Homotopy Group of a Loop Space

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1 Introduction.

In algebraic topology, homotopy group is a very important concept and is used to classify topological spaces.

The equivalence class of a path f on a topological space X under the equivalence relation of homotopy is called the homotopy class of f. All homotopy classes of loops $f : S^k \to X$ at the basepoint x_0 of X usually form a homotopy group. Topological spaces with different homotopy groups are never equivalent (homeomorphic). Even if the statement is not vice versa, at least to some extent, we can understand properties of certain spaces X from its homotopy groups, which makes their investiagations meaningful.

Also to probe a space X, we sometimes look at maps from $S^1 \to X$. When we fix a basepoint $x_0 \in X$, we can define the loop space ΩX as the space of all maps from S^1 to X taking the north pole to be x_0 (formal definition is given below). So loop spaces of X will provide ways to investigate X.

Therefore, in this paper, we would like to combine these two important ways of understanding a space X, i.e. to investigate the property of homotopy groups of loop spaces.

2 Preliminaries.

Let Y^X be set of continuous mappings $f : X \to Y$ with compact-open topology, then we have the following definitions used to investigate the homotopy group of a loop space.

Definition 1. *(Compact-Open topology) Given the set Y^X as defined above, the compact-open topology on the space Y^X of maps $f : X \to Y$ has a subbasis consisting of the sets $M(K, U)$ of mappings taking a compact set $K \subset X$ to an open set $U \subset Y$.*

The focus of this paper is the loop space which will be defined below in the form of Y^X.

Definition 2. *(Loop Space) Let (X, x_0) be a based topological space. The (based) loop space of*

X is the topological space

$$\Omega X := (X, x_0)^{(S^1, pt)}$$

with compact-open topology.

In other words, as a set, ΩX is the set of all continuous maps from the based unit circle S^1 to (X, x_0) such that the given basepoint of S^1 is sent to x_0. The basepoint of ΩX is the constant loop at the basepoint x_0.

Definitions below are related to the Loop-Suspension Adjunction which will be introduced in the next section. They will lead to the kernel of the investigation.

Definition 3. *(Reduced Suspension) Let (X, x_0) be a based topological space. The reduced suspension of X is*

$$\Sigma X = X \times I/(X \times \partial I \cup \{x_0\} \times I).$$

Definition 4. *(Smash Product) The smash product $X \wedge Y$ is defined to be the quotient*

$$X \times Y/X \vee Y.$$

Definition 5. *(One-point Compactification) The one-point compactification of X is the topological space X^* defined as follows:*

- *The underlying set of X^* is $X \cup \{\infty\}$.*
- *$U \subset X^*$ is open if and only if*
    1. *U is an open subset of X, or*
    2. *$U = V \cup \{\infty\}$, where V is an open subset of X and $X - V$ is compact.*

3 Loop-Suspension Adjunction.

Loop-Suspension Adjunction describes the corresponding relationship between basepoint-preserving maps $\Sigma X \to Y$ and basepoint preserving maps $X \to \Omega Y$:

$$[\Sigma X, Y] = [X, \Omega Y].$$

This adjunction is a categorical result which indicates that Σ and Ω are associated nicely with each other: up to homotopy, maps out of suspension spaces is homeomorphic to maps in to loop spaces.

It is the kernel of finding the homotopy groups of a loop space. We will prove this adjunction applied to compactly generated spaces in the following steps.

First, for all theorems below, Let X, Y, Z all be topological spaces. The lemma below is the most common understanding for reduced suspension of a topological space.

Lemma 1. *Let X be a topological space, then the suspension of X is $X \wedge S^1$.*

Pf/ We first view S^1 as an interval with the end points identified, i.e. $S^1 = I/\partial I$, then we can write $S^1 = I/\partial I$. It follows that

$$\begin{aligned} X \wedge S^1 &= X \times S^1 / X \vee S^1 \\ &= X \times I / X \vee S^1 \cup X \times \partial I \\ &= X \times I / \{x_0\} \times I \cup X \times \partial I \\ &= \Sigma X \end{aligned}$$

where x_0 is the point on x_0 identified with the point $0 \sim 1$ on S^1 in the wedge product. □

Proposition 1, Lemma 2 and Theorem 1 given below are all discussed in Appendix A in [1], which will give rise to the proof of the Loop-Suspension Adjunction.

Proposition 1 provides three useful properties of the compact-open topology from the view point of algebraic topology.

Proposition 1. *1. The evaluation map $e : X^Y \times Y \to X, e(f,y) = f(y)$, is continuous If Y is locally compact.*

2. If $f : Y \times Z \to X$ is continuous then so is the map $\hat{f} : Z \to X^Y$, $\hat{f}(z)(y) = f(y,z)$.

3. The converse to 2. holds when Y is locally compact. [1]

Pf/

1. For $(f,y) \in X^Y \times Y$, let U be an open neighbourhood of $f(y)$ in X. As Y is locally compact, there is a compact neighborhood $K \subset Y$ of y such that $f(K) \subset U$ by the continuity of f. It follows that $M(K,U) \times K$ is an open neighbourhood of (f,y) in $X^Y \times Y$ such that $e : M(K,U) \mapsto U$. Therefore e is continuous.

2. Suppose $f : Y \times Z \to X$ is continuous. To show that $\hat{f}$ is continuous, it suffices to show that for a subbasic set $M(K,U) \subset X^Y$, the set $\hat{f}^{-1}(M(K,U)) = \{z \in Z | f(K,z) \subset U\}$ is open in Z.

   Consider $z \in \hat{f}^{-1}(M(K,U))$. By definition of compact-open topology, U is open in X. Then by the continuity of f, $f^{-1}(U)$ is open in $Y \times Z$. Then by the definition of f and $\hat{f}$, $f^{-1}(U)$ is actually the open neighbourhood of the compact set $K \times \{z\} \in Y \times Z$. It follows that there exists an open set $V \subset Y$ and an open set $W \subset Z$ such that $V \times W \subset f^{-1}(U)$. So W is an open neighbourhood of z in $f^{-1}(M(K,U))$ and then $f^{-1}(M(K,U))$ is open in Z.

3. Note that f can be written as

   $$e \circ \mathbb{I} \times \hat{f} : Y \times Z \to Y \times X^Y \to X.$$

   When Y is locally compact, e is continuous by 1. and $\hat{f}$ is continuous by assumption. So $f : Y \times Z \to X$ is continuous. □

Note that, part 2. and part 3. of Proposition 1 provide the point-set topology verifying the Loop-Suspension Adjunction stated below. This is because they imply that a map $\Sigma \to Y$ is continuous if and only if the associated map $X \to \Omega Y$ is continuous, and similarly for homotopis of such maps.

We now prove the last step that would give the adjunction.

Lemma 2. *Let X be a normal topological space, then for each closed set C and each open set O containing C, there is another open set O' containing C whose closure is contained in O.*

Pf/ Let C be a closed set, and O be an open set in X. Then denote the complement of O as C' which is also a closed set. We now apply the normality property to C and C', there exists an open set O' containing C which is disjoint from an open set containing C'. So the closure of O' is contained in O. □

Theorem 1. *The map $X^{Y\times Z} \to \left(X^Y\right)^Z, f \mapsto \hat{f}$ is a homeomorphism if Y is locally compact Hausdorff and Z is Hausdorff. [1]*

Pf/ Proposition 1 part 2. implies that there is a well-defined function (map) from $X^{Y\times Z} \to (X^Y)^Z$ sending f to $\hat{f}$ and this function is injective. Part 3. of the proposition implies that this function is surjective if Y is locally compact, since for each $\hat{f}$, there is a corresponding map f. This gives that this map is a bijection, which is useful for later argument.

We first show that a subbabsis for $X^{Y\times Z}$ is formed by the sets $M(A\times B, U)$ with A and B range over compact sets in Y and Z respectively and U ranges over open sets in X. Given a compact set $K \subset Y \times Z$ and a map $f \in M(K, U)$, let K_Y and K_Z be the projections of K onto Y and Z. Note that $K_Y \times K_Z$ is a compact Hausdorff product space as Y and Z and K are all compact Hausdorff, and hence $K_Y \times K_Z$ is normal.

Applying Lemma 2, we let $C = k \in K$, and O be $K_Y \times K_Z \cap f^{-1}(U)$. It is obvious that O contains k and k is both in $K_Y \times K_Z$ by the definition of the product space and in $f^{-1}(U)$ as f maps K to U. It follows that there exists an open neighbourhood $O' \subset K_Y \times K_Z$ of k, whose closure is contained in O and then contained in $f^{-1}(U)$. We write this O' as $O' = V_k \times W_k \subset K_Y \times K_Z$, and its closure is written as a compact neighborhood $A_k \times B_k \subset f^{-1}(U)$ of k in $K_Y \times K_Z$.

Then the sets $V_k \times W_k$ for varying $k \in K$ form an open cover of the compact set K and so a finite number of the products $A_k \times B_k$ cover K. It follows that focusing these finite products $A_k \times B_k$, we have

$$f \in \cap_k M(A_k \times B_k, U) \subset M(K, U),$$

which indicates that the sets $M(A \times B, U)$ form a subbasis for $X^{Y\times Z}$ as desired.

Under the bijection argued above between $X^{Y\times Z}$ and $(X^Y)^Z$, the sets $M(A\times B, U)$ correspond to the sets $M(B, M(A, U))$, so it suffices to show the latter sets form a subbasis for $(X^Y)^Z$ to prove the homeomorphism. We will prove a more general result here:

> For any space Q, a subbasis for Q^Z is formed by the sets $M(K, V)$ as V ranges over a subbasis for Q and K ranges over compact sets in Z, assuming that Z is Hausdorff.

Note that if we take $Q = X^Y$ with subbaiss sets $M(A, U)$, the general statement gives back what we want.

Let $f \in M(K,U)$ such that K is compact in Z and U is open in Q. Write $U = \cup_\alpha U_{alpha}$ with each $U_\alpha = \cap_j V_{\alpha,j}$ of the given subbasis of Q. The cover of K by the open sets $f^{-1}(U_\alpha)$ has a finite subcovr, say by the open sets $f^{-1}(U_i)$. Since K is compact Hausdorff, it is normal and so we can write $K = \cup K_i$ with each $K_i \subset f^{-1}(U_i)$ compact. In other words, for each $k \in K$, it has a compact neighborhood K_k contained in some $f^{-1}(U_i)$ with $k \in f^{-1}(U_i)$. So the compactness of K implies that finitely many of these sets K_k cover K and we let K_i be the union of those contained in $f^{-1}(U_i)$. Now we have

$$f \in M(K_i, U_i) = M(K_i, \cap_j V_{ij}) = \cap_j M(K_i, V_{ij}),$$

for all i. Hence

$$f \in \cap_{i,j} M(K_i, V_{ij}) = \cap_i M(K_i, U_i) \subset M(K,U).$$

Since $\cap_{i,j} M(K_i, V_{ij})$ is a finite intersection, the sets $M(K,V)$ form a subbasis for Q^Z. $\square$

Here comes the Loop-Suspension Adjunction. Sketchily speaking, two functors are said to form an adjunction if they are dual to each other and $\Sigma - \Omega$ adjunction is an example of Eckmann-Hilton duality.

Theorem 2. *(Loop-Suspension Adjunction) If X is Hausdorff, then there is a homeomorphism of sets*

$$[\Sigma X, Y] \cong [X, \Omega Y].$$

Pf/ We rewrite Theorem 1 by modifying their labels:

> The map $Y^{Z\times X} \to \left(Y^Z\right)^X, f \mapsto \hat{f}$ is a homeomorphism if Z is locally compact Hausdorff and X is Hausdorff.

We now argue that there exists a natural homeomorphism

$$Y^{X\wedge Z} \cong Y^{X\times Z}.$$

There is clearly a bijection since for every continuous map f in $Y^{X\wedge Z}$, there exists another continuous map h in $Y^{X\times Z}$ such that $h = f \circ g$, where g is the quotient map from $X \times Z$ to $X \wedge Z$. The bijection is open and continuous because both f and g are continuous and open. So such homeomorphism exists.

Then by Theorem 1 and the above argument, we have

$$\left(Y^Z\right)^X \cong Y^{Z\wedge X}$$

for Z locally compact Hausdorff and X Hausdorff.

Now we take $Z = S^1$, which is locally compact Hausdorff, we have

$$[X, \Omega Y] \equiv (\Omega Y)^X \equiv \left(Y^{S^1}\right)^X \cong T^{S^1 \wedge X} = Y^{\Sigma X} \equiv [\Sigma X, Y].$$

$\square$

4 Homotopy Group.

After all the preparations above, this is the part to apply things above to find the homotopy group for a loop space.

But before investigating it, we first investigate the reduced suspension of and smash products between n-spheres S^n's, which appear a lot when we describe homotopy groups.

Lemma 3. *Let X be a commpact Hausdorff space and x_0 be some arbitrary point in X, then we have X homeomorphic to the one-point compactification of X with x_0 removed, i.e.*

$$X \cong (X - \{x_0\})^*.$$

Pf/ Consider the function

$$f : X \longrightarrow (X - x_0)^*$$

such that $f(x) = x$ for $x \neq x_0$ and $f(x_0) = \infty$.

f is clearly a bijection as it is almost an identity function with one point in X mapped to its 'replaced' point in $(X - x_0)^*$.

We now prove that f is an open map. Consider an open subset U in X. If $x_0 \notin U$, then U is clearly open in $X - \{x_0\}$, then by Definition 5, $f(U) = U$ is also open in $(X - x_0)^*$. If $x_0 \in U$, then we write $U = V \cup \{x_0\} \subset X$ and it follows that V is open in $X - \{x_0\}$. We also have that $f(U) = f(V) \cup f(x_0) = V \cup \{\infty\} \subset (X - \{x_0\})^*$. Then since X is Hausdorff compact and U is open, we know that $X - \{x_0\} - V = X - U$ is compact. Therefore by Definition 5 again, U is also open in $(X - \{x_0\})^*$. So f is an open map as an open subset in X is always open in $(X - \{x_0\})^*$.

Finally we prove that f is continuous. Consider an open set U in $(X - \{x_0\})^*$. If $\{\infty\} \notin U$, then $f^{-1}(U) = U$, and by Definition 5, is open in $X - \{x_0\}$. It follows that $f^{-1}(U)$ is open in X. If $\{\infty\} \in U$, then we write $U = V \cup \{\infty\}$, with V an open subset of $X - \{x_0\}$ and $X - \{x_0\} - V$ compact. Then $f^{-1}(U) = f^{-1}(V) \cup f^{-1}(\infty) = V \cup \{x_0\}$. It is clear that V is then also open in X, and so $f^{-1}(U)$ is also open in X.

Therefore f is a homeomorphism. □

Lemma 4. *If X and Y are compact Hausdorff spaces then*

$$X \wedge Y \cong ((X - \{x_0\}) \times (Y - \{y_0\}))^*,$$

where x_0 and y_0 are chosen basepoints of X and Y respectively.

Pf/ Let X and Y be two compact Hausdorff spaces. Then it is clear that $X \times Y$ is compact Hausdorff and $X \vee Y$ is a closed subset. It follows that every point is closed in $X \times Y/X \vee Y$. So the equivalence relation is a closed relation. Thus $X \wedge Y$ is a compact Hausdorff space as it is the quotient of the compact Hausdorff space $X \times Y$ by a closed equivalence relation.

Now let $Z = (X - \{x_0\}) \times (Y - \{y_0\})$, and we construct an inclusion that $f : Z \hookrightarrow X \times Y$ and a quotient map $g : X \times Y \to X \wedge Y$. We compose these two maps can get $g \circ f : Z \to X \wedge Y$. Based on this composition, we consider a similar map

$$h : Z \to X \wedge Y - \{p\},$$

where p is the basepoint where x_0 gets adjoined with y_0 in $X \vee Y$.

To prove that $X \wedge Y \cong Z^*$, i.e. $X \wedge Y$ is a one-point compactification of Z, it suffices to prove that h is a homeomophism.

We first argue that h is a continuous injection. f is naturally a continuous injection as it is almost an identity function. Even if several points related to basepoints are removed from the domain, they will not affect continuity and injectivity. We now consider $g' : X \times Y \to X \wedge Y - \{p\}$. It is again almost an identity function with the only ambiguous point removed, which gives an injection. As for continuity, for an open set U in $X \wedge Y - \{p\}$, it is also open in $X \wedge Y$ as the smash product is compact Hausdorff. Then as the quotient map g is continuous, U is also open in $X \times Y$. So g' is also a continuous injection and thus $h = f \circ g'$ is a continuous injection.

We now argue h is surjective. For any points in the range of h, we can find corresponding points in the domain because all points related to basepoints x_0 and y_0 are either removed or in some equivalence class.

Finally we argue h is an open map. Let $U \subset Z$ be open. Then $f(U) = U$ is also open in $X \times Y$ since Z is open in $X \times Y$. Also $g^{-1}(g(U)) = U$. Then as the quotient map g is continuous, $g(U) = g \circ f(U)$ is open in $X \wedge Y$. Since $X \wedge Y - \{p\}$ is open in $X \wedge Y$, $h(U) = g \circ f(U)$ is open in $X \wedge Y - \{p\}$.

Therefore h is a homeomorphism. □

Theorem 3. *Let S^n be an n-sphere and S^m be a m-sphere, then we have the smash product of these two spheres is homeomorphic to S^{n+m}, i.e.*

$$S^m \wedge S^n \cong S^{m+n}.$$

Pf/ Since S^m and S^n are both compact Hausdorff spaces, by Lemma 4, we have

$$S^m \wedge S^n \cong ((S^m - \{p\}) \times (S^n - \{q\}))^*.$$

Note that an n-sphere with one point removed is homeomorphic to a real plane (i.e. $S^n - \{p\} \cong \mathbb{R}^n$) by stereographic projection from the removed point.

Then it follows that

$$\begin{aligned} S^m \wedge S^n &\cong (\mathbb{R}^m \times \mathbb{R}^n)^* \\ &\cong (\mathbb{R}^{m+n})^* \\ &\cong (S^{m+n} - \{p\})^* \\ &\cong S^{m+n}, \end{aligned}$$

where the last homeomorphism comes from Lemma 3. □

Corollary 1. *There exists an homeomorphism between the reduced suspension of the n-sphere and the $(n+1)$-sphere*

$$\Sigma S^n \cong S^{n+1}.$$

Pf/From Lemma 1 and Theorem 3, we have

$$\Sigma S^n = S^n \wedge S^1 = S^{n+1}.$$

□

We now consider the homotopy group of a loop space which almost directly follows from the loop-suspension adjunction. Passing from a space to its loopspace has the effect of shifting homotopy groups down a dimension. Formally,

Theorem 4. *Let X be a based topological space, there is shift in homotopy groups of X and ΩX, i.e.*

$$\pi_k(\Omega X) = \pi_{k+1}(X).$$

Pf/ By definition of homotopy groups of a topological space, we have

$$\pi_{k+1}(X) := [S^{k+1}, X] = [\Sigma S^k, X] = [S^k, \Omega X] =: \pi_k(\Omega X),$$

where the first and last equality comes from the definition of homotopy groups. The second ineqlaity comes from Theorem 1 and the third inequality comes from loop-suspension adjunction (S^k is Hausdorff and so the adjunction can be applied). □

5 Application

Given a certain space $X = S^1$, we can use Theorem 4 to find the homotopy group of its loop space. Given that $\pi_k(X) = \pi_k(S^1) = \mathbb{Z}\delta_{k1}$, we see

$$\pi_k(\Omega S^1) = \pi_{k+1}(S^1) = \mathbb{Z}\delta_{(k+1)1} = \mathbb{Z}\delta_{k0}.$$

Specifically $\pi_0(\Omega S^1) = \mathbb{Z}$.

This makes sense because roughly speaking, each connected component in ΩS^1 and $\mathbb{Z}$ is contractible and there are countably many connected components in both spaces. So it is largely possible that ΩS^1 is homotopy equivalent to $\mathbb{Z}$ (It is actually correct with detailed proofs based on constructing fibration and applying Whitehead's theorem). From this point of view,

$$\pi_0(\Omega S^1) = \pi_0(\mathbb{Z}) = \mathbb{Z},$$

where the last equality comes from the fact that $\pi_0(X)$ represents the set of path components of X and $\mathbb{Z}$ is discrete.

6 Conclusion

Through Loop-Suspension Adjunction, we have investigated an important property for the homotopy group of a loop space ΩX:

$$\pi_{k+1}(X) = \pi_k(\Omega X).$$

Using this property, to know the k^{th}-homotopy group of ΩX, it suffices to know the $(k+1)^{\text{th}}$-homotopy group of X.

Also note that for the homotopy group of a n^{th}-loop space $\Omega^n X$, the result would be $\pi_{k+n}(X) = \pi_k(\Omega^n X)$ by applying Loop-Suspension Adjunction n times.

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References

[1] A. Hatcher, *Algebraic topology.* Tsinghua University Press, 2005.