28th Feb

Ring: A ring R is a set with 2 binary operations called addition $(+)$ and multiplication $(\cdot)$ which satisfies the foll. prop.

(1) $(R, +)$ is abelian group with identity 0.

(2) multiplication is associative i.e. $(ab)c = a(bc)$ $\forall\ a, b, c \in R$

(3) $\exists\ 1 \in R$ s.t. $1 \cdot a = a \cdot 1 = a\ \forall\ a \in R$ is called multiplicative identity.

(4) Distributive law

$(a+b)c = ac + bc$ &

$c(a+b) = ca + cb$ $\forall\ a, b, c \in R$

(*) If $ab = ba$ $\forall\ a, b \in R$ then R is called commutative ring.

Example: (1) $\mathbb{Z}, \mathbb{R}, \mathbb{Q}$ are all commutative ring

(2) $M_n(R)$ = set of all $n \times n$ matrices over R It is a ring which is not commutative for $n \geq 2$.

(3) $C(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid f$ is continuous$\}$ is a ring w.r.t

$(f+g)(x) = f(x) + g(x)$ &

$(fg)(x) = f(x) \cdot g(x)$ which is commutative

④ Let R be a ring.

$R[x]$ = The set of all polynomials in x with coeff in R.

If R is commutative, Then $R[x]$ is a commutative ring.

⑤ Any field is a ring.

⑥ The zero Ring $R = \{0\}$ consists of a single element 0. In which the multiplicative identity is same as the additive identity.

Remark: Let R be a ring in which $1 = 0$ (add. id. = mult. id.). Then R is the 0 ring. (using associativity)

Let $a \in R$. $a = a \cdot 1 = a \cdot 0 = 0$

$\therefore R$ is a zero ring

Defn: An element $0 \neq u \in R$ is called a unit in R if there is some $v \in R$. s.t. $uv = vu = 1$.

The set of units in R is denoted by $R^\times$.

Examples: ① The units in $\mathbb{Z}$ are $1, -1$.

② $\mathbb{R}[x]$ the units are non zero constant polynomials.

Definition: Let R be a Ring. A non zero element $a \in R$ is called a zero divisor if $\exists$ a non zero elt. $b \in R$ s.t. $a \cdot b = 0$ or $ba = 0$.

Ex Consider the Ring $\mathbb{Z}/6\mathbb{Z}$.

$$\bar{2} \cdot \bar{3} = \bar{0}$$

$$\bar{2} \neq \bar{0}, \quad \bar{3} \neq 0.$$

$\therefore \bar{2}$ is a zero divisor.

Observe that a zero divisor can never be a unit. Suppose that a is a unit and a zero divisor. Then $\exists$ non zero b s.t. $ab = 0$. If a is a unit. Then $\exists\ a' \in R$ s.t. $aa' = a'a = 1$

$$\therefore a'(ab) = 0$$
$$\Rightarrow a'a(b) = 0$$
$$\Rightarrow b = 0 \implies\impliedby.$$

Definition: A subring of a ring R is a subgroup of R which is closed under multiplication and contain 1.

Example: ① $\mathbb{Z}$ is a subring of $\mathbb{Q}$ and $\mathbb{Q}$ is a subring of $\mathbb{R}$.

Proposition: Let R is a ring. Then

(i) $0 \cdot a = a \cdot 0 = 0 \quad \forall\ a \in R$

(ii) $(-a) \cdot b = a \cdot (-b) = -ab \quad \forall\ a, b \in R$

(iii) $(-a) \cdot (-b) = ab \quad \forall\ a, b \in R$

(iv) $(-a) = (-1) \cdot (a) \quad \forall\ a \in R$

Ring Homomorphism

Let R and S be 2 rings.
Then a map $\phi : R \longrightarrow S$ is called a ring homomorphism.
if $\phi(a+b) = \phi(a) + \phi(b)$

$\phi(ab) = \phi(a)\phi(b)$

and $\phi(1_R) = 1_S$. $\longrightarrow$ [It is not used in all books]

Def^n: Let $\phi : R \longrightarrow S$ be a ring homo.
Then $\ker\phi = \{a \in R \mid \phi(a) = 0\}$

Q^n Is $\ker\phi$ a subring of R?

5th March

Let $\phi : R \to S$ be a ring homomorphism

$\ker \phi = \{ a \in R \mid f(a) = 0 \}$

A bijection homo is called an isomorphism

Proposition: Let $\phi : R \to S$ be a ring homo.

① The image of ϕ is a subring of S.

② The kernel of ϕ is not a subring of R unless it is the whole ring R.

Furthermore, if $\alpha \in \ker\phi$ then $r\alpha$ and $\alpha r \in \ker\phi$ $\forall\, r \in R$.

Proof: For any $r \in R$ & $\alpha \in \ker\phi$.

$\phi(\alpha r) = \phi(\alpha)\phi(r) = 0 \cdot \phi(r) = 0$

$\therefore \alpha r \in \ker\phi$.

Similarly $r\alpha \in \ker\phi$.

Definition: Let R be a ring and I be a subset of R. Then I is said to be a left Ideal if I is a subgroup of $(R,+)$ and I is closed under left multiplication

i.e. $rI \subseteq I$.

and I is said to be a right ideal if I is a subgp of $(R,+)$ and I is closed under right multiplication

i.e. $Ir \subseteq I$

ample: $n = 1800 = 2^3 \cdot 3^2 \cdot 5^2$

A subset I which is both a left ideal and a right ideal is called an ideal of R

Example

① $f : \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z}$.

$f(m) = \bar{m}$

check that f is a ring homomorphism

② consider $f : \mathbb{R}[x] \longrightarrow \mathbb{R}$

$f\left(\sum_{i=0}^{n} a_i x^i\right) = \sum_{i=0}^{n} a_i \alpha^i$ where $\alpha \in \mathbb{R}$

check f is a ring homomorphism

③ Define $\phi : \mathbb{Q}[x] \longrightarrow \mathbb{R}$

$\phi(f(x)) = f(\sqrt{2})$

$\ker\phi = \{f(x) \in \mathbb{Q}[x] \mid \phi(f(x)) = 0\}$

Claim: $\ker\phi = \{(x^2-2)q(x) \mid q(x) \in \mathbb{Q}[x]\}$

Note that: if $f \in \ker\phi$ then f cannot be a linear polynomial.

Also $(x^2-2) \in \ker\phi$.

Let $g(x) \in \ker\phi$.

& $g(x) = (x^2-2)\,q(x) + r(x)$

where degree $r(x) < 2$ or $r(x) = 0$.

$r(x) = g(x) - (x^2-2)\,q(x)$

$\Rightarrow r(\sqrt{2}) = 0 \quad \Rightarrow r(x) \in \ker\phi.$

$\Rightarrow r(x) = 0.$

$\Rightarrow g(x) = (x^2-2)(q(x))$

$\therefore \ker\phi = \{(x^2-2)\,q(x) \mid q(x) \in \mathbb{Q}[x]\}$

In any ring R, the set of multiples of a particular element 'a' forms an ideal called principal ideal generated by a. and is denoted by

$(a) = \{ar \mid r \in R\}$ [Here R is commutative]

Note: Assume now-onwards that the rings are commutative

In a ring of Integers the Ideals are of the form $n\mathbb{Z}$, or are formed by the single element

We may consider the ideal generated by a set of elements $a_1, a_2, \ldots a_n \in R$ which is defined to be the smallest ideal containing these elements and it is denoted by $(a_1, a_2, \ldots a_n) = \{r_1a_1 + \ldots r_na_n \mid r_i \in R\}$

In any ring R, the set consisting of zero alone is an ideal called the zero ideal and it is the called the unit ideal denoted by (1).

The unit ideal is the only ideal which contains an unit

An ideal I is said to be a proper Ideal if it is not (0) or (1).

There is no proper ideal in a Field.

Proposition: ① Let F be a field. The ideals of F are zero and unit ideal.
② Conversely, if a ring R has exactly 2 ideals then R is a field.

Proof: ② Assume that R has exactly two ideals.

WTS: $1 \neq 0$

If $1 = 0$ then it is the zero ring and then it will have only one ideal which is a contradiction.
$\therefore 1 \neq 0$. Then (1) and (0) are 2 diff ideals. Let $a \neq 0$ be an element of R
$(a) = (1)$. i.e. $\exists\ b \in R$.
S.T. $ab = 1$ $\therefore$ a is a unit

Proposition: Let F be a field and R' be a non zero ring and $\varphi : F \to R'$ be a ring homo.

Proof: Since F is a field therefore $\ker\phi$ is either (0) or (1).
But if $\ker(\phi) = (1)$ then R' will be a zero ring

$\Rightarrow \Leftarrow$

$\therefore \ker\phi = (0)$. Thus φ is injective

Definition: A ring in which all ideals are principal ideal is called a principal ideal ring (PIR).

Proposition: Every ideal in $\mathbb{Z}$ is principal ideal.

Proposition Let F is a field. Every ideal in $F[x]$ is a principal ideal.

Proof: Let $0 \subseteq I \subseteq F[x]$ be a proper ideal. $\exists$ a polynomial of (+ve degree in I.

Let $f(x) \in I$ having smallest positive degree. Let $g(x) \in I$. Then by division algo we have

$$g(x) = f(x)q(x) + r(x)$$

where $\deg(r(x)) < \deg f(x)$ or $r(x) = 0$

$$r(x) = g(x) - f(x)q(x) \in I.$$

$\Rightarrow r(x) = 0$ [by minimality of $\deg f(x)$]

$\therefore g(x) = f(x)q(x) \in (f(x))$.

Remark: $\mathbb{Z}[x]$ is not a Principal Ideal.

$(2, x) = (f)$ whose $f \in \mathbb{Z}[x]$.

Since $2 \in (f)$. $\therefore 2 = f \cdot g$.

Since $\deg(2) = 0 \Rightarrow \deg(f) = 0$

$\Rightarrow f \mid 2 \Rightarrow f = \pm 1, \pm 2$.

Since $(1) = (-1) = \mathbb{Z}[x] \neq I$

$\therefore f = \pm 2$, but ± 2 doesn't divide x.

So $x \notin (2)$ or (-2).

$\therefore (2, x)$ is not a principal ideal.

Remark: $\mathbb{Z}$, $F[x]$ where F is a field are PIR.

Integral Domains

An integral domain R is a non zero ring having no zero divisor

i.e. if $ab = 0$ then $a = 0$ or $b = 0$.

Example: (1) Fields are Int domain.

(2) $\mathbb{Z}$ is an Integral domain.

(3) $F[x]$ where F is a field is an Integral domain.

(4) $C(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ is cont}^{n} \}$

$C(\mathbb{R})$ is not an Integral domain.

Define: $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \begin{cases} 0 & \forall\ x \leq 0 \\ x & \forall\ x > 0 \end{cases}$

$g : \mathbb{R} \to \mathbb{R}$ by $g(x) = \begin{cases} x & \forall\ x < 0 \\ 0 & \forall\ x \geq 0 \end{cases}$

Then $f \cdot g = 0$ but $f \neq 0$ & $g \neq 0$

Proposition: Let R be an integral domain then R[x] is also an Integral domain.

Proposition: Let R be an integral domain if $ab = ac$ and $a \neq 0$ then $b = c$

Proof: $ab = ac \Rightarrow a(b-c) = 0$

$\because$ R is integral domain

$\Rightarrow a \neq 0$

$\Rightarrow b - c = 0 \Rightarrow \underline{b = c}$

Proposition: A finite Integral domain is a field

Proof. Let R be a finite Integral domain

Let $0 \neq x \in R$. WTS x is a unit.

Consider the element $x, x^2, x^3 \ldots$

$\because$ R is a finite set $\exists\ r > s$ s.t.

$$x^r = x^s$$

$\Rightarrow x^s(x^{r-s} - 1) = 0$

$\Rightarrow x^{r-s} - 1 = 0$ (As it is an integral domain $x^s \neq 0$)

$\Rightarrow x^{r-s} = 1$

$\Rightarrow x \cdot x^{r-s-1} = 1$

$\therefore$ x has an inverse

$\Rightarrow$ R is a field

Quotient Ring: Let R be a ring and I be an ideal of R. Then we have already seen that the set of all cosets of I form a group R/I. Now want to show R/I has a ring structure. Want to define multiplication of two cosets.

$$(a+I)(b+I) = (ab+I)$$

Well Defined: $a+I = a'+I$

$b+I = b'+I$

WTS: $ab+I = a'b'+I$

$\Rightarrow a-a' \in I \quad \& \quad b-b' \in I$

Say $a = a'+i_1$ & $b = b'+i_2$.

$\Rightarrow ab = (a'+i_1)(b'+i_2)$

$= a'b' + a'i_2 + i_1b' + i_1i_2$

$= a'b' + i$

$\Rightarrow ab - a'b' = i$

$\Rightarrow ab - a'b' \in I.$

1st Isomorphism Theorem

Let $f: R \longrightarrow S$ be a surjective ring homomorphism.

Let $I = \ker f$. Then $R/\ker f \cong S$.

Proof: $\bar{f}: R/I \longrightarrow S$

$$\bar{f}(a+I) = f(a)$$

Exercise Check $\bar{f}$ is well defined WTS

$$\bar{f}((a+I)(b+I)) = \bar{f}(a+I)\,\bar{f}(b+I)$$

$$\bar{f}((a+I)(b+I)) = \bar{f}(ab+I)$$

$$= f(ab) = f(a)f(b)$$

$$= \bar{f}(a+I)\,\bar{f}(b+I)$$

$\therefore \bar{f}$ is a Ring homomorphism.

Since f is surjective so $\bar{f}$ is surjective

$$\ker \bar{f} = \{a+I \in R/I \mid \bar{f}(a+I) = 0\}$$

$$= \{a+I \in R/I \mid f(a) = 0\}$$

$$= \{I\} \quad \therefore \bar{f} \text{ is injective}$$

(Take 0 as Representative)

Hence $\bar{f}$ is an isomorphism.

$\therefore R/I \cong S$

Example: $\mathbb{R}[x]$.

$$\phi: \mathbb{R}[x] \longrightarrow \mathbb{C}$$

$$\phi(r) = r \quad \forall\, r \in \mathbb{R}$$

$$\phi(x) = i$$

ϕ is a ring homomorphism

ϕ is a surjective ring homomorphism

$$\ker\phi = \{ f(x) \in \mathbb{R}[x] \mid f(i) = 0 \}$$

Note that $(x^2+1) \in \ker\phi$.

Let $g(x) \in \ker\phi$.

By division algorithm.

$$\Rightarrow g(x) = (x^2+1)\,q(x) + r(x)$$

where either $\deg r(x) < 2$ or $r(x) = 0$

$$r(x) = g(x) - (x^2+1)\,q(x)$$

$$r(i) = g(i) - 0 \cdot q(i)$$

$$r(i) = 0$$

$\Rightarrow r(x) \in \ker\phi$.

$r(x) = 0$. Since $\deg r(x) < 2$.

$\therefore g(x) = (x^2+1)\,q(x)$

$\therefore$ By first isomorphism theorem.

$$\mathbb{R}[x]/_{x^2+1} \cong \mathbb{C}.$$

$\hookrightarrow x^2+1 = 0 \Rightarrow x^2 = -1$

$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

$\hookrightarrow a + bx$

$$\frac{\mathbb{R}[x]}{(x^2)} \qquad \frac{\mathbb{R}[x,y]}{(y^3)}$$

Ex. $\phi : \mathbb{Z}[x] \longrightarrow \mathbb{Z}[i] := \{a+bi \mid a, b \in \mathbb{Z}\}$

$\downarrow$

Gaussian integers

$\phi(f(x)) = f(i)$

Ex: show that $\frac{\mathbb{Z}[i]}{1+3i} \cong \frac{\mathbb{Z}}{10\mathbb{Z}}$

$\mathbb{Z}[i] = a + bi$

$\mathbb{Z}[i]/1+3i =$

Homomorphism Theorem

Let $f: R \to S$ be a subjective ring homo. Then $K = \ker f$

$\exists$ a bijection b/w the set

{all ideals of R containing K} $\longleftrightarrow$ {all ideals of S}

$J \rightsquigarrow f(J)$

$f^{-1}(I) \hookleftarrow I$

Proof step 1: WTS $f(J)$ is an ideal of S. clearly $f(J)$ is a subgp of S. Let $s \in S$ & $a \in f(J)$. WTS $sa \in f(J)$

since f is surj. $\exists\ x \in R$ s.t. $f(x) = s$

& $\exists\ b \in J$ s.t. $a = f(b)$

$sa = f(x).f(b) = f(xb) \in f(J)$

2nd step $f^{-1}(I) = \{a \in R \mid f(a) \in I\} \supseteq K$

WTS. $f^{-1}(I)$ is an ideal.

Clearly $f^{-1}(I)$ is a subgp of R.

Let $a \in R$ and $x \in f^{-1}(I)$

WTS: $ax \in f^{-1}(I)$

$f(ax) = f(a) f(x) \in I$

$\downarrow \in I$

Thus $f^{-1}(I)$ is an ideal of R.

(We have shown that a Ring Homomorphism maps to an ideal too. Also the preimage of an ideal on a Ring homo maps to an ideal. Note: Homo need not be invertible)

Note:

3rd step: WTS $f^{-1}(f(J)) = J$ & $f(f^{-1}(J)) = I$

It is clear that $J \subseteq f^{-1}(f(J))$

WTS $f^{-1}(f(J)) \subseteq J$

Let $a \in f^{-1}(f(J))$

$\Rightarrow f(a) \in f(J)$

$\Rightarrow f(a) = f(y) \quad y \in J$

$\Rightarrow f(a-y) = 0$

$\Rightarrow a - y \in K \subseteq J$ (K → Kernel)

$\Rightarrow a \in J$ (Because Kernel Ideal is contained in the Ideal)

It is clear that $f(f^{-1}(I)) \subseteq I$

WTS: $I \subseteq f(f^{-1}(I))$

Let $a \in I$.

$\because f$ is surjective

$\exists\ x \in R$

s.t $f(x) = a \in I$

$\Rightarrow x \in f^{-1}(I)$.

$\Rightarrow \because I \subseteq f(f^{-1}(I))$.

Theorem: Let R be a Ring and I be an ideal of R. Then $R \xrightarrow{\pi} R/I$ is a surjective Ring homomorphism.

$\pi(a) = a + I$

By previous Thm, $\exists$ a bijection

$\{\text{Ideals of } R \supseteq \ker\pi\} \leftrightarrow \{\text{ideal of } R/I\}$.

Exercise: Let $R = \mathbb{Z}$ and $I = 6\mathbb{Z}$. Then find the ideals of $R/6I$.

Thm: Let $f: R \to S$ be an surj. ring homo and $J \subseteq S$ be an ideal. Then $f^{-1}(J)$ is an ideal of R and

$$R/f^{-1}(J) \cong S/J.$$

Proof: $R \xrightarrow{f} S \xrightarrow{\pi} S/J$

Then $\pi \circ f : R \to S/J$ is a surj. ring homo

$$\ker(\pi \circ f) = \{a \in R \mid \pi \circ f(a) \in J\}$$

$f(a) + J \in S$

$$= \{a \in R \mid f(a) \in J\}$$

$$\ker(\pi \circ f) = f^{-1}(J)$$

Ring 1st Isomorphism Thm

$$R/f^{-1}(J) \cong S/J$$

Remark: Let $\pi : R \to R/I$ be the natural Ring Homomorphism defined by $\pi(a) = a + I$.

Let $J \supseteq I$. $J/I = \{a + I \mid a \in J\}$

By previous Theorem,

$$R/I \,/\, J/I \cong R/J$$

because $\pi^{-1}(J/I) = J$.

Remark: Ideals of R/I have the form $J/I = \{b+I \mid b \in J\}$ where J is the ideal of $R \supseteq I$.

Construction of Rationals from Integers

Consider the set of all ordered pair

$f = \{(a,b) \mid a,b \in \mathbb{Z}, b \neq 0\}$

Define a Relation of f

$(a,b) \sim (c,d)$ $ad - bc = 0$

$(a_1,b_1) \sim (a_2,b_2)$ ~~(a_3,b_3)~~

and $(a_2,b_2) \sim (a_3,b_3)$

WTS $(a_1,b_1) \sim (a_3,b_3)$

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} \Rightarrow a_1b_2 - a_2b_1 = 0 \quad -(1)$$

$$a_2b_3 - b_2a_3 = 0 \quad -(2)$$

$$a_1b_2b_3 - a_2b_1b_3 = 0$$

$$a_2b_1b_3 - b_2b_1a_3 = 0$$

$$\Rightarrow a_1b_2b_3 - b_2b_1a_3 = 0$$

$$\Rightarrow b_2(a_1b_3 - b_1a_3) = 0$$

As $b_2 \neq 0$

$$\Rightarrow a_1b_3 - b_1a_3 = 0$$

$\therefore \sim$ is an equivalence relation

* Rational numbers are nothing but the equivalence classes.

Equivalence class of (a,b) is denoted by $\frac{a}{b}$.

$\mathbb{Q}$ is the set of all equivalence classes.

Define '+' & '.' in $\mathbb{Q}$ by

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

Thm: Let R be an integral domain. Then $\exists$ a field F and a inj ring homo $R \to F$

Pf: Consider the $f = \{(a,b) \mid a,b \in R, b \neq 0\}$

Define a Relation on f by

$(a,b) \sim (c,d)$ iff $ad - bc = 0$

$\sim$ is an equivalence relation.

Equivalence classes of $(a,b) \in f$ is denoted by $\frac{a}{b}$.

F = The set of all equivalence classes

Define (+) & (·) as follows:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \quad \& \quad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

Check that they are well defined.

F is a ring wrt '+' & '·' defined with

$\frac{0}{1}$ and $\frac{1}{1}$

↓ add identity ↓ mult. identity

~~0 is the 0 of~~

WTS: F is a Field.

Let $\frac{a}{b} \neq \frac{0}{1}$. $\Rightarrow a \neq 0$.

Then $(b, a) \in F$. which is the inverse of (a, b) or $\frac{a}{b}$.

$\therefore$ F is a Field.

$\phi : R \longrightarrow F$

$\phi(r) = \frac{r}{1}$ Check ϕ is a Ring homo.

$\ker \phi = \{r \in R \mid \frac{r}{1} = 0\}$

$\ker \phi = \{0\}$

$\Rightarrow \phi$ is injective.

$\therefore$ We identify R with $\phi(R) = \{\frac{r}{1} \mid r \in R\} \subset F$

Defn: F is called the quotient field of R and is denoted by Q.

Example: ① $\mathbb{Q}$ is the quotient field of $\mathbb{Z}$.

② $k[x]$. the quotient field is the field of Rational functions

$$k(x) = \left\{ \frac{f(x)}{g(x)} \mid g(x) \neq 0 \right\}$$

Philosophical Question Let R be a ring and I be an ideal of R. When is R/I an integral domain?

Let $a, b \in R$ and set. $\bar{a} = a + I$, $\bar{b} = b + I$

i.e. $\bar{a}, \bar{b} \in R/I$. Let $\bar{a} \cdot \bar{b} = \bar{0}$

$$\Rightarrow (a+I)(b+I) = 0$$

$$\Rightarrow ab + I = \bar{0} + \bar{I}$$

$$\Rightarrow ab \in I$$

if R/I is an Integral domain

$\Rightarrow$ if $ab \in I \Rightarrow$ either $a \in I$ or $b \in I$.

Proof

Definition: An Ideal $I \subsetneq R$ is called a prime ideal if $ab \in I$ then either $a \in I$ or $b \in I$.

Proposition: Let R be a rong. Then an ideal P is prime ideal of R iff R/P is an Integral domain.

$\mathbb{Z}/n\mathbb{Z}$

Example: ① $n\mathbb{Z}$ is a prime ideal in $\mathbb{Z}$ iff n is a prime number.

② Let us consider the polynomial ring $k[x]$ where k is a field.

A polynomial with coeff in k is said to be irreducible over k if it is non constant and can not be factored into the product of two more non-constant polynomials with coeff in k.

③ When an ideal $(f(x)) \subset k[x]$ is a prime ideal?

$(f(x))$ is a prime ideal ie. if $gh \in f(x)$ then either $g \in (f(x))$ or $h \in (f(x))$

If $f(x) \mid gh \Rightarrow$ Eith $f \mid g$ or $f \mid h$.

WTS If $f(x)$ is irreducible then $(f(x))$ is a prime ideal.

If $f(x) \nmid g$ then $\gcd(f, g) = 1$

$1 = fp + gq$

$\Rightarrow h = fph + ghq = f \cdot p \cdot h + fqq'$

$h = f(ph + qq')$

$\Rightarrow h \in f(x)$

$\therefore$ If f is irreducible polynomial then $(f(x))$ is a prime ideal.

$r(I) = \{x \in A \mid x^n \in I\}$ radical

(i) $I \subseteq r(I)$

(ii) $\phi : A \longrightarrow A/a$

$\phi^{-1}(n(A/a)) = r(a)$

$\phi : \mathbb{Z} \to \mathbb{R}$ $(\mathbb{Z}/12\mathbb{Z})$

$\phi : \mathbb{Z} \longrightarrow \mathbb{Z}/12\mathbb{Z}$

$\phi : \mathbb{R} \longrightarrow \mathbb{R}$

↓

ring homo

$\phi(x_1 + x_2) = \phi(x_1) + \phi(x_2)$

Show for $\mathbb{Q}$ $f(m/n) = \frac{m}{n}$

if $a \geq 0$

$\phi(a) \geq 0$

$a = \sqrt{a}\sqrt{a}$

$\phi(a) = \phi(\sqrt{a})(\sqrt{a})$

$\Rightarrow \phi(a) = (\phi(\sqrt{a}))^2$

$\rightarrow \phi(a) \geq 0.$

A ring

(i) x = nilradical = set of nilpotent elements
$= \bigcap_{P \text{ prime ideals}} P$

(ii) Maximal Ideal.

Read
① Zorn's Lemma,
② Posets.

$\Sigma = \{ I \mid I \text{ is a proper ideal of } A \}$

$I_1 \leq I_2 \iff I_1 \subseteq I_2$.

$0 \in \Sigma$

$I_1 \subseteq I_2 \subseteq \cdots \Longrightarrow$

To show $\bigcup I_i$ is a proper ideal.

Pf $x, y \in \bigcup I_i$, $\exists\, n$ s.t. $xy \in I_n$.
$x + y \in I_n \subseteq I_i$

Exercise: Every maximal ideal is a prime ideal.

A/I is an integral domain

$\Updownarrow$

I is a prime ideal

(2) A/I is a field

$\Updownarrow$

I is a maximal ideal

$\bigcap_{\substack{m \\ \text{maximal} \\ \text{ideal}}} m = J(A)$ Jacobson radical

(3) $x \in J(A) \iff 1 - xy$ is a unit $\forall\ y \in A$.

(4) Collection of prime ideals have a minimal element

$I \in \cup I_i$

$X = \text{Spec}(A) \longrightarrow$ Affine scheme of Ring

$= \{$ collection of all prime ideal of $A\}$

$E \subseteq A$

$V(E) = \{p \in X \mid p \supseteq E\}$

(1) A^a is ideal generated by E,

$V(E) = V(a)$

$V(0) = X$

$V(1) = \phi$

(iii) E_i, $V(\cup E_i) = \cap V(E_i)$

(iv) $V(a \cap b) = V(a) \cup V(b) = V(ab)$

X topology defined by closed sets of the form $V(E)$. ↓ Zariski Topology in X.

Prime Ideal

Let $k[x]$ be a poly. ring. In a poly ring $k[x]$ every ideal is gen by a poly. and if the poly. is irreducible then the ideals becomes a prime ideal

Proposition: A ring R is an Integral domain iff (0) is a prime ideal.

Example: In $\mathbb{Z}[i]$ consider the ideal (2) is not a prime ideal because $2 = (1+i)(1-i)$ & $(1+i) \notin (2)$ & $(1-i) \notin (2)$. Suppose $(1+i) \in (2)$

$1+i = 2(a+ib)$

$\Rightarrow 1 = 2a, 2b = 1$ which is not possible

Q// When is R/I is a field?

Now we investigate the surj. homo $\varphi : R \longrightarrow F$ where F is a field & R is a Ring.

Defⁿ: An ideal M of a ring R is called maximal ideal if whenever $M \subset I \subset R$ then either $I = M$ or $I = R$.
i.e M is not contained in any ideal other than M and R.

Zorn's Lemma:
Let S be a partially ordered set. If every totally ordered subset of S has an upper bound then S contains a maximal element.

Proposition: Maximal ideal exists in a non-zero ring.

Proof: Consider the set $S = \{ I \mid I$ is a proper ideal $\}$
under inclusion S is a partially ordered set. Let T be a totally ordered subset of S i.e. $\forall\ I, J \in T$

i.e. either $I \subset J$ or $J \subset I$.

Let $u = \bigcup \{ I \mid I \in T \}$

↓ due to chain of Ideal.

This u is an ideal of R and u is proper

By Zorn's lemma, S has a maximal ideal.

Proposition: An ideal M of R is a maximal ideal iff R/M is a field.

Proof: Let M be a maximal ideal

R/M has only 2 ideals $(\bar{0})$ and $R/M = (\bar{1})$

↓ (Because M is the maximum possible ideal)

since R/M has exactly 2 ideals then

R/M is a field.

Ex: Prove the converse

REMARK

Every maximal ideal is a prime ideal but the converse is not true.

Eg. in $\mathbb{Z}$, (0) is a prime ideal but not a maximal ideal as $(0) \subsetneq (n)$.

Converse Ex: $\mathbb{R}[x,y]$. Then (x) is not a maximal ideal

$$\mathbb{R}[x,y]/(x) \cong \mathbb{R}[y].$$

$\therefore (x)$ is a prime ideal but not a maximal ideal.

(0). The zero ideal is a maximal ideal iff R is a field.

Ex: ① In $\mathbb{Z}$, the maximal ideals are generated by prime ideals.

② Consider the $k[x]$.
Every maximal ideal is generated by an irreducible polynomial

$f(x) \subset m$
$(f(x)) \subset (g(x))$
$f(x) = g(x)h(x)$.

In $\mathbb{C}[x]$ every maximal ideal is generated by a linear polynomial $(x-a)$.

$m_a = (x-a) \longleftrightarrow \mathbb{C}$

$\{\text{max. ideals}(\mathbb{C}[x])\} \longleftrightarrow \mathbb{C}$.

$\mathbb{C}[x_1, x_2, \ldots x_n] \longleftrightarrow \mathbb{C}^n$

Ex What is the structure of the maximal ideals in $\mathbb{C}[x_1, x_2, \ldots, x_n]$.

Thm HILBERT'S NULLSTELLENSATZ

The maximal ideals of the polynomial ring $R = \mathbb{C}[x_1, x_2, \ldots x_n]$ are in 1-1 correspondence with the pts in $\mathbb{C}^n$.

A pt $a = (a_1, \ldots a_n) \in \mathbb{C}^n$

$\mathbb{C}[x_1, x_2, \ldots x_n] \longrightarrow \mathbb{C}^n$.

$f(x_1, \ldots x_n) \rightsquigarrow f(a_1, \ldots a_n)$

kernel of this map $= (x_1 - a_1, \ldots, x_n - a_n)$

Infact every maximal ideal of $\mathbb{C}[x_1, \ldots x_n]$ is of the form

$m_a = (x_1 - a_1, \ldots, x_2 - a_n)$

Defn: **Variety**: Let V be a subset of $\mathbb{C}^n$ If V can be defined as the set of common zeros of a finite number of polynomials in n variables, then V is called an algebraic variety.

$$V = Z(f_1, \ldots f_n)$$

Examples: In $\mathbb{C}^2$ every point is a variety because $Z(x-a, y-b)$.

Complex line in $\mathbb{C}^2$ is $Z(ax + by + c)$ is a variety.

Theorem: Let $f_1, \ldots f_k$ be polynomials in $\mathbb{C}[x_1, \ldots x_n]$ and $V = Z(f_1 \ldots f_n)$ Let $I = (f_1 \ldots f_n)$, The maximal ideals of the quotient rings $R = \frac{\mathbb{C}[x_1, \ldots x_n]}{I}$ is in bijective correspondence with the pts of V.

Proof: The maximal ideals of R corresponds to those maximal ideals of $\mathbb{C}[x_1, \ldots x_n]$ which contain I. Any maximal ideal is of the form $(x_1 - a_1, \ldots x_n - a_n)$ Now $(x_1 - a_1, x_2 - a_2, \ldots x_n - a_n) \supset I = (f_1 \ldots f_k)$

The maximal ideal contains I iff it contains the generator of $f_1, \ldots f_n$ of I. so if $f_i \in m_a$ iff $f_i(\underline{a}) = 0 \ \forall \ i$ which implies that $\underline{a} \in V(I)$.

Corollary: Let $f_1, \ldots, f_n$ be polynomials in $\mathbb{C}[x_1, \ldots, x_n]$. If the system of equations $f_1 = f_2 = \ldots = f_n = 0$, has no solution in $\mathbb{C}^n$, Then $1 = \sum_{i=1}^{n} g_i f_i$ where $g_i \in \mathbb{C}[x_1, \ldots x_n]$

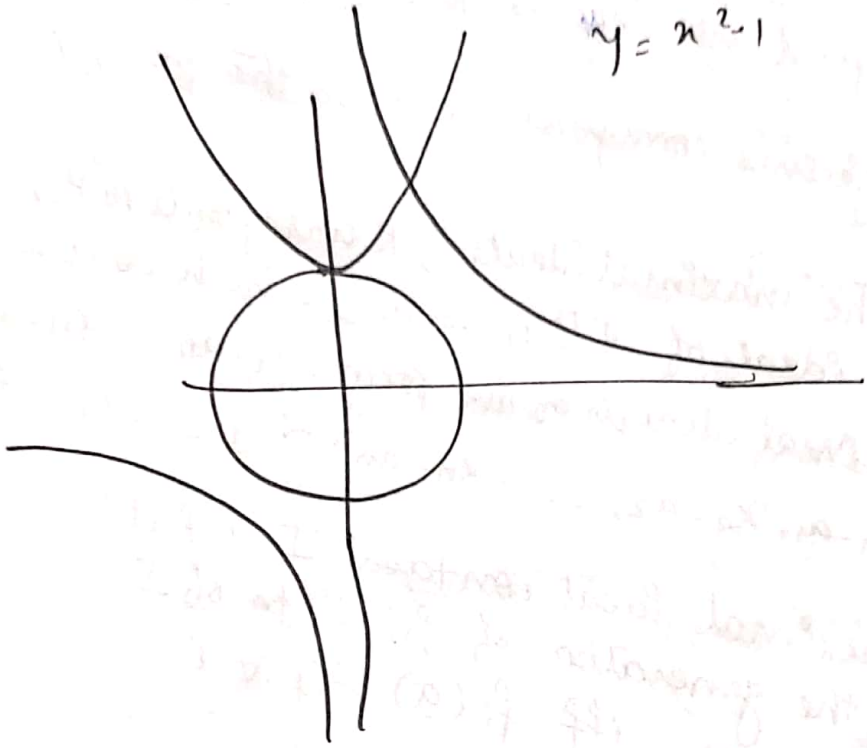
Proof: $V = Z(f_1, \ldots, f_n) = \phi$ which means there is no maximal ideal containing I. Therefore I is the whole ring i.e. $1 \in I$.

$\therefore 1 = \sum g_i f_i$

Example: $f_1 = x^2 + y^2 - 1$

$f_2 = x^2 - y + 1$

$f_3 = xy - 1$



Here $V(f_1, f_2, f_3) = \phi$

$\Rightarrow 1 \in (f_1, f_2, f_3)$

Unique Factorization Domain: (consider only Integral Domain)

Let R be an integral domain throughout

Definition: We say an elt a divides another elt b ($a|b$) if $b = aq$ for some $q \in R$.

The element a is a proper divisor of b if neither a nor q is a unit.

A non zero element a of R is called irreducible if a is not a unit and has no proper divisors. (i.e $a = bc$ then either b or c is a unit)

We say an element a is prime if (a) is a prime ideal.

Proposition Let R be an integral domain and $0 \neq a \in R$. If a is prime element then it is irreducible.

(a) is a prime ideal. If $a = bc$, then either $b \in (a)$ or $c \in (a)$. WLOG say $b \in (a)$

$\Rightarrow b = ad$. where $d \in R$.

$\therefore a = bc = adc$

$\Rightarrow \boxed{dc = 1}$ ($\because R$ is integral domain)

Example: ① The irreducible elements of $\mathbb{Z}$ are prime elements.

② The irreducible elements of $k[x]$ are irreducible polynomial.

Example: $R = \mathbb{Z}[\sqrt{-3}] = \{a + b\sqrt{-3} \mid a, b \in \mathbb{Z}\}$

Consider 2.

$$2 = (a + ib\sqrt{3})(c + id\sqrt{3})$$

$$2 = ac - 3bd + (bc + ad)\sqrt{3}i$$

$$2.\overline{2} = (a^2 + 3b^2)(c^2 + 3d^2)$$

$$4 = (a^2 + 3b^2)(c^2 + 3d^2)$$

$\therefore (a^2 + 3b^2)$ must divide 4 and $(a^2 + 3b^2)$ cannot be 2

Hence $(a^2 + 3b^2) = 4$ & $c^2 + 3d^2 = 1$

$\Rightarrow d = 0, c = \pm 1$

∴ one of the factors of 2 is a unit.
Thus 2 is irreducible elt.

2. $(1+\sqrt{-3})(1-\sqrt{-3}) = 4 \in (2)$.

But neither $(1+\sqrt{-3})$ or $(1-\sqrt{-3}) \in 2$

∴ (2) is not a prime ideal.

4 can have two representations | factorizations and it restricts the ~~prime~~ irreducible elements to generate a prime ideal.

UNIQUE FACTORIZATION DOMAIN

UFD is an integral domain satisfying that

1. Every non zero element $0 \neq a \in R$ can be written as product of irreducible factors say $p_1 \cdots p_n$ upto namely

$$a = x p_1 \cdots p_n .$$

2. The above factorization is unique if

$$a = u p_1 \cdots p_n = v q_1 \cdots q_m$$

are two factorizations into irreducible elements

p_i & q_j with units u, v then ~~n = m~~ $n = m$ & p_i & q_j are associates

Def: Two elements $a, b \in R$ are called associates if $a = ub$ for some unit $u \in R$.

Examples:

① $\mathbb{Z}$, $k[x]$ are UFD.

Proof: In an UFD an elt a is irreducible iff a is prime

Prime: Let a be an irreducible elt $\neq 0$.

WTS: (a) is prime

Let $bc \in (a)$

$\Rightarrow bc = ad$ for some $d \in R$

$\Rightarrow$ Since R is an UFD, we can decompose b, c, d into irreducible elts.

$$v\, b_1 \cdots b_s\, w\, c_1 \cdots c_t = a \cdot k\, d_1 \cdots d_r$$

$(r+1 = s+t)$

Since the factorization is unique a must be associated to c_i or d_i

$\Rightarrow a$ divides b or c

$\Rightarrow b \in (a)$ or $c \in (a)$

$\Rightarrow (a)$ is prime ideal.

Proposition :- Let R be an integral domain and $a, b \in R$. Then

(i) a is a unit iff $(a) = R$

(ii) a and b are associates iff $(a) = (b)$.

(iii) $a \mid b$ iff $(b) \subset (a)$.

(iv) a is a proper divisor of b iff

(b) $\subsetneq (a) \subsetneq R$.

(v) a is irreducible iff (a) is maximal among proper principle ideals of R

Defn : An int. domain R is called Factorization domain if every non zero element of R can be written as product of irreducible elt.

Proposition : Let R be an int domain

TFAE

(1) For every nonzero elt $b \in R$ which is not a unit, the process of factoring b terminates after finitely many steps and result a factorizations of b into irreducible elements

(2) R doesn't contain an infinite increasing chain of principal ideals

$(a_1) \subset (a_2) \subset (a_3) \subset \cdots$

Noetherean Ring

Proof: suppose R contains an infinite increasing sequence of principal ideals

$(a_1) \subset (a_2) \subset \cdots \subset (1)$

Since $(a_{n-1}) \subset (a_n)$ for all n.

$\therefore a_{n-1} = a_n b_n$

where a_n and b_n are not units.

$a_1 = a_2 b_2 = a_3 b_2 b_3 = a_4 b_2 b_3 b_4 = \cdots$

$(a_2 = a_3 b_3)$

Exercise Do the converse.

Example: Consider the poly ring $k[x_1]$.

Let $R = k[x_1, x_2, x_3, \ldots]$

where $x_2^2 = x_1$ $x_3^2 = x_2$ $x_4^2 = x_3$

& so on. $\therefore$ We can factor x_1 indefinitely in the Ring R and get infinite chain of principle ideal

$(x_1) \subset (x_2) \subset (x_3) \subset \cdots$

In $\mathbb{Z}$, $a, b \in \mathbb{Z}$

$(a, b) = (d) \longrightarrow \gcd(a, b)$

$(a, b) = (1)$

$1 = ar + bs$

In $\mathbb{Z}[x]$

$\gcd(2, x) = 1$

Proposition: Let a and b be 2 non zero elements of an UFD and let $a = u p_1^{e_1} \cdots p_n^{e_n}$,

$b = v p_1^{f_1} \cdots p_m^{f_m}$ are unique factorization

into irreducible factors where u, v are units and p_i's are distinct and $e_i, f_i \geq 0$. Then the element

$$d = p_1^{\min(e_1, f_1)} \cdots p_n^{\min(e_n, f_n)}$$

is a gcd of a and b.

Proof: Ex.

Defn: An integral domain R is called **principal ideal domain** (PID) if every ideal of R is principal.

Example: $\mathbb{Z}$, $K[x]$ are PID.

Proposition: In a PID irreducible elements are prime.

Proof: Let p be an irreducible element

WTS: (p) is a prime ideal

Let $ab \in p$

$\Rightarrow p \mid ab$

Let $p \nmid a$. Then $(p) \subsetneq (p, a)$

Since p is maximal among all the principal ideals.

$$(p, a) = (1)$$

$\Rightarrow 1 = pc + ad$ for $c, d \in R$

$b = pcb + abd$

$\Rightarrow b = p(cb + d'd) \in (p)$

$\Rightarrow b \in (p)$

$\therefore p$ is a prime ideal.

$\Rightarrow$ Every irreducible element is a prime element in a PID.

Prop 4: Every non zero prime ideal in a PID is a maximal ideal.

Proof: Let (p) be a non zero prime ideal. and (p) is not a maximal ideal.

There exists a maximal ideal S.T. $(p) \subset (m)$

$\Rightarrow p = rm$ for some $r \in R$

Since (p) is a prime ideal and $rm \in (p)$ then either $r \in (p)$ or $m \in (p)$. If $m \in p$ then $(p) = (m)$ is maximal ideal.

If $r \in (p)$, then $r = p \cdot s$ for some $s \in R$

$\therefore p = psm \Rightarrow sm = 1$ ($\because R$ is int domain)

$\Rightarrow m$ is a unit

Hence the proof.

corollary: Let R be any ring and $R[x]$ is a PID, then R is a field.

$R[x]/(x) \cong R$ $\quad R \subset R[x]$

$\downarrow$ is int domain

But

$\Rightarrow (x)$ is a prime ideal.

Since $R[x]$ is a PID so (x) is a maximal ideal. Hence R is a field.

Exercise: A PID is an UFD.

25th March

Proposition:

WTS: Existence of factorization in R. which is equivalent to show that R contains no infinite increasing chain of principle ideals.

suppose $(a_1) \subset (a_2) \subset (a_3) \ldots$ is an infinite chain of principal ideals.

Let I be the union of this chain of principal ideals. Then I is an ideal since R is a PID so I will be generated by a single element.

so $I = (b)$

Now since b is in the union of this ideals (a_n) it

$\Rightarrow b \in (a_n)$

On the other hand

$(a_n) \subset (a_{n+1}) \subset (b) \ldots$

$\therefore (a_n) = (a_{n+1}) = (b)$.

which is a contradiction.

Therefore every element of R can be written as product of irreducible elements.

Since every irreducible is prime so R is an UFD.

Example

$\mathbb{Z}[x]$ is a UFD but not a PID.

(How?)

EUCLIDEAN DOMAIN

Let us now abstract the procedure of division with remainder. To do this we need the notion of size of an element of a ring. In general a size fⁿ on an integral domain R will be any fⁿ.

$N: R \setminus \{0\} \longrightarrow \{0, 1, 2, 3, \ldots\}$

from the set of non zero elements of R to the set of all non negative integers.

Defⁿ: An integral domain R is an Euclidean Domain if there is a size function N on R S.T. for all $a, b \in R$ S.T. $b \neq 0$ there are elements $q, r \in R$ S.T. $a = bq + r$ and either $r = 0$ or $N(r) \leq N(b)$, where q is called the quotient and r is the remainder.

Example (1) Fields are E.D. where if $N(a) = 0 \ \forall\ a$ the

$$a = qb + 0$$

where $q = ab^{-1}$

(2) $\mathbb{Z}$ is an E.D. with $N(a) = |a|$.

(3) if F is a field then $F[x]$ is E.D. with $N(f(x)) = \deg f(x)$.

(4) $\mathbb{Z}[i]$ = Ring of Gaussian Integers

$$= \{a + bi \mid a, b \in \mathbb{Z}\}$$

is an E.D. with $N(a+ib) = a^2 + b^2$.

Let $\alpha = a + ib$ and $\beta = c + id \neq 0$.

$$\alpha/\beta = \frac{a+ib}{c+id} = \frac{(a+ib)(c-id)}{c^2+d^2}$$

$$\alpha/\beta = \frac{ac + bd + (bc - ad)i}{c^2 + d^2}$$

$$\alpha/\beta = r + is \in \mathbb{Q}[i].$$

Let p and q be integers closest to r & s.

$\therefore |r-p|$ & $|s-q|$ are atmost $1/2$

$$\theta = (r-p) + i(s-q)$$

$$\boxed{\gamma = \beta\theta} = \beta[(r-p) + i(s-q)]$$

$$= \beta[(r+is) - (p+iq)]$$

α $\qquad$ $\in \mathbb{Z}[i]$

$\Rightarrow \alpha = \beta(p+iq) + \nu$

$N(\nu) = N(\beta\theta) \overset{\text{Exercise}}{=} N(\beta)\,N(\theta)$

$= N(\beta)\left[(r-p)^2 + (s-q)^2\right]$

$\leq N(\beta)\left[\frac{1}{4} + \frac{1}{4}\right]$

$\leq \frac{N(\beta)}{2}$

$\therefore \boxed{N(\nu) < N(\beta)}$

$\therefore \mathbb{Z}[i]$ is an E.D.

Thm: Let R be an E.D. then R is a PID.

Exercise: Find the units in $\mathbb{Z}[i]$.

26th March

Field ⊂ E.D. ⊂ PID ⊂ UFD ⊂ Integral Domain

$\mathbb{Z}$ is a ED but not a field

unit! → $\mathbb{Z}[(1+\sqrt{-19})/2]$ is a PID but not ED.

- $\mathbb{Z}[x]$ is an UFD but not a PID
- $\mathbb{Z}[\sqrt{-3}]$ is an integral domain but not UFD

$\mathbb{Z}[i]$ - Ring of Gaussian Integers.

If $u \in \mathbb{Z}[i]$ is a unit in $\mathbb{Z}[i]$ then $N(u) = 1$ and the units are $\pm 1, \pm i$

Definition: A prime elt in $\mathbb{Z}[i]$ is called a Gaussian Prime

Proposition: 1) If $N(a,b) = a^2 + b^2 = p$ is a prime number then $a+ib$ is a Gaussian Prime

2) If π is a gaussian prime then $N(\pi) = \pi\bar{\pi}$ is either a prime number or square of a prime number

Pf ① Let $\alpha = a + Pb$ S.T. $N(\alpha) = p$

WTS α is a prime elt.

Since $\mathbb{Z}[i]$ is an E.D. so it is an UFD. have prime is equivalent to irreducible element. WTS α is an irreducible elt. Let $\alpha = \beta\nu$ where $\beta, \nu \in \mathbb{Z}[i]$

Then $N(\alpha) = N(\beta)\,N(\nu) = p$

Hence either $N(\beta)$ or $N(\gamma)$ is unit.
Hence either β or γ is unit.
Thus α & β irreducible element. Hence prime element

(2) Let π be a gaussian prime

WTS: $N(\pi)$ is either a prime number a square of a prime number.

$(\pi) \cap \mathbb{Z} \neq (0)$. since $\pi\bar{\pi} \in \mathbb{Z}$. $\neq 0$.

Note that $(\pi) \cap \mathbb{Z}$ is an ideal of $\mathbb{Z}$.

Ex: show that $(\pi) \cap \mathbb{Z}$ is a prime ideal in $\mathbb{Z}$.

Hence $(\pi) \cap \mathbb{Z} = (p)$ for some prime n.p.

$\therefore p \in (\pi)$

$\therefore p = \pi\mu$ where $\mu \in \mathbb{Z}[i]$.

$p^2 = N(p) = N(\pi)\, N(\mu)$

$N(\pi) = p$ or p^2

Thm: The following are equivalent for prime element p.

(1) $p = \pi\bar{\pi}$ where π is a gaussian integer

(2) $p = a^2 + b^2$ for $a, b \in \mathbb{Z}$

(3) $x^2 \equiv -1 \pmod{p}$ has an integer solution

(4) $p = 2$ or $p \equiv 1 \pmod 4$

Proof:

① ⇒ ②

$p = \pi\bar{\pi}$

Let $\pi = a + ib$

$\Rightarrow \pi\bar{\pi} = a^2 + b^2 = p$

② ⇒ ③

$p = a^2 + b^2$ for $a, b \in \mathbb{Z}$.

$a, b \neq 0$.

~~$p = a^2 + b^2$~~

$a^2 \equiv -b^2 \pmod{p}$

Since $\mathbb{Z}/p\mathbb{Z}$ is a field. 13

$\therefore (ab^{-1})^2 \equiv -1 \pmod{p}$

$\Rightarrow x^2 \equiv -1 \pmod{p}$

③ ⇒ ④

Let p be an odd prime.

Let $a \in \mathbb{Z}/p\mathbb{Z}$ and $a^2 \equiv -1 \pmod{p}$.

Then $o(a) = 4$ in the multiplicative gp $(\mathbb{Z}/n\mathbb{Z})^\times$

Hence $4 \mid p-1$

$\Rightarrow p \equiv 1 \pmod{4}$.

④ ⇒ ③

For $p=2$ 1 is solution to

$$x^2 \equiv -1 \pmod{p}$$

Now let $p \equiv 1 \pmod 4$

$\Rightarrow 4 \mid p-1$

Consider the gp $(\mathbb{Z}/p\mathbb{Z})^*$.

Since $4 \mid p-1$, so there is a subgroup of order 4 ~~is~~ in $(\mathbb{Z}/p\mathbb{Z})^*$ (Sylow's Theorem)

Let H be the subgroup of $(\mathbb{Z}/p\mathbb{Z})$ of order 4 and let H ~~is~~ a cyclic group.

so $\exists$ a s.t. $a^4 \equiv 1 \pmod p$.

$\Rightarrow p \mid a^4-1$

$\Rightarrow p \mid (a^2-1)(a^2+1)$

$\Rightarrow$ If $p \mid a^2-1$ then $a^2 \equiv 1 \pmod p$ which says that $O(a) = 2$ Assumed which is a contradiction. ($O(H) = 4$)

$\Rightarrow p \mid a^2+1$

$\Rightarrow a^2 \equiv -1 \pmod p$

$\Rightarrow a$ is a solution of $x^2 \equiv -1 \pmod p$

Claim: $(\mathbb{Z}/p\mathbb{Z})^*$ contains a unique element of order 2.

If $m^2 \equiv 1 \pmod p \Rightarrow p \mid m^2-1 = (m+1)(m-1)$

$\Rightarrow p \mid m+1$ or $p \mid m-1$

$\Downarrow$ $m \equiv -1 \pmod p$ $\quad$ $\Downarrow$ $m \equiv 1 \pmod p$ ✗

So ± 1 is the unique residue class of order 2 in $(\mathbb{Z}/p\mathbb{Z})^*$.

(3) $\Rightarrow$ (1).

$$\mathbb{Z}[i] \cong \frac{\mathbb{Z}[x]}{(1+x^2)}$$

$$\frac{\mathbb{Z}[i]}{(p)} \cong \frac{\mathbb{Z}[x]}{(p,\, 1+x^2)}$$

$$\longrightarrow \quad \frac{\mathbb{Z}[x]/(1+x^2)}{p\,\mathbb{Z}[x]/(1+x^2)} \cong \frac{\mathbb{Z}[x]/(1+x^2)}{(p, x^2+1)/(x^2+1)}$$

Ex $\dfrac{\mathbb{Z}[x]}{p\mathbb{Z}[x]} \cong (\mathbb{Z}/p\mathbb{Z})[x]$

$$\cong \frac{\mathbb{Z}[x]}{(p, x^2+1)}.$$

$$\cong \frac{\mathbb{Z}[x]/p\mathbb{Z}[x]}{(p, x^2+1)/p\mathbb{Z}[x]}$$

$$\cong \frac{\mathbb{Z}/p\mathbb{Z}[x]}{(x^2+\bar{1})}$$

We have $x^2 \equiv -1 \pmod p$ has a solution

$\exists\ a$ s.t. $a^2 \equiv -1 \pmod p$.

Thus p is not an irreducible element in $\mathbb{Z}[i]$.

Let $p = \pi\delta$ where δ and π are not unit

$\therefore N(p) = N(\pi)N(\delta) = p^2$

Thus $N(\pi) = p = \pi\bar{\pi}$.

Corollary (Fermat's two square Thm)
Let p be prime number then p is a sum of two square iff $p = 2$ or $p \equiv 1 \pmod 4$

Corollary The irreducible elements of $\mathbb{Z}[i]$ are

1. $(1+i)$ and its associates.
2. Rational primes p s.t. $p \equiv 3 \pmod 4$.
3. If $a+ib$, $a-ib$ the distinct irreducible factors of $p = a^2+b^2$ for the prime $p \equiv 1 \pmod 4$

Polynomials over UFD

(Q) Let R be an UFD. Is R[x] an UFD?

We know that F[x] is UFD where F is a field. Let K be the quotient field of R. Then R is a subring of K[x]. Since K[x] is an UFD, we try to relate factorization in R[x] and K[x].

$\mathbb{Z}[x]$ $\mathbb{Q}[x]$.

(Q) Let $f(x) \in \mathbb{Z}[x]$ is reducible then is it reducible in $\mathbb{Q}[x]$?

(Q) Is an irreducible polynomials over $\mathbb{Z}[x]$ remains irreducible over $\mathbb{Q}[x]$?

Eg: Consider the poly. $2x \in \mathbb{Z}[x]$

Then $2x = 2 \cdot x$ in $\mathbb{Z}[x]$ and 2 is not an unit so $2x$ is reducible But $2x$ is irreducible over $\mathbb{Q}[x]$ because 2 is a unit in $\mathbb{Q}$.

Defn: Let R be an UFD. The content of $f(x) \in R[x]$ denoted by $c(f)$ is the gcd of coeff. of $f(x)$. If $c(f) = 1$ we say $f(x)$ is primitive

$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$

Thm: Let R be an UFD. If $f(x), g(x) \in R[x]$ are primitive then $f(x)g(x)$ is primitive

Pf: Suppose $f(x)g(x)$ is not primitive

$\Rightarrow \gcd(fg) \neq 1$

Let p is a prime in R dividing the coefficients of $f(x)g(x)$.

Consider the following map

$\pi: R[x] \longrightarrow R/p\,[x]$

$\pi\left(\sum a_n x^n\right) = \sum \bar{a}_n x^n$.

($\phi: R \to R/p$ then $\phi(a_n) = \bar{a}_n$)

Then $\overline{f(x)g(x)} = \bar{0}$

R/p is an integral domain.

$\Rightarrow \bar{f}(x)\,\bar{g}(x) = \bar{0}$

$\Rightarrow \bar{f}(x) = 0$ or $\bar{g}(x) = 0$

$\Rightarrow \exists\, p$ that divides the content of either $f(x)$ or $g(x)$

$\Rightarrow$ ~~content~~ ($c(f) \neq 1$ or $c(g) \neq 1$

$\Rightarrow \Leftarrow$ (contradiction)

Corollary:

For $f(x), g(x) \in R[x]$ we have

$$c(fg) = c(f) \cdot c(g).$$

Proof Let $c(f) = a$ and $c(g) = b$

Then $f(x) = a \cdot f_1(x)$

$g(x) = b \cdot g_1(x)$.

$\Rightarrow f(x)\, g(x) = a \cdot b \cdot f_1(x)\, g_1(x)$

From last Thm f_1, g_1 are primitive since f, g are primitive

$f_1 g_1$ is primitive

$\Rightarrow c(fg) = ab$

$\Rightarrow c(fg) = c(f)\, c(g)$

Proposition

$f(x), g(x) \in R[x]$ both primitive and $f(x)$ & $g(x)$ are associates in $K[x]$.

Is $f(x)$ & $g(x)$ are associates in $R[x]$

Proposition Let R be an UFD with quotient field K. If $f(x), g(x) \in R[x]$ are primitive and associates in $K[x]$ then they are associates in $R[x]$.

Pf: Let $f(x) = \frac{a}{b} g(x)$ where $a, b \in R$, $b \neq 0$

Then $b f(x) = a g(x)$.

Since $f(x)$ and $g(x)$ are primitive

so $c(bf) = b$ & $c(ag) = a$.

Since in a UFD the gcd is unique upto multiple by unit.

so $a = ub$, where u is a unit in R.

$\therefore f(x) = u g(x)$, where $u \in R$ is a unit.

Propⁿ [Gauss lemma] Let R be an UFD with quotient field K and let $p(x) \in R[x]$. If $p(x)$ is reducible in $K[x]$ then $p(x)$ is reducible in $R[x]$. More precisely, if $p(x) = A(x)B(x)$ for some non constant polynomials $A(x), B(x) \in K[x]$, then there are non-zero elts $r, s \in K$ s.t. $rA(x) = a(x)$ and $sB(x) = b(x)$ and $p(x) = a(x)b(x)$ where $a(x), b(x) \in R[x]$ is a factorization in $R[x]$

~~Q) Is R[x] an UFD?~~

2nd April

Proof: Let $p(x) = A(x)B(x)$, where $A(x), B(x) \in K[x]$.

Now multiplying by common denominator for all of these coefficients, we obtain

$$d\,p(x) = a'(x)b'(x)$$

where $a'(x), b'(x) \in R[x]$ and d is a non zero element of R. If d is a unit in R, then the proposition is done (multiply both sides by d^{-1}). Assume that d is not an unit in R.

Since R is an UFD so let $d = p_1 \cdots p_n$ where p_i's are irreducible elts of R.

$$p_1 p_2 \cdots p_n\, p(x) = a'(x)b'(x)$$

(p_1) is a prime ideal of R

Have $R/p_i[x]$ is an integral domain.

$\overline{0} = \overline{a'(x)}\,\overline{b'(x)}$ [Reducing the equation modulo p_i]

which implies

either $\overline{a'(x)} = \overline{0}$ or $\overline{b'(x)} = \overline{0}$.

say $\overline{a'(x)} = 0$

$\Rightarrow$ all the coefficients of $a'(x)$ are divisible by p_i.

so $\frac{1}{p_i} a'(x) \in R[x]$.

Therefore by proceeding in the same direction and cancelling all the irreducible elements $(p_1, p_2 \ldots p_n)$ from both side we have the eqn.

$\Rightarrow f(x) = a(x)\,b(x)$

where $a(x), b(x) \in R[x]$

and the relation b/w $A(x)$ & $a(x)$

is $\frac{d}{p_i} A(x) = a'(x)$ and

$\frac{1}{p_i} a'(x) = a(x)$

$\therefore \frac{d}{p_i} A(x) = a(x)$ i.e

$\gamma A(x) = a(x)$

for some $\gamma \in K$

Remark: $a(x), b(x)$ need not be R multiples of $A(x), B(x)$

Obs: The elements of the ring R becomes units in UFD $K[x]$. eg. $7x$ factor into two irreducible 7 and x in $\mathbb{Z}[x]$ but is irreducible in $\mathbb{Q}[x]$ because 7 is a unit in $R[x]$.

Corollary: Let R be an UFD and K be the quotient field of R and let $p(x) \in R[x]$ with $c(p(x)) = 1$. Thus $p(x)$ is irreducible in $R[x]$ iff $p[x]$ is irreducible in $K[x]$

Proof: By Gauss's lemma if $p(x)$ is reducible over $K[x]$ then d is reducible over $R[x]$. Conversely let $p[x]$ is reducible over $R[x]$, i.e $p(x) = a(x)\,b(x)$ are non constant polynomials $\in R[x]$ i.e. $a(x)$ & $b[x]$ are not units in $K[x]$. Thus the same factorization shows that $p(x)$ is reducible over $K[x]$.

Thm: R is a UFD iff $R[x]$ is an UFD.

Proof: Let R be a UFD

WTS: $R[x]$ is a UFD.

Let K be the quotient field of R and $p(x) \in R[x]$ be non zero elt

WLOG: we may assume $p(x)$ is primitive (if $p(x)$ is not primitive we can write $p(x) = d\,p'(x)$ where $d \in R$ and $p'(x)$ is primitive, so d has unique factor) and $p(x)$ is not unit in $R[x]$ i.e. $\deg p(x) > 0$. Since $K[x]$ is an UFD, $p(x)$ can be factored uniquely into

irreducibles in $K[x]$.

By Gauss' Lemma

Such a factorization implies, there is a factorization of $p[x]$ in $R[x]$. Since $p[x]$ is primitive each factor of $p[x]$ is primitive. By previous corollary each factors is irreducible in $R[x]$. This shows that $p(x)$ can be written as a finite product of irreducibles in $R[x]$.

Uniqueness

Let $p(x)$ can be factorized into 2 different factors in $R[x]$

$$p(x) = q_1(x) \cdots q_r(x) = q_1'(x) \cdots q_s'(x)$$

Now viewing it as a factorization in $K[x]$ we get $r = s$ and q_i is associate to some q_i'. Then by previous proposition, they are associates. Therefore $R[x]$ is an UFD.

(8) $R[x,y] = (R[x])[y] \longrightarrow$ Is an UFD

$\downarrow$

UFD.

Proposition

Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \in \mathbb{Z}[x]$

If $\frac{r}{s} \in \mathbb{Q}$, $(r,s) = 1$ is a root of $p(x)$ then $r \mid a_0$ and $s \mid a_n$, In particular if $p(x)$ is monic and $p(d) \neq 0$ for all integer d dividing the constant term of $p(x)$, then $p(s)$ has no root in $\mathbb{Q}$.

Proof: $p(r/s) = 0$

$$a_n \frac{r^n}{s^n} + a_{n-1}\left(\frac{r}{s}\right)^{n-1} + \cdots + a_0 = 0$$

$$a_n r^n + a_{n-1} r^{n-1} s + \cdots + a_0 s^n = 0.$$

Thus $a_n r^n = -s(a_{n-1} r^{n-1} + \cdots + a_0 s^{n-1})$

$\Rightarrow s \mid a_n$ as $\gcd(s,r) = 1$

Similarly $r \mid a_0$.

Example The polynomial $x^3 - 3x - 1 \in \mathbb{Z}[x]$ This is not irreducible over $\mathbb{Q}[x]$. The possible roots are ± 1. None of them satisfy the eqn so it is irreducible.

Propn: Let I be a proper ideal in UFD R. and let $p(x)$ be a non constant monic polynomial in $R[x]$. If the image of $p(x)$ in $R/I(x)$ can not be factored into 2 polynomials of smaller deg than $p(x)$ is irreducible in $R[x]$.

Proof: Suppose $p[x]$ can not be factored in $R/I[x]$ but $p(x)$ is reducible in $R[x]$ i.e. $p(x) = a(x)\,b(x)$ where $a(x), b(x)$ monic poly in $R[x]$

Now reducing the coeffs mod I gives a factorization in $(R/I)[x]$ with non constant factors which is a contradiction.

(since $p(x)$ is monic $1 \notin I$ then $\bar{1}$ does not vanish)

Example $x^3 + x + 1 \in \mathbb{Z}[x]$ is irreducible over $\mathbb{Z}[x]$ as it is reducible over $\mathbb{Z}/2\mathbb{Z}[x]$.

Example $x^2 + 1 \in \mathbb{Z}[x]$ in $\mathbb{Z}/2\mathbb{Z}[x]$ the polynomial $f(x)$ is not irreducible.

In $\mathbb{Z}/3\mathbb{Z}$ $f(x)$ is irreducible.

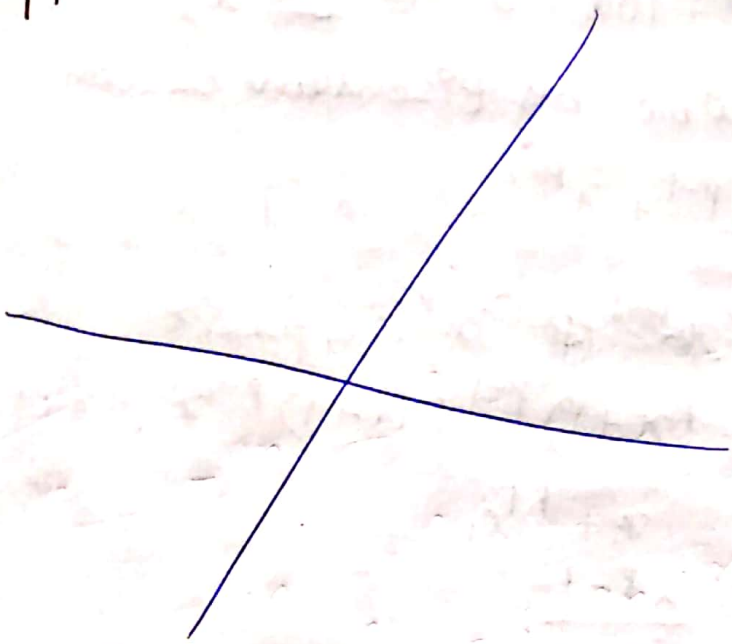
$\therefore$ $f(x)$ is irreducible over $\mathbb{Z}[x]$.

Propn: Eisenstein's Criterion.

Let P be a prime ideal of an UFD R and let $f(x) = a_n x^n + \cdots + a_0$ be a poly in $R[x]$. suppose $a_{n-1}, a_{n-2}, \ldots a_0 \in P$ and $a_n \notin P$ and $a_0 \notin P^2$. Then $f(x)$ has no divisor of deg d and $1 \le d \le n-1$ i.e. $f(x)$ is irreducible over $F[x]$ and if f is monic then f is irreducible over $R[x]$.

Proposition

Let P be a prime ideal in an UFD R.
and let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots a_0 \in P[x]$
suppose $a_0, \dots a_{n-2}, a_{n-1} \in P$.



Then $f(x) = p(x)q(x)$ where $p(x) \neq q(x)$ are given deg $\in R[x]$

Proof. Let $f(x) = p(x)q(x)$ where $p(x) \neq q(x)$ are given deg $\in R[x]$
Reduce the coeffs modulo P. Then

$$\bar{a}_n x^n = \bar{f}(x) = \bar{p}(x)\bar{q}(x)$$

Thus the leading coefficients of $p(x)$ & $q(x)$ do not belong to P. It also follows that $p(0), q(0) \in P$ which implies $f(0) = p(0)q(0) \in P^2$.

Observe clearly $f(x)$

Let $p(x) = a_0 x^2 + a_1 x + a_2$ ($a_2 \notin P$)
$q(x) = b_0 x^2 + b_1 x^3$ ($b_1 \in (P)$)
$p(x)q(x) = a_0 b_0 x^4 + (a_0 b_1 + b_0 a_1) x^2$
$+ a_2 b_0 x + b_1 a_2$ ($\notin P$)

Hence both the constant terms should $\in (P)$.

but $f(0) \notin P^2$ which is contradiction.
Thus $f(x)$ is irreducible

Example:

$f(x) = x^4 + 10x + 5 \in \mathbb{Z}[x]$

is irreducible by Eisenstein criterion

② $\phi_p(x) = x^{p-1} + x^{p-2} + \dots x + 1$

~~$\phi_p'(x) = (p-1)x^{p-2} + (p-2)x$~~

$\phi_p(x+1) = (x+1)^{p-1} + (x+1)^{p-2} + \dots x + 1$

$= x + {}^{p+1}C_1$

$\phi_p(x) = \dfrac{x^p - 1}{x - 1}$

$\phi_p(x+1) = \dfrac{(x+1)^p - 1}{x}$

$\phi_p(x+1) = x^{p-1} + {}^pC_1 x^{p-2} + \dots + {}^pC_{p-1}$

Then by EC with prime ideal p, $f(x)$ is irreducible. Thus $\phi_p(x)$ is irreducible

Exercise: By EC show that polynomial $f(x) = x^4 + 1$ is irreducible over $\mathbb{Z}[x]$.

Example: Let R be an integral domain and $f(x) \in R[x]$ is a monic irreducible polynomial, then $f(x)$ need not be irreducible over $K[x]$ where K is the Quotient field of R

Example: let $R = \mathbb{Z}[2i] = \{a + b2i \mid a, b \in \mathbb{Z}\}$ subring of an Integral Domain $\Rightarrow$ Integral Domain

Consider the polynomial $f(x) = x^2 + 1 \in \mathbb{Z}[2i][x]$

Note that $f(x)$ is irreducible over $R(x)$.

let K be the quotient field of R.

Then $K = Q(i)[x]$.

Thus $f(x)$ is reducible over $K[x]$.

Exercise Prove that $\mathbb{Z}[2i]$ is not an UFD.

8th April

Chinese Remainder Theorem

G_1, G_2 $G_1 \times G_2 = \{(a,b) \mid a \in G_1, b \in G_2\}$

$(a_1, b_1) \cdot (a_2, b_2) = (a_1 a_2, b_1 b_2)$

$(a_1, b_1) * (a_2, b_2) = (a_1 * a_2, b_1 * b_2)$

If R_1 and R_2 are two rings then $R_1 \times R_2$ has a ring structure.

In $\mathbb{Z}$, $\gcd(a,b) = 1$, then $1 = ra + sb$ where $r, s \in \mathbb{Z}$ i.e $(a) + (b) = 1$

Definition

We say that

Def

Let R be a ring and I and J are 2 proper ideals in R.
we say I & J are comaximal if $I+J=R$

$$\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}/mn\mathbb{Z} \quad \text{iff } (m,n)=1$$

Thm: Chinese Remainder Theorem

Let R be a ring and $I_1, \dots I_n$ be ideals of R s.t. $I_i + J_j = R$ for all $1 \le j, i \le n$ and $i \neq j$. Then

$$R/\bigcap_{i=1}^{n} I_i \cong R/I_1 \times R/I_2 \times \dots \times R/I_n$$

Pf: $f : R \longrightarrow R/I_1 \times R/I_2 \times \dots \times R/I_n$

$$a \xmapsto{f} (a+I_1, a+I_2, \dots, a+I_n)$$

Exercise: Check that f is a Ring Homomorphism

$$\ker f = \{a \in R \mid a \in I_i \ \forall i = 1, \dots n\}$$

$$= \bigcap_{i=1}^{n} I_i$$

WTS f is surjective, i.e. want to find $a \in R$ s.t. $f(a) = (a_1 + I_1, a_2 + I_2 \dots, a_n + I_n)$

For $j > 1$ we have by assumption

$$I_1 + I_j = R.$$

$\therefore \exists \; b_j \in I_1$ and $d_j \in I_j$

s.t. $b_j + d_j = 1$ for $j = 2, \dots n.$

$$\Rightarrow \prod_{j=2}^{n} (b_j + d_j) = 1$$

$$(b_2+d_2)(b_3+d_3)\dots(b_n+d_n)$$

$$(b_2b_3 + d_3b_2 + d_2b_3 + d_2d_3)\dots(b_n+d_n)$$

($b_2b_3 + d_3b_2 + d_2b_3 \in I_1$, $d_2d_3 \in I_2 I_3$.)

If we multiply, then we see that all the elements $\in I_1$ except $\prod_{j=2}^{n} d_j$

$$1 = \prod_{j=2}^{n} d_j \in \prod_{j=2}^{n} I_j$$

Thus $c_1 \equiv 1 \pmod{I_1}$

and $c_1 \equiv 0 \pmod{I_j}$ for $j \neq 1$

More generally, for all i we can find c_i with $c_i \equiv 1 \pmod{I_i}$

and $c_i \equiv 0 \pmod{I_j}$ for $j \neq i$

Now the arbitrary elts $a_1, a_2 \dots a_n \in R$ and set

$$a = a_1c_1 + a_2c_2 + \dots + a_nc_n.$$

Take (mod) I_i $\forall$ i

Then $f(a) = (a_1 + I_1, a_2 + I_2 \dots a_n + I_n)$

Ex =

$\therefore$ f is surjective ring homo

$a - a_i \equiv 0 \pmod{I_i}$

Thm: Hilbert's Nullstellensatz

The maximal ideals of the polynomial ring $R = \mathbb{C}[x_1, x_2 \ldots x_n]$ are in 1-1 corresponding with the points in $\mathbb{C}^n$.

For $n=1$ Maximal ideals in $\mathbb{C}[x_1]$ are in 1-1 correspondence with the pts of $\mathbb{C}$.

Here every maximal ideal of $\mathbb{C}[x]$ will be generated by an irreducible polynomials and irreducible polynomials are linear polynomials of the form say $x-a$, then this maximal ideal corresponds to the point $a \in \mathbb{C}$

Proof of the Theorem:

Consider a point $a = (a_1, a_2, \ldots a_n)$

consider the ideal $m_a = (x_1 - a_1, x_2 - a_2 \ldots x_n - a_n)$

First we will show that m_a is maximal ideal of R and then we show that every maximal ideal of R is of the form m_a for some $a \in \mathbb{C}^n$.

1st step : $s_a : \mathbb{C}[x_1, \ldots, x_n] \longrightarrow \mathbb{C}$

where $a \in \mathbb{C}^n$. $f(x_1, \ldots x_n) \rightsquigarrow f(a_1, \ldots a_n)$

$(a_1, a_2, \ldots a_n)$

By 1st Isomorphism Thm.

$$R/\ker(s_a) \cong \mathbb{C}$$

(Field)

Thus $\ker(s_a)$ is a maximal ideal.

Writing down the taylor series expansion of $f(x_1, x_2, \ldots x_n)$ around $(x_1, \ldots x_n)$ we get

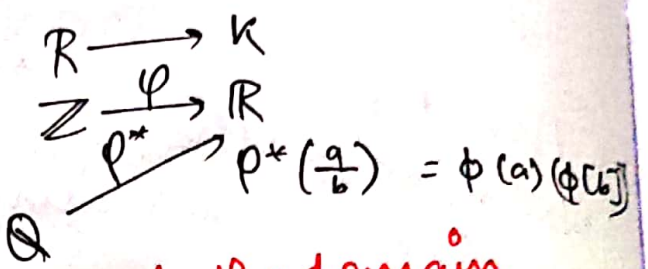
$$f(x_1, \ldots x_n) = f(a_1, a_2, \ldots a_n) + \sum c_i (x_i - a_i) + \sum_{i,j} c_{ij} (x_i - a_i)(x_j - a_j) \ldots$$

Let $f \in \ker s_a$ then $f(a_1, \ldots a_n) = 0$

$\therefore f(x) \in (x_1 - a_1, x_2 - a_2, \ldots x_n - a_n)$

$\therefore \ker s_a = (x_1 - a_1, x_2 - a_2, \ldots x_n - a_n)$

Proposition: Let R be an integral domain with quotient field F and let $\phi : R \to K$ be any injective homomorphism of R to a field K. Then there is a unique $\phi^* : F \to K$ which is the extension of ϕ



Note: Trying to extend the domain to quotient field. To show such an extension is unique

Proof: Define $\phi^* : F \to K$

$$\phi^*\left(\frac{a}{b}\right) = \phi(a)[\phi(b)]^{-1}$$

[$\because \phi$ is injective for $b \neq 0$, $\phi(b) \neq 0$ hence $[\phi(b)]^{-1}$ exists in K]

Check: well defined

Let $\frac{a}{b} = \frac{c}{d}$ in F

To show: $\phi(a)[\phi(b)]^{-1} = \phi(c)[\phi(d)]^{-1}$

We have: $ad - bc = 0$

$$\phi(ad - bc) = 0$$

$$\Rightarrow \phi(a)\phi(d) - \phi(b)\phi(c) = 0$$

$$\Rightarrow \phi(a)[\phi(b)]^{-1} = \phi(c)[\phi(d)]^{-1}$$

Uniqueness

Let $g : F \longrightarrow K$ be a homomorphism extending ϕ.

To show: $g\left(\frac{a}{b}\right) = \phi^*\left(\frac{a}{b}\right) \quad \forall \frac{a}{b} \in F$

$$g\left(\frac{a}{b}\right) = g(ab^{-1}) = g(a)\,g(b)^{-1}$$
$$= \phi(a)\,\phi(b)^{-1}$$
$$= \phi\left(\frac{a}{b}\right)$$

Hence $g = \phi^*$

Thm: $R = \mathbb{C}[x_1, x_2, \ldots x_n]$ Then $\exists$ a 1-1 correspondence b/w the maximal ideal of R

Proof: Let $a = (a_1, a_2, a_3, \ldots a_n) \in \mathbb{C}^n$

Then $ma = (x_1 - a_1,\ x_2 - a_2,\ x_3 - a_3 \ldots x_n - a_n)$ is a maximal ideal of R

WTS: Every maximal ideal of R is of the form ma for some $a \in \mathbb{C}^n$.

Let M be any maximal ideal of R and $K = \phi(x_1, \ldots x_n)$ denote the field. We have a natural surjective ring homomorphism

$$\Pi : \mathbb{C}[x_1, x_2 \ldots, x_k] \longrightarrow K$$

Let us denote by

$$\pi_1 : \mathbb{C}[x_1] \to K$$

The restriction of π to the subring $\mathbb{C}[x_1]$

Claim $\ker \pi_1$ is either zero or maximal ideal.

Assume $\ker \pi_1 \neq \{0\}$

Since K is not the zero ring, $\ker \pi_1$ is not the whole ring
$\therefore \exists$ non zero polynomial
$f \in \ker \pi_1$ s.T. $\deg f > 0$.

Then $f = g(x_1 - a_1)$ where $g(x) \in \mathbb{C}[x_1]$
since $f \in \ker \pi_1$ so $\pi_1(f) = 0$
$\Rightarrow \pi_1(g)\,\pi_1(x - a_1) = 0$ in K.
$\Rightarrow$ Either $g \in \ker \pi_1$ or $(x_1 - a_1) \in \ker \pi_1$

By repeating the process we can show that $(x_1 - a_1) \in \ker \pi_1 \quad \therefore (x_1 - a_1) \in \ker \pi = M$

Similarly we can show that there are complex numbers $a_2, \ldots a_n \in \mathbb{C}$ such that $x_i - a_i \in M$ for $i = 1, 2, \ldots m$

$(x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n) \subset M$
is a maximal ideal
$\therefore M = (x_1 - a_1, \ldots, x_n - a_n)$

WTS

$\ker \pi_1 \neq (0)$

Let us assume $\ker \pi_1 = (0)$ Then we have an injective ring homo. $\pi_1 : \mathbb{C}[x_1] \longrightarrow K$. Then by proposition (today) π_1 can be extended to a homo $\pi_1^* : C[x_1] \longrightarrow K$ where $\mathbb{C}[x_1]$ (not $\mathbb{C}[x_1]$) is the quotient field of $\mathbb{C}[x_1]$ which is the set of all rational f'n of form $f(x)/g(x)$, $f, g \in \mathbb{C}[x]$ $g \neq 0$

$$K = \frac{\mathbb{C}[x_1, \ldots, x_n]}{M}$$

The monomials $x^i = (x_1^{i_1}, x_2^{i_2}, \ldots x_n^{i_n})$ form a basis of $\mathbb{C}[x_1, x_2 \ldots x_n]$ as a vector space over $\mathbb{C}$. Since K is the gradient ring of $\mathbb{C}[x_1, \ldots x_n]$ the residue class of the monomials form a basis of K which is a countable. In $\mathbb{C}(x_1)$, we have uncountably many linearly independent elts, Thus $\mathbb{C}(x_1)$ can't be isomorphic to a subgp of K. Thus $\ker \pi_1 \neq (0)$

Quotient Field of R[[x]]

Let R be a field

$f = \sum_{n \geq 0} a_n x^n \in R[[x]]$

f will be a unit if $a_0 \neq 0$ (check)

Quotient field of $R[[x]]$

$= \left\{ \frac{f(x)}{g(x)} \mid g(x) \neq 0 \text{ and } f, g \in R[[x]] \right\}$

$g(x) = \sum_{n \geq 0} b_n x^n$

If $b_0 \neq 0$ then $g(x)$ is a unit in $R[[x]]$

Then $f(x)/g(x) \in R[[x]]$

If $b_0 = 0$, $g(x) = x^{n_0} h(x)$ where $h(x) \in R[[x]]$ & is a unit.

$$\frac{f(x)}{g(x)} = \frac{f(x)}{x^{n_0} h(x)} = \frac{1}{x^{n_0}} f_1(x), \quad (f_1 = fh^{-1})$$

where $f_1 \in R[[x]]$

$$\frac{f(x)}{g(x)} = \sum_{n=-\infty}^{\infty} c_n x^n \text{ : (Laurent series)}$$

If R is an integral domain, go back to $a_0 \neq 0$, hence this proof is stuck.

Gauss lemma

R be an integral domain. F be the quotient field of R.

Let $f(x) \in R[x]$, we need to understand the factorization of $f(x)$ in $R[x]$ and in $F[x]$

Let $f(x) \in R[x]$ be a poly, If $f(x)$ is irreducible over $F[x]$ then $f(x)$ is irreducible over $R[x]$.

$M = (p, f(x))$ where $f(x)$ is irreducible over $\mathbb{Z}/p\mathbb{Z}[x]$ then M is a maximal ideal of $\mathbb{Z}[x]$

$$\mathbb{Z}[x]/M \cong \frac{\mathbb{Z}[x]/p\mathbb{Z}}{(p, f(x))/p\mathbb{Z}} \cong \frac{\mathbb{Z}/p\mathbb{Z}[x]}{(\bar{f}(x))}$$

Let M be any maximal ideal of $\mathbb{Z}[x]$

Case1 $M \cap \mathbb{Z} \neq 0$

$M \cap \mathbb{Z}$ is an ideal of $\mathbb{Z}$.

Claim: $M \cap \mathbb{Z}$ is a prime ideal of $\mathbb{Z}$.

$M \cap \mathbb{Z} = (p)$ for some $p \in \mathbb{Z}$.

$M \cap \mathbb{Z} = p\mathbb{Z}$.

$\mathbb{Z}[x] \longrightarrow \mathbb{Z}/p\mathbb{Z}[x]$

$\therefore \bar{M} = (\bar{f}(x))M \longrightarrow \bar{M}$

where $f(x)$ is irreducible over $\mathbb{Z}/p\mathbb{Z}$.

$\bar{M}$ is a maximal ideal in $\mathbb{Z}/p\mathbb{Z}[x]$ by correspondence Thm.

$\mathbb{Z}\left[\frac{1+\sqrt{-19}}{2}\right]$ is a PID but not an E.D.

Let R be an integral domain and $\hat{R} = R^* \cup \{0\}$

$\uparrow$ units of R

An element $u \in R \setminus \hat{R}$ is called a universal side divisor if for every $x \in R$ there is some $z \in \hat{R}$ s.t. u divides $x - z$ i.e.

$x = qu + z$ where z is either zero or unit.

(No Restriction of Norm)

Proposition: Let R be an integral domain that is not a field. If R is an ED then there are universal side divisors.

Proof: Suppose R is an E.D. wrt Norm N and let $u \in R \setminus \hat{R}$. For any $x \in R$ we have $x = qu + r$ with $N(r) < N(u)$ (← of minimal norm) or $r = 0$. In either case the minimality of u implies $r \in \hat{R}$. $\therefore u$ is a universal divisor.

Claim: Let $R = \mathbb{Z}\left[\frac{1+\sqrt{-19}}{2}\right]$ is not E.D.

WTS: There is no universal division in R.

$\theta = \frac{1+\sqrt{19}}{2}$ then $\theta\bar{\theta} = 5$.

$N(a+\bar{\theta}b) = (a+\theta b)(a+\bar{\theta}b) = a^2+ab+5b^2$

The units of R is ± 1.

Ex: 2 and 3 are irreducible elements in R.

Suppose there is a norm say d s.t. R is an E.D wrt d.

Let $m \in R \setminus \tilde{R}$ s.t. $d(m)$ is smallest in $R \setminus \tilde{R}$.

Now $2 = mq+r$ with $d(r) < d(m)$ or $r = 0$

This implies $r = 0$ or ± 1

If $r = 0$ then $m \mid 2$ so $m = \pm 2$ since 2 is irreducible and m is not a unit.

Similarly if

$r = -1$ then $m = \pm 3$.

$\Rightarrow r = 1$ is not possible as m is a non unit

($m \mid 1 \Rightarrow m$ is a unit $\Rightarrow \Leftarrow$)

Next divide θ by m we get

$\theta = mq'+r'$ with $d(r') < d(m)$ or $r' = 0$

Thus $r' = 0$ or ± 1

Then one of $0, 0+1, 0-1$ is divisible by m.

But $m = \pm 2, \pm 3$ and none of $0, 0+1, 0-1$ is divisible which is a contradiction. $\therefore$ R is not an ED.