

ARMA Models

5

All models are wrong, but some are useful.
George Box (The Father of Modern Time Series)

A great place to start when using statistical models is to understand that we are most likely *not* going to be able to identify or define the exact process that is generating the data we are analyzing. In fact, we will probably not be able to define the process mathematically because it is just too complex. However, the process can often be approximated in a very *useful* way using statistical models! If we can find a “reasonable” statistical model, we will find that we can calculate probabilities, forecasts, and other items of interest. In addition, statistical models often offer insight into outside forces that may be driving changes in the response, which may be our principal interest. A big take-away here is to not stress about finding the “right” model out of tens, hundreds, thousands or more potential models... it is likely true that they are all wrong! Our aim is to find a *useful* model, and that is largely the focus of the remainder of this text. Box and Jenkins (1970) (yes, the Box whose quote is at the beginning of this chapter) is the landmark book that introduced a practical and integrated approach to model building that has been in use for the last half century. While this book (that is, the one you are reading) provides updated, and in many cases, more accessible approaches to model-building, the “Box-Jenkins” procedure is at the heart of what we do.

Stationary time series play a fundamental role in time series data analysis. Even when seasonal, trending, or other (nonstationary) features are present in the data, most time series models include a stationary component. Section 5.1 introduces a very important tool in the time series practitioner’s tool kit for analyzing stationary time series data, the autoregressive, AR (p), model. Section 5.2 discusses the autoregressive-moving average (ARMA (p,q)) model which is an important generalization of the AR model.

5.1 THE AUTOREGRESSIVE MODEL

The AR(p) model is similar to a multiple regression model in which the “independent variables” are lagged versions of X_t .

Definition 5.1: The process X_t is said to satisfy an AR(p) model if

$$X_t = \beta + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + a_t \quad (5.1)$$

where ϕ_k , $k = 1, \dots, p$ are real constants, $\beta = (1 - \phi_1 - \phi_2 - \cdots - \phi_p)\mu$, $\phi_p \neq 0$, and a_t is a white noise process with zero mean and finite variance σ_a^2 .

The formula in (5.1) says that the value of the process at time t is a linear combination of the p previous values plus a random noise component a_t . We begin our discussion of AR models by discussing their properties, including stationarity conditions and the behavior of autocorrelations and spectral densities for

specific models. Before discussing the general AR(p) model, we begin our discussion with the simplest AR model, the AR(1) model.

5.1.1 The AR(1) Model

The time series, X_t , is said to satisfy an AR(1) model if

$$X_t = \beta + \phi_1 X_{t-1} + a_t, \quad (5.2)$$

where ϕ_1 is a real, nonzero constant, and a_t is a white noise process with finite variance σ_a^2 . The constant $\beta = (1 - \phi_1)\mu$, is called the *moving average constant*. In essence, the AR(1) model specifies that the value of the process at time t depends on the value of the process at time $t - 1$, plus a random noise component, a_t , and a constant, β . This seems like a reasonable way to describe how a time series might progress in time. In fact, (5.2) has the appearance of a simple linear regression, but it differs from the standard regression model because the independent variable, X_{t-1} , is a “lagged” version of the dependent variable X_t . Theorem 5.1 states a key result concerning AR(1) processes.

Theorem 5.1: An AR(1) process is stationary if and only if $|\phi_1| < 1$.

Proof: See Woodward et al. (2017)

As a result of Theorem 5.1, the models in the left list below are stationary and those on the right are nonstationary.

Stationary Models

$$\begin{aligned} X_t &= .6X_{t-1} + a_t \\ X_t &= 10 - .8X_{t-1} + a_t \\ X_t &= 50 + .99X_{t-1} + a_t, \end{aligned}$$

Nonstationary Models

$$\begin{aligned} X_t &= 1.2X_{t-1} + a_t \\ X_t &= 10 - 3.1X_{t-1} + a_t \\ X_t &= X_{t-1} + a_t. \end{aligned}$$

Thus, by simple examination of an AR(1) model, it is very easy to determine whether it describes a stationary process. We will see that for higher order AR(p) models, that is, where $p > 1$, assessing stationarity by simple examination of the coefficients can be difficult or impossible. See Example 5.5.

5.1.1.1 The AR(1) in Backshift Operator Notation

The AR(1) model is sometimes expressed using the *backshift operator* defined by $BX_t = X_{t-1}$. Note that $Bc = c$ for a constant c . The AR(1) model in (5.2), that is the model $X_t = (1 - \phi_1)\mu + \phi_1 X_{t-1} + a_t$, can be written as

$$X_t - \mu - \phi_1(X_{t-1} - \mu) = a_t,$$

or, in backshift operator notation

$$(1 - \phi_1 B)(X_t - \mu) = a_t, \quad (5.3)$$



QR 5.1
Backshift Notation

or,

$$\phi(B)(X_t - \mu) = a_t,$$

where $\phi(B)$ is the operator

$$\phi(B) = 1 - \phi_1 B.$$

Key Points

1. In this book, we often work with the simplified zero-mean forms of the models in (5.2) and (5.3).
2. If an AR(1) process, X_t , has mean μ , then we can create a zero-mean process, \tilde{X}_t , by setting $\tilde{X}_t = X_t - \mu$.
3. Given a set of data, we typically analyze it as a zero-mean process by first estimating μ by \bar{X} , subtracting \bar{X} from each observation, and modeling the remaining data using the “zero-mean” model.

5.1.1.2 The AR(1) Characteristic Polynomial and Characteristic Equation

Associated with the operator $\phi(B) = 1 - \phi_1 B$ in (5.3) is the *characteristic polynomial*, $\phi(z) = 1 - \phi_1 z$, which is not an operator but is an algebraic expression where z is a real number. The equation $\phi(z) = 1 - \phi_1 z = 0$ is called the *characteristic equation*. Notice that the root of the characteristic equation obtained by solving $1 - \phi_1 z = 0$ is $z = 1/\phi_1$. Because the root of the characteristic equation of an AR(1) is the reciprocal of the coefficient, ϕ_1 , the result in Theorem 5.1 can be restated as follows.

Theorem 5.1 (restatement): An AR(1) process is stationary if and only if the root, r , of the characteristic equation, satisfies $|r| = |1/\phi_1| > 1$.

Key Point: While Theorem 5.1 applies to AR(1) processes, Theorem 5.3 will give stationary conditions for general AR(p) models. These conditions will also be based on the roots of the characteristic equation.

5.1.1.3 Properties of a Stationary AR(1) Model

In this section, we discuss the properties of a stationary AR(1) process of the form $X_t = (1 - \phi_1)\mu + \phi_1 X_{t-1} + a_t$, with $|\phi_1| < 1$. See Appendix A5.1 for a derivation of the properties listed below. Let X_t be a stationary AR(1) process satisfying (5.3).

- (a) The mean of X_t is given by $E[X_t] = \mu$ (a constant)

Note that if $\mu = 0$, then (5.2) simplifies to

$$X_t = \phi_1 X_{t-1} + a_t, \quad (5.4)$$

which, in operator notation, is

$$(1 - \phi_1 B) X_t = a_t. \quad (5.5)$$

The AR(1) model expressed in (5.4) and (5.5) is called the *zero-mean form* of the AR(1) process which, as stated earlier, we will often use for simplicity.

- (b) The variance of X_t is given by

$$\text{var}[X_t] = \frac{\sigma_a^2}{1 - \phi_1^2}. \quad (5.6)$$

By (5.6), $\text{var}[X_t]$ is a constant, which we will denote σ_X^2 , and is finite because σ_a^2 is assumed to be finite and $|\phi_1| < 1$.

- (c) The autocovariance between X_t and X_{t+k} , is given by

$$\text{cov}[X_t, X_{t+k}] = \phi_1^k \sigma_X^2. \quad (5.7)$$

From (5.7) it follows that the autocovariance between X_t and X_{t+k} depends only on k , and thus can be denoted by γ_k . Recall that if X_t is a stationary process then $\gamma_0 = \text{cov}[X_t, X_t] = \sigma_X^2$, so (5.7) can be written as

$$\text{cov}[X_t, X_{t+k}] = \phi_1^k \gamma_0.$$

Also, because $\rho_k = \gamma_k / \gamma_0$, it follows that

$$\rho_k = \phi_1^k, \quad k > 0. \quad (5.8)$$

Note that because $\gamma_k = \gamma_{-k}$ and consequently, $\rho_k = \rho_{-k}$, then for $k = 0, \pm 1, \pm 2, \dots$, (5.7) and (5.8), respectively, become

$$\gamma_k = \phi_1^{|k|} \gamma_0$$

and

$$\rho_k = \phi_1^{|k|},$$

for $= 0, \pm 1, \pm 2, \dots$



(5.9)

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Properties



5.1.1.4 Spectral Density of an AR(1)

The true (model-based) autocorrelations for an AR(1) model with parameter ϕ_1 are given in (5.9). Recall from Section 4.2 that if the true autocorrelations are known, then the true spectral density can be calculated. In the case of the AR(1) model, the spectral density is given by

$$\begin{aligned} S_x(f) &= \frac{\sigma_a^2}{\gamma_0} \frac{1}{|1 - \phi_1 e^{-2\pi i f}|^2} \\ &= \frac{1 - \phi_1^2}{|1 - \phi_1 e^{-2\pi i f}|^2}, \end{aligned} \quad (5.10)$$

see Woodward et al. (2017). Using Euler's formula, $e^{-2\pi i f} = \cos 2\pi f - i \sin 2\pi f$, the denominator of (5.10) becomes

$$\begin{aligned} |1 - \phi_1 e^{-2\pi i f}|^2 &= |1 - \phi_1 (\cos 2\pi f - i \sin 2\pi f)|^2 \\ &= |(1 - \phi_1 \cos 2\pi f) - i \phi_1 \sin 2\pi f|^2 \\ &= (1 - \phi_1 \cos 2\pi f)^2 + (\phi_1 \sin 2\pi f)^2, \end{aligned}$$

because $|a + bi|^2 = a^2 + b^2$. Thus, (5.10) can be written as

$$S_x(f) = \frac{1 - \phi_1^2}{(1 - \phi_1 \cos 2\pi f)^2 + (\phi_1 \sin 2\pi f)^2}. \quad (5.11)$$

In (5.11) we have simplified the formula for $S_x(f)$ in (5.10) to emphasize the fact that $S_x(f)$ is a real number that is easy to calculate. However, in practice we will use **tswge** functions **plots.true.wge** and **true.arma.spec.wge** to calculate model-based AR spectral densities.

5.1.1.5 AR(1) Models with Positive Roots of the Characteristic Equation

Consider the AR(1) models

$$X_t - .9X_{t-1} = a_t \quad (5.12)$$

and

$$X_t - .7X_{t-1} = a_t. \quad (5.13)$$

where a_t is zero-mean white noise with variance $\sigma_a^2 = 1$. Using the backshift operator, these models can be written as $(1 - .9B)X_t = a_t$ and $(1 - .7B)X_t = a_t$, with characteristic equations $1 - .9z = 0$ and $1 - .7z = 0$, respectively. Both models are stationary because the roots, 1.11 and 1.43, are greater than one in

absolute value. (This could also be determined by simple examination of the models using Theorem 5.1). Figure 5.1(a) shows the model-based autocorrelations for the AR(1) model with $\phi_1 = 0.9$. From (5.9), the model-based autocorrelations for this AR(1) model should be of the form of a damped exponential curve defined by $\rho_k = .9^k$. The autocorrelations in Figure 5.1(a) exhibit the expected behavior. The model-based spectral density plotted in Figure 5.1(b) is simply a plot of the function in (5.10) with $\phi_1 = .9$. The dominant feature of the spectral density is a peak at $f = 0$.

Figures 5.2(a) and (b) show that the AR(1) model with $\phi_1 = .7$ has exponentially damping autocorrelations that damp more quickly than for the $\phi_1 = .9$ case. Also, the peak in the spectral density is again at $f = 0$, but the peak is not as “sharp”.

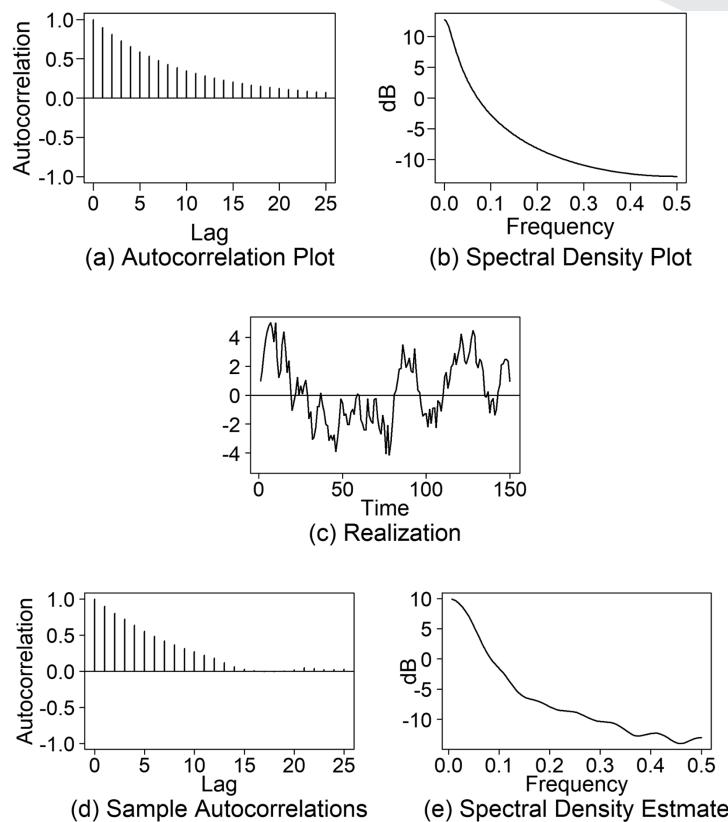


FIGURE 5.1 (a) Model-based autocorrelations and (b) spectral density for $(1 - .9B)X_t = a_t$, (c) realization of length $n = 150$ from this AR(1) model, (d) sample autocorrelations and (e) Parzen spectral density estimate based on the realization in (c).



QR 5.3
Characteristic
Equation of
AR(1)

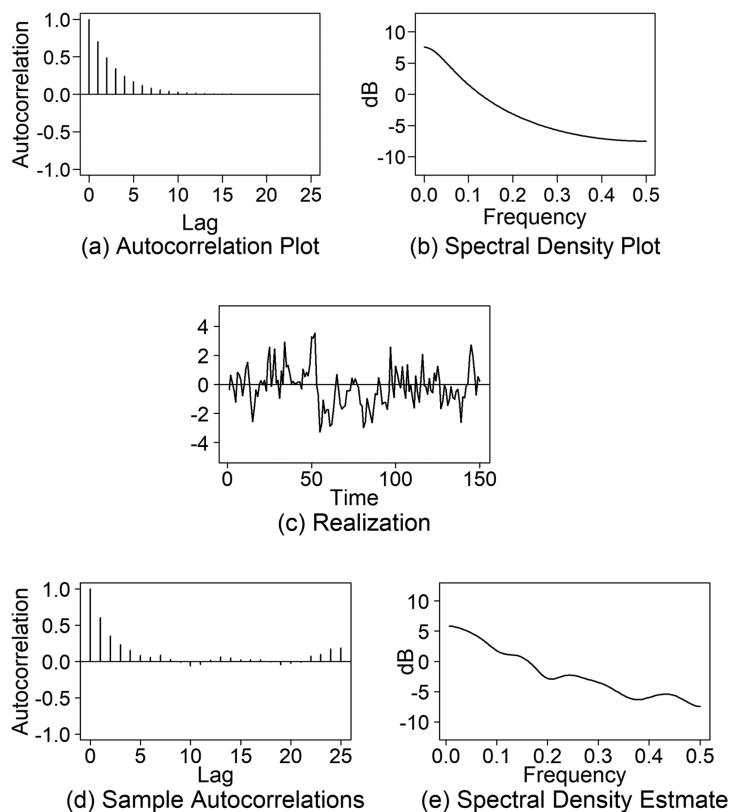


FIGURE 5.2 (a) Model-based autocorrelations and (b) spectral density for $(1 - .7B)X_t = a_t$, (c) realization of length $n = 150$ from this AR(1) model, (d) sample autocorrelations and (e) Parzen spectral density estimate based on the realization in (c).

Figures 5.1(a) and (b) and 5.2(a) and (b) were obtained using the commands

```
plotts.true.wge(phi=.9,plot.data=FALSE)
plotts.true.wge(phi=.7,plot.data=FALSE)
```

Key Points

1. It is important to note that Figures 5.1(a)–(b) and Figures 5.2(a)–(b) show the “theoretical” autocorrelations and spectral densities based on the mathematical models in (5.12) and (5.13), respectively.
2. In practice, given a set of data, we “estimate” the autocorrelations using the sample autocorrelations defined in (3.12) and the spectral density using a spectral density estimate such as the Parzen spectral density estimate (see Section 4.2.3).

Figures 5.1(c)–(e) and Figures 5.2(c)–(e) show realizations generated from these AR(1) models along with sample-based estimates of the autocorrelations and spectral density. The sample-based estimates in Figures 5.1(d)–(e) and Figures 5.2(d)–(e) have the same general appearance as the model-based quantities in Figures 5.1(a)–(b) and Figures 5.2(a)–(b), respectively. However, they are not exact replicas.

The realizations, sample autocorrelations, and Parzen spectral densities in Figures 5.1(c)–(e) were obtained using the commands

```
x=gen.arma.wge(n=150,phi=.9,sn=20)
plots.sample.wge(x)
```

while those in Figures 5.2(c)–(e) were produced by

```
x=gen.arma.wge(n=150,phi=.7,sn=130)
plots.sample.wge(x)
```

Key Points

1. Although the realizations in Figure 5.1(c) and Figure 5.2(c) were generated from the AR(1) models in (5.12) and (5.13), respectively, the sample autocorrelations and spectral density estimates were calculated without reference to a given model.
 - The sample autocorrelations and spectral density estimates are similar to, but not exactly the same as, the model-based versions.
2. A particular model may turn out to be “useful” if the sample autocorrelations and spectral density estimates calculated from the data have the general appearance of the corresponding model-based quantities.

One consequence of the autocorrelation structure for the $\phi_1 = .9$ case is that there will be a strong positive correlation, $\rho_1 = .9$, between adjacent observations in realizations from this model. That is, realizations should exhibit a tendency for the value of the realization at time t to be fairly close to the value at time $t+1$. This positive correlation associated with $\phi_1 = 0.9$ remains fairly strong (holds up) for observations at time t and time $t+k$ for values of k larger than $k = 1$. For example, the correlation between observations separated by seven time points is $\rho_7 = .9^7 = .48$. This implies that, for example, if the process takes on a value substantially below the mean at time t , then the values of the process at times in the neighborhood of t are also likely to be below the mean. As noted in Section 3.3.1 this type of correlation structure induces a random wandering behavior in the realizations. By construction, the mean of the process is zero. Notice in Figure 5.1(c), for example, several x_t values for t near $t = 125$ tend to remain above the mean. The random, aperiodic wandering behavior is reflected by a peak in the spectral density at $f = 0$. For $\phi_1 = .9$, this peak is fairly “sharp” in both the model-based values and the sample-based Parzen estimates.

Now notice that the correlation structure for $\phi_1 = .7$ dies out rather quickly. For example, while the correlation between values of the process at times t and $t+1$ is $.7^1 = .7$, the correlation between times t and $t+7$ is $.7^7 = .08$. That is, there is almost no correlation between values separated by seven units of time, and the “lingering” behavior above and below the mean as seen for $\phi_1 = .9$ would not be expected to occur. The realization in Figure 5.2(c) reflects this fact. Additionally, the process variance is also noticeably smaller for $\phi_1 = .7$ than for $\phi_1 = .9$. This can be seen from (5.6). Because $\sigma_a^2 = 1$ in the generated realizations, the variance for $\phi_1 = .7$ is $1/(1 - .7^2) = 1.96$ and for $\phi_1 = .9$ is $1/(1 - .9^2) = 5.26$. Figures 5.2(b) and (e), i.e. the model-based and the sample-based spectral densities in the $\phi_1 = .7$ case, both have mild peaks at or about $f = 0$.

Key Points

1. The realizations in Figures 5.1(c) and 5.2(c) represent an “artificial case” in which the realization is actually generated from the underlying AR(1) model.
2. Recalling the George Box quote in the opening paragraph to this chapter, “real” data will in all likelihood not be a realization from any AR (or ARMA, ARIMA, ...) model.

Try This: Submit the following commands *several times* to get an idea of the variability in realizations, sample autocorrelations, and spectral density estimates that you might expect to see from realizations generated from the models.¹

```
x=gen.arma.wge(n=150,phi=.9)
plotts.sample.wge(x)
```

Because no seed was set with **sn**, you will get a new randomly selected realization each time you submit the **gen.arma.wge** command above.

Repeat the procedure by replacing **phi=.9** with **phi=.7**.



QR 5.4 Investigating
AR(1) Realizations

5.1.1.6 AR(1) Models with Roots Close to +1

Figure 5.3 shows realizations from two AR(1) models with roots close to one in absolute value. Specifically we show realizations from AR(1) models $(1 - \phi_1 B)(X_t - 10) = a_t$, with $\phi_1 = .95$ and $\phi_1 = .99$. Noting that the theoretical mean of the model from which these realizations were generated is $\mu = 10$ (which is plotted as a horizontal line in the plots), it can be seen that there is considerable aimless, aperiodic wandering above and below the mean, especially for $\phi_1 = .99$. We describe this by saying that for $\phi_1 = 0.95$ and 0.99 , the autocorrelations, $\rho_k = \phi_1^k$, are strong (or *persistent*). That is, the autocorrelations remain substantial even when k is moderately large. As an example, for $\phi_1 = .99$, amazingly $\rho_{50} = .99^{50} = .61$. That is, there is a correlation of .61 between observations 50 time periods apart! The sustained wandering behavior increases as values ϕ_1 increase toward $\phi_1 = 1$. At $\phi_1 = 1$ the model is not a stationary AR(1) model because the absolute value of the root is equal to one instead of greater than one. We will discuss this model later in Section 5.1.1.8 and in Chapter 7. The data for Figure 5.3 were generated using the commands

```
gen.arma.wge(n=150,phi=.95,sn=305)
gen.arma.wge(n=150,phi=.99,sn=404)
```

¹ See Appendix 5A for a discussion of tswge function **gen.arma.wge**

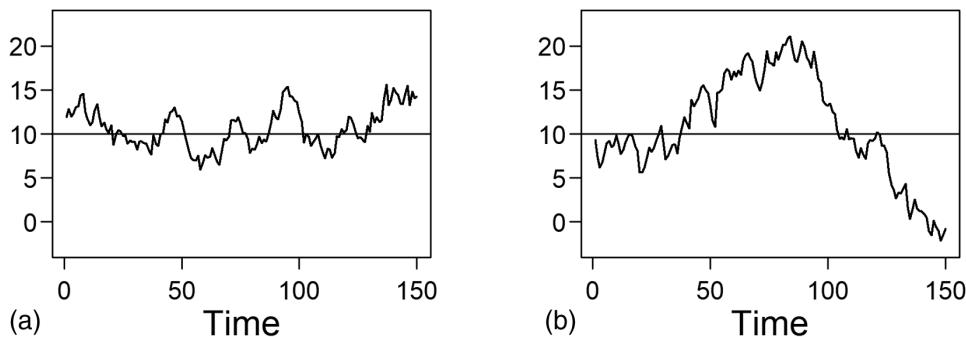


FIGURE 5.3 Realizations from two AR(1) models with roots close to one in absolute value.

Example 5.1 Dow Jones Stock Market Data

Figure 5.4(a) shows the monthly Dow Jones closing averages for March 1985 through December 2020 (data set `dow1985.ts` in `tswge`). (Source: Federal Reserve Bank). Of course, in this case we do not know the “actual” model, the model-based autocorrelations, or spectral density. Consequently, we must analyze the data based on the realization, and the corresponding sample-based estimates of autocorrelations and spectral densities. It can be seen that the time series has the typical wandering appearance of an AR(1) process with positive ϕ_1 . The sample autocorrelations and Parzen spectral density estimate are shown in Figures 5.4(b) and (c), where it can be seen that the autocorrelations are slowly damping, and the spectral density has a peak at zero. By comparing Figure 5.4(a) with Figure 5.3 it can be seen that ϕ_1 appears to be close to one, that is, near nonstationary (or maybe nonstationary with $\phi_1 = 1$). We will return to the analysis of data such as the Dow Jones data when we discuss the topics of nonstationarity, model identification, and parameter estimation.

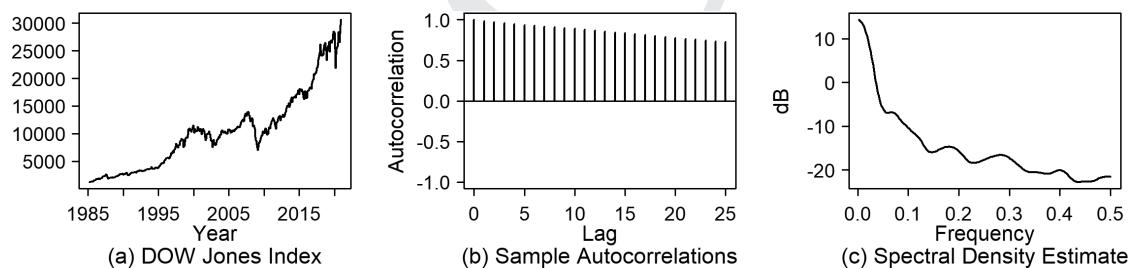


FIGURE 5.4 (a) Dow Jones monthly closing averages from March 1985 through December 2020, (b) sample autocorrelations, and (c) Parzen spectral density estimate.

5.1.1.7 AR(1) Models with Negative Roots of the Characteristic Equation

To describe the behavior of an AR(1) model associated with a negative root of the characteristic equation, we consider the AR(1) model

$$X_t + .7X_{t-1} = a_t. \quad (5.14)$$

Note that $\phi_1 = -.7$ and the characteristic equation is $1 + .7z = 0$ which has root $-1/.7 = -1.43$. Thus the root is negative and since $|-1.43| > 1$, the process is stationary. Using (5.8) the model-based autocorrelations for positive k are given by $\rho_k = (-.7)^k = (-1)^k (.7)^k$. So,

$$\rho_1 = -.7$$

$$\rho_2 = (-1)^2 (.7)^2 = .49$$

$$\rho_3 = (-1)^3 (.7)^3 = -.343$$

⋮

That is, the autocorrelations follow a damped, oscillating behavior, and if a value of the process at time t is above the mean, then the value at time $t + 1$ will likely be below the mean, and vice versa. Similarly, because $\rho_2 = .49$, the value at time $t + 2$ will tend to be on the same side of the mean as the value at time t . This oscillating behavior describes a period of length two (that is, the frequency is .5, the Nyquist frequency). Consequently, we expect the spectral density to have a peak at $f = .5$. The model-based autocorrelations in Figure 5.5(a) illustrate the expected damped oscillating behavior. The dominant feature of the model-based spectral density plotted in Figure 5.5(b) (a plot of the function in (5.10) with $\phi_1 = -.7$) is a peak at $f = .5$.

Figure 5.5(c) shows a realization of length $n = 100$ generated from $(1 + .7B)X_t = a_t$, along with sample-based estimates of the autocorrelations and spectral density. The expected oscillatory behavior in the realization is quite apparent in Figure 5.5(c). The sample autocorrelations oscillate, and the Parzen spectral density estimate has some “wiggle” but its main feature is a peak at $f = .5$. As in the case of the AR(1) with positive roots, whenever $|\phi_1|$ gets closer to one, these behaviors become more pronounced (sample autocorrelations damp more slowly but are still oscillatory, the peak in the spectral density is higher, etc.) See Problem 5.1(b). The following *tswge* commands create the plots in Figure 5.5.

```
plottts.true.wge(n=100,phi=-.7,plot.data=FALSE)
x=gen.arma.wge(n=150,phi=-.7,sn=967)
plottts.sample.wge(x)
```

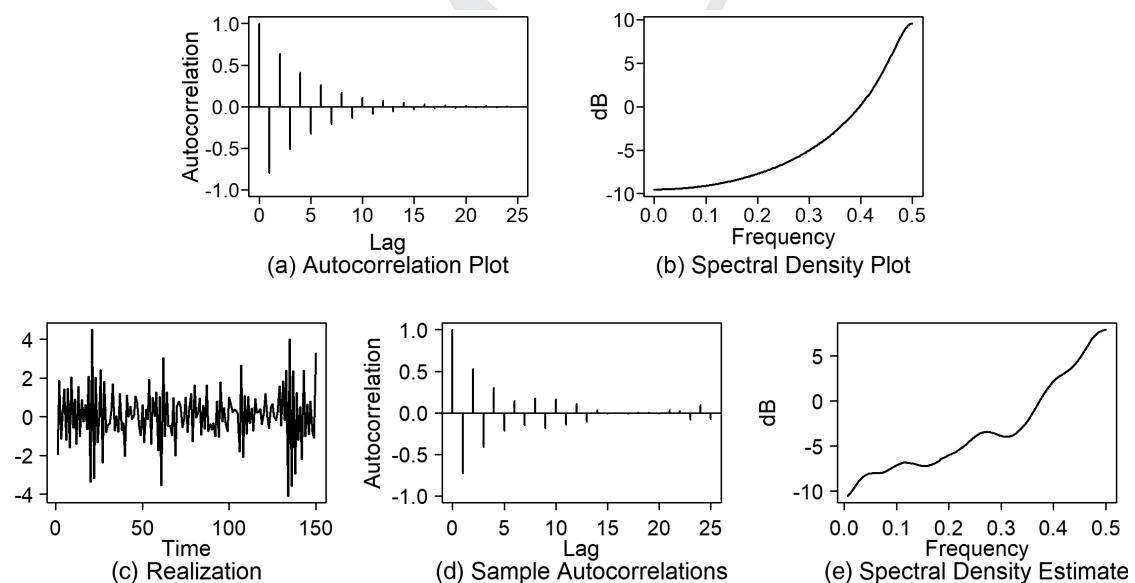


FIGURE 5.5 (a) Model-based autocorrelations and (b) spectral density for $(1 + .7B)X_t = a_t$. (c) Realization of length $n = 100$ from this model, (d) sample autocorrelations and (e) Parzen spectral density estimate based on the realization in (c).

5.1.1.8 Nonstationary 1st-order Models

In Figure 5.6, realizations are shown from the 1st-order model in (5.3) with $\phi_1 = 1$ and $\phi_1 = 1.1$ where $\sigma_a^2 = 1$. In neither case is it true that $|\phi_1| < 1$, so these are realizations from *nonstationary* models. Examination of the realizations shows that for $\phi_1 = 1$, the “wandering or quasi-linear behavior” is similar to, but more pronounced than, that in the realizations shown in Figure 5.3 for both $\phi_1 = 0.95$ and 0.99 . In Chapter 7 we will find that the model,

$$(1 - B)(X_t - \mu) = a_t,$$

is useful for modeling real datasets. The realization in Figure 5.6(a) is generated from the model (5.3) with $\phi_1 = 1$ and $\mu = 0$. Note that there does not seem to be an attraction to the mean value $\mu = 0$ which does not even show in the plot.

The realization associated with $\phi_1 = 1.1$ demonstrates what Box et al. (2008) refer to as “explosive” behavior. The random wandering seen in Figure 5.6(a) associated with $\phi_1 = 1$ is typical of certain real data series such as stock prices as mentioned in Section 5.1.1.6. However, the explosive realization associated with $\phi_1 = 1.1$ is not similar to that of any datasets that we will attempt to analyze, being better modeled as a deterministic process such as $X_t = -e^{(at+bt^2+ct^3)}$. The plot in Figure 5.6(a) was obtained using the *tswge* command

```
x=gen.arima.wge(n=150, d=1, sn=3065)
```

Because *tswge* will not generate realizations associated with roots inside the unit circle, the plot in Figure 5.6(b) was obtained by straightforward computation of the generating equation. The following code produces the plot in Figure 5.6(b).

```
a=gen.arma.wge(n=500, sn=10, plot=FALSE) #generates 500 N(0,1) white noise
xx.spin=rep(0, 500)
xx=rep(0, 150)
for(i in 1:499) {xx.spin[i+1]=1.1*xx.spin[i]+a[i]}
xx[1:150]=xx.spin[350:499] # generates 500 and keeps the last 150
```

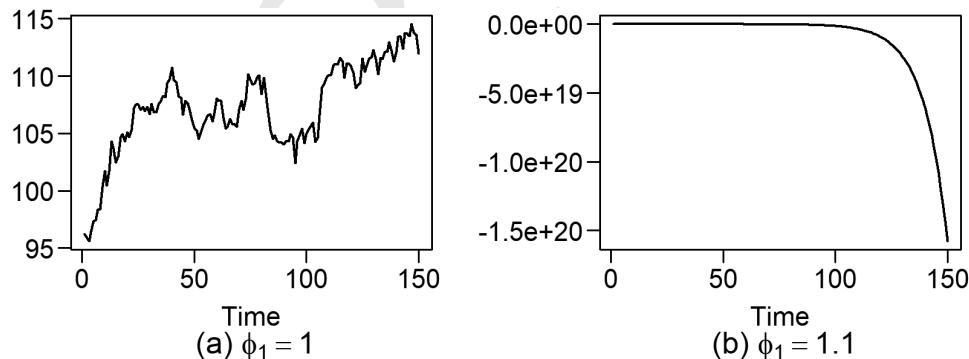


FIGURE 5.6 Realizations from 1st-order nonstationary models, (a) $(1 - B)X_t = a_t$, and (b) $(1 - 1.1B)X_t = a_t$.

5.1.1.9 Final Comments Regarding AR(1) Models

While the AR(1) is a useful model, it has its limitations. The peak in the spectral density is either at $f = 0$ or $f = .5$, implying that realizations will either display aimless (aperiodic) wandering or will oscillate back

and forth. Many datasets, for example, data like the DFW monthly average temperatures and business-related data such as the Air Passenger data shown previously and in Figure 5.7, clearly require more complex models to explain their behavior.

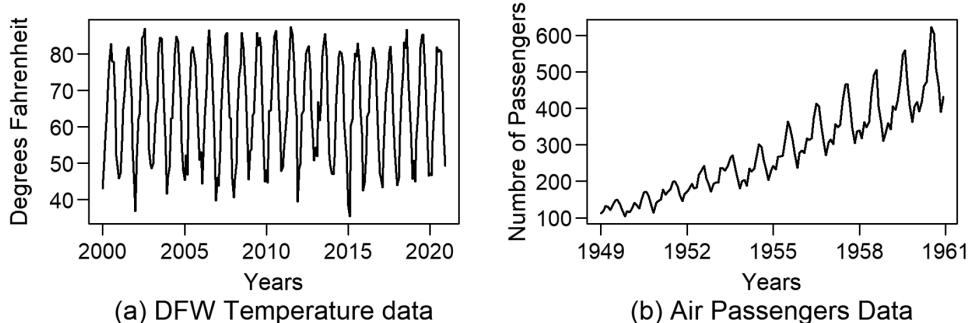


FIGURE 5.7 (a) DFW monthly temperature data (January 1990–December 2020) and (b) Monthly number of international airline passengers (January 1949–December 1960).

Key Points

1. As $\phi_1 \uparrow 1$, realizations from an AR(1) model begin to appear similar to realizations from a nonstationary model.
2. As $\phi_1 \rightarrow 0$, the AR(1) model approaches white noise.

5.1.2 The AR(2) Model

Understanding the features of the AR(1) and AR(2) models is critical for gaining insight into the more general AR(p) model. We have already discussed the AR(1) model, so, in this section, we consider the AR(2) model

$$X_t = \beta + \phi_1 X_{t-1} + \phi_2 X_{t-2} + a_t, \quad (5.15)$$

where ϕ_1 and ϕ_2 are real constants, $\beta = (1 - \phi_1 - \phi_2)\mu$, $\phi_2 \neq 0$, and a_t is a white noise process with zero mean and finite variance σ_a^2 .

Analogous to the AR(1) model, the AR(2) model in (5.15) has the appearance of a multiple regression model with two independent variables. Again, the “independent variables” are values of the dependent variable at the two previous time periods. In essence, the model specifies that the value of the process at time t is a linear combination of values at the two previous time periods plus a random noise component that enters the model at time t .

As in the case of the AR(1) model, it will often be useful to use the zero-mean form, and express the model as

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = a_t. \quad (5.16)$$

5.1.2.1 Facts about the AR(2) Model

- (a) $E[X_t] = \mu$, for the “non-zero mean” form of the AR(2) model in (5.15)
- (b) The process variance is given by $\sigma_x^2 = \gamma_0 = \frac{\sigma_a^2}{1 - \phi_1\rho_1 - \phi_2\rho_2}$, which is a finite constant.
- (c) The autocorrelations satisfy the equations

$$\rho_k = \phi_1\rho_{k-1} + \phi_2\rho_{k-2}, \quad k > 0. \quad (5.17)$$

See Appendix 5B.

Setting $k = 1$ and $k = 2$ in (5.17), we obtain the 2nd-order *Yule-Walker equations*

$$\begin{aligned}\rho_1 &= \phi_1\rho_0 + \phi_2\rho_1 \\ \rho_2 &= \phi_1\rho_1 + \phi_2\rho_0\end{aligned} \quad (5.18)$$

where, of course, $\rho_0 = 1$. Knowing ϕ_1 and ϕ_2 allows us to solve this 2×2 system of equations for ρ_1 and ρ_2 . Autocorrelations ρ_k , $k > 2$ can be calculated using the recursion $\rho_k = \phi_1\rho_{k-1} + \phi_2\rho_{k-2}$. These autocorrelations are also computed by *tswge* functions `true.arma.aut.wge` and `plotts.true.wge`.

The model-based spectral density of an AR(2) model is given by

$$S_x(f) = \frac{\sigma_a^2}{\gamma_0} \frac{1}{\left|1 - \phi_1 e^{-2\pi i f} - \phi_2 e^{-4\pi i f}\right|^2}.$$

5.1.2.2 Operator Notation and Characteristic Equation for an AR(2)

Recall that when discussing the AR(1) model, we defined the backshift operator $BX_t = X_{t-1}$. More generally, the backshift operator is defined as

$$B^k X_t = X_{t-k}$$

Using the more general form of the backshift operator, we can write the AR(2) model in (5.16) as

$$X_t - \phi_1 BX_t - \phi_2 B^2 X_t = a_t$$

or

$$(-\phi_1 B - \phi_2 B^2) X_t = a_t. \quad (5.19)$$

We often use the “shorthand” notation $\phi(B)X_t = a_t$, where $\phi(B)$ is the operator $\phi(B) = 1 - \phi_1 B - \phi_2 B^2$. Converting the operator $\phi(B)$ to the algebraic quantity $\phi(z)$ as in the AR(1) case, we obtain the AR(2) characteristic polynomial $\phi(z) = 1 - \phi_1 z - \phi_2 z^2$. The corresponding AR(2) characteristic equation is given by

$$\phi(z) = 1 - \phi_1 z - \phi_2 z^2 = 0. \quad (5.20)$$



It should be noted that the characteristic equation has two roots, r_1 and r_2 . These roots are either both real or are complex conjugate pairs, i.e. $r_1 = a + bi$ and $r_2 = r_1^* = a - bi$, where z^* denotes the complex conjugate of the complex number z .

Key Point: The behavior of the autocorrelations and spectral density of an AR(2) model depend on whether the roots of the characteristic equation are real or complex. See Woodward et al. (2017).

Theorem 5.2 gives the stationarity conditions for an AR(2) model and is an extension of the result given in Theorem 5.1(restatement) for AR(1) models.

Theorem 5.2 An AR(2) is stationary if and only if all of the roots of the characteristic equation are greater than one in absolute value.

Complex numbers that satisfy the condition $|z| = |a + bi| = \sqrt{a^2 + b^2} = 1$ are said to fall on the *unit circle* on the complex plane. A common way to express the condition in Theorem 5.2 is that *the roots of the characteristic equation fall outside the unit circle*.

Key Point: Theorem 5.2 can be restated as:

An AR(2) model is stationary if and only if all of the roots of the characteristic equation fall *outside the unit circle*.

Consider the following AR(2) models. According to the results of Theorem 5.2, which of these models are stationary?

- (a) $X_t - .2X_{t-1} - .63X_{t-2} = a_t$
- (b) $X_t - 1.7X_{t-1} + 1.2X_{t-2} = a_t$
- (c) $X_t - 1.6X_{t-1} + .8X_{t-2} = a_t$

Unlike in the AR(1) case, by only examining the size of the coefficients it is not straightforward to determine whether the models are stationary. In the following example we will examine each of these models.

Example 5.2 Checking AR(2) Models for Stationarity

Model (a) $X_t - .2X_{t-1} - .63X_{t-2} = a_t$

This model can be written in operator notation as $(1 - .2B - .63B^2)X_t = a_t$ and the resulting characteristic equation is $1 - .2z - .63z^2 = 0$. In order to find the roots, we factor this 2nd-order polynomial as $(1 - .9z)(1 + .7z) = 0$. It follows that the roots are $r_1 = 1/.9 = 1.11$ and $r_2 = 1/(-.7) = -1.43$, and because both of these roots are greater than one in absolute value, Model(a) is a stationary AR(2) model. The *tswge* function **unit.circle** displays points in the complex plane along with the unit circle. Figure 5.8 is obtained using the *tswge* command

```
unit.circle.wge(real=c(1.11,-1.43),imaginary=c(0,0))
```

The figure shows that both roots are real (fall along the horizontal real axis) and that both roots fall outside the unit circle (which is shown as the circle in the plot).

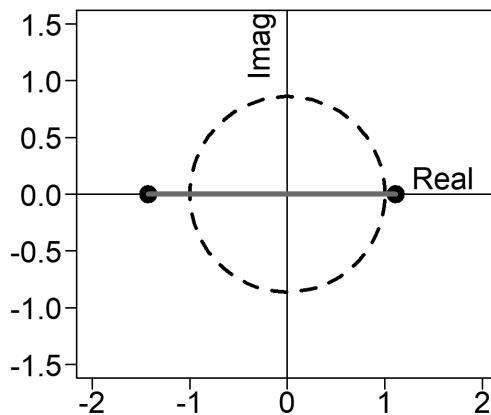


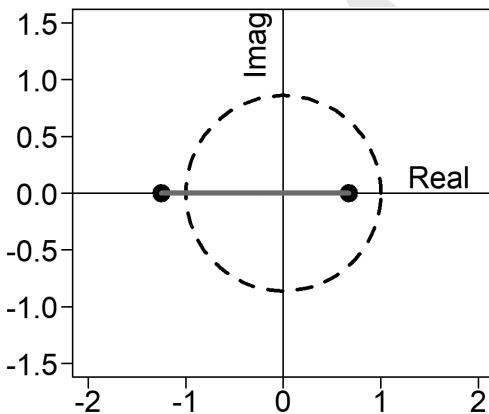
FIGURE 5.8 Plots of $r_1 = 1.11$ and $r_2 = -1.43$ in the complex plane.

Model (b) $X_t - .7X_{t-1} + 1.2X_{t-2} = a_t$

The operator form for this model is $(1 - .7B + 1.2B^2)X_t = a_t$, and the resulting characteristic equation is $1 - .7z + 1.2z^2 = 0$. By factoring, we get $(1 - 1.5z)(1 + .8z) = 0$, for which the roots are thus $r_1 = 1/1.5 = .67$ and $r_2 = -1/.8 = -1.25$. Since $r_1 = .67 < 1$, this is *not* a stationary AR(2) model. Figure 5.9, obtained using the command

```
unit.circle.wge(real=c(-1.25,.67),imaginary=c(0,0))
```

is a plot of the roots which illustrates that $r_1 = .67$ falls inside the unit circle and thus the model exhibits “explosive” behavior such as that seen in Figure 5.6(b), even though $r_2 = -1.25$ lies outside the unit circle.



QR 5.5
Characteristic
Equation of AR(2)

FIGURE 5.9 Plots of $r_1 = .67$ and $r_2 = -1.25$ in the complex plane.

Model (c) $X_t - 1.6X_{t-1} + .8X_{t-2} = a_t$

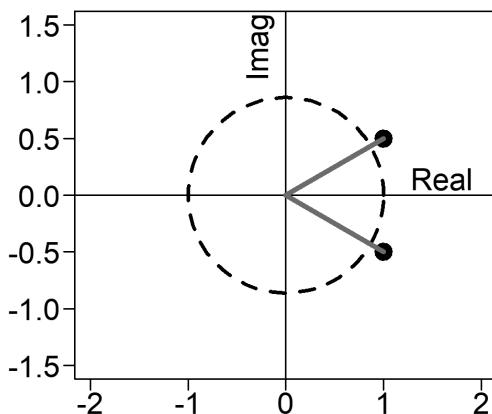
The model in operator form is $(1 - 1.6B + .8B^2)X_t = a_t$, and the characteristic equation is $1 - 1.6z + .8z^2 = 0$. This 2nd-order polynomial cannot be factored into two 1st-order factors with real coefficients, so we need to use the quadratic formula, from which we know that the roots of $az^2 + bz + c = 0$ are $r_1, r_2 = r_1^* = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. In our case, $a = .8$, $b = -1.6$, and $c = 1$. Going through the mathematics, we find that

$r_1, r_1^* = \frac{1.6 \pm \sqrt{(-1.6)^2 - 4(.8)(1)}}{2(.8)} = 1 \pm .5i$, and in both cases, $|r| = |r^*| = \sqrt{1^2 + .5^2} = 1.12$. Thus, by Theorem

5.2, Model (c) is a stationary AR(2) model. Figure 5.10, obtained using the command

```
unit.circle.wge(real=c(1,1),imaginary=c(.5,-.5))
```

shows that both roots are outside the unit circle.



QR 5.6
Characteristic
Equation with
Complex Roots

FIGURE 5.10 Plots of complex conjugate roots $1 \pm .5i$ in the complex plane.

5.1.2.3 Stationary AR(2) with Two Real Roots

Model (a) is an example of a stationary AR(2) model for which the characteristic equation has two real roots, 1.11 and -1.33. In this case the roots have opposite signs, but they could also both be positive or both be negative. The factored characteristic polynomial for Model (a) is $(1 - .9z)(1 + .7z)$, and each of these 1st-order factors are typical of AR(1) models. For example, the AR(1) model $(1 - .9B)Y_t = a_t$, for some Y_t , has the characteristic equation $1 - .9z = 0$. We examined such a model in Section 5.1.1.5 where we saw that the model $(1 - .9B)Y_t = a_t$ will have realizations with wandering behavior, exponentially damping autocorrelations, and a spectral density with a peak at $f = 0$. In Section 5.1.1.7, we discussed the AR(1) model with characteristic equation $1 + .7z = 0$. There we saw oscillating behavior in the realization, oscillating autocorrelations, and a peak in the spectral density at $f = .5$.

Interestingly, the AR(2) (Model (a))

$$X_t - .2X_{t-1} - .63X_{t-2} = (1 - .9B)(1 + .7B)X_t = a_t \quad (5.21)$$

will show a “combination” of these two 1st-order behaviors. The model-based autocorrelations and spectral density are shown in Figure 5.11(a) and (b), respectively. As a result of the discussion above, it follows that the spectral density in Figure 5.11(b) should show peaks at both $f = 0$ and $f = .5$. (Compare with Figures 5.1(b) and 5.5(b)). Consistent with the discussion in Section 5.1.1.5, because the 1st-order factor, $1 - .9B$, is associated with a root closer to the unit circle than $1 + .7B$, the peak of the spectral density at $f = 0$ is higher than it is at $f = .5$. The autocorrelations in Figure 5.11(a) show a damped exponential with a slight indication of oscillating behavior associated with the $(1 + .7B)$ factor.

Figure 5.11(c) shows a realization of length $n = 150$ generated from $(1 - .9B)(1 + .7B)X_t = a_t$ along with sample-based estimates of the autocorrelations and spectral density. The realization displays some “wandering behavior” with superimposed “oscillatory behavior”. (Compare with Figure 5.1(c) and Figure 5.5(c).) The sample autocorrelations at early lags are similar to the model-based autocorrelations, and the dominant features of the Parzen spectral density estimate in Figure 5.11(e) are peaks at $f = 0$ and at $f = .5$. The plots in Figure 5.11 are obtained using the *tswge* commands

```
plotts.true.wge(phi=c(.2, .63), plot.data=FALSE)
x=gen.arma.wge(n=150, phi=c(.2, .63), sn=13)
```

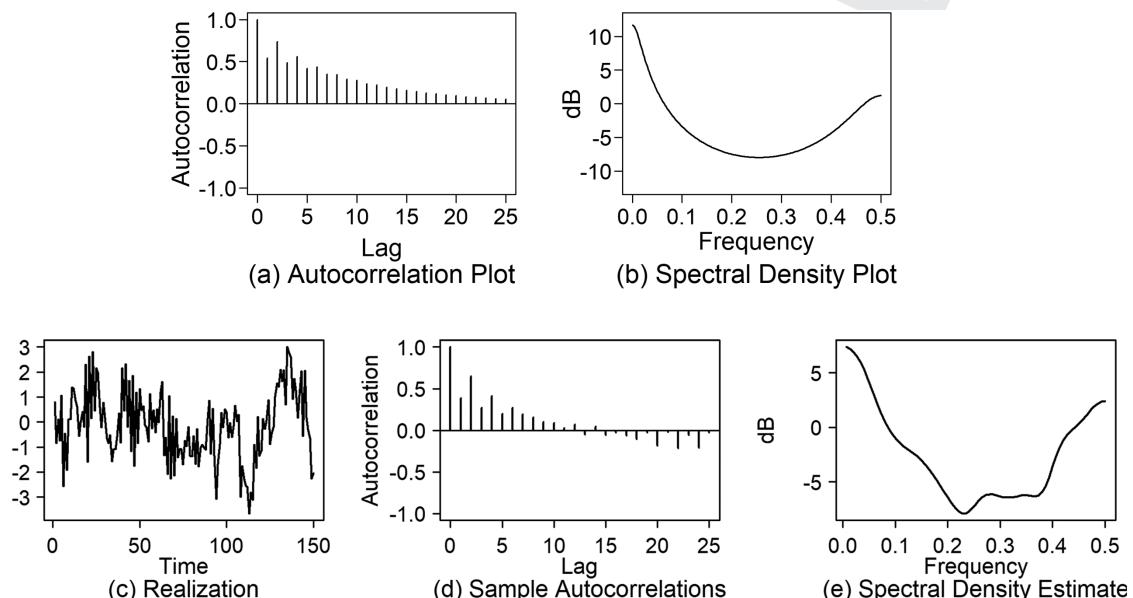


FIGURE 5.11 (a) Model-based autocorrelations and (b) spectral density for the model $(1 - .9B)(1 + .7B)X_t = a_t$. (c) Realization of length $n = 150$ from this model, (d) sample autocorrelations and (e) Parzen spectral density estimate based on the realization in (c).

In the AR(2) model (5.21), the positive real root ($1/.9$) is closer to one in absolute value than is the negative real root ($1/(-.7)$). The consequence of this, which we will re-visit in the case of the AR(p) model in Section 5.1.3.5, is that the behavior associated with the positive real root is “stronger”. The model $(1 - 1.6B + .63B^2)X_t = (1 - .9B)(1 - .7B)X_t = a_t$ has two positive real roots while the model $(1 + 1.6B + .63B^2)X_t = (1 + .9B)(1 + .7B)X_t = a_t$ has two negative real roots. The behavior of these two models is examined in Problem 5.2.

5.1.2.4 Stationary AR(2) with Complex Conjugate Roots

Returning to Model (c) in Example 5.2, we note that Model (c) is stationary because the associated characteristic equation for the model, $1 - 1.6z + .8z^2 = 0$, has the complex conjugate roots $1 \pm .5i$ which are both greater than one in absolute value.

The following are facts concerning AR(2) models associated with complex roots.

- (a) A stationary AR(2) model $(1 - \phi_1 B - \phi_2 B^2)X_t = a_t$ with complex conjugate roots has a model-based autocorrelation function, ρ_k , which has the appearance of a damped sinusoidal curve with system frequency²

$$f_0 = \frac{1}{2\pi} \cos^{-1} \left(\frac{\phi_1}{2\sqrt{-\phi_2}} \right)^3. \quad (5.22)$$

- (b) Realizations will tend to be pseudo-sinusoidal and associated with frequency approximately f_0 .
(c) The model-based spectral density

$$S_X(f) = \frac{\sigma_a^2}{\sigma_x^2 |1 - \phi_1 e^{-2\pi i f} - \phi_2 e^{-4\pi i f}|^2} \quad (5.23)$$

will have a peak at about f_0 . The spectral density in (5.23) can be calculated using *tswge* functions **true.arma.spec.wge** and **plotts.true.wge**.

AR(2) Model (c):

For Model (c), $\phi_1 = 1.6$ and $\phi_2 = -0.8$ (again – watch the signs!), so f_0 in (5.22) is given by

$$\begin{aligned} f_0 &= \frac{1}{2\pi} \cos^{-1} \left(\frac{\phi_1}{2\sqrt{-\phi_2}} \right) \\ &= \frac{1}{2\pi} \cos^{-1} \left(\frac{1.6}{2\sqrt{0.8}} \right) \\ &= .0738. \end{aligned} \quad (5.24)$$

Figures 5.12(a)–(b) display the model-based autocorrelations and spectral density for Model (c). The model-based autocorrelations in Figure 5.12(a) have a damped sinusoidal appearance with period length of about 14, which is consistent with the expected system frequency $f_0 = 1/14 \approx .07$. Note that the true spectral density in Figure 5.12(b) has a peak slightly below $f = .1$.

Figure 5.12(c) shows a realization of length $n = 150$ generated from Model (c). Figures 5.12(d) and (e) show the sample-based estimates of the autocorrelations and spectral density, respectively. We first note that the sample autocorrelations are similar to the model-based autocorrelations in that both show a damped sinusoidal behavior with a period of about 14 lags, which is consistent with a frequency of about .07. The Parzen spectral density estimate based on the realization in Figure 5.12(e) shows a distinct peak at about $f = .07$, which is not as sharp as the peak in Figure 5.12(b). The realization has the appearance of a pseudo-sinusoidal curve that goes through about 11–12 periods in the realization of length $n = 150$.

² When we use the term “system frequency”, we are referring to f_0 as given in (5.22). Specifically, as we discuss data in terms of their frequency behavior, we will use the notation f , unless reference is specifically made to the frequency calculated in (5.22).

³ Note that (5.22) states that $\cos(2\pi f_0) = \phi_1 / (2\sqrt{-\phi_2})$. Also note that $2\pi f_0$ is in radians.

That is, the period length is about $150/11.5 = 13$, which corresponds to a frequency of $f = 1/13 = .077$. Figures 5.12(a)–(e) were obtained using the commands

```
plotts.true.wge(phi=c(1.6,-.8),plot.data=FALSE)
x=gen.arma.wge(n=150,phi=c(1.6,-.8),sn=19)
plotts.sample.wge(x)
```

Another AR(2) model associated with complex roots:

Figures 5.13(a) and (b) show model-based autocorrelations and spectral density, respectively, for the model $(1 + .5B + .8B^2)X_t = a_t$. For this model, the roots of the characteristic equation $1 + .5z + .8z^2 = 0$ are $-0.3 \pm 1.1i$, which are complex and outside the unit circle. Consequently, this is also a stationary model.

Using (5.22) we obtain $f_0 = \frac{1}{2\pi} \cos^{-1}\left(\frac{-0.5}{2\sqrt{0.8}}\right) = .3$. The model-based autocorrelations in Figure 5.13(a)

have a damped sinusoidal appearance with period length about 3, which is consistent with the expected system frequency $f_0 \approx .33$. Because of the short period length, the sinusoidal nature of the autocorrelations is not apparent and has an oscillating appearance. The spectral density in Figure 5.13(b) has a peak at about $f = .3$.

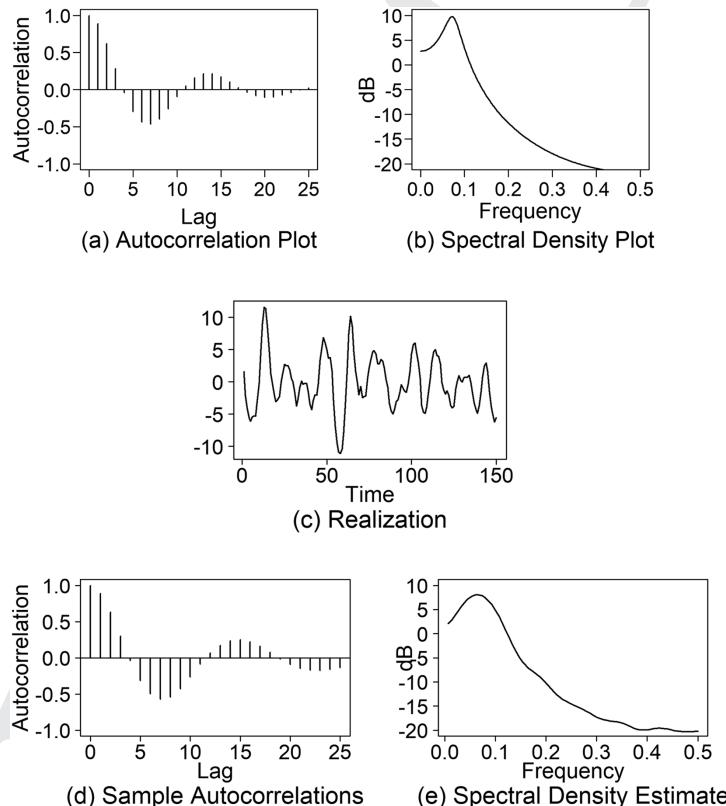


FIGURE 5.12 (a) Model-based autocorrelations and (b) spectral density for $(1 - 1.6B + .8B^2)X_t = a_t$, (c) realization of length $n = 150$ from $(1 - 1.6B + .8B^2)X_t = a_t$, (d) sample autocorrelations and (e) Parzen spectral density estimate based on the realization in (c).

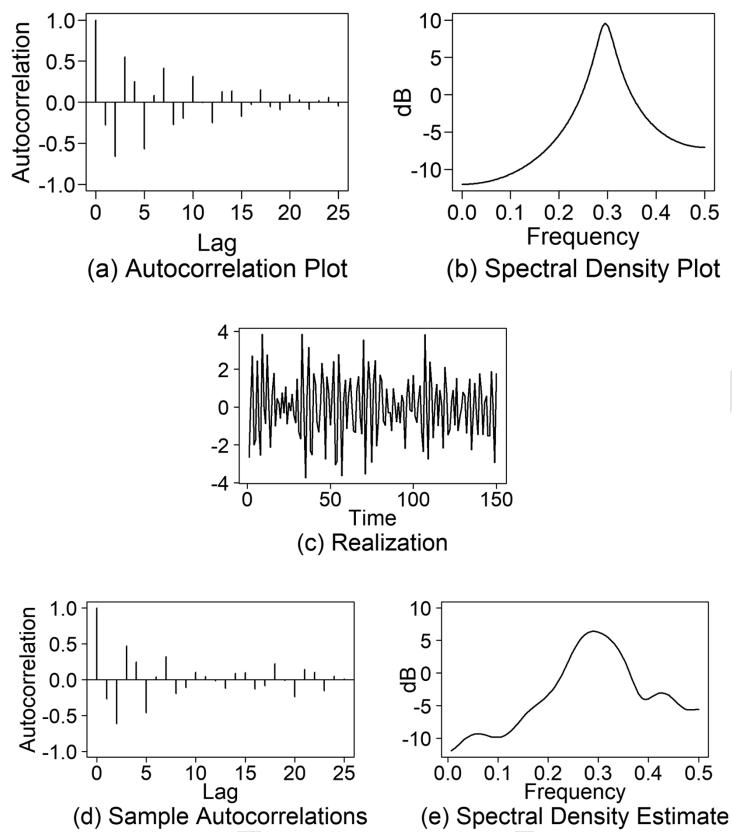


FIGURE 5.13 (a) Model-based autocorrelations and (b) spectral density for $(1 + .5B + .8B^2)X_t = a_t$, (c) realization of length $n = 150$ from $(1 + .5B + .8B^2)X_t = a_t$, (d) sample autocorrelations and (e) Parzen spectral density estimate based on the realization in (c).

Figures 5.13(c)–(e) show a realization of length $n = 150$ generated from $(1 + .5B + .8B^2)X_t = a_t$ along with sample-based estimates of the autocorrelations and spectral density, respectively. The realization has high-frequency behavior, and there are about 49 periods (do you agree?). That is, the period length is about $150/49=3.1$ and $f \approx 1/3.1 \approx .3$, which is consistent with the calculated f_0 . The sample autocorrelations in Figure 5.13(d) have a damped oscillating appearance (again, the sinusoidal nature is not obvious) with period length of about 3, and the Parzen spectral density estimate has a peak at about $f = .3$, which is consistent with the model-based spectral density in Figure 5.13(b). Figures 5.13(a)–(e) were obtained using the *tswge* commands

```
plotts.true.wge(phi=c(-.5, -.8), plot.data=FALSE)
x=gen.arma.wge(n=150, phi=c(-.5, -.8), sn=19)
plotts.sample.wge(x)
```

When discussing AR(1) and AR(2) models, we have typically plotted the model-based autocorrelations and spectral densities. We have then plotted realizations from the model of interest, sample autocorrelations, and Parzen spectral density estimates based on the generated realizations. (See, for example, Figure 5.12 and Figure 5.13.) We went through this detail to show what the “theoretical values” are and how these were estimated from data. We have seen that the sample autocorrelations and Parzen spectral density estimates have done a good job of estimating these “theoretical/model-based” quantities.

Key Points

- When analyzing real data, we will not “know” model-based or theoretical autocorrelations and spectral densities.
- From this point on when discussing models, we will typically not plot model-based autocorrelations and spectral densities unless they are needed for clarity.

Example 5.3 Canadian Lynx Data

Figure 5.14(a) (dataset `lynx` in `tswge`) shows the annual number of Canadian Lynx trapped in the Mackenzie River district of the Northwest Canada for the period 1821–1934 and Figure 5.14(b) shows the log (base 10) of the annual numbers trapped. These plots were previously given in Figures 1.11 and 1.12, respectively. As mentioned previously, this dataset that has interested researchers because of the cyclic behavior with cycle lengths of about 10 years. See, for example, Tong (1977), Bhansali (1979), Woodward and Gray (1983), and Woodward et al. (2017). The log of the lynx data is usually used for analysis because the resulting peaks and troughs of the cycles behave in similar manners, and thus the data are more AR-like. Also shown in Figure 5.14(c) and (d) are the sample autocorrelations and the Parzen spectral density estimate of the log-lynx data. The sample autocorrelations have a damped sinusoidal appearance with period of about 10, and the spectral density estimate in Figure 5.14(d) has a peak at about $f = .10$. The log-lynx data have the appearance of AR(2) data with a pair of complex conjugate roots associated with system frequency of about $f_0 = .10$. If an AR(2) model is fit to the data, the model is $X_t - 1.38X_{t-1} + .75X_{t-2} = a_t$. Using (5.22), the system frequency associated with this AR(2) model is $f_0 = .10$.

The plots in Figure 5.14 can be obtained using the commands

```
data(tswge)
plotts.wge(lynx)
llynx=log10(lynx)
plotts.sample.wge(llynx)
```

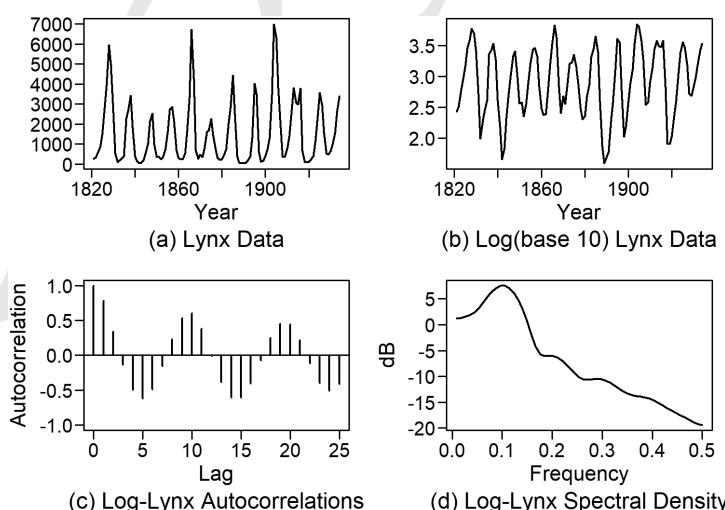


FIGURE 5.14 (a) Number of lynx trapped, (b) natural log of the data in (a), (c) sample autocorrelations of the data in (b), and (d) Parzen spectral density estimate of the data in (b).

5.1.2.5 Summary of AR(1) and AR(2) Behavior

Before proceeding to further examination of general AR(p) models, we summarize the information we have learned about AR(1) and AR(2) models:

- (a) *Facts about the AR(1) model $X_t - \phi_1 X_{t-1} = a_t$.*
 - (i) An AR(1) process is stationary if and only if $|\phi_1| < 1$, or equivalently, if the root $r_1 = \phi_1^{-1}$ of the characteristic equation is greater than one in absolute value.
 - (ii) The model-based autocorrelation function of a stationary AR(1) process is given by $\rho_k = \phi_1^{|k|}$. This autocorrelation function is a damped exponential if $\phi_1 > 0$ and an oscillating damped exponential if $\phi_1 < 0$.
 - (iii) Realizations from an AR(1) model with $\phi_1 > 0$ tend to be aperiodic with a general “wandering” behavior. When $\phi_1 < 0$, the realizations will tend to oscillate back and forth across the mean.
 - (iv) The spectral density $S_x(f)$ has a peak at $f = 0$ if $\phi_1 > 0$ and at $f = 0.5$ if $\phi_1 < 0$.
- (b) *Facts about an AR(2) model $X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = a_t$.*

An AR(2) process is stationary if and only if the roots of the characteristic equation $1 - \phi_1 z - \phi_2 z^2 = 0$ lie outside the unit circle.

The features of the AR(2) process depend on the nature of the roots of the characteristic equation.

Case 1: The roots of the characteristic equation are real.

Each real root corresponds to “1st-order behavior” in the realizations, autocorrelations, and spectral density.

Case 2: The roots of the characteristic equation are complex.

- (i) The model-based autocorrelation function is a damped sinusoidal with system frequency

$$f_0 = \frac{1}{2\pi} \cos^{-1} \left(\frac{\phi_1}{2\sqrt{-\phi_2}} \right) \text{ given in (5.22).}$$

- (ii) Realizations from an AR(2) model with complex conjugate roots will tend to be pseudo-sinusoidal with frequency f_0 given in (5.22), that is, with period $1/f_0$.
- (iii) The spectral density will have a peak near f_0 given in (5.22).

Key Point: As we will see in Section 5.1.3, knowing the behavior of AR(1) and AR(2) models is the key to understanding the behavior of an AR(p) model.

5.1.3 The AR(p) Models

We are now ready to discuss the general AR(p) model

$$X_t = \beta + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + a_t \quad (5.25)$$

defined in Definition 5.1. Note again that the AR(p) “looks” like a multiple regression equation, where in this case, the “independent variables” are the p previous values of the “dependent variable” X_t .

Another way to write (5.25), after rearranging terms, is

$$X_t - \mu - \phi_1(X_{t-1} - \mu) - \phi_2(X_{t-2} - \mu) - \cdots - \phi_p(X_{t-p} - \mu) = a_t. \quad (5.26)$$

As in the case of AR(1) and AR(2) models, we will frequently express the AR(p) in the zero-mean form

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} - \cdots - \phi_p X_{t-p} = a_t. \quad (5.27)$$

Equations (5.25)–(5.27) give the impression that an AR(p) model will be much more complicated to deal with than an AR(1) or AR(2) model. While this is somewhat true, as mentioned in the above Key Point, an understanding of the characteristics of AR(1) and AR(2) models leads directly to understanding the behavior of an AR(p) model. We will discuss this shortly.

5.1.3.1 Facts about the AR(p) Model

- (a) $E[X_t] = \mu$, for the “non-zero mean” form of the AR(p) model in (5.25) and (5.26).
- (b) The process variance is $\sigma_x^2 = \gamma_0 = \frac{\sigma_a^2}{1 - \phi_1\rho_1 - \phi_2\rho_2 - \cdots - \phi_p\rho_p}$, which is constant and finite when X_t is stationary.
- (c) The autocorrelations of an AR(p) process satisfy

$$\rho_k = \phi_1\rho_{k-1} + \phi_2\rho_{k-2} + \cdots + \phi_p\rho_{k-p}, \quad k > 0. \quad (5.28)$$

Equation (5.28) is a generalization of (5.17) for the AR(2) case, and it leads to the $p \times p$ *Yule-Walker Equations*

$$\begin{aligned} \rho_1 &= \phi_1 + \phi_2\rho_1 + \cdots + \phi_p\rho_{p-1} \\ \rho_2 &= \phi_1\rho_1 + \phi_2 + \cdots + \phi_p\rho_{p-2} \\ &\vdots \\ \rho_p &= \phi_1\rho_{p-1} + \phi_2\rho_{p-2} + \cdots + \phi_p. \end{aligned} \quad (5.29)$$

Analogous to the AR(2) case, knowing $\phi_1, \phi_2, \dots, \phi_p$ allows us to solve this $p \times p$ system of equations for ρ_k , $k = 1, \dots, p$. Model-based autocorrelations, ρ_k , $k > p$, can be computed using the recursion $\rho_k = \phi_1\rho_{k-1} + \phi_2\rho_{k-2} + \cdots + \phi_p\rho_{k-p}$. Not surprisingly, we use computer functions to perform these calculations. The **tswge** functions **true.arma.aut.wge** and **plotts.true.wge** use the Durbin Levinson algorithm to calculate model-based autocorrelations, ρ_k , $k = 1, 2, \dots$. See Durbin (1960), Levinson (1947), and Woodward et al. (2017).

- (d) The spectral density of an AR(p) model is given by

$$S_X(f) = \frac{\sigma_a^2}{\gamma_0} \frac{1}{\left|1 - \phi_1 e^{-2\pi i f} - \phi_2 e^{-4\pi i f} - \cdots - \phi_p e^{-2p\pi i f}\right|^2}. \quad (5.30)$$



Key Points

- As in the AR(1) and AR(2) cases, the behavior of the realizations, autocorrelations and spectral density depend on the roots of the characteristic equation defined below for the AR(p) model.
- If the coefficients $\rho_1, \rho_2, \dots, \rho_p$ of an AR(p) model are known, then the Yule-Walker equations can be used to solve for the model-based autocorrelations, $\phi_1, \phi_2, \dots, \phi_p$.
- Key Point 2 indicates that, using (5.30), the spectral density of an AR(p) process can be calculated if $\rho_1, \rho_2, \dots, \rho_p$ are known.

5.1.3.2 Operator Notation and Characteristic Equation for an AR(p)

The AR(p) model in (5.26) can be written in operator notation as

$$(1 - \phi_1 B - \phi_2 B^2 - \dots - \phi_p B^p)(X_t - \mu) = a_t \quad (5.31)$$

or by using a “shorthand” notation, $\phi(B)(X_t - \mu) = a_t$, where $\phi(B)$ is the p th order operator $\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \dots - \phi_p B^p$. Converting the operator $\phi(B)$ to the algebraic quantity $\phi(z)$ results in the general AR(p) characteristic polynomial $\phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \dots - \phi_p z^p$. The corresponding AR(p) characteristic equation is

$$\phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \dots - \phi_p z^p = 0. \quad (5.32)$$

The characteristic equation has p roots r_1, r_2, \dots, r_p which are real and/or complex, where the complex roots appear as conjugate pairs and some roots may be repeated.

Theorem 5.3 is the fundamental result regarding the stationarity of AR(p) processes. This result is the extension of Theorems 5.1 and 5.2 to AR(p) processes.

Theorem 5.3 An AR(p) process is stationary if and only if all of the roots of the characteristic equation are greater than one in absolute value.

Example 5.4 An AR(4) Model

Consider the AR(4) model

$$X_t - .13X_{t-1} - 1.4414X_{t-2} + .0326X_{t-3} + .8865X_{t-4} = a_t, \quad (5.33)$$

where $\sigma_a^2 = 1$. The operator notation for this model is

$$(1 - .13B - 1.4414B^2 + .0326B^3 + .8865B^4)X_t = a_t, \quad (5.34)$$

and the corresponding characteristic equation is

$$1 - .13z - 1.4414z^2 + .0326z^3 + .8865z^4 = 0. \quad (5.35)$$

Figure 5.15(a) is a realization of length $n = 200$ from (5.33), and Figures 5.15(b) and (c) are the associated sample autocorrelations and Parzen spectral density estimate, respectively. These plots were obtained using the commands

```
x=gen.arma.wge(n=200,phi=c(0.1300,1.4414,-.0326,-.8865),sn=9310,plot=FALSE)
plots.sample.wge(x)
```

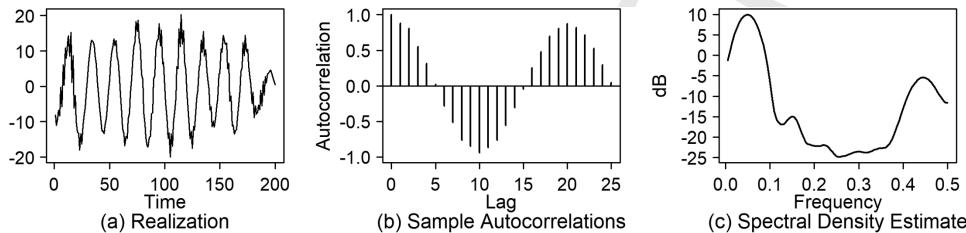


FIGURE 5.15 (a) Realization from the AR(4) model in (5.33), (b) sample autocorrelations, and (c) Parzen spectral density estimate calculated from the data in (a).

Note that Figure 5.15(a) was previously shown in Figure 4.20(a). The `gen.arma.wge` code specifies that $\phi_1 = .13$, $\phi_2 = 1.4414$, $\phi_3 = -.0326$, and $\phi_4 = -.8865$. These correspond to the coefficients in (5.33) (again –be careful about signs). The seed is set to 9310, and by selecting default values, $\mu = 0$ and $\sigma_a^2 = 1$. As noted in Chapter 4, this realization has a low-frequency behavior of about $f = .05$ and a high-frequency content at approximately $f = .45$.

Section 5.1.2.5 summarizes information about how the behaviors of realizations, autocorrelations, and spectral densities are related to the coefficients of AR(1) and AR(2) models. This leads us to the following question.

Question: What is it about the coefficients in (5.33) that created the behavior of the realization, sample autocorrelations, and spectral density in Figure 5.15?

We begin to answer this question by factoring the AR(p) characteristic polynomial.

5.1.3.3 Factoring the AR(p) Characteristic Polynomial

The roots of a quadratic equation can be found by using the quadratic formula. However, things get more “sticky” for polynomial orders greater than two. The cubic equation

$$1 - 2.1z + 1.6z^2 - .3z^3 = 0$$

can be factored into the form $(1 - .5z)(1 - 1.6z + .8z^2) = 0$. (Do you remember how to do this?) Based on this factorization, the roots are found by setting $(1 - .5z) = 0$, in which case $r_1 = 1/.5 = 2$, and by setting $(1 - 1.6z + .8z^2) = 0$, in which case the roots are $r_2 = 1 + .5i$ and $r_3 = 1 - .5i$. That is, this AR(3) will have 1st-order behavior associated with $1 - .5B$, 2nd-order cyclic behavior with system frequency $f_0 = .07$, and the process is stationary because all roots are outside the unit circle.

The 5th-order polynomial equation

$$1 - 2z + 1.94z^2 - 1.32z^3 + .72z^4 - .16z^5 = 0,$$



can be factored as $(1 - .4z)(1 - 1.6z + .8z^2)(1 + .5z^2) = 0$. (Good luck with this one!) The resulting roots are $r_1 = 1/.4 = 2.5$, $r_2 = 1 + .5i$, $r_3 = 1 - .5i$, $r_4 = \sqrt{2}i$, and $r_5 = -\sqrt{2}i$. Realizations will have behavior that is a combination of 1st-order factor $1 - .5B$ and cyclic behavior associated with the system frequencies of the 2nd-order factors $1 - 1.6B + .8B^2$ ($f_0 = .07$) and $1 + .5B^2$ ($f_0 = .25$). The process is stationary because all roots are outside the unit circle. (It is a useful exercise to check this.)

Higher order polynomial equations, such as the 5th-order equation above, must be solved using numerical methods. See for example, Press, Teukolsky, Vetterling, and Flannery (2007).

Note that in each case above, we

- factored the polynomial (into first- and/or second-order factors), and
- found the roots based on setting each of these factors equal to zero.

Theorem 5.4 below provides a generalization of these observations.

Theorem 5.4: The p th order polynomial $1 - \phi_1 z - \phi_2 z^2 - \dots - \phi_p z^p$ can always be factored as a product of

- 1st-order (linear) factors which are associated with real roots
- 2nd-order (quadratic) factors for which the roots are complex conjugate pairs⁴

Key Points

- $\phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \dots - \phi_p z^p = 0$ can always be factored as a product of
 - 1st-order (linear) factors – associated with the real roots
 - 2nd-order polynomial (quadratic) factors – associated with complex conjugate pairs
- Roots of $\phi(z) = 0$ can be found by factoring the polynomial into the 1st- and 2nd-order factors and then setting each factor equal to zero.
- A 2nd-order factor associated with a complex conjugate pair is referred to as an *irreducible 2nd-order factor*.
- While 1st- and 2nd-order equations can be solved easily, polynomial equations of 3rd-order and above are difficult or impossible to solve algebraically (that is, by using an equation like the quadratic equation). For this reason, such higher order polynomials are solved numerically using a computer.

5.1.3.4 Factor Tables for AR(p) Models

Theorem 5.4 states that any p th-order polynomial can be expressed as a product of 1st-order and/or irreducible 2nd-order factors. Understanding the 1st- and 2nd-order factors is the “key” to understanding the AR(p) model.

⁴ A 2nd-order factor associated with complex conjugate root pairs can be factored further to a product of two linear factors, but these two factors have complex (not real) coefficients.

Key Points

- 1st-order and 2nd-order factors like $(1 - \alpha_1 B)$ and $(1 - \alpha_1 B - \alpha_2 B^2)$ serve as building blocks (or the DNA) of an AR(p) model.
- Features of an AR(p) model are simply a combination of 1st- and 2nd-order features.
 - That is, there is no new “third-order feature.”

Example 5.4 (Revisited)

Consider again the AR(4) model in (5.33):

$$(1 - .13B - 1.4414B^2 + .0326B^3 + .8865B^4)X_t = a_t.$$

The associated characteristic equation is

$$1 - .13z - 1.4414z^2 + .0326z^3 + .8865z^4 = 0.$$

The factored form (obtained numerically) is

$$(1 - 1.89B + .985B^2)(1 + 1.76B + .9B^2) = 0.$$

The **tswge** function **factor.wge** uses numerical methods to produce these factors and outputs them in a “Factor Table”. The factor table is a very useful tool for quickly summarizing the “DNA” of an AR(p) model regarding the 1st- and 2nd-order factors. The command

```
factor.wge(phi=c(.13, 1.4414, -.0326, -.8865))
```

produces the output:

AR Factor Table				
Factor	Roots	Abs Recip	System	Freq
$1 - 1.8900B + 0.9850B^2$	$0.9594 + -0.3079i$	0.9925	0.0494	
$1 + 1.7600B + 0.9000B^2$	$-0.9778 + -0.3938i$	0.9487	0.4391	



Table 5.1 shows the format that will be used to display factor tables in the book.

QR 5.7
Factored Models

TABLE 5.1 Factor Table for Model (A): $X_t - .13X_{t-1} - 1.4414X_{t-2} + .0326X_{t-3} + .8864X_{t-4} = a_t$

FACTOR	ROOTS	$ r ^{-1}$	f_0
$1 - 1.89B + .985B^2$	$.96 \pm .31i$.99	.05
$1 + 1.76B + .9B^2$	$-.98 \pm .39i$.95	.44

The first thing we notice is that the factor table shows the (1st- and 2nd-order) factors of the model. In this case, there are two 2nd-order factors, $1 - 1.89B + .985B^2$ and $1 + 1.76B + .9B^2$, as noted above. In the following, we summarize the wealth of information contained in the factor table.

Factor Table Format

For each 1st-order or irreducible 2nd-order factor of the model, the factor table is displayed in four columns:

Column 1: The first and/or irreducible 2nd-order factors.

Column 2: Roots of the characteristic equations associated with the factors.

Column 3: The absolute reciprocal of the roots. This measure provides two key pieces of information:

- (a) *Stationarity:* If the roots are all outside the unit circle, the process is stationary. Because we table the *reciprocal* of the roots, the check for stationarity is whether all roots have reciprocals less than one.
- (b) *A measure of how close the roots are to the unit circle.* (The closer the reciprocal is to one, the closer the root is to the unit circle.) We will see that roots closest to the unit circle dominate the behavior of a stationary AR process.

Column 4: System frequencies f_0 . For stationary models:

- (a) $f_0 = 0$ for 1st-order factors with positive real roots
- (b) $f_0 = .5$ for 1st-order factors with negative real roots
- (c) f_0 is given by (5.22) for irreducible 2nd-order factors

Example 5.5 Consider the following AR(p) models.

- (A) $X_t - 1.95X_{t-1} + 1.85X_{t-2} - .855X_{t-3} = a_t$
- (B) $X_t - 2.6X_{t-1} + 3.34X_{t-2} - 2.46X_{t-3} + .9024X_{t-4} = a_t$
- (C) $X_t - 2.85X_{t-1} + 3.24X_{t-2} - 2.03X_{t-3} + .6X_{t-4} = a_t$

Figure 5.16 shows realizations from these three models given in random order. The code that generated the realizations will not be given here because that would defeat the purpose of the following question. Sorry!

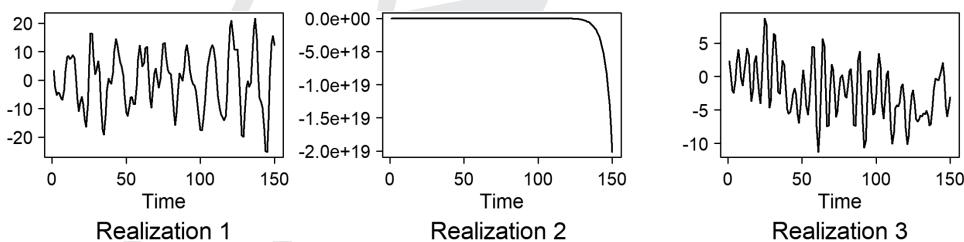


FIGURE 5.16 Realizations from AR(p) Models (A), (B), and (C) in random order.

Question: By examining the coefficients of the models, can you identify which models are stationary and which realizations correspond to which model?

You Can't? Don't Worry. Neither Can We! The coefficients themselves are not very informative.

Our Solution: Use `factor.wge` to solve the equations and provide key information.

Model (A): $X_t - 1.95X_{t-1} + 1.85X_{t-2} - .855X_{t-3} = a_t$

The command

```
factor.wge(phi=c(1.95,-1.85,.855))
```

produces the following factor table.

TABLE 5.2 Factor Table for Model (A): $X_t - 1.95X_{t-1} + 1.85X_{t-2} - .855X_{t-3} = a_t$

FACTOR	ROOTS	$ r ^{-1}$	f_0
$1 - .95B$	$1.05i$.95	.00
$1 - B + .9B^2$	$.55 \pm .90i$.95	.16

The factorization shows that the properties of this AR(3) model will be a combination of 1st-order behavior associated with $1 - .95B$ (that is, aperiodic wandering behavior with a “system frequency” of $f_0 = 0$) and 2nd-order behavior associated with $1 - B + .9B^2$, that is, pseudo-sinusoidal realizations associated with system frequency $f_0 = \frac{1}{2\pi} \cos^{-1}\left(\frac{1}{2\sqrt{.9}}\right) = .16$. The root associated with the 1st-order factor is $1/.95 = 1.05$, and the roots associated with the 2nd-order factor are $.55 \pm .90i$. The factor table shows that the absolute value of the reciprocal of all three roots is .95, which indicates stationarity because the reciprocals are all less than one in absolute value.

Based on the factor table in Table 5.2 and the above discussion, this model should produce realizations that show random wandering (associated with $1 - .95B$) and pseudo-sinusoidal behavior with frequency of about .16 (or period $1/.16 = 6$). Realization 3 in Figure 5.16 has pseudo-sinusoidal behavior along a wandering path. By counting, we see that there are about 23 cycles in the realization of length $n = 150$, implying that the period length is about $150 / 23 = 6.5$, which is consistent with a system frequency of $f_0 = .16$.

CONCLUSION: REALIZATION 3 IS FROM MODEL (A).

Figure 5.17 shows Realization 3, along with its sample autocorrelations and Parzen spectral density estimate. In addition to the realization behavior noted above, the sample autocorrelations have a sinusoidal behavior (2nd-order) that is not symmetric about zero, but instead seems to follow a damped exponential path (1st-order). Also the spectral density estimate has a peak at zero (associated with a 1st-order factor with positive real root) and a peak slightly below $f = .2$. These are consistent with the information in the factor table. Figure 5.17 can be obtained using the commands

```
x=gen.arma.wge(n=150,phi=c(1.95,-1.85,.855),sn=129,plot=FALSE)
plotts.sample.wge(x)
```

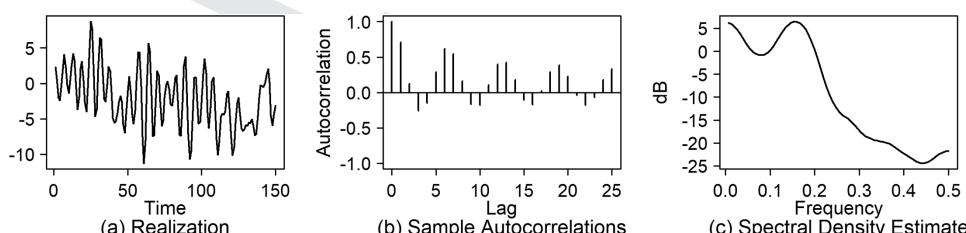


FIGURE 5.17 (a) Realization 3 in Figure 5.16, (b) sample autocorrelations, and (c) Parzen spectral density estimate calculated from this realization.

Model (B): $X_t - 2.6X_{t-1} + 3.34X_{t-2} - 2.46X_{t-3} + .9024X_{t-4} = a_t$

The command

```
factor.wge(phi=c(2.6,-3.34,2.46,-.9024))
```

produces the factor table in Table 5.3.

TABLE 5.3 Factor Table for Model (B): $X_t - 2.6X_{t-1} + 3.34X_{t-2} - 2.46X_{t-3} + .9024X_{t-4} = a_t$

FACTOR	ROOTS	$ r ^{-1}$	f_0
$1 - 1.8B + .96B^2$	$.94 \pm .40i$.98	.065
$1 - .8B + .94B^2$	$.43 \pm .94i$.97	.182

In this case, the 4th-order polynomial factors into two irreducible 2nd-order polynomials. That is, the characteristic equation has two pairs of complex conjugate roots. The factor table shows that the factored form is $(1 - 1.8B + .96B^2)(1 - .8B + .94B^2)X_t = a_t$. The model is stationary because all values in column 3 are less than one. The factor table indicates that the data will have cyclic behavior associated with frequencies $f_0 = .065$ and $f_0 = .182$. Realization 1 in Figure 5.16 goes through about 10 cycles in the 150 time points, so the period is about 15 and the associated frequency is about $f = 1/15 = .067$. Also, there appears to be a higher frequency behavior, but it is difficult to assess the period length. Realization 1 in Figure 5.16 is the only realization showing characteristics consistent with Model (B). **Conclusion: Realization 1 is from Model (B).**

Examining the components: In order to further understand AR(4) Model (B), Figure 5.18(a) shows a realization from an AR(2) model associated with the factor $1 - 1.8B + .96B^2$, which as can be seen from the factor table, is associated with a frequency $f = .065$ (or period length of about 15). The realization is cyclic and goes through about 10 cycles across 150 data points; that is, the period is about 15. The sample autocorrelations in Figure 5.18(b) are a smooth, slowly damping sinusoid again with period of about 15, while the Parzen spectral density estimate in Figure 5.18(c) has a peak slightly below $f = .10$.

Figure 5.18(d) shows a realization from the AR(2) model associated with the factor $1 - .8B + .94B^2$, which is associated with a frequency $f = .182$ (or period length of about 5.5). The realization has higher frequency behavior, going through about 28 cycles in 150 data values, indicating a period length of about $150/28 = 5.36$, and a frequency of $f = .19$. The sample autocorrelations have a damping sinusoidal behavior with period of about 5. The Parzen spectral density estimate has a peak at about $f = .20$.

Figure 5.18(g) displays Realization 1 from Figure 5.16 which we concluded was from AR(4) Model (B). The Parzen spectral density estimate has two peaks: (1) one slightly below $f = .1$ and (2) one at about $f = .2$. The sample autocorrelations in Figure 5.18(h) are very similar to those in Figure 5.18(b) for the AR(2) model associated with the lower frequency and show no indication of the higher frequency.

The plots in Figure 5.18 can be obtained using the following commands

```
xa=gen.arma.wge(n=150,phi=c(1.8,-.96,0,.001),mu=0,sn=3233)
plotts.sample.wge(xa) # plots (a-c)
xd=gen.arma.wge(n=150,phi=c(.8,-.94,0,.001),mu=0,sn=3233)
plotts.sample.wge(xd) # plots (d-f)
xg=gen.arma.wge(n=150,phi=c(2.6,-3.34,2.46,-.9024),mu=0,sn=3233)
plotts.sample.wge(xg) # plots (g-i)
```

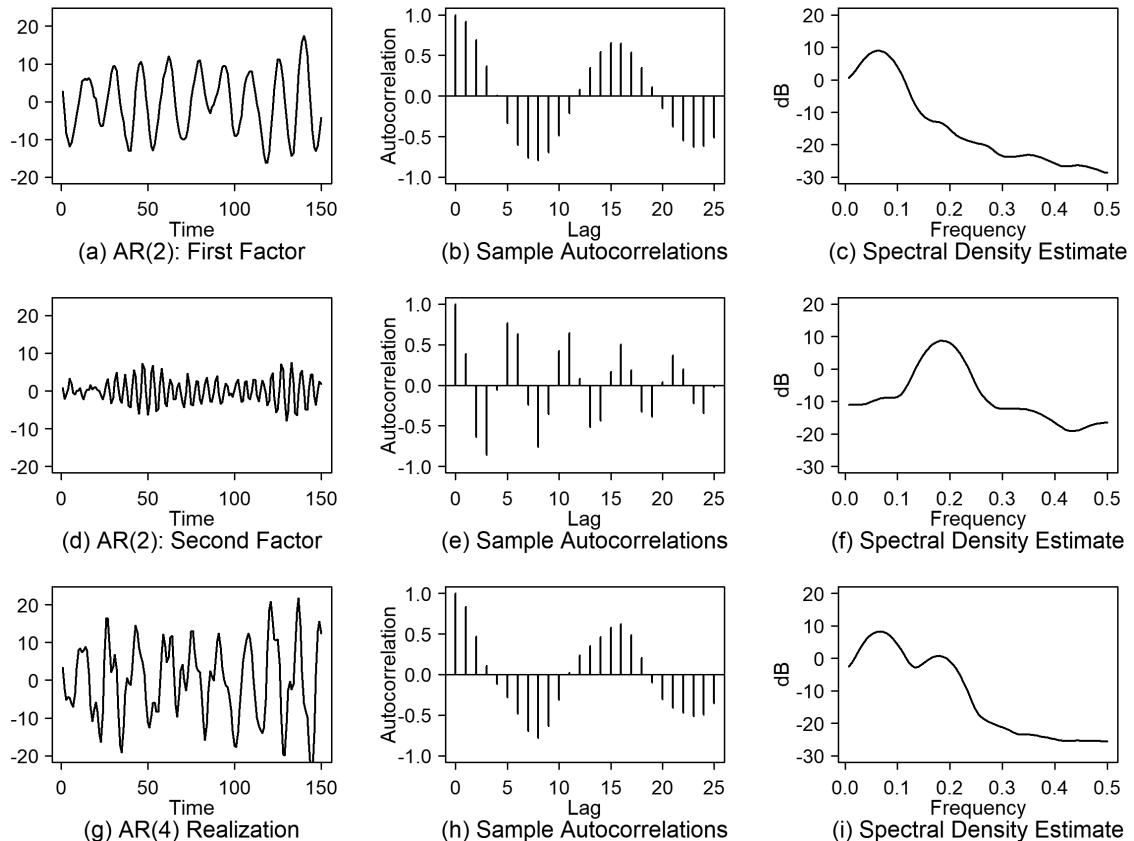


FIGURE 5.18 (a) Realization of length $n = 150$ from AR(2) model $(1 - 1.8B + .96B^2)X_t = a_t$, (b) sample autocorrelations and (c) Parzen spectral density estimate based on the realization in (a). (d) realization of length $n = 150$ from AR(2) model $(1 - .8B + .94B^2)X_t = a_t$, (e) sample autocorrelations and (f) Parzen spectral density estimate based on the realization in (d). (g) realization of length $n = 150$ from AR(4) model $(1 - 1.8B + .95B^2)(1 - .8B + .94B^2)X_t = a_t$, (h) sample autocorrelations and (i) Parzen spectral density estimate based on the realization in (g).

Model (C): $X_t - 2.85X_{t-1} + 3.24X_{t-2} - 2.03X_{t-3} + .6X_{t-4} = a_t$
The command

```
factor.wge(phi=c(2.85,-3.24,2.03,-.6))
```

produces the factor table in Table 5.4.

TABLE 5.4 Factor Table for Model (C): $X_t - 2.85X_{t-1} + 3.24X_{t-2} - 2.03X_{t-3} + .6X_{t-4} = a_t$.

AR-FACTOR	ROOTS	$ r ^{-1}$	f_0
$1 - 1.25B$.80	1.25	0
$1 - .80B$	1.25	.80	0
$1 - .80B + .60B^2$	$.67 \pm 1.11i$.77	.16

As a first check for stationarity, we scan the third column and see a value of 1.25, which is not less than 1. Consequently, the model is nonstationary. (Equivalently, the first-order factor $1 - 1.25B$ is

associated with the root $1/1.25=.8$ which is inside the unit circle.) We see that Realization 2 has “explosive behavior”.⁵

Conclusion: Realization 2 is from Model (C) (No further analysis is appropriate.)

mult.wge: A useful tswge command

The **tswge** function **factor.wge** has been shown to be valuable for revealing the underlying factors and associated behavior of an AR model. The vector **phi** is input to the **factor.wge** function, and it factors the model into first- and/or irreducible 2nd-order factors, and prints a factor table.

Consider the case in which you have an AR(5) model in factored form, for example,

$$(1 - 1.8B + .95B^2)(1 + .6B)(1 + .8B^2)X_t = a_t,$$

and you want to generate a realization from it. The function **gen.arma.wge** requires the autoregressive model to be specified by its model coefficients in the vector **phi**. Before generating a realization from this model, the AR coefficients must be available in *unfactored form*.

The **tswge** package provides a function, **mult.wge**, for multiplying the factors and outputting the model coefficients. Considering the AR(5) model above, the **mult.wge** command that produces the model coefficients is

```
cf=mult.wge(fac1=c(1.8,-.95),fac2=-.6, fac3=c(0,-.8))
```

The output from this command includes the vector

```
cf$model.coef [1] 1.200 -0.670 0.390 0.104 -0.456
```

which contains the parameters of the model. The model can now be written as

$$X_t - 1.2X_{t-1} + .67X_{t-2} - .39X_{t-3} - .104X_{t-4} + .456X_{t-5} = a_t.$$

Either of the two commands

```
x=gen.arma.wge(n=200,phi=cf$model.coef,sn=10)
```

or

```
x1=gen.arma.wge(n=200,phi=c(1.2,-.67,.39,.104,-.456),sn=10)
```

can be used to generate the same realization from the desired model.

Notes:

- (1) See Appendix 5A for more information on the **mult.wge** command.
- (2) Several **tswge** commands require model parameters as input.

⁵ **gen.arma.wge** will not generate realizations from a nonstationary process. The realization in Figure 5.16(c) was generated directly from the equation $X_t = 2.85 X_{t-1} - 3.24 X_{t-2} + 2.03 X_{t-3} - .6X_{t-4} + a_t$, where a_t is Normal(0,1) white noise.

5.1.3.5 Dominance of Roots Close to the Unit Circle

Factors associated with roots closer to the unit circle tend to be more dominant in the combined behavior of an AR(p) model. In order to further illustrate this concept, we consider the following models given in factored form:

Model (A): Previously studied Model (A) (complex and real roots equidistant from unit circle)

$$X_t - 1.95X_{t-1} + 1.85X_{t-2} - .855X_{t-3} = a_t, \text{ or in factored form:}$$

$$(1 - .95B)(1 - B + .9B^2)X_t = a_t.$$

Model (A_real): Real root closer to the unit circle than complex root

$$X_t - 1.71X_{t-1} + 1.22X_{t-2} - .475X_{t-3} = a_t, \text{ or in factored form:}$$

$$(1 - .95B)(1 - .76B + .5B^2)X_t = a_t.$$

Model (A_complex): Complex root closer to the unit circle than real root

$$X_t - 1.70X_{t-1} + 1.60X_{t-2} - .63X_{t-3} = a_t, \text{ or in factored form:}$$

$$(1 - .70B)(1 - B + .9B^2)X_t = a_t.$$

In order to understand these models, we compare their factor tables given in Table 5.5.

TABLE 5.5 Factor Tables for Models (A), (A_real), and (A_complex)

	FACTOR	ROOTS	$ r ^{-1}$	f_0
Model (A)	$1 - .95B$	1.05	.95	0
	$1 - B + .9B^2$	$.55 \pm .90i$.95	.16
Model (A_real)	$1 - .95B$	1.05	.95	0
	$1 - .76B + .5B^2$	$.76 \pm .90i$.70	.16
Model A_complex	$1 - .70B$	1.43	.70	0
	$1 - B + .9B^2$	$.55 \pm .90i$.95	.16

While examining Table 5.5 notice the following:

- (a) All three models are AR(3) models with a 1st- and 2nd-order factor
- (b) All three models have system frequencies 0 and .16
- (c) Model (A): $|r|^{-1} = .95$ for both real and complex roots
 - consequently, both real and complex roots are equally close to the unit circle
- (d) Model (A_real): $|r|^{-1} = .95$ for the real root, and $|r|^{-1} = .70$ for the complex roots
 - the real root is closer to the unit circle than are the complex roots
- (e) Model (A_complex): $|r|^{-1} = .70$ for the real root, and $|r|^{-1} = .95$ for the complex roots
 - the complex roots are closer to the unit circle than is the real root

To illustrate the fact that roots closer to the unit circle dominate, we generate realizations from each of the three models. These realizations, sample autocorrelations, and Parzen spectral density estimates are shown in Figure 5.19. The plots in Figure 5.18 can be obtained using the following commands:

```
A=gen.arma.wge(n=150,phi=c(1.95,-1.85,.855), sn=3847)
plotts.sample.wge(A) # plots (a-c)
A_real=gen.arma.wge(n=150,phi=c(1.71,-1.22,.475), sn=327)
plotts.sample.wge(A_real) # plots(d-f)
A_complex=gen.arma.wge(n=150,phi=c(1.70,-1.60,.63), sn=2813)
plotts.sample.wge(A_complex) # plots(g-i)
```

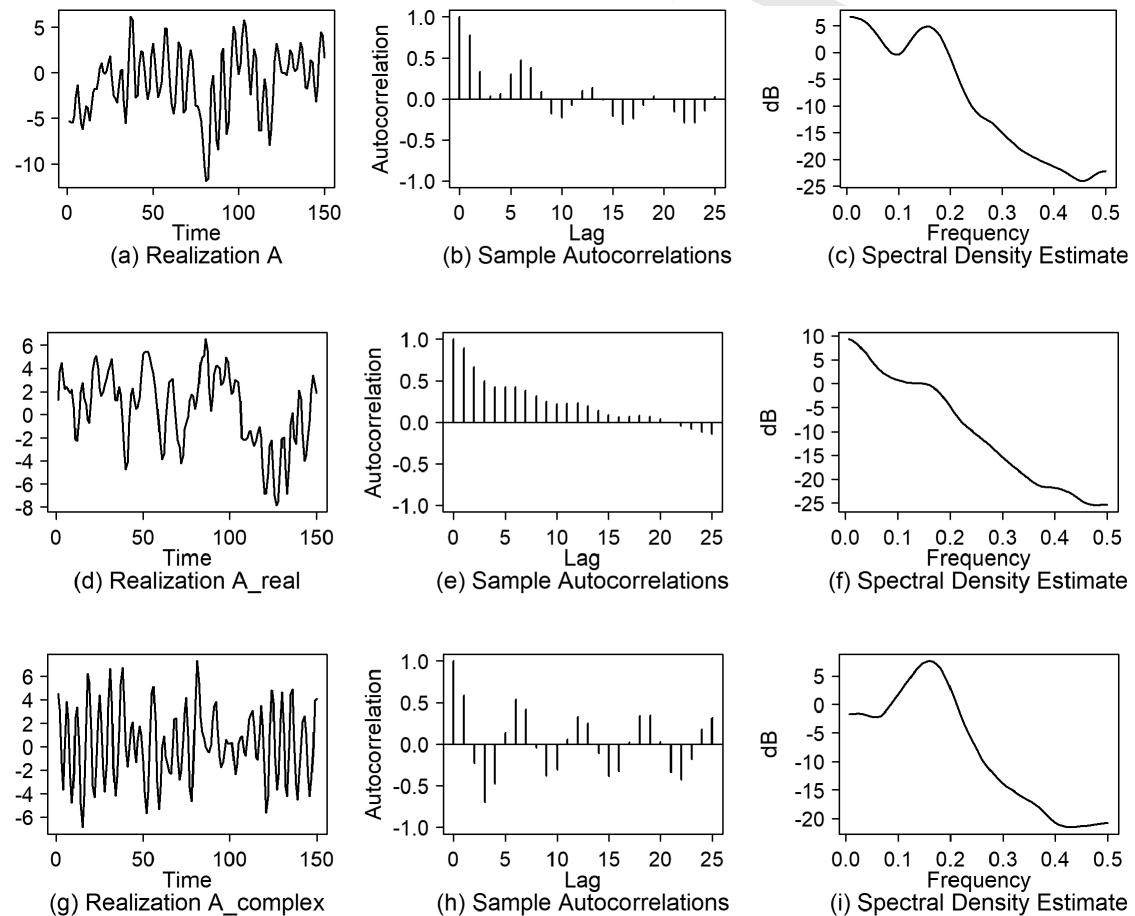


FIGURE 5.19 (a) realization, (b) sample autocorrelations, and (c) Parzen spectral density estimate from Model (A); (d) realization, (e) sample autocorrelations, and (f) Parzen spectral density estimate from Model (A_real); (g) realization, (h) sample autocorrelations, and (i) Parzen spectral density estimate from Model (A_complex).

We begin with Figures 5.19(a)–(c), which are based on a realization generated from Model (A): $(1-.95B)(1-B+.9B^2)X_t = a_t$, using *tswge* command **gen.arma.wge**. The data in Figure 5.19(a) are based on a different seed than the one used to generate the data from Model (A) shown, among other places, in Figure 5.17(a). Take a few minutes to compare Figures 5.17 and Figure 5.19(a)–(c). In both cases, the realization shows a pseudo-sinusoidal behavior associated with period of about 6, along a wandering path. The sample autocorrelations show damped sinusoidal behavior along a damped exponential path. Finally, the Parzen spectral density estimates in Figure 5.17(c) and Figure 5.19(c) both show

peaks at zero and at some frequency slightly below $f = .2$. An important take-away from this example is that the real and complex roots are equidistant from the unit circle, and consequently both behaviors are equally apparent in the figures.

Figure 5.19(d) shows a realization from Model(A_real). It was noted that the real root in Model(A_real) is substantially closer to the unit circle than are the complex roots. Consequently, in Figure 5.19(d) we see wandering behavior with only slight periodic behavior. The sample autocorrelations in Figure 5.19(e) show only a damped exponential behavior and the Parzen spectral density estimate in Figure 5.19(f) has a peak at $f = 0$ with a slight inflection point (but not really a peak) at $f = .16$. This illustrates the fact that the real root (which is the root closest to the unit circle) dominates the behavior.

Figure 5.19(g) shows a realization from Model (A_complex) in which the complex roots are substantially closer to the unit circle than is the real root. Figure 5.19(g) shows pseudo-sinusoidal behavior, but instead of this behavior being along a wandering path as in Figure 5.19(a), the pseudo-sinusoidal behavior is basically centered along the horizontal axis. The sample autocorrelations in Figure 5.19(h) show a damped sinusoidal behavior that is fairly symmetric around the zero axis. That is, there is not much evidence of the damped exponential behavior. Finally, the spectral density estimate in Figure 5.19(i) has a peak at about $f = .16$ and only a weak upward tendency, but not really a peak, around $f = 0$.



QR 5.8
Dominance

Key Points

1. Factors associated with roots closer to the unit circle tend to be more dominant in the combined behavior of an AR(p) model.
2. Much of the behavior of an AR process can be traced to its dominant factors, that is, those closest to the unit circle.
3. The factor table is an invaluable tool for inspecting the factors of an AR model, for understanding its composition, and ultimately for interpreting the physical process generating the data.

As we discuss the analysis of actual time series data in later chapters, the validity of these Key Points will become apparent. In the following example, we illustrate the use of the factor table to learn about the underlying features of an AR(9) model fit to the sunspot data.

Example 5.6 Sunspot Data

The sunspot data (data set **sunspot2.0** in **tswge**) is based on the new counting procedure for the years 1700 through 2020, Clette et al. (2016). Figures 5.20(a)–(c) show the sunspot data from 1700–2020, the sample autocorrelations, and the Parzen spectral density estimate, respectively. As mentioned previously, the cyclic behavior with periods of about 10–11 years has been the topic of extensive investigation. A reasonable model for these data is an AR(9) (take our word for it for now). We will discuss model identification of the sunspot data in Example 6.8.

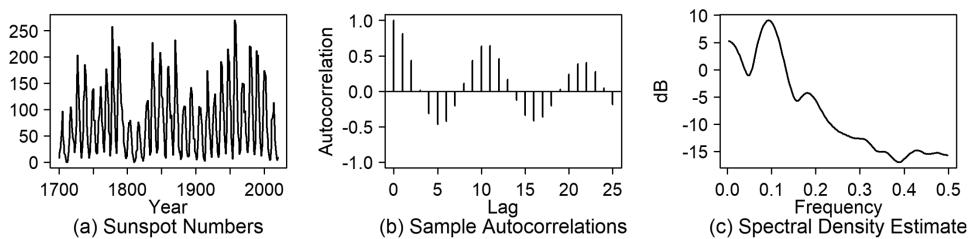


FIGURE 5.20 (a) sunspot data, (b) sample autocorrelations, and (c) Parzen spectral density estimate.

The AR(9) model fit to the (zero-mean form) of the sunspot data is⁶

$$(1 - 1.17B + .41B^2 + .13B^3 - .10B^4 + .07B^5 - .01B^6 - .02B^7 + .05B^8 - .22B^9)(X_t - 78.52) = a_t \quad (5.36)$$

While no information can be gained from the actual coefficients of the AR(9) model, we use the factor table in Table 5.6 to understand the underlying features of the model.

TABLE 5.6 Factor Table for the AR(9) Model Fit to sunspot2.0 Data

FACTOR	ROOTS	$ r ^{-1}$	f_0
$1 - 1.61B + .95B^2$	$0.85 \pm 0.58i$	0.97	0.094
$1 - 0.94B$	1.06	0.94	0.000
$1 - 0.60B + 0.72B^2$	$0.42 \pm 1.10i$	0.85	0.192
$1 + 1.44B + 0.59B^2$	$-1.21 \pm 0.46i$	0.77	0.442
$1 + 0.56B + 0.59B^2$	$-0.48 \pm 1.22i$	0.76	0.309

The factor table shows, not surprisingly, that the dominant component is a 2nd-order factor associated with $f_0 = .094$ (e.g. periods of about $1/0.094=10.6$ years). The second component (which also has a root fairly close to the unit circle) is associated with $f_0 = 0$, which is based on the long term or aperiodic behavior of the amplitude heights. The other factors are associated with more subtle features of the data. Note that none of these interpretations are possible by examination of the model coefficients alone.

5.1.4 Linear Filters, the General Linear Process, and AR(p) Models

Linear filters play a major role in time series data analysis. The linear filter produces an output process which is a linear combination of lagged values of an input process. Given a set of input data, x_t , the linear filter is defined as

$$y_t = \sum_{j=-\infty}^{\infty} h_j x_{t-j}, \quad (5.37)$$

where x_t is the observed (input) data, y_t is the output, and the h_j s are real-valued constants. A centered moving average is an example of a linear filter. Specifically, a 5th-order centered moving average smoother

⁶ We will discuss techniques for estimating the parameters and identifying the order, p , in Section 6.1.2. The estimates shown here are maximum likelihood estimates. See Section 6.1.1.



is defined as $s_t = (x_{t-2} + x_{t-1} + x_t + x_{t+1} + x_{t+2})/5$, which can be expressed in the form of (5.37) where $h_j = 1/5$ for $j = -2, -1, 0, 1, \text{and } 2$, and $h_j = 0$, elsewhere.

Key Points

1. If the linear filter “starts” at $j = 0$, it is called a *realizable* filter and has the property that X_t at time t depends only on present and past values of a_t .
2. A centered moving average is *not* a realizable filter because its value at time t depends on past, present, and future values of t .

A stationary AR(p) process is a special case of a particular type of a realizable linear filter called a *general linear process* which is defined in Definition 5.2.

Definition 5.2: (General Linear Process) The process X_t given by

$$X_t - \mu = \sum_{j=0}^{\infty} \psi_j a_{t-j}, \quad (5.38)$$

is called a *general linear process* (GLP) if a_t is a white noise process with zero mean and finite variance and if $\sum_{j=0}^{\infty} |\psi_j| < \infty$.

Theorem 5.5

- (1) A GLP satisfying Definition 5.2 is a stationary process.
- (2) The variance of a GLP as defined in Definition 5.2 is given by $\sigma_x^2 = \sigma_a^2 \sum_{j=0}^{\infty} \psi_j^2$.

Key Points

1. A GLP as defined by Definition 5.2 is a *stationary process*.
2. For a general discussion of the GLP, see Woodward et al. (2017) and Brockwell and Davis (1991).

As mentioned above, stationary AR(p) models are special cases of the general linear process (GLP). In this section we illustrate this concept using an AR(1) model and then give a general result for AR(p) processes.

5.1.4.1 AR(1) in GLP Form

For the AR(1) process $(1 - \phi_1 B)(X_t - \mu) = a_t$, the operator $(1 - \phi_1 B)$ is associated with an inverse operator, $(1 - \phi_1 B)^{-1}$, for which $(1 - \phi_1 B)^{-1}(1 - \phi_1 B)X_t = X_t$. Consequently,

$$(1 - \phi_1 B)^{-1}(1 - \phi_1 B)(X_t - \mu) = (1 - \phi_1 B)^{-1}a_t,$$

or

$$X_t - \mu = (1 - \phi_1 B)^{-1} a_t.$$

A natural question is, “How is $(1 - \phi_1 B)^{-1}$ defined?” Brockwell and Davis (1991) show that inverse operators are defined analogously to their algebraic counterparts. The algebraic counterpart in this case is $(1 - \phi_1 z)^{-1}$, where z is a real number. Note that $(1 - \phi_1 z)^{-1} = \frac{1}{1 - \phi_1 z}$, which can be solved by long division,

$$\begin{array}{r} 1 + \phi_1 z + \phi_1^2 z^2 + \phi_1^3 z^3 + \dots \\ \hline 1 - \phi_1 z) \overline{1} \\ \underline{1 - \phi_1 z} \\ + \phi_1 z \\ + \phi_1 z - \phi_1^2 z^2 \\ + \phi_1^2 z^2 \\ \pm \phi_1^2 z^2 + \phi_1^3 z^3 \\ + \phi_1^3 z^3 \end{array}$$

That is,

$$\begin{aligned} (1 - \phi_1 z)^{-1} &= \frac{1}{1 - \phi_1 z} \\ &= 1 + \phi_1 z + \phi_1^2 z^2 + \phi_1^3 z^3 + \dots \\ &= \sum_{k=0}^{\infty} \phi_1^k z^k. \end{aligned}$$

Converting back to operator notation (that is substituting the operator, B , for the numerical quantity, z) it follows that the operator $(1 - \phi_1 B)^{-1} = \sum_{k=0}^{\infty} \phi_1^k B^k$. Consequently, the AR(1) model, $(1 - \phi_1 B)X_t = a_t$, can be written as

$$\begin{aligned} X_t - \mu &= (1 - \phi_1 B)^{-1} a_t, \\ &= \left(\sum_{k=0}^{\infty} \phi_1^k B^k \right) a_t \cdots = \sum_{k=0}^{\infty} \phi_1^k a_{t-k} \\ &= \sum_{k=0}^{\infty} \psi_k a_{t-k} \end{aligned} \tag{5.39}$$

where $\psi_k = \phi_1^k$. The last equation in (5.39) is a GLP, by Definition 5.2, because $\sum_{k=0}^{\infty} |\psi_k| = \sum_{k=0}^{\infty} |\phi_1|^k < \infty$ for $|\phi_1| < 1$ (see Theorem 2.3, Woodward et al. (2017)).⁷

5.1.4.2 AR(p) in GLP Form

Equation (5.39) shows that a stationary AR(1) model, $\phi(B)(X_t - \mu) = a_t$, where $\phi(B) = 1 - \phi_1 B$, can be expressed as $X_t - \mu = \phi^{-1}(B)a_t = \sum_{k=0}^{\infty} \psi_k a_{t-k}$, where the ψ_k s are defined as the coefficients of the inverse operator $\phi^{-1}(B) = 1/\phi(B)$. In the AR(1) case, the ψ_k s are simply given by $\psi_k = \phi_1^k$ and $1/\phi(B) = 1/(1 - \phi_1 B) = \left(\sum_{k=0}^{\infty} \phi_1^k B^k\right)$. The generalization of this result applies to stationary AR(p) models. It can be shown (see

Brockwell and Davis, 1991 and Woodward, et al., 2017) that if $\phi(B) = (1 - \phi_1 B - \dots - \phi_p B^p)X_t = a_t$ is a stationary AR(p) process, that is, if all roots of $\phi(z) = 0$ are outside the unit circle, then X_t can be written in GLP form. In other words, for stationary AR(p) processes, X_t , it follows that $X_t = \phi^{-1}(B)a_t = \sum_{k=0}^{\infty} \psi_k a_{t-k}$ where $\sum_{k=0}^{\infty} |\psi_k| < \infty$. Again, the ψ -weights can be found by division as shown in Example 5.7.

Example 5.7 GLP form for $(1 - 1.6B + .8B^2)X_t = a_t$

In order to find the ψ -weights associated with this AR(2) model, we note that

$$X_t = (1 - 1.6B + .8B^2)^{-1} a_t$$

$$= \frac{1}{1 - 1.6B + .8B^2} a_t.$$

As in the AR(1) case, the operator $1/(1 - 1.6B + .8B^2)$ is defined analogously to its algebraic counterpart, $1/(1 - 1.6z + .8z^2)$. This ratio is an infinite-order polynomial, the coefficients of which are the ψ -weights.

Specifically, the ratio $\frac{1}{1 - 1.6z + .8z^2}$ can be obtained by long division as follows:

$$\begin{array}{r} 1+1.6z+1.76z^2+\dots \\ \hline 1-1.6z+.8z^2)1 \\ \quad 1-1.6z+.8z^2 \\ \quad \quad +1.6z-.8z^2 \\ \quad +1.6z-2.56z^2+1.28z^3 \\ \quad \quad \quad +1.76z^2-1.28z^3 \end{array}$$



QR 5.9
Polynomial
Division

⁷ We could have used the fact $\frac{1}{1 - \phi_1 z}$ is the sum of a geometric series $\sum_{k=0}^{\infty} (\phi_1 z)^k$.



From this long division procedure, we see that $\psi_0 = 1$, $\psi_1 = 1.6$, $\psi_2 = 1.76$, . . . A much easier way to find, say the first five ψ -weights beginning with ψ_1 , is to issue the **tswge** command

```
psi.weights.wge(phi=c(1.6,-.8),lag.max=5)
```

which gives $\psi_1 = 1.6$, $\psi_2 = 1.76$, $\psi_3 = 1.536$, $\psi_4 = 1.0496$, and $\psi_5 = .4056$. Recall that it is always true that $\psi_0 = 1$, so this is not included in the output list of ψ -weights from **psi.weights.wge**.

5.2 AUTOREGRESSIVE-MOVING AVERAGE (ARMA) MODELS

“Our goal will be to derive models possessing maximum simplicity and the minimum number of parameters consonant with representational adequacy.”

—Box and Jenkins (1970)

In Section 5.1 we discussed the autoregressive (AR) models, which are useful models for analyzing stationary time series data. Box and Jenkins (1970) popularized the use of a broader class of models, the autoregressive moving average (ARMA(p, q)) model, of which the autoregressive model is a special case. Specifically, the AR(p) model is the case of an ARMA(p, q) model where $q = 0$. However, we are getting ahead of ourselves because we haven't yet defined the ARMA(p, q) model. While the AR models can be used successfully to provide useful models, the best fitting AR model may have a large number of parameters; that is, p may be large. The quote at the beginning of this section implies that Box and Jenkins preferred models with the fewest parameters necessary to represent the data. They referred to this as a preference for *parsimonious* models. Given a set of stationary data, it is often the case that the “best” ARMA model will have fewer parameters than the “best” AR model. The large number of parameters is not always bad, and some time series giants (among them, Emmanuel Parzen) preferred the exclusive use of AR models for modeling stationary data.

Recall that the AR(p) model,

$$X_t = \beta + \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + a_t,$$

expresses the value of the process at time t as a linear combination of the process at times $t - 1, t - 2, \dots, t - p$, plus a random noise component, a_t , that enters the model at time t . The ARMA(p, q) model specifies that the value of the process at time t is a linear combination of the process at times $t - 1, t - 2, \dots, t - p$, plus a random noise component at time t , and a linear combination of random noise components that enter the model at times $t - 1, t - 2, \dots, t - q$. Specifically, we have the following definition.

Definition 5.3: Autoregressive-Moving Average Model

A time series X_t is said to satisfy an ARMA(p, q) model if

$$X_t = \beta + \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q}, \quad (5.40)$$

where ϕ_k , $k = 1, \dots, p$ and θ_k , $k = 1, \dots, q$ are real constants, $\beta = (1 - \phi_1 - \phi_2 - \dots - \phi_p)\mu$, $\phi_p \neq 0$, $\theta_q \neq 0$, and a_t is a white noise process with zero mean and finite variance σ_a^2 .⁸

Notes:

- (1) The ARMA(p, q) model can be written in operator notation as $\phi(B)(X_t - \mu) = \theta(B)a_t$ where $\phi(B) = 1 - \phi_1 B - \dots - \phi_p B^p$ and $\theta(B) = 1 - \theta_1 B - \dots - \theta_q B^q$ and where $\phi(B)$ and $\theta(B)$ have no common factors. See Example 5.11.
- (2) An AR(p) model is a special case of an ARMA(p, q) model with $q = 0$. Another special case of an ARMA(p, q) model is the *moving average* (MA) model which is the case with $p = 0$. We discuss MA(q) models next.

Caution: Be careful with the signs! We have adopted the notation used by Box, Jenkins, and Reinsel (2008), while other authors may change the signs, *especially for the MA coefficients*. Be careful how the authors have defined the ARMA model as you read different books, journal articles, and websites.

5.2.1 Moving Average Models

Moving average (MA) models are used to model stationary data but are not as useful as AR models. MA models are most useful in combination with the AR model to create the ARMA models.

Definition 5.4: Moving Average Model

A time series X_t is said to satisfy an MA(q) model if

$$X_t - \mu = a_t - \theta_1 a_{t-1} - \dots - \theta_q a_{t-q}, \quad (5.41)$$

where the θ_j 's are real constants, $\theta_q \neq 0$, and a_t 's are white noise with mean zero and variance σ_a^2 .

Recall that the autoregressive model says that the current value of X_t is a linear combination of past values of the process, plus noise. This is a natural way to describe how time series data might evolve in time. Note, however, that the MA(q) model specifies that X_t at time t is a linear combination of present and past noise components. This is not intuitive to many. Nevertheless, the MA(q) model can be useful in maintaining the performance of an AR model through a more parsimonious (fewer parameters) ARMA model. We begin with a discussion of the MA(1) (or ARMA(0,1)) model.

5.2.1.1 The MA(1) Model

The MA(1) model is the special case of (5.41) with $q = 1$, that is,

$$X_t - \mu = a_t - \theta_1 a_{t-1}. \quad (5.42)$$

⁸ An additional requirement for X_t to be ARMA(p, q) is given in Note (1).

The MA(1) can be expressed in operator notation as $X_t - \mu = \theta(B)a_t$, where $\theta(B) = 1 - \theta_1 B$. The corresponding MA(1) characteristic function and characteristic equation are given by $\theta(z) = 1 - \theta_1 z$ and $\theta(z) = 1 - \theta_1 z = 0$, respectively. The following are facts about the MA(1) model, X_t .

- (a) $E[X_t] = \mu$
- (b) The process variance is given by $\sigma_x^2 (= \gamma_0) = \sigma_a^2 (1 + \theta_1^2)$
- (c) The autocorrelations are given by



QR 5.10
Properties of MA(1)

(5.43)

$$\rho_0 = 1$$

$$\rho_1 = \frac{-\theta_1}{(1 + \theta_1^2)}$$

$$\rho_k = 0, k > 1$$

It can be shown that $|\rho_1| \leq .5$ for an MA(1) process.

The model-based spectral density of an MA(1) process is given by

$$S_X(f) = \frac{\sigma_a^2}{\sigma_x^2} |1 - \theta_1 e^{-2\pi i f}|^2.$$

See Woodward et al. (2017).

Example 5.8 Two MA(1) Models

We consider two MA(1) models below.

- (a) $X_t = a_t - .99a_{t-1}$. For this model we have

$$\theta_1 = .99 \text{ (be careful about signs!)}$$

$$\rho_1 = \frac{-\theta_1}{1 + \theta_1^2} = \frac{-0.99}{1 + 0.99^2} = -.49997$$

$$\rho_k = 0, k > 1$$

TABLE 5.7 MA-Factor Table for $X_t = a_t - .99a_{t-1}$

FACTOR	ROOTS	$ r ^{-1}$	f_0
$1 - .99B$	1.01	.99	0

Figure 5.21(a) and (b) show the model-based autocorrelations and spectral density for the MA(1) model $X_t = a_t - .99a_{t-1}$. The autocorrelation plot shows a single nonzero autocorrelation at lag one

($\rho_1 \approx -.5$) with all others being zero (except, of course, at lag zero).⁹ It is important to note that system frequencies in an MA factor table are represented by “dips” in the spectral density rather than peaks. Specifically, the spectral density in Figure 5.21(b) has a dip at $f = 0$. The plots below can be plotted using the *tswge* command

```
plottts.true.wge(phi=0,theta=.99,lag.max=25,vara=1,plot.data=FALSE)
```

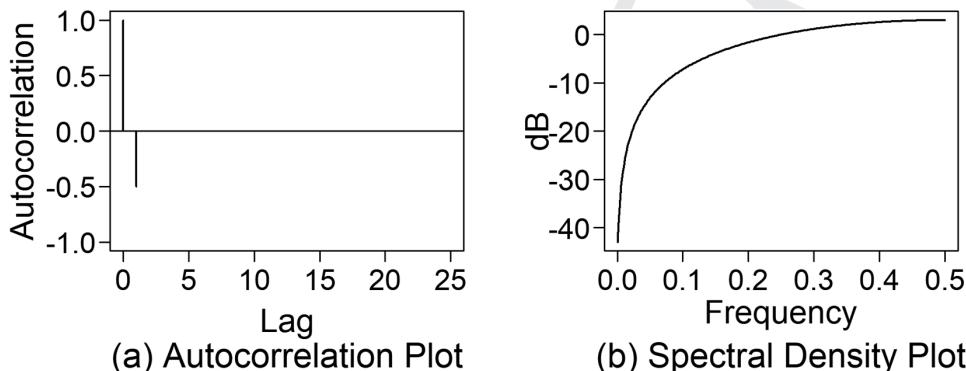


FIGURE 5.21 Model-based (a) autocorrelations and (b) spectral density for the MA(1) model $X_t = a_t - .99a_{t-1}$.

Key Point: System frequencies in an MA factor table are represented by “dips” in the spectral density rather than peaks.

Figure 5.22(a) shows a realization from the MA(1) model $X_t = a_t - .99a_{t-1}$, while Figures 5.22(b) and (c) display sample autocorrelations and Parzen spectral density estimates for this realization. Notice that $\hat{\rho}_1 \approx -.5$, which is consistent with the model-based autocorrelation. All other model-based autocorrelations are zero, and the sample autocorrelations for lags greater than one are all small in magnitude. The realization shows an oscillatory behavior consistent with a negative sample autocorrelation at lag one, and near-zero sample autocorrelations for lags greater than one. The main feature of the Parzen spectral density estimate in Figure 5.22(c) is the “dip” at $f = 0$. Figure 5.22 can be obtained using the command

```
x=gen.arma.wge(n=150,theta=.99,sn=53)
plottts.sample.wge(x)
```

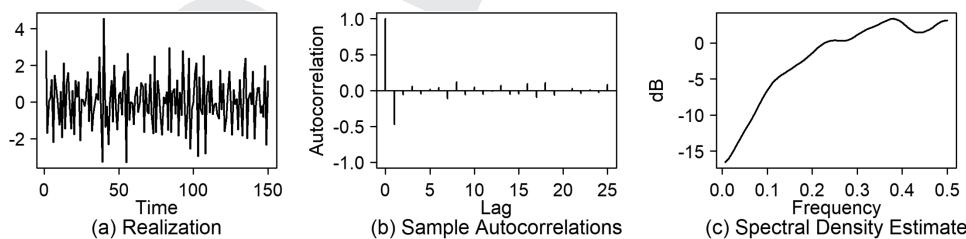


FIGURE 5.22 (a) Realization of $n = 150$ from $X_t = a_t - .99a_{t-1}$, (b) sample autocorrelations and (c) Parzen spectral density estimate based on the realization in (a).

⁹ If you are having trouble visualizing a situation in which $\rho_1 \neq 0$ while $\rho_k = 0$ for $k > 1$, you are not alone!

(b) $X_t = a_t + .8a_{t-1}$. For this model, we have

$$\theta_1 = -0.8 \text{ (again - be careful about signs!)}$$

$$\rho_1 = \frac{-\theta_1}{1 + \theta_1^2} = \frac{-0.8}{1 + 0.8^2} = 0.488$$

$$\rho_k = 0, k > 1$$

TABLE 5.8 MA-Factor Table for $X_t = a_t + .8a_{t-1}$

FACTOR	ROOTS	$ r ^{-1}$	f_0
$1 + 0.8B$	-1.25	0.80	0.50

Figure 5.23(a) shows a realization from the MA(1) model $X_t = a_t + .8a_{t-1}$. Figures 5.23(b) and (c) show sample autocorrelations and Parzen spectral density estimates for this realization. Notice in Figure 5.23(b) that $\hat{\rho}_1$ is slightly less than .5, which is consistent with the model-based correlation of $\rho_1 = 0.488$. All other sample autocorrelations are near zero, and the main feature of the Parzen spectral density estimate in Figure 5.23(c) is the “dip” at $f = .5$ which is the associated system frequency in the factor table. Figure 5.23 can be produced using the commands

```
x=gen.arma.wge(n=150,theta=-.8,sn=363)
plotts.sample.wge(x)
```

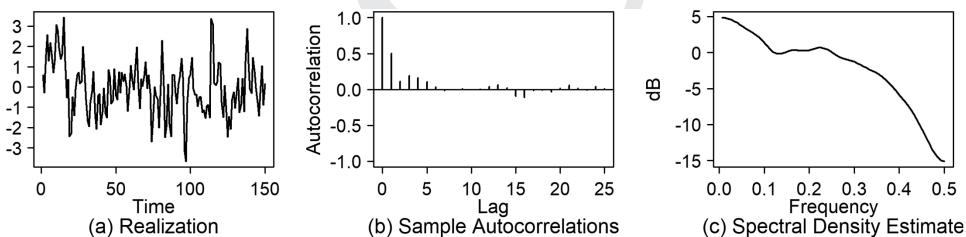


FIGURE 5.23 (a) Realization of $n = 150$ from $X_t = a_t + .8a_{t-1}$, (b) sample autocorrelations and (c) Parzen spectral density estimate based on the realization in (a).

5.2.1.2 The MA(2) Model

The MA(2) model is the special case of (5.41) with $q = 2$, that is,

$$X_t - \mu = a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2}. \quad (5.44)$$

The MA(2) model can be expressed in operator notation as $X_t - \mu = \theta(B)a_t$, where the operator $\theta(B)$ is defined by $\theta(B) = 1 - \theta_1 B - \theta_2 B^2$. The corresponding MA characteristic equation is given by $\theta(z) = 1 - \theta_1 z - \theta_2 z^2 = 0$. The following are facts about the MA(2) model, X_t .

- (a) $E[X_t] = \mu$
 (b) The process variance is given by.
 (c) The autocorrelations are given by

$$\rho_0 = 1$$

$$\rho_1 = \frac{-\theta_1 + \theta_1 \theta_2}{(1 + \theta_1^2 + \theta_2^2)}$$

$$\rho_2 = \frac{-\theta_2}{(1 + \theta_1^2 + \theta_2^2)}$$

$$\rho_k = 0, k > 2$$



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Properties
of MA(2)

- (d) The model-based spectral density of an MA(2) process is given by

$$S_X(f) = \frac{\sigma_a^2}{\sigma_x^2} \left| 1 - \theta_1 e^{-2\pi i f} - \theta_2 e^{-4\pi i f} \right|^2.$$

See Woodward et al. (2017).

Example 5.9 An MA(2) Model

We consider the MA(2) model below.

$$X_t = a_t - .4a_{t-1} + .9a_{t-2}.$$

For this model, we have

$$\theta_1 = .4, \theta_2 = -.9$$

$$\rho_1 = \frac{-\theta_1 + \theta_1 \theta_2}{1 + \theta_1^2 + \theta_2^2} = \frac{-0.4 + (-0.9)(0.4)}{1 + 0.4^2 + (-0.9)^2} = -0.3858$$

$$\rho_2 = \frac{-\theta_2}{1 + \theta_1^2 + \theta_2^2} = \frac{0.9}{1 + 0.4^2 + (-0.9)^2} = 0.4569$$

$$\rho_k = 0, k > 2$$

TABLE 5.9 MA-Factor Table for $X_t = a_t - .4a_{t-1} + .9a_{t-2}$

FACTOR	ROOTS	$ r ^{-1}$	f_0
$1 - .4B + .9B^2$	$.22 \pm 1.03$.95	.22

Figures 5.24(a) and (b) show the model-based autocorrelations and spectral densities for the MA(2) model $X_t = a_t - .4a_{t-1} + .9a_{t-2}$. The autocorrelation plot shows an autocorrelation a little greater than $-.5$ at lag one and an autocorrelation a little less than $.5$ at lag two, and $\rho_k = 0, k > 2$. Note that the factor table

has a system frequency of $f = .22$ and there is a *dip* in the spectral density at about this frequency. Output from the command

```
ma2=plottts.true.wge(phi=0, theta=c(.4, -.9), plot.data=FALSE)
```

includes

```
ma2$aut1 [1] 1.0000000 -0.3857868 0.4568528 0.0000000 0.0000000 0.0000000
```

The zeroes continue indefinitely, so it follows that $\rho_0 = 1, \rho_1 = -0.386, \rho_2 = 0.457$, and $\rho_k = 0, k > 2$.

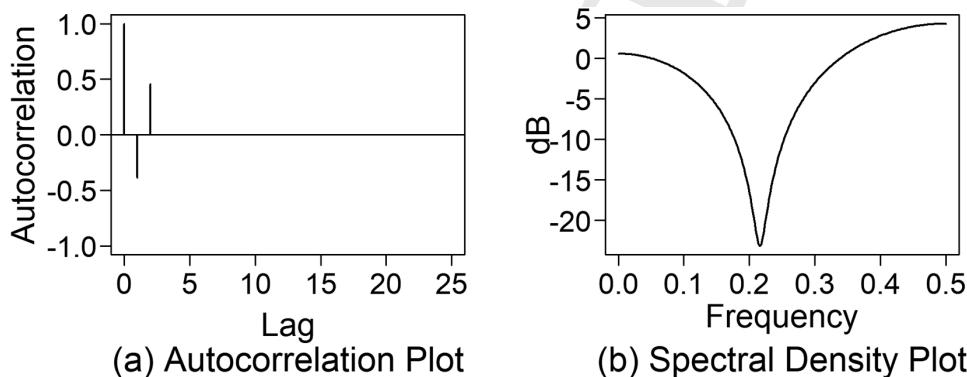


FIGURE 5.24 Model-based (a) autocorrelations and (b) spectral density for the MA(2) model $X_t = a_t - .4a_{t-1} + .9a_{t-2}$.

Figure 5.25(a) shows a realization from the MA(2) model $X_t = a_t - .4a_{t-1} + .9a_{t-2}$. Figures 5.25(b) and (c) show sample autocorrelations and Parzen spectral density estimates for this realization. Notice that $\hat{\rho}_1 \approx -.4$ and $\hat{\rho}_2$ is slightly less than .5 which is consistent with the model-based autocorrelation. All other model-based autocorrelations are zero, and the sample autocorrelations for lag greater than two are all small in magnitude. The realization shows oscillatory behavior and some wandering, but is difficult to describe. That is, what does “the lack of frequency behavior at $f = .22$ ” look like? *Hmm*. The main feature of the Parzen spectral density estimate in Figure 5.25 is the *dip* at about $f = .22$. Figure 5.25 can be obtained using the commands

```
x=gen.arma.wge(n=150, theta=c(.4, -.9), sn=65)
plottts.sample.wge(x)
```

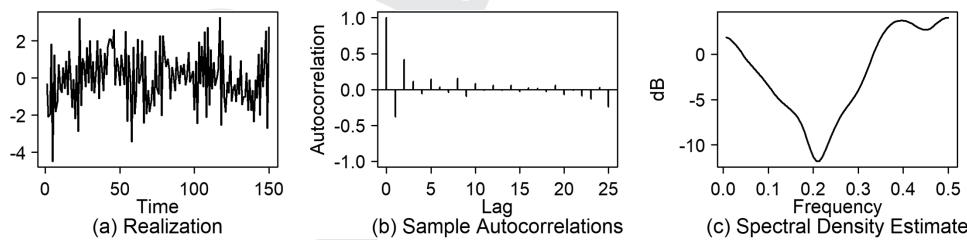


FIGURE 5.25 (a) Realization of $n = 150$ from $X_t = a_t - .4a_{t-1} + .9a_{t-2}$, (b) sample autocorrelations and (c) Parzen spectral density estimate based on the realization in (a).

5.2.1.3 The General MA(q) Model

It should be noted that an $\text{MA}(q)$ is a finite general linear process (GLP) and is *always stationary*. Recall that a GLP is given by $X_t - \mu = \sum_{j=0}^q \psi_j a_{t-j}$ so that for an $\text{MA}(q)$, $\psi_0 = 1, \psi_1 = -\theta_1, \dots, \psi_q = -\theta_q, \psi_j = 0, j > q$. For example, the $\text{MA}(2)$ model $X_t = a_t - .4a_{t-1} + .9a_{t-2}$ is already in GLP form with $\psi_0 = 1, \psi_1 = -.4, \psi_2 = .9$, and $\psi_j = 0, j > 2$.

(1) Mean, Variance, and Autocorrelations of an $\text{MA}(q)$

- (a) $E[X_t] = \mu$
- (b) The process variance is given by $\sigma_x^2 (= \gamma_0) = \sigma_a^2 (1 + \theta_1^2 + \dots + \theta_q^2)$
- (c) The autocorrelations have the property that $\rho_q \neq 0$ and $\rho_k = 0, k > q$. See Woodward et al. (2017) for a general expression.

(2) Operator Notation and the MA Characteristic Equation

The $\text{MA}(q)$ model in (5.41) can be written as

$$X_t - \mu = (1 - \theta_1 B - \dots - \theta_q B^q) a_t, \quad (5.45)$$

or by using the shorthand notation, $X_t - \mu = \theta(B) a_t$, where $\theta(B) = 1 - \theta_1 B - \dots - \theta_q B^q$.

Converting $\theta(B)$ to the algebraic quantity $\theta(z)$, we obtain the MA characteristic polynomial given by $\theta(z) = 1 - \theta_1 z - \dots - \theta_q z^q$ and the MA characteristic equation

$$\theta(z) = 1 - \theta_1 z - \dots - \theta_q z^q = 0. \quad (5.46)$$

The MA-factor table can be obtained using `tswge` function, `factor.wge`. For example, for the $\text{MA}(2)$ example discussed above, the MA-factor table shown in Table 5.9 can be obtained using `factor.wge(theta=c(.4, -.9))`. Recall that the system frequencies in an MA-factor table are frequencies at which there are *dips* in the spectral densities. That is, these frequencies tend to *not be* in the realization.

Key Points

1. The authors have encountered few real data-sets that are well modeled using an $\text{MA}(q)$ model.
2. In making an analogy in which the $\text{ARMA}(p,q)$ model is a meal, then the AR component is the entrée while the MA part is the salt and pepper.
3. The general $\text{ARMA}(p,q)$ model, which has both AR and MA components, is quite useful and is the topic of the remainder of this chapter.

5.2.1.4 Invertibility

We have seen that when $\text{AR}(p)$ processes are stationary, they can be written in GLP form. That is, the model $\phi(B)(X_t - \mu) = a_t$ can be written as $X_t - \mu = \phi^{-1}(B) a_t = \sum_{k=0}^{\infty} \psi_k a_{t-k}$ where $\sum_{k=0}^{\infty} |\psi_k| < \infty$. Note that essentially this says that a stationary AR process can be written as an *infinite order MA process*. A



question then arises concerning whether there are conditions under which an MA process can be written as an infinite order AR process. In operator notation, the question is whether $X_t - \mu = \theta(B)a_t$ can be written as

$$\theta^{-1}(B)(X_t - \mu) = a_t. \quad (5.47)$$

Proceeding as in the AR case, we expand $\theta^{-1}(z) = 1/\theta(z)$ in powers of z usually written as $\theta^{-1}(z) = \sum_{j=0}^{\infty} \pi_j z^j$, and formally replace z by the operator B . It follows that

$$\begin{aligned} \theta^{-1}(B)(X_t - \mu) &= \left(\sum_{k=0}^{\infty} \pi_k B^k \right) (X_t - \mu) \\ &= \sum_{j=0}^{\infty} \pi_j (X_{t-j} - \mu), \end{aligned} \quad (5.48)$$

and (5.48) becomes

$$\sum_{j=0}^{\infty} \pi_j (X_{t-j} - \mu) = a_t, \quad (5.49)$$

which is an *infinite order AR process*. The coefficients are, not surprisingly, called the π -weights. To find the π -weights for the MA(2) model $X_t = a_t - .4a_{t-1} + .9a_{t-2}$, the **tswge** command is

```
pi.weights.wge(theta=c(.4, -.9), lag.max=6)
```

A process that can be written as an infinite order AR is called an *invertible* process. Definition 5.5 gives a more precise definition of invertibility.

Definition 5.4: If an MA(q) process, X_t , can be expressed as in (5.49) where $\sum_{j=0}^{\infty} |\pi_j| < \infty$, then X_t is said to be *invertible*.

The following theorem provides conditions for an MA(q) process to be invertible.

Theorem 5.6 An MA process, X_t , is invertible if and only if the roots of the characteristic equation, $\theta(z) = 1 - \theta_1 z - \dots - \theta_q z^q = 0$ all lie outside the unit circle.

Key Points

1. To check an MA(q) process for invertibility, use the factor table.
2. For example, Table 5.9 shows that the model $X_t = a_t - .4a_{t-1} + .9a_{t-2}$ is invertible because the two roots (complex conjugate roots, $.22 \pm 1.030$) are outside the unit circle since each root has $|r|^{-1} = .95$.

(1) Reasons for Imposing Invertibility

When using an MA(q) model to analyze time series, we will restrict our focus to invertible models. There are two basic reasons for doing this.

(a) *Imposing Invertibility Removes Model Multiplicity*

Consider the MA(1) model $X_t = (1 - \theta_1 B)a_t$. From (5.43), we know that $\rho_1 = -\theta_1 / (1 + \theta_1^2)$ and $\rho_k = 0$, when $|k| > 1$. Now consider the MA(1) model

$$X_t = \left(1 - \frac{1}{\theta_1} B\right)a_t. \quad (5.50)$$

For the model in (5.50), ρ_i is given by

$$\begin{aligned}\rho_1 &= \frac{\frac{-1}{\theta_1}}{\left(1 + \frac{1}{\theta_1^2}\right)} \\ &= \frac{-\theta_1}{(1 + \theta_1^2)}.\end{aligned}$$

Also, we know that $\rho_k = 0$ for $k > 1$ for model (5.50). In other words, the two MA(1) models $X_t = (1 - \theta_1 B)a_t$ and $X_t = \left(1 - \frac{1}{\theta_1} B\right)a_t$ have the *exact same autocorrelations*.

For example, the models $X_t = (1 - .5B)a_t$ and $X_t = (1 - 2B)a_t$ have the same autocorrelations. However, $X_t = (1 - .5B)a_t$ is invertible, but $X_t = (1 - 2B)a_t$ is not. In general, if $\rho_k, k = 1, 2, \dots$ are the autocorrelations of an MA(q) process, there are more than one MA(q) models with these same autocorrelations, but only one of them is invertible. By requiring an MA(q) model to be invertible, a *unique* model is associated with a given set of MA(q) autocorrelations.

(b) *Invertibility Assures that Present Events Are Associated with the Past in a Sensible Manner*

Consider the MA(1) model $X_t = a_t - \theta_1 a_{t-1}$, where for convenience we will assume $\mu = 0$. Rewriting this equation as $a_t = X_t + \theta_1 a_{t-1}$, it follows that $a_{t-1} = X_{t-1} + \theta_1 a_{t-2}$, so that

$$\begin{aligned}a_t &= X_t + \theta_1 (X_{t-1} + \theta_1 a_{t-2}) \\ &= X_t + \theta_1 X_{t-1} + \theta_1^2 a_{t-2} \\ &= X_t + \theta_1 X_{t-1} + \theta_1^2 X_{t-2} + \theta_1^3 a_{t-3}.\end{aligned}$$



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Invertibility

Continuing, we see that

$$a_t = X_t + \theta_1 X_{t-1} + \theta_1^2 X_{t-2} + \dots + \theta_1^k X_{t-k} + \theta_1^{k+1} a_{t-k-1},$$



and rearranging terms,

$$X_t = -\theta_1 X_{t-1} - \theta_1^2 X_{t-2} - \cdots - \theta_1^k X_{t-k} + a_t - \theta_1^{k+1} a_{t-k-1}. \quad (5.51)$$

If $|\theta_1| > 1$, then the associated MA(1) process is not invertible, and (5.51) shows that X_t is *increasingly dependent on the distant past*. In most cases, this is not a physically acceptable model.

For example, for the noninvertible model $X_t = (1 - 2B)a_t$ and letting $k = 20$ in (5.51), we have

$$X_t = -2X_{t-1} - 4X_{t-2} - \cdots - 1048576X_{t-20} + a_t - 2097152a_{t-21},$$

and the increasing dependence on terms in the distant past is evident.

Key Points

1. All MA(q) models are stationary (because $\sum_{j=0}^{\infty} |\psi_j| = 1 + \sum_{j=0}^q |\theta_j| < \infty$).
2. All AR(p) models are invertible (because $\sum_{j=0}^{\infty} |\pi_j| = 1 + \sum_{j=0}^p |\phi_j| < \infty$).

5.2.2 ARMA(p,q) Models

We now study the general ARMA(p,q) process in (5.40) where $p \geq 0$ and $q \geq 0$. For the nonzero mean formula in (5.40), it follows that $E[X_t] = \mu$. However, expressions for the autocorrelations and variance of ARMA(p,q) processes are more complicated (see Woodward et al. (2017)). The **tswge** functions **plotts.true.wge** and **true.arma.aut.wge** calculate these quantities.

We will often express the ARMA(p,q) model in (5.40) in operator notation as

$$(1 - \phi_1 B - \cdots - \phi_p B^p)(X_t - \mu) = (1 - \theta_1 B - \cdots - \theta_q B^q)a_t,$$

or more compactly, $\phi(B)(X_t - \mu) = \theta(B)a_t$.

5.2.2.1 Stationarity and Invertibility of an ARMA(p,q) Process

We will only be interested in processes of the form (5.40) that are *stationary* and *invertible*. In this section, we discuss conditions under which an ARMA(p,q) process is stationary and invertible.

(1) Stationarity Conditions for an ARMA(p,q) Process

Theorem 5.7 specifies the conditions for an ARMA(p,q) process to be stationary.

Theorem 5.7 The ARMA(p, q) process, $\phi(B)X_t = \theta(B)a_t$, with $p > 0$, is stationary if and only if all the roots of $\phi(z) = 0$ fall outside the unit circle.

Theorem 5.7 says that the stationarity of an ARMA(p, q) process depends only on the roots of the characteristic equation associated with the autoregressive side of the equation. In this case, we can write the model in the infinite order moving average (GLP) form $X_t = \sum_{k=0}^{\infty} \psi_k a_{t-k}$, where $\sum_{j=0}^{\infty} |\psi_j| < \infty$. The ψ -weights can be found by operating on the left-hand and right-hand sides of $\phi(B)X_t = \theta(B)a_t$ by $\phi^{-1}(B)$, to obtain $\phi^{-1}(B)\phi(B)X_t = \phi^{-1}(B)\theta(B)a_t$. It follows that $X_t = \phi^{-1}(B)\theta(B)a_t$; that is, $X_t = \sum_{k=0}^{\infty} \psi_k a_{t-k}$, where the ψ_k s are obtained from the polynomial division $\theta(z)/\phi(z)$, as illustrated in Example 5.7 for the case in which $\theta(z) = 1$.

As in the AR(p) case, we recommend finding the ψ -weights using `tswge` function `psi.weights.wge`.

For example, consider stationary ARMA(2,1) model $(1 - 1.6B + .8B^2)X_t = (1 - .9B)a_t$. Then the command `psi.weights.wge(phi=c(1.6, -.8), theta=-.9, lag.max=6)` calculates the first six ψ -weights from this model. The output from the above command is

```
[1] 2.50000 3.20000 3.12000 2.43200 1.39520 0.28672
```



QR 5.13 ARMA as GLP

Key Points

1. The `psi.weights.wge` function outputs $\psi_1, \dots, \psi_{lag.max}$. The user should recall that in all cases, $\psi_0 = 1$.
2. The ψ -weights are useful for obtaining prediction limits for forecasts. This will be discussed in Section 6.3.1.

(2) Invertibility Conditions for an ARMA(p, q) Process

Theorem 5.8 gives the conditions for an ARMA(p, q) process to be invertible.

Theorem 5.8 The ARMA(p, q) model, $\phi(B)X_t = \theta(B)a_t$, with $q > 0$, is invertible if and only if all the roots of $\theta(z) = 0$ fall outside the unit circle.

Theorem 5.8 states that the condition of invertibility of an ARMA(p, q) model depends on the roots of the MA characteristic equation. If the ARMA(p, q) process is invertible, then again using properties of inverse operators, we can write $\theta^{-1}(B)\phi(B)X_t = a_t$, which is an infinite autoregressive model, usually written as $\sum_{j=0}^{\infty} \pi_j X_{t-j} = a_t$, where $\sum_{j=0}^{\infty} |\pi_j| < \infty$. To find π -weights for the ARMA(2,1) model $(1 - 1.6B + .8B^2)X_t = (1 - .9B)a_t$, the command is

```
pi.weights.wge(phi=c(1.6, -.8), theta=.9, lag.max=6)
```



QR 5.14 ARMA as AR

Key Points

1. In this book, whenever X_t is said to satisfy an ARMA(p, q) process, the tacit assumption is made that the process is *both stationary and invertible*. That is, all the roots of $\phi(z) = 0$ and $\theta(z) = 0$ lie outside the unit circle.
2. A stationary ARMA(p, q) model can be written as an infinite order MA model where $\sum_{j=0}^{\infty} |\pi_j| < \infty$.
3. An invertible ARMA(p, q) model can be written as an infinite order AR model where $\sum_{j=0}^{\infty} |\psi_j| < \infty$.

5.2.2.2 AR versus ARMA Models

In Chapter 6, we will discuss fitting models to data. In some cases, we may find more than one model that appears to be an appropriate fit. For example, it may be that either an AR(10) or an ARMA(3,1) would be appropriate models for a given dataset. As mentioned, some time series analysts prefer to restrict their attention to AR models even if it means the use of a large order of p , such as $p = 10$ or above. If you, however, prefer to use the model with the fewest number of parameters needed to appropriately model the data, then you might choose the ARMA(3,1) model. The fact that a (stationary and invertible) ARMA(3,1) model might be similar to an AR(10) model follows from the fact that the stationary and invertible ARMA(p, q) model can be expressed as an infinite order autoregressive (the π -weight formula given above). Because $\sum_{j=0}^{\infty} |\pi_j| < \infty$, it follows that $|\pi_j| \rightarrow 0$ as $j \rightarrow \infty$. That is, the coefficients of the “infinite order AR” may become negligible beyond a point.

Example 5.10 ARMA(3,1) Model $(1 - 2.57B + 2.50B^2 - .92B^3)X_t = (1 - .92B)a_t$

We consider the ARMA(3,1) model $(1 - 2.57B + 2.50B^2 - .92B^3)X_t = (1 - .92B)a_t$ and illustrate the contributions of the 3rd-order AR and 1st-order MA factors to the final ARMA(3,1) model. The factor table for this model is shown below.

TABLE 5.10 Factor Table for ARMA(3,1) Model $(1 - 2.57B + 2.50B^2 - .92B^3)X_t = (1 - .92B)a_t$			
AR-FACTOR	ROOTS	$ r ^{-1}$	f_0
$1 - 1.6B + .95B^2$.844 ± .586i	.973	.097
$1 - .97B$	1.0295	.970	.000
MA-FACTOR	ROOTS	$ r ^{-1}$	f_0
$1 - .92B$	1.111	.900	.000

From the factor table, we see that the irreducible 2nd-order AR-factor is associated with frequency $f_0 = .097$ (or a period length of about 10). The second factor is a 1st-order factor associated with a positive real root. Roots associated with both factors are close to the unit circle. We expect cyclic behavior in the realizations with periods of about 10 units and a wandering behavior associated with the positive real root. The realization from this AR(3) model shown in Figure 5.26(a) has the expected behaviors. The sample autocorrelations in Figure 5.26(b) have a damped sinusoidal behavior along a damped exponential path. Note that most of the autocorrelations are positive. The spectral density shown in Figure 5.26(c) has a peak at $f = 0$ and at about $f = .1$, as expected.

The MA-factor is a 1st-order factor with a positive real root. Recall that the “system frequency” of $f_0 = 0$ in an MA model is associated with a dip in the spectral density at $f_0 = 0$. Figure 5.26(d) shows a realization from $X_t = (1 - .92B)a_t$. Figure 5.26(e) shows that the lag-one autocorrelation is about $-.5$ and all other autocorrelations are quite small. The true lag-one autocorrelation is $\rho_1 = -.92 / (1 + .92^2) = -.498$, and the other true autocorrelations for this MA(1) process are zero. Also the Parzen spectral density estimate in Figure 5.26(f) has a dip at $f = 0$ with a rise in power (but not a peak) at the higher frequencies, which is consistent with spectral densities shown in Figures 5.21(b) and 5.22(c) for MA(1) processes with positive real root. This suggests that realizations from the MA(1) model will lack low frequency behavior and tend to be higher frequency. We see this behavior in the MA(1) realization in Figure 5.26(d).

DON'T LOOK at Figures 5.26(g), (h), and (i) yet. They show a realization, the sample autocorrelations, and Parzen spectral density estimate of the full ARMA(3,1) model $(1 - 2.57B + 2.50B^2 - .9215B^3)X_t = (1 - .92B)a_t$. First, let's discuss what we would expect for the behavior of the ARMA(3,1) model. This model should incorporate the features of both the AR and MA parts. That is, we expect the realization to be cyclic with a period of about 10. We also expect that the peak at $f = 0$ for the AR(3) part will be partially cancelled out by the dip at $f = 0$ in the MA(1) component.

NOW LOOK: The realization from the ARMA(3,1) model in Figure 5.26(g) has a cyclic behavior associated with a frequency of about $f = .1$ (period of about 10), but the wandering behavior is missing and the cyclic behavior centers around a horizontal center line. The sample autocorrelations in Figure 5.26(h) have a damped sinusoidal behavior similar to that of an AR(2) model. Finally, the Parzen spectral density estimate in Figure 5.26(i) has a peak at about $f = .10$ but only a small hint of a peak at $f = 0$.

The plots in Figure 5.26 can be obtained using the commands

```
fig5.26a=gen.arma.wge(n=150,phi=c(2.57,-2.50,.92),sn=65)
plotts.sample.wge(fig5.26a) # this plots Figure 5.26(a-c)
fig5.26d=gen.arma.wge(n=150,phi=0,theta=.92,sn=65)
plotts.sample(fig5.26d) # this plots Figure 5.26(d-f)
fig5.26g=gen.arma.wge(n=150,phi=c(2.57,-2.50,.92),,theta=.92,sn=65)
plotts.sample.wge(fig5.26g) # this plots Figure 5.26(g-i)
```

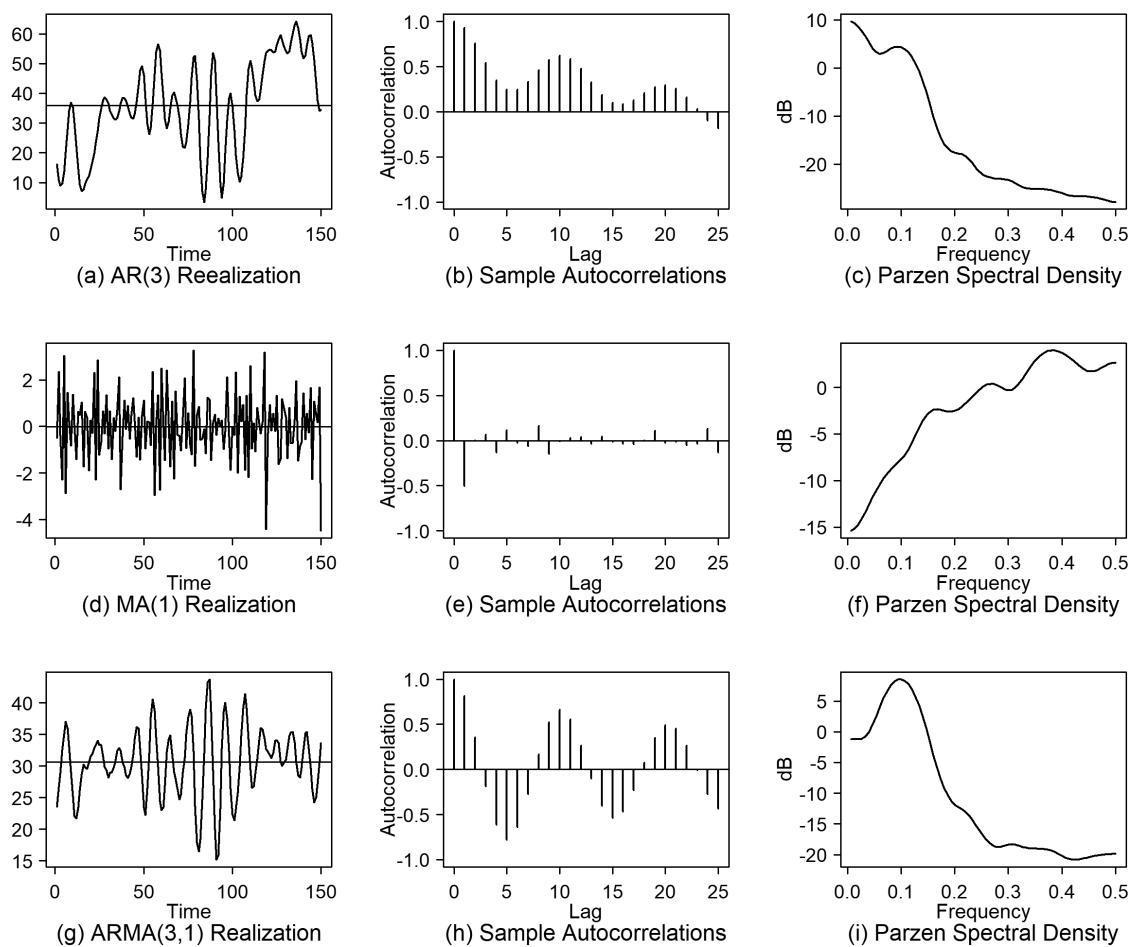


FIGURE 5.26 (a) Realization, (b) sample autocorrelations, and (c) Parzen spectral density estimate for ARMA(3,0): $(1 - 2.57B + 2.50B^2 - .92B^3)X_t = a_t$; (d) Realization, (e) sample autocorrelations, and (f) Parzen spectral density estimate for MA(1): $X_t = (1 - .92B)a_t$; (g) realization, (h) sample autocorrelations, and (i) Parzen spectral density estimate for ARMA(3,1): $(1 - 2.57B + 2.50B^2 - .92B^3)X_t = (1 - .92B)a_t$.

Example 5.11 $(1 - 1.3B + .4B^2)X_t = (1 - .8B)a_t$

We next examine the model $(1 - 1.3B + .4B^2)X_t = (1 - .8B)a_t$. Using the `tswge` command `true.arma.aut.wge(phi=c(1.3, -.4), theta=.8)`, we obtain the autocorrelations for this model that are viewed in Table 5.11 below:

TABLE 5.11 Autocorrelations for the model: $(1 - 1.3B + .4B^2)X_t = (1 - .8B)a_t$

LAG(K)	ρ_k
0	1
1	.5
2	.25
3	.125
:	:

Note that $\rho_1 = .5 = .5^1$, $\rho_2 = .25 = .5^2$, $\rho_3 = .125 = .5^3, \dots$, and clearly $\rho_k = .5^k$, which we recognize as the model-based autocorrelations of the AR(1) process $(1 - \phi_1 B)X_t = a_t$, with $\phi_1 = .5$. Consequently, the autocorrelations of $(1 - 1.3B + .4B^2)X_t = (1 - .8B)a_t$ are the same as those for the AR(1) model $(1 - .5B)X_t = a_t$. So, what's going on? In a situation like this, we recommend checking the factor table. (By now, surely you have noticed that this is a common recommendation!) The `tswge` command `factor.wge(phi=c(1.3, - .4), theta=.8)` gives Table 5.12.

TABLE 5.12 Factor Table for ARMA(2,1) Model $(1 - 1.3B + .4B^2)X_t = (1 - .8B)a_t$

AR-FACTOR	ROOTS	$ r ^{-1}$	f_0
$1 - .8B$	1.25	.8	.00
$1 - .5B$	2.00	.5	.00
MA-FACTOR	ROOTS	$ r ^{-1}$	f_0
$1 - .8B$	1.25	.8	.00

From the table, we see that the factor $(1 - .8B)$ appears on the autoregressive and moving average sides of the equation, and thus violates the condition that $\phi(B)$ and $\theta(B)$ share no common factors. Note that if we factor the model as $(1 - .8B)(1 - .5B)X_t = (1 - .8B)a_t$, and pre-multiply both the left-hand and right-hand sides by $(1 - .8B)^{-1}$, we obtain

$$(1 - .8B)^{-1}(1 - .8B)(1 - .5B)X_t = (1 - .8B)^{-1}(1 - .8B)a_t$$

or $(1 - .5B)X_t = a_t$. Consequently, since $(1 - .8B)^{-1}$ and $(1 - .8B)$ are canceling operators, the model specified in the example is *not an* ARMA(2,1) but is actually an AR(1).

Key Point: It is worth repeating that we recommend checking the factor table (for both the AR and MA parts) of an ARMA model in order to fully understand it.

- for example, the factor table can identify potential canceling or nearly canceling factors.

Example 5.12 Nearly Canceling Operators

Consider the AR(1) model $(1 - .95B)X_t = a_t$. A realization, sample autocorrelations, and Parzen spectral density estimate are shown in Figures 5.27(a)–(c). There we see wandering behavior in the realization, exponentially damping sample autocorrelations with $\hat{\rho}_k$ relatively strong (and positive) for small k , and a peak in the Parzen spectral density estimate at $f = 0$.

We next consider the ARMA(1,1) model $(1 - .95B)X_t = (1 - .8B)a_t$. This is clearly an ARMA(1,1) model because the factors do not cancel. However, both factors are associated with positive real roots, and Figures 5.27(d)–(f) show a realization, sample autocorrelations, and Parzen spectral density estimate for the ARMA(1,1) model $1 - .95B + X_t = a_t$; we see that the moving average factor has “damped” the effect of the AR part. In fact, the realization, sample autocorrelations, and Parzen spectral estimator appear similar to white noise, which would have been the case if the model were $(1 - .95B)X_t = (1 - .95B)a_t$. So, in this

case, $(1 - .8B)$ on the MA side “nearly” canceled the effect of the autoregressive part. Figure 5.27 was plotted using the commands

```
fig5.27a=gen.arma.wge(n=150,phi=.95,sn=20)
plotts.sample.wge(fig5.27a) # this plots Figure 5.27(a-c)
fig5.27d=gen.arma.wge(n=150,phi=.95,theta=.8,sn=20)
plotts.sample(fig5.27d) # this plots Figure 5.27 (d-f)
```

Key Point: The take-away from this example is that “near cancellation” may “hide” some of the autoregressive characteristics.

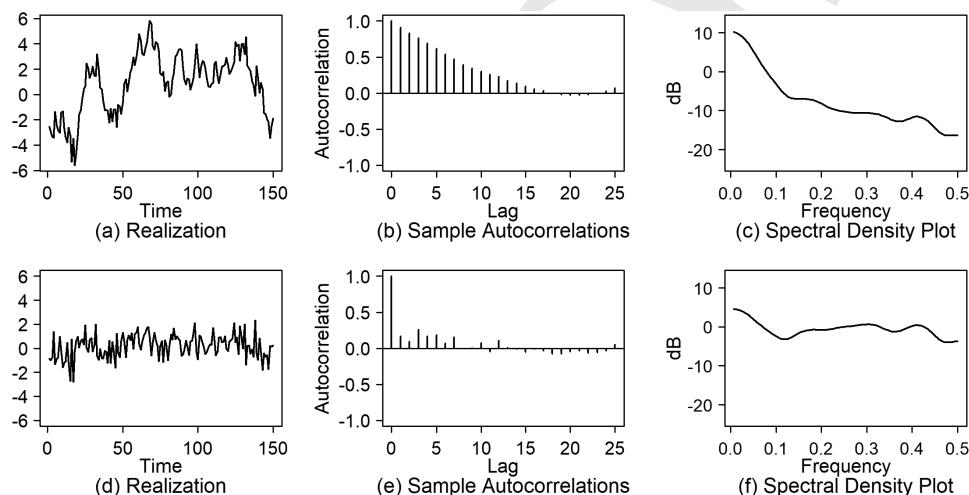


FIGURE 5.27 (a) Realization, (b) sample autocorrelations, and (c) Parzen spectral density estimate for AR(1) model $(1 - .95B)X_t = a_t$; (d) Realization, (e) sample autocorrelations, and (f) Parzen spectral density estimate for ARMA(1,1) model $(1 - .95B)(1 - .8B)X_t = a_t$.

5.3 CONCLUDING REMARKS

What a crucial and informative chapter! As was stated in the introduction, the concepts, definitions, and methodologies described in this chapter are at the heart of the analysis of time series data.

The autoregressive AR(1) and AR(2) models were introduced; realizations, autocorrelation function plots, and spectral density plots from both types were compared. These models were generalized to the AR(p) model, for which it was illustrated that by factoring, an AR(p) is (fortunately!) comprised of the simpler AR(1) and AR(2) components.

The AR(p) process was then extended to the autoregressive moving average ARMA(p,q) process, which considers the addition of a moving average component, MA(q). While not commonly used by itself to model real data, the MA(q) component is a valuable addition which interestingly often enhances an AR(p) model.

A novel idea that assists the analyst in extracting and organizing these components of an AR(p) model is the factor table, which will be a frequent consideration in various types of analyses in future chapters.

APPENDIX 5A

TSWGE FUNCTIONS

- (a) **factor.wge(phi, theta)** is a function that takes an AR, MA, or ARMA polynomials, factors them into their 1st-order and irreducible 2nd-order polynomial factors, and prints a factor table.

phi is a vector specifying the coefficients of the AR part of the model
theta is a vector specifying the coefficients of the MA part of the model

Example: Consider the model

$$X_t + .3X_{t-1} - .44X_{t-2} - .29X_{t-3} + .378X_{t-4} + .648X_{t-5} = a_t + .5a_{t-1} + .7a_{t-2}.$$

Then define **phi5=c(-.3, .44, .29, -.378, -.648)** and **theta2=c(-.5, -.7)**, and issue the command

factor.wge(phi=phi5, theta=theta2)

You will obtain the following factor tables:

AR Factor Table				
Factor	Roots	Abs Recip	System	Freq
1-1.6000B+0.9000B^2	0.8889+-0.5666i	0.9487	0.0903	
1+0.9000B	-1.1111	0.9000	0.5000	
1+1.0000B+0.8000B^2	-0.6250+-0.9270i	0.8944	0.3444	

MA Factor Table				
Factor	Roots	Abs Recip	System	Freq
1+0.5000B+0.7000B^2	-0.3571+-1.1406i	0.8367	0.2983	

- (b) **'gen.arma.wge(n, phi=0, theta=0, mu=0, var=1, plot=TRUE, sn=0)** is a function that generates a realization of length n from a given stationary AR, MA, or ARMA model.

n is the realization length

phi is a vector of AR parameters (default is **phi=0**)

theta is a vector of MA parameters (using signs as in this text) (default is **theta=0**)

mu is the process mean (**default=0**)

var is the white noise variance (default=1)

plot=TRUE (default) produces a plot of the generated realization

sn determines the seed number used in the simulation. (See Note (5) below)

Notes

- (1) This function finds p and q as the length of **phi** and **theta**, respectively. For example, if either **phi=0** or **theta=0** (also the default), the values for p and q are set to zero, respectively.
- (2) **gen.arma.wge** calls the base R function **arima.sim**, which uses the same signs as in this text for the AR parameters but opposite signs as in this text for MA parameters. The appropriate adjustments are made within **gen.arma.wge**, so that input parameters **phi** and **theta** for **gen.arma.wge** should contain parameters using the same signs as in this text. However, if you use **arima.sim** directly (which has options not employed in **gen.arma.wge**), then you must remember that the signs needed for the MA parameters have opposite signs as those in **gen.arma.wge**.
- (3) It will be useful to call the utility function **mult.wge** if you only know the factors of the model and not the coefficients.
- (4) By default, the white noise is zero-mean, normal noise with variance **vara=1**.
- (5) **sn=0** (default) produces results based on a randomly selected seed. If you want to reproduce the same realization on subsequent runs of **gen.arma.wge**, then set **sn** to the same positive integer value on each run.

Example: The command

```
x=gen.arma.wge(n=100,phi=c(1.6,-.9),theta=.9,mu=30,vara=1)
```

generates and plots the realization from the model $X_t - 1.6X_{t-1} + .9X_{t-2} = a_t - .9a_{t-1}$ and stores it in the vector **x**. The mean of the process is $\mu = 30$ and the white noise is normally distributed with mean zero and variance one, and the seed is randomly selected.

- (c) **mult.wge(fac1, fac2, fac3, fac4, fac5, fac6)**. The above functions dealing with ARMA processes require the coefficients **phi** and **theta** as input. If you only know your model in factored form, say for example,

$$(1 - 1.6B + .9B^2)(1 + .9B)(1 + B + .8B^2)X_t = a_t,$$

then without multiplying the associated polynomials you do not know the components of the vector **phi** used in the function calls. The **tswge** function **mult.wge** can be used to multiply up to six factors to obtain **phi** and/or **theta**.

fac1 through **fac6** are vectors specifying the factors of the AR or MA part of the model. Note: In the above example with three AR factors, the call statement is

```
mult.wge(fac1=c(1.6,-.9),fac2=-.9,fac3=c(-1,-.8))
```

The function outputs the coefficients of the polynomial resulting from the multiplication of the 1st- and 2nd-order factors. The output is in the form of a vector, and also included is the associated characteristic polynomial. For the example above, the output is

```
$char.poly
1 + 0.3*x - 0.44*x^2 - 0.29*x^3 + 0.378*x^4 + 0.648*x^5
$model.coef
[1] -0.300 0.440 0.290 -0.378 -0.648
```

Note This function uses the CRAN package **PolynomF**

- (d) **plotts.true.wge(n,phi,theta,lag.max,vara,sn,plot.data)** For a given ARMA(p,q) model, this function generates a realization (optional), calculates the true autocorrelations and spectral density (in dB), and plots the graphs

n is the realization length ($x(t)$, $t=1, \dots, n$)

phi is a vector of AR parameters (using signs as in this text). (default=0)

theta is a vector of MA parameters (using signs as in this text) (default=0)
lag.max is the maximum lag at which the autocorrelations and autocovariances will be calculated (default=25)
vara is the white noise variance (default=1)
plot.data specifies whether to plot a realization from the model (**default=TRUE**)
sn and **plot.data**. If **plot.data=TRUE**, then the function generates and plots a realization from the specified model. The parameter **sn** plays the same role as in **gen.arma.wge**

Notes

- (1) $\max(p, q+1) \leq 25$
- (2) This function uses a call to the Base R function **arima.sim**, which uses the same signs as in this text for the AR parameters but opposite signs for MA parameters. The appropriate adjustments are made here so that **phi** and **theta** should contain parameters using the signs as in this text. See discussion of **arima.sim** in the description of **gen.arma.wge**

Example: The command

```
test=plottts.true.wge(n=100,phi=c(1.6,-.90),theta=.9,lag.max=25,vara=1)
```

generates a realization, and computes the true autocorrelations (and autocovariances) and spectral density, and plots the results.

- test\$data** contains the realization if **plot.data=TRUE**
test\$aut1 contains the true autocorrelations (lags 0, 1, ..., **lag.max**). Note:
test\$aut1[1]=1, that is, the autocorrelation at lag 0.
test\$acv contains the true autocovariances (lags 0, 1, ..., **lag.max**). Note:
test\$acv[1] is the process variance, i.e. **gvar=g[1]**
test\$sd is the process standard deviation
test\$spec contains the 251 spectral density values associated with $f = 0, 1/500, 2/500, \dots, 250/500$ used in the plot.
- (e) **true.arma.aut.wge(phi, theta, lag.max, vara, plot)** calculates the true autocorrelations and autocovariances associated with a given AR, MA, or ARMA model.
- phi** is a vector of AR parameters (using signs as in this text)
theta is a vector of MA parameters (using signs as in this text)
lag.max is the maximum lag at which the autocorrelations and autocovariances will be calculated (default=25)
vara is the white noise variance (default=1)
plot is a logical variable: TRUE=plot autocorrelations, FALSE=no plot

Note: $\max(p, q+1) \leq 25$

Example: The command

```
traut=true.arma.aut.wge(phi=c(1.6,-.9),theta=.9,lag.max=25,vara=1)
```

calculates and plots the true autocorrelations, $\rho_k, k = 0, 1, \dots, 25$ associated with the ARMA model $X_t - 1.6X_{t-1} + .9X_{t-2} = a_t - .9a_{t-1}$. The autocorrelations are stored in the vector **traut\$acf** (which has 26 elements). Similarly the autocovariances $\gamma_k, k = 0, 1, \dots, 25$ are stored in the vector **traut\$acv**. The process variance is given by **traut\$acv[1]**.

- (f) **true.spec.wge(phi,theta,plot)** calculates the spectral density associated with a given AR, MA, or ARMA model.

phi is a vector of AR parameters (using signs as in this text)

theta is a vector of MA parameters (using signs as in this text)

plot is a logical variable: TRUE=plot spectral density, FALSE=no plot

Note: $\max(p, q+1) \leq 25$

Example: The command

```
trsperc=true.arma.spec.wge(phi=c(1.6,-.9),theta=.9)
```

calculates and plots the true spectral density associated with the ARMA model $X_t - 1.6X_{t-1} + .9X_{t-2} = a_t - .9a_{t-1}$. The vector **trsperc** contains the 251 spectral density values associated with $f = 0, 1/500, 2/500, \dots, 250/500$ used in the plot.

APPENDIX 5B

STATIONARITY CONDITIONS OF AN AR(1)

In the following, we derive the following properties of the stationary AR(1) model

$$X_t = (1 - \phi_1)\mu + \phi_1 X_{t-1} + a_t,$$

which, by Theorem 6.1, satisfies $|\phi_1| < 1$.

- (a) $E[X_t] = \mu$ (a constant)

Taking expectations of the terms in (6.2), we have

$$\begin{aligned} E[X_t] &= E[\beta + \phi_1 X_{t-1} + a_t] \\ &= E[(1 - \phi_1)\mu] + \phi_1 E[X_{t-1}] + E[a_t] \\ &= (1 - \phi_1)\mu + \phi_1 E[X_{t-1}] + E[a_t] \\ &= (1 - \phi_1)\mu + \phi_1 E[X_{t-1}], \end{aligned}$$

since $E[a_t] = 0$. Now, since X_t is stationary, $E[X_t] = E[X_{t-1}]$, so we will rewrite the equation above as $E[X_t] = (1 - \phi_1)\mu + \phi_1 E[X_t]$. So the following computations follow:

$$E[X_t] - \phi_1 E[X_t] = (1 - \phi_1)\mu,$$

$$E[X_t](1 - \phi_1) = (1 - \phi_1)\mu,$$

$$\begin{aligned} E[X_t] &= (1 - \phi_1)/ (1 - \phi_1) \mu \\ &= \mu, \text{ since } (1 - \phi_1) \neq 0, \text{ by the assumption of stationarity.} \end{aligned}$$

- (b) $\text{Var}[X_t] = \sigma_a^2 / (1 - \phi_1^2)$. See (c) below.
(c) The autocorrelation between X_t and X_{t+k} depends only on k .

Without loss of generality, we use the zero-mean form of the AR(1) model. In order to find ρ_k for an AR(1) process, we first find γ_k by evaluating the expectation $E[X_t X_{t+k}] = E[X_{t-k} X_t]$. Recall that we are using the zero-mean form, that is, $\mu = E[X_t] = 0$, for all t . Multiplying both sides of the equality in (6.3) by X_{t-k} , where $k > 0$, yields

$$X_{t-k} [X_t = \phi_1 X_{t-1} + a_t].$$

Taking the expectation of both sides, we obtain

$$E[X_{t-k} X_t] = \phi_1 E[X_{t-k} X_{t-1}] + E[X_{t-k} a_t].$$

The fact that $k > 0$ implies that X_{t-k} occurs prior to time t , and is thus uncorrelated with the noise term a_t that enters the model at time t . That is, for $k > 0$, $E[X_{t-k} a_t] = E[X_{t-k}] E[a_t] = 0$, because $E[X_{t-k}] = E[a_t] = 0$. Because $E[X_{t-k} X_t] = \gamma_{t-(t-k)} = \gamma_k$ and $E[X_{t-k} X_{t-1}] = \gamma_{t-1-(t-k)} = \gamma_{k-1}$, it follows from (6.5) that $\gamma_k = \phi_1 \gamma_{k-1}$ for $k > 0$, and

$$\begin{aligned} \gamma_1 &= \phi_1 \gamma_0 \\ \gamma_2 &= \phi_1 \gamma_1 = \phi_1^2 \gamma_0, \end{aligned}$$

so that, in general,

$$\gamma_k = \phi_1^k \gamma_0. \quad (6B.1)$$

It follows that $\rho_k = \phi_1^k \gamma_0 / \gamma_0 = \phi_1^k$ for $k > 0$.

The variance $\sigma_x^2 = \gamma_0$ can be obtained by pre-multiplying both sides of (6.3) by X_t and taking expectations as before, to obtain

$$E[X_t^2] = \phi_1 E[X_t X_{t-1}] + E[X_t a_t].$$

Thus,

$$\gamma_0 = \phi_1 \gamma_1 + E[X_t a_t].$$

The term $E[X_t a_t]$ is found using an analogous procedure, this time by post-multiplying both sides of the zero-mean AR(1) model by a_t and taking expectations, which gives

$$E[X_t a_t] = \phi_1 E[X_{t-1} a_t] + E[a_t^2]$$

Previously, it was shown that $E[X_{t-1} a_t] = 0$, and therefore, $E[X_t a_t] = E[a_t^2] = \sigma_a^2$. Substituting into (6.3) yields

$$\gamma_0 = \phi_1 \gamma_1 + \sigma_a^2$$

so that using (6.6), we have

$$\gamma_0 = \frac{\sigma_a^2}{1 - \phi_1^2}, \quad (6B.2)$$

which is a constant and is finite, since $|\phi_1| < 1$.

PROBLEM SET

- 5.1 (a) Generate realizations of length $n = 80$ from the following AR(1) models, where a_t is zero-mean white noise with $\sigma_a^2 = 1$.

$$(1 - .5B)X_t = a_t$$

$$(1 - .8B)X_t = a_t$$

$$(1 - .95B)X_t = a_t$$

- (i) For each model, print the factor table.
- (ii) For each model, plot the realization, sample autocorrelations, and Parzen spectral density estimate. Describe the behavior of the model shown in these plots.
- (iii) Explain how the behaviors of the plots in (i) change as the root of the characteristic equation approaches the unit circle.

- (b) Repeat (a) for the models

$$(1 + .5B)X_t = a_t$$

$$(1 + .8B)X_t = a_t$$

$$(1 + .95B)X_t = a_t$$

- 5.2 Generate realizations of length $n = 150$ from the following AR(2) models, where a_t is zero-mean white noise with $\sigma_a^2 = 1$.

$$(a) (1 - 1.6B + .63B^2)X_t = a_t$$

$$(b) (1 + 1.6B + .63B^2)X_t = a_t$$

- (i) For each model, print the factor table.
- (ii) What are the roots of the characteristic equations?
- (iii) Write the model in factored form.
- (iv) For each model, plot the realization, sample autocorrelations, and Parzen spectral density estimate. Describe the behavior of the model shown in these plots and the relationship between the factor table information and the plots.

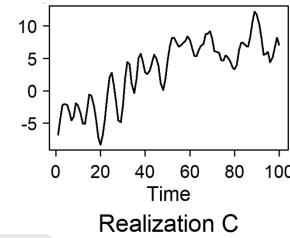
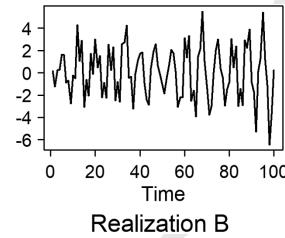
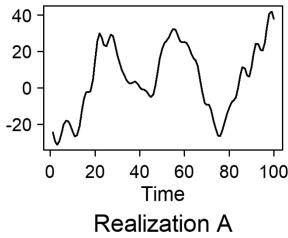
- 5.3 Determine whether the following models are stationary, and explain your answers.

$$(a) X_t - 1.55X_{t-1} + X_{t-2} - .25X_{t-3} = a_t$$

$$(b) X_t - 2X_{t-1} + 1.76X_{t-2} - 1.6X_{t-3} + .77X_{t-4} = a_t$$

- (c) $X_t - 1.9X_{t-1} + 2.3X_{t-2} - 2X_{t-3} + 1.2X_{t-4} - .4X_{t-5} = a_t$
 (c) $X_t - 2.5X_{t-1} + 3.14X_{t-2} - 2.97X_{t-3} + 2.016X_{t-4} - .648X_{t-5} = a_t$

5.4 Consider the following realizations:



This problem is similar to Example 5.5. Each of these realizations was generated from one of the three models below, given in random order.

- (a) $X_t - 2.9X_{t-1} + 3.72X_{t-2} - 2.63X_{t-3} + .828X_{t-4} = a_t$
 (b) $X_t - 1.99X_{t-1} + 1.69X_{t-2} - .693X_{t-3} = a_t$
 (c) $X_t - .28X_{t-1} - .134X_{t-2} + .8924X_{t-3} = a_t$

Work this problem in the stated order.

- Find factor tables for Models (a)–(c), and use this information to match the realizations to the models from which they were generated. Explain your answers.
- Generate the realizations for each of the three models, using the information below:
 For model (a): `a=gen.arma.wge(n=100, phi=c(2.9, -3.72, 2.63, -.828), sn=21)`
 For model (b): Use the `gen.arma.wge` function, inputting the appropriate coefficients with `sn=229`
 For model (c): Use the `gen.arma.wge` function, inputting the appropriate coefficients with `sn=26`

For each realization, use `plots.sample.wge` to display the realization, sample autocorrelations, and spectral density. How does the information in these plots compare with the information in the factor tables? Are there instances in which information is better presented in the factor tables versus the sample plots, or vice-versa?

- For each model, generate another realization (that is, use a different value for `sn`) and again create the `plots.sample.wge` plots. Do these plots reinforce your answers in (i)? Why or why not?

- 5.5 Consider the models $X_t = (1 - .5B)a_t$, and $X_t = (1 - 2B)a_t$, and compare the autocorrelations. Note that $X_t = (1 - .5B)a_t$ is invertible.
 5.6 Determine whether the following models are stationary and/or invertible and explain your answers.

- (a) $X_t - 1.6X_{t-1} + X_{t-2} = (1 - .9B)a_t$
 (b) $X_t - 2.6X_{t-1} + 1.5X_{t-2} - .38X_{t-3} - .72X_{t-4} = a_t - 1.9a_{t-1} + 1.7a_{t-2} - .72a_{t-3}$
 (c) $X_t = a_t - 2.9a_{t-1} + 2.7a_{t-2} - 1.52a_{t-3}$

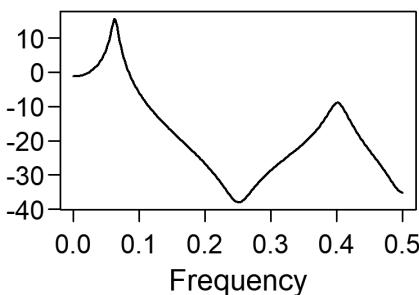
(d) $X_t - 1.9X_{t-1} + 2.5X_{t-2} - 2.24X_{t-3} + 1.36X_{t-4} - .576X_{t-5} = a_t - .9a_{t-1}$

5.7 The figure below is the true spectral density from one of the three models below. Use the corresponding factor tables to identify the appropriate model and explain your answers.

(a) $X_t - 1.9X_{t-1} + 2.5X_{t-2} - 2.24X_{t-3} + 1.36X_{t-4} - .576X_{t-5} = a_t - .9a_{t-1}$

(b) $X_t - .3X_{t-1} - .9X_{t-2} - 1X_{t-3} + .8X_{t-4} = a_t + .9a_{t-1} + .8a_{t-2} + .72a_{t-3}$

(c) $X_t - .6X_{t-1} + .2X_{t-2} - .54X_{t-3} + .81X_{t-4} = a_t - .99a_{t-1}$



5.8 Generate realizations from the following two models.

(a) $(1 - 1.6B + .9B^2)(X_t - 20) = a_t$

(b) $(1 - 1.6B + .9B^2)(X_t - 20) = (1 - .9B)a_t$.

For each realization, display the results of `plots.sample.wge`. Discuss the effect of the factor $(1 - .9B)a_t$.

PROOF