Eigenvalues & Eigenvectors.

Let $A \in M_n(F)$

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in M_2(\mathbb{R})$$

$(A - \lambda I)x = 0$ $\quad\quad$ λ : Eigenvalue $\in F$

x : Eigenvector $(\neq 0)$

$\Rightarrow$ $y = Ax$ $\quad$ A transforms x to some other vector y

$\rightarrow$ If $A\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \alpha \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, then α is called eigenvalue

$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \Rightarrow$ Eigenvector

Defn :- A scalar $\lambda \in F$ is called Eigenvalue of A if $\exists$ a non zero vector $x \in F^n$ s.t. $Ax = \lambda x$. The vector x is called Eigenvector of A corresponding to the eigenvalue λ.

Eg. $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in M_2(\mathbb{R})$

Then you cannot find a scalar $\lambda \in \mathbb{R}$ s.t. $Ax = \lambda x$ for some scalar vector $x \in \mathbb{R}^2$

If $A \in M_2(\mathbb{R})$, then $\lambda = \pm i$

\# Let $A \in M_n(F)$. Let λ be a eigenvalue of A. Then $\det(A - \lambda I) = 0$

$\Rightarrow$ Since λ is an eigenvalue, there exists a non-zero vector x s.t. $Ax = \lambda x$.

$$\Rightarrow (A - \lambda I)x = 0$$

$(A - \lambda I)y = 0 \longrightarrow$ System of eq^{n}s.

$$\det(A - \lambda I) = 0$$

Q. Let $A \in M_n(F)$ & $\lambda \in F$. Let $\det(A-\lambda I) = 0$, then λ is an eigenvalue of A.

$\Rightarrow$ Since $\det(A-\lambda I) = 0$, so $(A-\lambda I)y = 0$ has non-trivial solⁿ.
There exists a non-zero vector $x \in F^n$ s.t. $(A-\lambda I)x = 0$
$\Rightarrow Ax = \lambda x$

Q. Let $A \in M_n(F)$ and $\lambda \in F$. Then λ is an eigenvalue of A iff $\det(A-\lambda I) = 0$

Defⁿ :- Let $A \in M_n(F)$. Then $\det(A-xI)$ is called characteristic polynomial of A.
$\Rightarrow \det(A-xI) = 0 \longrightarrow$ characteristic eqⁿs.

\# A is similar to B if $\exists$ a non-singular matrix P s.t. $P^{-1}AP = B$.
\# $\det(A-xI) = \det(B-xI)$. $\therefore$ Characteristic eqⁿ for similar matrices are same.

Eigenvalues / Eigenvectors of a LT :-
$\rightarrow$ Finite dim.

$T: V \to V$

Let $\lambda \in F$ is called eigenvalue of T if $T(x) = \lambda x$ for some non zero vector at $x \in V$.
x is eigenvector of T corresponding to 'λ' eigenvalue.

$T(x) = \lambda x \Rightarrow (T-\lambda I)x = 0$
$T-\lambda I : V \to V$ is a LT.
If $T-\lambda I$ is not one-one for some $\lambda \in F$, then λ is an eigenvalue.

$V \to$ Finite / infinite dim.
$T: V \to V$

Let $\lambda \in F$ and λ is called eigenvalue of T if $T-\lambda I$ is not one-one.

$\Rightarrow$ V → Finite dim. vector space.

one-one $\Leftrightarrow$ onto.

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V: Vector space over the field F.
$T: V \to V$.
Let $\lambda \in F$, then λ is called an eigenvalue
not one-one.

$T-\lambda I$ is not one-one $\Rightarrow \exists$ a non-zero ve
$(T-\lambda I)x = 0$.

① $geo(\lambda_i) =$ Nullity of $(T-\lambda_i I)$
② λ is an eigen value of T iff $T-\lambda I$ is not one-one.
③ A, B are similar, $|A| = |B|$
char. eqⁿ A = char. eqⁿ B.

$T(x) = \lambda x$ Then x is an eigenvector of T corresponding to the eigenvalue λ.

S = Set of all real sequences.
$T: S \to S$ s.t. $T(x_1, x_2, \ldots, \cancel{x_n}) = (0, x_1, \ldots\ldots)$

Then T doesn't have eigenvalue. Right shift operator.

(Exercise) $T_1: S \to S$ $T(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$
Left shift operator

$T: \mathbb{R}^2 \to \mathbb{R}^2$ $T(x_1, x_2) = (x_2, -x_1)$

$T(x_1, x_2) = \lambda(x_1, x_2)$.
or, $(T-\lambda I)(x) = T(x_1, x_2) - \lambda(x_1, x_2)$.

$(x_2, -x_1) - \lambda(x_1, x_2) = 0$
$(x_2 - \lambda x_1, -x_1 - \lambda x_2) = 0$

$x_2 - \lambda x_1 = 0$.
$\lambda x_1 + \lambda^2 x_2 = 0$.
$x_2(1+\lambda^2) = 0$. $\therefore x_2 = 0$, $x_1 = 0$.

If A & B are similar then,
① They have same dets.
② They have same char. polynomials.
③ They have same set of eigen values.

$V \to$ FDV over the field F.

$T : V \to V$. $\quad \det(T) = ?$

Let B be a basis for V.

$[T]_B = A \qquad \det(T) = \det(A)$.

B_1 - basis $\qquad [T]_{B_1} = C \qquad C = P^{-1}AP$

$\det(C) = \det(A)$.

\# $V \to$ FDV $\quad T : V \to V$

Characteristic polynomial of T.

$\det(T - xI)$

Characteristic eqn $\quad \det(T - xI) = 0$

\# If two matrices are similar then they have same characteristic polynomial.
But converse is not true.

Example. $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

$P^{-1}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} P = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. $\qquad x^2$ is the char. poly.

$\Rightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = P\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}P^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \rightarrow$ Not possible.

\# $A \in M_n(\mathbb{R})$

Does A always have an eigenvalue? No.

\# $A \in M_n(\mathbb{C})$

$\det(A - xI) =$ polynomial with complex coeff. A always have atleast one root.

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Let $A \in M_n(\mathbb{C})$ Let λ be an eigenvalue of A.

Algebraic multiplicity of λ $\rightarrow$ no. of times λ appears in $\det(A - xI)$.

S = set of all eigenvectors of A corresponding to the eigen value λ.

S is not a subspace. $E_\lambda = S \cup \{0\} \rightarrow$ subspace of $\mathbb{C}^n$.

E_λ is eigenspace corresponding to the eigenvalue λ.

$\mathbb{C}^n \rightarrow$ FD

E_λ is a subspace of $\mathbb{C}^n$.

E_λ is finite dimensional. $\dim(E_\lambda)$ is called geometric multiplicity of λ.

$A \in M_n(\mathbb{C}) \longrightarrow$ Eigen values $\lambda_1, \lambda_2, \ldots, \lambda_k$

$\lambda_1 \downarrow m_1$, $\lambda_2 \downarrow m_2$, ..., $\lambda_k \downarrow m_k$ $\qquad m_1 + m_2 + \ldots + m_k = n$

$E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_k} \Rightarrow$ Subspace of $\mathbb{C}^n$.

and, $E_{\lambda_i} \cap E_{\lambda_j} = \{0\}$

alg. $(\lambda) \geq$ geo. (λ).

Let geo $(\lambda) = m$.

$\dim(E_\lambda) = m$. Let B be a basis of E_λ.

$B = \{x_1, x_2, \ldots, x_m\}$.

Extend B to the basis of V.

$B' = \{x_1, \ldots, x_m, x_{m+1}, \ldots, x_n\}$.

$\hookrightarrow$ is a basis for $\mathbb{C}^n$

$$Ax_1 = \lambda x_1 = \lambda x_1 + 0x_2 + \cdots + 0x_n.$$

$$\vdots$$

$$Ax_m = \lambda x_m = 0x_1 + \cdots + \lambda x_m + \cdots 0x_n$$

$$Ax_{m+1} = c_{11}x_1 + c_{12}x_2 + \cdots + c_{1n}x_n.$$

$$\vdots$$

$$Ax_n = c_{n1}x_1 + c_{n2}x_2 + \cdots + c_{nn}x_n.$$

$$\cancel{A(x_1, x_2, \ldots, x_n) =}$$

$$A\underbrace{(x_1\ x_2 \ldots x_n)}_{P} = (x_1 \ldots x_n)\left(\begin{array}{ccc|c} \lambda & 0 & 0 & \\ 0 & \lambda & 0 & \\ \vdots & \vdots & \lambda & \\ 0 & 0 & 0 & \end{array}\right)$$

$$= \underbrace{(x_1 \ldots x_n)}_{P}\left(\begin{array}{c|c} \begin{matrix}\lambda & & \\ & \ddots & \\ & & \lambda\end{matrix} & * \\ \hline O & * \end{array}\right)$$

$$P^{-1}AP = \underbrace{\left(\begin{array}{c|c} \begin{matrix}\lambda & & \\ & \ddots & \\ & & \lambda\end{matrix} & * \\ \hline 0 & * \end{array}\right)}_{B}$$

P^{-1} exists as all column vectors are l.i.

$$D = \left(\begin{array}{c|c} A & C \\ \hline O & B \end{array}\right) = \det(D - xI) = \det(A - xI)\det(B - xI)$$

$$\det(A - xI) = \det(B - xI) = (x-\lambda)^m \det(* - xI)$$

$\therefore$ alg.$(\lambda) \geq m$. atleast m λ's are there for A.

$A \in M_n(F)$, then λ is eigen value of A in F, iff $A - \lambda I$ is singular.

Given,
① $alg(\lambda_i) = geo(\lambda_i)$ then, T is diagn.
② $\bigoplus_{i=1}^{k} E_{\lambda_i} = \mathbb{C}^n$
③ $alg.(\lambda_i) \geq geo(\lambda_i)$
④ Eigenvectors from different eigen values are l.i.

$\Rightarrow alg(\lambda) \geq m = geo(\lambda)$.

We have seen,

$$alg(\lambda_i) \geq geo(\lambda_i) \quad \text{for } i = $$

now, $m_1 + m_2 + m_3 + \dots + m_n = n$.

if $alg(\lambda_i) = geo(\lambda_i)$ for $i = 1, 2, \dots,$

$\dim(E_{\lambda_i}) = m_i$

then, $\dim(E_{\lambda_1} \oplus E_{\lambda_2} \oplus \dots \oplus E_{\lambda_k}) = n$

$E_{\lambda_1} \oplus E_{\lambda_2} \oplus \dots \oplus E_{\lambda_k} = \mathbb{C}^n$.

Let, $B_1 = \{x_{11}, x_{12}, \dots, x_{1m_1}\}$ → Basis for E_{λ_i}

$B_2 = \{x_{21}, x_{22}, \dots, x_{2m_2}\}$ $i = 1, 2, \dots, k$.

$B_3 = \{x_{31}, x_{32}, \dots, x_{3m_3}\}$

$\vdots$

$B_k = \{x_{k1}, x_{k2}, \dots, x_{km_k}\}$.

$A(x_{11}\ x_{12} \dots x_{1m_1} \dots x_{k1} \dots x_{km_k})$

$$= (x_{11}\ x_{12} \dots x_{1m_1} \dots x_{k1} \dots x_{km_k}) \begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_{m_1} & & & \\ & & & \lambda_k & & \\ & & & & \ddots & \\ & & & & & \lambda_{m_k} \end{pmatrix}$$

m_1 times, m_2 times, m_k times

Note. :- Eigenvectors from different eigenvalues are l.i.

$\underbrace{B_1 \cup B_2 \cup B_3 \cup \dots \cup B_k}_{\text{Cardinality} = n}$ is basis for $\mathbb{C}^n$.

$$diag(\underbrace{\lambda_1, \dots, \lambda_1}_{m_1 \text{ times}}, \underbrace{\lambda_k, \dots, \lambda_k}_{m_k \text{ times}})$$

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Let T be a L.T. on the FDVS V. We say that T is diagonalizable if there is a basis for V each vector of which is a characteristic vector of T.

Let A ∈ $M_n(\mathbb{C})$ A is said to be diagonalizable if ∃ a non singular matrix P s.t. $P^{-1}AP$ = Diagonal matrix.
If A is similar to a diagonal matrix then A is called diagonalizable.

* A is diagonalizable.

$$P^{-1}AP = \begin{pmatrix} c_1 & 0 & 0 & \cdots \\ \vdots & c_2 & & \vdots \\ & & \ddots & \\ 0 & \cdots & & c_n \end{pmatrix}.$$

$P = (x_1, x_2 \cdots, x_n)$ ⟶ Column vector.

$$AP = P\begin{pmatrix} c_1 & \cdots & 0 \\ \vdots & c_2 & \vdots \\ 0 & \cdots & c_n \end{pmatrix}$$

$$A(x_1 \ x_2 \quad x_n) = (x_1 \ x_2 \quad x_n)\begin{pmatrix} c_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & c_n \end{pmatrix}$$

$$(Ax_1 \ Ax_2 \quad Ax_n) = (c_1x_1 \quad c_2x_2 \ \cdots \ c_nx_n)$$

$$\left.\begin{array}{l} Ax_1 = c_1x_1 \\ Ax_2 = c_2x_2 \\ \vdots \\ Ax_n = c_nx_n \end{array}\right\} c_1, c_2, \cdots, c_n \text{ are eigenvalues of } A.$$

Each column vector of P is eigen~~value~~ vector of A.

$A \in M_n(\mathbb{C})$

$\lambda_1 \neq \lambda_2$ eigenvalue of A.

$\downarrow \quad \downarrow$

$x_1 \quad x_2$. $\{x_1, x_2\}$ is LI.

$\lambda_1 \cdots \lambda_k$.

$x_1 \cdots x_k$ $\{x_1, \cdots, x_k\}$ are L.I.

① If A is diagonalizable, then $\text{alg}(\lambda_i) = \text{geo}(\lambda_i)$ for $i = 1, 2, \ldots, k$.

$A \longrightarrow \lambda \begin{smallmatrix} \nearrow x_1 \\ \searrow x_2 \end{smallmatrix}$ They might be any

$A \in M_n(\mathbb{C})$

$\lambda_1 \quad \ldots \quad \lambda_n \qquad \lambda_i \neq \lambda_j \quad i \neq j$

$x_1 \quad \ldots \quad x_n$

A is diagonalizable $P = (x_1, \ldots, x_n)$.

\# Let $A \in M_n(\mathbb{C})$ has distinct eigenvalues $\lambda_1, \ldots, \lambda_k$

Assume that $\text{alg}(\lambda_i) = \text{geo}(\lambda_i)$ for $i = 1, \ldots, k$

Then, A is diagonalizable.

and if A is diagonalizable, then $\text{alg}(\lambda_i) = \text{geo}(\lambda_i)$

$\exists$ a non sing. matrix P.

s.t. $P^{-1}AP = \begin{pmatrix} \lambda_1 & & & & \\ & \ddots & & & \\ & & \lambda_1 & & \\ & & & \ddots & \\ & & & & \lambda_k \\ & & & & & \ddots \\ & & & & & & \lambda_k \end{pmatrix}$

$AP = P \begin{pmatrix} & \\ & \end{pmatrix}$. $\qquad P = (x_1 \ldots x_n)$

$Ax_1 = \lambda_1 x_1$.	$Ax_{m_1+1} = \lambda_2 x_{m_1+1}$	$Ax_{n_k+1} = \lambda_k x_{n_k+1}$
$\vdots$	$\vdots$	$\vdots$
$Ax_{m_1} = \lambda_1 x_{m_1}$	$Ax_{m_2} = \lambda_2 x_{m_2}$	Ax_{m_k}.

$x_1 \quad x_{m_1} \qquad x_{m_1+1} \quad x_{m_2} \qquad x_{m_{k-1}+1} \quad x_{m_k}$.

Eigen vectors corr. to λ_1 $\qquad$ Eigenvectors corr. to λ_2

A is diagonalizable iff $E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_k} = \mathbb{C}^n$.

We know that, $m_1 + m_2 + \cdots + m_k = n$.
and $\dim(E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_k}) \leq n$.
$\therefore$ $E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_k}$ is a subspace of $\mathbb{C}^n$.

Now, if A is diagonalizable, then alg. (λ_i) = geo(λ_i)
then $\dim(E_{\lambda_i}) = m_i$, then $\dim\left(\bigoplus_{i=1}^{k} E_{\lambda_i}\right) = n$

$$\boxed{\text{So, } \bigoplus_{i=1}^{k} E_{\lambda_i} = \mathbb{C}^n}$$

Let $A \in M_n(\mathbb{C})$ and let λ be eigenvalue of A.
Let x be an eigenvector of A corresponding to λ.

Extend $\{x\}$ to a basis for $\mathbb{C}^n$. (By using extension thm.)
$\{x, x_1, x_2, \ldots, x_{n-1}\}$

$P = (x \; x_1 \; x_2 \; \ldots \; x_{n-1})$ P → non-singular column matrix.
↓
elements are l.i.

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & * & * & \cdots & * \\ 0 & * & & & \vdots \\ 0 & * & & & \vdots \\ \vdots & \vdots & & & \vdots \\ 0 & * & \cdots & \cdots & * \end{pmatrix}$$

λ_1 λ_2.

$x_1 \ldots x_k$ $x_{k+1} \ldots x_{k+m}$ geo$(\lambda_1) = k$, geo$(\lambda_2) = m$.

$\{x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_{k+m}\}$ extend this to the basis of $\mathbb{C}^n$.

$\Rightarrow$ $\{x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_{k+m}, y_1, \ldots, y_p\}$
$p + m + k = n$.

$$\boxed{P = (x_1 \; x_2 \; \ldots \; x_k \; x_{k+1} \; \ldots \; x_{k+m} \; y_1 \; \ldots \; y_p) \quad p + m + k = n}$$

$$
\begin{array}{c}
\begin{array}{ccccccccc} 1 & & & k & k+1 & & k+m & & \end{array} \\
P^{-1}AP = \begin{pmatrix}
\lambda_1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & * & * \\
0 & \lambda_1 & 0 & & \vdots & \vdots & & \vdots & * & * \\
0 & 0 & \lambda_1 & & \vdots & \vdots & & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & & 0 & \vdots & & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & & \lambda_1 & 0 & & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & & 0 & \lambda_2 & & 0 & \vdots & \vdots \\
0 & 0 & 0 & & \vdots & \vdots & & \lambda_2 & * & * \\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & &
\end{pmatrix}
\end{array}
$$

$P^{-1}AP$ has atleast k times λ_1 and m times λ_2.

\# $A \in M_n(\mathbb{C})$

trace $(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$ det $(A) = \lambda_1 \cdots \lambda_n$.

$\lambda_1, \ldots, \lambda_n$ are roots of det $(A - xI)$.

$$\det(A - xI) = \prod_{i=1}^{n} (x - \lambda_i) \text{ or maybe } (\lambda_i - x)$$

$$\begin{vmatrix} a_{11}-x & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22}-x & a_{23} & \cdots & a_{2n} \\ a_{31} & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ a_{n1} & \cdots & \cdots & \cdots & a_{nn}-x \end{vmatrix} = x^n - (\lambda_1 + \lambda_2 + \cdots + \lambda_n) x^{n-1} + \cdots + (-1)^n \prod_{i=1}^{n} \lambda_i$$

$\therefore$ Coeff. of x^{n-1} = Sum of eigen values = trace (A)

Substitute $x = 0 \Rightarrow$ det $(A) = \lambda_1 \lambda_2 \cdots \lambda_n$.

Note :-

trace $(AB) = tr(BA)$

$\therefore$ Similar matrices have same trace!

say, A & B are similar matrices.

$\boxed{P^{-1}AP = B} \Rightarrow A = PBP^{-1}$

$$tr(A) = tr((PB)(P^{-1})) = tr((P^{-1})(PB)) = tr(B)$$

Rank of a T = Rank of $[T]_B$
$\downarrow$ $\quad$ $\hookrightarrow$ no. of l.i cols. or rows.
dim of range of T.

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Let $A \in M_n(\mathbb{C})$ $\quad T: V \to V$
Let $\lambda_1, \ldots, \lambda_k$ be distinct eigenvalues of T.

Let m_i be algebraic multiplicity of λ_i for $i = 1, 2, 3, \ldots, k$.
The following are equivalent :-
① A is diagonalizable.
② A has n linearly independent eigen vectors.
③ $m_i = geo(\lambda_i)$ for $i = 1, 2, 3, \ldots, k$
④ $\mathbb{C}^n = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_k}$

$V \to$ FDV over F
$T: V \to V$

Def^n :- T is said to be diagonalizable if $\exists$ a basis B of V s.t. each vector in B is eigenvector of T.

If $\nexists$ such type of basis of V, then T is not diagonalizable

B of V
which n vectors
$B = \{x_1, x_2, \ldots, x_n\}$
$T(x_i) = \lambda_i x_i$ for $i = 1, 2, \ldots, n$

$\Rightarrow$ $T: V \to V$ $\quad \dim(V) = n$
T is diagonalizable.
B of V s.t. each vector of B is e.v. of T.
Let $B = \{v_1, v_2, \ldots, v_n\}$

where $T(v_i) = \lambda_i v_i$ $\quad i = 1, 2, \ldots, n$.

$$[T]_B = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ \vdots & \lambda_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & \lambda_n \end{bmatrix}$$

Choose any basis of V say B, find $[T]_B = A$.
if A is diagonalizable then T is diagonalizable.

* B_1 is a basis of V. $\exists$ a non-singular matrix P
$[T]_{B_1} = A$ s.t. $P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & & \lambda_n \end{bmatrix}$

Q.) Check if $T: \mathbb{R}^3 \to \mathbb{R}^3$
$T(x_1, x_2, x_3) = (x_1 - x_2, x_2, x_3)$ is diagonalizable.

A.) Choose Basis $B = \{(1,0,0), (0,1,0), (0,0,1)\}$
Find $[T]_{BB}$

$T(1,0,0) = (1,0,0)$
$T(0,1,0) = (-1,1,0)$
$T(0,0,1) = (0,0,1)$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [T]_B$$

No

Now, check whether $[T]_B$ is diagonalizable or not.

Eigen values of T are 1, 1, 1 geometric mult. (1) = 2.
alg(1) $\neq$ geo(1)

Q.) $P(x, F)$ = Set of all polynomials with coeff. from F.

$p, q \in P(x, F)$ V → FDV, F
$T: V \to V$

$(p+q)(T) = p(T) + q(T)$
$(pq)(T) = p(T)\, q(T)$

$P(T) = a_1T^3 + a_2T^2 + a_3T + a_4 I$.

Let λ be an eigen value of T.
Then $P(\lambda)$ is an eigenvalue of $P(T)$.

$\exists\ x \neq 0 \in V$ s.t. $T(x) = \lambda x$.

$$\boxed{P(T) = a_m T^m + a_{m-1} T^{m-1} + \cdots + a_0 I.}$$

This is also a new operator from $V \to V$.

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$$(P(T))(x) = a_m (T \circ T \circ \cdots \circ T)(x) + \cdots + a_0 I(x)$$

$$= \underbrace{(a_m \lambda^m + a_{m-1}\lambda^{m-1} + \cdots + a_0)}_{\text{Eigenvalue of } P(T)} x.$$

Q.) λ is an eigenvalue of $P(T)$ and corresponding eigenvector x

Exercise. What can you say about T.

Let $B = a_3 A^3 + a_2 A^2 + \cdots + a_0 I$.

λ is an eigenvalue of B and corr. eigenvector x.

Is λ is eigenvalue of A??

$V \to$ FDV

$T : V \to V$

Let $P(x) \in \mathbb{P}(x, F)$

Def^n :- P is said to be annihilating polynomial of T if $P(T) = 0$.(Zero transformation)

Q.) $T : \mathbb{R}^3 \to \mathbb{R}^3$

$T(x_1, x_2, x_3) = (x_1 - x_2, x_2, x_3)$.

Supply an annihilating polynomial of T.

Can you always have an A.P. for any T.

$V \to$ FDV $-$ F $\dim(V) = n$.

$T : V \to V$

$\dim(\mathcal{L}(V, V)) = n^2$

Now, $I, T, T^2, T^3, \ldots, T^{n^2}$ L.D.

So, we have a non zero comb^n $\to$ $c_1 I + c_2 T + \cdots + c_{n^2+1} T^{n^2} = 0$

$\exists\, c_i \neq 0$ $P(x) = c_1 + c_2 x + \cdots + c_{n^2} x^{n^2}$

Then, there is atleast one annihilating polynomial for T.

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$V \to$ FDV, F $\quad T: V \to V$

$S_T = \{ P(x) \in \mathbb{P}(x, F) \mid P(T) = 0 \}$

$S_T \neq \phi$

* S is subspace of $\mathbb{P}(x, F) \longrightarrow$ Finite dim.

For each $T: V \to V$

$P_T(x) \in S_T$ $\quad$ B-Basis of V.

$P_T(x) \in S_T$

$P(\neq 0) \in S_T$ which is least degree.

$P \in S_T$

$Q \notin S_T$ s.t $\deg(Q) < \deg(P)$

Monic polynomial :-

A polynomial is said to be Monic polynomial if the coefficient of highest degree term of P is 1. $\quad x^2+1 \to$ Monic

$2x^2+1 \to$ Not monic.

Minimal Polynomial :-

$V \to$ FDV, F

$T: V \to V$

Same defⁿ for $A \in M_n(F)$

A Monic polynomial $P(x) \in \mathbb{P}(x, F)$ is said to be minimal polynomial of T if $P \in S_T$ and P is least degree polynomial.

① Monic.
② $P(T) = 0$ or $P \in S_T$
③ Least Degree

P is the Minimum Polynomial of T. $\quad (m_T(x))$

Q.) $T: \mathbb{R}^3 \to \mathbb{R}^3$
$T(x_1, x_2, x_3) = (x_1 - x_2, x_2, x_3)$ Find $m_T(x)$.

A.) $\{(1,0,0), (0,1,0), (0,0,1)\} \to B$

$$[T]_B = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T, T^2 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Try,
$T^2 - 2T + I = O_T$.

$$\therefore m_T(x) = (x-1)^2$$

Thm. :- $T : V \to V$
Then, $m_T(x)$ divides $P_T(x)$

$m_T(x)$ → minimal polynomial
$P_T(x)$ → characteristic polynomial.

⇒ If you are able to show that each root of m_T is root of P_T.

⇒ Each root of P_T is eigenvalue of T.

Claim :- λ is root of m_T iff λ is eigenvalue of T.

⇒ Let λ be a root of m_T.

$$m_T = (x - \lambda)\, q(x), \quad \deg(q) = \deg(m_T) - 1$$

$$\Rightarrow m_T(T) = (T - \lambda I)\, q(T)$$

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if, $q(T)$ is identically zero, then?

$(T-\lambda I)\, q(T) = 0$ | $q(T) = 0 \quad q(T)(v) = 0 \quad \forall v \in V$

$q(T) \neq 0$ ⟨because m_T is minimal poly.⟩

$q(T) : V \to V$ → linear operator.

$\exists$ a vector $v \in V$ st. $q(T)(v) \neq 0$.

$w = q(T)(v) \in V$.

$\Rightarrow (T-\lambda I)\, w = 0 \quad \langle w \neq 0 \rangle$

$\therefore \lambda$ is an eigen value of T.

Other way,

$\Rightarrow$ Let λ be an eigen value of T.

$\exists\, v \neq 0$ in V such that $(T-\lambda I)\, v = 0$

~~We consider the subspace $LS(v)$.~~

~~Then $(x-\lambda)$ is annihilating polynomial to T on $LS(v)$.~~

~~It's~~

$m_T(T) = 0 \quad \Rightarrow m_T(T)(v) = 0$

$(a_0 + a_1 T + \cdots + T^n)(v) = 0$

Suppose v is eigenvector of λ (λ is eigenvalue of T).

$T(v) = \lambda v$

$\Rightarrow (a_0 + a_1\lambda + \cdots + \lambda^n)(v) = 0$

v is eigenvector $(v \neq 0)$

$a_0 + a_1\lambda + \cdots + \lambda^n = 0$

Hence, a root to $m_T(x)$.

* $m_T(x)$ divides $P_T(x)$

$T : \mathbb{R}^3 \to \mathbb{R}^3$

$T(x_1, x_2, x_3) = (x_1 - x_2,\ x_2,\ x_3)$

$P_T(x) = (x-1)^3$

$(x-1), \quad (x-1)^2, \quad (x-1)^3$

$\times \quad$ $(x-1)^2$ → minimal polynomial.

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$T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$

Let $B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

$T(A) = AB$ Find minimal polynomial.

$T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

$T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$.

$T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

$T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$ $\therefore P_T(x) = (x-2)^2(x-1)^2$

$m_T(x) = (x-2)(x-1)$

Why minimal polynomial?

Thm:- T is diagonalizable iff m_T is product of unique linear factors.

T is diagonalizable.

$\Updownarrow$

$m_T(x) = (x-\lambda_1)(x-\lambda_2)\dots(x-\lambda_k)$ $\lambda_i \neq \lambda_k$, $i \neq k$.

$T^2 = T$ Is T diagonalizable?

$T^2 - T = 0 \Rightarrow q(x) = x^2 - x$ $q(T) = 0$

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$q \in S_T$ $\quad m_T$ is minimal polynomial.
then m_T divides q. (each root of m_T is also a root of q_T)

So, Possibilities of $m_T^{(x)}$ are x, $x-1$, $x(x-1)$.

if $m_T(x) = x \Rightarrow T \equiv 0$ diagonal.

$m_T(x) = x-1 \Rightarrow T \equiv I$ diagonal

$m_T(x) = x(x-1) \Rightarrow T$ is diagonalizable.

$\deg(r) < \deg(m_T)$.

$q = m_T q_1(x) + r(x)$

$\Rightarrow r(T) = 0 \Rightarrow r \equiv 0$

$$\underset{\Downarrow \atop 0}{q(T)} = \underset{\Downarrow \atop 0}{m_T(T)}\, q_1(T) + r(T)$$

$\boxed{\therefore q = q_1 m_T.}$

$\boxed{T^2 = T.}$ $\rightarrow$ Idempotent operator

$T^3 = I$ is it diagonalizable.

$T^3 - I = 0$ $\quad \therefore q = x^3 - 1$ $\quad q(T) = 0$

$\therefore q \in S_T$

now, m_T divides q. (x^3-1)

$m_T(x)$ can be $(x-1)$, (x^2+x+1) or $(x-1)(x^2+x+1)$

For, $m_T(x) = (x-1) \Rightarrow T = I$ diagonal

$m_T(x) = x^2+x+1 \Rightarrow T$ is diagonalizable.

as, $m_T(x) = (x-\alpha)(x-\bar{\alpha})$
$\hookrightarrow$ ~~coefficient~~ ω, ω^2

Product of unique linear factors
$(x-\omega)(x-\omega^2)$.

$m_T(x) = (x-1)(x-\omega)(x-\omega^2)$

Product of unique linear factors.

minimal   characteristic

Hence, m_T divides P_T

λ is root of m_T iff λ is root of P_T 〈excluding multiplicity〉

For, $T^3 = 0$

$q(x) = x^3$

Now, $m_T(x)$ can be $x, \; x^2, \; x^3$

$x \to T=0$; x^2 ✗, x^3 ✗ not diagonalizable.

0 is the only eigenvalue of this operator.

λ is an eigenvalue of T, $P(\lambda)$ is also an eigenvalue of $P(T)$.

$P(x) = x^3$ λ^3 is an eigenvalue of $P(T) \equiv 0$.

$\lambda^3 = 0, \; \lambda = 0$

$T^2 = T$.   either 0 or 1 or 0, 1.

$T^3 = I$

Supply a matrix $A \in M_3(\mathbb{R})$ s.t ω, ω^2 are the only distinct eigen values.

$\mathbb{C}$

Real coeff. polynomial P, the P can't have odd no. of complex roots.

Invariant Subspace :-

$V \rightarrow VS$

$T : V \rightarrow V$

A subspace W of V is said to be invariant subspace under T if $T(W) \subseteq W$.

$T(W) = \{T(x) \mid x \in W\}$ $\quad T(W) \subseteq W$

Ex. $\quad V \rightarrow VS \quad T : V \rightarrow V$

$W = \{0\}$ – invariant under T.

$W = V$ – invariant under T.

$y = T(T(x)) \quad T(x) \in V \quad y \in W \quad T(W) \subseteq W$.

$W = \text{Null}(T)$ – Invariant under T.

$T(W) = \{T(x) \mid x \in \text{Null}(T)\} = \{0\} \subseteq W$.

$W = \text{Range}(T) = \{T(x) \mid x \in V\} \rightarrow$ invariant under T.

$T(W) = \{T \circ T(x) \mid x \in V\} \quad y \in T(W) \quad y = T \circ T(x)$ for some $x \in V$.

$T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad T(x_1, x_2) = (-x_2, x_1) \rightarrow$ Linear operator doesn't have eigenvalue.

Find all invariant subspaces under T.

$V \rightarrow$ invariant $\quad V = \mathbb{R}^2$

$\{0,0\}$, "

$\mathbb{R}^2$, $\{(0,0)\}$

Invariant under T.

$\rightarrow (x,y) \neq (0,0)$

W of dimension 1. $W = LS((x,y))$

If W is invariant under T.

$T(W) \subseteq W$.

$T(x,y) = \alpha(x,y)$, $\alpha \in \mathbb{R}$. α is an eigenvalue of T. This is not possible.

W is not invariant.

$T: V \to V$ If V has one-dimensional invariant subspace under T, then T must have an eigenvalue.

$T: V \to V$ — linear operator.

W — invariant subspace under T.

$W \to VS$ $T_W : W \to W$ — linear operator.

V — FDV, W — FDV $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ be a basis of W.

$B = \{\alpha_1, \alpha_2, \ldots, \alpha_k, \alpha_{k+1}, \ldots, \alpha_n\}$ basis for V.

$[T]_B = A$ in V.

$T(\alpha_1) = c_1\alpha_1 + \cdots + c_k\alpha_k + 0\alpha_{k+1} + \cdots + 0.\alpha_n$.

$T(\alpha_2) = b_1\alpha_1 + \cdots + b_k\alpha_k + 0\alpha_{k+1} + \cdots + 0.\alpha_n$

$\vdots$

$T(\alpha_k) = d_1\alpha_1 + \cdots + d_k\alpha_k + 0\alpha_{k+1} + \cdots + 0.\alpha_n$.

$T(\alpha_{k+1}) = t_1\alpha_1 + \cdots + t_k\alpha_k + t_{k+1}\alpha_{k+1} + \cdots + t_n\alpha_n$.

$$A = \left[\begin{array}{cccc|ccc} c_1 & b_1 & \cdots & d_1 & t_1 & \cdots & s_1 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ c_k & b_k & \cdots & d_k & t_k & \cdots & s_k \\ \hline 0 & 0 & & 0 & t_{k+1} & \cdots & s_{k+1} \\ 0 & 0 & & 0 & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & t_n & \cdots & s_n \end{array}\right] = \left[\begin{array}{c|c} B & D \\ \hline 0 & C \end{array}\right]$$

$B = [T_W]_{\{\alpha_1, \alpha_2, \ldots, \alpha_k\}}$

Characteristic and minimal polynomials of T and T_W.

$\det(A - xI)$ $\qquad$ $\det(B - xI)$.

Eigenvalues of B are also eigenvalues of A except multiplicity.

$\boxed{\det(A - xI) = \det(B - xI)\det(C - xI)}$.

Characteristic polynomial of T_W divides characteristic poly of T.

$P(x) = x^2+1$ $\qquad$ $P(A) = \begin{bmatrix} P(B) & * \neq P(D) \\ 0 & P(C) \end{bmatrix}$

$\Rightarrow P(B) = 0$

and $P(C) = 0$.

$m_T(x) =$ minimal polynomial of T.

$m_T(T) = 0$ $\qquad$ $m_T(A) = 0 \Rightarrow m_T(B) = 0 \Rightarrow$ minimal polynomial of B divides m_T.

* minimal polynomial of T_W divides minimal polynomial of T.

22/10/19 Cayley-Hamilton thm. Let f be characteristic polynomial of T, then $f(T) = 0$.

Thm. :- T is diagonalizable iff m_T is product of distinct (unique) linear factors.

$A \in M_n(\mathbb{C})$ $\qquad$ for $A \in M_n(\mathbb{R})$

$A^* = A^T$

$A^* = (\bar{A})^T = \overline{(A^T)}$

$\hookrightarrow$ adjoint of A

$A \in M_n(\mathbb{C}) \Rightarrow A^* = (\bar{A})^T = \overline{(A^T)}$ adjoint of $A = A^*$

$A \in M_n(\mathbb{R}) \Rightarrow A^* = A^T$

① Symmetric : A matrix $A \in M_n(\mathbb{R})$ is said to be symmetric if $A = A^T$. (Real Eigen values)

② Hermitian :- A matrix $A \in M_n(\mathbb{C})$ is said to be Hermitian if $A^* = A$. (Real Eigen values)

③ Unitary :- A matrix $A \in M_n(\mathbb{C})$ is said to be unitary if $A^* A = A^* A = I$. (Eigen values are of unit modulus)

④ Orthogonal :- $A \in M_n(\mathbb{R})$ is said to be orthogonal if $AA^T = A^TA = I$. (Eigen values are of unit modulus)

⑤ Normal :- $A \in M_n(\mathbb{C})$ is said to be normal if $A^*A = AA^*$

⑥ Skew symmetric :- $A^T = -A$.

⑦ Skew Hermitian :- $A^* = -A$

(Eigen values are zero or purely Imaginary)

Symmetric, Skew-symmetric, orthogonal, are special case of real normal matrix. $A \in M_n(\mathbb{R})$

Hermitian, Skew-Hermitian, Unitary are special case of complex normal matrix. $A \in M_n(\mathbb{C})$.

For Skew-Hermitian A, Eigenvalues are 0 or purely imaginary

$$A^* = -A.$$

Let λ be an eigenvalue of corresponding eigenvector $x \in \mathbb{C}^n$

$$Ax = \lambda x$$
$$(Ax)^* = (\lambda x)^*$$
$$x^* A^* = \bar{\lambda} x^*.$$
$$-x^* A = \bar{\lambda} x^* \Rightarrow -x^*Ax = \bar{\lambda} x^* x$$
$$-x^* \lambda x = \bar{\lambda} x^* x.$$
$$\therefore \bar{\lambda} = -\lambda.$$

$\therefore \lambda$ is 0 or purely img.

Hermitian	Skew Hermitian	Unitary
Real e.vs.	Imaginary or 0 e.vs.	Unit modulus e.v.s.

A — Hermitian

$\lambda_1 \neq \lambda_2$ — eigenvalues

$\downarrow \quad \downarrow$

$x_1 \quad x_2 \longrightarrow$ LI. $\quad \{x_1, x_2\}$ are orthogonal.

$Ax_1 = \lambda_1 x_1$.

$(Ax_1)^* = (\lambda_1 x_1)^* \Rightarrow x_1^* A^* = \bar{\lambda}_1 x_1^*$

$x_1^* A^* x_2 = \bar{\lambda}_1 x_1^* x_2. \Rightarrow x_1^* A x_2 = \bar{\lambda}_1 x_1^* x_2.$

$\lambda_2 x_1^* x_2 = \lambda_1 x_1^* x_2$. (as λ_1 is real.)

$\Rightarrow (\lambda_2 - \lambda_1) x_1^* x_2 = 0$ $\quad$ But, $\lambda_1 \neq \lambda_2$.

$\therefore x_1^* x_2 = 0$ $\quad$ x_1 and x_2 are orthogonal

$\langle x, y \rangle = x^* y$ Standard inner product on $\mathbb{C}^n$.

Hermitian (Symmetric) $\longrightarrow A \longrightarrow \lambda \begin{matrix} \nearrow x_1 \\ \searrow x_2 \end{matrix}$ } Not always orthogonal.

Thm. Let $A \in M_n(\mathbb{C})$. Then there exists a unitary matrix S such that $S^* A S = U \rightarrow$ (Upper triangular) (Schur thm.)

A is a complex matrix so, it has atleast one eigen value.

Let x be an eigenvector of A corresponding to λ.

Extend x to $\{x_0, x_1, x_2, x_3, \dots, x_{n-1}\}$ — basis of $\mathbb{C}^n$.

$\hookrightarrow$ Transform to orthogonal set.

~~$P = (x_1, x_1, x_2, \dots, x_{n-1})$~~. $P = (y_1, y_2, \dots, y_n)$.

$$AP = P \begin{pmatrix} \lambda_1 & c_{11} & \cdots & c_{1n-1} \\ 0 & c_{21} & & \vdots \\ \vdots & \vdots & & \\ 0 & c_{n-11} & \cdots & c_{n-1n-1} \end{pmatrix}$$

$$\Rightarrow P^* A P = \begin{pmatrix} \lambda_1 & c_{11} & \cdots & c_{1n-1} \\ 0 & c_{21} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & c_{n-11} & \cdots & c_{n-1n-1} \end{pmatrix}$$

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$$C = \begin{pmatrix} C_{21} & \cdots & C_{2n-1} \\ \vdots & \ddots & \vdots \\ C_{n-11} & \cdots & C_{n-1n-1} \end{pmatrix}_{n-1 \times n-1}$$

$$P_1^* C P_1 = \left(\begin{array}{c|c} \lambda_2 & \cdots \\ \hline 0 & \\ \vdots & B_{n-2 \times n-2} \\ 0 & \end{array}\right) \quad P_1 \to (n-1 \times n-1)$$

$r_1, \ldots, r_{n-1} \to$ rows of P_1 $\to$ orthonormal.

$$\left.\begin{array}{l} (1,0,\ldots 0) \\ (0 \;\; r_1) \\ (0 \;\; r_2) \\ \vdots \\ (0 \;\; r_{n-1}) \end{array}\right\} \text{orthonormal} \quad S = \left(\begin{array}{c|c} 1 & 0 \cdots 0 \\ \hline 0 & \\ \vdots & P_1 \\ 0 & \end{array}\right)$$

$S^* S = S S^* = I.$

$S \to$ from P_1 s.t. $S^* P^* A P S = \left(\begin{array}{c|c|c} \lambda_1 & * & * \cdots * \\ \hline 0 & \lambda_2 & * \cdots * \\ \hline \vdots & 0 & \\ 0 & \vdots & B \\ & 0 & \end{array}\right)$

Fact :-

$S^* S = S S^* = I \iff$ Set of column vectors is orthonormal
Set of row vectors is orthonormal.

Thm. $A \in M_k(\mathbb{C})$ not true for Real matrices
$\exists$ a unitary matrix S s.t. $S^* A S = U \to$ upper Δ matrix.

Proof By mathematical induction,

$n = 1$, $[a] \to$ Upper triangular matrix.

This statement is true for $n = k$.

$\left[A \in M_n(\mathbb{C}), \exists \text{ a unitary matrix } S \text{ s.t. } S^* A S = U \right]$

To show this statement is true for $n = k+1$, $A \in M_{n+1}(\mathbb{C})$.

Def^n. A matrix A is called unitarily diagonalizable if A is similar to a diagonal mat. D with a unitary matrix P.

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A has atleast one eigenvalue λ. Let x be an eigenvector corresponding to λ. Extend x to $\{x, x_1, x_2, \dots, x_k\}$ basis for $\mathbb{C}^{k+1}$.

By using Gram Schmidt's process, transform $\{x, x_1, x_2, \dots x_k\}$ to orthonormal set $\{y_1, y_2, \dots, y_{k+1}\}$

$$\Rightarrow P = (y_1, y_2, \dots, y_{k+1}) \longrightarrow \text{unitary}.$$

$$AP = P\left(\begin{array}{c|ccc} \lambda_1 & c_{11} & \cdots & c_{1k} \\ 0 & c_{21} & \cdots & c_{2k} \\ \vdots & \vdots & & \vdots \\ 0 & c_{k1} & & c_{kk} \end{array}\right). \qquad P^*AP = \left(\begin{array}{c|c} \lambda & \cdots \\ \hline 0 & \\ \vdots & C \\ 0 & \end{array}\right)$$

By Induction Hypothesis,

$\exists$ a unitary matrix P_1 s.t. $P_1^* C P_1 = U_1 \longrightarrow$ upper Δular

$$Q = \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & P_1 & \\ 0 & & & \end{array}\right) \longrightarrow \text{unitary}.$$

$$Q^* P^* A P Q = Q^* \left(\begin{array}{c|ccc} \lambda & c_1 & \cdots & c_k \\ \hline 0 & & & \\ 0 & & C. & \\ \vdots & & & \\ 0 & & & \end{array}\right) Q = \underbrace{\left(\begin{array}{c|ccc} \lambda & c_1 & \cdots & c_k \\ \hline 0 & & & \\ \vdots & & U_1 & \\ 0 & & & \end{array}\right)}_{\text{upper } \Delta\text{ular}}$$

Hence, proved.

Q.) By the above thm, Prove that Hermitian matrix is diagonalizable
Unitary matrix is unitarily diagonalizable

[Every normal matrix is Unitarily diagonalisable.] $\rightarrow$ Spectral thm.

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Every Hermitian matrix is unitarily diagonalizable. $U^* = U$ $\langle U$ – Hermitian + Upper Δ^{ular} U – diagonal$\rangle$

" Unitary matrix " " " . $UU^* = U^*U = I$. $\langle U$ – unitary + upper Δ^{ular} U – diagonal$\rangle$

" Normal matrix " " " . $\langle U$ – normal + upper Δ^{ular} U – diagonal$\rangle$ $U^*U = UU^*$