Frequency - Response

Theory

$$U(s) \longrightarrow \boxed{G(s)} \longrightarrow Y(s)$$

For a linear Invariant system represented by $G(s)$

Input: $u(t) = sen(\omega . t)$, $t > \emptyset$

sinusoidal transfer function

$$\boxed{G(j\omega) = |G(j\omega)| \; \underline{/G(j\omega)}}$$

Amplitude Ratio of output and input

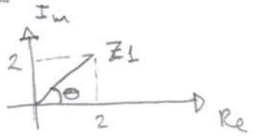
Phase shift between the input and the output

output: $\left[\, y(t) = |G(j\omega)| \cdot sen\left[\omega t + \underline{/G(j\omega)}\right] \,\right]$

Complex Numbers

Modulos — Argument

- $Z_1 = 2 + 2i$



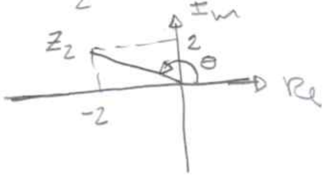
$|Z_1|^2 = 2^2 + 2^2$

$|Z_1| = \sqrt{2^2 + 2^2}$

$|Z_1| = \sqrt{8}$

$arg(Z_1) = tg^{-1}\left(\frac{2}{2}\right) = \frac{\pi}{4} = 45^\circ$

- $Z_2 = -2 + 2i$



$|Z_2|^2 = (-2)^2 + 2^2$

$|Z_2| = \sqrt{8}$

$arg(Z_2) = \pi - tg^{-1}\left(\frac{2}{2}\right) = \frac{3}{4}\pi$ ou 135°

- $Z_3 = -2 - 2i$

$|Z_3|^2 = (-2)^2 + (-2)^2$

$|Z_3| = \sqrt{8}$

$arg(Z_3) = -\pi + tg^{-1}\left(\frac{2}{2}\right) = -\frac{3}{4}\pi = -135^\circ$

- $Z_4 = 2 - 2i$

$|Z_4|^2 = 2^2 + (-2)^2$

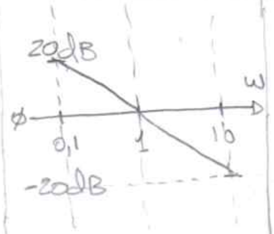
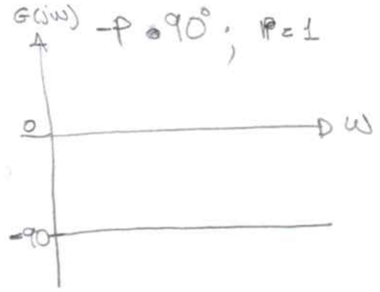
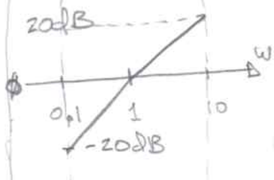
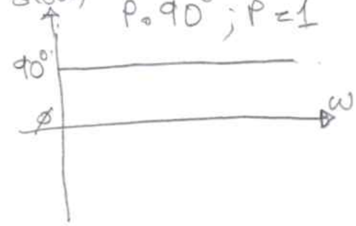
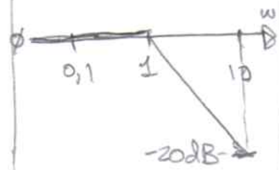
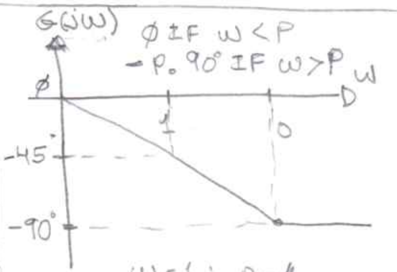
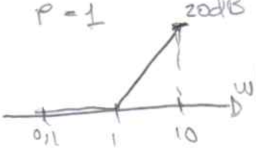
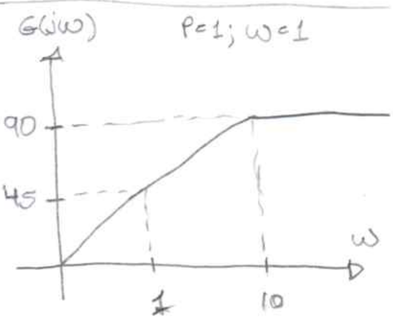
$|Z_4| = \sqrt{8}$

$arg(Z_4) = -tg^{-1}\left(\frac{2}{2}\right) = -\frac{\pi}{4} = -45^\circ$

Frequency Response

theory

Basic Factors of $G(j\omega)$

type of term	$G(j\omega)$	slope (dB/deg)	Magnitude	Phase Angle (Degrees)
Gain K_B	K_B	$\emptyset$	$2\emptyset \log K$	0° IF $K_B > \emptyset$ -180° IF $K_B < \emptyset$
Pole at origin	$\frac{1}{(j\omega)^p}$	$-20 \times p$	 p=1 $-20 \cdot p \cdot \log\omega$	$-p \cdot 90^\circ$; p=1 
Zero at origin	$(j\omega)^p$	$20 \times p$	p=1 $20 \cdot p \cdot \log\omega$ 	$p \cdot 90^\circ$; p=1 
First order Pole	$\frac{1}{\left(1+\frac{j\omega}{p}\right)^p}$	$\emptyset$ IF $\omega < p$ $-20 \cdot p$ IF $\omega > p$	p=1 ω=1 	$\emptyset$ IF $\omega < p$ $-p \cdot 90^\circ$ IF $\omega > p$  ω=1; p=1
First order Zero	$\left(1+\frac{j\omega}{z}\right)^p$	$\emptyset$ IF $\omega < z$ $+20 \cdot p$ IF $\omega > z$	ω=1 p=1 	p=1; ω=1  $\emptyset$ IF $\omega < z$ $+p \cdot 90^\circ$ IF $\omega > z$

Exercicio extra 1:

$$\boxed{G(s) = \frac{a}{s}} \Rightarrow \text{Pole at origin}$$

- Replace "s" by "jw":

$$G(j\omega) = \frac{a}{j\omega}$$

- $|G(j\omega)|$: [$a = 1$]

$$20 \cdot Log\,|G(j\omega)| = 20\,Log(|a|) - 20\,Log(|j\omega|)$$

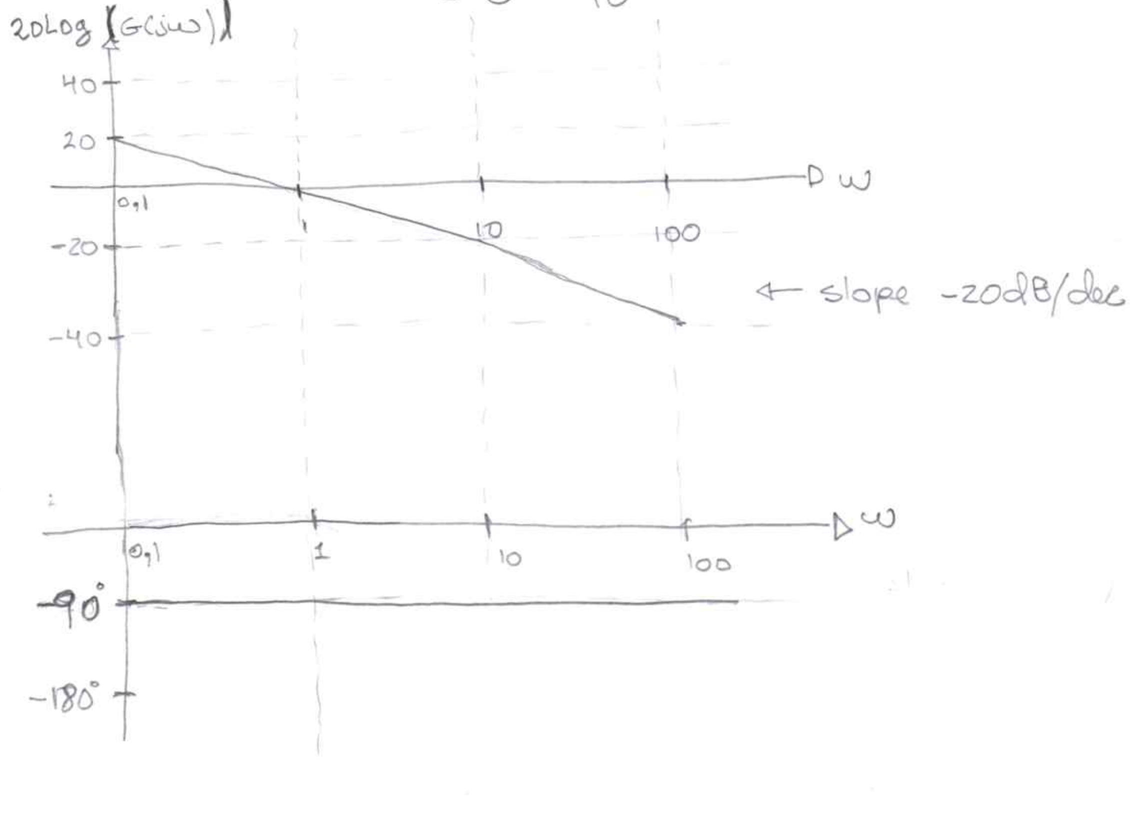
$$= 0 - \underbrace{20\,Log(\omega)}_{\text{Pole at origin}}$$

G(jw)	slope dB/dec	Magnitude dB	Phase
$\frac{1}{j\omega}$	-20	-20log(w)	-90

- $\angle G(j\omega)$:

$$G(j\omega) = arg\left(\frac{1}{j\omega}\right) = arg(1) - arg(j\omega)$$

$$= 0 - 90^\circ$$



Exercicio extra 2:

$\boxed{G(s) = s}$ $\Rightarrow$ Zero at origin

- Replace "s" by "jω":

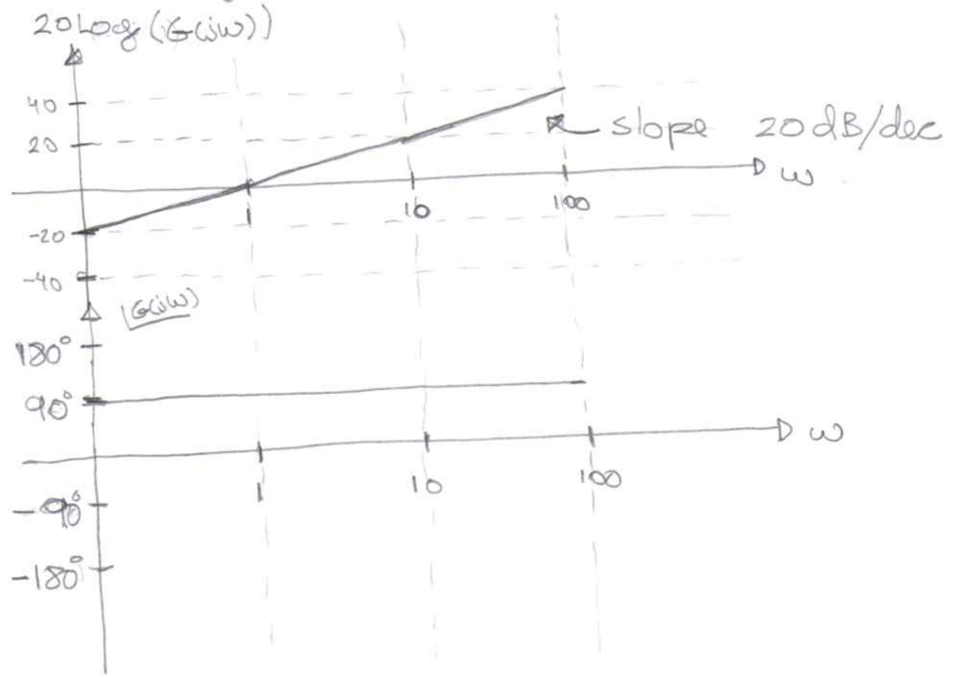
$G(j\omega) = j\omega$

- $|G(j\omega)|$:

$20 \cdot Log|G(j\omega)| = 20\,Log(\omega)$ $\Rightarrow$

	$G(j\omega)$	Slop dB/dec	Magnitude dB	Phase
Zero at origin	$j\omega$	20	$20 Log\omega$	90°

$\underline{/G(j\omega)} = arg(j\omega) = 90°$



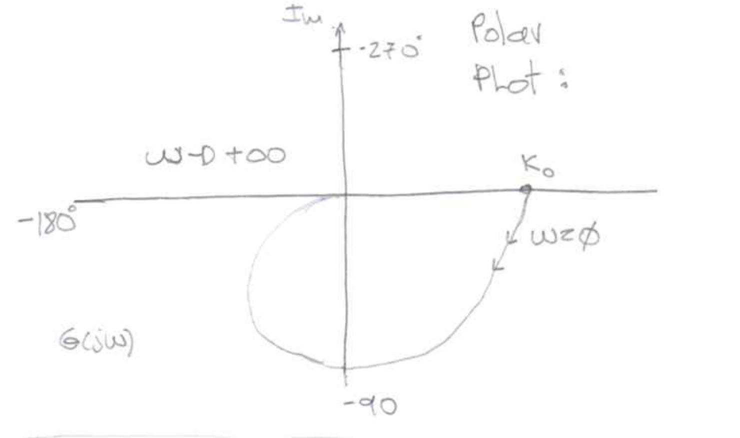
Frequency Response — theory

Polar Plots

Type $\emptyset$ system:

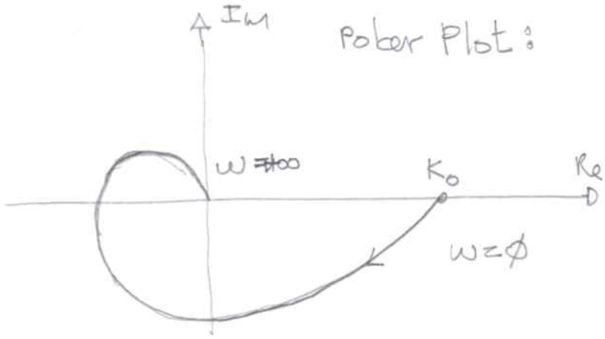
- $$G(j\omega) = \frac{K_o}{\left(1+\frac{j\omega}{p_1}\right)\cdot\left(1+\frac{j\omega}{p_2}\right)}$$

$$G(j\omega) \Rightarrow \begin{cases} K_o \angle 0^\circ & \omega \to 0^\circ \\ \emptyset \angle -180^\circ & \omega \to \infty \end{cases}$$



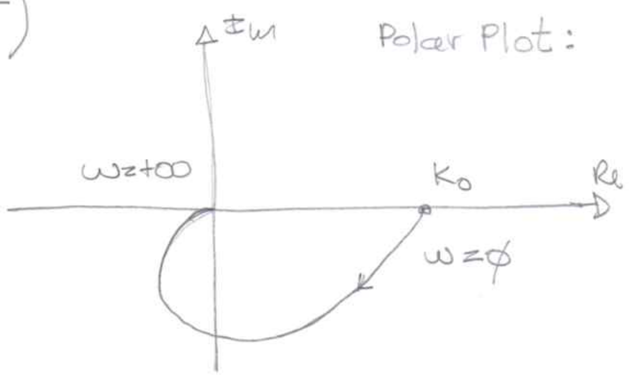
- $$G(j\omega) = \frac{K_o}{\left(1+\frac{j\omega}{p_1}\right)\cdot\left(1+\frac{j\omega}{p_2}\right)\cdot\left(1+\frac{j\omega}{p_3}\right)}$$

$$G(j\omega) \Rightarrow \begin{cases} K_o \angle 0^\circ & \omega \to 0^+ \\ \emptyset \angle -270^\circ & \omega \to \infty \end{cases}$$



- $$G(j\omega) = \frac{K_o\left(1+\frac{j\omega}{z_1}\right)}{\left(1+\frac{j\omega}{p_1}\right)\cdot\left(1+\frac{j\omega}{p_2}\right)\cdot\left(1+\frac{j\omega}{p_3}\right)}$$

$$G(j\omega) \Rightarrow \begin{cases} K_o \angle 0^\circ & \omega \to 0^+ \\ \emptyset \angle -180 & \omega \to \infty \end{cases}$$



Frequency Response

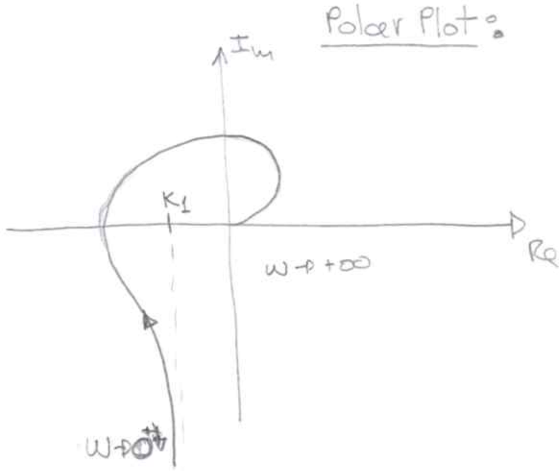
theory

Type 1 systems:

- $G(j\omega) = \dfrac{K_1}{j\omega\left(1+\frac{j\omega}{p_1}\right)\cdot\left(1+\frac{j\omega}{p_2}\right)\cdot\left(1+\frac{j\omega}{p_3}\right)}$

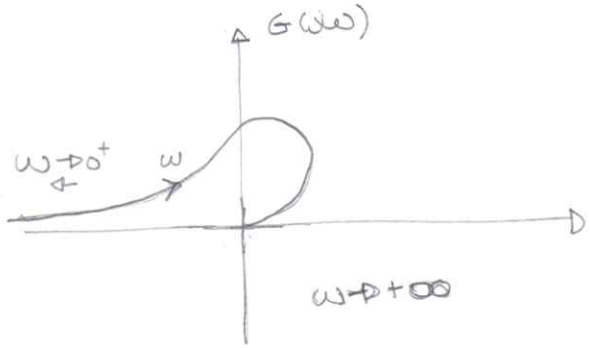
$$G(j\omega) \to \begin{cases} K_1 \ \underline{\lfloor -90^\circ} & \omega \to 0^+ \\ \phi \ \underline{\lfloor -360^\circ} & \omega \to \infty \end{cases}$$

Polar Plot:

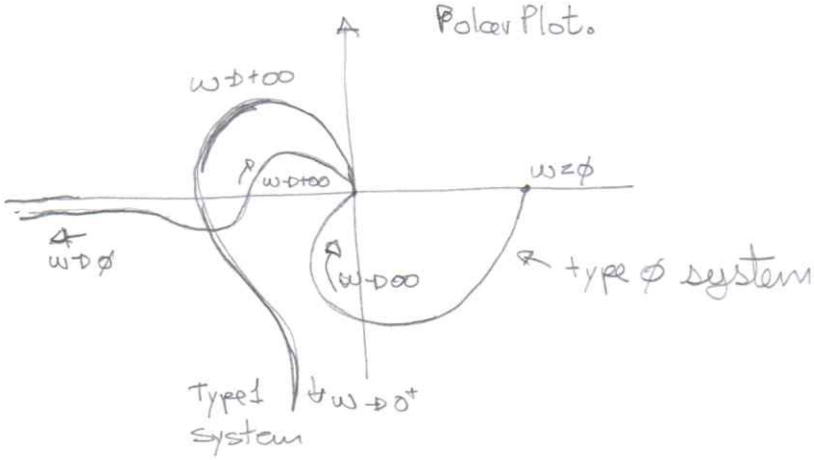


Type 2 systems:

- $G(j\omega) = \dfrac{K_2}{(j\omega)^2\left(1+\frac{j\omega}{p_1}\right)\left(1+\frac{j\omega}{p_2}\right)}$



Polar plots Base on the system



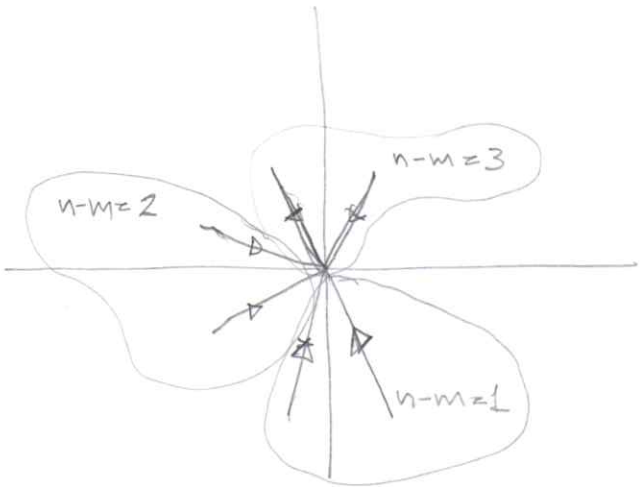
Retirado de "Modern control Engineering"

Frequency Response

theory

Polar plots in the high-frequency range.

$$G(j\omega) = \frac{b_0 (j\omega)^m + \dots}{a_0 (j\omega)^n + \dots}$$



Frequency Response

theory

Base Factors of $G(j\omega)$

type of term	$G(j\omega)$	slope (dB/dec)	Magnitude	Phase angle degrees
transport Lag	$e^{-j\omega T}$	ϕ	ϕ dB	$\angle G(j\omega) = -\omega T$ (rad) 6a) $\angle G(j\omega) = -57,3\,\omega T$ (degrees)

Quadractic factors see in the theoretical slides

Frequency Response

theory

Polar Plots

Extra exercise 1:

$$G(s) = \frac{1}{(s+1)(s+2)}$$

- Rewrite the T.F as a product of basic factors:

$$G(s) = \frac{1}{(s+1)} \cdot \frac{1}{(s+2)}$$

- convert into standard time constant form:

$$G(s) = \frac{1}{(s+1)} \cdot \frac{1}{2(\frac{s}{2}+1)}$$

$$= \frac{1}{2} \cdot \frac{1}{(s+1)} \cdot \frac{1}{(\frac{s}{2}+1)}$$

- Replace "s" by "jω":

$$G(j\omega) = \frac{1}{2} \cdot \frac{1}{(j\omega+1)} \cdot \frac{1}{(\frac{j\omega}{2}+1)}$$

- Evaluate $|G(j\omega)|$ and $\underline{/G(j\omega)}$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{\omega^2+1^2}} \cdot \frac{1}{\sqrt{(\frac{\omega}{2})^2+1^2}} \qquad \Big| \; 0 - tg^{-1}\left(\frac{\omega}{1}\right) - tg^{-1}\left(\frac{\omega}{2}\right)$$

ω	$\|G(j\omega)\|$	$\underline{/G(j\omega)}$
∅	$\frac{1}{2}$	0°
0,1	0,497	−8,6°
1	0,316	−71,6°
2	0,158	−108,4°
⋮		—
∞	∅	−180°

Polar Plot: