Modified Four Tank System
Nonlinear State Estimation

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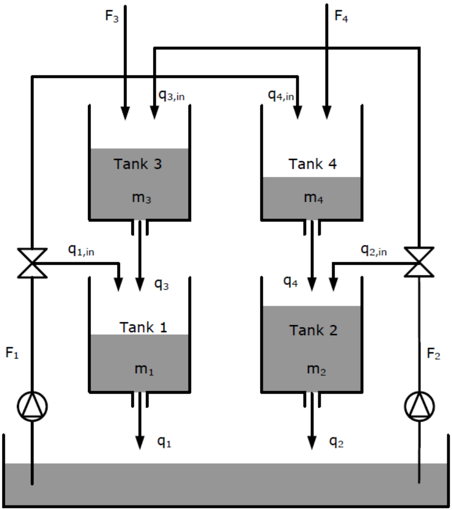
Overview

Illustration: Modified Four Tank System



Nonlinear Model - Simulation and Control Models

- **Simulation Model:**
  The simulation of the modified four tank system (MFTS) is governed by a stochastic differential equation (SDE) model

$$\begin{aligned} d\boldsymbol{x}_S(t) &= f_S(t, \boldsymbol{x}_S(t), u(t), d_S(t), \theta_S)dt + \sigma_S(t, \boldsymbol{x}_S(t), u(t), d_S(t), \theta_S)d\boldsymbol{\omega}_S(t), \\ \boldsymbol{y}(t_k) &= g_S(t, \boldsymbol{x}_S(t), u(t), d_S(t), \theta_S) + \boldsymbol{v}_S(t_k), \end{aligned}$$

  where the process noise $\boldsymbol{\omega}_S(t)$ denotes a standard Wiener process and $\boldsymbol{v}_S(t_k) \sim \mathcal{N}(0, R_S)$ denotes the normally distributed measurement noise.

- **Control Model:**
  The control model for the MFTS is similarly described by an SDE model

$$\begin{aligned} d\boldsymbol{x}(t) &= f(t, \boldsymbol{x}(t), u(t), d(t), \theta)dt + \sigma(t, \boldsymbol{x}(t), u(t), d(t), \theta)d\boldsymbol{\omega}(t), \\ \boldsymbol{y}_C(t_k) &= g(t, \boldsymbol{x}(t), u(t), d(t), \theta) + \boldsymbol{v}(t_k), \end{aligned}$$

  where the process noise $\boldsymbol{\omega}(t)$ denotes a standard Wiener process and $\boldsymbol{v}(t_k) \sim \mathcal{N}(0, R)$ denotes the normally distributed measurement noise.

Simulation Model - Drift

The drift term is computed as

$$f_S(t, \mathbf{x}_S, u(t), d_S(t), \theta_S) = \begin{bmatrix} f_1(t, \mathbf{x}_S, u(t), d_S(t), \theta_S) \\ f_2(t, \mathbf{x}_S, u(t), d_S(t), \theta_S) \\ f_3(t, \mathbf{x}_S, u(t), d_S(t), \theta_S) \\ f_4(t, \mathbf{x}_S, u(t), d_S(t), \theta_S) \end{bmatrix} = \begin{bmatrix} \rho\left(q_{in,1}(t) + q_3(t) - q_1(t)\right) \\ \rho\left(q_{in,2}(t) + q_4(t) - q_2(t)\right) \\ \rho\left(q_{in,3}(t) - q_3(t)\right) \\ \rho\left(q_{in,4}(t) - q_4(t)\right) \end{bmatrix},$$

where

$$\begin{bmatrix} q_{in,1}(t) \\ q_{in,2}(t) \\ q_{in,3}(t) \\ q_{in,4}(t) \end{bmatrix} = \begin{bmatrix} \gamma_1 F_1(t) \\ \gamma_2 F_2(t) \\ (1-\gamma_2)\, F_2(t) + F_3(t) \\ (1-\gamma_1)\, F_1(t) + F_4(t) \end{bmatrix}, \qquad \begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \\ q_4(t) \end{bmatrix} = \begin{bmatrix} a_1\sqrt{2gH_1(t)} \\ a_2\sqrt{2gH_2(t)} \\ a_3\sqrt{2gH_3(t)} \\ a_4\sqrt{2gH_4(t)} \end{bmatrix},$$

where $H_i(t) = m_i(t)/(\rho A_i)$ for $i \in \{1,2,3,4\}$, a_i and A_i for $i \in \{1,2,3,4\}$ are the cross-sectional areas of the drainage pipes and cyllindrical tanks respectively, and γ_i for $i \in \{1,2\}$ are the split-values of the in-flow pipes.

Simulation Model - Diffusion

The process noise will be added to the disturbances of the system, such that the in-flows, $q_{in}(t)$, are described by

$$\begin{bmatrix} q_{in,1}(t) \\ q_{in,2}(t) \\ q_{in,3}(t) \\ q_{in,4}(t) \end{bmatrix} dt = \begin{bmatrix} \gamma_1 F_1(t) \\ \gamma_2 F_2(t) \\ (1-\gamma_2)\, F_2(t) \\ (1-\gamma_1)\, F_1(t) \end{bmatrix} dt + \begin{bmatrix} 0 \\ 0 \\ F_3(t) \\ F_4(t) \end{bmatrix} dt + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \rho\sigma_1 & 0 \\ 0 & \rho\sigma_2 \end{bmatrix} \begin{bmatrix} d\boldsymbol{\omega}_1(t) \\ d\boldsymbol{\omega}_2(t) \end{bmatrix},$$

which leads to the diffusion model

$$\sigma_S(t, \boldsymbol{x}_S(t), u(t), d_S(t), \theta_S) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \rho\sigma_1 & 0 \\ 0 & \rho\sigma_2 \end{bmatrix}.$$

Simulation Model - Full Model

The full simulation model thus becomes

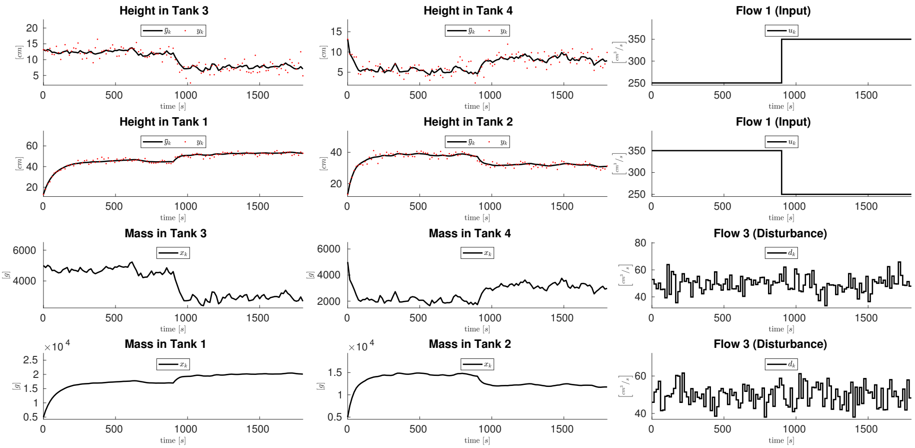
$$
\begin{aligned}
d\boldsymbol{x}_S(t) &= f_S(t, \boldsymbol{x}_S, u(t), d_S(t), \theta_S)dt + \sigma_S(t, \boldsymbol{x}_S(t), u(t), d_S(t), \theta_S)d\boldsymbol{\omega}_S(t) \\
&= \begin{bmatrix} \rho\left(q_{in,1}(t) + q_3(t) - q_1(t)\right) \\ \rho\left(q_{in,2}(t) + q_4(t) - q_2(t)\right) \\ \rho\left(q_{in,3}(t) - q_3(t)\right) \\ \rho\left(q_{in,4}(t) - q_4(t)\right) \end{bmatrix} dt + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \rho\sigma_1 & 0 \\ 0 & \rho\sigma_2 \end{bmatrix} d\boldsymbol{\omega}_S(t)
\end{aligned}
$$

where

$$
\begin{bmatrix} q_{in,1}(t) \\ q_{in,2}(t) \\ q_{in,3}(t) \\ q_{in,4}(t) \end{bmatrix} = \begin{bmatrix} \gamma_1 F_1(t) \\ \gamma_2 F_2(t) \\ (1-\gamma_2)\, F_2(t) + F_3(t) \\ (1-\gamma_1)\, F_1(t) + F_4(t) \end{bmatrix}, \qquad \begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \\ q_4(t) \end{bmatrix} = \begin{bmatrix} a_1\sqrt{2gH_1(t)} \\ a_2\sqrt{2gH_2(t)} \\ a_3\sqrt{2gH_3(t)} \\ a_4\sqrt{2gH_4(t)} \end{bmatrix},
$$

where $H_i(t) = m_i(t)/(\rho A_i)$ for $i \in \{1, 2, 3, 4\}$, a_i and A_i for $i \in \{1, 2, 3, 4\}$ are the cross-sectional areas of the drainage pipes and cyllindrical tanks respectively, and γ_i for $i \in \{1, 2\}$ are the split-values of the in-flow pipes.

Closed-loop Simulation



Control Model - Drift

In the control model, we augment the state vector with the disturbances for the state estimation algorithm to reject them

$$f(t, \boldsymbol{x}, u(t), d(t), \theta) = \begin{bmatrix} f_1(t, \boldsymbol{x}, u(t), d(t), \theta) \\ f_2(t, \boldsymbol{x}, u(t), d(t), \theta) \\ f_3(t, \boldsymbol{x}, u(t), d(t), \theta) \\ f_4(t, \boldsymbol{x}, u(t), d(t), \theta) \\ f_5(t, \boldsymbol{x}, u(t), d(t), \theta) \\ f_6(t, \boldsymbol{x}, u(t), d(t), \theta) \end{bmatrix} = \begin{bmatrix} \rho \left(q_{in,1}(t) + q_3(t) - q_1(t) \right) \\ \rho \left(q_{in,2}(t) + q_4(t) - q_2(t) \right) \\ \rho \left(q_{in,3}(t) - q_3(t) \right) \\ \rho \left(q_{in,4}(t) - q_4(t) \right) \\ \lambda_1 \left(\bar{F}_3(t) - \boldsymbol{F}_3(t) \right) \\ \lambda_2 \left(\bar{F}_4(t) - \boldsymbol{F}_4(t) \right) \end{bmatrix},$$

where

$$\begin{bmatrix} q_{in,1}(t) \\ q_{in,2}(t) \\ q_{in,3}(t) \\ q_{in,4}(t) \end{bmatrix} = \begin{bmatrix} \gamma_1 F_1(t) \\ \gamma_2 F_2(t) \\ (1-\gamma_2)\, F_2(t) + \boldsymbol{F}_3(t) \\ (1-\gamma_1)\, F_1(t) + \boldsymbol{F}_4(t) \end{bmatrix}, \qquad \begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \\ q_4(t) \end{bmatrix} = \begin{bmatrix} a_1 \sqrt{2gH_1(t)} \\ a_2 \sqrt{2gH_2(t)} \\ a_3 \sqrt{2gH_3(t)} \\ a_4 \sqrt{2gH_4(t)} \end{bmatrix},$$

where $H_i(t) = \boldsymbol{m}_i(t)/(\rho A_i)$ for $i \in \{1, 2, 3, 4\}$, a_i and A_i for $i \in \{1, 2, 3, 4\}$ are the cross-sectional areas of the drainage pipes and cyllindrical tanks respectively, and γ_i for $i \in \{1, 2\}$ are the split-values of the in-flow pipes.

Control Model - Diffusion

In the control model, we include the disturbances as states for the state estimator to reject the disturbances. We model the disturbances as SDEs on the form

$$
\begin{aligned}
d\boldsymbol{F}_3(t) &= \lambda_1 \left(\bar{F}_3(t) - \boldsymbol{F}_3(t)\right) dt + \sigma_1 d\boldsymbol{\omega}_1(t), \\
d\boldsymbol{F}_4(t) &= \lambda_2 \left(\bar{F}_4(t) - \boldsymbol{F}_4(t)\right) dt + \sigma_2 d\boldsymbol{\omega}_2(t).
\end{aligned}
$$

which leads to the diffusion model

$$
\sigma(t, \boldsymbol{x}(t), u(t), d(t), \theta) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}.
$$

Control Model - Full Model

The full simulation model thus becomes

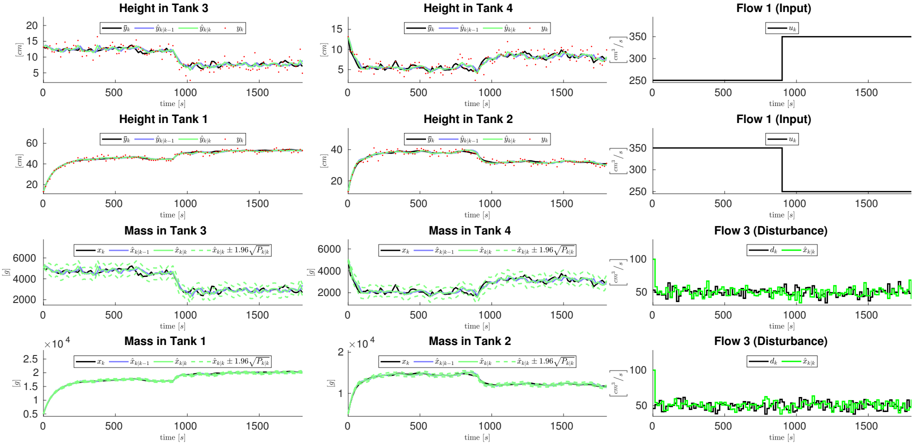
$$
\begin{aligned}
d\boldsymbol{x}(t) &= f(t, \boldsymbol{x}, u(t), d(t), \theta)dt + \sigma(t, \boldsymbol{x}(t), u(t), d(t), \theta)d\boldsymbol{\omega}(t) \\
&= \begin{bmatrix} \rho\left(q_{in,1}(t) + q_3(t) - q_1(t)\right) \\ \rho\left(q_{in,2}(t) + q_4(t) - q_2(t)\right) \\ \rho\left(q_{in,3}(t) - q_3(t)\right) \\ \rho\left(q_{in,4}(t) - q_4(t)\right) \\ \lambda_1\left(\bar{F}_3(t) - \boldsymbol{F}_3(t)\right) \\ \lambda_2\left(\bar{F}_4(t) - \boldsymbol{F}_4(t)\right) \end{bmatrix} dt + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} d\boldsymbol{\omega}(t)
\end{aligned}
$$

where

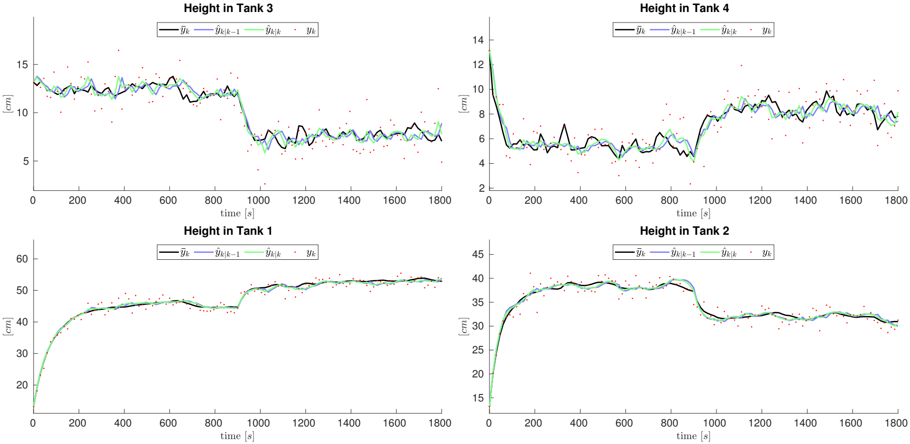
$$
\begin{bmatrix} q_{in,1}(t) \\ q_{in,2}(t) \\ q_{in,3}(t) \\ q_{in,4}(t) \end{bmatrix} = \begin{bmatrix} \gamma_1 F_1(t) \\ \gamma_2 F_2(t) \\ (1-\gamma_2)\, F_2(t) + \boldsymbol{F}_3(t) \\ (1-\gamma_1)\, F_1(t) + \boldsymbol{F}_4(t) \end{bmatrix}, \qquad \begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \\ q_4(t) \end{bmatrix} = \begin{bmatrix} a_1\sqrt{2gH_1(t)} \\ a_2\sqrt{2gH_2(t)} \\ a_3\sqrt{2gH_3(t)} \\ a_4\sqrt{2gH_4(t)} \end{bmatrix},
$$

where $H_i(t) = m_i(t)/(\rho A_i)$ for $i \in \{1,2,3,4\}$, a_i and A_i for $i \in \{1,2,3,4\}$ are the cross-sectional areas of the drainage pipes and cyllindrical tanks respectively, and γ_i for $i \in \{1,2\}$ are the split-values of the in-flow pipes.

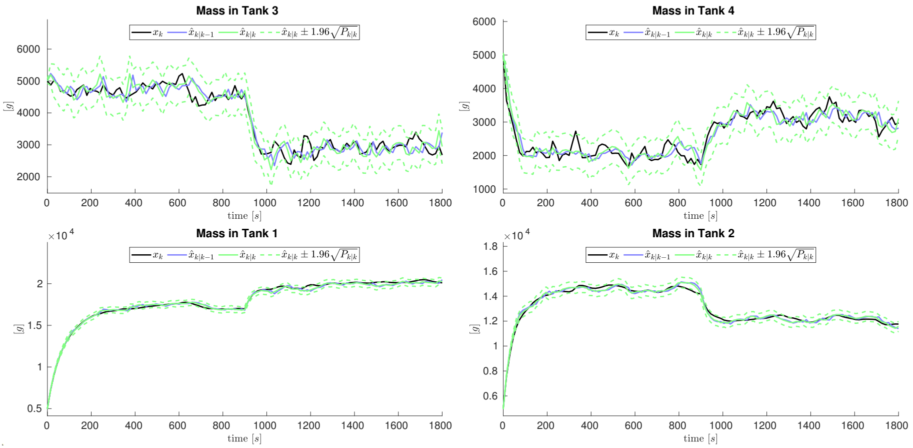
Closed-loop Simulation - State Estimation



State Estimation - Measurements



State Estimation - States



State Estimation - Inputs/Disturbances