V – set of all continuous funcs from $\mathbb{R}$ to $\mathbb{R}$
↳ infinite dimensional

$T: V \to V$ defined by

$$T(f(x)) = \int_0^x f(t)\,dt$$

$f_1\ f_2 \in V$

$$T = c_1 f_1(x) + c_2 f_2(x)$$

$$= \int_0^x (c_1 f_1(t) + c_2 f_2(t))\,dt$$

$$= c_1 T_1(f_1) + c_2 T_2(f_2)$$

Q. st T has no eigenvalue (characteristic values)

Sol: $T(f(x)) = \lambda f(x) = \int_0^x f(t)\,dt$

$$\Rightarrow \quad \lambda f'(x) = f(x)$$

$$\int \frac{f'(x)}{f(x)} = \int \frac{1}{\lambda}$$

$$\ln|f(x)| = \alpha + \frac{x}{\lambda}$$

$$f(x) = A e^{x/\lambda}$$

$$A\lambda e^{x/\lambda} = \lambda A e^{x/\lambda} - A\lambda$$

0 is not eigen value:

$$T(f(x)) = 0 f(x)$$

$$\Rightarrow \int_0^x f(t)\,dt = 0 \quad \forall x \in \mathbb{R}$$

$$\Rightarrow f(x) = 0 \quad \forall x \in \mathbb{R}.$$

Then $\exists$ $(c-\delta, c+\delta)$ → $f(x) > 0$ → $\int_{c-\delta}^{c+\delta} > 0$

→ $f(x) < 0$

$$\int_0^{c+\delta} - \int_0^{c-\delta} = 0 \Rightarrow \int_{c-\delta}^{c+\delta} = 0$$

But → $\int_{c-\delta}^{c+\delta} > 0$ → $\int_{c-\delta}^{c+\delta} < 0$

$\therefore \nexists c$ st $f(c) \neq 0$.

$\therefore f(x) = 0 \ \forall x \in \mathbb{R}$

But Eigen vector $\neq 0$.

$\lambda \neq 0$ is an eigen value:

$T(f(x)) = \lambda f(x)$

for some non-zero $x \in V$

$\Rightarrow \int_0^x f(t)dt = \lambda f(x) \quad \forall x \in \mathbb{R}$.

$f(0) = 0$

$f(x) = \lambda f'(x)$

$f(x) = ce^{x/\lambda}$

But $f(0) = 0 \Rightarrow f(x) = 0$

But $f(x) \neq 0$.

$V = M_3(\mathbb{R})$

let $A \in M_3(\mathbb{R})$

$T: V \to V$, st $T(B) = AB - BA$

show that if A is diagonalisable then T is diagonalisable.

Sol: $\exists P$ st $P^{-1}AP = D$.

$S: V \to V$

$S(B) = P^{-1}BP \quad \forall B \in M_3(F)$

$S(B) = 0 \Rightarrow \boxed{B=0}$

$\forall B \in V \quad \exists X \ S(X) = B$. ✓ $\quad X = PXP^{-1}$

since, A is diagonalisable $\exists P$ st

$P^{-1}AP = D$ - diagonal

$T(B) = DB - BD$

$U = S \circ T \circ S^{-1}$

$S^{-1}(B) = PBP^{-1}$

$T(S^{-1}(B)) = DPBP^{-1} - PBP^{-1}D$

$S(T(S^{-1}(B))) = P^{-1}(DPBP^{-1} - PBP^{-1}D)P$

$\cancel{= PDPBP^{-1}P^{-1}}$

$= P^{-1}DPB - BP^{-1}DP$

$S(B) = PBP^{-1}$ ✓

$S^{-1}(B) = P^{-1}BP$

$T(S^{-1}(B)) = D(P^{-1}BP) - (P^{-1}BP)(D)$

$S(T(S^{-1}(B))) = P(DP^{-1}BP - P^{-1}BPD)P^{-1}$

$= \cancel{PDB}P \quad PDP^{-1}BPP^{-1} - BPDP^{-1}$

$= PDP^{-1}B - BPDP^{-1}$

$= \underline{\underline{AB - BA}}$

$\therefore \boxed{S \circ T \circ S^{-1} = U}$

$\mathcal{B} = \{E_{ij}\}$

$S \circ T \circ S^{-1} : V \to V$

$S^{-1}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} =$

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~~E[U] =~~ 

$(V, \langle , \rangle)$ — FIP

$f : V \to F$, linear functional

Then there exists a unique vector $\beta \in V$

st $f(\alpha) = \langle \alpha, \beta \rangle \ \forall \alpha \in V$.

let $\{\alpha_1, \ldots, \alpha_n\}$ be orthonormal basis.

$$\beta = \sum_{i=1}^{n} \overline{f(\alpha_i)}\, \alpha_i$$

$$\langle \alpha_i, \beta \rangle = \overline{\overline{f(\alpha_i)}} = f(\alpha_i) \qquad \alpha = \sum_{i=1}^{n} c_i \alpha_i$$

$$\begin{aligned} \langle \alpha, \beta \rangle &= \sum c_i f(\alpha_i) \\ &= \sum f(c_i \alpha_i) \\ &= f\left(\sum c_i \alpha_i\right) = \underline{f(\alpha)} \end{aligned}$$

suppose there exists two such β. say β_1 and β_2.

$$\langle \alpha, \beta_1 \rangle = \langle \alpha, \beta_2 \rangle \quad \forall \alpha \in V$$

$$\Rightarrow \langle \alpha, \beta_1 - \beta_2 \rangle = 0 \quad \forall \alpha \in V$$

$$\boxed{\beta_1 - \beta_2 = 0} \quad \therefore \boxed{\beta_1 = \beta_2}$$

not always true for infinite dimensional space.

$f: R^3 \longrightarrow R$

$f(x_1, x_2, x_3) = 2x_1 + 5x_2 + 3x_3$

$\beta = [2\ 5\ 3]^T$

$(V, <,>)$ IPS

$T: V \to V$ linear operator

A linear operator $T_1: V \to V$ is said to be adjoint of T if $<T(\alpha), \beta> = <\alpha, T_1(\beta)> \quad \forall\ \alpha, \beta \in V$

Adjoint of T is denoted by T^*

$T: R^2 \to R$

$T(x_1, x_2) = (x_1 + x_2, x_1 - x_2)$

Find adj(T).

$<T(\alpha), \beta> = <\alpha, T_1(\beta)>$

$T_1(\beta) = <c_1\beta_1 + c_2\beta_2, d_1\beta_1 + d_2\beta_2>$

$<\alpha, T_1(\beta)> = \alpha_1(c_1\beta_1 + c_2\beta_2) + \alpha_2(d_1\beta_1 + d_2\beta_2>$

$<T(\alpha), \beta> = (\alpha_1 + \alpha_2)\beta_1 + (\alpha_1 - \alpha_2)\beta_2$

$\therefore\ T_1(\beta) = <\beta_1 + \beta_2, \beta_1 - \beta_2)$

1. Does T always have T^*

2. If T has T^*, then is it unique?

$T: V \to V$, linear operator, then T always have a unique adjoint operator.

May not be true for infinite dimensional.

Take $\beta \in V$ define

$$f_\beta(\alpha) = \langle \alpha, \beta \rangle \quad \forall \alpha \in V$$

f_β is linear functional for each $\beta \in V$.

$! \exists \beta_1 \in V$ st

$$f_\beta(\alpha) = \langle \alpha, \beta_1 \rangle \quad \forall \alpha \in V.$$

$T_1 : V \to V$ such that

$$T_1(\beta) = \beta_1 \quad \forall \beta \in V.$$

①. T_1 is linear

②. $\langle T(\alpha), \beta \rangle = \langle \alpha, T_1(\beta) \rangle \quad \forall \alpha, \beta \in V.$

<u>step</u> for each $\beta \in V$

$$f_\beta(\alpha) = \langle T(\alpha), \beta \rangle$$

for f_β $\exists$ unique $\beta_1 \in V$ st

$$f_\beta(\alpha) = \langle \alpha, \beta_1 \rangle \text{ for all } \alpha \in V.$$

$T_1 : V \to V \qquad T_1(\beta) = \beta_1 \quad \forall \beta \in V.$

$\beta', \beta'' \in V$

$T_1(\alpha_1\beta' + \alpha_2\beta'') = \alpha_1\beta_1' + \alpha_2\beta_2'$

$f_{\beta'}(\alpha) = \langle T(\alpha), \beta' \rangle \ \forall \alpha \in V$

$f_{\beta''}(\alpha) = \langle T(\alpha), \beta'' \rangle \ \forall \alpha \in V$

$(V, \langle , \rangle)$ — FDIPS

$T: V \to V$ linear operator

then T has adjoint operator T^*

Let $\beta \in V$

$f: V \to F$ define

$f(\alpha) = \langle T(\alpha), \beta \rangle$

$\exists \beta' \in V$ st $f(\alpha) = \langle \alpha, \beta' \rangle \ \forall \alpha \in V$

$T_1: V \to V$ defined by,

$T_1(\beta) = \beta'$ for all $\beta \in V$.

we want to show that T_1 is adjoint of T.

$f_{\beta_1}(\alpha) = \langle T(\alpha), \beta_1 \rangle \quad \forall \alpha \in V$

$f_{\beta_2}(\alpha) = \langle T(\alpha), \beta_2 \rangle \quad \forall \alpha \in V$

$\exists$ unique β_1' st $f_{\beta_1}(\alpha) = \langle \alpha, \beta_1' \rangle$

$\exists$ unique β_2' st $f_{\beta_2}(\alpha) = \langle \alpha, \beta_2' \rangle$

$\langle \alpha, T_1(\alpha_1\beta_1 + \alpha_2\beta_2) \rangle =$ ~~$\langle T(\alpha), \alpha_1 \beta_1 \ldots$~~

$= \bar{\alpha}_1 \langle \alpha, \beta_1' \rangle + \bar{\alpha}_2 \langle \alpha, \beta_2' \rangle$

$= \bar{\alpha}_1 \langle T(\alpha), \beta_1 \rangle + \bar{\alpha}_2 \langle T(\alpha), \beta_2 \rangle = \langle \alpha, \alpha_1\beta_1' + \alpha_2\beta_2' \rangle$

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$(V, \langle\ \rangle_1)$ – FDIPS

$(V, \langle\ \rangle_2)$ – FDIPS

$T: V \to V$

T^* is adjoint of T wrt to $\langle , \rangle_1$

T^* is also adjoint for T wrt $\langle , \rangle_2$ or not?

No. Adjoint of an operator depends on IP.

$(V, \langle , \rangle) \longrightarrow (V, \langle , \rangle)$

infinite

May or may not have adjoint.

* FDIPS

$\{\alpha_1, \ldots, \alpha_n\}$ – orthonormal basis.

$T: V \to V$ linear operator.

$[T]_B = A$

$A_{ij} = \langle T(\alpha_j), \alpha_i \rangle$

$$\alpha = \sum_{i=1}^{n} \langle \alpha, \alpha_i \rangle \alpha_i$$

$$T(\alpha_j) = \sum_{i=1}^{n} A_{ij} \alpha_i$$

$$\langle T(\alpha_j), \alpha_i \rangle = A_{ij}$$

$T: V \to V$ $(V, \langle\ \rangle)$ FDIP

T has adjoint T^*

$T^*: V \to V$ linear operator.

$[T]_B = A$ $[T^*]_B = B$

$A_{ij} = \langle T(\alpha_j), \alpha_i \rangle$

$B_{ij} = \langle T_i(\alpha_j), \alpha_i \rangle = \langle \alpha_j, T(\alpha_i) \rangle = \overline{\langle T(\alpha_i), \alpha_j \rangle} = [T^*]_{B(ji)}$

$\boxed{B = A^*}$

Not true if basis is not orthonormal.

<u>Def</u>: $T: V \to V$

T is called self adjoint if $T = T^*$

e.g; $T: R^2 \to R^2$

$T(x_1, x_2) = (x_1 + x_2, x_1 - x_2)$

<u>Th</u>: $(V, \langle\ \rangle)$ FDIPS

$B = \{\alpha_1, \ldots, \alpha_n\}$ $[T] = A$

$T: V \to V$ linear

T is self adjoint iff $A^* = A$.

T is self adjoint iff A is hermitian.

$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \to$ symmetric $\to$ orthonormal

$(0,1)$ $(1,0)$

$A = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \to$ Not symmetric $\to$ not orthonormal

Def^n $T: V \to V$ $(V, \langle\ \rangle)$ FDIP

$T, U: V \to V$, linear operator

(i). $(T+U)^* = T^* + U^*$

(ii). $(cT)^* = \bar{c}\, T^*$

(iii). $(T \circ U)^* = U^* \circ T^*$

Def^n $(V, \langle , \rangle)$ FDIP $T: V \to V$ LT

Then T is called unitary operator if and only if

$T^*T = TT^* = I$

Th: T is unitarry iff A is unitary.

Example of unitary operator:

I - identity operator.

Eigenvalues of unitary operator are unit modulus.

Th: $T: V \to V$ unitary operator

$\langle T(x), T(y) \rangle = \langle x, y \rangle$

$\langle x, T^*T(y) \rangle = \langle x, y \rangle$

unitary operator preserves the inner product.

Def^n: $(V, \langle , \rangle)$ FDIP

$T: V \to V$ LO. Then T is called normal operator

if $T^*T = TT^*$.

Th: T is normal iff A is normal.

Normal operator is always diagonalisable.

Example of one normal operator which is neither self adjoint nor unitary.

1. Jordan cannonical form:

2. Definiteness of Hermitian matrix

$A \in M_n(\mathbb{C})$

$x \in \mathbb{C}^n$

$x^* A x$

$q(x_1, x_2) = x_1^2 + x_1 x_2 + x_2^2 = x^T A x$

$\hookrightarrow$ write as a matrix!

$$(x_1 \;\; x_2)\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1^2 + x_1 x_2 + x_2^2$$

$$(x_1 \;\; x_2)\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} =$$

$$q(x_1, \ldots, x_n) = a_{11}x_1^2 + a_{22}x_2^2 + \ldots + a_{nn}x_n^2$$
$$+ a_{12}x_{12}^2 + \ldots$$

$c_1 x_1^2 + \ldots + c_n x_n^2$ — Diagonal form

$P^* A P = D = \mathrm{diag}(\lambda_1, \ldots \lambda_n)$

Definiteness: $q \geq 0$, $q > 0$, $q < 0$, $q \leq 0$.

A - Hermitian

$x^*Ax \longrightarrow$ Real

$A \in M_n(\mathbb{C})$

x^*Ax is real for any $x \in \mathbb{C}^n$

then A is Hermitian.

$(e_i + ie_j)^* A (e_i + ie_j) = a_{ii} + a_{jj} - ia_{ji} + ia_{ij} =$ real

$(e_i + e_j)^* A (e_i + e_j) = a_{ii} + a_{jj} + a_{ij} + a_{ji} =$ real

$e_i^* A e_i = a_{ii}$ — real

for $i = 1, 2, \ldots, n$

(1) $\leftarrow a_{ij} + a_{ji} =$ real

$\leftarrow ia_{ij} - ia_{ji} =$ real

$a_{ij} = x_1 + iy_1$

$a_{ji} = x_2 + iy_2$

$\Rightarrow y_1 + y_2 = 0$

$x_1 - x_2 = 0 \Rightarrow x_1 = x_2$

$a_{ij} = \overline{a_{ji}}$ $\quad i = 1, 2, 3, \ldots$
$\quad j = 1, 2, 3, \ldots$
$\quad i \neq j$

$A \in M_n(\mathbb{C})$

$A^* = A \longrightarrow$ Hermitian

$A \in M_n(\mathbb{R})$

$A^T = A \longrightarrow$ Real symmetric

$A \in M_n(\mathbb{C})$

$A^T = A \longrightarrow$ complex symmetric

Ex: Whatever result true for Hermitian are those true for complex symmetric matrix.

def^n: A-Hermitian

A is said to be positive definite or PS if

$x^*Ax > 0$ for all non-zero $x \in \mathbb{C}^n$

Thm:

A Hermitian A is PD iff all eigenvalues of A are positive.

$\exists$ a unitary matrix U such that $U^*AU = D$.

$$x^*Ax = x^*UDU^*x$$
$$= (x^*U)D(xU^*)^*$$
$$= yDy^* = \sum_{i=1}^{n} \lambda_i |y_i|^2 > 0$$

$\therefore$ A is positive definite.

$\lambda_i \quad x_i$

$$x_i^*Ax_i = \lambda_i x_i^* x_i \; (\neq 0)$$
(>0)

$\Rightarrow \boxed{\lambda_i > 0} \quad \forall \; i = 1, 2, \ldots, n$

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Th: A-Hermitian, A is PD iff $\exists$ a non-singular $B \in M_n(\mathbb{C})$ such that $A = B^*B$.

$\Rightarrow$ A is PD

$\exists$ a unitary U such that $A = UDU^*$

$A = UDU^* = B^*B$

$= UD^{1/2}D^{1/2}U^* = B^*B$

$$D = \begin{pmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ \cdots & \cdots & \cdots \end{pmatrix} \quad \sqrt{D} = \begin{pmatrix} \sqrt{d_{11}} & 0 & 0 \\ 0 & \sqrt{d_{22}} & 0 \\ 0 & \cdots & \end{pmatrix}$$

$\boxed{B = \sqrt{D}U^*}$

$\Leftarrow$ $x^*Ax = x^*B^*Bx$

$= y^*y > 0$

for non-zero $x \in \mathbb{C}^n$

A is PD.

Th: A-Hermitian

A PD iff all principle minor of A is positive.

check definition of principle minors!

Th: A-Hermitian

A is PD iff leading principal minors are positive.

$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3 < 0$

Not PD

Defn: A - Hermitian

A is said to be positive semidefinite $x^* A x \geq 0$

$\forall$ non-zero $x \in \mathbb{C}^n$.

PD $\Rightarrow$ PSP
$\not\Leftarrow$

A - Hermitian

A is PSD iff:

i). all the eigen values of A are non-negative

ii). $A = B^* B$ for some $B_{n \times n}$.

iii). All possible minor of A are non-negative.

*iv). All leading principal minor of A are non-negative

Negative definite:

if $x^* A x < 0$ $\forall$ non-zero $x \in \mathbb{C}^n$

A is ND iff

i). All the eigen values of A are negative

ii). is not true for ND.

iii). is not true for ND

iv). is not true for ND