OPERATIONS RESEARCH

Optimisation, Theory & Applications By S.S. Rao.

Engineering Optimization: Theory & Practice. By S. S. Rao.

What you do in optimization?

maximize or minimize using constraints etc.

We have done for non-linear.

Here, we are doing for linear.

LPP - (Linear Programming Problem)

① max : $Z = 2x_1 + 3x_2$

s.t. $3x_1 + 2x_2 = 6$

$x_1, x_2 \geq 0$

We want 1 basic solution.

$x_1 = 0$, $x_2 = 3$, $Z = 9$ This max

② min $Z = 2x_1 + 3x_2$

s.t. $3x_1 + 2x_2 = 6$

$x_1, x_2 \geq 0$

$x_1 = 2$, $x_2 = 0$, $Z = 4$ min

General Problem

max. $Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$

s.t. $a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$, $b_1 \geq 0$

$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$, $b_2 \geq 0$

$\vdots$

$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$, $b_m \geq$

$x_1, x_2, \dots x_m \geq 0$

Model I :

$$\max : Z = \sum_{j=1}^{n} c_j x_j$$

$$\text{s.t.} \quad \sum_{j=1}^{n} a_{ij} x_j = b_i \,, \quad i = 1, 2, \ldots, m \quad b_i \geq 0$$

$$x_j \geq 0 \,, \quad j = 1, 2, \ldots, n$$

$$\max : Z = C^T X$$

$$\text{s. to} \quad AX = b$$

$$X \geq 0$$

$$C = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

Model 2 :

$$\min \; Z = \sum_{j=1}^{n} c_j x_j$$

$$\text{s. to.} \quad \sum_{j=1}^{n} a_{ij} x_j = b_i \,, \quad b_i \geq 0 \;\; \forall i \quad i = 1, 2, \ldots$$

$$x_j \geq 0 \;\; \forall j \quad j = 1, 2, \ldots, n$$

Example (slides)

$$\max : Z = 10x_1 + 20x_2 + 30x_3$$

subject to $\quad x_1 + x_2 + x_3 = 60 \qquad$ 2 constraints

$$x_1 + 5x_2 + 10x_3 = 410$$

$$x_1, x_2, x_3 \geq 0$$

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an: $Z = 10x_1 + 20x_2 + 30x_3$

s.to $2x_1 - 3x_2 + 5x_3 = 30$

$3x_1 + 5x_2 - 2x_3 = 40$

$x_1, x_2, x_3 \geq 0$

Check whether they are linearly dependent or independent

i) Non-basic variables are always 0.

sic
sible solⁿ

NBV	BV (Basic variable)	solⁿ
$x_1 = 0$	$x_2 = 38$, $x_3 = 22$	$Z = 1420 \rightarrow$ max
$x_2 = 0$	$x_1 = 190/9$, $x_3 = 350/9$	$z = \frac{12400}{9}$
$x_3 = 0$	$x_1 = -55/2$, $x_2 = 175/2$	$\rightarrow$ Not a BFS so don't calculate value of Z.

BFS

$x_2 + 5x_3 = 30$

$x_2 - 2x_3 = 40$

$\frac{\begin{vmatrix} 30 & 5 \\ 40 & -2 \end{vmatrix}}{\begin{vmatrix} -3 & 5 \\ 5 & -2 \end{vmatrix}} = \frac{-260}{-19} = \frac{260}{19}$ BFS

$\frac{1900}{9} + \frac{10500}{9}$

$\frac{2950}{9}$

$3 \; \frac{\begin{vmatrix} -3 & 30 \\ 5 & 40 \end{vmatrix}}{\begin{vmatrix} -3 & 5 \\ 5 & -2 \end{vmatrix}} = \frac{-120-150}{-19} = \frac{-270}{-19}$

$x_3 = 0$

$2x_1 - 3x_2 = 30$

$3x_1 + 5x_2 = 40$

$\frac{\begin{vmatrix} 30 & -3 \\ 40 & 5 \end{vmatrix}}{\begin{vmatrix} 2 & -3 \\ 3 & 5 \end{vmatrix}}$ $\frac{150+120}{19}$ $\frac{270}{19}$

$\begin{vmatrix} 2 & 30 \\ 3 & 40 \end{vmatrix} = \frac{-10}{19}$

$x_2 = 0$ $2x_1 + 5x_3 = 30$

$3x_1 - 2x_3 = 40$

$x_1 = \frac{\begin{vmatrix} 30 & 5 \\ 40 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & 5 \\ 3 & -2 \end{vmatrix}} = \frac{-260}{-19} = \frac{260}{19}$

$\frac{\begin{vmatrix} 2 & 30 \\ 3 & 40 \end{vmatrix}}{\begin{vmatrix} 2 & 5 \\ 3 & -2 \end{vmatrix}} = \frac{10}{19}$

$z =$

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BFS means everything has to be non negative.
Basic solutions means that it can be anything.

LPP either has 1
none
or infinitely many soln

Operation Research by
H A Taha

Linear programming vol 1
George B Dantzig & Mu
T
L.P vol 2

Study of Business operations to improve the efficiency in industry is known as OR.

variance - 2nd moment.

normal Distribution

lognormal distribution

central limit theorem

finding maximum profit / production / performance
or finding minimum cost / loss / risk of some
world decision making problems.

Type	Capital	Rajdhani	Hostel eqn
T	20	10	200
D	18	24	300
C	25	50	500
Cost per package	Rs 20,000	Rs 30,000	—

$20x + 18y + 25z = 20{,}000$

$10x + 24y + 50z = 30{,}000$ ~~VTA~~ Not these variables

$x \geq 200$ $y \geq 300$ $z \geq 500$

Let x = no. of units of furniture from Capital

Let y = no. of units of furniture from Rajdhani

These variables are known as control variables (under control of the mathematician)

LP model can be formulated as :

Min cost in Rs $= 20{,}000x + 30{,}000y$

$$20x + 10y \geq 200 \quad (1)$$

$$18x + 24y \geq 300 \quad (2)$$

$$25x + 50y \geq 500 \quad (3)$$

$$x, y \geq 0 \text{ and integers} \quad \text{-- } (4)$$

Restrictions imposed on problem.

Inequalities

$$2x + y \geq 20 \quad - (5)$$

$$3x + 4y \geq 50 \quad - (6)$$

$$x + 2y \geq 20 \quad - (7)$$

Not solving $\leftarrow$ $\boxed{AX = b}$ $AX = 0$

Only checking

Let's say not a sq sol^n or no sol^n

This system of linear equation.

Always has a trivial sol^n

$m = 3 \quad n = 5$

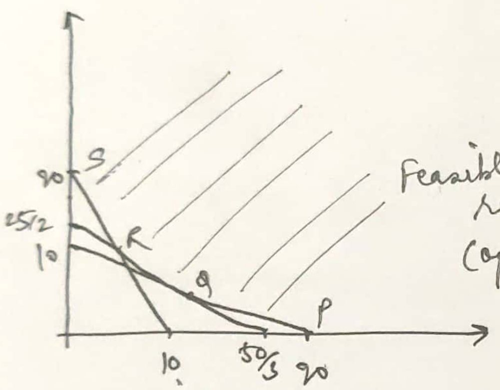
Out of 5, any 3 non-zero, remaining 0

$5 - 3 = 2$

Any 2 variables are 0.

$^5C_2 = {^5C_3} = \frac{5 \times 4}{2} = 10$

① Choosing control variables
② Whether to max. or min.
③ According to requirement, we go for constraints.



Extreme points are
P(20, 0) Q(10, 5) R(6, 8) S(0, 20)

At P, cost of furniture = ₹ 400,000

Q, —"— = 350,000 → min value at $x = 10$, $y = 5$

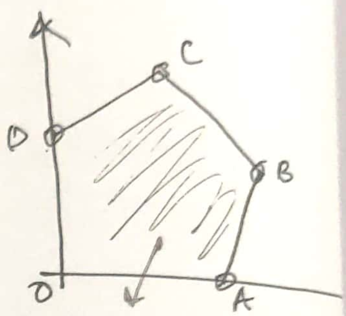
R, —"— = 360,000

S, —"— = 600,000

seeing if it satisfies inequ...

T - 250
D - 300
C - 500

at a cost of ₹ 350,000



Infinite pts. in feasible region

every BFS - correspon... extreme pt. + optimum solⁿ oc... at the extrem... only.

It has 1, no or infinitely ma... solⁿ like a linear system

Finding Basic Feasible Solutions of a Linear System Equations

$AX = b$ be of m linear equations with n variables

$(n > m)$ where A is a real matrix of size m by n,

X is a column vector having n elements, ie

$X = (x_1, x_2, x_3 \ldots x_n)^T$, and b is a non-zero

column vector having m elements

$(n > m)$

The system is consistent if

(BV)

If $AX = b$ is consistent, select any m variables and set remaining $(n-m)$ variables to zero.

After setting $(n-m)$ variables to zero the system $AX = b$ becomes $BX_B = b$ where B is a non-singular matrix of order m (i.e $|B| \neq 0$) and X_B is a column vector with m elements.

If a sol^n

$$X_B = B^{-1} b$$

Basic sol^n

Max no. of possible basic sol^n

$$^nC_m = {}^nC_{n-m}$$

If all basic variables non-negative - Basic Feasible sol^n.

Max : $Z = x_1 + 4x_2$

$$x_1 + x_2 \le 10$$
$$x_1 + 4x_2 \le 16$$

Introducing slack variables x_3, x_4

$$x_1 + x_2 + x_3 = 10$$
$$x_1 + 4x_2 + x_4 = 16$$
$$x_1, x_2, x_3, x_4 \geq 0$$

S.No	Non Basic variables	Basic variables
1.	$x_1 = 0, x_2 = 0$	$x_3 = 10, x_4 = 6$
2.	$x_1 = 0, x_3 = 0$	$x_2 = 10, x_4 = -24$
3.	$x_1 = 0, x_4 = 0$	$x_2 = 4, x_3 = 6$
4.	$x_2 = 0, x_3 = 0$	$x_1 = 10, x_4 = 6$
	$x_2 = 0, x_4 = 0$	$x_1 = \ , x_3 =$
	$x_3 = 0, x_4 = 0$	$x_1 = \ , x_2 =$

6 Basic soln, Only 4 are BFS

optimal solⁿ is obtained

Sl. no. 3, i.e. $x_1 = 0, x_2 = 4$

6. ie $x_1 = 8, x_2 = 2$

with $Z = 16$

$$X^{(1)} = \begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$

$$X^{(2)} = \begin{pmatrix} x_1^{(2)} \\ x_2^{(2)} \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}$$

Exceptional

$x_1 + x_2 + x_3 = 10$

$2x_1 + 2x_2 + x_3 = 19$

$x_3 = 0$ NBV

$x_1 + x_2 = 10$

$2x_1 + 2x_2 = 19$

Not fulfilled, becau

Matrix B is a sing

$$x^* = \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ 4 \end{pmatrix} + (1-\lambda) \begin{pmatrix} 8 \\ 2 \end{pmatrix}$$

$z = 16$

$\lambda = 1/2 \quad \begin{pmatrix} 4 \\ 3 \end{pmatrix} \quad z = 16$

$\lambda = \ldots$

R. Model II

H.E. 6 hours continuous

work requirement given

x_1 = no. of H.E. in period 1

x_2 = ——— // ——— 2

x_3 = ——— // ——— 3

x_4 = ——— // ——— 4

$$\text{Min} = x_1 + x_2 + x_3 + x_4 \quad \text{(Mgmt. will want this)}$$

s.t.

$$x_1 \geq 8$$
$$x_1 + x_2 \geq 12$$
$$x_1 + x_2 + x_3 \geq 9$$
$$x_2 + x_3 + x_4 \geq 12$$
$$x_3 + x_4 \geq 10$$
$$x_4 \geq 7$$
$$x_1, x_2, x_3, x_4 = 0, 1, 2, 3 \ldots$$

n integers, sol^{n}s are not infinite (they are countable)

(B.F.S.)

OR Lab - I (Questions)

① Find the basic solution (s) and the Basic Feasible soln of the linear system $AX = b$ where $n > m$.

Also write a program in C or C++ to find the Basic soln and the B.F.S. of $AX = b$.

$$A = (a_{ij})_{m \times n} \qquad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

② a) $\max Z = C^T X$ s.t. $AX = b$, $X \geq 0$

b) $\min Z = C^T X$ s. to. $AX = b$, $X \geq 0$

Find the B.F.S. of $AX = b$. Then find the optimum soln of (a) and (b). Also write a program in C or C++ to find the optimal soln.

③ Find the optimal soln of

a) $\max : Z = C^T X$ s. to. $AX \leq b$, $X \geq 0$

$\boxed{\begin{matrix} AX < b \\ AX > b \end{matrix}}$ X

b) $\min : Z = C^T X$ s. to. $AX \geq b$, $X \geq 0$

c) $\max : Z = C^T X$ s. to. $AX \, (*) \, b$, $X \geq 0$

$* \, (\leq, \geq, =)$

d) $\min : Z = C^T X$ s. to. $AX \, (*) \, b$, $X \geq 0$ $\quad * \Rightarrow \leq = \geq$

-ily you can change to integer programming problem.
All the variables are integers.

, Find $\underset{\sim}{X} = (x_1, x_2, \ldots, x_n)^T$

so as to

$$\min: Z = c^T X + d$$

$$\text{s.to. } AX \geq b$$

$$X \geq 0$$

$$x_j = 0, 1, 2, \ldots$$ Integer programming problem

-imum seeking methods are known as Optimization Techniques. It is a part of Operations Research.

Linear model — Non-linear models

~~Im~~ If no restrictions, variables are free. (>0, <0,)

-ax: $Z = 2x_1 + 5x_2$

s.to $x_1 + x_2 \leq 100$

let $x_1 = y_1 - y_2$ $\quad y_1, y_2 \geq 0$

$x_2 = y_3 - y_4$ $\quad y_3, y_4 \geq 0$

$\Rightarrow$ max: $Z = 2y_1 - 2y_2 + 5y_3 - 5y_4$

s.to. $y_1 - y_2 + y_3 - y_4 \leq 100$

$y_1, y_2, y_3, y_4 \geq 0$

size of the matrix has increased.

The variables are restricted, but no. of variables have increased

LPP - I max : Z

LPP - II min : Z (minimization type problem can be solved using maximization type problem)

Generally, in LP – we solve max. prob.
NLP – we solve min. prob.

a) Linear Programming Problem (LPP)
b) Integer Programming Problem (IPP)
c) Quadratic Programming Problem (QPP)

$$\text{eg. max: } f(x) = X^T Q X + c^T X + d$$
$$\text{s.to } g(x) \leq b$$
$$X \geq 0$$

d) Geometric Programming Problem (GPP)
(has been developed using the inequality AM $\geq$ GM)
e) Goal Programming Problem (GPP)
f) Dynamic Programming Problem (DPP)
g) Separable Programming Problem (SPP)
h) Stochastic Programming Problem (SPP)
i) Fuzzy Programming Problem (FPP)
j) Binary Programming Problem (BPP) (variables can be linear / non-linear but they are binary)

Interior Point Method (1984) – developed for large size optimization problem.
(Karmarkar)

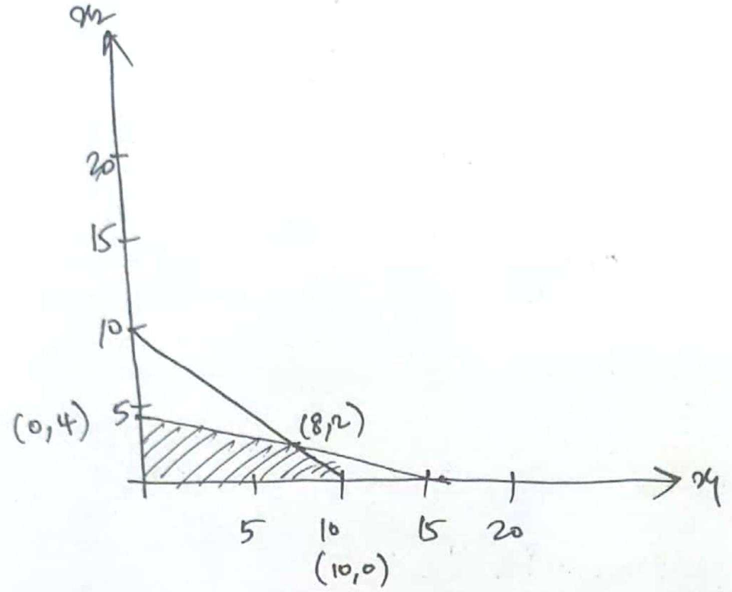
Graphical method for solving LPP :.

$$\max : Z = x_1 + 3x_2$$

subject to $\quad x_1 + x_2 \le 10$

$$x_1 + 4x_2 \le 16$$

$$x_1, x_2 \geq 0$$



S.No.	x_1	x_2	Z	
1.	0	0	$Z=0$	min
2.	10	0	$Z=10$	
3.	6	4	$Z=12$	
4.	8	2	$Z=14$	max

max. value of $Z^* = 14 \qquad x_1^* = 8 \quad x_2^* = 2$

Only one optimal solution

imal is feasible, dual is feasible.

~~min~~ Duality thm.

$$\min : z_1 \geq \max : z$$

S.D.T

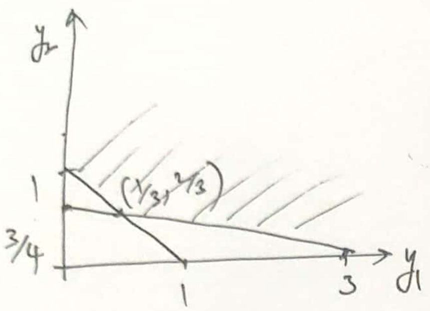
$$\min \cdot z_1 = \max z = 0$$

Dual LPP

$\min: z_1 = 10y_1 + 16y_2$

s. to. $y_1 + y_2 \geq 1$

$y_1 + 4y_2 \geq 3$



z_1	y_1	y_2
16	0	1
14	1/3	2/3
30	3	0

Complementary slagness principle

$(x_1 + x_2 - 10)y_1 = 0$

$(x_1 + 4x_2 + 16)y_2 = 0$

$(y_1 + y_2 - 1)x_1 = 0$

$(y_1 + 4y_2 - 3)x_2 = 0$

$y_1^* = 1/3 \quad y_2^* = 2/3$

$x_1^* = 8 \neq 0$

$x_2^* = 2 \neq 0$

Primal Program (1)

$\max: z = x_1 + 3x_2$

s.t. $x_1 + x_2 \leq 100$

$x_1 + 2x_2 \leq 110$

$x_1 + 4x_2 \leq 160$

$x_1, x_2 \geq 0$

(Finding in dual variables)

3.diff dual variables $y_1, y_2, y_3 \geq 0$
(because 3 constraints)

$\min z_1 = 100y_1 + 110y_2 + 160y_3$

a) max : $Z = \sum_{j=1}^{n} c_j x_j$

s.to $\sum_{j=1}^{n} a_{ij} x_j \le b_i \quad i = 1, \ldots m$

$x_j \geq 0 \quad j = 1, \ldots n$

b) min : $Z = \sum_{j=1}^{n} c_j x_j$

s. to - $\sum_{j=1}^{n} a_{ij} x_j \le b_i \quad i = 1, \ldots m$

$x_j \geq 0 \quad j = 1, \ldots n$

variables are known as control variables

n no. of variables $= (x_1, x_2, \ldots x_n)^T$

m —" ——— $=$ $\quad x_j \geq 0 \; \forall j$

Let x_j be a free variable.

Let $x_j = x_j' - x_j''$ $\quad x_j' \geq 0$ | non-negative

$\Downarrow$ free $\quad x_j'' \geq 0$

In OR. sometimes continuous, sometimes discrete.

$r(A) = r(A|b) = m < n$ consistent

If $m = n$ infinite sol^n

$$\left.\begin{array}{l} \text{If } r(A) = m \\ r(A|b) = m+1 \end{array}\right\} \text{inconsistent.}$$

Example

$$\boxed{\begin{array}{l} x_1 + x_2 + x_3 = 10 \\ x_1 + 2x_2 + x_4 = 11 \\ x_1 + 4x_2 \ldots + x_5 = 16 \end{array}}$$

$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_5 \end{pmatrix}$

$A = (a_{ij})_{3 \times 5}$

$b = \begin{pmatrix} 10 \\ 11 \\ 16 \end{pmatrix}$

No. of basic variables $\Rightarrow r(A) = m = 3$

No. of Non-basic variables $n-m = 5-3 = 2$

S.No.	B.V.	N.B.V.	Rem
1.	$x_3=10, x_4=11, x_5=16$	$x_1=0, x_2=0$	B.F.S
2.	$x_2=4, x_3=6, x_4=3$	$x_1=0, x_5=0$	B.F.S
3.			
4.			
5.			
6.			
7.			
8.			
9.			
10.			

Single objective Optimization

Linear Optimization - Many types of programming.

Mathematical formulation of a problem.

— " ——— ~~of~~ ~~L~~ $\Rightarrow$ Linear prog

Optimization is an act of obtaining best results under given restrictions.

Find $\underset{\sim}{X} = (x_1, x_2, \ldots, x_n)^T \rightarrow$ Decision vector (control variables)

so as to

Objective function $\rightarrow$ Max : $Z = c^T X + d$, $d = 0$

s. to. $AX \leq b$ } set of constraints

$X \geq 0 \rightarrow$ Natural restrictions

LINEAR PROGRAMMING PROBLEM

s. to. $y_1 + y_2 + y$

$y_1 + 2y_2$

Solving x_1,

Any One is Pa

① (

②

③

Now

$y_1 + y_2 + y_3 \geq 1$

$y_1 + 2y_2 + 4y_3 \geq 3$

olving x_1, x_2, z

$x_1^* = \cancel{25}\ 60 \quad x_2^* = 25 \qquad z^* = 135$

One is Primal, Another is Dual

① $(x_1 + x_2 - 100)y_1 = 0$

② $(x_1 + 2x_2 - 110)y_2 = 0$

③ $(x_1 + 4x_2 - 160)y_3 = 0$

④ $(y_1 + y_2 + y_3 - 1)x_1 = 0$

⑤ $(y_1 + 2y_2 + 4y_3 - 3)x_2 = 0$

$x_1, x_2 \geq 0$

$y_1 + y_2 + y_3 = 1$

$y_1 + 2y_2 + 4y_3 = 3$

Now substitute values of x_1, x_2 in ①

$(60 + 25 - 100)y_1 = 0 \Rightarrow y_1 = 0$

$y_1 = 0$

$y_2 + y_3 = 1$

$2y_2 + 4y_3 = 3$

$y_2 = y_3 = 1/2$ is also an optimal sol^n

max $\qquad\qquad\qquad\qquad$ Dual

Primal

$$\max: Z = 2x_1 + 8x_2 \qquad\qquad y_1 + y_2 \geq 2$$

$$\text{s.to.} \quad x_1 + x_2 \leq 10 \qquad\qquad y_1 + 4y_2 \geq 8$$

$$x_1 + 4x_2 \leq 16$$

$$x_1, x_2 \geq 0$$

$\begin{pmatrix} 0 \\ 4 \end{pmatrix}$ $\begin{pmatrix} 8 \\ 2 \end{pmatrix}$ Infinite no. of solⁿ

Some Definitions and Theorem

Point in n-dimensional space: A point $X = (x_1, x_2, \dots x_n)$ n co-ordinates x_i, $i = 1, 2, 3, \dots n$. Each of them are

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad \underset{\sim}{X} \in \mathbb{R}^n$$

$x_1, x_2 \dots x_n$ are real numbers

In 2-D $\quad$ If $n = 2$ $\quad \underset{\sim}{X} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

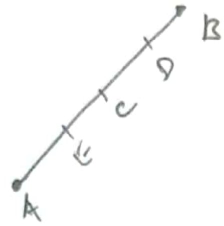
$$\underset{\sim}{X}^{(1)} = \begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \end{pmatrix} \qquad \underset{\sim}{X}^{(2)} = \begin{pmatrix} x_1^{(2)} \\ x_2^{(2)} \end{pmatrix}$$

$$\underset{\sim}{X}^{(k)} = \begin{pmatrix} x_1^{(k)} \\ x_2^{(k)} \end{pmatrix} \qquad X \in \mathbb{R}^2$$

If $n = 3$ $\quad \underset{\sim}{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ $\quad \underset{\sim}{X} \in \mathbb{R}^3 \quad \underset{\sim}{X}^{(1)} = \begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{pmatrix} \quad \underset{\sim}{X}^{(2)} =$

etc.

Line segment in n-dimensions:



$$X_p^{(1)} = \begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \\ \vdots \\ x_n^{(1)} \end{pmatrix} \qquad X_{20}^{(2)} = \begin{pmatrix} x_1^{(2)} \\ x_2^{(2)} \\ \vdots \\ x_n^{(2)} \end{pmatrix}$$

A is one of the extreme points

B —"— ———

There are infinite pts. on the line segment.

athematically, representing them

$$\boxed{L = \{ X(\lambda) \mid X(\lambda) = \lambda X_1^{(1)} + (1-\lambda) X_2^{(2)}, 0 \le \lambda \le 1 \}}$$

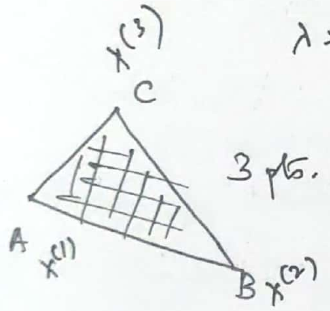
$\lambda = 0$, pt. B

$\lambda = 1$, pt. A

$\lambda = 1/2$, mid-pt. (C)

$\lambda = 3/4$, pt. D

$\lambda = 1/3$, pt. E



3 pts. in a plane. A $X^{(1)} = \begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \end{pmatrix}$ B $X^{(2)} = \begin{pmatrix} x_1^{(2)} \\ x_2^{(2)} \end{pmatrix}$

C $X^{(3)} = \begin{pmatrix} x_1^{(3)} \\ x_2^{(3)} \end{pmatrix}$

A, B, C are 3 extreme points

A is optimal sol^n, B is optimal sol^n $\Rightarrow$ AB ~~to~~ is the sol^n.

during the region S

$$S = \{ X(\lambda_1, \lambda_2, \lambda_3) \mid X(\lambda_1, \lambda_2, \lambda_3) = \lambda_1 X^{(1)} + \lambda_2 X^{(2)} + \lambda_3 X^{(3)}, \; \lambda_1, \lambda_2, \lambda_3 \ge 0, \; \lambda_1 + \lambda_2 + \lambda_3 = 1 \}$$

This is known as convex Hull.

Hyper-plane

A hyper-plane is defined as:

$$H = \{X \mid c^T X = b\}$$

$$\Rightarrow c_1x_1 + c_2x_2 + \dots + c_nx_n = b$$

A hyper-plane has $(n-1)$-dimensions in an n-dimensional ~~f~~ space. In 2-dimensional space hyper-place is a line. In 3-dimensional space it

eg. $\underset{\sim}{X} \in \mathbb{R}^3 \quad c \in \mathbb{R}^3$

$$2x_1 + 3x_2 + 4x_3 = 10$$

$$\underset{\sim}{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad c = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \quad c^T X = 10$$

$y = ax + b$

$\boxed{x_2 = +ax_1 + b}$

rep f^n of 1 variable rep ented

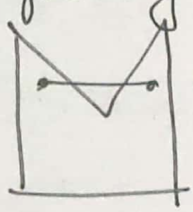
A hyper-plane divides the n-dimensional space into 2 closed half spaces as.

i) $c_1x_1 + c_2x_2 + \dots + c_nx_n \leq b$

ii) $c_1x_1 + c_2x_2 + \dots + c_nx_n \geq b$

Convex set: A convex set is a collection of points such that if x_1 and x_2 are any 2 points in the the line segment joining them is also in the set S.

A convex set.



Not convex. Some part is inside, some is outside.

Let, $x = \lambda x_1 + (1-\lambda) x_2 \quad 0 \leq \lambda \leq 1$

If $x_1, x_2 \in S$, then $x \in S$.

olyhedron and Polytope:

e Point : A pt. ~~on~~ in the convex set S which does not lie on a line segment joining 2 other points of set.

le ~~point~~ solution : In a LPP any solution X which satisfy $AX=b$ and $X \geq 0$ is called a feasible solⁿ.

implex :

$$\max: Z = x_1 + 2x_2 + 3x_3 + 100$$ const. doesn't matter.

s.to. $x_1 + x_2 + x_3 \leq 10$

$2x_1 + 2x_2 + x_3 \leq 20$

$x_1, x_2, x_3 \geq 0$

F.S. Method

$x_1 + x_2 + x_3 + x_4 = 10$, $x_4 \geq 0$

$2x_1 + 2x_2 + x_3 + x_5 = 20$, slack variable $x_5 \geq 0$

$x_1 + x_2 + x_3 + x_4 + 0x_5 = 10$ | $x_1, x_2, x_3 \geq 0$

$2x_1 + 2x_2 + x_3 + 0x_4 + x_5 = 20$ | $x_4, x_5 \geq 0$

B.V.	N.B.V. (zeroes)	Z
$x_1\ x_2$	x_3, x_4, x_5	10
$x_1 + x_2 = 10$, $2x_1 + 2x_2 = 20$, $x_1 + x_2 = 10$ } also has ∞ solⁿ; $x_1 = 10, x_2 = 0$; $x_2 = 10, x_1 = 0$; Degenerate solution	x_2, x_4, x_5	10
2. x_1, x_3; $x_1 = 10, x_3 = 0$	x_2, x_3, x_5	16
3. x_1, x_4; $x_1 = 10, x_4 = 0$		

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4.	$x_1 = 10,\ x_5 = 0$	20
5.	$x_2 = 10,\ x_3 = 0$ All these are	20
6.	$x_2 = 10,\ x_4 = 0$ degenerate	20
7.	$x_2 = 10,\ x_5 = 0$	20
8.	$x_3 = 0,\ x_4 = -10$ Not feasible	X
9.	$x_3 = 10,\ x_5 = 10$	30
10.	$x_4 = 10,\ x_5 = 20$	0

$$\max\ Z = 30 \quad \Bigg| \quad \begin{array}{ll} x_1^* = 0 & x_4^* = 0 \\ x_2^* = 0 & x_5^* = 10 \\ x_3^* = 10 & \end{array}$$

Now going by simplex method

$$\max: \ Z = x_1 + 2x_2 + 3x_3$$

$$x_1 + x_2 + x_3 \leq 10$$

$$2x_1 + 2x_2 + x_3 \leq 20$$

$$x_1, x_2, x_3 \geq 0$$

$$x_1 + x_2 + x_3 + Z_1 = 10, \quad Z_1 \geq 0 \quad Z_1 = x_4$$

$$2x_1 + 2x_2 + x_3 + Z_2 = 20 \quad Z_2 \geq 0, \ Z_2 = x_5$$

$$x_1 + 2x_2 + 3x_3 = Z$$

$$Z_1 = -x_1 - x_2 - x_3 + 10$$

$$Z_2 = -2x_1 - 2x_2 - x_3 + 20$$

$$Z = x_1 + 2x_2 + 3x_3 + 0$$

ial Simplex Tableau:

$-x_1$	$-x_2$	$-x_3$	1	
1	1	1 p	10	$= z_1$
2	2	1	20	$= z_2$
→ −1	−2	−3 ✓	0	$= z$

ior

Condensed simplex tableau

m is preferable)

B.F.S.

$$\left.\begin{array}{l} x_1=0,\ x_2=0,\ x_3=0 \\ z_1=10,\ z_2=20 \end{array}\right\} z=0$$

We will try to catch the hardcore criminal.

$$\min:\left\{\frac{10}{1},\ \frac{20}{1}\right\}$$

p → only +ve ratios.

$$\begin{array}{ccc} p & \boxed{} & q/r \\ & & s \end{array}$$

V & BV will be interchanged r/q
(z_1 & x_3)

eau 1

reciprocal of p
& divide by this p.

$q=10$

$s=20$

$-x_1$	$-x_2$	$-z_1$		
1/p = 1	1/p = 1	1/1	10	$= x_3$
1	1	−1	~~20~~ 10	$= z_2$
2	1	3	30	$= z$

We continue this until all are +ve

$$s^* = \frac{ps - qr}{p} = \frac{20-10}{1} = 10$$

$$\frac{1\times 0 - (-3\times 10)}{} = 30$$

BV $x_3 = 10$
$z_2 = 10$

NBV $x_1 = 0$, $x_2 = 0$, $z_1 = 0$

$x_1^* = 0$
$x_2^* = 0$
$x_3^* = 10$

$z^* = 30$

max : $Z = 3x_1 + 2x_2 + 3x_3$

s.t. $x_1 + x_2 + x_3 \le 10$

$2x_1 + 2x_2 + x_3 \le 20$

	$-x_1$	$-x_2$	$-x_3$	1	
	1	1	1	10	$= x_4$
	2	2	1	20	$= x_5$
	-3 ✓	-2	-3	0	$= Z$

$10/1$, $20/2$ → choose any 1

min $(10/1, 20/2)$

$x_1 = 0$ $x_2 = 0$ $x_3 = 0$ $x_4 = 10$ $x_5 = 20$

$Z = 0$

N.B.V.

	$-x_4$	$-x_2$	$-x_3$	1	
	$1/1$	$1/1$	$1/1$ p	$10/1$	$= x_1$
	-2	0	-1	0	$= x_5$
	3	1	0	30	$= Z$

IR ⇒ x-ve

This table is optimal

B.V.

$x_4 = 0$ $x_2 = 0$ $x_3 = 0$ $x_1 = 10$, $x_5 = 0$

$Z = 30$

degenerate sol^n

$x_1 = 10$
$x_2 = 0$
$x_3 = 0$
$x_4 = 0$
$x_5 = 0$

w.r.t. this 0, further iteration possible.

$0/-1$ → we will not take it
→ has to be -ve

$10/1$

$$X^{(1)} = \begin{pmatrix} 10 \\ 0 \\ 0 \end{pmatrix}$$

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$-x_4$	$-x_2$	$-x_1$	1	
1	1	1	10	$= x_3$
-1	1	$+1$	10	$= x_5$
3	1	0	30	$= z$

$$\left.\begin{array}{l} x_1 = 0 \\ x_2 = 0 \\ x_3 = 10 \end{array}\right\} \qquad z^* = 30 \qquad X_2 = \begin{pmatrix} 0 \\ 0 \\ 10 \end{pmatrix}$$

$x_4 = 0$

$x_5 = 10$

2 optimal sol^n, $\therefore$ infinitely many.

$$X^* = \lambda X^{(1)} + (1-\lambda) X^{(2)}$$

$$= \lambda \begin{pmatrix} 10 \\ 0 \\ 0 \end{pmatrix} + (1-\lambda) \begin{pmatrix} 0 \\ 0 \\ 10 \end{pmatrix} \qquad 0 \le \lambda \le 1$$

$$\lambda = 1/2 \qquad X^* = \begin{pmatrix} 5 \\ 0 \\ 5 \end{pmatrix}$$

problem was Now check if it fulfils constraints.

It does!

$\lambda = 1/3, \ldots$

Infinite no. of optimal sol^n.

2. max : $z = x_1 + 3x_2$

s.to $x_1 + x_2 \leq 10$

$x_1 + 2x_2 \leq 11$

$x_1 + 4x_2 \leq 16$

$x_1, x_2 \geq 0$

$x_1 + x_2 + z_1 = 10$

$x_1 + 2x_2 + z_2 = 11$

$x_1 + 4x_2 + z_3 = 16$

$z_1, z_2, z_3 \geq 0$

slack variable

$z_1 = 10 - x_1 - x_2$

$z_2 = -x_1 - 2x_2 + 11$

$z_3 = -x_1 - 4x_2 + 16$

Initial simplex tableau

$-x_1$	$-x_2$	1	
1	1	10	$z_1 = x_3$
1	2	11	$z_2 = x_4$
1	4 = p	16	$z_3 = x_5$
−1	−3	0	= z

IR

most −ve

min $\left(\frac{10}{1}, \right.$

16

$-x_1$	$-z_3$	1	
$3/4$	$-1/4$	$24/4 = 6$	$= z_1$
$2/4$	$-2/4$	3	$= z_2$
$1/4$	$1/4$ reci	4	row elements divided by 4 $= x_2$
$-1/4$	$3/4$	$48/4 = 12$	$= z$

column elements divided by -ve of p.

$z_1 = 6$ | $x_1 = 0$
$z_2 = 3$ | $z_3 = 0$
$x_2 = 4$

$$\min \left(\frac{6}{(3/4)}, \frac{3}{(2/4)}, \frac{4}{(1/4)} \right)$$

$$(8, 6, 16)$$

$-z_2$	$-z_3$	1	
$-3/2$	$1/2$	$3/2$	$= z_1$
2	-1	6	$= x_1$
$-1/2$	$1/2$	$5/2$	$= x_2$
$1/2$	$1/2$	$27/2$	$= z$

divide by p

$z_1 = 3/2$
$x_1^* = 6$
$x_2^* = 5/2$
$z^* = 27/2$

n Method : (Primal simplex Method)

find $x_1, x_2, \ldots x_n$ so as to

$$\max : z = \sum_{j=1}^{n} c_j x_j + d \quad \text{const}. \text{ linear objective function.}$$

s. to. $\left\{ \begin{array}{l} \sum_{j=1}^{n} a_{ij} x_j \le b_i , \quad i = 1, 2, \ldots m \quad b_i \geq 0 \ \forall i \\ x_1, x_2, \ldots x_n \geq 0 \quad 0 \le x_j < \infty \end{array} \right.$

tion

Dual simplex Method :

Find $x_1, x_2, \ldots x_n$ so as to

$$\min : Z = \sum_{j=1}^{n} c_j x_j + d$$

$$\text{subject to } \sum_{j=1}^{n} a_{ij} x_j \geq b_i, \quad i = 1, 2, \ldots m, \quad b_i \geq 0$$

→ Type II equation

$$x_j \geq 0, \quad j = 1, 2, \ldots n$$

$$0 \leq x_j < \infty, \quad j = 1, 2, \ldots n$$

Introducing slack variables

$$\max : Z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n + d$$

s.to.

$$a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n + z_1 = b_1$$

$$a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n + z_2 = b_2$$

$$\cdots\cdots\cdots\cdots$$

$$a_{i1} x_1 + a_{i2} x_2 + \ldots + a_{in} x_n + z_i = b_i$$

$$\cdots\cdots\cdots\cdots$$

$$a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n + z_m = b_m$$

$$x_1, x_2, \ldots, x_m \geq 0$$

$$z_1, z_2, \ldots, z_m \geq 0$$

Initial Simplex Tableau :

	$-x_1$	$-x_2$	$\cdots$	$-x_v$		$-x_n$	1	
	a_{11}	a_{12}		a_{1v}		a_{1n}	b_1	z_1
	a_{21}	a_{22}		a_{2v}		a_{2n}	b_2	z_2
	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\vdots$
	a_{u1}	a_{u2}	$\cdots$	a_{uv}	$\cdots$	a_{un}	b_u	$= z_u$
	a_{m1}	a_{m2}	$\cdots$	a_{mv}	$\cdots$	a_{mn}	b_m	$= z_m$
I.R →	$-c_1$	$-c_2$	$\cdots$	$-c_v$	$\cdots$	$-c_n$	d	$= Z$

$(m+1) \times (n+1)$

N.B.V. = n ; No. of B.V. = m

N.B.V. $x_1 = x_2 = x_3 \dots = x_v = \dots x_n = 0$

I: Select the most -ve element in the last row of the simplex tableau.

If no -ve element exists, then the maximum value of LPP is d and a maximizing point is $x_1 = x_2 \dots = x_n = 0$. Stop the method.

eg. min: $z = 2x_1 + 3x_2 + 4x_3 + 10$

$x_1 + x_2 + x_3 \leq 100$

$2x_1 + x_2 + 2x_3 \leq 150$

$x_1 - x_2 + x_3 \leq 80$

Max: $x_1, x_2, x_3 \geq 0$

$-z = -2x_1 - 3x_2 - 4x_3 - 10$

$-c_1 = 2, \; -c_2 = 3, \; -c_3 = 4, \; d = -10$

All +ve

$\therefore -z = -10 \rightarrow z = 10$

optimal pt.

*(If all are -ve - unbounded problem) - Stop

Step II: Suppose step I gives the element $-c_v$ at the bottom of the vth column. Form all positive ratios of the element in the last column to corresponding elements in the v-th column. That is form ratios ~~for roto~~ b_i/a_{iv} for which $a_{iv} > 0$.

The element say a_{uv} which produces the smallest ratio b_i/a_{uv} is called the pivotal element. *

Step III : Form a new simplex Tableau using

a) Interchange the role of x_v and z_u. Then relabel the row and column of the p[ivotal] element while keeping other labels [un]unchanged.

b) Replace the pivotal element $(p>0)$ by its reciprocal $1/p$ i.e. a_{uv} by $1/a_{uv}$.

c) Replace the other elements of the row of the pivotal element (row elements/pivotal element)

d) Replace the other elements of the column of the pivotal element $\left(-\frac{\text{column element}}{\text{pivotal element}}\right)$

e) Replace all other elements (say s) of the Tableau by the elements of the form

$$s^* = \frac{ps - qr}{p}$$

Step 3 leads to a new Tableau that prese[nts] an equivalent LPP.

Only for max. LPP where constraints ≤ 0.

Theorem 1: Intersection of any no. of convex sets is also conve[x]

Theorem 2: The feasible region of a LPP forms a convex set.

$$AX = b \quad , X \geq 0$$

Let $X_1 \in S$ $\quad X_1 \geq 0$

Let $X_2 \in S$ $\quad X_2 \geq 0$

$AX_1 = b \quad AX_2 = b$

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$X_\lambda = \lambda X_1 + (1-\lambda) X_2$

$0 \leq \lambda \leq 1 \qquad \lambda \geq 0$

$X_1 \geq 0 \qquad 1-\lambda \geq 0$

$X_2 \geq 0$

$X_\lambda \geq 0$

$X_\lambda = \lambda X_1 + (1-\lambda) X_2$

$$A X_\lambda = A(\lambda X_1 + (1-\lambda) X_2)$$
$$= \lambda A X_1 + (1-\lambda) A X_2$$
$$= \lambda b + (1-\lambda) b = b$$

$\Rightarrow A X_\lambda = b \qquad X_\lambda \geq 0$

S Implies that it fulfils the feasible region.

...orem 3: In general a LPP has either one optimal solution or no optimal solution or infinite no. of optimal solⁿ.

Any local min/max solⁿ is a global min/max solⁿ of LPP.

...orem 4: Every B.F.S. is an extreme point of the convex set of the feasible region.

$AX = b \qquad X \geq 0$

$X = \begin{pmatrix} X_B \\ X_N \end{pmatrix} \qquad B X_B = b$

$X_B = B^{-1} b \qquad |B| \neq 0$

Scanned by CamScanner

$$X_B = \begin{pmatrix} \bar{b}_1 \\ \bar{b}_2 \\ \vdots \\ \bar{b}_m \end{pmatrix} \geq 0$$

$$X = \begin{array}{l} \\ \text{B.V.} \\ \\ \\ \text{N.B.V} \\ \\ \end{array} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \\ \hline x_{m+1} \\ x_{m+2} \\ \vdots \\ x_n \end{pmatrix} \qquad Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{m+1} \\ \vdots \\ y_n \end{pmatrix} \qquad Z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \\ \vdots \\ z_n \end{pmatrix}$$

Let $Y \geq 0 \quad Z \geq 0$

$\lambda \geq 0 \quad (1-\lambda) \geq 0$

$X = \lambda Y + (1-\lambda) Z$

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} y_{m+1} \\ y_{m+2} \\ \vdots \\ y_n \end{pmatrix} + (1-\lambda) \begin{pmatrix} z_{m+1} \\ z_{m+2} \\ \vdots \\ z_{m+n} \end{pmatrix}$$

$\lambda \geq 0 \quad y_{m+1} \geq 0$

$(1-\lambda) \geq 0 \quad z_{m+1}$

$\lambda y_{m+1} + (1-\lambda) z_{m+1} = 0$

$y_{m+1} = 0 = z_{m+1}$

$\lambda y_{m+2} + (1-\lambda) z_{m+2} = 0$

$y_{m+2} = z_{m+2} = 0$

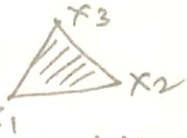
$\vdots$

$\lambda y_n + (1-\lambda) z_n = 0 \qquad y_n = z_n = 0$

$Y = Z = X_B$ Only possible

So X is an extreme point by contradiction

15: Let S be a close bounded convex polyhedron with p number of extreme point X_i, $i = 1, 2, \dots, p$. Then any vector $X \in S$ can be written as:

eg 

$$X = \sum_{i=1}^{p} \lambda_i X_i \ , \quad \sum_{i=1}^{p} \lambda_i = 1 \ , \quad \lambda_i \geq 0$$

$X = \lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3$

$\lambda_1, \lambda_2, \lambda_3 \geq 0$

$\lambda_1 + \lambda_2 + \lambda_3 = 1$

n6: Let S be a close & convex polyhedron. Then the min. of a linear f^n over S is attained at an extreme point of S.

<u>Proof</u>:

$$\boxed{\begin{aligned} \min:\ & f = c^T X \\ \text{s. to.}\ & AX = b \\ & X \geq 0 \end{aligned}}$$

P_2: max : $f = c^T X$

s. to. $AX = b$

$X \geq 0$

We prove by contradiction.

Let X^* be an optimal solution

Min. does not occur at an optimal point.

$$\boxed{\begin{aligned} c^T X^* &< c^T X_1 \\ c^T X^* &< c^T X_2 \\ c^T X^* &< c^T X_3 \end{aligned}}$$

$$c^T X^* < c^T X_i \ , \quad i = 1, 2, \dots p$$

For $0 \leq \lambda_i \leq 1$, $\lambda_i c^T X^* < \lambda_i c^T X_i$ $\quad i = 1, 2, \dots p$

$$\sum_{i=1}^{p} \lambda_i c^T X^* < \sum_{i=1}^{p} c^T \lambda_i X_i \ , \quad i = 1, 2, \dots p$$

ppose $\lambda_i = \lambda_i^*$, & $X^* = \sum_{i=1}^{p} \lambda_i^* X_i$ $\quad$ so, $\sum_{i=1}^{p} \lambda_i^* c^T X^* < \sum_{i=1}^{p} \lambda_i^* c^T X_i$

$$c^T X^* \sum_{i=1}^{p} \lambda_i < c^T \sum \lambda_i^* X_i \rightarrow c^T X^* < c^T X^* \text{ (contradiction}$$

X^* has to be an extreme pt.

eg. 2

max: $Z = 2x_1 + 3x_2$

s. to. $\cancel{2x_1 + 3x_2 = 12}$

$\cancel{2x_1 + x_2 \geq 8}$

$\cancel{x_1, x_2 \geq 0}$

$x_1 + 2x_2 \leq 20$

$x_1 + x_2 = 15$ → must include variable Neither slack nor surplus but an artificial variable (FOR SIMPLEX)

$3x_1 + 2x_2 \geq 30$

$x_1, x_2 \geq 0$

$x_1 + 2x_2 + s_1 = 20$, $s_1 \geq 0$ (slack variable)

$x_1 + x_2 + s_2 = 15$, $s_2 \geq 0$ (If you cannot make $s_2 = 0$, the problem is infeasible)

min: s_2 to zero

$3x_1 + 2x_2 - x_3 + s_3 = 30$,

$x_3 \geq 0$ (surplus variable)

Artificial variable

min: s_3 to zero

So we have to min: $(s_2 + s_3)$ ⟶ min = 0

If 0, it is ideal.

Let M be a large +ve real number (Big M)

max: $Z = 2x_1 + 3x_2 - M(s_2 + s_3)$

↳ This M is driving out the artificial variables

$s_2 = 15 - x_1 - x_2$

$s_3 = 30 - 3x_1 - 2x_2 + x_3$

$$\begin{aligned} \text{max}: Z &= 2x_1 + 3x_2 - M(s_2 + s_3) \\ &= 2x_1 + 3x_2 - M(45 - 4x_1 - 3x_2 + x_3) \\ &= x_1(2 + 4M) + x_2(3 + 3M) - x_3 M - 45M \end{aligned}$$

Artificial variables - basic variable.

Initially, surplus variable cannot be a basic variable.

C_B	C_N → / $B \backslash N$	2 x_1	3 x_2	0 x_3	X_B	
0	s_1	1	2	0	20	
$-M$	s_2	1	1	0	15	15/1
$-M$	s_3	$\boxed{3}$	2	-1	30	30/3
		$-4M$ -2	$-3M$ -3	M	$-45M$	

I.R. →

Most -ve

C_B	C_N → / $B \backslash N$	$-M$ s_3	3 x_2	0 x_3	X_B	
0	s_1	$-1/3$	$\boxed{4/3}$	$1/3$	10	7.5
$-M$	s_2	$-1/3$	$1/3$	$1/3$	5	15
2	x_1	$1/3$	$2/3$	$-1/3$	10	15
		$\frac{4M+2}{3}$	$\frac{-M-5}{3}$	$\frac{-M-2}{3}$	$-5M+20$	

C_B	C_N → / $B \backslash N$	$-M$ s_3	0 s_1	0 x_3	X_B	
3	x_2	$-1/4$	$3/4$	$1/4$	$15/2$	30
$-M$	s_2	$-1/4$	$-1/4$	$\boxed{1/4}$	$5/2$	10
2	x_1	$1/2$	$-1/2$	$-1/2$	5	
		$\frac{5M+1}{4}$	$\frac{M+5}{4}$	$\cancel{\frac{130-10M}{4}}$ $\frac{-M-1}{4}$	$\frac{130-10M}{4}$	

$1+8+M+5$

$\frac{13M+13}{\cancel{3}4}$

$\frac{15M+3}{9}$ $\frac{5M+1}{\cancel{3}4}$

$\frac{M-8+M+5}{9}$

$\frac{-3M-3}{9} = \frac{-M-1}{\cancel{3}4}$

$\frac{1}{3}+\frac{M}{3}+M$ $\frac{4M+2}{3}$

$M-9+8M+4$

$1-4M-2$

$-135M+120M+60$

$-\frac{15M+60}{3}$

$-5M+20$

$\frac{M+5}{\cancel{3}} \times \frac{\cancel{3}}{4}$

$\frac{-20M+80+10M+50}{3}$

$\frac{130-10M}{\cancel{3}4}$

M	1	M+1	35

$x_1^* = 10, \; x_2^* = 5, \; x_3^* = 10$

$s_1 = s_2 = s_3 = 0$.

In a program, we have to pass on a value of M. To av that, we go by 2-phase simplex method.

Two-Phase Simplex Method

$$\max: f = 2x_1 + 3x_2 \quad \text{- O.F.}$$

s. to.

$$x_1 + 2x_2 \leq 20$$
$$x_1 + x_2 = 15$$
$$3x_1 + 2x_2 \geq 30$$
$$x_1, x_2 \geq 0$$

Phase-I Method

$$x_1 + 2x_2 + s_1 = 20$$
$$x_1 + x_2 + a_1 = 15$$
$$3x_1 + 2x_2 - x_3 + a_2 = 30$$

$s_1, a_1, a_2, x_3 \geq$

Artificial O.F. $\min: A = a_1 + a_2$

$\rightarrow \boxed{\max: -A = -a_1 - a_2}$

	C_N	0	0	0		
C_B	B \ N	x_1	x_2	x_3	X_B	
0	s_1	1	2	0	20	20
-1	a_1	1	1	0	15	15
-1	a_2	3	2	-1	30	10
		-4	-3	1	-45	

$-a_1-a_2+0x_1+0x_2+0x_3$

After iterations, phase I will terminate. If $-a_1-a_2=0$ the feasible, if not infeasible.

	C_N	-1	0	0		
C_B	B \ N	a_2	x_2	x_3	X_B	
0	s_1	-1/3	$\boxed{4/3}$	1/3	10	7.5
-1	a_1	-1/3	1/3	1/3	5	15
0	x_1	1/3	2/3	-1/3	10	15
		4/3	-1/3 ✓	-1/3	-5	

As soon as you have removed a_1 & a_2 or this 0, we move to phase II.

	C_N	-1	0	0	
	B \ N	a_2	s_1	x_3	X_B
0	x_2	-1/4	3/4	1/4	15/2
-1	a_1	-1/4	-1/4	$\boxed{1/4}$	5/2
0	x_1	1/2	-1/2	-1/2	5
		5/4	1/4	-1/4	-5/2

we drop these a.v. columns.

		-1	0	-1	
	B \ N	$\cancel{a_2}$	s_1	$\cancel{a_1}$	X_B
0	x_2	0	1	-1	5
0	x_3	-1	-1	4	10
0	x_1	0	-1	2	10
		✗	0	✗	0

We have removed the artificial variables.

$Z = 2x_1 + 3x_2$

In the previous table Phase II table

1 st

		0	
C_B	B \ N	s_1	X_B
3	x_2	1	5
0	x_3	-1	10
2	x_1	-1	10
		1	35

since it is +ve, it is optimum.

$x_1^* = 10$

$x_2^* = 5$ $\qquad f^* = 35$

Complete

Duality theory for LPP

Primal Program (P):

$$\text{Max} : z = \sum_{i=1} c_j x_j$$

w.r.t. the above primal problem (P) we find dual problem (D) as D:

$$\text{min} : z' = \sum_{i=1}^{m} b_i y_i$$

s.to.

$$\sum_{i=1}^{m} a_{ij} y_i \geq c_j \quad , j = 1, 2, 3, \dots n$$

$$y_i \geq 0 \quad , i = 1, 2, 3, \dots m$$

respect to above primal problem (P)

ple :

Max: $Z = 2x_1 + 3x_2 + x_3$

s.to. $x_1 + x_2 + x_3 \le 5$

$4x_1 + 6x_2 + 2x_3 \le 15$

$x_1, x_2, x_3 \geq 0$

Primal LPP

check if $x_1 = 0$, $x_2 = 5/4$, $x_3 = 15/4$ is an optimal sol^n & $Z = \frac{15}{4} + \frac{15}{4} = 7.5$

Dual LPP

min: $Z = 5y_1 + 15y_2$

s.to. $y_1 + 4y_2 \geq 2$

$y_1 + 6y_2 \geq 3$

$y_1 + 2y_2 \geq 1$ $y_1, y_2 \geq 0$.

Complementary slackness principle

w.r.t. x_1, x_2, x_3, we find the value of y_1 & y_2.

If primal is feasible, dual is also feasible.

Primal has optimal, dual also does

solve this by graphical get value of Z.

$y_1^* = 0$, $y_2^* = 1/2$

$Z' = 15/2$

Purpose of this duality problem - check for the optimal sol^n (will be used in transportation problem)

LPP: $\max: Z = \sum_{j=1}^{n} c_j x_j$

Primal LPP s. to. $\sum_{j=1}^{n} a_{ij} x_j \le b_i$, $i = 1, 2, \ldots n$

$x_j \geq 0$, $j = 1, 2, \ldots n$

$$\min: -Z = -\sum_{j=1}^{n} c_j x_j$$

Constraints $\leq$ so $\lambda_1, \lambda_2 .. \lambda_m \geq 0$

If constraints $=$ λs will be

$$\sum_{j=1}^{n} a_{ij} x_j + s_i^2 - b_i = 0 \quad i = 1, 2, \ldots, m$$

$$s_i^2 \geq 0 \quad i = 1, 2, \ldots m$$

$$-x_j \leq 0 \Rightarrow -x_j + t_j^2 = 0, \quad j = 1, 2, \ldots m$$

$$t_j^2 \geq 0, \quad j = 1, 2, \ldots n, \quad \mu_1, \mu_2, \ldots \mu$$

$$\min: L(\underset{\sim}{x}, \underset{\sim}{s}, \underset{\sim}{\lambda}, \underset{\sim}{\mu}, \underset{\sim}{T} \qquad)$$

$$= -\sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{m} \lambda_i \left(\sum_{j=1}^{n} a_{ij} x_j + s_i^2 - b_i \right) + \sum_{j=1}^{n} \mu_j (-$$

$$\frac{\partial L}{\partial x_j} = 0, \quad j = 1, 2, \ldots n$$

$$\frac{\partial L}{\partial s_i} = 0, \quad i = 1, 2, \ldots m$$

$$\frac{\partial L}{\partial \lambda_i} = 0 \quad i = 1, 3 \ldots m$$

$$\frac{\partial L}{\partial \mu_j} = 0 \quad \cancel{j}\, i = 1, 2, \ldots n$$

$$\frac{\partial L}{\partial t_j} = 0 \quad j = 1, 2, \ldots n$$

Scanned by CamScanner

$$\frac{\partial L}{\partial x_j} = 0, \quad -g_j + \sum_{i=1}^{m} \lambda_i a_{ij} - \mu_j = 0 \;\forall j \quad \mu_j \geq 0 \;\forall j.$$

$$\Rightarrow -g_j + \sum_{i=1}^{m} \lambda_i a_{ij} \geq 0$$

$$① \;\boxed{\sum_{i=1}^{m} \lambda_i a_{ij} \geq g_j \quad j = 1, 2, \ldots n}$$

Dual

$$\Rightarrow \sum_{i=1}^{m} a_{ij} y_i \geq g_j \quad (j = 1, 2, \ldots n)$$

$$\frac{\partial L}{\partial s_i} = 0 \Rightarrow 2 s_i \lambda_i = 0 \;\forall i \Rightarrow 2 s_i^2 \lambda_i = 0$$

$$s_i^2 = b_i - \sum_{j=1}^{n} a_{ij} x_j$$

$$② \;\boxed{2\left(b_i - \sum_{j=1}^{n} a_{ij} x_j\right) y_i = 0} \;\forall i$$

$$\frac{\partial L}{\partial \lambda_i} = 0 \Rightarrow \sum_{j=1}^{n} a_{ij} x_j + s_i^2 - b_i = 0$$

$$\Rightarrow \boxed{\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, 2, \ldots m}$$

$$\frac{\partial L}{\partial \mu_j} = 0 \Rightarrow -x_j + t_j^2 = 0 \Rightarrow x_j \geq 0 \;\forall j$$

$$\boxed{t_j^2 = x_j \;\forall j}$$

$$\frac{\partial L}{\partial t_j} = 0 \Rightarrow 2 t_j \mu_j = 0$$

$$\Rightarrow 2 t_j^2 \mu_j = 0 \Rightarrow \boxed{2 x_j \mu_j = 0}$$

$$\sum_{i=1}^{m} a_{ij} \lambda_i \geq g_j \;\forall j \Rightarrow g_j \leq \sum_{i=1}^{m} a_{ij} \lambda_i \quad j = 1, 2, \ldots n, \; x_j \geq 0$$

$$g_j x_j \leq x_j \left(\sum_{i=1}^{m} a_{ij} \lambda_i\right)$$

$$\sum_{j=1}^{n} c_j x_j \leq \sum_{j=1}^{n} x_j \left(\sum_{i=1}^{m} a_{ij} \lambda_i\right)$$

$$z = \sum_{j=1}^{n} c_j x_j$$

$$\sum_{j=1}^{n} c_j x_j \leq \sum_{i=1}^{m} \lambda_i \left(\sum_{j=1}^{n} a_{ij} x_j\right) \qquad \sum_{j=1}^{n} a_{ij} x_j \leq$$

$$z \leq \sum_{i=1}^{m} \lambda_i b_i$$

$$\max : z \leq \min \sum_{i=1}^{m} \cancel{\lambda_i b} \; y_i b_i = z' \text{ (Weak Duality)}$$

$$\max : z = \min z' \text{ (Strong duality)}$$

Dual LPP

$$\min z' = \sum_{i=1}^{m} \lambda_i b_i \qquad \lambda_i = y_i$$

$$i = 1, 2, \ldots m$$

$$\text{s.to.} \sum_{i=1}^{m} a_{ij} \lambda_i \geq c_j \quad j = 1, 2, \ldots n$$

$$\lambda_1, \lambda_2, \ldots, \lambda_m \geq 0$$

$$\left(\sum_{j=1}^{n} a_{ij} x_j - b_i\right) y_i = 0 \quad i = 1, 2, \ldots m$$

$$\left(\sum_{i=1}^{m} a_{ij} y_i - c_j\right) x_j = 0, \quad j = 1, 2, \ldots, n$$

Complementary pairs

max value of primal is the min. of the primal

... Dual of the dual is the primal problem

Scanned by CamScanner

Primal LPP

$\max: Z = C^T X$

s.to. $AX \le b$

$X \ge 0$

$\max: Z = \sum_{j=1}^{n} c_j x_j$

s.to. $\sum_{j=1}^{n} a_{ij} x_j \le b_i \quad i = 1, 2, \ldots n$

$x_1, x_2 \ldots x_n \ge 0$

Dual LPP

$\min: z' = b^T Y$

s.to. $A^T Y \ge C$

$Y \ge 0$

$\min: z' = \sum_{i=1}^{m} b_i y_i$

s.to. $\sum_{i=1}^{m} a_{ij} y_i \ge c_j \quad i = 1, 2, \ldots n$

$y_1 y_2, \ldots y_m \ge 0$

$\min: z' \ge \max: z$

(weak duality thm.)

$\min: z' = \max: z$

(strong duality thm.)

check if $\underset{\sim}{X} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 60 \\ 25 \end{pmatrix}$ is an optimal solⁿ of the LPP.

$\max z = \min: z'$

$(x_1 + x_2 - 100) y_1 = 0 \rightarrow y_1 = 0$

$(x_1 + 2x_2 - 110) y_2 = 0$

$(x_1 + 4x_2 - 160) y_3 = 0$

$(y_1 + y_2 + y_3 - 1) x_1 = 0$

$(y_1 + 2y_2 + 4y_3 - 3) x_2 = 0$

$\min z' = 100y_1 + 110y_2 + 160y_3$

s.to. $y_1 + y_2 + y_3 \ge 1$

$y_1 + 2y_2 + 4y_3 \ge 3$

$y_1, y_2, y_3 \ge 0$

$y_1 + y_2 + y_3 = 1$ $\quad$ $y_2 + y_3 = 1 \rightarrow$ $\quad$ $y_1 = 0$

$y_1 + 2y_2 + 4y_3 = 3$ $\quad$ $2y_2 + 4y_3 = 3$ $\quad$ $y_2 = 1/2$

$y_3 = 1/2$

$$z' = (110 + 160)\, 1/2 = 135$$

Example 2 Now changing the example

y_2 & y_3 will be free (equality conditions)

Example 3 : Primal problem (P)

min : $z = x_1 + 3x_2$

s. to. $x_1 + x_2 \geq 100$

$x_1 + 2x_2 \geq 110$

$x_1 + 4x_2 \geq 160$

$x_1, x_2 \geq 0$

For this problem, let's say I'm checking

60 & 25

$60 + 25 \ngeq 100$

Not fulfilled

Dual problem (D)

max $z' = 100y_1 + 110y_2 + 160y_3$

s. to. $y_1 + y_2 + y_3 \leq 1$

$y_1 + 2y_2 + 4y_3 \leq 3$

min : z = max : z' (140)

$(x_1 + x_2 - 100)\, y_1 = 0$

$(x_1 + 2x_2 - 160)\, y_2 = 0$

$(x_1 + 4x_2 - 160)\, y_3 = 0$

$(y_1 + y_2 + y_3 - 1)\, x_1 = 0$

$(y_1 + 2y_2 + 4y_3 - 3)\, x_2 = 0$

80 & 20

$y_2 = 0$

$y_1 + y_3 = 1$ $\quad$ $y_1 = 1/3$

$y_1 + 4y_3 = 3$ $\quad$ $y_3 = 2/3$

$$z' = 100 \times 1/3 + 160 \times 2/3 = \frac{420}{3} = 140$$

So it is an optimal sol^n

$\rightarrow \quad y_1 = 0$

$3 \quad y_2 = 1/2$

$y_3 = 1/2$

roblem,

I'm checking

$0 \& 25$

$+25 \not> 100$

ot fulfilled

$ini: z = max: z'$

$140)$

$y_1 = 1/3$

$-3 \quad y_3 = 2/3$

$\frac{20}{3} = 140$

l soln

---

max : $z = c_1x_1 + c_2x_2 + c_3x_3$

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \le b_1$ Type I inequation

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$

$-(a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \ge b_3)$

$x_1, x_2, x_3 \ge 0$ If at all x_1, x_2, x_3 are free

changing the inequation

so restricted

Dual:

min : $z' = b_1y_1 + b_2y_2 - b_3y_3$

$a_{11}y_1 + a_{21}y_2 - a_{31}y_3 \ge c_1$ All will be changed to equality

$a_{12}y_1 + a_{22}y_2 - a_{32}y_3 \ge c_2$

$a_{13}y_1 + a_{23}y_2 - a_{33}y_3 \ge c_3$

$y_1 \ge 0$, $y_3 \ge 0$, y_2 is free

$\cancel{y_1 \ge 0}, \cancel{y_3 \ge 0}$

max: $z = c_1x_1 + c_2x_2 + c_3x_3$

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$ $\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \le b_1 \\ -a_{11}x_1 - a_{12}x_2 - a_{13}x_3 \le -b_1 \end{cases}$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$ $\begin{cases} a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \le b_2 \\ -a_{21}x_1 - a_{22}x_2 - a_{23}x_3 \le -b_2 \end{cases}$

$x_1, x_2, x_3 \ge 0$

(~~Now they are nomore free~~)

Dual

$t_1, t_2, t_3, t_4 \ge 0$

let $y_1 = t_1 - t_2$

$y_2 = t_3 - t_4$

y_1, y_2 are free

min ; $z' = b_1(t_1 - t_2) + b_2(t_3 - t_4)$

s.t. $a_{11}(t_1 - t_2) + a_{21}(t_3 - t_4) \ge c_1$

$a_{12}(t_1 - t_2) + a_{22}(t_3 - t_4) \ge c_2$

$a_{13}(t_1 - t_2) + a_{23}(t_3 - t_4) \ge c_3$

Dual

$\min : z' = b_1 y_1 + b_2 y_2$

s.to. $a_{11} y_1 + a_{21} y_2 \geq c_1$

$a_{12} y_1 + a_{22} y_2 \geq c_2$

$a_{13} y_1 + a_{23} y_2 \geq c_3$

y_1, y_2 are free.

$\min : z = x_1 + 3x_2$

s.to. $x_1 + x_2 \geq 10$

$x_1 + 2x_2 \geq 11$

$x_1 + 4x_2 \geq 16$

$x_1, x_2 \geq 0$

If there is type I inequation, we can introduce slack variable.

$\max : -z = -x_1 - 3x_2$

$-x_1 - x_2 \leq -10$

$-x_1 - 2x_2 \leq -11$

$-x_1 - 4x_2 \leq -16$

$x_1, x_2 \geq 0$

$-x_1 - x_2 + x_3 = -10$

$-x_1 - 2x_2 + x_4 = -11$

$-x_1 - 4x_2 + x_5 = -16$

If x_1 & x_2 are NBV

$$\left.\begin{aligned} x_3 &= -10 \\ x_4 &= -11 \\ x_5 &= -16 \end{aligned}\right\} \text{Infeasible}$$

An indication that problem may be infeasible

0	x_5	-1	-4	-16
		1	3	0

$\min\left(\left|\frac{3}{-4}\right|, \left|\frac{1}{-1}\right|\right)$

We apply Dual simplex. In primal simplex, -ve in the last row & we take +ve ratios

In dual simplex, we take -ve ratios

	B \ N	x_1 (-1)	x_5 (0)	X_B
0	x_3	-3/4	-1/4	-6
0	x_4	-2/4	-2/4	-3
-3	x_2	1/4	-1/4	4
		1/4	3/4	-12

-ve, so not optimal

Basic variables have to be +ve

	B \ N	x_3 (0)	x_5 (0)	X_B
-1	x_1	-4/3	1/3	8
0	x_4	-2/3	-1/3	1
-3	x_2	1/3	1/3	2
		1/3	2/3	-14

$\begin{pmatrix} 8 \\ 2 \end{pmatrix}$

No Big M

No artificial variables.

We have used dual simplex method.

Primal simplex Method

IR → $z_j - c_j < 0$

$x_B \geq 0$

pivot element $(p) > 0$

Dual simplex Method

I.R → $z_j - c_j > 0$

$x_B \leq 0$

pivot element $(p) < 0$

Primal simplex Tableau

max: $z = x_1 + 3x_2$

s.to. $x_1 + x_2 + x_3 = 10$

$x_1 + 2x_2 + x_4 = 11$

$x_1 + 4x_2 + x_5 = 16$

$x_1, x_2 \geq 0$

	C_N	1	3	
C_B	B \ N	x_1	x_2	X_B
0	x_3	1	1	10
0	x_4	1	2	11
0	x_5	1	4	16
		-1	-3	0

In type 2, we must use artificial variables.

min: $z = 2x_1 + 3x_2 + 4x_3$

s.to. $x_1 + x_2 + x_3 \geq 100$

$2x_1 + x_2 + 3x_3 \geq 200$

$x_1, x_2, x_3 \geq 0$

To avoid use of artificial variables, it can be changed to

max: $-z = -2x_1 - 3x_2 - 4x_3$

$-x_1 - x_2 - x_3 + x_4 = -100$

Dual simplex

$-2x_1 - x_2 - 3x_3 + x_5 = -200$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

C_B	$B \backslash N$	x_1	x_2	x_3	X_B
	C_N	-2	-3	-4	
0	x_4	-1	-1	-1	-100
0	x_5	$\boxed{-2}$	-1	-3	-200 ✓
		2	+3	4	0

→ No -ve element here.

$$\min\left(\left|\frac{4}{-3}\right|, \left|\frac{3}{-1}\right|, \left|\frac{2}{-2}\right|\right) = \left(4/3, 3, 1\right)$$

Product ~~from~~ Form Inverse of a Matrix :

$$B = I_m = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ 0 & 0 & & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 1 \end{bmatrix} \qquad B^{-1} = B$$

$$\text{let } A = \begin{bmatrix} 5 & 1 \\ 9 & 2 \end{bmatrix} \qquad A^{-1} = ?$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = B^{-1}$$

column changed

$$B_1 = \begin{bmatrix} 5 & 0 \\ 9 & 1 \end{bmatrix} \qquad \text{let } B_1^{-1} = E_1 B^{-1} = E_1$$

Interested to know how E_1 & E_2 looks like.

entering variable to basic variable

outgoing variable to non-basic variable.

$$B_2 = \begin{bmatrix} 5 & 1 \\ 9 & 2 \end{bmatrix} \qquad B_2^{-1} = E_2 B_1^{-1} = E_2 E_1$$

$$B_2 = A \Rightarrow A^{-1} = E_2 E_1$$

$$e_1 = B^{-1} C_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 9 \end{pmatrix} = \begin{pmatrix} 5 \\ 9 \end{pmatrix}$$

$$\eta_1 = \begin{pmatrix} 1/5 \\ -9/5 \end{pmatrix}$$ → behaving like a pivot

let $E_1 = \begin{bmatrix} 1/5 & 0 \\ -9/5 & 1 \end{bmatrix} = B_1^{-1}$

$$e_2 = B_1^{-1} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1/5 & 0 \\ -9/5 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1/5 \\ 1/5 \end{pmatrix}$$

$$\eta_2 = \begin{pmatrix} -1 \\ 5 \end{pmatrix} \qquad E_2 = \begin{bmatrix} 1 & -1 \\ 0 & 5 \end{bmatrix}$$

$$A^{-1} = B_2^{-1} = E_2 E_1 = \begin{bmatrix} 1 & -1 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1/5 & 0 \\ -9/5 & 1 \end{bmatrix}$$

↓ represent as product form of 2 matrix.

$$= \begin{bmatrix} 2 & -1 \\ -9 & 5 \end{bmatrix}$$

NBV
↑

$$B = \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array} = I_3 = B^{-1}$$

We can change any 1 column.

→ If this becomes BV, it enters into either of the 3 columns.

$$B_1 = \begin{pmatrix} a_1 & 0 & 0 \\ a_2 & 1 & 0 \\ a_3 & 0 & 1 \end{pmatrix} \qquad B_1^{-1} = \,?$$

$a_1, a_2, a_3 \neq 0$

Generally, pivot element is non zero.

$$\text{or } B_1 = \begin{pmatrix} 1 & a_1 & 0 \\ 0 & a_2 & 0 \\ 0 & a_3 & 1 \end{pmatrix} \qquad B_1 = \begin{pmatrix} 1 & 0 & a_1 \\ 0 & 1 & a_2 \\ 0 & b & a_3 \end{pmatrix}$$

$$\begin{bmatrix} 1/a_1 & 0 & 0 \\ -a_2/a_1 & 1 & 0 \\ -a_3/a_1 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & -a_1/a_2 & 0 \\ 0 & 1/a_2 & 0 \\ 0 & -a_3/a_2 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & -a_1/a_3 \\ 0 & 1 & -a_2/a_3 \\ 0 & 0 & 1/a_3 \end{bmatrix}$$

Let B is the original <u>Basis Matrix</u> $(B = B^{-1})$

B_{new} be the new basis matrix which is identical to B except column r.

Let C is the rth column vector of B_{new}

Let $e = B^{-1}C$

Let $\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_m \end{pmatrix}$ where $\eta_i = \begin{cases} -\dfrac{e_i}{e_r}, & i \neq r \\ 1/e_r, & i = r \end{cases}$

$$\eta = \begin{pmatrix} -e_1/e_r \\ -e_2/e_r \\ \vdots \\ 1/e_r \\ -e_{r+1}/e_r \\ \vdots \\ -e_m/e_r \end{pmatrix} \qquad B^{-1}_{new} = E_r B^{-1}$$

$$= \begin{pmatrix} 1 & 0 & -e_1/e_r & 0 \\ 0 & 1 & -e_2/e_r & 0 \\ \vdots & \vdots & \vdots & \vdots \\ & & 1/e_r & \\ & & \vdots & \\ 0 & 0 & -e_{r+1}/e_r & \end{pmatrix} B^{-1}$$

where E_r is an identity matrix except its rth column. rth column is replaced by the η.

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$$A = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = B^{-1}$$

If writing a code, we consider 1, 2, 3
3, 2, 1 . . .

$$B_1 = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

identity matrix

$$B_1^{-1} = E_1 B^{-1} = E_1$$

$$B_2 = \begin{pmatrix} 2 & 1 & 0 \\ 2 & 3 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$B_2^{-1} = E_2 B_1^{-1} = E_2 E_1$$

$$B_3 = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix} \qquad A^{-1} = B_3^{-1} = E_3 B_2^{-1}$$
$$= E_3 E_2 E_1$$

$$A^{-1} = B_m^{-1} = E_m E_{m-1} \cdots E_1$$

$$e_1 = B^{-1} a_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

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$$\eta = \begin{pmatrix} 1/2 \\ -1 \\ -1/2 \end{pmatrix} \qquad E_1 = \begin{bmatrix} 1/2 & 0 & 0 \\ -1 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix}$$

$$e_2 = B_1^{-1} \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 & 0 \\ -1 & 1 & 0 \\ -1/2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 \\ 2 \\ 1/2 \end{pmatrix}$$

$$\eta_2 = \begin{pmatrix} -1/4 \\ 1/2 \\ -1/4 \end{pmatrix} \qquad E_2 = \begin{bmatrix} 1 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ 0 & -1/4 & 1 \end{bmatrix}$$

$$B_2^{-1} = E_2 B_1^{-1} = E_2 E_1$$

$$= \begin{pmatrix} 1 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ 0 & -1/4 & 1 \end{pmatrix} \begin{pmatrix} 1/2 & 0 & 0 \\ -1 & 1 & 0 \\ -1/2 & 0 & 1 \end{pmatrix}$$

$$B_3^{-1} = E_3 B_2^{-1} = E_3 E_2 E_1$$

We are not going to find inverse using this method. However we will be using it in Revised simplex method.

Type I inequalities. Plain simplex can be used..

max : $z = x_1 + 3x_2$

s.to. $x_1 + x_2 \le 10$ $\qquad$ $x_1 + x_2 + x_3 = 10$

$x_1 + 2x_2 \le 11$ $\qquad$ $x_1 + 2x_2 + x_4 = 11$

$x_1 + 4x_2 \le 16$ $\qquad$ $x_1 + 4x_2 + x_5 = 16$

$x_1, x_2 \ge 0$ $\qquad$ $x_3, x_4, x_5 \ge 0$

	C_N	1	3	0	0	0	
C_B	BV \ N	x_1	x_2	x_3	x_4	x_5	X_B
				Basis matrix			
0	x_3	1	1	1	0	0	10
0	x_4	1	2	0	1	0	11
0	✓x_5	1	$\boxed{4}$	0	0	1	16
		-1	-3				

$$B_{new} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{pmatrix}$$

$$E_1 B^{-1} = E_1 = \begin{pmatrix} 1 & 0 & -1/4 \\ 0 & 1 & -2/4 \\ 0 & 0 & 1/4 \end{pmatrix}$$