26/2/2020 ??

Spline Interpolation technique :

x	x_0	x_1	......	x_n.	$y = f(x)$
y	y_0	y_1	......	y_n.	

$p_n(x)$ by using all the $(n+1)$ data points, which is global interpolation polynomial for $y = f(x)$ valid in $\underbrace{I[x_0, x_1, \ldots, x_n]}_{\text{smallest interval containing } x_i\text{'s}}$.

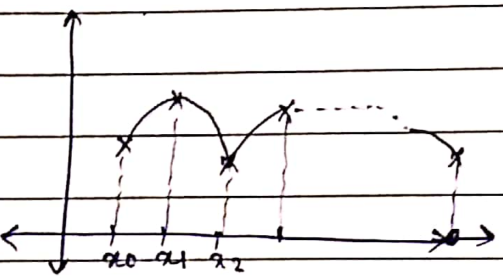
Piecewise interpolation polynomial :-

Instead of taking the whole set of data points and constructing the polynomial, we create polynomials b/w each interval.

$$S(x) = \{ p_k(x) \;,\; x_{k-1} \le x \le x_k \mid k = 1, 2, \ldots, n \}$$

$p_1(x), p_2(x), \ldots, p_n(x)$

$y \sim S(x) \qquad y \sim p_k(x)$ for $x_{k-1} \le x \le x_k \qquad k = 1, 2, \ldots, n-1$.



each $p_k(x)$ has constraints.
→ $p_k(x_{k-1}) = y_{k-1}$
and $p_k(x_k) = y_k$

Knot points are,

$x_1, x_2, \ldots, x_{n-1}$

If $p_k(x)$ are cubic polynomial Then the set of $S(x)$ is called the cubic spline and the representation of $y(x)$ by $S(x)$ is termed as Spline interpolation.

* Interpolation polynomial at $x_k = y_k$. (Given)

$y \sim p_m(x) \qquad y_k = p_m(x_k) \qquad k = 0, 1, 2, \ldots, n$.

$$p_m(x) = a_0 + a_1 x + \ldots + a_m x^m$$

$$a_0 + a_1 x_0 + a_2 x_0^2 + \ldots + a_m x_0^m = y_0$$
$$a_0 + a_1 x_1 + a_2 x_1^2 + \ldots + a_m x_1^m = y_1$$
$$\vdots$$
$$a_0 + a_1 x_n + \ldots + a_m x_n^m = y_n.$$

$(n+1)$ eqⁿ. $(m+1)$ variables $(a_i's)$.

If $m=n$, then we surely have a solⁿ.

$$A = \begin{vmatrix} 1 & x_0 & \cdots & x_0^n \\ 1 & x_1 & & x_1^n \\ \vdots & & & \vdots \\ 1 & x_n & & x_n^n \end{vmatrix} = \prod_{\substack{i,j=0 \\ i\neq j}}^{n} (x_i - x_j)$$

$\neq 0$ if $x_i \neq x_j$. $\forall\, i,j$.

If the node points $x_0, x_1, x_2, \ldots, x_n$ are distinct then, we have a unique solⁿ, hence the interpolation polynomial is unique and of degree n.

→ So, if you use all the data points then $\exists$ unique interpolating polynomial.

<u>To find an interpolating polynomial :-</u>

Let $y = f(x)$ s.t.

$y(x_m) = 0$ $m = 0, 1, 2, \ldots, k-1, k+1, \ldots, n$

$= y_k$, $m = k$.

Consider,

$y \sim p_n(x)$ s.t., $p_n(x_m) = 0$, $m = 0, 1, 2, \ldots, k-1, k+1, \ldots, n$

$p_n(x_k) = y_k$.

So, $p_n(x) = a_n (x-x_0)(x-x_1)\ldots(x-x_{k-1})(x-x_{k+1})\ldots(x-x_n)$

is a polynomial of degree n which interpolates $y = f(x)$ at $x_0, x_1, \ldots, x_n$ points.

$$a_n = \frac{y_k}{(x_k-x_0)(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}.$$

So, the unique interpolation polynomial is $p_n(x)$.

Let, $l(x) = (x-x_0)(x-x_1)\ldots(x-x_n)$.

then, $p_n(x) = \dfrac{y_k\, l(x)}{(x-x_k)\, l'(x_k)}$

$l'(x_k) = (x_k-x_0)(x_k-x_1)\ldots(x_k-x_{k-1})(x_k-x_{k+1})\ldots(x_k-x_n)$.

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$y_m = x_m \neq 0$, $m = 0, 1, 2, \dots, n$.

$$y_m = y_0 + y_1 + y_2 + \dots + y_n$$

Generalize the technique for the above curve $p_n(x)$ as :-

$$p_n(x) = \sum_{k=0}^{n} \frac{y_k \, l(x)}{(x - x_k)\, l'(x_k)}$$

$$l(x) = \prod_{i=0}^{n} (x - x_i)$$

$p_n(x)$ is a polynomial of degree n and $p_n(x_i) = y_i$ $i = 0, 1, 2, \dots$ which is unique interpolation polynomial, known as Lagrange's interpolation polynomial.

$y \sim p_k(x)$, $x_{k-1} \le x \le x_k$.

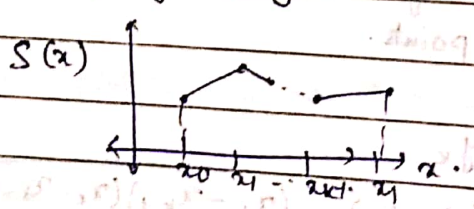
$p_k(x_{k-1}) = y_{k-1}$

$l(x) = (x - x_{k-1})(x - x_k)$ $\quad$ $p_k(x_k) = y_k$.

$$p_k(x) = \sum_{i=k-1}^{k} \frac{y_i \, l(x)}{(x - x_i)\, l'(x_i)}$$

$$= \frac{y_{k-1}(x - x_k)}{(x_{k-1} - x_k)} + \frac{y_k (x - x_{k-1})}{(x_k - x_{k-1})}$$

If we use Lagrange's polynomial, then each $p_k(x)$ is a straight line.



Spline interpolation, $p_k(x)$ is cubic polynomial.

S(x) is continuous, twice differentiable. i.e. at each knot points $p_k(x)$ has the same slope and curvature i.e.

$$p'_k(x_k) = p'_{k+1}(x_k)$$

$$p''_k(x_k) = p''_{k+1}(x_k) \qquad k = 1, 2, \dots, n-1.$$

$p_k(x_k) = y_k$, $p_k(x_{k+1}) = y_{k+1}$

$S(x)$ is the set of piecewise cubic polynomial, $S''(x)$ is piecewise linear. i.e. $p_k''(x)$ is linear polynomial with,

$$p_k''(x_k) = M_k \quad , \quad p_k''(x_{k+1}) = M_{k+1}.$$

where $p_k(x)$ is valid in $[x_k, x_{k+1}]$.

By Lagrange's formula $p_k''(x)$ is

$$p_k''(x) = \frac{M_k\,(x - x_{k+1})}{(x_k - x_{k+1})} + \frac{M_{k+1}\,(x - x_k)}{(x_{k+1} - x_k)}$$

Let the points are equiplaced

i.e. $h = x_{k+1} - x_k \quad , \quad k = 0, 1, 2, \ldots$

$$p_k''(x) = \frac{M_k}{h}\cdot(x - x_{k+1}) + \frac{M_{k+1}}{h}(x - x_k).$$

Integrate twice w.r.to x.

$$p_k(x) = \frac{M_k}{h}\cdot\frac{(x - x_{k+1})^3}{3!} + \frac{M_{k+1}}{h}\frac{(x - x_k)^3}{3!} + \underbrace{Ax + B}_{C_k(x - x_k) + D_k(x_{k+1} - x)}.$$

C_k & D_k are consts. of integration.

Now,

$$p_k(x_k) = y_k.$$

$$p_k(x_{k+1}) = y_{k+1}$$

$$p_k(x_k) = \frac{h^2}{6}M_k + D_k . h = y_k. \Rightarrow D_k = \frac{1}{h}\left(y_k - \frac{h^2}{6}M_k\right)$$

$$p_k(x_{k+1}) = \frac{h^2}{6}M_{k+1} + C_k h = y_{k+1} \Rightarrow C_k = \frac{1}{h}\left(y_{k+1} - \frac{h^2}{6}M_{k+1}\right)$$

$$p_k(x) = \frac{M_k}{6}\left[\frac{(x_{k+1} - x)^3}{h} - h\,(x_{k+1} - x)\right].$$

$$+ \frac{y_k}{h}(x_{k+1} - x) + \frac{M_{k+1}}{6}\left[\frac{(x - x_k)^3}{h} - h\,(x - x_k)\right].$$

$$+ \frac{1}{h}\,y_{k+1}\,(x - x_k).$$

We will apply the continuity ~~eq~~ rel^{n}s.

$$p_k'(x_{k+1}) = p_{k+1}'(x_k).$$

$$p_k'(x) = \frac{M_k}{6}\left[(-3)\frac{(x_{k+1}-x)^2}{h} + h\right] - \frac{y_k}{h}$$

$$+ \frac{M_{k+1}}{6}\left[\frac{3(x-x_k)^2}{h} - h\right] + \frac{y_{k+1}}{h}$$

$$p_k'(x_{k+1}) = \frac{M_k}{6}[h] - \frac{y_k}{h} + \frac{M_{k+1}}{6}\left[3h^2 - h\right] + \frac{y_{k+1}}{h}$$

$$= \frac{3M_{k+1}h^2}{6} - \frac{h^2}{6} + 1 =$$

$$p_k'(x_k) = p_{k-1}'(x_k) \qquad k = 1, 2, \ldots, n-1.$$

$$p_k'(x_k) = \frac{M_k}{6}(-2h) - \frac{M_{k+1}}{6}h + \frac{1}{h}\Delta y_k.$$

$$p_k'(x_{k+1}) = \frac{M_k}{6}h + \frac{M_{k+1}}{3}h + \frac{1}{h}\Delta y_k. \qquad p_k(x) \text{ in } [x_k, x_{k+1}]$$

replace $k \to k-1$

$$p_{k-1}'(x_k) = \frac{M_{k-1}}{6}h + \frac{M_k h}{3} + \frac{1}{h}\Delta y_{k-1}$$

So, $$-\frac{M_k h}{3} - \frac{M_{k+1}h}{6} + \frac{\Delta y_k}{h} = \frac{M_{k-1}h}{6} + \frac{M_k h}{3} + \frac{\Delta y_{k-1}}{h}$$

$$M_{k-1} + 4M_k + M_{k+1} = \frac{6}{h^2}\left[y_{k+1} - 2y_k + y_{k-1}\right]$$

$$k = 1, 2, \ldots, n-1.$$

for, $p_k(x)$ ~~=~~ for, $k = 0, 1, 2, \ldots, n-1$

involves, $M_0, M_1, M_2, \ldots, M_n \rightarrow$ i.e. $(n+1)$ M_k's.

where as (*) is a tri-diagonal system of $(n-1)$ eqns.

We need to impose two more Cond^{n}s for the 2nd derivative M

1) $M_0, M_n \rightarrow$ given. ② $M_0 = M_1$, $M_{n-1} = M_n \rightarrow$ periodic cond^n.

③ $M_0 = 0 = M_n \Rightarrow$ flat end cond^n.