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Higher order BVP :-

$$\frac{d^3y}{dx^3} + A(x)\frac{d^2y}{dx^2} + B(x)\frac{dy}{dx} + C(x)\,y = D(x),$$

B.Cs. $y(a) = y_a,\quad y'(a) = y'_a,\quad y'(b) = y'_b.$

well-posed BVP.

$$y_i''' = (y_i'')' = \frac{y''_{i+1} - y''_{i-1}}{2h} = \frac{1}{2h^3}\left[y_{i+2} - 2y_{i+1} + \cancel{y_i} - \cancel{y_i} + 2y_{i-1} - y_{i-2}\right]$$

$$y_i''' = \frac{1}{2h^3}\left[y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2}\right] + O(h^2).$$

Discretize the ODE :-

$$\frac{1}{2h^3}\left[y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2}\right] + A_i\left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right) + B_i\left(\frac{y_{i+1} - y_{i-1}}{2h}\right) + C_i y_i = d_i \quad \text{--- } (*)$$

Unknowns y_i's are $y_1, y_2, y_3, \ldots, y_n \rightarrow n$ unknowns.

$$\frac{-3y_0 + 4y_1 - y_2}{2h} = y'_a \quad \text{--- (i)} \qquad \frac{3y_n - 4y_{n-1} + y_{n-2}}{2h} = y'_b \quad \text{--- (ii)}$$

$(n-3) + 2 = n-1$ eqn. involving n unknowns.
(Not a compact system.)

System $(*)$ is not a tridiagonal system. → Very imp. property. (it's very fast of compute)

Let $z = \frac{dy}{dx}$ — (a)

$$\frac{d^2z}{dx^2} + A(x)\frac{dz}{dx} + B(x)\,z + C(x)\,y = D(x) \quad \text{--- (b)}$$

BCs $y(a) = y_a,\quad z(a) = y'_a,\quad z(b) = y'_b.$

Eqn. (a) and (b) are coupled eqn.

Integrate (a) b/w x_{i-1} to x_i to get.

$$\int_{x_{i-1}}^{x_i} dy - \int_{x_{i-1}}^{x_i} z\,dx = 0$$

apply trapezoidal formula.

$$y_i - y_{i-1} - \frac{\delta x}{2}\left[z_i + z_{i-1}\right] + O(h^3) = 0 \qquad \text{—— (i)}$$

$h = \delta x_{\cdot} = x_i -$

which is the discretization of (a).

as no local derivatives are involved.

Use Central difference scheme to discretize (b).

$$\frac{z_{i+1} - 2z_i + z_{i-1}}{h^2} + A_i \frac{z_{i+1} - z_{i-1}}{2h} + B_i z_i + C_i y_i = D_i \qquad \text{—— (ii)}$$

$i = 1, 2, 3, \dots, n-1$

The eq^n (i) & (ii) are linear algebraic eq^n for y_i and z_i.

B.C.s. $y_0 = y_a$, $z_0 = y_a'$, $z_n = y_b'$

System of eq^n. (i) & (ii) forms. $(n-1) + (n-1) = 2n - 2 = 2(n-1)$ eq^n. involving $y_1, y_2, \dots, y_{n-1}$ & $z_1, z_2, \dots, z_{n-1}$ i.e. of $2(n-1)$ unknowns. Thus, the system is compact.

Let, $X_i = \begin{pmatrix} y_i \\ z_i \end{pmatrix}$ is the vector of unknown at x_i.

Find, X_i $i = 1, 2, 3, \dots, n-1$.

Combine eq^n. (i) & (ii) into a matrix form as

$$\underline{a_i}\, X_{i-1} + \underline{b_i}\, X_i + \underline{c_i}\, X_{i+1} = \underline{d_i} \quad i = 1, 2, \dots, n-1.$$

$$-y_{i-1} + y_i - \frac{h}{2} z_{i-1} - \frac{h}{2} z_i = 0.$$

$$z_{i-1}\left(\frac{1}{h^2} - \frac{A_i}{2h}\right) + z_i\left(B_i - \frac{2}{h^2}\right) + z_{i+1}\left(\frac{1}{h^2} + \frac{A_i}{2h}\right) + C_i y_i = D_i$$

$i = 1, 2, \ldots, n-1.$

$$\underline{a_i} = \begin{pmatrix} -1 & -h/2 \\ 0 & \frac{1}{h^2} - \frac{A_i}{2h} \end{pmatrix}. \qquad \underline{b_i} = \begin{pmatrix} 1 & -h/2 \\ C_i & B_i - \frac{2}{h^2} \end{pmatrix}$$

$$\underline{c_i} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{h^2} + \frac{A_i}{2h} \end{pmatrix} \qquad d_i = \begin{pmatrix} 0 \\ D_i \end{pmatrix}.$$

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$X^T = [x_1 \; x_2 \; \cdots \; x_{n-1}]$

$$A = \begin{bmatrix} \underline{b}_1 & \underline{c}_1 & 0 & \cdots & 0 \\ \underline{a}_2 & \underline{b}_2 & \underline{c}_2 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \\ & & & \underline{a}_{n-1} & \underline{b}_{n-1} \end{bmatrix}$$

A is called block tri-diagonal matrix with that, the system of matrix eqn $(*)$ can be expressed as $\boxed{AX = D}$

$$D = \begin{bmatrix} \underline{d}_1 - a_1 x_0 \\ \underline{d}_2 \\ \vdots \\ \underline{d}_{n-1} - c_{n-1} x_n \end{bmatrix}$$

'A is a block tri-diagonal matrix whose elements are matrices $\underline{a}_i, \underline{b}_i, \underline{c}_i$.

For the block system, we apply an algo to reduce to the following form.

$$\begin{bmatrix} I & \underline{c}_1' & 0 & 0 & \cdots & 0 \\ 0 & I & \underline{c}_2 & 0 & \cdots & 0 \\ 0 & & \ddots & \ddots & & 0 \\ & & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & & & I \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{bmatrix} = \begin{bmatrix} \underline{d}_1' \\ \underline{d}_2' \\ \vdots \\ \underline{d}_{n-1}' \end{bmatrix}$$

Block-elimination algorithm.

Q.7 $\quad y''' + 4y'' + y' - 6y = 1 \qquad 0 < x < 1.$

Different types of B.Cs.

$\boxed{\text{b.c.} \quad y(0) = 0 \qquad y'(0) = 0 \qquad y'(1) = 0}$.

b.c. $\quad y'(0) = 0 \qquad y'(1), \; y(1) = \alpha.$

(Try to find a compact set of eq^{n}s.)

$$\frac{dy}{dx} = z$$

$$\int_{x_{i-1}}^{x_i} dy = \int_{x_{i-1}}^{x_i} z\,dx \quad \text{Apply trapezoidal formula.}$$

$$y_i - y_{i-1} - \frac{h}{2}\cdot(z_i + z_{i-1}) + O(h^3) = 0 \quad \text{——— (1)}$$

$$z'' + 4z' + z - 6y = 1.$$

$$\frac{z_{i+1} - 2z_i + z_{i-1}}{h^2} + \frac{\cancel{4}}{2}\cdot\frac{z_{i+1} - z_{i-1}}{\cancel{2}h} + z_i - 6y_i = 1.$$

$$z_{i+1}\left(\frac{1}{h^2} + \frac{1}{h}\right) - z_i\left(\frac{2}{h^2} - 1\right) + z_{i-1}\left(\frac{1}{h^2} - \frac{2}{h}\right) - 6y_i = 1. \quad \text{—— (2')}$$

and,
$$-\frac{h}{2}z_i - \frac{h}{2}z_{i-1} + y_i - y_{i-1} = 0. \quad \text{—— (1)}$$

$$\underline{a_i}X_{i-1} + \underline{b_i}X_i + \underline{c_i}X_{i+1} = \underline{d_i} \qquad i = 1, 2, \dots, n-1$$

$$X_i = \begin{pmatrix} y_i \\ z_i \end{pmatrix} \quad \underline{a_i} = \begin{pmatrix} -1 & -h/2 \\ 0 & \frac{1}{h^2} - \frac{2}{h} \end{pmatrix}. \qquad \underline{b_i} = \begin{pmatrix} 1 & -h/2 \\ -6 & 1 - \frac{2}{h^2} \end{pmatrix}$$

$$\underline{c_i} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{h^2} + \frac{1}{h} \end{pmatrix}. \qquad \underline{d_i} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

$i = 1$, $z_0 = 0$, $y_0 = 0$. (Given).

$i = n-1$, $z_n = 0$, y_n (Not given). $= \alpha$ (arbitrary choice).

We will vary $i = 1, 2, 3, \dots, n-1$. So, $\underline{n-1 \text{ eq}^n\text{s}}$.

$$X_1 = \begin{pmatrix} y_1 \\ z_1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} y_2 \\ z_2 \end{pmatrix}, \dots, X_{n-1} = \begin{pmatrix} y_{n-1} \\ z_{n-1} \end{pmatrix}$$

So, $\underline{n-1 \text{ variables}}$.

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We get a compact system., $n-1$ eq^{n}s and $n-1$ variables.
- Block-tri-diagonal form will be $(n-1 \times n-1)$ matrix.

$$A = \begin{bmatrix} b_1 & c_1 & 0 & 0 & \cdots & 0 \\ a_2 & b_2 & c_2 & 0 & \cdots & 0 \\ & & \ddots & \ddots & & \\ & & & & a_{n-1} & b_{n-1} \end{bmatrix}$$

Remark:-
as, in c_n place of y_n is 0, we can take any arbitrary choice of y_n.

$$D_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ldots\ldots, D_{n-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Q.5

$$y^{(4)} + 81y = 81x^2, \quad 0 < x < 1$$

$$y(0) = y(1) = y''(0) = y''(1) = 0.$$

$$z = y'' \Rightarrow y'' - z = 0. \quad \text{--- (1)}$$

$$z'' + 81y = 81x^2 \quad \text{--- (2)}$$

$$y(0) = y(1) = 0$$
$$z(0) = z(1) = 0 \quad \text{(Modified B.Cs).}$$

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - z_i = 0. \quad \text{--- (1)}$$

$$\frac{z_{i+1} - 2z_i + z_{i-1}}{h^2} + 81y_i = 81x_i^2 \quad \text{--- (2)}$$

$$\boxed{a_i x_{i-1} + b_i x_i + c_i x_{i+1} = d_i}. \qquad x_i = \begin{pmatrix} y_i \\ z_i \end{pmatrix}.$$

$$a_i = \begin{pmatrix} 1/h^2 & 0 \\ 0 & \frac{1}{h^2} \end{pmatrix}. \qquad b_i = \begin{pmatrix} -2/h^2 & -1 \\ 81 & -2/h^2 \end{pmatrix}$$

$$c_i = \begin{pmatrix} 1/h^2 & 0 \\ 0 & 1/h^2 \end{pmatrix} \qquad d_i = \begin{pmatrix} 0 \\ 81x_i^2 \end{pmatrix}$$

$Ax = d$ $\longrightarrow$ $A \to$ block tridiagonal matrix.

$i = 1, 2, 3, \ldots, n-1$. (n-1 eq^ns.)

we have, $X_0 = \begin{pmatrix} y_0 \\ z_0 \end{pmatrix}$ (Given).

and, $X_n = \begin{pmatrix} y_n \\ z_n \end{pmatrix}$ (Given)

$X_1, X_2, \ldots, X_{n-1}$ are $(n-1)$ unknowns.

$\therefore$ Compact system.

a_i, b_i, c_i's are $m \times m$ matrices and v_i, d_i are m-component vectors. Sol^n of this system.

$v_n = d_n'$, $v_i = D_i - C_i' v_{i+1}$ $\quad i = n-1, n-2, \ldots, 2, 1.$

(back substitution).

Convert it to upper Δular matrix.

$$\begin{bmatrix} B_1 & C_1 & 0 & \cdots & 0 \\ A_2 & B_2 & C_2 & \cdots & 0 \\ 0 & A_3 & B_3 & C_3 \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & & A_n & B_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} D_1 \\ D_2 \\ \vdots \\ \vdots \\ D_n \end{bmatrix}$$

where,

$C_1' = (B_1)^{-1} C_1,$

$D_1' = (B_1)^{-1} D_1.$

$B_i' = B_i - A_i C_{i-1}'$

$C_i' = (B_i')^{-1} C_i$

$D_i = (B_i')^{-1} \{ D_i - A D_{i-1}' \}$

$i = 2, 3, \ldots$

Example. $y^{(4)} - y^{(3)} + y = x^2$ $\quad y(0) = y'(0) = 0$

$y(1) = y'(1) = 0$

We can't use $Z = y''$, because of B.C.s.

$h = 0.25$.

$$y^{(4)} = (y'')'' = \frac{y_{i-1}'' - 2y_i'' + y_{i+1}''}{h^2} + O(h^2).$$

$$= \frac{1}{h^4} \left[y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2} \right]$$

Put, $i = 1, 2, 3, \ldots, n-1$.

Introduce two fictitious points. y_{-1} and y_{n+1}.

$y_0' = 0 \Rightarrow y_1 = y_{-1}.$

$y_n' = 0 \Rightarrow y_{n+1} = y_{n-1}.$

$$\left[y_i''' = (y_i'')' = \frac{y_{i+1}'' - y_{i-1}''}{2h}. \right]$$

Discretize the ODE for $n = 1, 2, \ldots, n-1$.