Discrete Mathematics (SC612)
Tutorial 7
25^{th} September, 2025

1. Count the number of positive integers relatively prime to 300 in the set $\{1, \ldots, 300\}$.

   Hint: This is an application of the principle of inclusion & exclusion.

2. Consider strings of length exactly 20 over the alphabet $\Sigma = \{9, 1\}$. A monotic block is a maximal substring of identical characters. Example $s = 00111001110011111010$. The monotic block lengths reading from left to right are: 2,3,2,3,2,5,1,1,1. The total is 20. Count the number of strings such that the block lengths are strictly increasing from left to right. The example we gave does not qualify, because for example the third block is shorter than the second block. Also the the 8^{th} block and 9^{th} block have identical length. It is allowed that there is only one block, and the strings may start with either 0 or 1.

3. We saw in the lecture that for a grid consisting of $m + 1$ vertical lines 0 to m, and $n + 1$ horizontal lines 0 to n, the number of shortest paths from $(0, 0)$ to (m, n) is $\binom{m+n}{m}$.

   (a) Find the smallest number of grid points (other than $(0, 0)$ and (m, n)) that must be deleted, such that exactly one of the original shortest paths remain.

   (b) Is it possible to delete nodes in such a way that all the original shortest paths (length $m + n$) between $(0, 0)$ and (m, n) are destroyed, but paths (of longer length) still remain. What is the smallest number of points to be deleted to achieve this?

4. Suppose you have a 6×6 grid (number of squares, not, number of parallel lines. Suppose you need to colour all the 36 squares with colours from $Col = \{Red, Blue, Green, Orange\}$ such that the squares immediately to the left, right, top and bottom of any square cannot be coloured the same colour as that square. How many ways are there to colour the entire grid? Can you first conclude that the number is a multiple of 24, even before making the actual calculation?

5. Suppose you are given a list of **ordered pairs** of integers, such that no two ordered pairs have the same integer in their first coordinates and also no two integers have the same integer in their second coordinates. You are required to make a hierarchical arrangement such that at the head must be the ordered pair with the smallest first coordinate. The remaining ordered pairs are to be partitioned into two disjoint sets, such that in one set all the elements have second coordinate smaller than the head's second coordinate, and the other set is such that all the elements have second coordinate larger than the head's second coordinate. You need to repeat the same proicedure on each of the two sets, until all the elements have been placed in the hierarchy. Perform this for the input: $(3, 8), (5, 6), (11, 2), (33, 29), (51, 22), (4, 43), (18, 18)$. In how many ways can this hierarchy be made in general?

6. In how many ways can n distinct integers be arranged in a sequence such that the longest number of consecutively increasing or decreasing elements is 2?