Discrete Mathematics (SC 612)
Second Insemester Exam

October 7^{th}, 2025

Time: 2 hours
Marks: 20 $\times$ 5

1. A binomial coefficient is defined as any number which can be written in the form $\binom{n}{r}$, for any positive integer n and any non-negative integer $r \leq n$. Example: 6 is a binomial coefficient because $\binom{4}{2} = \binom{6}{1} = \binom{6}{5} = 6$. It can be shown that 6 is not a binomial coefficient in any other way and is thus a binomial coefficient in exactly three ways.
   (a) Show that 1 is the only positive integer that is a binomial coefficient in infinitely many ways.
   (b) Show that every positive integer is a binomial coefficient in at least 1 way.
   (c) Show that 2 the only number is a binomial coefficient in exactly 1 way.
   (d) Show that prime numbers > 2 are binomial coefficients in exactly 2 ways.
   (e) Show that there are positive integers that are binomial coefficients in exactly 3 ways.
   (f) Show that there are positive integers that are binomial coefficients in exactly 4 ways.
   (g) Show that there are no integers that are binomial coefficients in 5 or more ways.
   (h) Characterise numbers that are binomial coefficients in exactly 3 ways.
   (i) Characterise numbers that are binomial coefficients in exactly 2 ways.
   (j) Characterise numbers that are binomial coefficients in exactly 4 ways.
2. Count the number of equivalence relations on the set $S = \{1, 2, 3, 4, 5, 6\}$.
3. How many 4 length strings can be generated using only the symbols a and b such that you are not allowed to use the same letter in three successive positions?

4. Let $f : S \to S$ be a function, where $|S|$ is finite.

   (a) What is the condition such that $\exists i, \forall x, f^i(x) = x$.

   (b) Give an example of such a function for $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ for which the smallest value of i such that $\forall x, f^i(x) = x$ is $i = 30$.

   Here $f^i(x)$ represents applying the function repeatedly i times, each time, on the answer to the previous time. That is $f^1(x) = f(x)$ and $f^i(x) = f(f^{i-1}(x))$, for $i \geq 2$.

5. Consider the set $S = \{10, 14, 22, 35, 55, 77\}$. Define a relation $R = \{(x, y) | x | y\}$.

   (a) Prove that this relation is a partial order.

   (b) Prove that it is neither an upper lattice nor a lower lattice.

   (c) What are the lengths of the longest chain and the longest anti-chain for this partial order?

   (d) Add exactly two elements to this set S to get a S', such that the identical relation on this set S' is a lattice. The choice of maximal element you add must be as small as possible.

   (e) Consider the maximum element in the set S' obtained in part (d). Add as many elements as possible to the set S' that are smaller than that maximum element to get a set S'' such that the relation defined is still a lattice.

   (f) Draw the Hasse Diagram for the partial order on the set S''.