Quantum Graphs on M_2

Björn Schäfer
Saarland University

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Introduction

Quantum graphs generalize classical graphs by replacing the commutative vertex algebra $C(V)$ with a finite-dimensional C^*-algebra.

Weaver (2010) defined them as special quantum relations.

Duan–Severini–Winter (2013) introduced them independently to study zero-error communication via quantum channels.

Musto–Reuter–Verdon (2018) presented a unified theory in categorical terms.

Recent research shows their significance for the theory of quantum groups:

- M. Wasilewski introduced quantum Cayley graphs (2024)
- Brannan–Gromada-Matsuda-Skalski-Wasilewski (2025): quantum Frucht theorem

This talk: General theory + Classification of quantum graphs on M_2 by Matsuda (2022), Gromada (2022) and Kiefer–S. (2025).

History (1: Weaver's quantum relations)

Idea: Classically, a relation is a subset of $X \times X$. The quantum analog of a set is a vN-algebra $\mathcal{M} \subset B(\mathcal{H})$.

Definition (Weaver 2010)

A quantum relation on a von Neumann algebra $\mathcal{M} \subset B(\mathcal{H})$ is a weak*-closed operator bimodule over $\mathcal{M}'$, i.e.

$$\mathcal{V} \subset B(\mathcal{H}) \quad \text{such that} \quad \mathcal{M}'\mathcal{V}\mathcal{M}' \subset \mathcal{V}.$$

Then $\mathcal{V}$ is called

- reflexive if $\mathcal{M}' \subset \mathcal{V}$,
- symmetric if $\mathcal{V} = \mathcal{V}^*$,
- antisymmetric if $\mathcal{V} \cap \mathcal{V}^* \subset \mathcal{M}'$,
- transitive if $\mathcal{V}^2 \subset \mathcal{V}$.

A **quantum graph** on $\mathcal{M}$ is a reflexive, symmetric quantum relation on $\mathcal{M}$.

History (2: Duan–Serverini–Winter's noncommutative graphs)

Let $N : X \to Y$ be a communication channel and lets its confusability graph G_N have vertex set X and an edge between $x, x' \in X$ if Bob might confuse the messages x and x'.

The **independence number** of G_N is the maximum number of letters that can be sent without confusion in a single use of the channel (one-shot zerror-error capacity).

A **quantum** channel is a CPTP map $\Phi : M_m \to M_n$. What is its confusability graph?

Let $\Phi : M_m \ni x \mapsto \sum_i E_i x E_i^* \in M_n$ with Kraus operators $E_i : \mathbb{C}^m \to \mathbb{C}^n$.
Duan-Severini-Winter (2013) define the **non-commutative confusability graph** of Φ as

$$V := \mathrm{span}\{E_i^* E_j \mid i, j\} \subset M_m.$$

Then, the **one-shot zero-error capacity** of Φ is the maximal size of a set of orthogonal vectors $\{|\phi_1\rangle, \ldots, |\phi_k\rangle\} \subset \mathbb{C}^m$ such that

$$|\phi_i\rangle \langle\phi_j| \in V^\perp \quad \text{for all } i \neq j.$$

There are more notions for a independence number of a quantum graph, as well a Lovasz theta functions due to various authors.

History (3: Musto–Reutter–Verdon's viewpoint)

Musto–Reutter-Verdon (2018) define a categorical framework for

- quantum sets,
- quantum functions between quantum sets, including bijections,
- quantum graphs on a quantum set,
- quantum graph homomorphisms and isomorphisms.

Everything takes place in a *concrete dagger category* (think: finite-dimensional Hilbert spaces with linear maps as morphisms and adjoint as dagger).

A **quantum set** is a *special symmetric dagger Frobenius algebra* in this category (think: finite-dimensional C*-algebra with a *special* tracial functional)

A **quantum function** between quantum sets B_1, B_2 is a map $T : \mathcal{H} \otimes B_1 \to B_2 \otimes \mathcal{H}$ for a Hilbert space $\mathcal{H}$ satisfying certain conditions.

A **quantum graph** is a particular endomorphism A of a quantum set (and this is equivalent to the previous definition).

Part I: General Theory

Definition

A quantum set is a pair (B, ψ) of a finite-dimensional C^*-algebra B and a faithful positive functional ψ such that

$$mm^* = \mathrm{Id}_B,$$

where $m^* : L^2(B) \to L^2(B) \otimes L^2(B)$ is the adjoint of the multiplication map w.r.t. the inner product induced by ψ.

- Equivalently: ψ is a state and a δ-form, i.e. $mm^* = \delta^2 \mathrm{Id}_B$ for some $\delta > 0$.
- For every finite C*-algebra B there is a unique **tracial** ψ as above.

Example

- $(\mathbb{C}^n, \tau)$, where τ is the counting measure ($\tau(e_g) = 1$ for all g).
- $(M_n, n\mathrm{Tr})$, where Tr is the un-normalized trace on M_n

Why $mm^* = \mathrm{Id}_B$? Consider $\mathbb{C}^n$ and an arbitrary positive functional

$$\psi(\mathbf{x}) = \sum_{i=1}^{n} p_i x_i, \quad p_i > 0.$$

Then,

$$m^*(e_i) = \frac{1}{p_i} e_i \otimes e_i$$
$$mm^*(e_i) = \frac{1}{p_i} e_i.$$

Thus, $mm^* = \mathrm{Id}_B$ implies $p_i = 1$ for all i.

There are three ways to define a quantum graph G on (B, ψ):

This corresponds classically to

Definition

1. Given $B \subset B(\mathcal{H})$, G is a $B' - B'$-bimodule $V \subset B(\mathcal{H})$.

   ... $\mathrm{span}\{E_{ij} \in M_n \mid i \sim j\} \subset M_n$

2. G is a projection $P \in B \otimes B^{\mathrm{op}}$

   ... $\sum_{i \sim j} e_i \otimes e_j \in \mathbb{C}^n \otimes \mathbb{C}^n$

3. G is a linear operator $A : L^2(B) \to L^2(B)$ which is Schur-idempotent and real, i.e.

   ... $A = (\delta_{i \sim j})_{i,j=1}^n : \mathbb{C}^n \to \mathbb{C}^n$.

$$m(A \otimes A)m^* = A, \quad A(b^*) = (A(b))^*.$$

Then A is called a quantum adjacency matrix.

Proposition (Weaver 2010)

Let $B_1 \subset B(\mathcal{H}_1)$ and $B_2 \subset B(\mathcal{H}_2)$ be isomorphic finite-dimensional C^*-algebras.
There is a 1:1-correspondence between the B_1'-B_1'-bimodules in $B(\mathcal{H}_1)$ and the B_2'-B_2'-bimdodules in $B(\mathcal{H}_2)$.

Weaver proved this more generally for arbitrary vN-algebras.

Idea of the proof.

It suffices to consider the case $B_2 = I_K \otimes B_1$ for a large enough Hilbert space K. Then the correspondence maps $V \subset B(\mathcal{H}_1)$ to $B(K) \otimes V$. $\square$

$$B = \begin{bmatrix} M_{n_1} & & \\ & M_{n_2} & \\ & & \cdot \end{bmatrix}$$

Example

A particular representation is the following: Let
$B = \bigoplus_i M_{n_i} \subset B(\bigoplus \mathbb{C}^{n_i})$.
Then $B' = \bigoplus_i \mathbb{C} I_{n_i}$, and $V \subset B(\bigoplus_i \mathbb{C}^{n_i})$ is a $B' - B'$-bimodule iff

$$V = \bigoplus V_{ij}$$

with $V_{ij} \subset M_{n_i,n_j}$.

$$B' = \begin{bmatrix} \mathbb{C} I_{n_1} & & \\ & \mathbb{C} I_{n_2} & \\ & & \cdot \end{bmatrix}$$

$$V = \begin{bmatrix} V_{1,2} & V_{1,3} & \cdot \\ V_{2,1} & V_{2,2} & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$$

Correspondence between bimodule $V \subset B(\mathcal{H})$ and projection $P \in B \otimes B^{\mathrm{op}}$ (1)

Problem: Find a 1:1-correspndence "bimodule $V \subset B(\mathcal{H}) \leftrightarrow$ projection $P \in B \otimes B^{\mathrm{op}}$".

Let $B = \bigoplus_i M_{n_i}$ and $\mathcal{V} = \bigoplus_{i,j} \mathcal{V}_{ij}$.

We need to find a projection $P \in B \otimes B^{\mathrm{op}} = \bigoplus_{i,j} M_{n_i} \otimes M_{n_j}^{\mathrm{op}}$ such that $\mathcal{V} \overset{1:1}{\leftrightarrow} P$.

For any i, j one has $\mathcal{V}_{ij} \subset M_{n_i,n_j}$ and an action of $M_{n_i} \otimes M_{n_j}^{\mathrm{op}}$ on M_{n_i,n_j} via

$$M_{n_i,n_j} \ni B \mapsto ABC \in M_{n_i,n_j}, \qquad A \in M_{n_i}, C \in M_{n_j}^{\mathrm{op}}.$$

Identify $M_{n_i,n_j} \cong \mathbb{C}^{n_i n_j}$ and $M_{n_i} \otimes M_{n_j}^{\mathrm{op}} \cong M_{n_i n_j}$ in the natural way. Then the action corresponds to matrix multiplication from the left.

Let $P_{ij} \in M_{n_i n_j} \cong M_{n_i} \otimes M_{n_j}^{\mathrm{op}}$ be the orthogonal projection onto the subspace $\mathcal{V}_{ij} \subset \mathbb{C}^{n_i n_j}$.

Then $P := \bigoplus_{i,j} P_{ij} \in B \otimes B^{\mathrm{op}}$ is the desired projection corresponding to $\mathcal{V}$.

In other words: It is $P = \sum_i a_i \otimes b_i^{\mathrm{op}}$ if $x \mapsto \sum_i a_i x b_i$ is the projection onto V.

This argument is a simplified version of the one for Proposition 2.23 from Weaver (2010).

Correspondence between bimodule $V \subset B(\mathcal{H})$ and projection $P \in B \otimes B^{\mathrm{op}}$ (2)

Note: The previous arguments do not involve the state ψ at all.

However, to obtain a full 1:1-correspondence $V \leftrightarrow P \leftrightarrow A$ in the case where ψ is not tracial, problems arise, because

- we need to be able to define a q. adjacency matrix A from either V or P,
- natural properties such as undirectedness, reflexivity, etc. need to fit together under the correspondences.

Daw's solution (2024): allow only bimodules V and projections P which satisfy an additional symmetry condition and use the modular conjugation on $L^2(B)$ to construct P from V.

Wasilewski's solution (2024): allow arbitary V's and P's and use the KMS-inner product on $L^2(B)$ to define the projection P starting from V.

Later: Wasilewski's approach.

Problem: Given a linear operator $A : L^2(B) \to L^2(B)$ with

$$m(A \otimes A)m^* = A, \quad A(b^*) = (A(b))^*.$$

find a projection $P \in B \otimes B^{\mathrm{op}}$.

Proposition

Assume that ψ is tracial, and let

$$\Psi : B(L^2(B)) \to B \otimes B^{\mathrm{op}}, \quad x\psi(y^* \cdot) = |x\rangle \langle y| \mapsto x \otimes y^{\mathrm{op}}.$$

Then $P := \Psi(A)$ is idempotent iff A is Schur-idempotent, and self-adjoint iff A is real. The map $A \overset{\Psi}{\mapsto} P$ is a 1:1-correspondence.

Correspondence between $P \in B \otimes B^{\mathrm{op}}$ and $A : L^2(B) \to L^2(B)$ (2: Wasilewski)

Proposition

Let

$$\Psi : B(L^2(B)) \to B \otimes B^{\mathrm{op}}, \quad |x\rangle\langle y| = x\langle y, \cdot\rangle_{\mathrm{KMS}} \mapsto x \otimes y^{\mathrm{op}}.$$

Then $P := \Psi(A)$ is idempotent iff A is Schur-idempotent, and self-adjoint iff A is real.

The map $\Psi : A \mapsto P$ is a 1:1-correspondence.

If $\psi = \mathrm{Tr}(\rho\cdot)$, then

$$\begin{aligned}\langle x, y\rangle_{KMS} &= \psi(x^* \rho^{\frac{1}{2}} y \rho^{-\frac{1}{2}}) \\ &= \mathrm{Tr}(\rho^{\frac{1}{2}} x^* \rho^{\frac{1}{2}} y).\end{aligned}$$

Proof.

Observe $m(|b_1\rangle \otimes |b_2\rangle) = |b_1 b_2\rangle$, and consequently $(\langle a_1| \otimes \langle a_2|)m^* = \langle a_2 a_1|$. Therefore,

$$m(|b_1\rangle\langle a_1| \otimes |b_2\rangle\langle a_2|)m^* = |b_1 b_2\rangle\langle a_1 a_2| \overset{\Psi}{\mapsto} b_1 b_2 \otimes (a_2 a_1)^{\mathrm{op}} = (b_1 \otimes a_1^{\mathrm{op}})(b_2 \otimes a_2^{\mathrm{op}}),$$

i.e. Ψ turns Schur-product into C*-algebra product.
It remains to see "A is real iff P is self-adjoint". For that observe

$$A(b^*)^* = \left(\sum_i |x_i\rangle\langle y_i|\,|b^*\rangle\right)^* = \sum_i |x_i^*\rangle\langle b^*|\,|y_i\rangle = \sum_i |x_i^*\rangle\langle y_i^*|\,|b\rangle.$$

For $\langle b^*|\,|y_i\rangle = \langle y_i^*|\,|b\rangle$ we need the KMS-inner product!
Hence, $A(b^*)^* = A(b)$ iff $\sum_i x_i \otimes y_i^{\mathrm{op}} = \sum_i x_i^* \otimes (y_i^*)^{\mathrm{op}}$. □

Correspondence between $P \in B \otimes B^{\mathrm{op}}$ and $A : L^2(B) \to L^2(B)$ (3: Remarks)

Proposition

Let

$$\Psi : B(L^2(B)) \to B \otimes B^{\mathrm{op}}, \quad |x\rangle \langle y|_{KMS} = x\langle y, \cdot\rangle_{\mathrm{KMS}} \mapsto x \otimes y^{\mathrm{op}}.$$

Then $P := \Psi(A)$ is idempotent iff A is Schur-idempotent, and self-adjoint iff A is real.

The map $\Psi : A \mapsto P$ is a 1:1-correspondence.

- In the non-tracial case this correspondence is due to Wasilewski (2024) .
- In that case, Daws (2024) has an alternative correspondence.
- If ψ is tracial, $P = (A \otimes \mathrm{Id})m^*(1_B)$.
- ? $P \in B \otimes B^{\mathrm{op}}$ "doesn't remember ψ"!

What we have seen so far ...

If $B = M_n \subset M_n$, then $\mathrm{vec}(V) \subset \mathbb{C}^{n^2}$ is the image of a projection $\tilde{P} \in M_{n^2}$. Via $M_n \otimes M_n^{\mathrm{op}} \cong M_{n^2}$ one obtains the projection $P \in B \otimes B^{\mathrm{op}}$.

$B \otimes B^{\mathrm{op}} \ni a \otimes b^{\mathrm{op}} \mapsto |a\rangle\langle b| \in B(l^2(B))$ using the KMS-inner product on $L^2(B)$.

$V \subset B(\mathcal{H})$

- V is a B'-B'-bimodule
- $V = V^*$ (undirectedness)
- $I \in V$ (reflexivity) or $I \perp V$ (loopfreeness)

$P \in B \otimes B^{\mathrm{op}}$

- P is a projection
- $\sigma(P) = P$ for the flip map σ (undirectedness)
- $m(P) = P$ (reflexivity) or $m(P) = 0$ (loopfreeness)

$A : L^2(B) \to L^2(B)$

- A is Schur-idempotent and real
- $A = A^*$ (undirectedness)
- $m(A \otimes \mathrm{Id})m^* = \mathrm{Id}$ (reflexivity) or $m(A \otimes \mathrm{Id})m^* = 0$ (loopfreeness)

⚠ The correspondence $V \leftrightarrow P$ must account for the state ψ if it is not tracial.

Let $B = \bigoplus_i M_{n_i}$ and $\psi((x_i)) = \sum_i \mathrm{Tr}(\rho_i x_i)$.

Assume $\mathcal{H} = \bigoplus_i \mathbb{C}^{n_i}$ and for $(x_{ij})_{i,j} \in B(\mathcal{H})$ let

$$\psi^{-1}(x) := \sum_i \mathrm{Tr}(\rho_i^{-1} x_{ii}) I_i \in B'.$$

- ψ' makes $B(\mathcal{H})$ a B'-B'-bimodule.

Lemma

The map

$$\begin{aligned} \Phi : B \otimes B^{\mathrm{op}} &\to_{B'} B(\mathcal{H})_{B'} \\ a \otimes b^{\mathrm{op}} &\mapsto a \cdot \sigma_{-i/2}(b), \end{aligned}$$

is an isomorphism of vN-algebras, where

$$\sigma_z(b) := \rho^{iz} b \rho^{-iz}, \quad b \in B, z \in \mathbb{C}.$$

This is almost the desired correspondence $P \leftrightarrow V$.

Theorem

The map

$$B \otimes B^{\mathrm{op}} \ni P \mapsto V := \sigma_{\frac{i}{4}}^{\psi^{-1}}(\Phi(P)) \subset B(\mathcal{H})$$

is a 1:1-correspondence between projections P and B'-B'-bimodules V, where

$$\sigma_z^{\psi^{-1}}(b) := \rho^{-iz} b \rho^{-iz}, \quad b \in B, z \in \mathbb{C}.$$

Arbitrary representations "look like" the above after tensoring with $B(\mathcal{K})$.

Let G be a quantum graph on the **tracial** quantum set (B, ψ) with edge space $V \subset B(\mathcal{H})$, quantum edge projection $P \in B \otimes B^{\mathrm{op}}$ and quantum adjacency matrix A.

- G is **undirected** if A is self-adjoint $\Leftrightarrow$ P is preserved by flipping tensor legs $\Leftrightarrow V = V^*$.
  - If ψ is not tracial, then in Wasilewski's setting there are two notions: GNS-undirected and KMS-undirected. For GNS-undirectedness, P and V need to satisfy additional conditions.
- G is **loopfree** if $m(A \otimes \mathrm{Id}_B)m^* = 0 \Leftrightarrow m(P) = 0 \Leftrightarrow I_{\mathcal{H}} \in V^{\perp}$.
- G is **reflexive** if $m(A \otimes \mathrm{Id}_B)m^* = \mathrm{Id}_B \Leftrightarrow m(P) = 1_B \Leftrightarrow I_{\mathcal{H}} \in V$.
- G is d-**regular** if $A(1_B) = d1_B$ for some $d \geq 0$.

Example (empty quantum graph)

$$V = \{0\}, \quad P = 0, \quad A = 0.$$

Example (complete quantum graph)

$$V = B(\mathcal{H}), \quad P = 1_B \otimes (1_B)^{\mathrm{op}}, \quad A = \psi(\cdot)1_B.$$

Example (all-loops quantum graph)

$$V = \mathbb{C}I_{\mathcal{H}}, \quad P = \sum_i e_i \otimes e_i^{\mathrm{op}}, \quad A = \mathrm{Id}_B,$$

where $\{e_i\}$ is any orthonormal basis of $L^2(B)$ (with respect to the KMS-inner product if ψ is not tracial).

Examples of Quantum Graphs (2: Some QGraph on M_2 from Gromada)

Let us pick some projection $P \in M_2 \otimes M_2 = M_4 = M_2(M_2)$.

$$P = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}.$$

By transposing the second leg this gives a projection in $M_2 \otimes M_2^{\mathrm{op}}$.

Via $a \otimes b \leftrightarrow |a\rangle \langle b^t|_{2\mathrm{Tr}}$ this corresponds to the map

$$e_{11} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_{12} \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad e_{21} \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_{22} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We can write A as a matrix with respect to the basis $\{e_{11}, e_{12}, e_{21}, e_{22}\}$:

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Evidently, $A = A^*$, so the quantum graph is undirected.

Selected Literature

- Nik Weaver: Quantum Relations (2010).
- Duan, Severini, Winter: Zero-Error Communication via Quantum Channels, Noncommutative Graphs, and a Quantum Lovász Number (2013).
- Musto, Reutter, Verdon: A compositional approach to quantum functions (2018).

- Daniel Gromada: Some Examples of quantum graphs (2022).
- Junichiro Matsuda: Classification of quantum graphs on M2 and their quantum automorphism groups (2022).

- Matthew Daws: Quantum graphs: Different perspectives, homomorphisms and quantum automorphisms (2024).
- Matteusz Wasilewski: On quantum Cayley graphs (2024).

- Matthew Daws: Quantum graphs in infinite-dimensions: Hilbert–Schmidts and Hilbert modules (this Monday on the ArXiv!).

Part II: Quantum Graphs on M_2

Independently, Matsuda and Gromada (both 2022) classified all loop-free/reflexive undirected quantum graphs on the quantum set $(M_2, 2\mathrm{Tr})$.

Matsuda also classified non-tracial quantum graph on (M_2, ψ_q) + their quantum automorphism groups + quantum isomorphic classical graphs.

Kiefer–S. (2025) extended the classification to directed quantum graphs on M_2.

Different approaches: Gromada uses the space V for the classification, while Matsuda uses the projection P.

This talk: Gromadas approach.

II.1 Undirected Tracial Quantum Graphs on M_2

Let $B = M_2 \subset B(\mathbb{C}^2) = M_2$. Then $B' = \mathbb{C}I_2$ and any vector space $V \subset M_2$ is a B'-B'-bimodule.

$\boxed{V \to P}$ First, let $\tilde{P} : M_2 \to M_2$ be the orthogonal projection onto V with respect to the inner product $\langle a, b\rangle_{(2\mathrm{Tr})^{-1}} = \frac{1}{2}\mathrm{Tr}(a^* b)$.

Let

$$\Phi : M_2 \otimes M_2^{\mathrm{op}} \to B(M_2), x \otimes y^{\mathrm{op}} \to x \cdot y$$

Then V and P are related via $\Phi(P) = \tilde{P}$.

$\boxed{P \to A}$ Let

$$\Psi : B(M_2) \to M_2 \otimes M_2^{\mathrm{op}}, \quad |x\rangle \langle y|_{2\mathrm{Tr}} \mapsto x \otimes y^{\mathrm{op}}.$$

Then A and P are related via $\Psi(A) = P$.

Equivalent definitions for quantum graphs on $(M_2, 2\mathrm{Tr})$ (2: Concrete viewpoint)

Assume we have given $V \subset M_2$. How can we compute $A : M_2 \to M_2$?

Let $\{X_i \mid i\}$ be an ONB of V with respect to $\langle a, b\rangle_{(2\mathrm{Tr})^{-1}} = \frac{1}{2}\mathrm{Tr}(a^* b)$.
Then the projection onto V is given by

$$\tilde{P} : M_2 \to M_2, \quad b \mapsto \sum_i x_i \langle x_i, b\rangle_{(2\mathrm{Tr})^{-1}}$$

We have

$$\tilde{P}(b) = \frac{1}{2}\sum_i x_i \mathrm{Tr}(x_i^* b) = \frac{1}{2}\sum_{i,k,\ell} x_i e_{k\ell} x_i^* b e_{\ell k} = \frac{1}{2}\sum_{i,k,\ell} (x_i e_{k\ell} x_i^*)\, b\, (e_{\ell k}).$$

Thus, P is given by

$$P = \Phi^{-1}(\tilde{P}) = \frac{1}{2}\sum_{i,k,\ell} x_i e_{k\ell} x_i^* \otimes (e_{\ell k})^{\mathrm{op}} \in M_2 \otimes M_2^{\mathrm{op}}.$$

Equivalent definitions for quantum graphs on $(M_2, 2\mathrm{Tr})$ (2: Concrete viewpoint)

Assume we have given $V \subset M_2$. How can we compute $A : M_2 \to M_2$?

Thus, P is given by

$$P = \Phi^{-1}(\tilde{P}) = \frac{1}{2} \sum_{i,k,\ell} x_i e_{k\ell} x_i^* \otimes (e_{\ell k})^{\mathrm{op}} \in M_2 \otimes M_2^{\mathrm{op}}.$$

Recall,

$$\Psi : B(M_2) \to M_2 \otimes M_2^{\mathrm{op}}, \quad |x\rangle \langle y|_{2\mathrm{Tr}} \mapsto x \otimes y^{\mathrm{op}}.$$

Therefore, A is given by

$$A := \frac{1}{2} \sum_{i,k,\ell} |x_i e_{k\ell} x_i^*\rangle \langle e_{\ell k}|_{2\mathrm{Tr}} : M_2 \to M_2.$$

Observe,

$$A(e_{k\ell}) = 2 \cdot \frac{1}{2} \sum_i x_i e_{k\ell} x_i^* = \sum_i x_i e_{k\ell} x_i^*.$$

Thus,

$$A = \sum_i x_i \cdot x_i^*.$$

By representing A as an element in $M_2 \otimes M_2$ we get

$$A = \sum_i X_i \otimes X_i^*.$$

When are two quantum graphs on $(M_2, 2\mathrm{Tr})$ isomorphic?

Let G_1, G_2 be quantum graphs on $(M_2, 2\mathrm{Tr})$ with edge spaces $V_1, V_2 \subset M_2$.

Definition

G_1 and G_2 are **isomorphic** if there is a $*$-automorphism π of M_2 such that

$$2\mathrm{Tr} \circ \pi = 2\mathrm{Tr}, \quad V_2 = \pi(V_1).$$

- Classically: two graphs G_1, G_2 on vertex set $\{1, \ldots, n\}$ are isomorphic iff there exists a permutation matrix P such that $A_{G_2} = PA_{G_1}P^t$.
- in other words: $G_1 \cong G_2$ iff there is a unitary $U \in M_2$ such that $V_2 = UV_1U^*$.
- rephrasing for quantum adjacency matrices: $A_1 \circ \pi = \pi \circ A_2$.

Classification of undirected reflexive quantum graphs on $(M_2, 2\mathrm{Tr})$ (1)

Goal: Classify all undirected reflexive quantum graphs on $(M_2, 2\mathrm{Tr})$ up to isomorphism.

Let $V_1, V_2 \subset M_2$ be the edge spaces of two undirected reflexive quantum graphs on $(M_2, 2\mathrm{Tr})$, i.e.

$$V_i = V_i^*, \quad \text{and} \quad I \in V_i.$$

Then V_i is completely determined by $\tilde{V}_i := V_i \cap \{\text{self-adjoint matrices}\}$.

In the Pauli basis we can express

$$\tilde{V}_i = \{x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3 \mid (x_1, x_2, x_3) \in S_i \subset \mathbb{R}^3\}$$

for some real space $S_i \subset \mathbb{R}^3$.

We have

$$\exists U \in \mathrm{SU}(2): \; V_1 = UV_2U^* \quad \Leftrightarrow \quad \exists R \in \mathrm{SO}(3): \; S_2 = RS_1.$$

Need to classify all subspaces $S \subset \mathbb{R}^3$ up to rotations!

Goal: Classify all undirected reflexive quantum graphs on $(M_2, 2\mathrm{Tr})$ up to isomorphism.

Need to classify all subspaces $S \subset \mathbb{R}^3$ up to rotations!

However, this is easy: Any two subspaces of $\mathbb{R}^3$ of the same dimension are equivalent up to a rotation!

Theorem (Matsuda, Gromada (2022))

There are exactly four undirected reflexive quantum graphs on $(M_2, 2\mathrm{Tr})$ up to isomorphism.

Their quantum adjacency matrices are

$$A_1 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 2 & & & \\ & 0 & & \\ & & 0 & \\ & & & 2 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 1 & & & 2 \\ & 1 & & \\ & & 1 & \\ 2 & & & 1 \end{pmatrix}, \qquad A_4 = \begin{pmatrix} 2 & & & 2 \\ & 0 & 0 & \\ & 0 & 0 & \\ 2 & & & 2 \end{pmatrix}.$$

Let's look at the calculation of A_2:

$$V_2 := \operatorname{span}\left\{ I, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

The matrices I and σ_z form an ONB with respect to $\langle \cdot, \rangle_{(2\mathrm{Tr})^{-1}}$.

Recall

$$A := \sum_{i,k,\ell} |x_i e_{k\ell}\rangle \langle x_i^* e_{\ell k}|_{\mathrm{Tr}} : M_2 \to M_2.$$

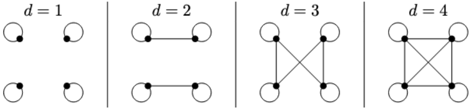
Thus,

$$\begin{aligned} A_2 &= \sum_{k,\ell} |I e_{k\ell}\rangle \langle I\, e_{\ell k}| + \sum_{k,\ell} |\sigma_x e_{k\ell}\rangle \langle \sigma_z^* e_{\ell k}| \\ &= \sum_{k,\ell} |e_{k\ell}\rangle \langle e_{\ell k}| + |e_{11}\rangle \langle e_{11}| + |e_{22}\rangle \langle e_{22}| \\ &\quad - |e_{12}\rangle \langle e_{21}| - |e_{21}\rangle \langle e_{12}| \\ &= I \otimes I + \sigma_z \otimes \sigma_z \end{aligned}$$

Quantum isomorphic classical graphs

Theorem (Matsuda (2022))

The above quantum graphs are quantum isomorphic to the following classical graphs:



Definition

Two quantum graphs A_1, A_2 on $(B_1, \psi_1), (B_2, \psi_2)$ are quantum isomorphic if there is a *-homomorphism

$$\pi : L^2(B_1) \to L^2(B_2) \otimes B(\mathcal{H})$$

such that

$$\tilde{\pi} : \mathcal{H} \otimes L^2(B_1) \to L^2(B_2) \otimes \mathcal{H}, \quad \xi \otimes b_1 \mapsto (\mathrm{Id} \otimes \xi^\dagger)\pi(b_1)$$

is a unitary map, **and**

$$\pi \circ A_1 = A_2 \circ \pi.$$

- Brannan et al. (2020): Let $\mathcal{O}(G_1, G_2)$ be the universal *-algebra generated by the entries of a matrix $p = (p_{ij})$ such that

$$\pi : b_j^{(1)} \mapsto \sum_i b_i^{(2)} \otimes p_{ij}$$

  is as above.
- Brannan et al. (2020): Quantum isomorphism entails that the QAut-groups are monoidally equivalent.

Idea of Matsuda's proof.

The classical graphs are found by comparing spectra. Then, $\mathcal{O}(\cdot,\cdot)$ can be described as universal C*-algebra, and one can find a non-trivial representation. □

Matsuda's classification of non-tracial quantum graphs on M_2

$2\mathrm{Tr}$ is the unique *tracial* quantum set functional on M_2, i.e. satisfying $mm^* = \mathrm{Id}$.

The non-tracial quantum set functionals on M_2 are

$$\psi_q : M_2 \to \mathbb{C}, \quad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto (1+q^2)(q^{-2}a_{11} + a_{22}), \quad q \in (0,1].$$

Matsuda (2022) classifies all undirected reflexive quantum graphs on (M_2, ψ_q) up to isomorphism:

Theorem (Matsuda (2022))

There are exactly four undirected reflexive quantum graphs on (M_2, ψ_q) up to isomorphism. Their quantum adjacency matrices are with $\delta := q^{-1} + q$

$$A_1^q = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \qquad A_2^q = \begin{pmatrix} q^{-1}\delta & & & \\ & 0 & & \\ & & 0 & \\ & & & q\delta \end{pmatrix},$$

$$A_3^q = \begin{pmatrix} 1 & & & \delta \\ & 1 & & \\ & & 1 & \\ \delta & & & 1 \end{pmatrix}, \qquad A_4^q = \begin{pmatrix} q^{-1}\delta & & & \delta \\ & 0 & 0 & \\ & 0 & 0 & \\ \delta & & & q\delta \end{pmatrix}.$$

II.2 Directed Tracial Quantum Graphs on M_2

Classification problem for directed quantum graphs on $(M_2, 2\mathrm{Tr})$

Goal: Classify all directed loopfree quantum graphs on $(M_2, 2\mathrm{Tr})$ up to isomorphism.

As before, we need to classify all subspaces $V \subset M_2$ up to conjugation by unitaries,

or equivalently, all subspaces $S \subset \mathbb{C}^3$ up to a rotation from $\mathrm{SO}(3)$.

Not any two *complex* subspaces of the same dimension are equivalent up to a real rotation.

Lemma

For all one-dimensional $V \subset \mathbb{C}^3$ there is a unique $\beta \in [0,1]$ such that

$$V \cong R\,\mathrm{span}\left\{\begin{pmatrix}1\\ i\beta\\ 0\end{pmatrix}\right\}$$

for some $R \in \mathrm{SO}(3)$.

Proof idea: Let $V = \mathrm{span}\left\{\begin{pmatrix}a\\ b\\ c\end{pmatrix} + i\begin{pmatrix}a'\\ b'\\ c'\end{pmatrix}\right\}$ for $a,\dots,c' \in \mathbb{R}$.

By applying a suitable rotation we can achieve

$$RV = \mathrm{span}\left\{\begin{pmatrix}a\\ b\\ c\end{pmatrix} + i\begin{pmatrix}a'\\ 0\\ 0\end{pmatrix}\right\},$$

and

$$R'V = \mathrm{span}\left\{\begin{pmatrix}a\\ b\\ 0\end{pmatrix} + i\begin{pmatrix}a'\\ 0\\ 0\end{pmatrix}\right\} = \mathrm{span}\left\{\begin{pmatrix}x\\ y\\ 0\end{pmatrix}\right\}.$$

The number $\frac{x}{y} \in \hat{\mathbb{C}}$ completely determines the plane. After using a Möbius transformation the action of $\mathrm{O}(2)$ looks as follows:

- spin around the axis through 0 and ∞
- swap upper and lower half sphere

Thus, up to this action the plane is completely determined by the latitude in the upper half sphere.

More precisely, for every V there is a unique $\beta \in [0,1]$ such that

$$V \cong \mathrm{span}\left\{\begin{pmatrix}1\\ i\beta\\ 0\end{pmatrix}\right\}.$$

Lemma

For all one-dimensional $V \subset \mathbb{C}^3$ there is a unique $\beta \in [0,1]$ such that

$$V \cong R\,\mathrm{span}\left\{ \begin{pmatrix} 1 \\ i\beta \\ 0 \end{pmatrix} \right\}$$

for some $R \in \mathrm{SO}(3)$.

Theorem (Kiefer-S. (2025))

Any loopfree quantum graph on $(M_2, 2\mathrm{Tr})$ with one edge is isomorphic to one of

$$A^{(1B)}_{0,\beta} = \frac{1}{1+\beta^2} \begin{pmatrix} 0 & 0 & 0 & (1+\beta)^2 \\ 0 & 0 & 1-\beta^2 & 0 \\ 0 & 1-\beta^2 & 0 & 0 \\ (1-\beta)^2 & 0 & 0 & 0 \end{pmatrix}$$

for $\beta \in [0,1]$.

Proof.

There is a unique $\beta \in [0,1]$ such that

$$V = \mathrm{span}\left\{\sigma_x + i\beta\sigma_y\right\} = \mathrm{span}\left\{ \begin{pmatrix} 0 & 1+\beta \\ 1-\beta & 0 \end{pmatrix} \right\}$$

up to isomorphism.
This matrix has

$$\|T\|^2_{(2\mathrm{Tr})^{-1},KMS} = 1+\beta^2$$

and the quantum adjacency matrix is (represented as an element of in $M_2 \otimes M_2$)

$$\frac{1}{\|T\|^2_{\dots}} \begin{pmatrix} 0 & 1+\beta \\ 1-\beta & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1-\beta \\ 1+\beta & 0 \end{pmatrix}.$$

$\square$

Directed quantum graphs on $(M_2, 2\mathrm{Tr})$ (final remarks)

- There are $[0,1]$-many loopfree quantum graphs on $(M_2, 2\mathrm{Tr})$ with one edge.
- Loopfree quantum graphs with two or three edges are obtained as complements of the one-edge graphs.
- On non-tracial quantum sets (M_2, ψ_q) there are even more directed quantum graphs.
  - Some of them are KMS-undirected but not GNS-undirected.

Further questions:

- Classify quantum graphs on M_3.
- What is the smallest quantum graph that is not quantum-isomorphic to a classical graph?

Thank you!