Let V be a finite dimensional V.S. Let $\dim(V) = n$. Assume that $B = \{u_1, u_2, \ldots, u_n\}$ is an ordered basis of V.

Define $f_i(x) = c_i$ for n $\quad x = \sum c_i u_i$

Then,

$$\{f_1, f_2, \ldots, f_n\} \text{ is basis of } V^*.$$

$$\left.\begin{array}{l} V = \mathbb{R}^3 \\ V^* \end{array}\right\} \begin{array}{l} \dim(V) = 3 \\ \dim(V^*) = 3 \end{array} \Big\} \text{ They have same dimension}$$

V is isomorphic to V^*.

Let V is finite dim. VS. Then V is isomorphic to V^*.

(Not true for infinite dimensional)

For isomorphism : Find A LT. $T: V \to V^*$ which is bijective.

$b_1 f_1 + b_2 f_2 + b_3 f_3 = 0 \quad \forall\, x \in V$

$x = 0\cdot(1,0,0) + 0\cdot(0,1,1) + 1\cdot(0,0,1)$

$b_3 = 0$

$V \to VS \quad F \to$ Field

$V^* =$ Set of LT's from $V \to F$. <linear functional>

$\hookrightarrow V^*$ is the VS over the field F.

$\hookrightarrow$ Dual space of V.

$V \to$ Finite Dimensional $\quad \dim(V) = n$.

$B = \{\alpha_1, \alpha_2, \dots, \alpha_n\} \longrightarrow$ ordered basis of V.

$S = \{f_1, f_2, \dots, f_n\} \to$ This the basis of V^*.

$f_i(\alpha_j) = \delta_{ij}$ | $\delta_{ij} = 1$ if $i=j$, $= 0$ if $i \neq j$

$\downarrow$ Dual basis of B.

$\dim(V) = n = \dim(V^*)$.

$B' = \{u_1, u_2, \dots, u_n\} \longrightarrow$ Basis (new)

$S' = \{f_1', f_2', \dots, f_n'\} \longrightarrow$ Dual basis of B'.

$f_i'(\alpha_j) = \beta \quad , \beta \in F$

$V = \mathbb{R}^3 \quad B = \{(1,0,0), (0,1,1), (0,0,1)\}$

$\downarrow \alpha_1 \quad \downarrow \alpha_2 \quad \downarrow \alpha_3$.

$x = (x_1, x_2, x_3)$.

$f_1(x) = c_1x_1 + c_2x_2 + c_3x_3$ is the only possible L.C. from $\mathbb{R}^3$ to $\mathbb{R}$.

To find out <u>Dual Basis</u>, we have to find f_1, f_2, f_3.

$f_1 : \mathbb{R}^3 \longrightarrow \mathbb{R}$, $f_2 : \mathbb{R}^3 \longrightarrow \mathbb{R}$, $f_3 : \mathbb{R}^3 \longrightarrow \mathbb{R}$.

Take $(x, y, z) \in \mathbb{R}^3$

$(x, y, z) = c_1(1,0,0) + c_2(0,1,1) + c_3(0,0,1)$.

$c_1 = x \qquad c_2 = y \qquad c_3 = z - y$.

$$f_1(x,y,z) = f_1\left(x(1,0,0) + y(0,1,1) + (z-y)(0,0,1)\right)$$

$$= x \underbrace{f_1(1,0,0)}_{1} + y \underbrace{f_1(0,1,1)}_{0} + (z-y) \underbrace{f_1(0,0,1)}_{0}$$

$f_1(v) = x$, $v \in \mathbb{R}^3$.

Do yourself!!! $f_2(x, y, z) =$

$V = P_2(x)$ = Set of all real coeff. polynomials of degree ≤ 2.

$B = \{1, x, x^2\}$.

find $\{f_1, f_2, f_3\}$. ??

Take $ax^2+bx+c \in P_2(x) = ax^2 + bx + c.1.$

$f_1(ax^2+bx+c) = a f_1(x^2) + b f_1(x) + c f_1(1)$
$= c$

$\therefore f_2 = b \quad , f_3 = a.$

$V \to VS$

$\{f_1, f_2, \dots, f_n\}$ is dual basis of B.

Can you find out B?

$V = P_1(x) \qquad f_1(P(x)) = \int_0^1 P(x)\,dx$

$f_2(P(x)) = \int_0^{-1} P(x)\,dx.$

$c_1 \int_0^1 P(x)\,dx + c_2 \int_0^{-1} P(x)\,dx = 0 \qquad \forall P(x) \in \mathbb{R}^3$

$c_1 - c_2 = 0. \qquad P(x) = 1.$

$c_1 + c_2 = 0 \qquad P(x) = x.$

$c_1 = c_2 = 0 \quad \therefore$ l.i.

dim$(V^*) = 2$ as, dim$(V) = 2$.

and two l.i. elts. will span the whole V^*.

Check, f_1, f_2 are L.T.

f_1, f_2 are L.I or not.

Basis has a one-one mapping to Dual Basis.

Let Basis of $P_1(x) = \{\alpha_1, \alpha_2\}$. $\alpha_1 = a_1 + b_1 x$

$\alpha_2 = a_2 + b_2 x$

Now, any $P \in P_1(x)$

$$P = a\alpha_1 + b\alpha_2.$$

~~$f_1(P) = \int_0^1 P(x)\,dx = a\int_0^1 \alpha_1 dx + b\int$~~

$$f_1(P) = a f_1(\alpha_1) + b f_1(\alpha_2).$$

for now.

$$f_1(\alpha_1) = 1, \quad f_1(\alpha_2) = 0, \quad f_2(\alpha_1) = 0, \quad f_2(\alpha_2) = 0.$$

$$a_1 + \frac{b_1}{2} = 1, \quad a_2 + \frac{b_2}{2} = 0, \quad -a_1 + \frac{b_1}{2} = 0, \quad -a_2 + \frac{b_2}{2} = 0$$

$a_1 = 1/2, \quad b_1 = 1$. $\quad \Rightarrow a_1 - \frac{b_1}{2} = 0, \quad \Rightarrow a_2 - \frac{b_2}{2} = 1$

$a_2 = 1/2, \quad b_2 = -1$.

$$\therefore \text{Basis} = \left\{ \tfrac{1}{2} + x, \ \tfrac{1}{2} - x \right\}.$$

\# $V \to VS, F$ $T, S : V \to V$ (Linear Transformation)

$T \circ S : V \to V$ $T\circ S(v) = T(S(v)) \to$ LT from $V \to V$.

\# $V = \mathbb{R}^3$ $W = \mathbb{R}^2 \longrightarrow \{(1,1), (0,1)\}$

$B = \{(1,0,0), (0,1,1), (0,0,1)\}$

$$T : V \to W \quad T(x,y,z) = (x+y, y-z).$$

$T(1,0,0)$ $T(0,0,1)$ $T(0,1,1)$

$T(1,0,0) = (1,0) = 1(1,1) - 1(0,1)$

$T(0,1,1) = (1,0) = (1,1) - (0,1)$

$T(0,0,1) = (0,-1) = 0(1,1) - (0,1)$

$$A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & -1 \end{pmatrix}$$

$(1,2,1) = 1(1,0,0) + 2(0,1,1) - (0,0,1)$

$\begin{pmatrix}1\\2\\-1\end{pmatrix} \rightarrow$ coordinate of $(1,2,1)$

$T(1,2,1) = (3,1) = 3(1,1) - 2(0,1)$

$\therefore \begin{pmatrix}3\\-2\end{pmatrix} \rightarrow$ coordinate of $\begin{pmatrix}3\\1\end{pmatrix}$.

$$\therefore \underbrace{\begin{pmatrix}1 & 1 & 0\\-1 & -1 & -1\end{pmatrix}}_{A} \underbrace{\begin{pmatrix}1\\2\\-1\end{pmatrix}}_{x} = \underbrace{\begin{pmatrix}3\\-2\end{pmatrix}}_{T(x)}.$$

$\boxed{\therefore T(x) = Ax.}$

$F \rightarrow V$, $F \rightarrow W$ } finite dimensional.

$\dim(V) = n$ $\quad$ $\dim(W) = m$

$\downarrow$ $\quad$ $\downarrow$

$B = \{v_1, v_2, \ldots, v_n\}$ $\quad$ $B' = \{w_1, w_2, \ldots, w_m\}$.

$T: V \rightarrow W$

$T(v_1) = c_{11}w_1 + c_{21}w_2 + \cdots + c_{m1}w_m$

$T(v_2) = c_{12}w_1 + c_{22}w_2 + \cdots + c_{m2}w_m$

$\vdots$

$T(v_n) = c_{1n}w_1 + \cdots + c_{mn}w_n$

$$A = \begin{pmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1n} \\ \vdots & & & & \vdots \\ c_{m1} & \cdots & & \cdots & c_{mn} \end{pmatrix} \Rightarrow [T]_{BB'}$$

$V = \mathbb{R}^3$ $\quad B = \{(1,1,0), (0,1,0), (0,0,1)\}$.
$B' = \{(1,0,0), (0,1,0), (0,0,1)\}$.

$T(x,y,z) = (x, y+z, x-z)$

$T(1,1,0) = (1,1,1)$, $T(0,1,0) = (0,1,0)$.
$T(0,0,1) = (0,1,-1)$.

$(1,1,1) = (1,0,0) + (0,1,0) + (0,0,1)$
$(0,1,0) = 0.(1,0,0) + (0,1,0) + 0.(0,0,1)$
$(0,1,-1) = 0.(1,0,0) + (0,1,0) - (0,0,1)$

$$T_{BB'} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}.$$

Find the image of $v = (1,2,1)$
i.e. $T(1,2,1)$

$(1,2,1) = (1,1,0) + (0,1,0) + (0,0,1)$. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

$$\therefore T(v) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$$

$V = \mathbb{R}^3$ $\quad W = \mathbb{R}^2$

$$B = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 1 & 1 \end{pmatrix}.$$

any $T(x,y,z)$
$$= x\begin{pmatrix} 1 \\ 2 \end{pmatrix} + y\begin{pmatrix} -3 \\ 1 \end{pmatrix} + z\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\therefore T(x,y,z) = (x-3y, 2x+y+z)$

w.r.to standard basis.

Date
Page

5/9/19

F
V W — Finite dim. $\dim(V) = n$ $\dim(W) = m$

$T : V \to W$ | B is basis for V.
B' " " " W.

Then we have matrix A
$A = [T]_{BB'}$ $m \times n$ size.

Let $A \in M_{m \times n}(F)$ Can you find a LT from V to W. s.t $[T]_{BB'} = A$ for some Bases B for V and B' for W.

$B = \{u_1, u_2, \dots, u_n\}$.
$B' = \{v_1, v_2, \dots, v_m\}$

$T : V \to W$ defined by.

$T(u_1) = a_{11} v_1 + a_{21} v_2 + \dots + a_{m1} v_m$
$\vdots$
$T(u_n) = a_{1n} v_1 + a_{2n} v_2 + \dots + a_{mn} v_m$

Now, if we have a LT and Bases then we can find a LT.

$\left.\begin{array}{l} M_{m \times n}(F) \\ \mathcal{L}(V, W) \end{array}\right\}$ they are isomorphism

$T : V \to W$. LT. $T : V \to F$ Linear Functional
$T : V \to V$ Linear Operator.

$\mathcal{L}(V, W)$ is isomorphic to $M_{m \times n}(F)$ where $\dim(V) = m$ and $\dim(W) = n$.

Can you supply an isomorphism $\mathcal{L}(V, W)$ to $M_{m \times n}(F)$?

$S : \mathcal{L}(V, W) \to M_{m \times n}(F)$ which is isomorphism.
$B \to V$ and $B \to V'$ (Basis)
$S(T) = [T]_{BB'}$

① Well defined or not. $T_1, T_2 \in V$, $T_1 = T_2 \Rightarrow S(T_1) = S(T_2)$

② Linear or not. $S(\alpha_1 T_1 + \alpha_2 T_2) = [\alpha_1 T_1 + \alpha_2 T_2]_{BB'}$

③ Bijective or not. $= \alpha_1 [T_1] + \alpha_2 [T_2]$

$\ker(S) = \{0\} \longrightarrow S$ is one-one.

$T \in \ker(S)$ $\quad S(T) = 0 \quad [T]_{BB'} = 0 \Rightarrow T = 0$

$A \in M_{m \times n}(F)$ $\quad T(u_1) = a_{11} v_1 + \cdots + a_{m1} v_m$

$\vdots$

$T(u_n) = a_{1n} v_1 + \cdots + a_{mn} v_m$.

$\therefore$ $T \in \mathcal{L}(V, W)$.

s.t. $S(T) = A = [T]_{BB'}$

$\left.\begin{array}{l} T : V \to V \\ S : V \to V \end{array}\right\}$ $T \circ S : V \to V$ $\quad (T \circ S)(v) = T(S(v))$

$V \to$ finite dim. $\quad B, B'$ two bases in V.

$[T \circ S]_{BB'} = [T]_{BB'} [S]_{BB'}$

Q.) $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$T(x, y, z) = (x + y, y, y - z)$.

$B = \{(1,0,0), (0,1,0), (0,0,1)\}$ $\quad B' = \{(1,1,0), (0,1,0), (0,0,1)\}$

$[T]_B \leftrightarrow [T]_{BB}$

$(x, y, z) = c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1)$

$= x(1,0,0) + y(0,1,0) + z(0,0,1)$.

$[T]_B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} = A_1$ $\quad [T]_{B'} = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix}$.

So, LT matrix changes with different bases.

$M_{m \times n}(F) \longrightarrow$ well known.

Using the properties of matrices, we want to find out props. of T.

There is a relation b/w A_1 and A_2.

$rank(T) = dim(Im(T))$

$rank(A)$ = no. of l.i. rows or columns.

If A, B are similar, then $rank(A) = rank(B)$.

$det(A) = det(B)$.

$P^{-1}AP = B$ (P is non singular matrix.)

9/9/19

V, W → VS over field F.

$T: V \to W$ is a LT.

T is called invertible if T is bijective.

T^{-1} is inverse of T.

$T^{-1}: W \to V$ is a LT.

$T: \mathbb{R}^3 \to \mathbb{R}^3$

$T(x, y, z) = (3x, x-y, 2x+y+z)$

Check whether T is invertible or not.

$T^{-1}(x, y, z) = (x/3, x/3 - y, y + z - x)$

$(x, y, z) \in \mathbb{R}^3$ You have to find a vector u in W, such that $T^{-1}(u) = (x, y, z)$.

Find T^{-1}'s form.

$V, W \to$ FVS (Equal dim.) over F.

$T: V \to W$ — LT. Then the following are equivalent.

1) T is invertible. 3) T is onto.

2) T is one-one

Let B be a basis of V and B' be a basis of W.

$[T^{-1}]_{B'B} = [T]^{-1}_{BB'}$ To prove.

$\forall \; T: V \to W$ invertible.

$B = \{u_1, u_2, \dots, u_n\}$

$B' = \{v_1, v_2, \dots, v_n\}$

$T: V \to W$

$[T]_{BB'} = (a_{ij}) \qquad T(u_i) = \sum_{j=1}^{n} a_{ji} v_j$

$T^{-1}(v_1), T^{-1}(v_2) \dots \dots T^{-1}(v_n)$.

$T^{-1}(v_i)$ = l.c. of u_i's.

$[T^{-1}]_{B'B} \qquad T^{-1}\left(\sum_{j=1}^{n} a_{ji} v_j\right) = u_i$

$\Rightarrow \sum a_{ji} T^{-1}(v_i) = u_i$

$$\boxed{\begin{aligned} u_1 &= \sum a_{j1} T^{-1}(v_j) \\ u_2 &= \sum a_{j2} T^{-1}(v_j) \\ &\vdots \\ u_n &= \sum a_{jn} T^{-1}(v_j) \end{aligned}}$$

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = A \begin{bmatrix} T^{-1}(v_1) \\ T^{-1}(v_2) \\ \vdots \\ T^{-1}(v_n) \end{bmatrix}$$

$T^{-1}(v_1) = b_{11}\overset{u_1}{\hat{}} + b_{12}u_2 + b_{13}u_3 + \dots + b_{1n}u_n$

$$\begin{bmatrix} T^{-1}(v_1) \\ \vdots \\ T^{-1}(v_n) \end{bmatrix} = A^{-1} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}. \qquad A^{-1} = B.$$

$$B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ \vdots & \ddots & & \vdots \\ b_{n1} & \dots & \dots & b_{nn} \end{bmatrix}$$

classmate

Date

Page

$T: V \to W$ $B = \{u_1, u_2, \ldots, u_n\}$ $B' = \{v_1, v_2, \ldots, v_n\}$

$\downarrow B \quad \downarrow B'$

$[T]_{BB'} = A = (a_{ij})$

$B = \{u_1, u_2, \ldots, u_n\}$ $B' = \{v_1, v_2, \ldots, v_n\}$

$[T]_{BB'} = A = (a_{ij})$

$T(u_1) = a_{11} v_1 + \ldots + a_{n1} v_n$.

$T(u_2) = a_{12} v_1 + \ldots + a_{n2} v_n$

$\vdots$

$T(u_n) = a_{1n} v_1 + \ldots + a_{nn} v_n$

$u_1 = T^{-1}(a_{11} v_1 + \ldots + a_{n1} v_n)$.

$\vdots$

$u_n = T^{-1}(a_{1n} v_1 + \ldots + a_{nn} v_n)$

$[u_1 \; u_2 \ldots u_n] \, \underset{B}{\underset{\parallel}{A^{-1}}} = [T^{-1}(v_1) \; T^{-1}(v_2) \ldots T^{-1}(v_n)]$

$T^{-1}(v_1) = b_{11} u_1 + b_{21} u_2 + \ldots + b_{n1} u_n$.

$\vdots$

$T^{-1}(v_n) = b_{1n} u_1 + b_{2n} u_2 + \ldots + b_{nn} u_n$.

$$[T^{-1}]_{B'B} = \begin{pmatrix} b_{11} & \ldots & b_{1n} \\ \vdots & & \\ b_{n1} & \ldots & b_{nn} \end{pmatrix}$$

$$T: \underset{B}{V} \to \underset{B'}{W}$$

$S: W \to U \to B''$ $ST: V \to U$

$$[ST]_{BB''} = [S]_{B'B''} [T]_{BB'}$$