Simple PDE Problems for Physics-Informed Neural Networks (PINNs)

Dr. Bivas Bhaumik

Solve the **five partial differential equation (PDE) problems** using **Physics-Informed Neural Networks (PINNs)** implemented in **PyTorch** library.
Each problem includes the governing PDE equation, domain, initial/boundary conditions, and the **exact analytical solution** for validation.

Problem 1: One-Dimensional Heat Equation

Solve the heat equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

on the domain:

$$x \in [0, 1], \quad t \in [0, 1].$$

Initial condition:

$$u(x, 0) = \sin(\pi x). \tag{2}$$

Boundary conditions:

$$u(0, t) = 0, \quad u(1, t) = 0. \tag{3}$$

Exact solution:

$$u(x, t) = e^{-\pi^2 t} \sin(\pi x). \tag{4}$$

Problem 2: One-Dimensional Wave Equation

Solve the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \tag{5}$$

on the domain:

$$x \in [0, 1], \quad t \in [0, 1].$$

Initial conditions:

$$u(x, 0) = \sin(\pi x), \tag{6}$$

$$\frac{\partial u}{\partial t}(x, 0) = 0. \tag{7}$$

Boundary conditions:

$$u(0, t) = 0, \quad u(1, t) = 0. \tag{8}$$

Exact solution:

$$u(x, t) = \sin(\pi x) \cos(\pi t). \tag{9}$$

Problem 3: Burgers' Equation (Viscous)

Solve Burgers' equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}, \tag{10}$$

with viscosity:

$$\nu = 0.01.$$

Domain:

$$x \in [-1, 1], \quad t \in [0, 1].$$

Initial condition:

$$u(x, 0) = -\sin(\pi x). \tag{11}$$

Boundary conditions:

$$u(-1, t) = 0, \quad u(1, t) = 0. \tag{12}$$

Exact solution (Cole–Hopf transformation):

$$u(x, t) = -\frac{2\nu\pi\sin(\pi x)}{\cosh(\pi^2\nu t) + \cos(\pi x)}. \tag{13}$$

Problem 4: Poisson Equation

Solve the Poisson equation:

$$\frac{\partial^2 u}{\partial x^2} = -\pi^2\sin(\pi x), \tag{14}$$

on the domain:

$$x \in [0, 1].$$

Boundary conditions:

$$u(0) = 0, \quad u(1) = 0. \tag{15}$$

Exact solution:

$$u(x) = \sin(\pi x). \tag{16}$$

Problem 5: First-Order Transport Equation

Solve the linear transport equation:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0, \tag{17}$$

on the domain:

$$x \in [0, 1], \quad t \in [0, 1].$$

Initial condition:

$$u(x, 0) = \sin(2\pi x). \tag{18}$$

Boundary condition:

$$u(0, t) = \sin(-2\pi t). \tag{19}$$

Exact solution:

$$u(x, t) = \sin(2\pi(x - t)). \tag{20}$$

Remark

For each PDE, the PINN loss function typically consists of:

- PDE residual loss
- Initial condition loss
- Boundary condition loss

These problems are intentionally simple and ideal for learning, benchmarking, and teaching PINN implementations in PyTorch.