First order Logic and role of logic in proofs

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First order logic also known as **predicate logic** has two fundamental operators or connectives:

- **Existential quantifier:** denoted symbolically by $\exists$ and is used to quantify the presence of atleast one element over a set that satisfies some condition. The condition is usually a proposition, or a formula in propositional logic, but can also be from other contexts.
- **Universal quantifier:** denoted symbolically by $\forall$.

The two most important logical equivalences in first order logic are:

- $\neg(\forall x, P(x)) \equiv \exists x, \neg(P(x))$
- $\neg(\exists x, P(x)) \equiv \forall x, \neg(P(x))$

The first of the two above is used in the **proof technique** called **proof by counter example**. We used it to refute the dubious conjecture that for every positive multiple of 10, say $y = 10x$, the number of primes $\leq y$ is $4x$.

Some examples of use of first order logic in mathematics:

In the context of limits in calculus, consider

$$f(x) = \frac{x^2 - 6x - 7}{x - 7}$$

and

$$L = \lim_{x \to 7} f(x) = 8$$

This can be rephrased as

$$\forall \epsilon \in R, \exists \delta \in R | \forall x \in ((8 - \delta, 8 + \delta), |L - f(x)| \leq \epsilon$$

A second example of an application of first order logic is an alternating two player strategy game. A player P_1 to play has a winning strategy in the current game configuration against player P_2, if:

$$\exists m_1 | \forall m'_1 \exists m_2 | \forall m'_2 | \cdots | \exists m_k \text{ player } P_1 \text{ has a win}$$

This is an example of **nested quantifiers** (several quantifiers within a chain).

In propositional logic we have associated with $p \Rightarrow q$ what is known as its **converse**, namely $q \Rightarrow p$. Thse are not logically equivalent. In fact these are usually the two sides of a proof of a theorem of the **if and only if** type.

On the other hand, $((\neg q) \Rightarrow (\neg p)) \equiv (p \Rightarrow q)$. The formula on the left hand side of the equivalence is the **contrapositive** of the one on the right and **vice versa**. **Proof by contraposition** is a powerful proof technique.