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A minimum spanning tree with node index ≤ 2

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Abstract: This paper considers a connected graph *G(N,L),* where *N={1,2,…,* n*}* is a set of n nodes and *L={1,2,…,* k*}* is a set of k links joining these nodes to form a connected graph. The minimum spanning tree of a graph can be easily obtained by any existing method, where the node index at each node i is likely to vary and it can be $1 \leq n_i \leq (n-1)$. This paper develops a method to find the minimum spanning tree, such that the node index of each node n_i is such that $1 \leq n_i \leq 2, \forall i$. It is anticipated that such a spanning tree may have an application in determination of the travelling salesman tour.

Keywords: Connected graph; Minimum spanning tree; Node index; Travelling salesman problem.

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1. Introduction

Given a list of cities and their pair-wise distances, the minimum spanning tree (MST) of a given graph can be obtained iteratively by any greedy approach, which is linear in time and converges in $(n-1)$ iterations, where n is the number of nodes. There are many methods to find the MST, see for example [2]. The method arbitrarily starts from any node and connects that node to a nearest node, forming a spanning tree of the two nodes. In the next iteration, one more node is selected which is nearest to one of them and also not forming a loop with already selected nodes. Ties are resolved arbitrarily. After $(n-1)$ such iterations, all nodes and the selected $(n-1)$ links form a MST of the given graph. The index of a node in the MST is given by the number of arcs joining this node to other nodes. It is obvious that the index value for each node will be at least one and at most $(n-1)$, the maximum being realized when all nodes are connected to the same node. Let the index of the node i be denoted by n_i. In this paper, we find a MST when the index of a node i is subject to the condition that $n_i \leq 2, \forall i$. Such a MST may have applications in the determination of the travelling salesman tour (TST), which will be considered in a subsequent publication. The TST has applications in planning, logistics and microchip design in its purest form. Additional constraints such as limited resources or time windows, which make the TST considerably harder, have also been considered [10,11]. An excellent overview, applications, formulations and solution procedures have been discussed in [7].

The MST is one of the most well-known problems in combinatorial optimization [5]. According to [5], the MST is the shortest distance that is used to connect all the nodes in a network. A new approach based on multi-objective metrics and MST was designed for dense urban areas by [12]. Zhaocai et al. [14] defined the MST as a problem of finding the minimum edge connected subsets containing all the vertex of a given undirected graph. They have come up with a new and fast algorithm for solving the Minimum Spanning Tree problem based on DNA molecules computation. Concurring with both [14], [5] and [12], [1,2] also defined the MST as the problem of finding a spanning tree with minimum total cost such that each non-leaf node in the tree has a degree of at least d (d>2). The MST previously used to perform more comprehensive studies of asset returns correlations, can also be used to deduce the underlying ownership structure with reasonable accuracy [13]. The Euclidean MST-based evolutionary Algorithm to solve multi-object optimisation problems was proposed by [6].

In computational complexity theory the travelling salesman problem (TSP) belongs to the class of NP-complete problems, which means that in the worst-case running time of the TSP may increase exponentially with the number of cities [8,9]. An advantage of the proposed MST approach is that when it is applied to the TSP, it has potential to reduce complexity of the TSP, which is exponential to many problems but each being in the linear form.

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The supporting theory justifying the proposed MST method is discussed in Section 2. The algorithm is presented in Section 3. Numerical illustrations have been given in Section 4 and finally the paper has been concluded in Section 5.

2. The problem statement and the mathematical support

2.1 The problem

For a given graph *G(N,L),* the MST obtained by any known greedy approach will have node indexes n_i, where $1 \leq n_i \leq (n-1)$ for node i and hence such MST will have to be modified to satisfy the condition $1 \leq n_i \leq 2, \forall i$. In this paper, it is assumed that the 'n' node network is a connected graph where each node has at least two arcs emanating from it.

2.2 Network definitions and modification theorems

2.2.1Definitions

Index of a node: is given by the number of arcs emanating from that node. Since total number of selected arcs in a MST will be $(n-1)$, the total index value of these selected arcs in a MST will be $2(n-1)$. In an extreme case, the total index number can be distributed such that $n_i = 1$ for $(n-1)$ nodes and for one node the index can be $n_i = (n-1)$, i.e. when all nodes are connected to one node. When the index has to satisfy $n_i \leq 2, \forall i$, however, it is clear that the selection of arcs forming the minimum connected graph will have to be readjusted. Since the MST will be comprised of all nodes and $(n-1)$ selected arcs, the number of nodes with index 2 will be at most $(n-2)$, and the remaining two nodes will have the index 1 to get the total index value of $2(n-1)$. Therefore, the selected arcs joining nodes with index > 2 will have to be replaced by other arcs to balance out the index requirement on each node. The network modification theorem given in this paper attains even distribution requirement of the index values.

Basis arc: An arc connecting two nodes i and j is said to be ***basic*** if $x_{ij} = 1$, i.e. it belongs to the MST solution. If $x_{ij} \neq 1$, then the arc is said to be ***non-basic***.

High and low index nodes: Since the number of basic arcs emanating from a node gives its index value, a node is called a *high index node* if its number of basic arcs is greater than two and a low index node if the number of basic arcs is one. We require a MST, where the index n_i at node i satisfies the condition $1 \leq n_i \leq 2, \forall i$.

Neighbouring arcs: These are arcs that emanate from neighbouring nodes. A node i is said to be a neighbour to node j if the two nodes i and j are connected by a single arc.

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In a completely connected graph, all nodes are neighbouring nodes as all pairs of nodes are directly connected by single arcs.

2.2.2 Balancing index by arc weight modification

Theorem 1 Adding the same constant to all arcs emanating from the same node does not change the relative merit of any given selected arc in the MST but can create alternatives. From the MST obtained by the greedy approach, we know the index for each node, and therefore, high and low index nodes can be easily identified. The balancing of index number will commence by adding constants to some selected arcs.

In a completely connected graph, each node has $(n-1)$ arcs emanating from it. In a connected graph, the number of arcs emanating from a node can be less than or equal to $(n-1)$. Consider that there are $(n-1)$ arcs emanating from a node i as shown in Figure 1.

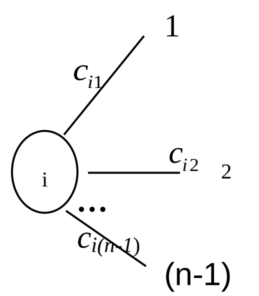

Figure 1: Arcs emanating from a node

Adding a constant μ to all arcs in Figure 1 generates the modified distances as shown in Figure 2.

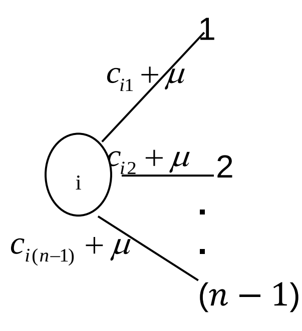

Figure 2: After adding a constant μ

Proof

When an arc length is changed by adding a constant quantity to each arc emanating from that node, it will have consequences on both sides of the arc, i.e. arc-weight distribution of two nodes will be affected. Therefore, two separate considerations are

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required at both ends of the arc. Consider that adding a constant modifies the arcs emanating from the node. The motivation for this modification is to create an alternative for the MST. When the node k is a candidate for modification as the index n_k is > 2; let the arc (k, p) be currently a MST member causing the imbalance. If length of the arc (k, p) is increased equal to an arc (p, l), which at present is not a member of the MST as the arc-weight (k, p) < arc-weight (p, l). After adding a constant, the arc (k, p) is made equal to arc weight (p, l); the arc (k, p) can now be replaced by the arc (p, l) in the MST. Thus altering an arc-weight of (k, p) brings the corresponding index value change at node k, as well as at node p. The index at node k goes down by 1 and the index at node p goes up by 1.

Consideration at the node *k*

Let the optimal MST be of length $(\mathrm{L}[\tau_o])$. It is a sum of $(n - 1)$ selected arc-weights in the given $n$node network. The MST under the index restriction can have at most two arcs emanating from node 'i'; one of them will give entry and the other will provide exit from that node. Since $(\mathrm{L}[\tau_o])$ is minimum, the same MST will remain minimum in the modified network as shown by relation (1).

$$L[\tau_\circ] + 2\mu \leq \min(\{L'[\tau_k] + 2\mu\}) \quad (1)$$

Note that relation (1) holds since (2) is true by the definition.

$$L[\tau_\circ] \leq \min\{L'[\tau_\circ]\} \quad (2)$$

Here $L[\tau_\circ]$ represents the minimum weight MST and $L'[\tau_\circ]$ represents the set of weights of other MSTs excluding the minimum weight.

The constant μ is a positive quantity that created an alternative without changing the relative merit of a given MST.

Consideration at the other end of the arc, i.e. at nodes $n - 1$

Since in a connected graph all arcs emanating from a node 'k' are changed, we have also changed arc weights from other nodes 'p' to this node 'k'. Thus, relative merits of the arcs from the node 'p'are changing. However, the affected arcs have no place in the MST as these other arcs were not belonging to the MST but just create only alternatives.

Thus, arc weights can be modified in the above manner, resulting in an equivalent network with alternative MSTs.

2.2.3 Balancing index by arc weight modification –Theorem 2

For any given MST solution the number of basic arcs emanating from node i can be altered by adding a constant to all arcs emanating from that node.

Proof

Let any two neighbouring nodes of node i be j and k as shown in Figure 3.

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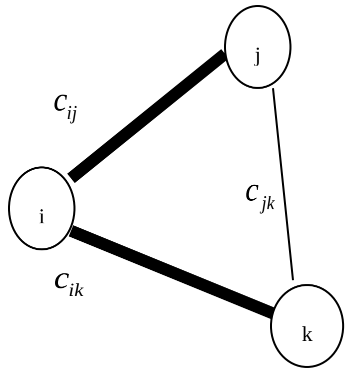

Figure3: Neighbouring nodes

In Figure 3 the arcs ($i;j$) and ($i;k$) are basic.

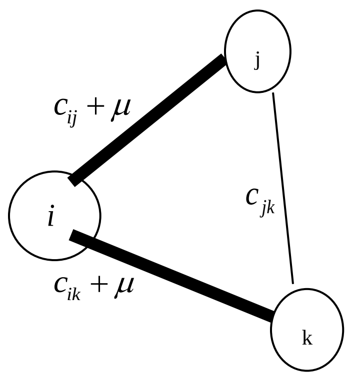

Figure 4: Addition of a constant μ

If μ is a positive quantity such that, $c_{ik}+\mu > c_{jk}$ or $c_{ij}+\mu > c_{jk}$, then the new MST solution becomes as shown in Figure 5 or in Figure 6.

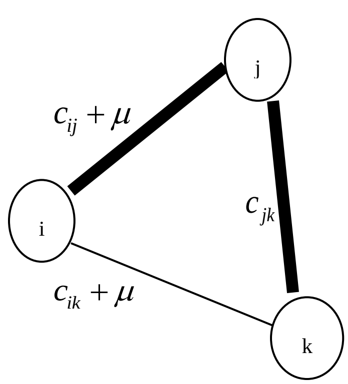

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Figure 5: New MST when $c_{ik} + \mu \geq c_{jk}$

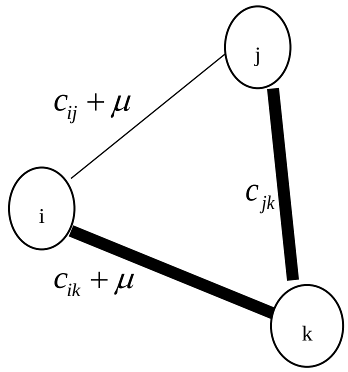

Figure 6: New MST when $c_{ij} + \mu > c_{jk}$

These diagrams show ways of reducing the number of basic arcs emanating from a given node i. Only three nodes and three arcs are used to illustrate the theorem. Method for any number of nodes greater than three is similar to the above case. The question is how to find the quantity μ.

The purpose of including an additional quantity to the existing arc weights is to create alternative arcs that can qualify to become basic as a member of the new MST. Thus, one can alter the number of basic arcs from a given node. The value of μ is the minimum difference to create an alternative arc to form a new MST.

2.2.4 MST path

The MST under the index condition$(1 \leq n_i \leq 2, \forall i)$ is a path. Note that theorem 1 is applicable to any high index node; its repeated applications can modify index value to a desired value, which in this case is 2. When $(n - 2)$ nodes have index 2, the remaining two nodes will have index 1; they form a path. This path may be useful for the TSP. Such a path will be also useful for the situation when a single truck is being used to deliver seeds to various centers and deliveries must be done before the season starts. It is assumed that seeds are being sourced from a supplier who is willing to deliver to any desired starting point. In this case the MST path will give a far better solution when seeds are delivered at the node with index value 1.

An example of a MST path is given in Figure 7.

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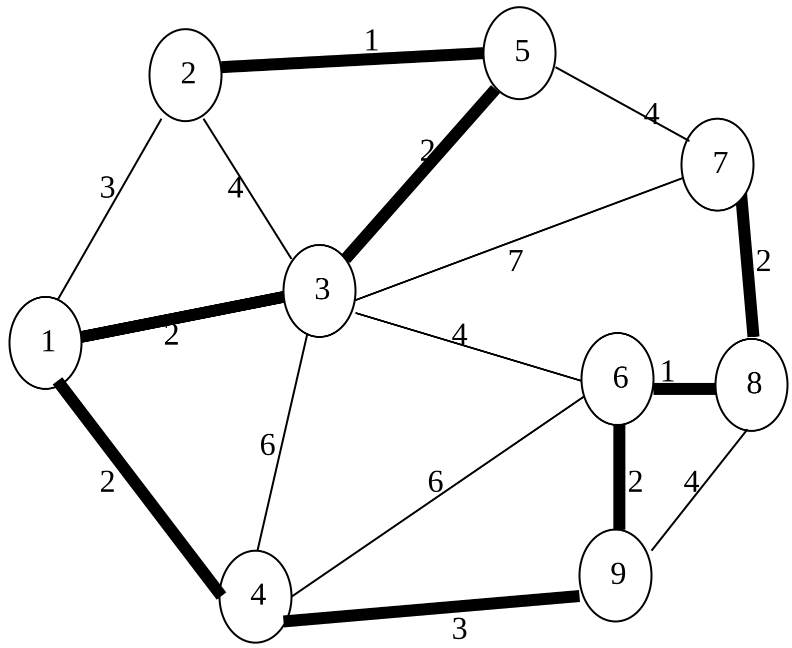

Figure 7: An MST path between nodes 2 and 7.

The MST solution in Figure 7 is a path. If the supplier is willing to deliver to any of the 9 centers represented by nodes then it makes sense to have the delivery either at node 7 or node 2. If the associated weights in this graph are in hours then the shortest time in which deliveries can be made to all nodes is:

$$L[MST] = 2+1+2+3+2+2+2+1 = 15\text{hours}. \quad (3)$$

Note that there is no other delivery time less than 15 hours.

3. The algorithm

The algorithm to find the MST with index restriction can be described as follows:

Step 1: Find MST of the given graph by any known method. If in the process of arc selection, a tie is experienced, always select the arc that does not increase degree of a node beyond 2. Go to Step 3.

Step 2: Find the MST of the modified network and go to Step 3.Once again ties are resolved as in step 1. As arc lengths are modified, more and more ties will be observed. Always select an arc that does not increase the index of a node beyond 2, if possible.

Step 3: Check if the MST obtained satisfies index conditions i.e. all nodes have index less than or equal to 2? If the answer is no, go to step 4 else go to step 5.

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Step 4: Select a node 'k' with index 3 or more. With the help of the neighbouring arcs, find the minimum value μ to reduce the index at the high index node 'k'. Reduction in index is achieved by adding an appropriate minimum quantity μ to all arcs emanating from the selected node 'k'. This way creates an alternative and change in the basic arc to reduce index of the node 'k'and increase index of a low index node. Go to step 2.

Step 5: The optimal MST is obtained when all index conditions are satisfied.

4. Numerical illustrations

To illustrate all features of the proposed algorithm, two examples have been presented.

Example 4.1
This is a simple but interesting example, where trivially many solutions can be identified. Consider that MST is required with index condition on the graph as shown in Table 1.

Table 1: Arc length of 6-node completely connected graph

j\i	1	2	3	4	5	6
1	-	1	1	1	1	1
2	1	-	2	2	2	2
3	1	2	-	3	3	3
4	1	2	3	-	4	4
5	1	2	3	4	-	5
6	1	2	3	4	5	-

Solution:

One can arbitrarily start from any node, say we commence from node 6 select the arc (6,1) as the first element of the MST. Other elements of the MST will be (1,2), (1,3), (1,4) and (1,5). This MST is presented in Figure 8.

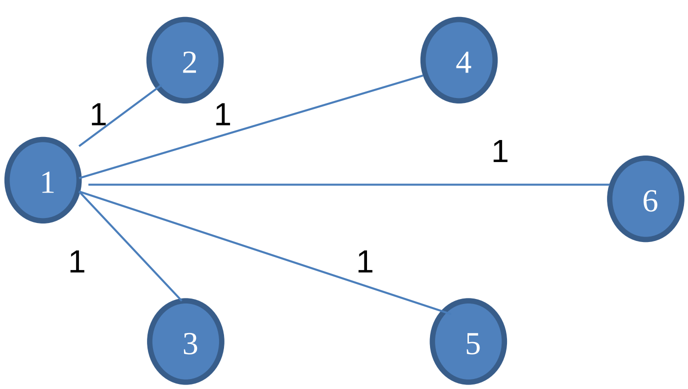

Figure 8: MST of the network in Table 1

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Since the MST in Figure 8 does not satisfy the index condition, the node 1 with index 5 is selected to reduce its index value. Since the maximum required index is 2, the node 1 has to be altered by shifting three arcs. Theorem 2 will apply. The degree of node 1 can be reduced by adding 1 to all arcs emanating from node 1. The modified network is shown in Table 2.

Table 2: Modified arc lengths

j\i	1	2	3	4	5	6
1	-	2M1	2M1	2M1	2M1	2M1
2	2M1	-	2	2	2	2
3	2M1	2	-	3	3	3
4	2M1	2	3	-	4	4
5	2M1	2	3	4	-	5
6	2M1	2	3	4	5	-

The entry 2M1 in the cell (1,2) in Table 2 indicates that the modified arc weight is 2 and the modification was carried out on all arcs emanating from the node 1. Now let us look at the reason for this modification. The nearest neighbouring arc that is not part of the MST is of length 2. If L[arc(1;4)] becomes the same as L[arc(2;4)] then we have the option of replacing the basic arc (1,4) by (2;4). Return to Step 2 and find the new MST. It is shown in Figure 9.

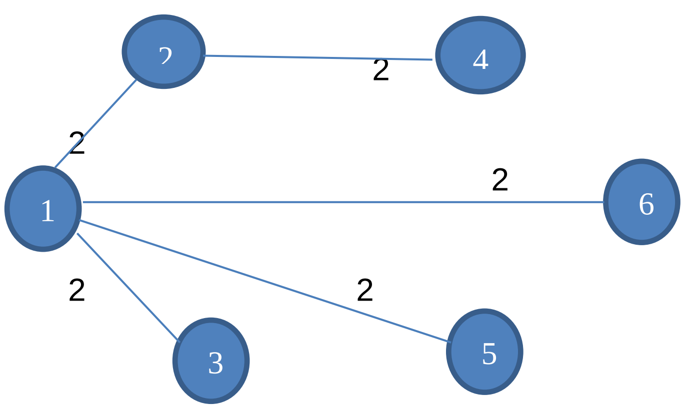

Figure 9: MST of the graph in Table 2

Still the index of node 1 is 4. We therefore carry out two more applications of Theorem 2 by increasing arc weights. First the minimum arc weight becomes 3 and in the next iteration it becomes 4. These modified arc weights are shown in Tables 3 and 4. Once again the element (3M1,2) in the cell indicates that 3 is a modified weight and modifications were carried out on all arcs emanating from nodes 1 and 2.

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Table 3: Modified distances

j\i	1	2	3	4	5	6
1	-	3M1,2	3M1,2	3M1,2	3M1,2	3M1,2
2	3M1,2	-	3M2	3M2	3M2	3M2
3	3M1,2	3M2	-	3	3	3
4	3M1,2	3M2	3	-	4	4
5	3M1,2	3M2	3	4	-	5
6	3M1,2	3M2	3	4	5	-

Once again, one more basic arc from node 1 can be removed by adding 1 to all arcs as shown in Table 4.

Table 4: Modified arc lengths

j\i	1	2	3	4	5	6
1	-	4M1,2,3	4M1,2,3	4M1,2,3	4M1,2,3	4M1,2,3
2	4M1,2,3	-	4M2,3	4M2,3	4M2,3	4M2,3
3	4M1,2,3	4M2,3	-	4M3	4M3	4M3
4	4M1,2,3	4M2,3	4M3	-	4	4
5	4M1,2,3	4M2,3	4M3	4	-	5
6	4M1,2,3	4M2,3	4M3	4	5	-

The MST is shown in Figure 10.

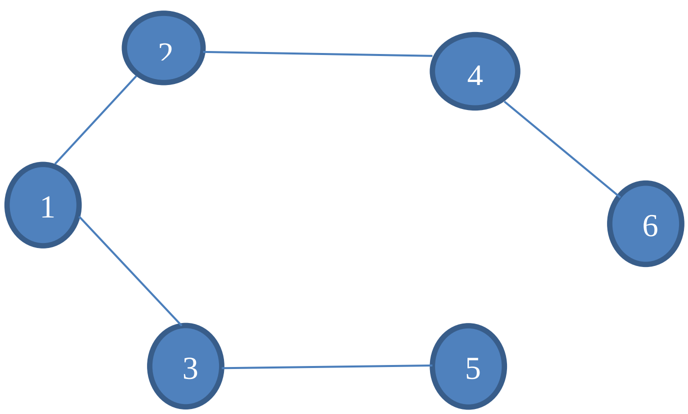

Figure 10: The MST of the network in Table 5

In Figure 10, each and every arc is of length 4. This problem has multiple solutions. In summary, the MST without restriction has length 5 and under the index restriction, it became 11.

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Example 4.2
Reconsider the 6-node completely connected network used by Cowen [3] for a TSP. It is given in Table 6 and the objective is to find an index restricted MST.

Table 5: Arc-weights considered by Cowen

From\To	**1**	**2**	**3**	**4**	**5**	**6**
1	-	11	9	9*	15	16
2	11*	-	14	10	10	15
3	9	14	-	6	13	11*
4	9	10	6*	-	9	10
5	15	10*	13	9	-	8
6	16	15	11	10	8*	-

The optimal tour as obtained by Cowen [3] is comprised of the following arcs: {(1,4), (4,3), (3,6), (6,5), (5,2), (2,1)}. These arcs have been indicated by a star mark in Table 5. The optimal tour length is given by {9+6+11+8+10+11 = 55}.

Since we can start from any node, we commence arbitrarily from node 6 and select the first arc (6,5) as part of the MST. Next selected arc will be either from the node 5 or the node 6, which is arc (5,4). We continue similarly, and select the third arc as (4,3). At the next stage, we have a tie. Two possibilities arise. They are arc (3,1) or (4,1). Note that the arc (4,1) together with the existing selected arcs will create three basic arcs from node 4, whereas the arc (3,1) does maintain index balance. Thus, arc (3,1) is selected. Selected arcs so far are: $\{(6,5)_1, (5,4)_2, (4,3)_3, (3,1)_4\}$. One more arc has to be selected to connect the node 2, which still is an isolated node. This is either link (4,2) or (5,2). If (4,2) is selected, it will increase the number of basic links at node 4 and similarly, if the link (5,2) is selected, it will increase the number of basic links at node 5. Hence in either case imbalance of basic arcs will arise at nodes 4 or 5. Therefore all links emanating from nodes 4 and 5 are altered by adding 1. These modified arc lengths are shown in Table 6.

Table 6: Modified arc lengths in rows 4 and 5

From\To	1	2	3	4	5	6
1	-	11	9	10M4	16M5	16
2	11	-	14	11M4	11M5	15
3	9	14	-	7M4	14M5	11
4	10M4	11M4	7M4	-	11M4,5	11M4
5	16M5	11M5	14M5	11M4,5	-	9M5
6	16	15	11	11M4	9M5	-

Once again, starting from node 6, the MST from Table 6 will be comprised of the arcs: $\{(6,5)_1, (5,4)_2, (4,3)_3, (3,1)_4\}$. Now there are three possibilities to connect the node 2 to form the part of the MST. They are links (1,2) or (4,2) or (5,2). Note once again that the link (4,2) will increase the number of basic arcs at node 4, the link (5,2) will increase the

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number of basic arcs at node 5 and hence the link selected for the MST is (1,2). Thus, the required MST will be given by:
$\{(6,5)_1, (5,4)_2, (4,3)_3, (3,1)_4, (1,2)_5\}$.
These selected arcs in the MST will give rise to the MST path as follows: 2 ->1 -> 3 -> 4 ->5 -> 6. Note that some of this path is contained in the tour obtained by Cowen.

5. Concluding remarks

In this paper arc-weight modification theorems have been developed to create alternative MSTs in a network and restricting the index value ≤ 2. A shortest path in a non-directed network has an alternative interpretation that it gives rise to an MST with index ≤ 2 of selected nodes, which lie on the shortest path. These MST paths may have an application in determination of the travelling salesman tour (TST). Since MST approach is linear and the TST is a NP-hard, the TST obtained through the MST is likely to reduce complexity of the TST. Obtaining a TST through the MST will be the subject matter of a subsequent paper.

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