Books:

1). Introduction to Real Analysis

Bartle and Sherbert

2). Mathematical Analysis by Apostal

3). Principles of mathematical analysis by walter rubin

4) by simmons

Set of Real numbers is a complete ordered field.

Field: $\mathbb{R} \times \mathbb{R} \xrightarrow[+,\cdot]{} \mathbb{R}$

$\mathbb{R}$ has the field axiom.

(i). $(a+b)+c = a+(b+c)$

(ii). $\exists\, 0 \in \mathbb{R}$ s.t. $a+0 = 0+b$

(iii). For $a \in \mathbb{R}$ there is some element $b \in \mathbb{R}$

such that $a+b = b+a = 0$

$b \rightarrow$ additive inverse.

(iv). $a+b = b+a$

(v). $a.(b.c) = (a.b).c$

(vi). $\exists\, 1 \in \mathbb{R}$ such that $1.a = a.1 = a \quad \forall a \in \mathbb{R}$

$1 \rightarrow$ multiplicative identity

(vii). For $a \in \mathbb{R}$, $a \neq 0$, we have $b \in \mathbb{R}$ s.t.

$a.b = b.a = 1$

$b \rightarrow$ multiplicative inverse

denoted by a^{-1}

(viii). $a.(b+c) = a.b + a.c$

(ix). $a.b = b.a$

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Order (Axiom) properties:

$\exists$ a non-empty subset $\mathbb{P}$ of $\mathbb{R}$
(there is)

s.t.

(i). $a, b \in \mathbb{P} \Rightarrow a+b \in \mathbb{P}$

(ii). $a, b \in \mathbb{P} \Rightarrow a.b \in \mathbb{P}$

(iii). $a \in \mathbb{R}$, exactly one of the following three are true:

$a = 0$

$a \in \mathbb{P}$

$-a \in \mathbb{P}$ (Trichotomy)

Def: a is positive (or $a>0$) if $a \in \mathbb{P}$

$a \geq 0$ if $a \in \mathbb{P} \cup \{0\}$

Def: $a>b$ (or a is greater than b)

if $a-b>0$, $a-b \in \mathbb{P}$

Another version of Trichotomy:

If $a, b \in \mathbb{R}$, then exactly one of the following is true—

1. $a=b$ 2. $a>b$ 3. $b>a$

Def: $a<b$ if $b-a \in \mathbb{P}$

Theorem: Natural numbers are positive real numbers.

$$n = \underbrace{1+1+\ldots+1}_{n \text{ times}}$$

Theorem: $1 \in \mathbb{P}$ pf: $1 = 1^2 \in \mathbb{P}$

Theorem:

If $a \in \mathbb{R}$, then $a^2 \in \mathbb{P}$

$\neq 0$

pf: $a \in \mathbb{P} \Rightarrow a.a \in \mathbb{P}$

$-a \in \mathbb{P} \Rightarrow (-a)^2 \in \mathbb{P}$

Lemma:

If $ab>0$ then either

$a>0$ $b>0$ or,
$a<0$ $b<0$

If $ab<0$ then either

$a>0$ $b<0$ or,
$a<0$ $b>0$

- If $ab>0$ prove that $a \neq 0$.

or $a=0 \Rightarrow ab=0$

Lemma:

1). $a>b, b>c \Rightarrow a>c$

2) $a>b \Rightarrow a+c>b+c$

3). $a>b, c>0 \Rightarrow ac>bc$

There is no smallest positive real number.

Theorem

If $a \in \mathbb{R}$ is such that $0 \leq a < \varepsilon$ for all positive real number, then $a=0$.

proof: suppose, $a>0$

We can expect $a > \frac{a}{2} > 0$

$\varepsilon = \frac{a}{2}$

A contradiction!

Completness Axiom

A non-empty set of real numbers which has an upper bound, also has a supremum.

upper bound:

Let S be a non-empty subset of Real numbers.

$u \in \mathbb{R}$ is an upper bound of S if $u \geq s \ \forall s \in S$

If such u exists, then say S is bounded above.

lower bound:

$S \neq \phi$ A real number l is a lower bound of S

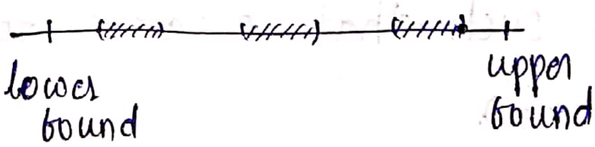
if $l \leq s \ \forall s \in S$.

If such l exists, then say is bounded below

If Both, S is Bounded.

Ex: $\{x \mid x \geq 0\}$

"Bounded Below"



Supremum:

S is a non-empty subset of real numbers

supremum (or) least upper bound.

$\sup S$ is real number u s.t.

(i). $u \geq s \ \forall s \in S$

(ii). $v < u \Rightarrow \exists s' \in S$ s.t. $v < s'$

$S = [0,1]$

$\sup S = 1$

$S = (-\infty, 0)$

$\sup S = 0$

sup S is unique if it exists.

<u>Infimum:</u>

S is a non-empty set of real numbers.

A lower bound l is called infimum if:

1). $l \le s \quad \forall s \in S$

2). $l' > l$, then $\exists s' \in S$ s.t. $l' > s'$

$S = [0,1]$

$\inf S = 0$

<u>Lemma:</u>

Let S be a non-empty set of real numbers. A real u is the sups if,

(i). u is the upper bound

(ii). If v is upper bound, then $v \ge u$.

<u>Lemma:</u>

A upper bound u of S is the supremum ~~S = [0,1]~~ if for any $\varepsilon > 0$ $\exists s_\varepsilon \in S$ s.t. $u - \varepsilon < s_\varepsilon$

Examples:

(1). $S = \{-1, 0, 1, \sqrt{2}\}$

$\sup S = \sqrt{2}$

$\inf S = -1$

(2). $S = [0, 1)$

$\inf S = 0$

$\sup S = 1$

$1 \geq s \quad \forall s \in S$

Take $v < 1$

we want $s' \in S$, ~~such that~~ $s' > v$

$s' = \max\left\{v + \frac{1-v}{2}, \frac{1}{2}\right\} \in S$

$s' > v$

(3). $\left\{\frac{1}{n} \middle| n \in \mathbb{N}\right\} = S$

$\sup S = 1$

$\inf S = 0$

$s \leq 1 \quad \forall s \in S$

~~Take~~

~~need to prove $\exists s_\epsilon \; \forall \epsilon > 0$ s.t. $1 - \epsilon < s_\epsilon$~~

~~$s_\epsilon = \max\{0,$~~

To prove s.t. for $v < 1$ there exist s' s.t. $s' > v$.

$s' = \frac{1}{\lfloor \frac{1}{v} \rfloor}$; $s' > v$

$0 < \frac{1}{n} \quad \forall n$

$\Rightarrow$ 0 is lower bound

$\Rightarrow \inf S \geq 0$

we have to s.t if $x > 0$, $\exists\, n$ s.t. $x > \frac{1}{n}$.

(4). $S = \{n \in \mathbb{N}\}$

For Bounded above,

prove that $\exists x$ s.t $\forall n$; $x > n$.

To contradict, prove that $\forall x \in \mathbb{R}$ $\exists n$ s.t. $n > x$.

For (3), $x > \frac{1}{n} \Leftrightarrow n > \frac{1}{x}$ $y = \frac{1}{x}$

$y \in \mathbb{R}$, $\exists n$ s.t. $n > y$.

<u>Archimedian property</u>:

For $y \in \mathbb{R}$, there $\exists\, n_y \in \mathbb{N}$ s.t. $n_y > y$.

<u>proof</u>: Assume, $\forall n \in \mathbb{N}$, $n \leq y$.

y is a upper bound

So, By completenees axiom. Let z be the supremum.

Take $z-1$ $\exists\, n_1 \in \mathbb{N}$ s.t. $n_1 > z-1$

$\Rightarrow n_1 + 1 > z$

contradiction!

So, the set of natural numbers is unbounded above in the set of real numbers.

For(3), To show $\inf S = 0$

We need for $x > 0$, $\exists n$ s.t. $x > \frac{1}{n}$.

Corollary: If $y>0$, then there is a natural no. such that $n-1 \leq y < n$.

Well ordering principle:

Any non-empty ~~The~~ set of natural numbers has a least element.

Principle of Mathematical Induction:

Induction:

Let S be a set of natural numbers s.t.,

1) $1 \in S$

2) If $n \in S$ then, $n+1 \in S$

Then $S = \mathbb{N}$

proof:

$E_y = \{m \in \mathbb{N} \mid m > y\}$

$E_y \neq \varphi$ (By Archimedian)

By well ordering property,

E_y has a least element, say n. $\Rightarrow n > y$

$n-1 \notin E_y$

$\Rightarrow n-1 \leq y$

$\Rightarrow n-1 \leq y < n$

Theorem: $\sqrt{2}$ is not a rational number.

Theorem: $\exists\, x \in \mathbb{R}$, s.t. $x^2 = 2$

proof: $S = \{x \in \mathbb{R} \mid x^2 < 2\}$ $0, 1 \in S$

S is bounded above $x < 3 \ \forall x \in S$

So, S has a supremum, say y.

Then $y > 1$

as $1.25 \in S \Rightarrow y \geq 1.25 \Rightarrow y > 1$

claim: $y^2 = 2$

case 1: $y^2 \nless 2$

case 2: $y^2 \ngtr 2$

case 1: $y^2 \nless 2$

If $y^2 < 2$, we shall find $n \in \mathbb{N}$ s.t. $y + \frac{1}{n} \in S$

$$\left(y+\frac{1}{n}\right)^2 = y^2 + \frac{1}{n^2} + \frac{2y}{n} \leq y^2 + \frac{2y}{n} + \frac{1}{n} = y^2 + 2\left(\frac{y+1}{n}\right)$$

we want to choose n such that, $2\left(\frac{y+1}{n}\right) < 2 - y^2$

Then $y + \frac{1}{n} \in S$

$$2\left(\frac{y+1}{n}\right) < 2 - y^2 \Rightarrow n > \frac{2(y+1)}{(2-y^2)}$$

n exists By archimedian principle

So case-1 is True (Contradiction of negation)

case-2: $y^2 \ngtr 2$ (prove in the same way)

Thus, $y^2 = 2$

Theorem: The set of real numbers is uncountable.

Corollary: The set of irrational numbers in uncountable.

Theorem: If $x<y$ are two real numbers $\exists$ a rational number r such that $x<r<y$.

Corollary: If $x<y$ are two real numbers $\exists$ a irrational number z such that $x<z<y$.

proof: We can assume, $0<x<y$.

Take $y-x>0$

Then $\exists n \in N$ s.t. $y-x>\frac{1}{n}$

$\Rightarrow ny-nx>1$

$\Rightarrow ny>nx+1$

$r=\frac{m}{n}$ $\quad \exists m \in N$ st

$m-1 \leq nx < m$

$nx<m \leq nx+1<ny$

$\therefore nx<m<ny$

$x<\frac{m}{n}<y$

where, $r=\frac{m}{n}$.

Cor. proof: $\frac{x}{\sqrt{2}}<\frac{y}{\sqrt{2}}$ $\exists$ a rational number r such that $\frac{x}{\sqrt{2}}<r<\frac{y}{\sqrt{2}} \Rightarrow x<r\sqrt{2}<y$.

Theorem: The set of Real numbers is uncountable

Enough to prove $[0,1]$ is uncountable.

Nested Intervals:

Intervals: $[a,b]$ etc.,

Nested Intervals:

$\langle I_n \rangle_n$ is a seq of intervals s.t. $I_1 \supset I_2 \supset I_3 \supset \ldots\ldots$

~~Nested Intersection~~

Nested Interval Theorem:

If I_n is a nested sequence of closed and bounded interval then $\bigcap\limits_{i=1}^{\infty} I_n$ is not empty.

Note: $I_n = \left(0, \frac{1}{n}\right)$ (example) (not closed)

$I_n \supset I_{n+1}$ as $\frac{1}{n+1} < \frac{1}{n}$

$I_n = [n, \infty)$ $\quad \bigcap I_n = \varphi$

proof: $\langle a_n \rangle_{n \in \mathbb{N}}$ is bounded above as,

$$a_n \le b_n \le b_1$$

$\langle a_n \rangle$ has supremum of $\underline{x}$

claim: To show, $x \le b_n$ as $x \ge a_n$ then enough to show that each b_n is an upper bound.

m,n: To show that $a_m \le b_n$.

First case $m<n$: $a_m \le a_n \le b_n$

Second case $m>n$: $a_m \le b_m \le b_n$.

$\therefore x \le b_n$ and $x \ge a_n$ $\forall n$ $\therefore$ $x \in \bigcap\limits_{i=1}^{n} I_i$

Property: 1). Set of rational numbers is dense in $\mathbb{R}$.
For every $x<y$ real there exists a rational z such that $x<z<y$.

2). Set of irrational numbers is dense in $\mathbb{R}$, i.e., for $x<y$ real numbers, $\exists$ an irrational z s.t. $x<z<y$.

3). Set of real numbers is not countable.

4). Set of irrationals is also not countable.

Thm 3:

We shall prove $I=[0,1]$ is uncountable.

Suppose if possible $[0,1]$ is countable. We can write $[0,1]=\langle x_n\rangle_{n\in\mathbb{N}}$

proof: $I_1=[0,1]$

Form an interval I_2 such that $I_2\subset I_1$.

$I_2 \not\ni x_1$

If $x_1=0$, then take $I_2=[\frac{1}{2},1]$

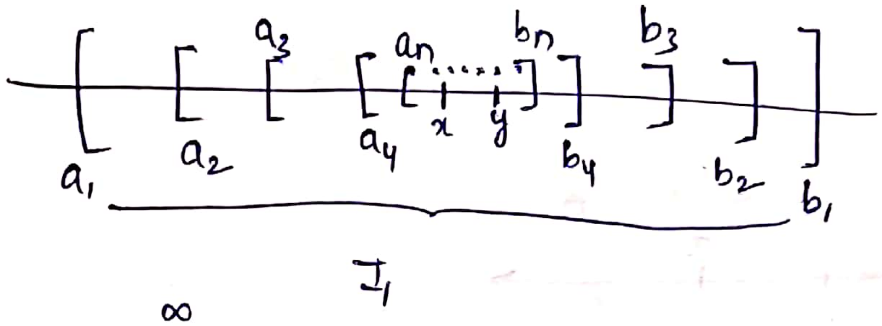
If $x_1\neq 0$, then take $I_2=[0,\frac{x_1}{2}]$

If I_n is obtained, get $I_{n+1}\subset I_n$ and $I_{n+1}\not\ni x_n$.

$\langle I_n\rangle$ is nested sequence.

$\bigcap_{n=1}^{\infty} I_n$ does not contain x_m $\forall m\in\mathbb{N}$.

$\bigcap_{n=1}^{\infty} I_n=\varphi$. contradiction



$[x, y] \subset \bigcap_{n=1}^{\infty} I_n$

$b_2 - a_2 \leq b_1 - a_1$

$b_{n+1} - a_{n+1} \leq b_n - a_n$

Thm:

If I_n is nested sequence of closed and ~~k~~ bdd intervals and ~~odd intervals~~.

and $\inf\{b_n - a_n \mid n \in \mathbb{N}\} = 0$

Then $\bigcap_{n=1}^{\infty} I_n =$ singleton set.

proof: $x = \sup\{a_n\}_{n\in\mathbb{N}}$

$y = \inf\{b_n\}_{n\in\mathbb{N}}$

claim:

$[x, y] = \bigcap_{i=1}^{n} I_n$

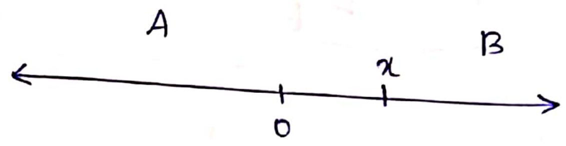
$0 \leq y - x \leq b_n - a_n$

as $[x, y] \subset I_n \cdot \forall n$

$\inf\{b_n - a_n\} = 0$

$\Rightarrow y - x = 0 \Rightarrow y = x$

Dedekind definition of real numbers or Dedekind cut:



The dedekind cut is a partition of the set of rationals Q into A and B.

(1). A is a non-empty proper subset of Q.

(2). $x \in A, y \in B \Rightarrow x < y$

(3). A has no greatest element.

<u>Example</u>:

1). If r is a rational number, then

$$A = \{y \in Q \mid y < r\}$$
$$B = \{y \in Q \mid y \geq r\}$$

2) $x = \sqrt{2}$

$x^2 < 2$ or $x \leq 0$

$$A = \{y \in Q \mid x < \sqrt{2}\}$$
$$B = \{y \in Q \mid x^2 \geq 2 \text{ and } x \geq 0\}$$

$x_n \longrightarrow \pi$, $x_n < \pi$ $x_n \in Q$

$$A = \{y \in Q \mid y \leq x_n \text{ for some } n\}$$

If A is a subset of Q s.t.

1). A is non-empty proper subset of Q

2). $x, y \in Q$, $x < y$ and $y \in A \Rightarrow x \in A$.

3). A ~~contains~~ countains no greatest element.

Then A represents a real number.

Sequence

Theorem: The set of real numbers is not countable.

Def: A function defined on $\mathbb{N}$, where range is contained by $\mathbb{R}$.

Remark:

$(x_n \mid n \in \mathbb{N})$ — a sequence

$\{x_n \mid n \in \mathbb{N}\}$ — set of values of a sequence

Ex: 1) $(n \mid n \in \mathbb{N})$ $\quad f : \mathbb{N} \xrightarrow[Id]{} \mathbb{N}$

2). $(b, b, \ldots\ldots)$. $\quad b \in \mathbb{R}$

3) $\left(\frac{1}{n} \mid n \in \mathbb{N}\right)$ $\quad$ Take $\varepsilon > 0$

By Archimedian property,

$\exists N \in \mathbb{N}$ such that $N > \frac{1}{\varepsilon}$

$\Rightarrow n > \frac{1}{\varepsilon} \quad \forall n > N$

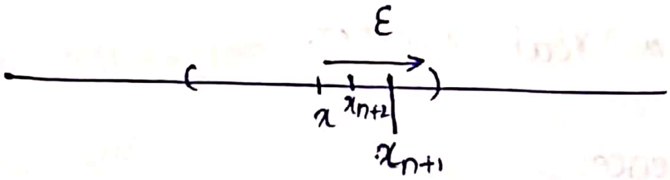
$\Rightarrow \left|\frac{1}{n} - 0\right| < \varepsilon \quad \forall n \geq N$

Limit of a sequence:

$(x_n \mid n \in \mathbb{N})$ is said to converge to a real number x

if for $\varepsilon > 0$, $\exists$ ~~real~~ natural number $N(\varepsilon)$ such that,

$|x_n - x| < \cancel{N}(\varepsilon) \quad \forall n \geq N(\varepsilon)$



x is a limit of $(x_n \mid n \in \mathbb{N})$

If no, such x exists, then $(x_n \mid n \in \mathbb{N})$ is said to be

<u>divergent.</u>

<u>Ex:</u> 1) $(n \mid n \in \mathbb{N})$ is divergent

Homework: By Archimedian.

2) $(b^n \mid n \in \mathbb{N})$; $b > 0$

case-1: $0 < b < 1$

Assume, $x = 0$

$|b^n - 0| < \varepsilon \quad \forall n \geq N(\varepsilon)$. There $N(\varepsilon)$ natural s.t.

let, $N(\varepsilon) > \log_b^{\varepsilon} \quad \forall \varepsilon > 0$

$\therefore$ sequence is convergent. to 0. (unique)

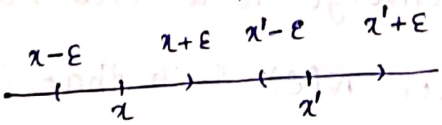
case-2: $b > 1$

say limit $= L \Rightarrow \quad |b^n - L| < \varepsilon \quad \forall n \geq N(\varepsilon)$

Theorem:

A sequence in $\mathbb{R}$ can have atmost one limit.

Suppose it has two limits



If $\varepsilon < \frac{x' - x}{2}$; They are disjoint

All But finitely many terms are in the interval.

So finite interval is $(x' - \varepsilon, x' + \varepsilon)$ but it must contain infinite elements. So contradiction.

Taking $\varepsilon = \frac{1}{2}|x - x_1|$

$\exists N(\varepsilon)$ such that $|x_n - x| < \varepsilon \quad \forall\, n \geq N(\varepsilon)$

$\exists N'(\varepsilon)$ such that $|x_n - x_1| < \varepsilon \quad \forall n \geq N'(\varepsilon)$

let $N = \max\{N(\varepsilon), N'(\varepsilon)\}$

$\Rightarrow \exists N$ such that $|x_n - x| < \varepsilon$ and $|x_n - x_1| < \varepsilon \quad \forall n \geq N.$

$$\therefore \quad |x - x_1| \leq |x_n - x| + |x_n - x_1|$$
$$< 2\varepsilon$$

But $|x - x_1| = 2\varepsilon$; contradiction.

Ex: 2). case-2: $b > 1$

Observation: Sequence is unbounded

"Range is unbounded"

proof: Take $M > 0 \quad \exists n$ s.t. $b^n > M$

$\exists n \in \mathbb{N}$ s.t. $n > \log_b^M \Rightarrow b^n > M \quad \exists n \in \mathbb{N} \quad \forall M$

$\exists\, n_0(\varepsilon)$ such that $b^{n_0(\varepsilon)} > \varepsilon + L \quad \forall \varepsilon$

$\Rightarrow |b^{n_0(\varepsilon)} - L| > \varepsilon$ contradiction!

$$\overset{x}{\overline{\quad (\quad | \quad) \quad}}$$

Theorem: Convergent sequence is Bounded.

proof: Take $\varepsilon = 1 \quad \exists n \in \mathbb{N}$,

such that $|x_n - x| < \varepsilon \quad \forall n \geq N$

$$\Rightarrow |x_n| \leq |x_n - x| + |x|$$
$$< \varepsilon + |x| = 1 + |x|$$

$M = \max\{|x_1| \ldots\ldots |x_n|, |x|+1\}$ Then, $|x_n| < M \quad \forall n \in \mathbb{N}$

$\Rightarrow \{x_n \mid n \in \mathbb{N}\}$ is bounded

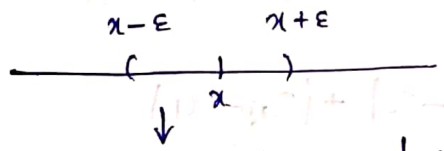
(~~But its unbounded!~~)

$(b^n \mid n \in \mathbb{N})$ is unbounded for $b>1$.

convergent if $0<b<1$

divergent if $b>1$

Equivalent definition of limit of a sequence:



$$V_\varepsilon(x) = \{ y \in \mathbb{R} \mid |x-y| < \varepsilon \}$$

Let (x_n) be a sequence and $x \in \mathbb{R}$. Then the following are equivalent:

(a). (x_n) converges to x

(b). For every $\varepsilon>0$, $\exists N(\varepsilon)$ such that,

$$x-\varepsilon < x_n < x+\varepsilon \quad \forall n \geq N(\varepsilon)$$

(c). For every ε-nbhd $V_\varepsilon(x)$ there is a natural number $N(\varepsilon)$ such that $x_n \in V_\varepsilon(x)$ $\forall n \geq N(\varepsilon)$

$(x_n) = \left(\frac{1}{n} \middle| n \in \mathbb{N}\right)$ converges to 0

$(x_n') = \left(\frac{1}{10+n} \middle| n \in \mathbb{N}\right)$ converges to 0

$(y_n) = (n \mid n \in \mathbb{N})$ diverges

$(y_n') = (100+n \mid n \in \mathbb{N})$ diverges m-tail of a sequence (x_n):

Let (x_n) be sequence and $m \in \mathbb{N}$.

Then m-tail of (x_n) is another sequence of X_m defined by $(X_m)_n = (x_{m+n} | n \in \mathbb{N})$

Theorem:

If $X = (x_n)$ is a convergent sequence, then its m-tail X_m is again convergent and

$$\lim_{n\to\infty} X = \lim_{n\to\infty} X_m$$

Methods of finding limit:

Theorem:

(a). $X = (x_n)$ and $Y = (y_n)$; $c \in \mathbb{R}$ converge to x and y respectively. Then $X+Y = (x_n + y_n)_n$ converges to $x+y$.

$X \cdot Y$ converges to $x \cdot y$

$cX = c(x_n)_n$ converges to cx

(b). If $X = (x_n)$ converges to x

$Y = (y_n)$ converges to y.

m-tail Y_m is a sequence of non-zero number & $y \neq 0$

Then $\frac{X}{Y_m} = \left(\frac{x_n}{y_{n+m}}\right)_n$ converges to $\frac{x}{y}$.

Squeeze Theorem:

$(x_n), (y_n), (z_n)$ are sequence such that,

$x_n \leq y_n \leq z_n$

If (x_n) and (z_n) are convergent.

$(x_n) = (-2)$ $(y_n) = 2$ $(z_n) = (-1)^n$

x and y are convergent. z is not convergent.

Then $(y_n)_n$ is convergent and $\lim y_n = \lim x_n = \lim z_n$

Applications:

1). $z_n = (a^n + b^n)^{1/n}$; $0 < a < b$

$z_n = b\left(1 + \frac{a^n}{b^n}\right)^{1/n}$

$c = \frac{a}{b}$; $0 < c < 1$; $z_n = b(1+c^n)^{1/n}$

Let us assume limit is 'b'

$|b(1+c^n)^{1/n} - b| < \epsilon$ $\forall n \geq N(\epsilon)$

$|(1+c^n)^{1/n} - 1| < \frac{\epsilon}{b}$ $\forall n \geq N(\epsilon)$

$< \epsilon'$ $\forall n \geq N(\epsilon)$ (Don't do)
(Do squeeze)

Let, $x_n = 1 + y_n$ and $x_n = \frac{z_n}{b}$

Then, $(1+c^n)^{1/n} = 1 + y_n$

$\Rightarrow 1 + c^n = (1+y_n)^n \geq 1 + ny_n$

$\Rightarrow c^n \geq ny_n$

$\Rightarrow 1 > c^n \geq ny_n \geq 0$

$\Rightarrow \frac{1}{n} \geq y_n \geq 0$ $\forall n$ $\Rightarrow$ By squeeze theorem
$\lim y_n = 0$

$\Rightarrow \lim x_n = 1$

$\Rightarrow \lim z_n = \underline{\underline{b}}$

2). $\left(\frac{\sin x}{x}\right)_n = (x_n)$

$$-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n} \quad \forall n$$

$$\lim\left(\frac{1}{n}\right) = 0 \; ; \; \lim\left(\frac{-1}{n}\right) = 0 \Rightarrow \lim\left(\frac{\sin n}{n}\right) = 0$$

<u>Converges</u>

<u>Theorem</u>:

If (x_n) converges to x then $(|x_n|)_n$ converges to $|x|$.

counter example for converse: $(-1, 1, -1, \ldots) \rightarrow (1, 1, \ldots\ldots)$

1. Show that non-empty finite subset of S $S \subseteq R$ countains a supremum.
its

Sol: Let us solve by induction;

Let T be number of elements in S, And let S countaining n elements has supremum.

We want to prove that $T = \mathbb{N}$

(1). Base case: # 1 element $\{a\}$ → supremum exists.

(2). Let any subset of k elements.

∴ Any $S' = \{a_1, \ldots, a_k\}$ $\#S = k$

If $\#S = k+1$, $S = \{a_1 \ldots, a_k, a_{k+1}\}$

$$\sup S = \max\{\sup S', a_{k+1}\} \in S$$

2. Let, $S \subseteq R$, $S = \emptyset$. Prove that if a number $u \in R$ has the property (i). for every $n \in N$ ~~$\epsilon \in R$~~ has the property $u - \frac{1}{n}$ is not upper bound of S, (ii). for every $n \in \mathbb{N}$, $u + \frac{1}{n}$ is an upper of S. prove that $u = \sup S$.

Sol:

$\forall n \ \exists s \in S$ s.t. $u - \frac{1}{n} < s \leq u + \frac{1}{n}$ then $s = u$

First, show u is the upper bound.

$s \in S$, $s \leq u$

$\exists s' \in S$ st $s' > u \Rightarrow s' - u > 0$

$s' = (s' - u) + u$

$\exists N \in \mathbb{N}$ st $s' - u > \frac{1}{N} \Rightarrow s' > u + \frac{1}{N}$ (contradiction)

If $v<u$, v is not an upper bound.

Let us assume v is still an upper bound

~~$\exists s' \in S$ st~~ ~~$\nexists s' \in S$ s.t~~ ~~$u>s'>v$~~

$\cancel{u}=s'$ Take $u-v>0 \Rightarrow \frac{1}{u-v} < N'$ $N' \in \mathbb{N}$

$\Rightarrow u-v > \frac{1}{N'}$

But $u-\frac{1}{N'}$ is not upper bound.

3. Given any $x \in \mathbb{R}$ prove that $\exists n \in \mathbb{N}$ such that $n-1 \leq x < n$.

4. If $u>0$ $x<y$. Then, there is a rational number r such that $x<ur<y$.

Sol: The set $\{ru \mid u>0, r \text{ is rational}\}$ is dense in $\mathbb{R}$.

$\frac{x}{u} < \frac{y}{u}$; already proved.

5. $K_n := (n,\infty)$. Then $\bigcap_{i=1}^{\infty} K_n = \varphi$.

Let $x \in \bigcap_{i=1}^{\infty} K_n$ $\Leftrightarrow$

Howerver By Archimedian $\exists n \in \mathbb{N}$ s.t. $n>x$

$\Rightarrow x \notin K_n \Rightarrow x \notin \bigcap_{n=1}^{\infty} K_n$ $\therefore \bigcap_{i=1}^{\infty} K_n = \varphi$

6. Let X and Y be non-empty sets in $\mathbb{R}$.

Let $h: X \times Y \to \mathbb{R}$ have bounded range

i.e. $\exists M>0$ st $|h(x,y)|<M$ $\forall x,y$.

Let F and G is defined by,

$$F(x) := \sup \{ h(x,y) \mid y \in Y \}$$
$$G(y) := \sup \{ h(x,y) \mid x \in X \}$$

Then prove that,

$$\sup \{ h(x,y) \mid x \in X, y \in Y \} = \sup \{ F(x) \mid x \in X \}$$
$$= \sup \{ G(y) \mid y \in Y \}$$

<u>Sol:</u> Let,

$$u = \sup \{ h(x,y) \mid x \in X, y \in Y \}$$

$$u_1 = \sup_{x \in X} F(x)$$

u_1 is an upper bound of for $F(x)$ $\forall x \in X$

u_1 is an upper bound of $h(x,y)$ $\forall x \in X$, $\forall y \in Y$

$$u_1 \geq F(x) \geq h(x,y)$$

$$\Rightarrow u_1 \geq u$$

If, $u \geq h(x,y)$ $\forall x, y$

<u>Fix x</u> $\quad u \geq F(x)$ $\forall x$

$$u \geq u_1$$

$$\therefore \boxed{u = u_1}$$

Monotone Sequence:

Ex:

$$x_{n+1} = \frac{x_n}{2} + 1$$

suppose this is convergent.

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \left(\frac{x_n}{2}+1\right) = x$$

$$\Rightarrow x = \frac{x}{2} + 1 \Rightarrow \boxed{x=2} \quad \lim_{n\to\infty} x_n = 2$$

$x_1 = 8$ $(5, 3.5, 2.75, 2.375, 2.1875, \ldots)$

This is a decreasing sequence. & convergent.

$x_n = (n \mid n \in \mathbb{N})$

This is increasing and divergent.

Def: $X = (x_n)$ is monotonically

(i). increasing if $x_1 \le x_2 \ldots\ldots$

(ii). decreasing if $x_1 \ge x_2 \ldots\ldots$

X is monotone if its monotonically increasing or decreasing

Ex: $(a^n \mid n \in \mathbb{N})$ is increasing if $a>1$
decreasing if $a<1$

Monotone convergence Theorem:

A monotone sequence (x_n) of real number is convergent if and on if its bounded.

(a). If (x_n) is decreasing and bounded, then
$\lim(x_n) = \inf\{x_n \mid n \in \mathbb{N}\}$

(b). If (x_n) is increasing and bounded, then
$\lim(x_n) = \sup\{x_n \mid n \in \mathbb{N}\}$

conter examples:

1. $(-n \mid n \in \mathbb{N}) \to$ decreasing

$\to$ Bounded (not bounded)

$\to$ Divergent

2. $((-1)^n \mid n \in \mathbb{N}) \to$ Bounded

$\to$ Not Monotonic

$\to$ Divergent

proof:

$(x_n) \to$ monotonically decreasing

Bounded

Then (x_n) is convergent and

$\lim x_n = \inf\{x_n \mid n \in \mathbb{N}\}$

Let $x = \inf\{x_n \mid n \in \mathbb{N}\}$

for $\varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. $x_N \in (x-\varepsilon, x+\varepsilon)$

$x \le x_n \le x_N$ as x_n is decreasing.

$\Rightarrow x_n \in (x-\varepsilon, x+\varepsilon) \quad \forall n \ge N$

$\Rightarrow \lim\limits_{n\to\infty} x_n = x$

Application:

(1) $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$

(x_n) is increasing as $x_{n+1} - x_n = \frac{1}{n+1} > 0$

To show, x_n is unbounded above

$$x_{2^n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2^n}$$

$$\ge 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \dots + \frac{1}{2^n}$$

$$> 1 + \frac{1}{2} + 2\left(\frac{1}{2}\right) + 4\left(\frac{1}{4}\right) + \dots + 2^{n-1}\left(\frac{1}{2^{n-1}}\right)$$

$$> 1 + n/2$$

$\therefore$ (x_n) is unbounded above, so it is divergent.

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(2). Let $x_1 > 1$ and $x_{n+1} = 2 - \frac{1}{x_n}$

Firsty $x_n > 1$. $\forall n$.
(can be proved by induction)

$$x_n - x_{n+1} = x_n - 2 + \frac{1}{x_n} = \frac{x_n^2 - 2x_n + 1}{x_n} = \frac{(x_n - 1)^2}{x_n} > 0$$

$\therefore$ $x_n > x_{n+1}$ Decreasing sequence
and also Bounded Below.

It is convergent!

$x_n > 1 \Rightarrow \inf(x_n | n \in \mathbb{N}) \geq 1$

Therefore, $x \geq 1$

$x = 2 - \frac{1}{x} \Rightarrow \boxed{x = 1}$

Euler's Number:

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

$$e_n = \left(1 + \frac{1}{n}\right)^n$$

First prove, $e_n \leq e_{n+1}$ $\forall n$
Then $2 \leq e_n \leq 3$

$$\cancel{e_{n+1} - e_n = \left(1 + \frac{1}{n+1}\right)^{n+1} - \left(1 + \frac{1}{n}\right)^n}$$

$$\cancel{= \left(1 + \frac{1}{n+1}\right)\left(1 + \frac{1}{n+1}\right)^n - \left(1 + \frac{1}{n}\right)^n}$$

$$\cancel{= \left(\left(1 + \frac{1}{n+1}\right)^n - \left(1 + \frac{1}{n}\right)^n\right) + \frac{1}{n+1}\left(1 + \frac{1}{n+1}\right)^n}$$

$$e_n = \left(1 + \frac{1}{n}\right)^n = 1 + n \cdot \frac{1}{n} + {}^nC_2 \cdot \frac{1}{n^2} + \ldots\ldots {}^nC_n \cdot \frac{1}{n^n}$$

$$= 1 + 1 + \frac{1}{2}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \ldots + \cancel{\left(1 - \frac{n-r}{n}\right)}$$

$$e_{n+1} = \left(1+\frac{1}{n+1}\right)^{n+1}$$

$$= 1+1+\frac{1}{2!}\left(1-\frac{1}{n+1}\right)+\frac{1}{3!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)+\dots$$

k^{th} term in e_n:

$$\frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\left(1-\frac{k-1}{n}\right)$$

k^{th} term in e_{n+1}:

$$\frac{1}{(k!)}\left(1-\frac{1}{n+1}\right)\dots\dots\left(1-\frac{k-1}{n+1}\right)$$

$$1-\frac{i}{n} \leq 1-\frac{i}{n+1} \Rightarrow e_n \leq e_{n+1}$$

$$2 < e_n < 1+1+\frac{1}{2}+\dots+\frac{1}{2^{n-1}} < 3$$

$$\therefore 2 \leq e_n < 3$$

Bounded above and monotonically increasing.

→ Subsequence and Bolzano - Weierstrass Theorem:

Example: $x_n = c^{1/n}$; $c>1$

$x_{n+1} < x_n$

$c.c^{1/n} > c \Rightarrow c^{n+1/n} > c$

$\Rightarrow c^{1/n} > c^{1/n+1}$

and $x_n > 1$

By monotone convergence theorem;

$x_n = c^{1/n}$ is convergent

let $\lim\limits_{n\to\infty} c^{1/n} = x$

$c_{2n} = c^{1/2n}$

$c_{2n}^2 = c_n$

$\lim\limits_{n\to\infty} c_{2n}^2 = \lim\limits_{n\to\infty} c_n$

Don't know whether $\lim\limits_{n\to\infty} c_{2n}$ exists.

Assume, $\lim\limits_{n\to\infty} c_{2n} = x = \lim\limits_{n\to\infty} c_n$

$\Rightarrow x^2 = x$

$\Rightarrow x = 1$ or $x = 0$

infimum $\{x_n \mid n\in\mathbb{N}\} \geq 1$

$\|$

$\lim(x_n)$ as x_n is decreasing $\Rightarrow x \geq 1 \Rightarrow \boxed{x=1}$

Subsequence: $n_1 < n_2 < \dots$

~~$x_1 < x_2 < x_3 < \dots$~~

Let $X = (x_n)$ and ~~$n_1 < n_2 < \dots n_k < \dots$~~ be a strictly increasing sequence of natural numbers

Then $X' = (x_{n_1}, x_{n_2} \dots)$ is a subsequence of (x_n).

x_n $\quad (x_{2n})$ $2 < 4 < 6 < 8 < \dots$

(x_{3n}) $3 < 6 < 9 < 12 < \dots$

Example:

m-tail of (x_n) $\quad (x_{m+1}, x_{m+2}, \dots)$

Here, $n_1 = m+1$, $n_2 = m+2, \dots$; $n_k = m+k$

Theorem: If (x_n) is converging to x then any subsequence (x_{n_k}) also converges to x.

proof: For any $\varepsilon > 0$, $\exists$ a natural no. N such that $|x_n - x| < \varepsilon \quad \forall n \geq N$

$n_N \geq N$ $\left(\begin{array}{l} n_1 \geq 1 \\ n_2 > n_1 \Rightarrow n_2 \geq 2 \end{array} \right)$

$n_k \geq k \geq N$ if $\forall k \geq N$

$\Rightarrow |x_{n_k} - x| < \epsilon \quad \forall k \geq N$

Divergence Criteria:

(1). If a sequence X has two subsequences X' and X'', where, X' is converging to x' and X'' is converging to x''

$x' \neq x''$ Then X is divergent

(2). If X is unbounded. Then X, is divergent.

Example: $(-1)^n = X$

$X' = (-1)^{2n}$ $\quad x' = +1$

$X'' = (-1)^{2n+1}$ $\quad x'' = -1$

$\therefore$ X is divergent.

Theorem-1:

A subsequence X' of a convergent sequence X is again convergent & $\lim X' = \lim X$.

07/08/19

Q. If all the convergent subsequences of a bdd sequence have the same limit x, then x is convergent ? → Yes

Ex: $(x_n)_n$ such that $x_{2n} = n$

$$x_{2n-1} = \frac{1}{n}$$

claim: The convergent subsequences of $(x_n)_n$ countains x_{2n-1} for $n \geq k$ (for some k) and the limit is 0.

Application:

1). $(-1)^n$ is divergent

2) $(\sin n)_n$ is divergent

Assume $(\sin n)_n$ is convergent.

∴ All subsequences are convergent to same limit.

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \quad ; \quad \sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}}$$

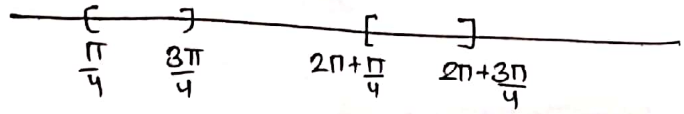
$$\sin x \geq \frac{1}{\sqrt{2}} \quad \text{if } x \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$$

length of $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right] > 1 \Rightarrow \exists N \in \mathbb{N}$ s.t

$$N \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$$

~~define~~ ~~$a_n = \{\sin n \mid n \in [$~~

$$\left[2k\pi + \frac{\pi}{4}, 2k\pi + \frac{3\pi}{4}\right]$$



$\sin x \geq \frac{1}{2}$ in $\left[2k\pi + \frac{\pi}{4}, 2k\pi + \frac{3\pi}{4}\right]$

$\hookrightarrow$ length $= \frac{\pi}{4} > 1$

$\therefore \left[2k\pi + \frac{\pi}{4}, 2k\pi + \frac{3\pi}{4}\right]$ countains atleast one natural no. n_k.

$(\sin n_k)_k$ — a subsequence of $(\sin n)_n$

$\downarrow$

convergent and limit $\geq \frac{1}{\sqrt{2}}$

$|\cancel{x} - \sin n_k| < \varepsilon \quad \forall k \geq N(\varepsilon)$

$|x - \sin n_k| < \frac{1}{n} \quad \forall k \geq N(\frac{1}{n})$

$\not\Rightarrow \quad \sin n_k > \frac{1}{\sqrt{2}}$

$\Rightarrow \quad |x| \geq \frac{1}{\sqrt{2}}$

Now take, $\left[2k\pi + \frac{7\pi}{6}, 2k\pi + \frac{11\pi}{6}\right] = J_k$

$\sin x \leq \frac{-1}{2} \qquad \forall x \in J_k$

J_k has length $= \frac{2\pi}{3} > 1$

J_k countains a natural number m_k'

$(\sin m_k)_k \rightarrow$ convergent limit $\leq \frac{-1}{2}$

So, $(\sin n)_n$ is divergent.

limit $\leq -\frac{1}{2}$ and also limit $\geq \frac{1}{\sqrt{2}}$

contradiction!

$|x_n - x| < \varepsilon \quad \forall n \geq N(\varepsilon)$

<u>Theorem</u>: Let $X = (x_n)$ be a sequence. The following are equivalent:

(1). X does not converge to x.

(2) There exists $\varepsilon_0 > 0$ such that

for any $k \in \mathbb{N}$, $\exists\, n_k \geq k$ s.t.

$$|x_{n_k} - x| > \varepsilon_0$$

(3). There exists $\varepsilon_0 > 0$ such that there is a subsequence (x_{n_k}) and $|x_{n_k} - x| > \varepsilon_0$

<u>proof:</u> (i) $\Rightarrow$ (ii) X does not converge to for any positive $\varepsilon > 0$, x, $\exists N(\varepsilon)$ s.t. $|x_n - x| < \varepsilon \quad \forall n \geq N(\varepsilon)$

There exists an $\varepsilon_0 > 0$ s.t. we cannot get $N(\varepsilon)$ with $|x_n - x| < \varepsilon \quad \forall n \geq N(\varepsilon)$

$N(\varepsilon) = 1$, then there is $n_1 \geq 1$ s.t.

$$|x_{n_1} - x| \geq \varepsilon_0$$

$\vdots$

$N(\varepsilon) = k$, then there is $n_k \geq k$ so that

$$|x_{n_k} - x| \geq \varepsilon_0$$

(ii). $\Rightarrow$ (iii) obvious

(iii). $\Rightarrow$ (ii). (x_{n_k}, k) is not convergent

so, (x_n) is not convergent to x.

Ex: $((-1)^n)_n$

$(-1)^{2n}$ and $(-1)^{2n+1}$ are convergent.

Bolzano- Weierstrass Theorem:

A Bounded sequence has a convergent subsequence

Theorem: If a bounded sequence X has the property,-
all its convergent subsequences have the same limit x,
then X converges to x

proof:

If possible suppose to x. X does not converge to
x By (iii) of previous theorem, $\exists$ a subseq $(x_{n_k})_k$
such that, $|x_{n_k} - x| > \varepsilon_0$

$X' = (x_{n_k})_k$ has a convergent subsequence X''.
(by B-W thm as $(x_{n_k})_k$ is bounded)

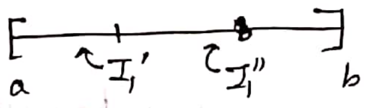
X'' converges to x.

So, X' converges to x, a $\rightarrow\leftarrow$ to (A).

~~Thm: The $X = (x_n)_n$~~

proof: $X = (x_n)$ is bounded. $\therefore x_n \in [a,b]$ $\forall n$

$n_1 = 1$

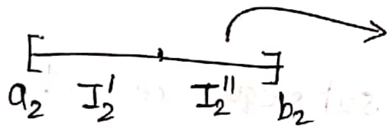


At least one of I_1' and I_1'' contains infinitely many elements(terms). Let I_1' contains infinitely many terms. Let n_2 be smallest natural number (>1) so that

$x_{n_2} \in I_1'$

call I_1' by $I_2 = [a_2, b_2]$

length of $I_2 = \frac{b-a}{2}$



contains infinitely many terms

choose n_3 to be smallest natural number such that $n_3 > \max\{n_1, n_2\}$ such that $x_{n_3} \in I_2''$

call I_2'' by I_3 length of $I_3 = \frac{b-a}{2^2}$

We can choose n_{k+1} such that $x_{k+1} \in I_{k+1}$

$I_1 \supset I_2 \supset \ldots\ldots \supset I_k \supset I_{k+1} \supset \ldots$

length of $I_k = \frac{b-a}{2^{k-1}}$

$\inf\{\text{length of } I_k\} = \frac{b-a}{2^{k-1}}$

$$\bigcap_{n=1}^{\infty} I_n = \{x\}$$ by nested interval theorem.

<u>claim:</u> $\lim_{K\to\infty} x_{n_K} = x$

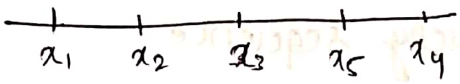
$$|x - x_{n_K}| \leq \text{length of } I_K = \frac{b-a}{2^{K-1}}$$

<u>Theorem:</u> Monotone subsequence Theorem:—

A sequence (x_n) of real numbers has a monotone subsequence. $x_{n_K} \leq x_{n_{K+1}}$ $\forall K$ or,

$x_{n_K} \geq x_{n_{K+1}}$ $\forall K$

<u>proof:</u>



suppose, m_1 be such that,

$x_m \geq x_{m_1}$ $\forall m \geq m_1$

$n_1 = m_1$

suppose we get $m_2 > m_1$ s.t.

$x_n \geq x_{m_2}$ $\forall n \geq m_2$

$x_{m_1} \leq x_{m_2}$

now, $x_{m_2} \leq x_{m_3}$ (similarly)

$x_{m_K} \leq x_{m_{K+1}} \leq \ldots$

$\hookrightarrow$ increasing sequence

Thus we get it if there are infinitely many bottoms.

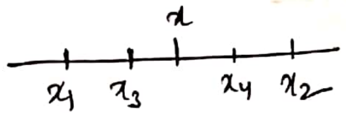
What if there is no such bottom?

x_{n_2} is not a bottom so $\exists\, n_3 > n_2$ s.t.

$x_{n_3} < x_{n_2}$

No such bottom we get a decreasing sequence.

Cauchy criterion:



May be $|x_3 - x_4| < |x_1 - x_4|$

$$|x_n - x_{n-1}| \longrightarrow 0$$

Def: A seq. $X = (x_n)$ is a cauchy sequence if for $\varepsilon > 0$, there is $K(\varepsilon) \in \mathbb{N}$ s.t.

$$|x_n - x_m| < \varepsilon \quad \forall n, m \geq K(\varepsilon)$$

Example:

(a). $\left(\frac{1}{n} \middle| n \in \mathbb{N}\right)$ is a cauchy sequence

$$|x_n - x_m| = \left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{1}{n} + \frac{1}{m}$$

$$\exists N \text{ s.t. } N > \frac{2}{\varepsilon} \Rightarrow \frac{1}{N} < \frac{\varepsilon}{2}$$

$$n, m \geq N \Rightarrow \frac{1}{n}, \frac{1}{m} \geq \frac{1}{N}$$

$$\Rightarrow |x_n - x_m| < \varepsilon$$

lemma:

Every convergent sequence is a cauchy sequence.

pf: suppose $(x_n) \to x$

Then for $\varepsilon/2 > 0$, we have $N(\varepsilon/2)$

s.t. $|x_n - x| < \varepsilon/2 \quad \forall n \geq N(\varepsilon/2)$

If $n, m \geq N(\varepsilon/2) \Rightarrow |x_n - x_m| \leq |x_n - x| + |x_m - x|$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

1. $\langle a_n \rangle$ - increasing sequence
$\langle b_n \rangle$ - decreasing sequence

such that $a_n \leq b_n \ \forall n \in \mathbb{N}$

$\lim(a_n) = \lim(b_n)$

Thereby deduce nested interval property from the monotone convergence theorem.

Sol: $\quad a_m \leq a_{m+1} \quad \forall m \in \mathbb{N}$

$\quad b_m \geq b_{m+1} \quad \forall m \in \mathbb{N}$

& $\quad a_m \leq b_m \quad \forall m \in \mathbb{N}$

$a_n \leq b_n \leq b_1$

By monotonic convergence theorem,

$\lim(a_n)$ exists. $\lim(a_n) = \sup\{a_n \mid n \in \mathbb{N}\} = a$

$b_m \geq a_m \geq a_1$

By monotonic convergence theorem,

$\lim(b_n)$ exists. $\lim(b_n) = \inf\{b_n \mid n \in \mathbb{N}\} = b.$

sequence of non-negative reals,

$b_n - a_n \geq 0$ $\quad$ Bounded below + Decreasing.

$\Rightarrow \lim(b_n - a_n) \geq 0 \Rightarrow b \geq a$

Method-II:

Let n be fixed.

If $m <= n$; $\quad a_m \leq a_n \quad$ $a_m \leq a_n \leq b_n$

$b_m \geq b_n$

If $m > n$; $\quad a_m \leq b_m \leq b_n$

$\therefore \; a_m \leq b_n \quad \forall m$

$\Rightarrow \lim(a_m) \leq \lim(b_n)$

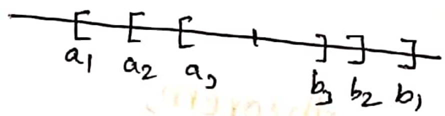
↳ exist by above proof.

Nested-interval property:

I & $[a_1, b_1] \supseteq [a_2, b_2] \supseteq \dots$

then $\bigcap_{n=1}^{\infty} [a_n, b_n]$ is non-empty.

proof:



$\langle a_n \rangle$ – monotonically increasing

$\langle b_n \rangle$ – monotonically decreasing

$a_m \leq b_n \quad \forall n.$

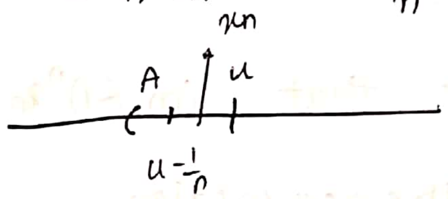
By the problem, $\lim(a_n) \leq \lim(b_n)$

$\bigcap I_n \equiv$ atleast singleton set

③. Let A be an infinite set that is bounded above and let $u = \sup(A)$. Show that there is a sequence $\{x_n\}$ in A $(x_n) \subseteq A$ such that $u = \lim(x_n)$

Sol: u is a supremum, so $u - \frac{1}{n}, n \in \mathbb{N}$ is an upper bound.

so, for $n \in \mathbb{N}$ $\exists x_n \in A$ $u - \frac{1}{n} < x_n$



④. Let $x_1 = a > 0$ & define $x_{n+1} = x_n + \frac{1}{x_n}$. Study convergence or divergence of $\langle x_n \rangle$.

Sol: $x_n \geq 0 \Rightarrow x_{n+1} \geq 0$

$x_{n+1} - x_n \geq 0 \Rightarrow$ monotonically increasing

$$\lim_{n\to\infty} (x_{n+1}) = \lim_{n\to\infty} \left(x_n + \frac{1}{x_n}\right)$$

$\Rightarrow \frac{1}{x_n} = 0$ (not correct) $\quad x_1 < x_2 < x_3 < \ldots\ldots$

prove unboundedness:

⑤. Suppose that every subsequence of $X=(x_n)$ has subsequence that converges to 0. Then, show that $\lim x_n = 0$.

Sol: (proof already done)

⑥. Let $\forall n \in \mathbb{N}$ $\lim (-1)^n x_n = 0$. Then show that x_n converges to 0.

Given an example to show that $\lim (-1)^n x_n = 0$ can't be dropped from the assumption.

Sol: $|(-1)^n x_n - 0| < \epsilon \quad \forall n \geq N$

$\Rightarrow \quad |x_n - 0| < \epsilon \quad \forall n \geq N$

$\therefore \quad \lim(x_n) = 0$

$x_n = (-1)^n$ ✓ – counter example

③. $s_n = \sup\{x_k : k \geq n\}$

$t_n = \inf\{x_k : k \geq n\}$

$\{s_n\}, \{t_n\}$ – monotone & convergent.

pt. if $\lim(s_n) = \lim(t_n) \Rightarrow (x_n)$ is convergent.

Sol: $A \subseteq B \subseteq \mathbb{R}$

$\sup(A) \leq \sup(B)$

$\sup(s_m) \geq \inf(s_n)$

$\therefore \quad \sup(s_1) \geq \sup(s_2) \geq \dots$

$\inf(t_1) \leq \inf(t_2) \leq \dots$

• s_n : decreasing and Bounded

t_n : increasing and Bounded.

$\cancel{\lim(s_n) =}$ $\cancel{\inf(s_n)}$

$\cancel{\lim(t_n) = \sup}(t_n)$

$\cancel{\text{state}} = S \Rightarrow \cancel{\inf(s_n) = \sup}(t_n)$

$\lim(s_n) = \lim(t_n) = s$ (say)

It is easy to see that,

$s_n \leq t_n$ let, $I_n = [s_n, t_n]$

to s.t; $\bigcap_{n=1}^{\infty} I_n = s$

$t_n \leq x_n \leq s_n$

$\lim(t_n) \leq \lim(x_n) \leq \lim(s_n)$

$\Rightarrow \lim(x_n) = s.$

Q. Let $\langle x_n \rangle$ be a bounded sequence and for $n \in \mathbb{N}$

let $s_n = \sup\langle x_k \mid k \geq n \rangle$;

$s = \inf\langle s_n \rangle$. S.T. there exists a subsequence of $\langle x_n \rangle$

that converges to $\underline{x}$.

Sol: $s = \inf\langle s_n \rangle$

$(s + \frac{1}{n})$ is not a lower bound for $\langle s_n \rangle$

there exists a n_1 such that $s_{n_1} < s + \frac{1}{n}$.

$\sup\langle s_k \mid k \geq n_1 \rangle < s + \frac{1}{n}$

$x_k < s + \frac{1}{n} \quad \forall k \geq n_1$ $\quad x_{k_n} < s + \frac{1}{n}$

$|x_{k_n} - s| < \frac{1}{n} \to 0$

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Not ex: $x_n = 1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}$

$$|x_m - x_n| = \frac{1}{m+1} + \frac{1}{m+2} + \dots + \frac{1}{n}$$

$\exists \varepsilon > 0$ for any $K \in \mathbb{N}$ $m, n \geq K$

$$|x_m - x_n| > \varepsilon$$

$$|x_m - x_n| > \frac{n-m}{n}$$

Take $n = 2m$ $|x_m - x_n| > \frac{1}{2}$

clearly, (x_n) is not a cauchy sequence.

Not convergent $\Rightarrow$ Not cauchy

Thm (cauchy convergence criteria)

A cauchy sequence is also convergent. A sequence (x_n) is a cauchy sequence iff (x_n) is convergent.

Property of cauchy sequence:

Lemma: Every cauchy sequence is Bounded.

Pf: Left as homework.

Proof of CCC:

(x_n) is cauchy sequence

$\Rightarrow$ (x_n) is bounded

$\Rightarrow$ (x_n) has a convergent subseq $(x_{n_k})_k$

limit is x (B-W Thm)

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For $\frac{\epsilon}{2}$, $\exists N\left(\frac{\epsilon}{2}\right) \in \mathbb{N}$ s.t.

$$|x_{n_k} - x| < \frac{\epsilon}{2} \qquad \forall k \geq N\left(\frac{\epsilon}{2}\right)$$

(x_n) is cauchy,

$$|x_m - x_n| < \frac{\epsilon}{2} \qquad \forall m, n \geq K\left(\frac{\epsilon}{2}\right)$$

$$N = \max\left\{ K\left(\frac{\epsilon}{2}\right), N\left(\frac{\epsilon}{2}\right)\right\}$$

$$|x_m - x_n| < \frac{\epsilon}{2} \qquad \forall m, n \geq N$$

$$|x_{n_k} - x| < \frac{\epsilon}{2} \qquad \forall k \geq N$$

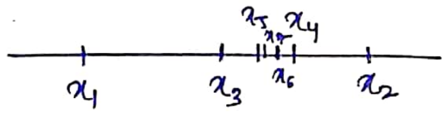
$$\Rightarrow \quad |x_m - x| < \epsilon \qquad \forall m \geq N$$

Application:

If $x_1 < x_2$ are arbitary real number and,

$$x_{n+1} = \frac{1}{2}(x_{n+1} + x_n)$$

what is the limit of x_n?



$$x_n - x_{n-1} = \frac{1}{2}(x_{n-1} + x_{n-2}) - \frac{1}{2}(x_{n-2} + x_{n-3})$$

$$= \frac{1}{2}(x_{n-1} - x_{n-3}) = -\frac{1}{2}(x_{n-1} - x_{n-2})$$

$$x_n - x_{n-1} = \left(\frac{-1}{2}\right)^{n-2}(x_2 - x_1)$$

$n > m$

$$|x_n - x_m| \leq |x_n - x_{n-1}| + |x_{n-1} + x_{n-2}| + \dots + |x_{m+1} - x_m|$$

$$\leq |x_2 - x_1| \frac{\left(\frac{-1}{2}\right)^{m-1}\left(1 - \left(\frac{-1}{2}\right)^{n-m}\right)}{(3/2)}$$

$$= \left(\left(\frac{1}{2}\right)^{n-2} + \left(\frac{1}{2}\right)^{n-3} + \dots + \left(\frac{1}{2}\right)^{m-1} \right) |x_2 - x_1|$$

$$= \left(\frac{1}{2}\right)^{m-1} \left(1 + \frac{1}{2} + \dots + \left(\frac{1}{2}\right)^{n-m-1} \right) |x_2 - x_1|$$

$$\leq \left(\frac{1}{2}\right)^{m-1} (2) |x_2 - x_1| \leq \left(\frac{1}{2}\right)^{m-2} |x_2 - x_1| < \epsilon$$

$$\forall m, n \geq K(\epsilon)$$

$$\left(\frac{1}{2}\right)^{m-2} < \frac{\epsilon}{|x_2 - x_1|} \leq \epsilon'$$

$$\Rightarrow \quad m > 2 + \log_2 \epsilon'$$

(archimedian principle)

→ Let $x = (x_n)$

$$x_n - x_1 = (x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + \dots \dots + (x_2 - x_1)$$

$$= \left(\frac{-1}{2}\right)(x_{n-1} - x_{n-2})$$

$$= \frac{1 - \left(\frac{-1}{2}\right)^{n-1}}{1 - \left(\frac{-1}{2}\right)} (x_2 - x_1)$$

$$x_n - x_1 = \frac{2}{3}\left(1 - \left(\frac{-1}{2}\right)^{n-1}\right)(x_2 - x_1)$$

Take limit,

$$x - x_1 = \frac{2}{3}(x_2 - x_1)$$

$$\Rightarrow \quad x = \frac{1}{3}(x_1 + 2x_2)$$

Another Application:

The poly $x^3-5x+1=0$

has root r, $0<r<1$. Find r.

Sol: start with x_1, $0<x_1\le 1$

$$x_{n+1} = \frac{1}{5}(x_n^3+1)$$

To show $(x_n)_n$ is a cauchy sequence:

$$|x_{n+1}-x_n| = \left|\frac{1}{5}(x_n^3+1) - \frac{1}{5}(x_{n-1}^3+1)\right|$$

$$= \frac{1}{5}|x_n^3 - x_{n-1}^3|$$

$$= \frac{1}{5}|x_n - x_{n-1}|\,|x_n^2 + x_{n-1}^2 + x_n x_{n-1}|$$

$$\le \frac{3}{5}|x_n - x_{n-1}|$$

$$\le \left(\frac{3}{5}\right)^{n-1}|x_2 - x_1|$$

$$|x_m - x_n| \le |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \dots + |x_{n+1} - x_n|$$

$$\le |x_2-x_1|\left(\left(\frac{3}{5}\right)^{m-2} + \dots + \left(\frac{3}{5}\right)^{n-1}\right)$$

$$\le |x_2-x_1|\left(\frac{3}{5}\right)^{n-1}\left(\frac{5}{2}\right)$$

$\therefore$ Its a cauchy sequence.

choose $$m > \frac{\ln\left(\varepsilon/\frac{5}{2}|x_2-x_1|\right)}{\ln\left(\frac{3}{5}\right)}$$

$$|x_n - x_{n-1}| \le \frac{3}{5}|x_{n-1} - x_{n-2}| \qquad \frac{3}{5}<1$$

~~Application of cauchy convergence theorem:~~

Def: A sequence (x_n) is contractive if $\exists c$, $0<c<1$ such that $|x_{n+1}-x_n| \le c|x_n - x_{n-1}|$

Theorem: Every contractive sequence is a cauchy sequence and hence, convergent.

$\lim x_n = x$

Then x is a root of the polynomial.

$$|x-x_m| \le \frac{c^{m-1}}{1-c}|x_2-x_1|$$

$$|x_n-x_m| \le \frac{c^{m-1}}{1-c}|x_2-x_1| \quad \forall n \ge m$$

Take $n\to\infty$ and c fixed,

$$|x-x_m| \le \frac{c^{m-1}}{1-c}|x_2-x_1|$$

In the example,

Fix, $x_1 = 0.5$

$x_2 = 0.225$

$|x_2-x_1| = 0.275$

$c = \frac{3}{5} = 0.6$

$$|x-x_m| \le \frac{(0.6)^{m-1}}{1-0.6}|x_2-x_1|$$

$$|x-x_6| \le \frac{(0.6)^5}{(0.4)}(0.5625) = \underline{0.11394}$$

$x_6 = \underline{0.20164}$

convergence of a series:

$$x_1 + x_2 + \dots\dots + x_n = \sum x_n$$

$(x_n) \rightarrow$ a sequence

$s_1 = x_1$

$s_1 = x_1 + x_2$

$\vdots$

$s_K = x_K + s_{K-1} = x_1 + x_2 + \dots\dots + x_K \leftarrow$ Pascal's sum of the series.

If the sequence (new one) s_n converges to s, then the series Σx_n converges to s.

Thm: Σx_n converges $\rightarrow \lim(x_n) = 0$

Ex: $\frac{1}{2} + \frac{1}{2} + \dots\dots$ is divergent.

Cauchy criteria for series:

The series Σx_n converges iff for any $\varepsilon > 0$, $\exists$ a natural number $M(\varepsilon)$ such that $n, m \geq M(\varepsilon)$ implies

$$|s_n - s_m| = |x_{m+1} + \dots\dots + x_n| < \varepsilon$$

completeness axiom:

A non-empty bounded above set has a supremum

$\Updownarrow$

A cauchy sequence is convergent

$\sqrt{2} \rightarrow$ not rational

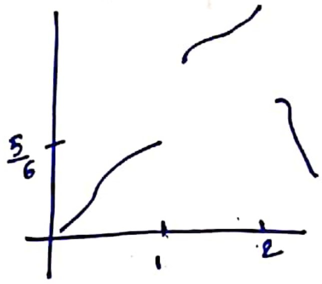
$\exists$ a set of rationals

(r_n) st $r_n \rightarrow \sqrt{2}$

The r_n is a cauchy sequence

But (r_n) is not convergent in the set of rationals.

Limit of a function:



If $x \to 1$, then where $f(x) \to$

$f: \mathbb{R} \to \mathbb{R}$

we have to define,

$f: A \to \mathbb{R}$

where, $A \subseteq \mathbb{R}$

we want $x \to 1$ means there will be points in A, which are close to 1, i.e., if we go very close to 1, there will be a point of A.

cluster point: (Limit point)

Let $A \subseteq \mathbb{R} \to$ non-empty subset. $c \in \mathbb{R}$ is a cluster point of A if ~~for~~ $\forall \varepsilon > 0$, $\exists x \in A$ s.t. $|x - c| < \varepsilon, x \neq c$ or, $0 < |x - c| < \varepsilon$

For any ε, $V_\varepsilon(c)$ countains a point other than c.

Exm:

(1). $A = (0, 1]$

set of cluster points of A is $[0, 1]$

(2). The set of cluster points of $\{0, -1, 1\}$ is

$\left(\frac{-1}{10}, \frac{1}{10}\right)$ contains no point of A other than a

$\Rightarrow$ 0 is not a cluster point.

~~(3). A discrete set does not have any cluster points~~

(4). Finite set doesn't have any cluster points.

(5). $\left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$ cluster point $= \{0\}$

Theorem:

$c \in \mathbb{R}$ is a cluster point of A if and only if there exists a sequence (x_n) such that $x_n \in A$ & $x_n \neq c$ $\forall n \in \mathbb{N}$

$\lim(x_n) = c$.

proof:

$c \in \mathbb{R}$ is a cluster point of A.

$\varepsilon = \frac{1}{n}$ $\quad \exists x_n \in A,\ x_n \neq c$ s.t. $|x_n - c| < \frac{1}{n}$

Take (x_n) - $x_n \in A,\ x_n \neq c\ \forall n$

$\lim x_n = c$

Def: (Limit of a function)$^{\text{at } c}$

For $\varepsilon > 0$, $\exists \delta(\varepsilon) > 0$ such that,

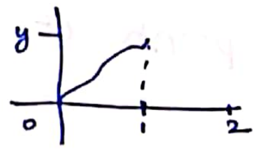
$|f(x) - L| < \varepsilon \quad \forall x \in A, |x - c| < \delta$

If such L exists, then $L =$ limit of f at c.

$(x_n) \to 1$

$(f(x_n)) \to y$ or L

where $\lim\limits_{x \to 1} f(x) = L$



$L = \lim\limits_{x \to c} f(x)$. If such limit L doesn't exist

f diverges at c.

Exm:

(1). $f(x) = 1 \quad \forall x \in A$, c is a cluster point of A.

$$\lim_{x \to c} f(x) = 1.$$

(2). If $f(x) = x$, $\lim_{x \to c} f(x) = c$.

proof: $|f(x) - c| = |x - c| < \varepsilon$

if $\delta(\epsilon) = \varepsilon$

then $|x - c| < \delta(\epsilon) = \epsilon$

$\Rightarrow |f(x) - c| < \epsilon$

(3). $f(x) = x^2 \quad \forall x \in A \qquad \lim_{x \to c} f(x) = c^2$

proof: $|f(x) - c^2| = |x - c||x + c| < \epsilon \quad \forall \quad |x - c| < \delta(\epsilon)$

$|x + c| < \delta(\epsilon) + 2c$

$\delta(\epsilon)\left(\delta(\epsilon) + 2c\right) < \epsilon$

$\delta(\epsilon)^2 + 2\cdot\delta(\epsilon).c + c^2 < \epsilon + c^2$

$\boxed{\delta(\epsilon) = \sqrt{\epsilon + c^2} - c} \quad \forall \epsilon \ \exists \delta(\epsilon)$

Sir's proof: Take $|x - c| < 1$

$|x| \leq |x - c| + |c| < |c| + 1$

$|f(x) - c^2| \leq (1 + 2|c|)|x - c|$

For $\epsilon > 0$, we need to find $\delta(\epsilon) > 0$ s.t.

$|x - c| < \delta(\epsilon)$, $x \in A$, $x \neq c$

$\Rightarrow |f(x) - c^2| < \epsilon$

$\delta'(\epsilon) = \dfrac{\epsilon}{1 + 2|c|^2}$; $\delta(\epsilon) = \min\left\{1, \dfrac{\epsilon}{1+2|c|}\right\}$

Exm:

$$\lim_{x\to c} \frac{1}{x} = \_\_\_\_\_ .$$

case-I: $c \neq 0$

$c<0$ $\quad \left|f(x) - \frac{1}{c}\right| = \left|\frac{1}{x} - \frac{1}{c}\right| = \frac{|x-c|}{|x||c|} < \epsilon$

$\forall\, |x-c| < \delta(\epsilon)$

we want to choose $|x| > \frac{1}{2}|c|$

$$\frac{|x-c|}{|x||c|} = \frac{2|x-c|}{|c|^2}$$

take,

$|x-c| < \frac{1}{2}|c|$

$\Rightarrow |x| \geq |c| - |x-c|$

as $|c| \leq |x-c| + |x|$

$|x| \geq |c| - \frac{1}{2}|c|$

$\geq \frac{|c|}{2}$

$$\frac{|x-c|}{|x||c|} < \frac{2|x-c|}{|c|^2} < \epsilon$$

if $|x-c| < \frac{|c|^2\epsilon}{2}$ $\quad$ Here, $\delta(\epsilon) = \frac{|c|^2\epsilon}{2}$

$$\delta(\epsilon) = \min\left\{\frac{|c|^2\epsilon}{2}, \frac{1}{2}|c|\right\}$$

$\epsilon < \frac{1}{|c|}$ $\quad$ $\epsilon > \frac{1}{|c|}$

Case-II: $c = 0$

Take $(x_n) = \left(\frac{1}{n}\right)$

$(f(x_n)) = (n)$

Thus does not have the limit.

theorem:

Let $A \subseteq \mathbb{R}$, c is a cluster pt of A, $f: A \to \mathbb{R}$

(a). If $L \in \mathbb{R}$, $L \neq \lim_{x \to c} f(x)$ iff there exists a sequence (x_n) in A, $x_n \neq c$ $\forall n$, $\lim(x_n) = c$ but $\lim(f(x_n) \neq L)$

(b). $f(x)$ does not have a limit at c iff $\exists$ a seq (x_n) $x_n \in A$, $x_n \neq c$ $\forall n$ st $(f(x_n))$ is not convergent.

proof:

First assume (a)

To prove (b)

suppose $f(x)$ does not have a limit.

$\Rightarrow$ 1 and -1 are not limits of f at c.

$\Rightarrow \exists$ a seq (x_n), $x_n \in A$, $x_n \neq c$

and $\lim_{n \to \infty} (f(x_n)) \neq 1$ (by (a))

$(f(x_n))_n$ is not convergent $\Rightarrow$ (b)

If $(f(x_n))_n$ is convergent,

then by (a), $1 \neq \lim_{n \to \infty} (f(x_n))$

say $\lim_{n \to \infty} (f(x_n))_n = L$

also, $L \neq \lim_{x \to c} f(x)$

$\Rightarrow \exists y_n$ - $y_n \in A$, $y_n \neq c$

$\lim_{n \to \infty} (y_n) = c$ but $\lim_{n \to \infty} (f(y_n)) \neq L$

Take $(z_n)_n$ st. $z_{2m} = x_m$ $z_{2m-1} = y_m$

$\lim_{n \to \infty} z_n = c$ and $(f(z_n))$ is not convergent.

Theorem: (sequential criterion)

$A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$, c is a cluster point of A, then,

$\lim_{x \to c} f(x) = L$ iff for any sequence (x_n), $x_n \in A$

$x_n \neq c$, $x_n \to c$, $(f(x_n)) \longrightarrow L$.

Proof: Fix $\epsilon > 0$,

$\lim f(x) = L$

$\Rightarrow x \to c$

$\Rightarrow \exists\, \delta > 0$ st for $x \in A, x \neq c$

$|x - c| < \delta \Rightarrow |f(x) - L| < \epsilon$

$(x_n) \longrightarrow c \Rightarrow \exists N(\delta) \in \mathbb{N}$

st $n \geq N(\delta) \Rightarrow |x_n - c| < \delta$

So, $n \geq N(\delta) \Rightarrow |f(x_n) - L| < \delta$

$\Rightarrow \lim_{n \to \infty} (f(x_n)) = L$

converse part:

suppose if possible, $\lim_{x \to c} f(x) \neq L$

$\Rightarrow \exists\, \epsilon_0 > 0$ and for any $\delta > 0$,

$\exists x \in A$, $x \neq c$

st $|x - c| < \delta$ & $|f(x) - L| \geq \epsilon_0$

Take $\delta = \frac{1}{n}$, $\Rightarrow x_n \in A$, $x_n \neq c$, $|x_n - c| < \delta$

but $|f(x_n) - L| \geq \epsilon_0$

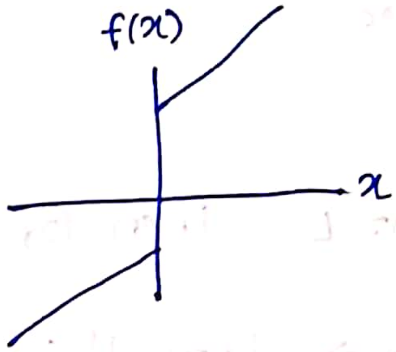
(x_n) is a sequence from A, $x_n \neq c$ $\forall n$

$(x_n) \to c$, but $\lim_{n \to \infty} f(x_n) \neq L$, a contradiction.

Application:

$$f(x) = x + sgx(x) \qquad sgn(x) = \begin{cases} 1 \; ; & x > 0 \\ -1 \; ; & x < 0 \\ 0 \; ; & x = 0 \end{cases}$$

$$\lim_{x \to 0} f(x) = \; ?$$



Take $x_n = \frac{1}{n}$, $y_n = \frac{-1}{n}$

$x_n \to 0 \qquad y_n \to 0$

$f(x_n) \to 1 \qquad f(y_n) \to -1$

$z_{2n} = \frac{1}{n} \qquad (z_n) \to 0$

$z_{2n} = \frac{-1}{n} \qquad f(z_n)$ is not convergent.

Properties of limit:

<u>Theorem</u>: Let $A \subseteq \mathbb{R}$ and c be a cluster point of A. $f: A \to \mathbb{R}$. If $\lim_{x \to c} f(x)$ exists, then f is bdd on the nbhd of c.

<u>Proof</u>: $\lim_{x \to c} f(x) = L$ Then for $\varepsilon = 1$, there is $\delta > 0$ st.

$0 < |x-c| < \delta,\ x \in A \Rightarrow |f(x) - L| < 1$

$$|f(x)| \leq |f(x) - L| + |L|$$
$$< 1 + |L| \quad , \ |x-c| < \delta,\ x \neq c$$

$$|f(x)| \leq \max\{|L|+1, |f(c)|\}, \quad |x-c| < \delta$$

<u>Def</u>: $f: A \to B$, c is a cluster point of A. f is said to be bdd on a nbhd of c if we have $\delta > 0$, $M > 0$ st

$$|f(x)| \leq M \quad \forall x \in (c-\delta, c+\delta) \cap A$$

<u>Theorem</u>: $A \subseteq \mathbb{R}$, c is a cluster point of A. $f, g: A \to \mathbb{R}$. Let $b \in \mathbb{R}$, Assume $\lim_{x \to c} f, g$ exists.

(a). $\lim_{x \to c} (f+g)(x) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$

(b). $\lim_{x \to c} (bf(x)) = b \lim_{x \to c} f(x)$

(c). If $|g(x)| \neq 0 \ \forall x \in A$. Then,

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$$

(d). $\lim_{x \to c} f(x) g(x) = \lim_{x \to c} f(x) \lim_{x \to c} g(x)$

counter example:

$g(x) = x \quad x \in \mathbb{R} - \{0\}$

$g(x) \neq 0 \quad \forall x$

$\frac{1}{g(x)}$ is defined $= \frac{1}{x}$

$\lim_{x \to 0} g(x) = x = 0 \qquad \lim_{x \to 0} \frac{1}{x}$ does not exist.

Applications:

1. $\lim_{x \to c} x^2 = c^2$

Theorem:

If $a \leq f(x) \leq b$

$f: A \to B$ and c is a cluster point of A &

$\lim_{x \to c} f(x)$ exists.

then $a \leq \lim_{x \to c} f(x) \leq b$

Theorem (squeeze Theorem):

Let $f, g, h: A \longrightarrow \mathbb{R}$, c is a cluster point, and $f(x) \leq g(x) \leq h(x) \quad \forall x \in A$. and $\lim_{x \to c} f(x) = \lim_{x \to c} h(x)$

$$\lim_{x \to c} g(x) = \lim_{x \to c} f(x) = \lim_{x \to c} h(x)$$

$\lim_{x \to 0} \sin \frac{1}{x}$ does not exist.

$$\lim_{x\to 0} x\sin\frac{1}{x}$$

$$-1 \le \sin\frac{1}{x} \le 1$$

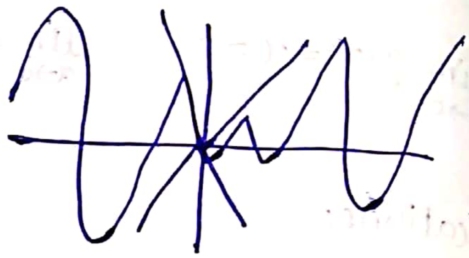
$x>0 \Rightarrow \quad -x \le x\sin\frac{1}{x} \le x$

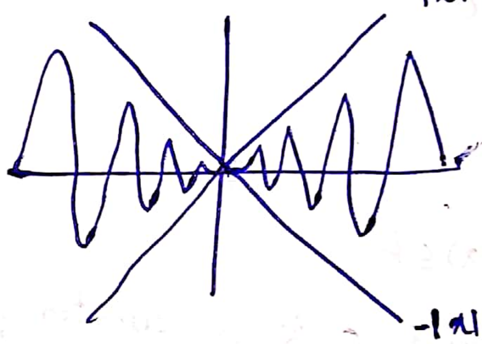
$x<0 \Rightarrow \quad x \le x\sin\frac{1}{x} \le -x$

$\Rightarrow \quad -|x| \le x\sin\frac{1}{x} \le |x|$

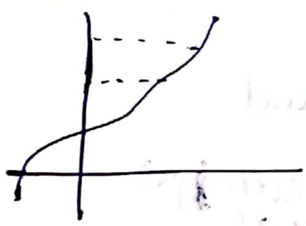
apply squeeze theorem.

$$\lim_{x\to 0} |x| = \lim_{x\to 0} -|x| = 0$$





Continuity:



$$\lim_{x\to c} f(x) = f(c)$$

Here we assume c is a cluster point of domain A.

For any $\epsilon>0$, there is an $\delta>0$ such that $|x-c|<\delta, x\in A, x\neq c$ implies

$$|f(x)-f(c)|<\epsilon$$

$\Rightarrow$
$x=c$

$$|f(x)-f(c)|<\epsilon \quad \forall \ |x-c|<\delta, \ x\in A$$

Assume c is not a cluster point.

$\exists \delta_1 > 0$ st, $(c-\delta_1, c+\delta_1) \cap A = \{c\}$

$\Rightarrow$ Any function should be continuos at $\underline{c}$.

<u>Tutorial - 3</u>

Q1. Let (x_n) be bounded for $n \in \mathbb{N}$ let

$s_n : \sup\{x_k \mid k \geq n\}$ and $s = \inf s_n$

show that there exists a subsequence of (x_n) that converges to s.

$s = \inf s_n$

$\exists s_{n(r)}$ such that $s < s_{n(r)} < s + \frac{1}{r}$, $r \in \mathbb{N}$

we want $n(r) > n(r-1)$

we have $s < s_{n(r-1)}$

choose $n(r)$ such that. $s < s_{n(r)} < \min\{ s_{n(r-1)}, s + \frac{1}{r}\}$

$s_{n(r)} = \sup\{x_k \mid k \geq n(r)\}$

$\Rightarrow \exists x_{n(r)}$ such that $s_{n(r)} > x_{n(r)} > s_{n(r)} - \frac{1}{2r}$

as $s_{n(r)}$ is the supremum.

we can choose $n(r) > n(r-1)$

$n_r := n(r)$ and $(x_{n_r}) \to s$ as

$|x_{n_r} - s| < \frac{1}{r}$

Q2. Show by definition that a bounded monotone sequence is a cauchy sequence.

$x_i \leq x_j \quad \forall i \leq j \quad // \quad x_i \geq x_j \quad \forall i \leq j$

$|x_m - x_n| < \epsilon \quad \forall m, n \geq N(\epsilon)$

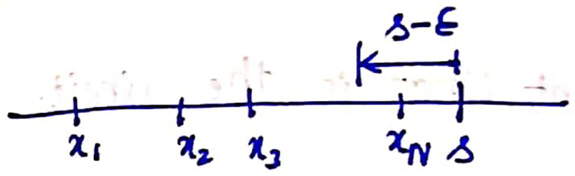
$\exists \epsilon_0$ ~~s.t.~~ $|x_m - x_n| \geq \epsilon_0$

$\forall m, n \geq N$

$|x_m| \geq \epsilon_0 + |x_n| \quad \exists m, n \geq N$

Scanned with CamScanner

First assume (x_n) is increasing.



s: $\sup\{x_n \mid n \in \mathbb{N}\}$

we get N s.t. $s-\epsilon < x_N < s$

$$s - \epsilon < x_N < s$$

$$|x_n - s| < \epsilon \quad \forall\, n \geq N(\epsilon)$$

$$|x_n - s| < \frac{\epsilon}{2} \quad \forall\, n \geq N\left(\frac{\epsilon}{2}\right)$$

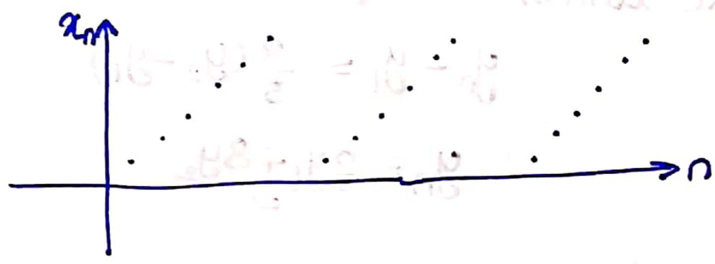
$$|x_m - s| < \frac{\epsilon}{2} \quad \forall\, m \geq N\left(\frac{\epsilon}{2}\right)$$

$$\Rightarrow |x_m - x_n| < \epsilon \quad \forall\, m, n \geq N\left(\frac{\epsilon}{2}\right)$$

Q3. Let p be a natural number. Give an example of a sequence (x_n), which is not cauchy, but

$$\lim_{n\to\infty} |x_{n+p} - x_n| = 0$$

sol: $n \pmod{p}$



$$x_{n+p} - x_n = 0$$

$$\lim_{n\to\infty} (x_{n+p} - x_n) = 0$$

> prove non-cauchy by subsequences.

$x_n = \sqrt{n}$

Q4. If $y_1 < y_2$ are two arbitary real no. and $y_n = \frac{1}{3}y_{n-1} + \frac{2}{3}y_{n-2}$. Show that y_n is convergent. What is the limit.

$$y_n - y_{n-1} = \frac{-2}{3}(y_{n-1} - y_{n-2})$$
$$= \left(\frac{-2}{3}\right)^2 (y_{n-2} - y_{n-3})$$
$$\dots = \left(\frac{-2}{3}\right)^{n-2} (y_{n-(n-2)} - y_{n-(n-1)})$$
$$= \left(\frac{-2}{3}\right)^{n-2} (y_2 - y_1)$$

$$y_n - y_1 = (y_n - y_{n-1}) + \dots + (y_2 - y_1)$$

$$\Rightarrow \quad y_n - y_1 = \left(\frac{-2}{3}\right)^{n-2}(y_2 - y_1) + \dots + (y_2 - y_1)$$

$$= (y_2 - y_1) \cdot \frac{1 - \left(\frac{-2}{3}\right)^{n-1}}{\frac{5}{3}} = \frac{3}{5}(y_2 - y_1)\left(1 - \left(\frac{-2}{3}\right)^{n-1}\right)$$

Take limits,

$$y_n - y_1 = \frac{3}{5}(y_2 - y_1)$$
$$y_n = \frac{2y_1 + 3y_2}{5}$$

Q5. Determine the limit $(3n)^{1/2n}$ using $\lim_{n\to\infty} (n)^{1/n} = 1$.

$x_n = n^{1/n}$

$x_{3n} = (3n)^{1/3n}$

$x_{3n} \to 1 \Rightarrow x_{3n}^{3/2} \to 1$

$\Rightarrow (3n)^{1/2n} \longrightarrow 1$

Continuity:

$f : A \to \mathbb{R}$

1). c is a cluster point of A

f is continuos at c, given $c \in A$

$\lim_{x \to c} f(x) = f(c)$

$\Rightarrow$ For $\varepsilon > 0$ $\exists \delta(\varepsilon) > 0$ st

$|f(x) - f(c)| < \varepsilon$ $\forall |x - c| < \delta(\varepsilon)$ $\begin{matrix} x \neq c \\ x \in A \end{matrix}$ (limit)

$\exists$ a nbhd $(c - \delta_1, c + \delta_1)$ st $(c - \delta_1, c + \delta_1) \cap A = \{c\}$

any function should be continuos at c.

Def^n: $f : A \to \mathbb{R}$, $c \in A$ f is continuos at c if for given $\varepsilon > 0$, $\exists \delta(\varepsilon) > 0$ such that $x \in A$.

$|x - c| < \delta(\varepsilon) \Rightarrow |f(x) - f(c)| < \varepsilon$

$x \in A \cap V_\delta(c) \Rightarrow f(x) \in V_\varepsilon(f(c))$

Remark: If c is a cluster point, then,

given f is continuos at $c \Rightarrow$

1). $\lim_{x \to c} f(x)$ exists

2). $\lim_{x \to c} f(x) = f(c)$

Exm:

(i). $f(x) = 1 \quad \forall x \in \mathbb{R}$

(ii). $f(x) = x \quad \forall x \in \mathbb{R}$

(iii). $f(x) = x^2 \quad \forall x \in \mathbb{R}$

$\Rightarrow \lim_{x \to c} x^2 = c^2 \quad \forall c \in \mathbb{R}$

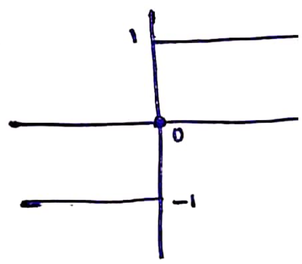
(iv). $f(x) = \frac{1}{x}, x \neq 0 \longrightarrow$ continuos at every $c \in \mathbb{R}$

$\lim_{x \to c} \frac{1}{x} = \frac{1}{c} = f(c)$

Def^n: $f: A \to \mathbb{R}$ is continuos at (on A) if f is continuos on each point of A.

Exm:

1). $f(x) = \text{sgn}(x)$



$|f(x) - f(0)| < 1 \quad \nexists \; \delta(1)$

limit does not exist.

we can show that subsequence does not converge to 0.

2). $x_{2n} = \frac{1}{n}$

$x_{2n-1} = \frac{-1}{n}$

$f(x_n) = \begin{cases} 1 & \text{if } n \text{ is even} \\ -1 & \text{if } x \text{ is odd} \end{cases}$

$f(x)$ is divergent

$\lim_{n \to \infty} \text{sgn}(x)$ does not exist.

Sequential criterion for discontinuity:

f is discont at $c \in A$ if and only if there is a sequence (x_n) st $x_n \in A$, $(x_n) \to c$ but $(f(x_n))$ doesn't converge to $f(c)$.

Sequential criterion for continuity:

f is continuos at $c \in A$ if and only if for every sequence (x_n) such that $x_n \in A$ and $(x_n) \to c$, we have $f(x_n) \longrightarrow f(c)$.

Exm of continuos functions:

suppose $f: A \to R$ is continuos.

Let $B \subseteq A$. Then we can define a function

$g: B \to R$

$g := f \quad \forall x \in B$

g is continuos on B.

For $\epsilon > 0$, $\exists \delta(\epsilon) > 0$ such that;

$x \in V_{\delta(\epsilon)}(c) \cap B$

$\Rightarrow f(x) \in V_{\epsilon}(f(c))$

$V_{\delta(\epsilon)}(c) = \{x \mid x \in (c - \delta(\epsilon), c + \delta(\epsilon))$

Not Example:

(1). Dirichlet's discontinuos functions:

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

let c be rational

let (x_n) be sequence of rationals st $x_n \to c$

$f(x_n) = 1 \quad \forall n$

$(c, c+\frac{1}{n})$ $\exists$ a rational

$c < r_n < c + \frac{1}{n}$

$r_n < r_{n-1}$

$c < r_n < \min\{ r_{n-1}, c + \frac{1}{n} \}$

$(y_n) \to$ a sequence of irrationals

st $(y_n) \to c$ and $(f(y_n)) = 0 \neq 1 = f(c)$

by sequential criterion, $f(x)$ is discontinuos at $x = c$

Exm of continuos functions:

(1). $f(x) = x$ $\quad f(x) = x^2$ $\quad f(x) = 1$

(2). $f(x) = \frac{1}{x}$, $x \neq 0$ is also continuos

$$\lim_{x \to c} \frac{1}{x} = \frac{1}{c}$$

(3). $f: A \to B$ is continuos, $B \subseteq A$

Then the restriction of f to B is also continuos.

$g: B \to R$ defined as $g(x) := f(x)$ $\forall x \in B$

Not examples:

(1). Dirichlet discont function:

$f: R \to R$

$f(x) = 1$ if x is rational

$= 0$ if x is irrational

(2). Let $sgn(x) = 1$ if $x > 0$

$= 0$ if $x = 0$

$= -1$ if $x < 0$

This is continuos at each $c \in \mathbb{R} - \{0\}$ and discontinuos at $c = 0$.

(3). Let $A = \{x \mid x > 0\}$

$$f(x) = \begin{cases} \frac{1}{n}; & \text{if } x \text{ is rational and } x = \frac{m}{n} \text{ (coprimes)} \\ 0; & \text{otherwise} \end{cases}$$

f is discontinuos at all rational numbers.

(x_n) be sequence of irrational st $(x_n) \to c$

$(f(x_n)) = 0$

$(f(x_n)) \to 0$

but $f(c) = \frac{1}{n}$.

choose $\epsilon > 0$ we need to choose $\delta > 0$ st,

$|x-c| < \delta \Rightarrow |f(x)-f(c)| < \epsilon$

$c \to$ irrational

$|x-c| < \delta \Rightarrow |f(x)| < \epsilon$

we need to get x such that, $f(x) = \frac{1}{n} < \epsilon$

First choose $n_0 \in \mathbb{N}$ st $\frac{1}{n_0} < \epsilon$.

Take $(c-1, c+1)$

claim: $(c-1, c+1)$ countains only finitely many rationals of the form $\frac{m}{n}$ where $n < n_0$

$$\frac{m}{n} < c+1 \Rightarrow m < (c+1)n$$
$$< (c+1)n_0$$

choose $\delta > 0$ such that $(c-\delta, c+\delta)$ doesn't countain rationals $\frac{m}{n}$ where $n < n_0$

$$x \in (c-\delta, c+\delta) \Rightarrow x = \frac{m}{n} \; ; \; n \geq n_0$$
$$\therefore f(x) = \frac{1}{n} \leq \frac{1}{n_0}.$$

(4). $f: \mathbb{R} \to \mathbb{R}$

$$f(x) = \sin\frac{1}{x} \quad , x \neq 0$$
$$= 0 \quad , x = 0$$

is not continuos at $x = 0$

when $x \neq 0$ $f(x)$ is continuos at x.

so, $\lim_{x \to 0} f(x)$ does not exist.

$\sin\frac{1}{x} = g \circ h$

$h(x) = \frac{1}{x}$, $x \neq 0$

$g(x) = \sin x$

<u>Theorem:</u> $f: A \to R$, $g: B \to R$. Image of $f \subseteq B$

Then, $g \circ f$ is defined.

If f and g are continuos, $g \circ f$ also continuos.

$\therefore \sin\frac{1}{x}$ is continuos at $x \neq 0$.

$\sin\frac{1}{x}$ is not continuos on $\mathbb{R}$.

$\sin\frac{1}{x}$ is restriction of f on $\mathbb{R} - \{0\}$

(5). $g: \mathbb{R} \to \mathbb{R}$

$g(x) = x\sin\frac{1}{x}$, $x \neq 0$

g is continuos on $\mathbb{R} - \{0\}$

$\lim_{x \to 0} g(x) = 0$

$G: \mathbb{R} \to \mathbb{R}$

$G(0) = 0, x = 0$ else, $G(x) = g(x)$

G is continuos on $\mathbb{R}$

G is the extension of g.

General method for a new continuous functions:

suppose $f: A \to B$ be a continuous function, c is a cluster point of A, If $\lim_{x \to c} f(x)$ exists.

Then we define a function,

$$F: A \cup \{c\} \to \mathbb{R} \qquad F(x) := f(x), \; x \in A$$
$$= \lim_{x \to c} f(x) \; ; \; x = c$$

F is an extension of f to $A \cup \{c\}$.

F is continuous.

<u>proof:</u> ($g \circ f$ continuity)

Let $c \in A$, $f(c) \in B$

For given $\epsilon > 0$, there exists $\delta_1 > 0$, such that,

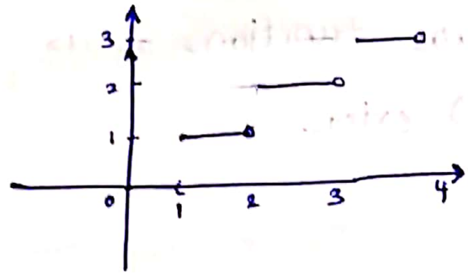
$$|y - f(c)| < \delta_1 \Rightarrow |g(y) - g(f(c))| < \epsilon \qquad y \in B$$

$$|x - c| < \delta \Rightarrow |f(x) - f(c)| < \delta_1 \qquad x \in A$$

$x \in A$, $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \delta_1$

$x \in A$, $|x - c| < \delta \Rightarrow |g(f(x)) - g(f(c))| < \epsilon$ ∎

(6). $f(x) = [x]$, $[x]$ is gif



$c \in \mathbb{Z} \rightarrow$ not continuos $\quad \lim_{x\to c+} f \neq \lim_{x \to c-} f$

$c \notin \mathbb{Z} \rightarrow$ continuos

Def (Right Hand limit)

$f : A \rightarrow \mathbb{R}$

Let c be cluster point $A \cap (c, \infty)$. Then $\lim_{x \to c+} f(x) = L$

if for given $\epsilon > 0$, there is $\delta > 0$ st $0 < x - c < \delta$

$\Rightarrow |f(x) - L| < \epsilon$ $\quad \lim_{x \to c-} f(x) = L \rightarrow$ similar defn

<u>Theorem:</u>

$f : A \rightarrow \mathbb{R}$ $\quad c$ is a cluster point of both $(c, \infty) \cap A$ and $(-\infty, c) \cap A$. Then $\lim_{x \to c} f(x)$ exists iff both left and right limits exist, and are equal.

Constructing new continuos functions:

<u>Theorem:</u> Let $f, g : A \rightarrow \mathbb{R}$, f and g are continuos, $c \in A$. Then,

(1). $f+g$, $f-g$ are continuos

(2). $g(x) \neq 0$. Then $\dfrac{f(x)}{g(x)}$ is continuos.

Application:

(1). $p(x) = a_n x^n + \dots\dots + a_0$

is continuos on $\mathbb{R}$

(2). $\frac{p(x)}{q(x)}$, $p(x), q(x) \longrightarrow$ polynomial

$\hookrightarrow$ Rational function.

Let $A = \{x \mid q(x) \neq 0\}$ Then, $\frac{p}{q}$ is continuos on A.

$A = \mathbb{R} - \langle q^{-1}(0) \rangle$

General Fact:

$f, g : A \to R$ f, g are continuos

Let $B = \{x \in A \mid g(x) \neq 0\}$. Then $\frac{f}{g}$ is continuos on B.

$f : A \to R \longrightarrow$ continuos

prove that

$|f|(x) = |f(x)|$ is also continuos

f is continuos $\Rightarrow$ $|f|$ is continuos.

$\rightarrow f(x) = 1$ if $x \in \mathbb{Q}$

-1 if $x \notin \mathbb{Q}$

f is discontinuos at each point but $|f|$ is continuos.

(3). $f : A \to R$ is continuos and $f > 0$,

then $\sqrt{f}$ is also continuos.

> $\cos(\sqrt{1+x^2})$ is continuos

Let $f, g : \mathbb{R} \to \mathbb{R}$ continuos such that,

$$f(r) = g(r) \quad \forall r \in \mathbb{Q}$$

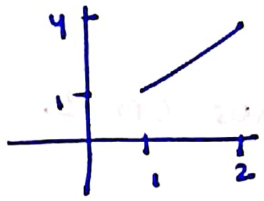
Then, $f(x) = g(x) \quad \forall x \in \mathbb{R}$ Let $x \in \mathbb{R} - \mathbb{Q}$

Let (r_n) be sequence of rationals such that $r_n \to x$.

Then $f(r_n) = g(r_n)$

$$\therefore \lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} g(r_n) \Rightarrow f(x) = g(x).$$

Image of a continuos function:



$f(x) = x^2$

Domain: $(1, \infty)$

~~Range~~ (Image) : $(1, \infty)$

we take the domain to be bounded.

on $[1,2] \cup [3,4]$ $f(x) = \frac{1}{x}$ is bounded

$[1,2] \to$ closed interval

contains all of its cluster points.

<u>Def</u>: A set A is closed if it contains all its cluster points.

<u>Exm</u>: $[0,1]$

Def: $A \neq \varphi$ be a subset of $\mathbb{R}$, A point ~~in~~ c is called a cluster point of A if for any $\epsilon > 0$, $\exists x \in A$ st $0 < |c - x| < \epsilon$.

(2). $\left\{ \frac{1}{n} \middle| n \in \mathbb{Z} \right\}$ not closed

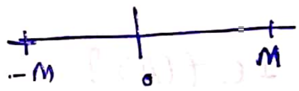
0 is cluster point

(3). $\mathbb{N}$ has no cluster points.

$\mathbb{N}$ is closed

(4). $(0, 1] \longrightarrow$ not closed.

Def: A function $f: A \to R$ is bounded if $\exists M > 0$ st $|f(x)| < M$ $\forall x \in A$.



Theorem: Let $f: A \to \mathbb{R}$ be a continuos function and A be a closed and bounded set. Then f is bounded on A.

proof: Let f be unbounded.

For a natural number n, $\exists$ a x_n st $|f(x_n)| > n$

So, $(f(x_n))$ is unbounded.

For any subsequence, (x_{n_r}) $|f(x_{n_r})| > n_r \geq r$

So, $f(x_{n_r})$ is unbounded.

By Bolzano-weierstrass:

(x_n) is bdd $\Rightarrow$ (x_n) has convergent subsequence.

let $(x_{n_K}) \to x$ $\quad x \in A$

If x is one of x_{n_K}, $x \in A$

If x is none of x_{n_K}, then $\exists K$ st

$$|x_{n_K} - x| < \epsilon \quad \forall k \geq K$$

A is closed set $\quad \Rightarrow x$ is cluster point of A.

so $x \in A$

$\Rightarrow \quad (f(x_{n_K})) \to f(x)$

$\Rightarrow (f(x_{n_K}))$ is bounded, contradiction!

Bounded $\Rightarrow$ Has supremum S & infimum I

When does $S \in f(A)$ & $I \in f(A)$?

Def: $f: A \to R$, f has an absolute maximum of A if $\exists a \in A$ st $f(a) \geq f(x)$ $\forall x \in A$ $\quad$ a is called absolute maximum point of A.

Theorem: Let A be closed and bounded set and $f: A \to R$ be continuos. ~~$f: A \to R$ be~~ Then f has absolute maximum and absolute mininum on A.

proof: By previous theorem, $f(A)$ is bounded, so, has a supremum M and an infimum m.

To show, $M = f(a)$ for some $a \in A$
$\quad m = f(b)$ for some $b \in B$

M is supremum of $f(A) \Rightarrow \exists s_n \in A$ st

$$f(s_n) \leq M - \epsilon \text{ (or) } M - \frac{1}{n} < f(s_n) \leq M$$

s_n is in A $\Rightarrow$ (s_n) is bounded.

By BW theorem, s_n has a convergent subsequence

(s_{n_r}) let $(s_{n_r}) \longrightarrow a$

$a \in A$ as A is closed.

To show $M = f(a)$.

$$M - \frac{1}{n_r} < f(s_{n_r}) \leq M$$

as $(s_{n_r}) \to a$, f is cont at A, so $(f(s_{n_r})) \longrightarrow f(a)$

By squeeze theorem $f(a) = M$.

Tutorial-4

1. $|g(x) - c^2| < 2a|x-c|$

$g : (0,a) \to \mathbb{R}$

$\lim_{x \to c} x^2 = c^2$

$0 < |x-c| < \delta, \ \delta = \frac{\epsilon}{2a} \Rightarrow |x^2 - c^2| < \epsilon$

3. show that $\lim_{x \to 0} \sin\left(\frac{1}{x^2}\right)$ doesn't exist.

To choose (x_n) st $\sin\frac{1}{x^2} = 1$

To choose (y_n) st $\sin\frac{1}{y^2} = 0$

$\frac{1}{x_n^2} = 2n\pi + \frac{\pi}{2} \Rightarrow x_n = \frac{\sqrt{2}}{\sqrt{(4n+1)\pi}}$

$\frac{1}{y_n^2} = n\pi \Rightarrow y_n = \frac{1}{\sqrt{n\pi}}$

define $(z_n)_n$ $\quad z_{2n} = x_n$

$z_{2n-1} = y_n$

$z_n \to 0$ but $f(z_n)$ does not exist.

2. $\lim_{x \to c} \sqrt{x} = \sqrt{c}, \ x > 0$

$|x-c| < \delta(\epsilon) \Rightarrow$

$|\sqrt{x} - \sqrt{c}| < \epsilon$

$|x-c| < 1$

$|x| < |c| + 1$

$\sqrt{x} < \sqrt{c+1}$

$|x-c| < \delta(\epsilon) \Rightarrow$

$\frac{|x-c|}{\sqrt{x}+\sqrt{c}} < \epsilon$

$|x-c| < \epsilon(\sqrt{c} + \sqrt{c+1})$

$\delta(\epsilon) = \min\{1, \epsilon(\sqrt{c}+\sqrt{c+1})\}$ or, $\delta(\epsilon) = \epsilon\sqrt{c}$

4.

(i). $|f(x)-0| = |x|$ if x is rational

$= 0$ if x is irrational

$|f(x)-0|<\epsilon$ if $0<|x-0|<\epsilon$

$\lim_{x\to 0} f(x) = 0$

$c\neq 0 \Rightarrow \lim_{x\to c} f(x)$ doesn't exist.

c is irrational

$\exists$ a sequence (x_n),

x_n is rational &

$(x_n)\to c$

Then $f(x_n) = (x_n) \to c$

$\exists (y_n)$, y_n irrational and $(y_n)\to c$

Then $f(y_n) = 0$ $(f(y_n)) \to 0$

7. prove that $\lim_{x\to 0} f(x)=0$

$f(0)=0$

$\lim_{x\to 0} f(x) = L$

$\Rightarrow$ For any sequence (x_n), $(x_n)\to 0$

$(f(x_n)) \to L$

$f(x_n) = f\left(2\,\frac{x_n}{2}\right) = 2f\left(\frac{x_n}{2}\right)$

let (y_n), $y_n = \frac{x_n}{2}$, $(y_n)\to 0$

$$f(x_n) = 2f(y_n)$$
$$\downarrow \qquad \downarrow$$
$$L \qquad 2L$$

$\Rightarrow L=0$

$|f(x)-f(c)|<\epsilon$

$= |f(x-c)| < \epsilon$

Take $y = x-c$ $\quad |y|<\delta \Rightarrow |f(x-c)-0|<\epsilon \Rightarrow |f(x-c)|<\epsilon$

Maximum–Minimum Theorem:

$m = \inf(f(A)) \quad M = \sup(f(A))$

Then, $m, M \in f(A)$

Problem: let $f:[a,b] \to \mathbb{R}$ be a continuos functions let $f(x)>0$ $\forall x \in [a,b]$. Then $\exists \alpha>0$ such that $f(x) \geq \alpha$ $\forall x \in [a,b]$

Sol: $I = [a,b]$

$f(I)$ is bounded

let $\alpha = \inf f(I)$

Then $\alpha \in f(I)$ $\quad \alpha > 0$

Bolzano's intermediate value Theorem:

Let I be an interval and $f: I \to R$ be continuos. If $a,b \in I$ $(a<b)$ and k is in between $f(a)$ and $f(b)$ there $\exists c \in (a,b)$ such that $f(c)=k$.

Interval: $I \subset R$

if $a,b \in I$ st $a<b$, then, $a<c<b \Rightarrow c \in I$

particular case: If $f(a)<0, f(b)>0$

$\exists$ $c \in (a,b)$ st $f(c)=0$

Location of root Theorem:

If $a, b \in I$ such that $f(a) < 0$, $f(b) > 0$ then $\exists c \in (a,b)$ such that $f(c) = 0$.

proof of intermediate value Theorem:

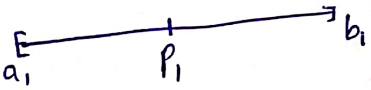
$$g(x) = f(x) - k$$

$$g : I \to \mathbb{R}$$

$$g(a) < 0 \quad g(b) > 0 \Rightarrow \exists c \in (a,b) \text{ st } f(c) = 0.$$

<u>proof:</u> $a_1 = a \quad b_1 = b \quad I = [a_1, b_1]$

Take $P_1 = \frac{a_1 + b_1}{2}$



If $f(p_1) = 0$ then $c = P_1$

Now, suppose $f(p_1) \neq 0$. Then $f(p_1) > 0$ or $f(p_2) < 0$.

If $f(p_1) > 0 \quad a_2 = a_1 \quad b_2 = P_1$

If $f(p_1) < 0 \quad a_2 = P_1 \quad b_2 = b_2$

repeat, $I_n \subset I_{n-1} \subset \cdots\cdots \subset I_2 \subset I_1$

$$P_{n+1} = \frac{a_n + b_n}{2}$$

we get a sequence of intervals,

$$\text{length of } I_n = \frac{2}{2^n}(b-a)$$

then $\bigcap_{n=1}^{\infty} I_n = \{p\}$ (by nested interval property)

<u>claim:</u> $p = c$ $\qquad (a_n) \to p$

$(b_n) \to p$

$p = \lim(a_n) = \lim(b_n) \qquad \cancel{\Rightarrow f(p) > 0}$

$f(a_n) \to f(p)$

$f(p) \le 0$

$f(b_n) \to f(p)$

$f(p) \ge 0$

$f(p) = 0$

Application:

show that $x = \cos x$ has a solution in $(0, \frac{\pi}{2})$ Use the bisection method to find an approximation with an error upto 10^{-2}

Sol: $f(x) = x - \cos x$

$f(0) = -1 < 0$

$f(\frac{\pi}{2}) = \frac{\pi}{2} > 0$

$a = 0 \quad b = \pi/2$

$a_1 = 0 \quad b_1 = \frac{0 + \pi/2}{2} = 0.785398$

$f(a_1) = -1$

$f(b_1) = \frac{\pi}{4} - \frac{1}{\sqrt{2}} > 0$

$a_2 = 0 \quad b_2 = \frac{\pi}{8}$

$f(a_2) = -1$

$f(b_2) = \frac{\pi}{8} - \cos(\frac{\pi}{8}) < 0$

$a_3 = \frac{\frac{\pi}{8} + \frac{\pi}{4}}{2} \quad b_3 = \frac{\pi}{4}$

$a_3 = \frac{3\pi}{16} \quad b_3 = \frac{\pi}{4}$

proof: $P_n = \frac{a_{n-1}+b_{n-1}}{2}$

$\{c\} = \cap I_n$

$|P_n - c| \le |b_n - a_n| = \frac{1}{2^n}|b-a|$

$p_n, c \in [a_n, b_n]$ $\quad = \frac{1}{2^n}\left(\frac{\pi}{2}\right)$

we need, $\frac{1}{2^n} \cdot \frac{\pi}{2} < 10^{-2} \Rightarrow n > 7$

take $n = 8$

n	a_n	b_n	P_n	$f(p_n)$
1	0	$\frac{\pi}{2}$	$\frac{\pi}{4}$	$0.783 > 0$
2	0	$\frac{\pi}{4}$	$\frac{\pi}{8}$	$-0.6073 < 0$
3	$\frac{\pi}{8}$	$\frac{\pi}{4}$	0.5890	$-0.2425 < 0$

preservation Theorem:

Thm: If I is an interval and $f: A \to R$ be a continuos

Then $f(I)$ is also an interval.

proof: intermediate value theorem:

If $f: I \to R$ and $a, b \in I$ and $a, b \in I$ and $f(a) \ne f(b)$

If k is a real number between $f(a)$ & $f(b)$ $\exists c \in (a,b)$

or (b,a) st $f(c) = k$.

Pf: $\alpha, \beta \in f(I)$ $\quad \alpha = f(a)$

$\beta = f(b)$

If $\alpha < k < \beta$ Then to show $k \in f(I)$

$k = f(c)$ $\quad c \in (a,b)$

Thm: If I is closed and bounded &

f is continuos then,

$f(I)$ is also closed and bounded.

Sol: $I \rightarrow$ closed and bounded

$\Rightarrow f(I)$ is bounded set

I is interval $\Rightarrow f(I)$ is interval

$f(I)$ is bounded $\Rightarrow$ M is supremum
m: is infimum

$f(I) \subseteq [m, M]$

$m, M \in f(I)$ maximum-minimum theorem

As $f(I)$ is interval $[m, M] \in f(I)$

counter-example:

(1). closed and unbounded:

$\tanh x$ Image is $(-1, 1)$

$\frac{1}{x} : [1, \infty) \longrightarrow (0, 1]$

(2). open and bounded:

$c : (0,1) \longrightarrow \mathbb{R}$ Image is $\{c\}$
$\downarrow$
not open

(3). open and unbounded interval:

constant function

Application:

Every odd degree real polynomial has a real root.

pf: $p(x) = a_{2n+1}x^{2n+1} + \dots\dots + a_0$

$$p(x) = x^{2n+1}\left(a_{2n+1} + \frac{a_{2n}}{x} + \dots + \frac{a_0}{x^{2n+1}}\right), x \neq 0$$

$$= x^{2n+1} r(x)$$

$$r(x) = a_{2n+1} + \frac{a_{2n}}{x} + \dots + \frac{a_0}{x^{2n+1}}$$

$r(x) \to a_{2n+1}$ as $x \to \infty, -\infty$

If $a_{2n+1} > 0$ $p(x) = x^{2n+1} a_{2n+1} \to \infty$ as $x \to \infty$

$p(x) \to -\infty$ as $x \to -\infty$

$p(x)$ takes both negative and positive values

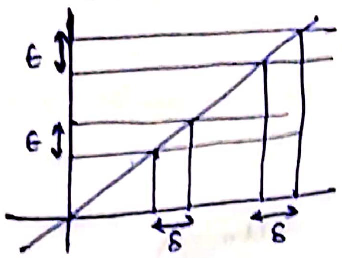
let $p(a) < 0$ and $p(b) > 0$

$a < 0$ $b > 0$

Then $\exists$ $c \in (a,b)$ $p(c) = 0$

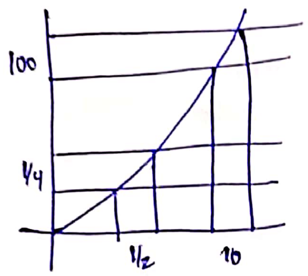
For given $M > 0$, $x > k \Rightarrow p(x) > M$.

uniform continuity:



δ is same for given ϵ at any x.

counter example:



$f(x)=x^2$

$\delta(\frac{1}{2})=\frac{1}{4}$

$\delta(10)=100$

(something wrong)

definition:

$f:A\to R$ if for any given $\epsilon>0$. There exists $\delta(\epsilon)$ st

$|x-c|<\delta(\epsilon) \Rightarrow |f(x)-f(c)|<\epsilon \quad \forall x,c\in A.$

Note: f is uniformly continuos on A

$\Rightarrow$ f is continuos on A

Ex: (1). $f(x)=kx$,

$$|f(x)-f(c)| = |k||x-c| = |k||x-c|$$

For any $\epsilon>0$, we can choose $\delta(\epsilon)=\dfrac{\epsilon}{|k|}$

def: $f:A\to R$ is uniformly continuos if for $\epsilon>0$, $\exists\delta(\epsilon)>0$ such that $|x-c|<\delta(\epsilon) \Rightarrow |f(x)-f(c)|<\epsilon$.

Lipschitz function:

2). $f: \mathbb{R} \to \mathbb{R}$ st

$|f(x)-f(c)| < k|x-c|$ is uniformly continuos.

$k \geq 0$, k is constant.

$\delta(\epsilon) = \frac{\epsilon}{k}$, $|x-c| < \frac{\epsilon}{k}$ $\quad |f(x)-f(c)| < k|x-c| < \epsilon$

$\frac{|f(x)-f(c)|}{|x-c|} < k$; slope should be less than k.

3). $f(x) = x^n$ ~~$x \in [0,k]$~~ $x \in A$ / Bounded set.

$|x^n - c^n| < \epsilon \Leftarrow |x-c| < \delta(\epsilon)$

show that $f(x)$ is lipschitz.

$|x| < M$.

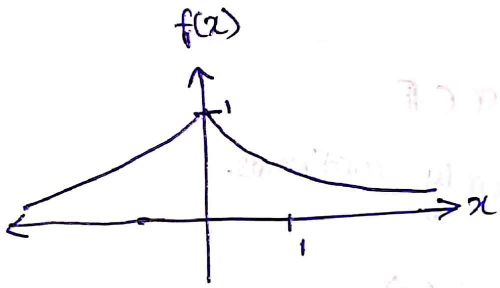
$\Rightarrow$ $|x-c| < M + |c|$

$|x^n - c^n| = |x-c| \, |x^{n-1} + x^{n-2}c + \dots + c^{n-1}|$

$< nM^{n-1}|x-c|$

$\delta(\epsilon) = \frac{\epsilon}{nM^{n-1}} \Rightarrow |x-c| < \delta(\epsilon) \Rightarrow |x^n - c^n| < \epsilon$

4). $f(x) = \frac{1}{1+x^2}$, $x \in \mathbb{R}$



Show that function f is lipschitz,

$\left| \frac{1}{1+x^2} - \frac{1}{1+c^2} \right| < \epsilon \Leftarrow |x-c| < \delta(\epsilon)$

$$\frac{|x-c||x+c|}{|1+x^2||1+c^2|} < \epsilon \quad \Leftarrow \quad |x-c| < \delta(\epsilon)$$

$$|x-c| < \frac{\epsilon |1+x^2||1+c^2|}{|x+c|}$$

$$< \frac{\epsilon |1+c^2|}{|x+c|}$$

$$|f(x)-f(c)| < \left| \frac{1}{1+x^2} - \frac{1}{1+c^2} \right|$$

$$< \frac{|x+c||x-c|}{(1+x^2)(1+c^2)}$$

$$< |x-c|\left(\frac{|x|}{(1+x^2)(1+c^2)} + \frac{|c|}{(1+x^2)(1+c^2)} \right)$$

$$< |x-c|\left(\frac{1}{1+c^2} + \frac{1}{1+x^2} \right) < 2|x-c|$$

$\therefore$ It is lipschitz function.

5). $f: A \to R$

$x \to x^n$ is bounded uniform continuity theorem:

Let A be closed and bounded set, If f is continuos on A. then f is uniformly continuos.

ot example:

1). $f(x) = x^2$, $x \in \mathbb{R}$

is not uniformly continuos.

2). $f(x) = \frac{1}{x}$, $x \in (0,1)$

is not uniformly continuos.

The following are equivalent:

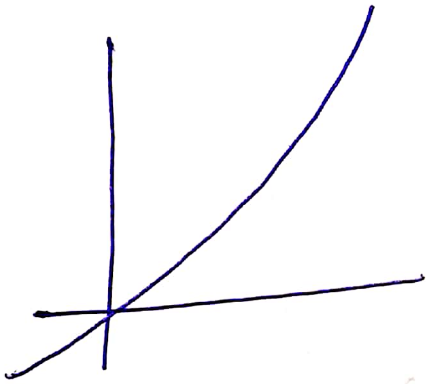
$f: A \to \mathbb{R}$

(a). f is not uniformly continuos

(b). $\exists\, \epsilon_0 > 0$ such that for each $\delta > 0$ there are x_δ, c_δ with $|x_\delta - c_\delta| < \delta$ but $|f(x) - f(c)| \geq \epsilon_0$.

(c). $\exists\, \epsilon_0 > 0$ and two sequences (x_n) and (c_n) such that $|x_n - c_n| \to 0$ but $|f(x_n) - f(c_n)| \geq \epsilon_0$.

$f(x) = x^2$



$x_n = n$

$c_n = n + \frac{1}{n}$

$|x_n - c_n| = \frac{1}{n} \to 0$

$|f(x_n) - f(c_n)| = \left|\left(n + \frac{1}{n}\right)^2 - n^2\right|$

$\geq 2 \qquad \epsilon_0 = 2$

Hence, not uniformly continuos.

Uniform continuity Theorem:

$f: A \to R$ where A is closed and bounded. Then f is uniformly continuos.

proof:

suppose if possibly, f is not uniformly continuos.

$\Rightarrow \exists (x_n)$ and (c_n) s.t $|x_n - c_n| \to 0$ but $|f(x_n) - f(c_n)| \geq \epsilon$

By Bolzano-Weirstrass then, as (x_n) is bdd, (x_n) has a convergent subsequence (x_{n_k})

similarly $(x_{n_k}) \to x$. So, $x \in A$ as A is closed.

$(c_{n_k}) \to x$ as $|x_{n_k} - c_{n_k}| \to 0$

$\Rightarrow f(x_{n_k}) \to f(x)$ and $f(c_{n_k}) \to f(x)$ (By continuity)

$\Rightarrow |f(x_{n_k}) - f(c_{n_k})| \to 0$, contradiction!

One property of uniform continuity:

$(x_n) \to x$, f is cont $\Rightarrow f(x_n) \to f(x)$

$f(x) = \frac{1}{x}$ $\quad f: A \to R$

$(x_n) = \left(\frac{1}{n}\right) \to$ cauchy

$(f(x_n)) = (n) \to$ not cauchy

Let (x_n) be a cauchy sequence.

For any given $\epsilon > 0$, $\exists \delta(\epsilon) > 0$ st

$|x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon$

$\exists K \in \mathbb{N}$ st $|x_n - x_m| < \delta$ if $m,n \geq k$

$m,n \geq k \Rightarrow |f(x_n) - f(x_m)| < \epsilon$

$\Rightarrow |f(x_n) - f(c_{n_k})| \to 0$, contradiction to $(*)$.

$\Rightarrow$

continuos Extension Theorem:

$f:(a,b) \longrightarrow \mathbb{R}$ is continuos can we extend f to

$F:[a,b] \to \mathbb{R}$

Exm: (a). $\frac{1}{x}$ $x \in (0,1)$

$\lim_{x\to 0} \frac{1}{x}$ doesn't exist. So, we can't extend.

$F(0) = \lim_{x\to 0} f(x) =$ doesn't exist

If we can extend to F.

$F:[a,b] \to \mathbb{R}$ is continuos and $F(x) = f(x)$ on (a,b)

$\lim_{x\to a} F(x) = \lim_{x\to a} f(x)$

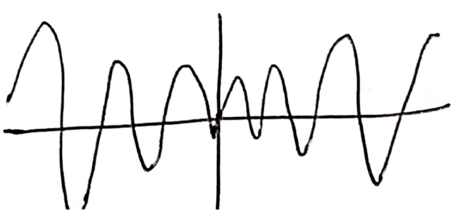
(b). $\sin\left(\frac{1}{x}\right)$, $x \in (0,1)$ can't be extended on $[0,1]$

In (a) & (b) the functions are not uniformly cont.

(c). $x\sin\left(\frac{1}{x}\right)$, $x \neq 0$

$$\lim_{x\to 0} \left| x\sin\left(\frac{1}{x}\right) \right| = 0$$

We can extend at 0, by putting $F(0) = 0$.



$x\sin\left(\frac{1}{x}\right)$

$x\sin\frac{1}{x}$ can be extended to a cont. function $F:[0,1] \to \mathbb{R}$

By uniform continuity theorem, F is uniformly continuos

Thm: $f:(a,b) \to \mathbb{R}$ is continuos, f is uniformly continuos if and only if we want extend f to a continuos function on $[a,b]$.

proof: suppose f is uniformly continuos.

To show $\lim_{x\to a} f(x)$, $\lim_{x\to b} f(x)$ exists.

After showing, we can define

$$F:[a,b] \to R$$

$$F(x) = f(x), x \in (a,b)$$

$$F(a) = \lim_{x\to a} f(x)$$

$$F(b) = \lim_{x\to b} f(b)$$

Then F is continuos on $[a,b]$

property: $f:A \to R$ is uniformly continuos. Then (x_n) is a cauchy sequence $\Rightarrow f(x_n)$ is cauchy.

proof: If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to b} f(x)$ exists. Then we can

define, $F:[a,b] \to R$

$$F(a) = \lim_{x\to a} f(x)$$

$$F(b) = \lim_{x\to b} f(x)$$

Hence, F is continuos.

we take a $x_n \to a$ $x_n \in (a,b)$

$\lim(f(x_n))$ exists.

$\cancel{(x_n) \to a}$, $\cancel{x_n \in (a,b)}$

$\cancel{\lim f(x_n)}$) $(x_n) \to a \Rightarrow (x_n)$ is a cauchy sequence

As f is uniformly continuos on A, $(f(x_n))$ is also a cauchy sequence. So, $(f(x_n))$ is convergent.

let $L = \lim_{x \to a} f(x_n))$

To use sequence criterion for limit, we have to show, for an

$(y_n) \to a$, $f(y_n) \to L$

Another sequence, $(y_n) \to a$

$(x_n) \to a \quad (y_n) \to a$

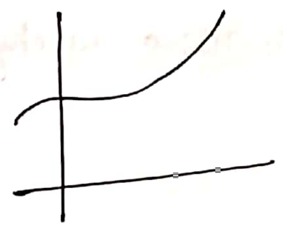
$|x_n - y_n| \to 0$

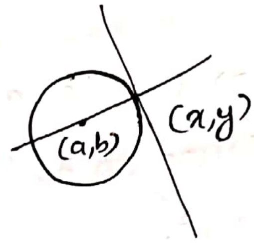
$\Rightarrow |f(x_n) - f(y_n)| \to 0$ (By uniform continuity)

$\Rightarrow \lim(f(y_n)) = L$

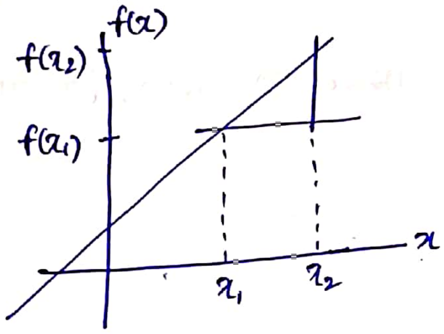
$\therefore \lim_{x \to a} f(x) = L$ by sequential criterion.

Differentiation:





$(x-a)^2+(y-b)^2=1$



Average rate of change:

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}$$

Instantaneous rate of change:

$$\lim_{x_2\to x_1}\frac{f(x_2)-f(x_1)}{x_2-x_1}$$

Def: $f: A\to R$, c is cluster point of A $(c\in A)$. We say f is differentiable at c and L is the derivative if for any $\epsilon>0$ $\exists\,\delta>0$ such that,

$$\left|\frac{f(x)-f(c)}{x-c}-L\right|<\epsilon \quad \forall\ 0<|x-c|<\delta$$

if such L exists

$$f'(c)=L \quad \Rightarrow \quad \lim_{x\to c}\frac{f(x)-f(c)}{x-c}=L$$

If $B\subset A$ is those points where f is differentiable, then $f': B\to R$ is a function.

Property:

(1). Caratheodory Theorem:

$f: A\to R$ f is differentiable at c iff $\exists$ a continuos function (φ) on A st $f(x)-f(c)=\varphi(x)(x-c)$

proof: If such φ exists, then,

$$\varphi(c) = \lim_{x \to c} \varphi(x)$$
$$= \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$$

$\therefore$ f is differentiable at c.

Now, suppose f is differentiable at c

$$\varphi(x) = \frac{f(x)-f(c)}{x-c}, x \neq c$$

$$\varphi(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$$

φ is continuos as,

$$\varphi(c) = \lim_{x \to c} \phi(x)$$

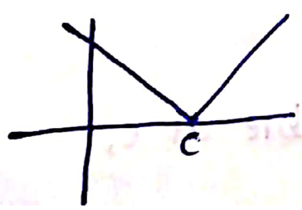
(2). f is diff at c $\Rightarrow$ f is continuos at c.

pf: $f(x)-f(c) = \frac{f(x)-f(c)}{x-c} \cdot x-c$

$$\lim_{x \to c} (f(x)-f(c)) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c} \lim_{x \to c} (x-c)$$
$$= (f'(c))(0)$$
$$= 0$$

$$\lim_{x \to c} f(x) = f(c)$$

converse is not true: $f(x) = |x-c|$ is not differentiable at c.



Ex: $f(x) = \frac{1}{x}, x \neq 0$

$$f(x) - f(c) = \varphi(x)(x-c) \qquad x \neq 0$$

$$\Rightarrow \frac{1}{x} - \frac{1}{c} = \varphi(x)(x-c) \Rightarrow \varphi(x) = \frac{-1}{xc}$$

$\varphi(x)$: continuos on $R - \{0\}$

$$\varphi(c) = \lim_{x \to c} \varphi(x) = \frac{-1}{c^2}.$$

Let $f, g : A \to R$ let f and g be differentiable at c. Then $f+g, f-g, f \cdot g, f/g, \alpha f$ ($g(x) \neq 0$ $\forall x \in A$ and $g'(c) \neq 0$) are differentiable at c

$$(f+g)'(c) = f'(c) + g'(c)$$

$$(\alpha f)'(c) = \alpha f'(c)$$

$$(f \cdot g)'(c) = f(c) g'(c) + f'(c) \cdot g(c)$$

$$(f/g)'(c) = \frac{g(c) f'(c) - f(c) g'(c)}{(g(c))^2}$$

$$(g \circ f)'(c) = \frac{(g \circ f)(x) - (g \circ f)(c)}{x - c}$$

$$= \frac{(g \circ f)(x) - (g \circ f)(c)}{f(x) - f(c)} \quad \frac{f(x) - f(c)}{x - c}$$

$$= g'(f(c)) f'(c)$$

chain rule:

$f : A \to R$ $g : B \to R$ $f(A) \subseteq B$, f is differentiable at c, g is differentiable at $f(c)$, then $g \circ f$ is differentiable at c, and $(g \circ f)'(c) = g'(f(c)) f'(c)$

proof:

f is differentiable at $c \Rightarrow \exists$ a continuos func. φ st,

$$f(x) - f(c) = \varphi(x)(x-c) \text{ by caratheodory}$$

g is differentiable at $f(c) \Rightarrow \exists$ a continuos func. ψ st,

$$\cancel{g(f(x)} \; g(y) - g(f(c)) = \psi(y)(y - f(c))$$

put $y = f(x)$,

$$g(f(x)) - g(f(c)) = \psi(f(x))(f(x) - f(c))$$
$$= \psi(f(x))\varphi(x)(x-c)$$

Here, $\psi(f(x))\varphi(x)$ is continuos,

thus $g \circ f$ is differentiable

$$(g \circ f)'(c) = \psi(f(x))\varphi(x)\Big|_{x=c} = \psi(f(c))\varphi(c)$$
$$= g'(f(c))\,\varphi'(c).$$

Application:

1). Let $f: A \to R$ be st $f(x) \neq 0 \;\; \forall x \in A$ f be diff on A,

$\frac{1}{f(x)}$ is differentiable ?

$$g(x) = \frac{1}{x}$$

$$(g \circ f)'(c) = \frac{-1}{(f(c))^2} f'(c)$$

2). At which points $|\sin x|$ is differentiable.

$\sin x : R \to R$ $\qquad g(y) = |y| : R \to \mathbb{R}$

$$|\sin x| = g \circ \sin x$$

let $c \in R$, $c \neq n\pi$

$f(c) \neq 0$, g is differentiable at $f(c)$

By chain rule, $g \circ \sin x$ is differentiable at c.

$$(g \circ \sin x)' = g'(\sin x)(\sin x)'|_c$$

$$g'(x) = \begin{cases} 1, & x>0 \\ -1, & x<0 \end{cases}$$

$$(g \circ \sin x)' = \begin{array}{ll} \cos c & \text{if } \sin c > 0 \\ -\cos c & \text{if } \sin c < 0 \end{array}$$

$f: A \to R$ is differentiable at each point of A. Then at what point $|f|$ is differentiable?

<u>Soln:</u> By chain rule, f is differentiable at c, g is diff at $f(c) \Rightarrow g \circ f$ is differentiable at c

So, if $f(c) \neq 0$ $g \circ f(x)$ is diff at c. Here $g(y) = |y|$.

$$(g \circ f)'(c) = \begin{cases} f'(c); & f(c) > 0 \\ -f'(c), & f(c) < 0 \end{cases}$$

(3), $f(x) = x \sin \frac{1}{x}$, $x \neq 0$
$\quad = 0$, $x = 0$

$$f'(0) = \lim_{x \to 0} \frac{x \sin \frac{1}{x} - 0}{x - 0} = \lim_{x \to 0} \sin \frac{1}{x}$$

Does not exist as f is not differentiable at $x = 0$.

#

$g = f^{-1} \Rightarrow g \circ f(x) = x$

$g'(f(c)) \cdot f'(c) = 1$

$\Rightarrow g'(f(c)) = \dfrac{1}{f'(c)}$ ~~$\Rightarrow g'(x) =$~~

g is differentiable at $f(c) \Rightarrow f'(c) \neq 0$

Theorem: $f: I \to R$, I is an interval

f is strictly monotone and continuos.

let $J = f(I)$ let $g: J \to R$ be ~~continuos~~ the inverse.

Then g is also strictly monotonic and continuos. If f is differentiable at c and, $f'(c) \neq 0$, then g is differentiable at $f(c)$ and $g'(f(c)) = \frac{1}{f'(c)}$

continuos theorem (continuos inverse theorem):

$f: I \to R$ is strictly monotonic and continuos. Then, f^{-1} exists, strictly monotonic, f^{-1} is continuos and $f(I)$ is an interval.

Exm: Take $f(x) = x^3 + 2x + 1$

f: strictly increasing

f: continuos $\Rightarrow$

f^{-1}: exists & is continuos

$f'(x) = 3x^2 + 2 > 0$

f^{-1} is differentiable at c and

$$(f^{-1})'(f(c)) = \frac{1}{f'(c)} = \frac{1}{2+3c^2}$$

Theorem:

Let $I \subseteq R$, $f: I \to R$ be a strictly monotone and continuos on I. Let $J := f(I)$. J is an interval and let $g: J \to R$ be the inverse of f. Then g is strictly monotone and continuos. If f is differentiable at c and $f'(c) \neq 0$ then g is differentiable at $f(c)$ then,

$$g'(f(c)) = \frac{1}{f'(c)}$$

Exm: (i). $x^3 + 3x + 1 = f(x)$

$$f'(c) = 3c^2 + 3 > 0$$

~~let~~, $f(\mathbb{R}) = \mathbb{R}$

let $g: R \to R$ be the inverse

Then g is strictly monotonic and continuos.

$$g'(f(c)) = \frac{1}{3(c^2+1)}$$

~~f is onto ⇒ g'(y) = 1~~

(ii). $f(x) = x^n$, n is odd, $n \neq 1$

f is strictly increasing & continuos

So, $f(\mathbb{R}) = \mathbb{R}$

$x_1 < x_2$

Then $x_1 < 0$, $x_1^n < 0$

$x_2 > 0$, $x_2^n > 0$

g is strictly monotone and continuos.

$$f'(c) = nc^{n-1}$$

If $c \neq 0$, g is diff at $f(c)$ and,

$$g'(f(c)) = \frac{1}{f'(c)} = \frac{1}{nc^{n-1}}$$

$$g'(y) = \frac{1}{n y^{(n-1/n)}} \quad , y \neq 0$$

$$c^n = y \Rightarrow c = y^{1/n}$$