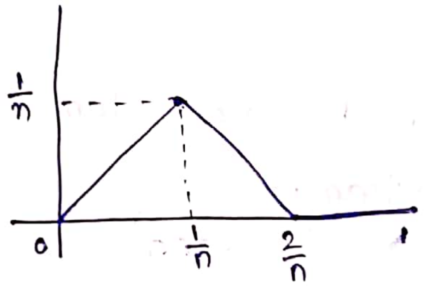
$f_n'(x) = x^{n-1}$

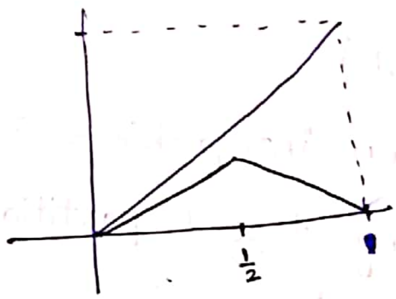
$f_n' \to g \qquad g(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}$

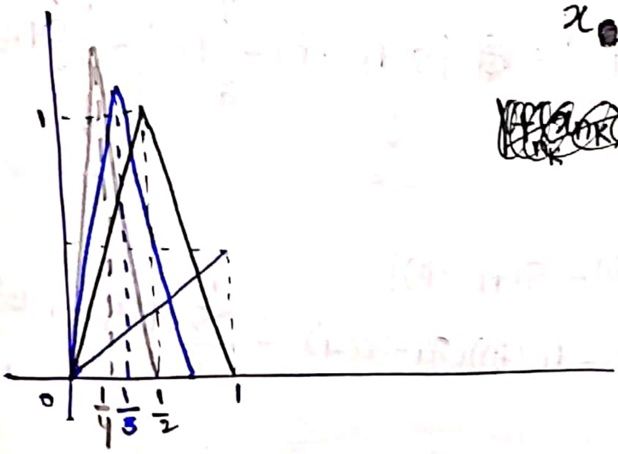
not uniformly continuos.

$f_n \to f$ when will $\lim_{n\to\infty} \int_a^b f_n = \int_a^b \lim_{n\to\infty} f_n$

Ex 1: $f_n(x) = \begin{cases} n^2 x & ; 0 \le x \le 1/n \\ -n^2\left(x - \frac{2}{n}\right) & ; \frac{1}{n} \le x \le \frac{2}{n} \\ 0 & ; \frac{2}{n} \le x \le 1 \end{cases}$







$x_K = \frac{1}{K}$ $\qquad K = 1, 2, 3, \ldots$

$n_K = K$

$x_{n_K} = \frac{1}{K}$

$|f_{n_K}(x_{n_K}) - f(x_{n_K})| = |K| > 1$

$\therefore$ not uniformly continuos.

Thm:

Each $f_n [a,b]$ is riemann integrable , $f_n \to f$ uniformly. Then, f is riemann integrable and $\lim_{n\to\infty} \int_a^b f_n \, dx = \int_a^b f \, dx$.

proof:

By cauchy criterion of uniform convergence,

For $\epsilon > 0$, $\exists H \in \mathbb{N}$ st $\forall m,n \geq H$,

$$\| f_m - f_n \| < \epsilon \qquad A = \int_a^b f$$

$$\Rightarrow -\epsilon < f_m(x) - f_n(x) < \epsilon \quad \forall x \in [a,b]$$

$$\Rightarrow \int_a^b (-\epsilon) < f_m(x) - f_n(x) < \int_a^b \epsilon$$

$$\Rightarrow -\epsilon(b-a) < \int_a^b f_m - \int_a^b f_n < \epsilon(b-a)$$

$$\Rightarrow \left| \int_a^b f_m - f_n \right| < \epsilon \quad \forall m,n \geq H.$$

$\Rightarrow \int_a^b f_n$ is cauchy convergent ~~on A~~

$\int_a^b f_n \to A \Rightarrow \exists K_1$ st $n \geq K_1$ $\left| \int_a^b f_n - A \right| < \epsilon$

let $K_2 = \max\{K_1, H\}$

f_{K_2} is riemann integrable $\Rightarrow \exists$ a partition P_2 finer than P_ϵ such that for any tagged partition $\dot{P}$ finer than P_ϵ,

$$|S(f_{K_2}; \dot{P}) - \int_a^b f_{K_2}| < \epsilon$$

$$|S(f,\dot{P}) - A| \leq |S(f;\dot{P}) - S(f_{K_2},\dot{P})| + \underbrace{|S(f_{K_2};\dot{P}) - \int_a^b f_{K_2}|}_{<\epsilon} + \underbrace{\left|\int_a^b f_{K_2} - A\right|}_{<\epsilon}$$

$$\leq$$

we need to show that $|S(f;\dot{P}) - S(f_{K_2};\dot{P})|$

$$= \left| \sum_{i=1}^{n} (f(t_i) - f_{K_2}(t_i))(x_i - x_{i-1}) \right| \leq \sum_{K=1}^{n} |f(t_i) - f_{K_2}(t_i)|(x_i - x_{i-1})$$

$|f(t_i) - f_{k_2}(t_i)| < \epsilon$

$\Rightarrow |S(f;\dot{P}) - S(f_{k_2};\dot{P})| \le \epsilon(b-a)$

$\Rightarrow$ f is Riemann integrable

$$A = \int_a^b f$$

(5). Use mean value theorem,

$$|\sin x - \sin y| \le |x-y| \quad \forall x, y$$

(6). $f(x) := 2x^4 + x^4 \sin\left(\frac{1}{x}\right);\ x \neq 0$

$f(0) := 0$

show that f has an absolute ~~max.~~ min, but its derivative has both +ve and −ve in every nbhd of 0.

Sol: $f(x) = 2x^4 + x^4 \sin\left(\frac{1}{x}\right) \ge 2x^4 - x^4 = x^4 \ge 0$

f has absolute minimum at $x = 0$.

$x \neq 0$, $f'(x) = 8x^3 + 4x^3 \sin\left(\frac{1}{x}\right) - x^2 \cos\left(\frac{1}{x}\right)$

$= x^2 \left(8x + 4x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)\right)$

$x = \frac{1}{2n\pi}$

$$f'(x) = \left(\frac{1}{2n\pi}\right)^2 \left(\frac{8}{2n\pi} - 1\right) < 0$$

$x = \frac{1}{(2n+1)\pi}$

$$f'(x) = \quad > 0.$$

(7). $f: I \to R$, is diff on I, If f' is never 0, then either $f'(x) > 0$, $\forall x \in I$ or $f'(x) < 0$ $\forall x \in I$.

(Darboux Theorem).

11-24-1)

(8). f, g diff on $\mathbb{R}$, $f(0)=g(0)$, $f'(x) \leq g'(x)$. show that $f(x) \leq g(x)$ $\forall x \geq 0$.

(11). f is cont on $[a,b]$ f'' exists on (a,b) suppose that the graph of f and the line segment joining $(a, f(a))$ and $(b, f(b))$ intersects at $(x_0, f(x_0))$, $a < x_0 \leq b$. show $\exists\, c \in (a,b)$ st $f''(c)=0$.

Sol:

$$\frac{y-f(b)}{x-b} = \frac{f(a)-f(b)}{a-b}$$

$$\frac{f(x_0)-f(b)}{x_0-b} = \frac{f(a)-f(b)}{a-b} = \frac{f(b)-f(a)}{b-a}$$

$$\Rightarrow f(x_0) = f(b) + \frac{(x_0-b)}{(b-a)}(f(b)-f(a))$$

define $g(x) := f(x) - \left(f(b) - \frac{(x-b)(f(b)-f(a))}{(b-a)}\right)$

$x=a$, $g(a)=0$, $g(b)=0$, $g(x_0)=0$

By MVT, $\exists\, c_1 \in (a, x_0)$ st

$g'(c_1)=0$

$\exists\, c_2 \in (x_0, b)$ st $\quad g'(c_2)=0$

$\therefore \quad \exists\, c_3 \in (c_1, c_2)$ st $\quad g''(c_3)=0$

$\Rightarrow f''(c_3)=0$

$f: I \to \mathbb{R}$, I is an interval, f is diff on I, $f''(a)$ exists

$$f''(a) = \lim_{h \to 0} \frac{f(a+h)+f(a-h)-2f(a)}{h^2}$$

Sol: For given $\epsilon>0$, $\exists\, \delta>0$ st

$$\left|\frac{f'(a+h)-f'(a)}{h} - f''(a)\right| < \epsilon$$

$$\left|\frac{f(a+h)+f(a-h)-2f(a)}{h^2} - f''(a)\right| \leq \left|\frac{f(a+h)+f(a-h)-2f(a)}{h^2} - \frac{f'(a+h)-f'(a)}{h}\right| + \left|\frac{f'(a+h)-f'(a)}{h} - f''(a)\right|$$

$0 < \theta < 1$

$$= \left| \frac{f(a+h) - f(a+\theta h) + f(a+\theta h) - f(a) + f(a-h) - f(a)}{h^2} - \left(\frac{f'(a+h) - f'(a)}{h}\right) \right|$$

$$= \left| \frac{\frac{f(a+h) - f(a+\theta h)}{h} - f'(a+h) + \frac{f(a+\theta h) - f(a)}{h} - f'(a) + \frac{f(a-h) - f(a)}{h} + f'(a)}{h} \right|$$

Theorem

If (f_n) be a sequence of functions such that (f_n) converges to f uniformly. $f_n \in R[a,b]$ and $\lim_{n\to\infty} \int_a^b f_n \, dx = \int_a^b f \, dx$

Theorem

$I = [a,b]$ let (f_n) be a sequence of functions on $I \to R$ that converges on I to f. Suppose each derivative f_n' is continuos on I and the sequence (f_n') is uniformly convergent to g on I. Then,

$$f(x) - f(a) = \int_a^x g(t)\,dt \text{ . So, } f'(x) = g(x) \quad \forall x \in I$$

Sol: Each $f_n' \in R[a,b]$ and f_n' is continuos. Previous theorem say $g \in R[a,b]$ and $\int_a^x g(t)\,dt = \lim_{n\to\infty} \int_a^x f_n'\,dt = \lim_{n\to\infty} (f_n(x) - f_n(a))$

(By Fundamental theorem of calculus)

$$= f(x) - f(a)$$

$$\Rightarrow \boxed{f'(x) = g(x)}$$

Ex: $f_n(x) = nx(1-x)^n$ $\to [0,1]$. Show that (f_n) converges nonuniformly to an integrable function f and $\int_0^1 f\,dx = \lim \int_0^1 f_n(x)\,dx$.

Sol: $\lim_{n\to\infty} nx(1-x)^n$

$$f_n(x) = nx(1-x)^n \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

let $1 - x = \frac{1}{1+y}$, $y > 0$ $\quad x = 0, 1.$

$$nx(1-x)^n = \frac{ny}{(1+y)^{n+1}}$$

$$(1+y)^{n+1} \geq \frac{n(n+1)}{2} y^2$$

$$\frac{ny}{(1+y)^{n+1}} \le \frac{ny}{\frac{(n+1)ny^2}{2}} = \frac{2}{y}\frac{1}{n+1}$$

$$\Rightarrow \frac{ny}{(1+y)^{n+1}} \longrightarrow 0$$

limit function is $f \equiv 0$.

$$c_n = \frac{1}{n+1} \qquad c_n \longrightarrow 0$$

$$f_n(c_n) = \frac{n}{n+1}\left(\frac{n}{n+1}\right)^n = \left(\frac{n}{n+1}\right)^{n+1} \longrightarrow \frac{1}{e}$$

$\Rightarrow$ convergence is not uniform.

Bounded convergence theorem:

Let (f_n) be a sequence in $R[a,b]$ that converges to $f \in R[a,b]$. Suppose $\exists B>0$ such that $|f_n(x)| \le B$ $\forall x \in [a,b]$ $\forall n \in \mathbb{N}$. Then,

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$$

Ex: $f_n(x) = \frac{nx}{1+nx}$ $\forall x \in [0,1]$ $= 0$ if $x = 0$

$f_n \to f \equiv 1$ $\forall x \in (0,1]$

Show that the convergence is not uniform.

Sol: $\int_0^1 f_n(x)\,dx = 1 - \frac{1}{n}\ln(n+1)$

$B = 1$

Dini's Theorem:

suppose that (f_n) is a monotone sequence of continuos funotions on $I = [a,b]$ that converges on I to a continuos functions f. Then the convergence is uniform.

midsem: 10-12, 13 m

Remarks:

1). We need I to be closed and bounded. Take

$$f_n(x) = \frac{x}{n} \quad \forall x \in [0,\infty)$$

$f_n \to$ decreasing sequence of continuos function.

$f_n \to f \equiv 0$ (continuos)

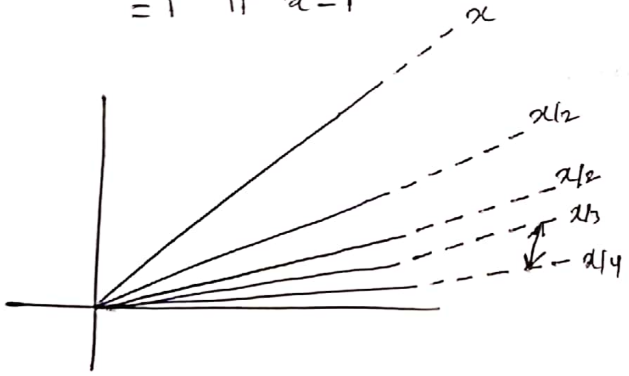
But the convergence is not uniform.

2). Limit function f needs to be cont,

$$f_n(x) = x^n, \; x \in [0,1]$$

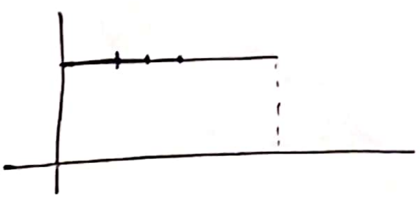
$f = 0$ if $x < 1$
$= 1$ if $x = 1$

convergence is not uniform.



3). Each f_n needs to be cont.

$$f_n(x) = \begin{cases} 1 & \text{if } x \in (0, 1/n) \\ 0 & o/w \end{cases}$$



$f_n \to f \equiv 0$

convergence is not uniform.

~~$\|f_n - f\|$~~ $\|f_n - f\| = 1$

Series of Functions:

Let $f_n : A \to R$ Fix $x_0 \in A$

$\sum f_n(x_0)$

sequence of partial sums:

$$s_1(x_0) = f_1(x_0)$$
$$s_2(x_0) = s_1(x_0) + f_2(x_0)$$
$$\vdots$$
$$s_n(x_0) = f_1(x_0) + f_2(x_0) + \dots + f_n(x_0)$$

Def:

1). Pointwise convergence:

$\{s_k(x)\}$ converges $\forall x \in A$.

2). Absolute convergence:

$s_k'(x_0) = |f_1(x_0)| + \dots + |f_n(x_0)|$

$\{s_k'(x_0)\}$ ~~c~~onverges $\forall x \in A$

3) Uniform convergence:

The sequence $\{s_k(x)\}$ converges on A uniformly.

(We denote that by $\sum_{n=1}^{\infty} f_n$)

Lemma: The sequence of functions (f_n) converges uniformly the function $f \equiv 0$.

proof: $s_n(x) \longrightarrow s(x)$ uniformly

$\exists \epsilon > 0,\ \exists N \in \mathbb{N}$ st $\|s_n(x) - s(x)\| < \epsilon$

$\Rightarrow |s_n(x) - s_{n+1}(x)| < \epsilon \quad \forall n \geq N \quad \forall x \in A.$

$\Rightarrow |f_{n+1}(x)| < \epsilon \quad \forall n \geq N \quad \forall x \in A.$

Properties of uniformly convergent series:

Theorem:

Each $f_n : A \to R$ is continuos A The series Σf_n converges uniformly to s. Then, s is ~~constant~~. continuos.

Theorem:

Each $f_n : A \to R$ is Riemann integrable. The series Σf_n converges uniformly to s. Then, s is also Riemann Integrable and

$$\sum_{n=1}^{\infty} \int_a^b f_n = \int_a^b s = \int_a^b \sum_{n=1}^{\infty} f_n$$

Theorem:

Bounded interval

Each f_n is differentiable on I, $\Sigma f_n'$ converges uniformly to function t.

At a point x_0,

$\Sigma f_n(x_0)$ converges. Then Σf_n converges uniformly to a function s and $s' = t$.

$$(\Sigma f_n)' = \Sigma f_n'$$

Cauchy criterion:

Each $f_n : A \to R$ Σf_n converges uniformly iff

for given $\epsilon > 0$ $\exists N \in \mathbb{N}$ st

$$\| f_{n+1} + f_{n+2} + \dots + f_m \| < \epsilon \quad \forall\, m > n \geq N \quad \forall \in \quad \forall x \in A$$

Comparision Test for series:

$0 \leq a_n \leq b_n$ $\forall n \in \mathbb{N}$ Σb_n converges $\Rightarrow$ Σa_n converges

If $|a_n|$ converges $\Rightarrow$ Σa_n converges.

Weierstrass M-test:

Each $f_n: A \to R$ For each $n \in \mathbb{N}$, $\exists M_n$ such that $|f_n(x)| \le M_n \ \forall x \in A$

If $\sum M_n$ converges, then the series $\sum f_n$ converges uniformly.

Ex:

1). $\sum a_n \sin nx$, $\sum |a_n|$ is convergent

$\sum |f_n|$ converges uniformly $\Rightarrow \sum f_n$ converges uniformly.

$\Downarrow$

$|f_{n+1}| + \dots + |f_m| < \epsilon \quad \forall m, n \ge N(\epsilon)$

$\forall x$

$\vee|$

$|f_{n+1} + \dots + f_m| \Rightarrow$ ~~Ab~~ $\sum f_n$ converges uniformly.

1) $f_n(x) = \dfrac{1}{n^2 + x^2} \quad \forall x \in \mathbb{R}$

$|f_n(x)| = \dfrac{1}{x^2 + n^2} \le \dfrac{1}{n^2}$

$\sum \frac{1}{n^2}$ converges so $\sum f_n$ converges.

2). $f_n(x) = \dfrac{x^n}{1+x^n}$ ~~$\forall x \in \mathbb{R}$~~ $\forall x \ge 0$

$x \ge 1 \quad \lim_{n \to \infty} f_n(x) = 1$

But for ~~uniform~~ Pointwise convergency for $n \ge 1$, $\sum f_n(x)$, $f_n \to 0$.

$\dfrac{x^n}{1+x^n} \le x^n$

$\sum x^n$ converges $\Rightarrow \sum \dfrac{x^n}{1+x^n}$ converges.

(By comparision test)

$\sum \dfrac{x^n}{1+x^n}$ converges uniformly on $[0, r]$ $r \le 1$.

$\|f_n(x)\| = \dfrac{x^n}{1+x^n} \le x^n \le r^n \quad x \in [0, r] \quad \sum r^n$ converges if $r < 1$.

$\Rightarrow$ By weirstrass M-test, f_n is UC on $[0, r]$

proof by cauchy criterion,

For $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $n > m \geq N$

$$|f_{m+1}(x) + \dots + f_n(x)| < \epsilon \quad \forall x \in A$$

Take $\varepsilon = 1/3$

$$\sup_{x \in [0,1]} f_{m+1}(x) = 1/2$$

$$\sup |f_{m+1}(x) + \dots + f_n(x)| \geq 1/2$$

$\therefore$ Not uniformly convergent.

Boundedly convergent: (For sequence of functions)

A sequence of functions (f_n) is said to be boundedly convergent on A if (f_n) is pointwise convergent and uniformly bounded on A.

~~Arzela's theorem:~~

$\exists M > 0$ $|f_n(x)| \leq M$ $\forall n \in \mathbb{N}$, $\forall x \in A$

Thm (Arzeli's):

Assume (f_n) is boundedly convergent on $[a,b]$ and suppose each $f_n \in R[a,b]$ ~~then~~ also assume $f \in R[a,b]$

Then, $\lim_{n \to \infty} \int_a^b f_n(x) = \int_a^b f(x)dx.$

Remark:

We can't drop the condition $f \in R[a,b]$

As let $\{r_1, r_2, \ldots, r_n, \ldots\}$ be the set of rationals.

$$f_n(x) = \begin{cases} 1 & \text{if } x = r_1, \ldots r_n \\ 0 & \text{o/w} \end{cases}$$

Each $f_n \in R[a,b]$ But the limit $f \equiv \begin{cases} 1 & \text{if } x \in Q \\ 0 & \text{otherwise} \end{cases}$ is not RI

(f_n) is boundedly convergent.

Dirichlet Test: of uniform convergency:

let $(f_n(x))$ be a sequence of functions. Let $F_n(x)$ denote the n^{th} partial sum. Assume (F_n) is uniformly bounded on A. Let g_n be a sequence of real ~~numbers~~ valued functions st $g_{n+1}(x) \le g_n(x)$ $\forall x \in A$ $\forall n$

Assume $g_n \to 0$ uniformly on A. Then $\sum f_n g_n$ converges uniformly.

Ex1). $\sum \frac{e^{inx}}{n}$

Here, $f_n(x) = e^{inx}$; $g_n(x) = \frac{1}{n}$

$$F_n(x) = e^{ix} + \ldots + e^{inx}$$

It can be shown that $|F_n(x)| \le \frac{1}{|\sin(\frac{x}{2})|}$ when $x \ne 2m\pi$

Take $x \in (\delta, 2\pi - \delta)$ so $|\sin(\frac{x}{2})| > |\sin(\frac{\delta}{2})|$

$$\Rightarrow |F_n(x)| \le \frac{1}{|\sin(\frac{\delta}{2})|}$$

$g_n \to 0$ ~~uniformly~~ $\to 0$ & $g_n(x) \le g_{n+1}(x)$

So, $\sum f_n g_n$ converges uniformly.

$$\Rightarrow \quad f''(a) = \lim_{h\to 0} \frac{f(a+h)+f(a-h)-2f(a)}{h^2}$$

$$\left| \frac{f'(a+h)-f(a)}{h} - f''(a) \right| < \epsilon \quad \forall\, |h| < \delta$$

$$\left| \frac{f(a+h)-f(a+\theta h)}{(1-\theta)h} - f'(a+h) + \frac{f(a+\theta h)-f(a)}{\theta h} - f'(a) + \left(\frac{f(a-h)-f(a)}{-h} - f'(a) \right) \right| < \epsilon$$

$$0 < \theta < 1$$

$$\Rightarrow \left| \frac{f(a+h)-f(a+\theta h)}{h} - (1-\theta) f'(a+h) + \frac{f(a+\theta h)-f(a)}{h} - \theta f'(a) + \frac{f(a-h)-f(a)}{h} + f'(a) \right| < (1-\theta)\varepsilon$$

$$\Rightarrow \left| \frac{f(a+h)+f(a-h)-2f(a)}{h} - (1-\theta)\left(f'(a+h)-f'(a)\right) \right| < (1-\theta)\varepsilon$$

$$\Rightarrow \left| \frac{f(a+h)+f(a-h)-2f(a)}{h^2} - (1-\theta)\frac{f'(a+h)-f'(a)}{h} \right| < \frac{(1-\theta)\varepsilon}{h} < \varepsilon$$

by taking $1-\theta < h$.

$$\Rightarrow \left| \frac{f(a+h)+f(a-h)-2f(a)}{h^2} - (1-\theta) f''(a) \right| < \varepsilon + (1-\theta)\varepsilon$$

$$< \underline{\underline{2\epsilon}}$$

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Assignment - 8

4). $f \in R[a,b]$ and if $f(x) = g(x)$ except for a finite number of points prove that $g \in R[a,b]$ and $\int_a^b f = \int_a^b g$.

Soln: $h(x) = f - g$

$h \equiv 0$ except for a finite number of points.

$h \in R[a,b] \Rightarrow g = f - h \Rightarrow g \in R[a,b]$

$\int_a^b h = 0 \Rightarrow \int_a^b f = \int_a^b g$

let $M = \max\{h(c_1), \dots h(c_n)\}$

$\dot{P}$ be any partition,

$S(h,\dot{P}) = \sum S(h,\dot{P}) + \sum S(h,\dot{P}) \leq M(2n)\|\dot{P}\| + 0$

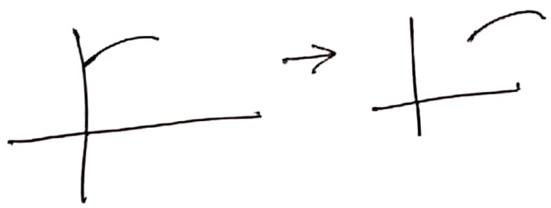
$c_1 \dots c_n$ as tag

Take $\|\dot{P}\| = \frac{\varepsilon}{2Mn}$

$\therefore |S(h,\dot{P}) - 0| < \epsilon$.

5). If $f \in R[a,b]$ and $c \in \mathbb{R}$ we define g on $[a+c, b+c]$ as $g(y) := f(y-c)$

prove $g \in R[a+c, b+c]$ and $\int_{a+c}^{b+c} g = \int_a^b f$.



6). Show that there exists a step function φ on $[a,b]$ st

$$S(\varphi, P) = \int_a^b f$$

Soln: $$\varphi(x) = \frac{\int_a^b f}{b-a}$$

5). If $c \in [a,b]$ then $\overline{\int_a^b} f = \overline{\int_a^c} f + \overline{\int_c^b} f$

Sol: $$\overline{\int_a^b} f = \inf_P U(P,f)$$

Take a partition P_1 of $[a,c]$ and P_2 of $[b,c]$

$P = P_1 \cup P_2$ is partition of $[a,b]$

$$U(P,f) = U(P_1,f) + U(P_2,f)$$
$$\geq \inf U(P_1,f) + \inf U(P_2,f)$$
$$\uparrow \qquad \qquad \uparrow$$
$$U(P_3,f) \qquad U(P_4,f)$$

$P_5 = P_3 \cup P_4$

we have to show that, $\overline{\int_a^b} f \leq \overline{\int_a^c} f + \overline{\int_c^b} f$

$$\overline{\int_a^b} f \geq \overline{\int_a^c} f + \overline{\int_c^b} f$$

Case 1: $c \in P$ $\quad P_1 = P \cap [a,c]$

$P_2 = P \cap [a,c]$

$$U(P,f) = U(P_1,f) + U(P_2,f)$$

case-2: $c \notin P$ $\quad$ let $P' = P \cup \{c\}$

$P_3' = P' \cap [a,c]$ $\quad P_4 = P' \cap [c,b]$

$$U(f,P) \geq U(f,P_3) + U(f,P_4) \geq \inf f + \inf f \geq \overline{\int_a^c} f + \int_c^5 f$$

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2). If $f \in R[a,b]$ and if $(\dot{P}_n)$ in any sequence of tagged partitions of $[a,b]$ such that $\|\dot{P}_n\| \to 0$ prove that $\int_a^b f = \lim_{n\to\infty} S(f;\dot{P}_n)$

sol: $f \in R[a,b] \Rightarrow$ for $\epsilon > 0$, $\exists\, \delta > 0$ st $\|\dot{P}\| < \delta$

$$|S(f;\dot{P}) - \int_a^b f| < \epsilon$$

$\|\dot{P}_n\| \to 0$ $\exists N \in \mathbb{N}$ st $n \geq N \Rightarrow \|\dot{P}\| < \delta$

$$\Rightarrow |S(f;\dot{P}_n) - \int_a^b f| < \epsilon \quad \forall n \geq N.$$

3). f is bounded on $[a,b]$ and $\exists$ two sequences of tagged partitions of $[a,b]$ such $\|\dot{P}_n\| \to 0$ $\|\dot{Q}_n\| \to 0$ but,

$\lim_{n\to\infty} S(f;\dot{P}_n) \neq \lim_{n\to\infty} S(f;\dot{Q}_n)$ Then, $f \notin R[a,b]$

Assignment-9

3). f and g are continuous on $[a,b]$ and $g(x)>0$ $\forall x \in [a,b]$

show that $\exists c \in (a,b)$ st $\int_a^b fg = f(c)\int_a^b g$. Show that the conclusion fails if we don't assume $g(x)>0$.

$$m \le \frac{\int_a^b fg}{\int_a^b g} \le M$$

$\therefore$ $\exists c$ st $f(c)\int_a^b g = \int_a^b fg$

$m \le f(x) \le M$

$mg(x) \le f(x)g(x) \le M(g(x))$

Here to Here only if $g(x)>0$ (or) $g(x)<0$.

4). f is continuous on $[a,b]$ $f(x) \ge 0$ $\forall x \in [a,b]$

let $M_n = \left(\int_a^b f^n\right)^{1/n}$

show $\lim M_n = \sup\{f(x) \mid x \in [a,b]\}$

Sol: let $K = \sup\{f(x) \mid x \in [a,b]\}$

$f(x) \le K$

$f^n(x) \le K^n$

$\Rightarrow \left(\int_a^b f^n\right) \le K^n(b-a)$

$\Rightarrow \left(\int_a^b f^n\right)^{1/n} \le K(b-a)^{1/n}$

$M_n \le K(b-a)^{1/n}$

$\Rightarrow \boxed{\lim M_n \le K}$

For $\epsilon>0$ $\exists x_0 \in [a,b]$ $f(x_0) > K-\epsilon$

$\exists$ a nbhd $(x_0-\delta, x_0+\delta)$ st $f(x) > K-\epsilon$ $\forall x \in (x_0-\delta, x_0+\delta)$

$f^n > (K-\epsilon)^n$ on $(x_0-\delta, x_0+\delta)$

$$\int_a^b f \geq \int_{x_0-\delta}^{x_0+\delta} f^n = (K-\epsilon)^n (2\delta)$$

$\lim M_n \geq (K-\epsilon) \quad \forall \epsilon$

$\Rightarrow \quad K-\epsilon \leq \lim M_n \leq K$

$\Rightarrow \quad \boxed{\lim M_n = K}$

6). $f:[a,b] \to R$ and $\phi_1\ \phi_2$ are two antiderivates of f on $[a,b]$ show that $\phi_1 - \phi_2$ is a constant.

Sol: $\int_a^x f - \int_c^x f$

$\phi_1' = \phi_2' = f$

$(\phi_1 - \phi_2)' = 0$

$\cancel{\phi_1(x)} = (\phi_1 - \phi_2)(x) - (\phi_1 - \phi_2)(a) = 0$

$\Rightarrow \quad \phi_1 - \phi_2 = \text{constant}$

mean value theorem

7). $f:[0,1] \to R$ $\quad \int_0^x f = \int_x^1 f$

St $f(x) = 0 \quad \forall x \in (0,1)$

Sol: $\int_0^x f = \frac{1}{2}\int_0^1 f$

$\left(\int_0^x f\right)' = f(x) = 0 \quad \forall x$

Assignment-9

5). $f(x) = (x) \quad |x| \geq 1$

$\quad = (-x) \quad |x| < 1$

$F(x) = \frac{1}{2}|x^2 - 1|$

$$F(x) = \begin{cases} \frac{1}{2}(x^2-1) ; & |x| \geq 1 \\ \frac{1}{2}(1-x^2); & |x| < 1 \end{cases}$$

$F'(x) = f(x) \quad \forall x \neq \pm 1$

$f(-1) = -1$

$f(1) = 1$

$f(x) = -x \quad \forall x \, (-1, 1)$

$f \equiv g$ except at $x = -1, 1 \qquad g(x) = x \quad \forall x \in [-1, 1]$

$\therefore g \in R[-1, 1] \qquad \therefore f \in R[-2, 3]$

$\therefore f \in R[-2, -1] \quad f \in R[-1, 1] \quad f \in [1, 3]$

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By fundamental theorem of calculus,

$$\int_{-2}^{3} f = F(3) - F(-2) = 5/2$$

2). show that $f(x) = \sin\frac{1}{x}, x \in (0,1]$
$= 0, x = 0$

$\in R[0,1]$

Sol:

If f is bounded by M on $[a,b]$ and if the restriction of f to every interval $[c,b]$, $c \in (a,b)$ is RI. Then $f \in R[a,b]$ and $\int_c^b f \to \int_a^b f$

squeeze theorem:

For $\varepsilon > 0$ $\exists\, \alpha_\varepsilon, \omega_\varepsilon$ st $\alpha_\varepsilon \le f \le \omega_\varepsilon$ and $\int_a^b (\omega_\varepsilon - \alpha_\varepsilon) < \varepsilon$

$$\alpha_\varepsilon^{(x)} = f(x) \quad x \in [c,b]$$
$$= -M, \quad x \in [a,c]$$

$$\omega_\varepsilon^{(a)} = f(x); \quad x \in [c,b]$$
$$= M, \quad x \in [a,c]$$

$$\int_a^b (\omega_\varepsilon - \alpha_\varepsilon)dx = 2M(c-a) < \epsilon$$

$$\cancel{M//A} \quad c - a < \frac{\epsilon}{2M}$$

$$\Rightarrow \boxed{c < a + \frac{\epsilon}{2M}}$$

$\therefore f \in R[a,b]$ by squeeze theorem.

Assignment - 10

1). $f_n(x) = \dfrac{x^n}{1+x^n}$ for $x \geq 0$ let $0<b<1$

uniform convergence on $[0,b]$ and not uniform of $[0,1]$

Sol: $f_n(x) = \dfrac{x^n}{1+x^n} \leq x^n$ $\lim\limits_{n\to\infty} f(x) = \begin{cases} 0, & x \neq 1 \\ 1/2; & x = 1 \end{cases}$

$\therefore$ convergence is not uniform on $[0,1]$.

$\|f_n(x) - 0\| = |x^n| \leq |b|^n$

if $|b|^n < \epsilon$

$\therefore$ convergence is uniform on $[0,b]$.

2). If $a>0$, $n^2x^2e^{-nx}$ CU on $[a,\infty)$ but not on $[0,\infty)$

Sol: $\cancel{f(x)}$ $f(x) \equiv 0$

$x = \frac{1}{n}$ $|f_n(x) - f(x)| \not\to 0$

$\therefore$ Not uniformly continuos on $[0,\infty)$

$|n^2x^2e^{-nx}| < \epsilon$

$\cancel{a^2 < \epsilon e^a}$

$\cancel{2 \ln a < \ln \epsilon + a}$

$2xn^2e^{-nx} - n^2x^2ne^{-nx} = 0$ $x > 2/n$ decreases.

$\Rightarrow$ $xn = 2$ $\boxed{x = 2/n}$

$n^2x^2e^{-nx} \leq \dfrac{6n^2x^2}{n^3x^3} \leq \dfrac{6}{nx} \leq \dfrac{6}{nb} < \epsilon$

if $n > 6/\epsilon b$

Q3). (f_n) converges uniformly on A to f and $|f_n(x)| \leq M \ \forall n \in \mathbb{N}$ and $x \in A$. If g is continuos on $[-M, M]$ show that gof_n converges uniformly to gof on A.

Sol: $|f(x)| \leq |f(x) - f_n(x)| + |f_n(x)|$

$< \epsilon + |f_n(x)| \quad \forall n \geq N$

Take $n = N$ $\quad |f(x)| < \epsilon + |f_N(x)| < \epsilon + M$

$$|gof_n(x) - gof(x)| = |g(f_n(x)) - g(f(x))|$$
$$< \epsilon \text{ if } |f_n(x) - f(x)| < \delta.$$

as g is uniformly continuos.

$|f_n(x) - f(x)| < \delta$ if $n \geq k$ for some $k \in \mathbb{N}$.

Q4). f_n is uniformly continuos on $\mathbb{R}$, $f_n(x) = f(x + \frac{1}{n})$. ST (f_n) converges to f uniformly.

Sol: $\left|f(x+\frac{1}{n}) - f(x)\right| < \epsilon \quad \forall \ |x + \frac{1}{n} - x| < \delta.$

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Q5). $f_n(x) = \dfrac{nx}{1+nx^2}$, $x \in A = [0,\infty)$

show that each f_n is bounded on A.

But the pointwise limit f is not bounded on f. Does f converge uniformly?

Sol: $(1+nx^2)n - nx(2nx) = 0$

$\Rightarrow 1+nx^2 = 2x^2n$

$\Rightarrow \boxed{x = \frac{1}{\sqrt{n}}}$

$1+nx^2 \geq nx^2$

$\dfrac{nx}{1+nx^2} \leq \dfrac{1}{x} \leq 1$ on $[1,\infty)$

$1+nx^2 \geq 1$ $\quad \dfrac{nx}{1+nx^2} \leq nx \leq n.$

Each f_n is bounded.

pointwise limit $f(x) = 0, \; x=0$
$\frac{1}{x}, \; x \neq 0$

$\therefore$ f is not bounded on $[0,\infty)$

$x = \frac{1}{n}$ $\quad |f_n(x) - f(x_0)| = \left|\frac{n}{n+1} - n\right|$

Q6). If $a>0$, show that $\lim\limits_{n\to\infty} \int_0^{\pi} \dfrac{\sin nx}{nx}\,dx = 0$

what happens if $a=0$.

Sol: $f_n(x) = \dfrac{\sin nx}{nx}$

$\|f_n(x) - f(x)\| = \left|\dfrac{\sin nx}{nx}\right| \leq \dfrac{1}{na}$

$n \geq \dfrac{1}{a\epsilon}$

$\lim\limits_{n\to\infty} \int_a^{\pi} \dfrac{\sin nx}{nx}\,dx = \int_a^{\pi} = 0$

Take $x = \frac{1}{n}$ $\qquad |f_n(x) - f(x)| = \frac{\sin 1}{n}$

not UC on $[0, \pi]$

$$\lim_{x \to 0} \frac{\sin nx}{nx} = 1$$

$f_n(x) = g_n(x)$ except $x = 0$

Fix n, $\lim_{n \to \infty} \frac{\sin nx}{nx} = 1$

$\left| \frac{\sin y}{y} - 1 \right| < 1$ for $|y - 0| < \delta$

$\Rightarrow |nx| < \delta$

$\Rightarrow |x| < \frac{\delta}{n}$