04/03/20

Linear second order PDE:

$$Au_{xx} + Bu_{xy} + Cu_{yy} + au_x + bu_y + cu = d(x,y)$$

$A, B, C, c, a, b, d \longrightarrow$ Functions of x, y

1). Hyperbolic: $B^2 - 4AC > 0$

$$(x,y) \longrightarrow (\xi, \eta)$$

$$u_{\xi\eta} = G(u, u_\xi, u_\eta, \xi, \eta)$$

$$u_{\alpha\alpha} - \gamma u_{\beta\beta} = G(u, u_\xi, u_\eta, \alpha, \beta)$$

2). Parabolic: $B^2 - 4AC = 0$

$$u_{\eta\eta} = G(u, u_\eta, \eta, u_\xi, \xi)$$

3). Elliptic PDE: $B^2 - 4AC < 0$ / $\xi, \eta \longrightarrow$ complex $\alpha \pm i\beta$

~~$u_{\xi\xi} + u_{\eta\eta}$~~

$$u_{\alpha\alpha} + u_{\beta\beta} = G(u, u_\alpha, u_\beta, \alpha, \beta).$$

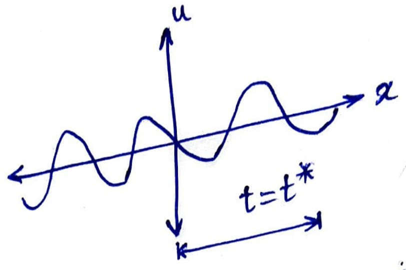
$(\xi, \eta) \longrightarrow$ canonical variable

Some realistic PDEs.

wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad , \quad t>0, \quad -\infty < x < \infty$$

Hyperbolic $\downarrow$ Toughest



$\alpha = x + ct \; ; \; \beta = x - ct$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \alpha} \cdot c + \frac{\partial u}{\partial \beta}(-c)$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}(u_\alpha) . c + (-c)\frac{\partial}{\partial t}(u_\beta)$$

$$= c\left(c u_{\alpha\alpha} - c u_{\alpha\beta}\right) + (-c)\left(c u_{\alpha\beta} - c_{\beta\beta}\right)$$

$$= c^2 u_{\alpha\alpha} + c^2 u_{\beta\beta} - 2c^2 u_{\alpha\beta}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \alpha} + \frac{\partial u}{\partial \beta}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(u_\alpha) + \frac{\partial}{\partial x}(u_\beta) = u_{\alpha\alpha} + u_{\beta\beta} + 2u_{\alpha\beta}$$

$\Rightarrow \quad c^2 u_{\alpha\beta} = 0 \Rightarrow \boxed{u_{\alpha\beta} = 0}$

$\Rightarrow u_\alpha = f(\alpha)$

$\Rightarrow \boxed{u = F(\alpha) + G(\beta)}$

$\Rightarrow u(x,t) = F(x+ct) + G(x-ct)$

$t=0, \; u(x,0) = f(x)$

$\Rightarrow f(x) = F(x) + G(x)$