Model Predictive Control

Lecture 5: Realization

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02619 Model Predictive Control

Learning Objectives

1. Convert a deterministic continuous-time transfer function to a discrete-time state space model

The Realization Problem

Consider the MIMO transfer function model

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} \frac{0.1133e^{-.715s}}{1.783s^2+4.48s+1} & \frac{0.9222}{2.071s+1} \\ \frac{0.3378e^{-0.299s}}{0.361s^2+1.09s+1} & \frac{-0.321e^{-0.94s}}{0.104s^2+2.463s+1} \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix}$$

Question

1. How do we simulate this model?
2. Design controllers based on this model

Answer: Convert it to a discrete-time state space model

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k + Du_k \end{aligned}$$

Question: How to compute (A, B, C, D).

SISO - Controller Canonical Realization

The SISO transfer function

$$Y(s) = G(s)U(s) \qquad G(s) = \frac{b_0 s^4 + b_1 s^3 + b_2 s^2 + b_3 s + b_4}{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4}$$

can be realized as the state-space equation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

in controller canonical form

$$A = \left[\begin{array}{ccc|c} -a_1 & -a_2 & -a_3 & -a_4 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \qquad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} b_1 - a_1 b_0 & b_2 - a_2 b_0 & b_3 - a_3 b_0 & b_4 - a_4 b_0 \end{bmatrix} \qquad D = b_0$$

SISO - Observer Canonical Realization

The SISO transfer function

$$Y(s) = G(s)U(s) \qquad G(s) = \frac{b_0 s^4 + b_1 s^3 + b_2 s^2 + b_3 s + b_4}{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4}$$

can be realized as the state-space equation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

in observer canonical form

$$A = \left[\begin{array}{c|ccc} -a_1 & 1 & 0 & 0 \\ -a_2 & 0 & 1 & 0 \\ -a_3 & 0 & 0 & 1 \\ \hline -a_4 & 0 & 0 & 0 \end{array}\right] \qquad B = \begin{bmatrix} b_1 - a_1 b_0 \\ b_2 - a_2 b_0 \\ b_3 - a_3 b_0 \\ b_4 - a_4 b_0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \qquad D = b_0$$

Realization of a Rational SISO Transfer Function

$$Y(s) = G(s)U(s) \qquad G(s) = \frac{\beta_0 s^4 + \beta_1 s^3 + \beta_2 s^2 + \beta_3 s + \beta_4}{\alpha_0 s^4 + \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4} \quad \alpha_0 \neq 0$$

$$\begin{aligned} G(s) &= \frac{\beta_0 s^4 + \beta_1 s^3 + \beta_2 s^2 + \beta_3 s + \beta_4}{\alpha_0 s^4 + \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4} \\ &= \frac{(\beta_0/\alpha_0) s^4 + (\beta_1/\alpha_0) s^3 + (\beta_2/\alpha_0) s^2 + (\beta_3/\alpha_0) s + (\beta_4/\alpha_0)}{s^4 + (\alpha_1/\alpha_0) s^3 + (\alpha_2/\alpha_0) s^2 + (\alpha_3/\alpha_0) s + (\alpha_4/\alpha_0)} \\ &= \frac{b_0 s^4 + b_1 s^3 + b_2 s^2 + b_3 s + b_4}{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4} \end{aligned}$$

$$\begin{aligned} a_i &= \frac{\alpha_i}{\alpha_0} \qquad i = 1, 2, 3, 4 \\ b_i &= \frac{\beta_i}{\alpha_0} \qquad i = 0, 1, 2, 3, 4 \end{aligned}$$

Discretization

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad x(t_0) = x_0$$

Solution

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

Zero-order-hold

$$u(t) = u_k \qquad t_k \leq t < t_{k+1} = t_k + T_s$$

Let $x(t_k) = x_k$. Then the solution $x(t_{k+1})$ at t_{k+1} can be expressed as

$$\begin{aligned} x_{k+1} = x(t_{k+1}) &= e^{A(t_{k+1}-t_k)}x_k + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)}Bu(\tau)d\tau \\ &= \left[e^{AT_s}\right] x_k + \left[\int_0^{T_s} e^{As}dsB\right] u_k \end{aligned}$$

Discretization

$$\dot{x}(t) = A_c x(t) + B_c u(t) \qquad x(t_0) = x_0$$
$$y(t) = C_c x(t) + D_c u(t)$$

$$x_{k+1} = A x_k + B u_k$$
$$y_k = C x_k + D u_k$$

$$A = \exp(A_c T_s)$$
$$B = \int_0^{T_s} \exp(A_c s) ds B_c$$
$$C = C_c$$
$$D = D_c$$

$$
\begin{aligned}
A &= \exp(A_c T_s) \\
B &= \int_0^{T_s} \exp(A_c s) ds B_c
\end{aligned}
$$

$$
\begin{bmatrix} A & B \\ 0 & I \end{bmatrix} = \exp\left(\begin{bmatrix} A_c & B_c \\ 0 & 0 \end{bmatrix} T_s\right)
$$

SISO - Realization Procedure

$$Y(s) = G(s)U(s) \qquad G(s) = \frac{\beta_0 s^4 + \beta_1 s^3 + \beta_2 s^2 + \beta_3 s + \beta_4}{\alpha_0 s^4 + \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4} \quad \alpha_0 \neq 0$$

1. Determine state dimension: $n = 4$
2. Convert to standard transfer function $G(s) = \frac{b_0 s^4 + b_1 s^3 + b_2 s^2 + b_3 s + b_4}{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4}$

$$a_i = \frac{\alpha_i}{\alpha_0} \quad i = 1, 2, 3, 4 \qquad b_i = \frac{\beta_i}{\alpha_0} \quad i = 0, 1, 2, 3, 4$$

3. Continuous-time realization

$$A_c = \left[\begin{array}{c|ccc} -a_1 & 1 & 0 & 0 \\ -a_2 & 0 & 1 & 0 \\ -a_3 & 0 & 0 & 1 \\ \hline -a_4 & 0 & 0 & 0 \end{array}\right] \qquad B_c = \begin{bmatrix} b_1 - a_1 b_0 \\ b_2 - a_2 b_0 \\ b_3 - a_3 b_0 \\ b_4 - a_4 b_0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \qquad D = b_0$$

4. Discretization with sampling time T_s and zero-order-hold

$$\begin{bmatrix} A & B \\ 0 & I \end{bmatrix} = \exp\left(\begin{bmatrix} A_c & B_c \\ 0 & 0 \end{bmatrix} T_s \right)$$

$$\begin{aligned} x_{k+1} &= A x_k + B u_k \\ y_k &= C x_k + D u_k \end{aligned}$$

The SISO transfer function

$$Y(s) = G(s)U(s) \qquad G(s) = \frac{b_0 s^4 + b_1 s^3 + b_2 s^2 + b_3 s + b_4}{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4} e^{-\lambda s}$$

can be realized as the delayed state-space equation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t-\lambda) \\ y(t) &= Cx(t) + Du(t-\lambda) \end{aligned}$$

in observer canonical form

$$A = \left[\begin{array}{c|ccc} -a_1 & 1 & 0 & 0 \\ -a_2 & 0 & 1 & 0 \\ -a_3 & 0 & 0 & 1 \\ \hline -a_4 & 0 & 0 & 0 \end{array}\right] \qquad B = \begin{bmatrix} b_1 - a_1 b_0 \\ b_2 - a_2 b_0 \\ b_3 - a_3 b_0 \\ b_4 - a_4 b_0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \qquad D = b_0$$

Delay Differential Equation

$$\dot{x}(t) = Ax(t) + Bu(t - \lambda) \qquad x(t_0) = x_0$$

has the solution

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau - \lambda)d\tau$$

which can also be expressed as

$$\begin{aligned} x_{k+1} = x(t_{k+1}) &= e^{A(t_{k+1}-t_k)}x_k + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)}Bu(\tau - \lambda)d\tau \\ &= e^{AT_s}x_k + \int_0^{T_s} e^{A\eta}Bu(t_{k+1} - \lambda - \eta)d\eta \end{aligned}$$

with $t_k = kT_s$

Let the time delay be expressed as

$$\lambda = lT_s - mT_s \qquad l \in \mathbb{N}_0 = \{0, 1, 2, \dots\} \quad 0 \leq m < 1$$

Zero-order-hold approximation

$$u(t) = u_k \qquad t_k \leq t < t_{k+1}$$

Then

$$\begin{aligned} x_{k+1} &= e^{AT_s}x_k + \int_0^{T_s} e^{A\eta}Bu(t_{k+1} - \lambda - \eta)d\eta \\ &= e^{AT_s}x_k + \int_0^{T_s} e^{A\eta}Bu(t_{k+1} - lT_s + mT_s - \eta)d\eta \\ &= e^{AT_s}x_k + \int_0^{mT_s} e^{A\eta}B\overbrace{u(t_{k+1} - lT_s + mT_s - \eta)}^{=u_{k+1-l}}d\eta \\ &\quad + \int_{mT_s}^{T_s} e^{A\eta}B\overbrace{u(t_{k+1} - lT_s + mT_s - \eta)}^{=u_{k-l}}d\eta \end{aligned}$$

$$
\begin{aligned}
x_{k+1} &= e^{AT_s}x_k + \int_0^{mT_s} e^{A\eta}B\overbrace{u(t_{k+1}-lT_s+mT_s-\eta)}^{=u_{k+1-l}}d\eta \\
&\quad + \int_{mT_s}^{T_s} e^{A\eta}B\overbrace{u(t_{k+1}-lT_s+mT_s-\eta)}^{=u_{k-l}}d\eta \\
&= \underbrace{\left[e^{AT_s}\right]}_{=\Phi}x_k + \underbrace{\left[\int_0^{mT_s} e^{A\eta}Bd\eta\right]}_{=\Gamma_2}u_{k+1-l} + \underbrace{\left[\int_{mT_s}^{T_s} e^{A\eta}Bd\eta\right]}_{=\Gamma_1}u_{k-l} \\
&= \Phi x_k + \Gamma_1 u_{k-l} + \Gamma_2 u_{k+1-l}
\end{aligned}
$$

$$
\begin{aligned}
\Phi &= e^{AT_s} \\
\Gamma_1 &= \int_{mT_s}^{T_s} e^{A\eta}Bd\eta \\
\Gamma_2 &= \int_0^{mT_s} e^{A\eta}Bd\eta
\end{aligned}
$$

$$x_{k+1} = \Phi x_k + \Gamma_1 u_{k-l} + \Gamma_2 u_{k+1-l}$$

$$\Phi = e^{AT_s}$$
$$\Gamma_1 = \int_{mT_s}^{T_s} e^{A\eta} B d\eta$$
$$\Gamma_2 = \int_0^{mT_s} e^{A\eta} B d\eta$$

$$\lambda = lT_s - mT_s \qquad l \in \mathbb{N}_0 \quad 0 \leq m < 1$$

Table: Cases

	$m = 0$	$0 < m < 1$
$l = 0$	I	
$l = 1$	II	IV
$l > 1$	III	V

Case I-III: $m = 0$

The delay is

$$\lambda = lT_s - mT_s = lT_s \qquad l \in \mathbb{N}_0$$

is an integer multiple of the sampling time and

$$\begin{aligned}
\Phi &= e^{AT_s} \\
\Gamma_1 &= \int_{mT_s}^{T_s} e^{A\eta} B d\eta = \int_0^{T_s} e^{A\eta} B d\eta \\
\Gamma_2 &= \int_0^{mT_s} e^{A\eta} B d\eta = \int_0^0 e^{A\eta} B d\eta = 0
\end{aligned}$$

Then

$$\begin{aligned}
x_{k+1} &= \Phi x_k + \Gamma_1 u_{k-l} + \Gamma_2 u_{k+1-l} \\
&= \Phi x_k + \Gamma_1 u_{k-l}
\end{aligned}$$

with

$$\begin{bmatrix} \Phi & \Gamma_1 \\ 0 & I \end{bmatrix} = \exp\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} T_s \right)$$

Case I: $m = 0$, $l = 0$ (no delay)

$$x_{k+1} = \Phi x_k + \Gamma_1 u_{k-l} = \Phi x_k + \Gamma_1 u_k$$

with

$$\begin{bmatrix} \Phi & \Gamma_1 \\ 0 & I \end{bmatrix} = \exp\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} T_s \right)$$

Discrete-time state space realization

$$\begin{aligned} x_{k+1} &= \Phi x_k + \Gamma_1 u_k \\ y_k &= C x_k + D u_k \end{aligned}$$

Case II: $m = 0$, $l = 1$ (Delay of one sample time)

$$x_{k+1} = \Phi x_k + \Gamma_1 u_{k-l} = \Phi x_k + \Gamma_1 u_{k-1}$$

with

$$\begin{bmatrix} \Phi & \Gamma_1 \\ 0 & I \end{bmatrix} = \exp\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} T_s \right)$$

Let

$$w_1(k) = u(k-1) \qquad w_1(k+1) = u(k)$$

Then

$$\begin{aligned} \begin{bmatrix} x \\ w_1 \end{bmatrix}_{k+1} &= \begin{bmatrix} \Phi & \Gamma_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ w_1 \end{bmatrix}_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k \\ y_k &= C x_k + D u_{k-1} \\ &= \begin{bmatrix} C & D \end{bmatrix} \begin{bmatrix} x \\ w_1 \end{bmatrix}_k + 0 u_k \end{aligned}$$

Case III: $m = 0$, $l > 1$ (integer delay)

$$x_{k+1} = \Phi x_k + \Gamma_1 u_{k-l}$$

with

$$\begin{bmatrix} \Phi & \Gamma_1 \\ 0 & I \end{bmatrix} = \exp\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} T_s\right)$$

Example: $l = 4$

$$\begin{aligned}
w_1(k) &= u(k-4) \qquad & w_1(k+1) &= u_(k-3) = w_2(k) \\
w_2(k) &= u(k-3) \qquad & w_2(k+1) &= u(k-2) = w_3(k) \\
w_3(k) &= u(k-2) \qquad & w_3(k+1) &= u(k-1) = w_3(k) \\
w_4(k) &= u(k-1) \qquad & w_4(k+1) &= u(k)
\end{aligned}$$

Case III: $m = 0$, $l > 1$ (integer delay)

Example: $l = 4$

$$x_{k+1} = \Phi x_k + \Gamma_1 u_{k-l}$$

$$\begin{bmatrix} x \\ w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}_{k+1} = \begin{bmatrix} \Phi & \Gamma_1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}_k + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_k$$

$$\begin{aligned} y_k &= C x_k + D u_{k-l} \\ &= \begin{bmatrix} C & D & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}_k + 0 u_k \end{aligned}$$

Case IV-V: $0 < m < 1$

$$\Phi = e^{AT_s}$$

$$\Gamma_1 = \int_{mT_s}^{T_s} e^{A\eta} B d\eta$$

$$\Gamma_2 = \int_0^{mT_s} e^{A\eta} B d\eta$$

Observe

$$\Phi = e^{AT_s} = \underbrace{e^{A(1-m)T_s}}_{=\Phi_1} \underbrace{e^{AmT_s}}_{=\Phi_2} = \Phi_1 \Phi_2$$

$$\Gamma_1 = \int_{mT_s}^{T_s} e^{A\eta} B d\eta = \int_0^{T_s - mT_s} e^{A(mT_s + \sigma)} B d\sigma$$

$$= \underbrace{e^{AmT_s}}_{\Phi_2} \underbrace{\int_0^{(1-m)T_s} e^{A\sigma} B d\sigma}_{=\tilde{\Gamma}_1} = \Phi_2 \tilde{\Gamma}_1$$

$$\Phi_1 = e^{A(1-m)T_s} \qquad \tilde{\Gamma}_1 = \int_0^{(1-m)T_s} e^{A\eta} B d\eta$$

$$\Phi_2 = e^{AmT_s} \qquad \Gamma_2 = \int_0^{mT_s} e^{A\eta} B d\eta$$

1. Compute
$$\begin{bmatrix} \Phi_1 & \tilde{\Gamma}_1 \\ 0 & I \end{bmatrix} = \exp\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} (1-m)T_s \right)$$
2. Compute
$$\begin{bmatrix} \Phi_2 & \Gamma_2 \\ 0 & I \end{bmatrix} = \exp\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} mT_s \right)$$
3. Compute
$$\Phi = \Phi_1 \Phi_2$$
$$\Gamma_1 = \Phi_2 \tilde{\Gamma}_1$$

$$x_{k+1} = \Phi x_k + \Gamma_1 u_{k-l} + \Gamma_2 u_{k-l+1}$$

Case IV: $0 < m < 1$, $l = 1$ (Delay less than 1 sample time)

$$x_{k+1} = \Phi x_k + \Gamma_1 u_{k-1} + \Gamma_2 u_k$$

$$w_1(k) = u(k-1) \qquad w_1(k+1) = u(k)$$

$$\begin{bmatrix} x \\ w_1 \end{bmatrix}_{k+1} = \begin{bmatrix} \Phi & \Gamma_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ w_1 \end{bmatrix}_k + \begin{bmatrix} \Gamma_2 \\ 1 \end{bmatrix} u_k$$

$$\begin{aligned} y_k &= C x_k + D u_{k-1} \\ &= \begin{bmatrix} C & D \end{bmatrix} \begin{bmatrix} x \\ w_1 \end{bmatrix}_k + 0 u_k \end{aligned}$$

Case V: $0 < m < 1$, $l > 1$

$$x_{k+1} = \Phi x_k + \Gamma_1 u_{k-l} + \Gamma_2 u_{k-l+1}$$

Example: $l = 4$

$$\begin{aligned}
w_1(k) &= u(k-4) & \qquad w_1(k+1) &= u(k-3) = w_2(k) \\
w_2(k) &= u(k-3) & \qquad w_2(k+1) &= u(k-2) = w_3(k) \\
w_3(k) &= u(k-2) & \qquad w_3(k+1) &= u(k-1) = w_4(k) \\
w_4(k) &= u(k-1) & \qquad w_4(k+1) &= u(k)
\end{aligned}$$

$$\begin{bmatrix} x \\ w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}_{k+1} = \begin{bmatrix} \Phi & \Gamma_1 & \Gamma_2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}_k + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_k$$

Case V: $0 < m < 1$, $l > 1$

$$
\begin{aligned}
x_{k+1} &= \Phi x_k + \Gamma_1 u_{k-l} + \Gamma_2 u_{k-l+1} \\
y_k &= C x_k + D u_{k-l}
\end{aligned}
$$

$$
\begin{bmatrix} x \\ w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}_{k+1} = \begin{bmatrix} \Phi & \Gamma_1 & \Gamma_2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}_k + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_k
$$

$$
y_k = \begin{bmatrix} C & D & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}_k + 0 u_k
$$

SISO system with delay - Realization Procedure

$$Y(s) = G(s)U(s) \quad G(s) = \frac{\beta_0 s^4 + \beta_1 s^3 + \beta_2 s^2 + \beta_1 s + \beta_0}{\alpha_0 s^4 + \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4} e^{-\lambda s}$$

1. Determine the order $(n = 4)$
2. Convert the rational part of the transfer function to the standard form: $G(s) = \frac{b_0 s^4 + b_1 s^3 + b_2 s^2 + b_3 s + b_4}{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4}$
3. Continuous-time realization of delay differential equation in observer canonical form: (A_c, B_c, C_c, D_c)
4. Determine $l \in \mathbb{N}_0$ and $0 \leq m < 1$ such that $\lambda = lT_s - mT_s$.
5. Discretization by computation of Φ, Γ_1, and Γ_2
6. Realization of the discrete-time system according to the five cases: (A, B, C, D)

$$x_{k+1} = Ax_k + Bu_k$$
$$y_k = Cx_k + Du_k$$

Computation of Impulse Response Coefficients

The following procedure can be used to compute the impulse response matrices for a MIMO system given a MIMO transfer function. For all input-output pairs do the following

1. Realize each SISO system as a discrete-time state-space system using the procedure just described (A, B, C, D).
2. Compute the impulse response coefficients for this SISO system

$$(H_0)_{i,j} = D$$
$$(H_k)_{i,j} = CA^{k-1}B \qquad k = 1, 2, \ldots$$

Let H_k be the MIMO impulse response coefficients. For a 2×2 system these matrices would be

$$H_k = \begin{bmatrix} (H_k)_{11} & (H_k)_{12} \\ (H_k)_{21} & (H_k)_{22} \end{bmatrix} \qquad k = 0, 1, 2, \dots$$

Define the Hankel matrix as

$$\mathcal{H}_{N,N} \triangleq \begin{bmatrix} H_1 & H_2 & \dots & H_N \\ H_2 & H_3 & \dots & H_{N+1} \\ \vdots & \vdots & & \vdots \\ H_N & H_{N+1} & \dots & H_{2N-1} \end{bmatrix}$$

The impulse response $\{H_i\}_{i=1}^{\infty}$ has a minimal realization $\Sigma_n(A, B, C, D)$ of order n if and only if

$$\text{rank}\{\mathcal{H}_{n+1,j}\} = \text{rank}\{\mathcal{H}_{n,j}\} = n \qquad j = n, n+1, \dots$$

This implies that $\{H_i\}_{i=1}^{\infty}$ has a minimal realization, $\Sigma_n(A, B, C, D)$, of order n if and only if there exists an $N \geq n+1$ such that a SVD-decomposition of $\mathcal{H}_{N,N}$ yields

$$\mathcal{H}_{N,N} = K\Lambda L' = \begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} L_1 & L_2 \end{bmatrix}' = K_1 \Lambda_1 L_1'$$

in which $\Lambda_1 \in \mathbb{R}^{n \times n}$ is a diagonal matrix

$$\Lambda_1 = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

with positive elements on the diagonal, $\lambda_i > 0$, and $\lambda_{i+1} \leq \lambda_i$ for $i = 1, 2, \dots, n$. K and L are orthogonal matrices.

Extended controllability matrix

$$\mathcal{C}_N = \begin{bmatrix} B & AB & A^2B & \dots & A^{N-1}B \end{bmatrix}$$

Extended observability matrix

$$\mathcal{O}_N = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{N-1} \end{bmatrix}$$

Markov parameters (MIMO impulse response coefficients)

$$H_i = \begin{cases} D & i = 0 \\ CA^{i-1}B & i = 1, 2, \dots \end{cases}$$

$$
\mathcal{H}_{N,N} \triangleq \begin{bmatrix} H_1 & H_2 & \dots & H_N \\ H_2 & H_3 & \dots & H_{N+1} \\ \vdots & \vdots & & \vdots \\ H_N & H_{N+1} & \dots & H_{2N-1} \end{bmatrix}
$$

$$
= \begin{bmatrix} CB & CAB & \dots & CA^{N-1}B \\ CAB & CA^2B & \dots & CA^N B \\ \vdots & \vdots & & \vdots \\ CA^{N-1}B & CA^N B & \dots & CA^{2N-2}B \end{bmatrix} = \mathcal{O}_N \mathcal{C}_N
$$

$$
\mathcal{O}_N \mathcal{C}_N = \mathcal{H}_{N,N} = K_1 \Lambda_1 L_1'
$$

$$
\mathcal{O}_N = K_1 \Lambda_1^{1/2}
$$

$$
\mathcal{C}_N = \Lambda_1^{1/2} L_1'
$$

$$\mathcal{O}_N = K_1 \Lambda_1^{1/2}$$
$$\mathcal{C}_N = \Lambda_1^{1/2} L_1'$$

$$\mathcal{C}_N = \begin{bmatrix} B & AB & A^2B & \dots & A^{N-1}B \end{bmatrix}$$

$$\mathcal{O}_N = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{N-1} \end{bmatrix}$$

$$B = (\mathcal{C}_N)_{:,1:m} = \Lambda_1^{1/2} [(L_1)_{1:m,:}]'$$
$$C = (\mathcal{O}_N)_{1:p,:} = (K_1)_{1:p,:} \Lambda_1^{1/2}$$

$$\tilde{\mathcal{H}}_{N+1,N+1} = \begin{bmatrix} H_2 & H_3 & \dots & H_{N+1} \\ H_3 & H_4 & \dots & H_{N+2} \\ \vdots & \vdots & & \vdots \\ H_{N+1} & H_{N+2} & \dots & H_{2N} \end{bmatrix}$$

$$\tilde{\mathcal{H}}_{N+1,N+1} = \begin{bmatrix} CAB & CA^2B & \dots & CA^NB \\ CA^2B & CA^3B & \dots & CA^{N+1}B \\ \vdots & \vdots & & \vdots \\ CA^NB & CA^{N+1}B & \dots & CA^{2N-1}B \end{bmatrix} = \mathcal{O}_N A \mathcal{C}_N$$

$$\tilde{\mathcal{H}}_{N+1,N+1} = \mathcal{O}_N A \mathcal{C}_N = K_1 \Lambda_1^{1/2} A \Lambda_1^{1/2} L_1'$$

$$A = \Lambda_1^{-1/2} K_1' \tilde{\mathcal{H}}_{N+1,N+1} L_1 \Lambda_1^{-1/2}$$

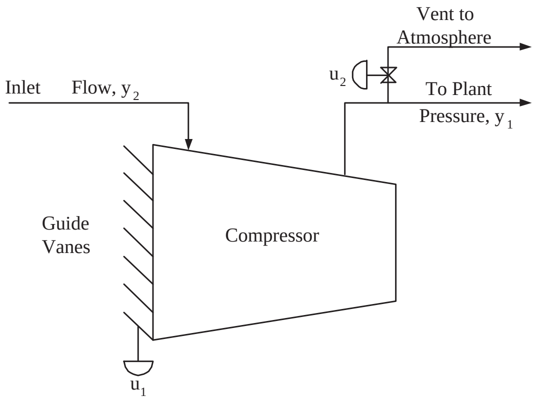
$$
\begin{aligned}
A &= \Lambda_1^{-1/2} K_1' \tilde{\mathcal{H}}_{N+1,N+1} L_1 \Lambda_1^{-1/2} \\
B &= (\mathcal{C}_N)_{:,1:m} = \Lambda_1^{1/2} [(L_1)_{1:m,:}]' \\
C &= (\mathcal{O}_N)_{1:p,:} = (K_1)_{1:p,:} \, \Lambda_1^{1/2} \\
D &= H_0
\end{aligned}
$$

This realization is balanced in the sense that

$$
\mathcal{O}_N' \mathcal{O}_N = \Lambda_1^{1/2} K_1' K_1 \Lambda_1^{1/2} = \Lambda_1
$$
$$
\mathcal{C}_N \mathcal{C}_N' = \Lambda_1^{1/2} L_1' L_1 \Lambda_1^{1/2} = \Lambda_1
$$

By construction, the obtained realization is a minimal realization, which implies that the realization is both controllable and observable.

Compressor Model



The compressor has the MIMO transfer function model

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} \frac{0.1133e^{-.715s}}{1.783s^2+4.48s+1} & \frac{0.9222}{2.071s+1} \\ \frac{0.3378e^{-0.299s}}{0.361s^2+1.09s+1} & \frac{-0.321e^{-0.94s}}{0.104s^2+2.463s+1} \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix}$$

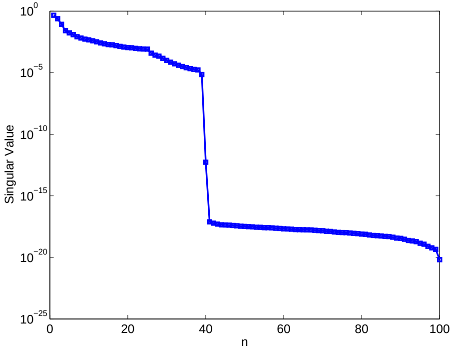
Matlab - Specification

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} \frac{0.1133e^{-.715s}}{1.783s^2+4.48s+1} & \frac{0.9222}{2.071s+1} \\ \frac{0.3378e^{-0.299s}}{0.361s^2+1.09s+1} & \frac{-0.321e^{-0.94s}}{0.104s^2+2.463s+1} \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix}$$

```
num = cell(2,2);
den = cell(2,2);
lambda = zeros(2,2);

num(1,1) = {0.1133}
den(1,1) = {[1.783 4.48 1]}
lambda(1,1) = 0.715

num(2,1) = {0.3378}
den(2,1) = {[0.361 1.09 1]}
lambda(2,1) = 0.299

num(1,2) = {0.9222}
den(1,2) = {[2.071 1]}
lambda(1,2) = 0.0

num(2,2) = {-0.321}
den(2,2) = {[0.104 2.463 1]}
lambda(2,2) = 0.94
```

Matlab - Realization

```
Ts = 0.05;
Nmax = 100;
tol = 1.0e-8;

[Ad,Bd,Cd,Dd,sH] = mimoctf2dss(num,den,lambda,Ts,Nmax,tol);
```



$$
\begin{aligned}
x_{k+1} &= A_d x_k + B_d u_k \\
y_k &= C_d x_k + D_d u_k
\end{aligned}
$$

$$A_d \in \mathbb{R}^{39\times 39}$$

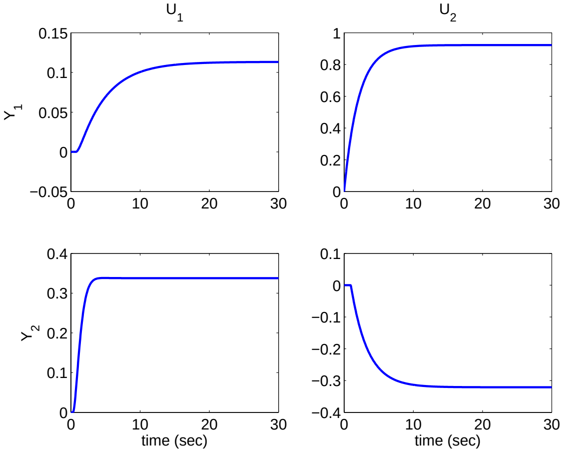
$$B_d \in \mathbb{R}^{39\times 2}$$

$$C_d \in \mathbb{R}^{2\times 39}$$

$$D_d = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{2\times 2}$$

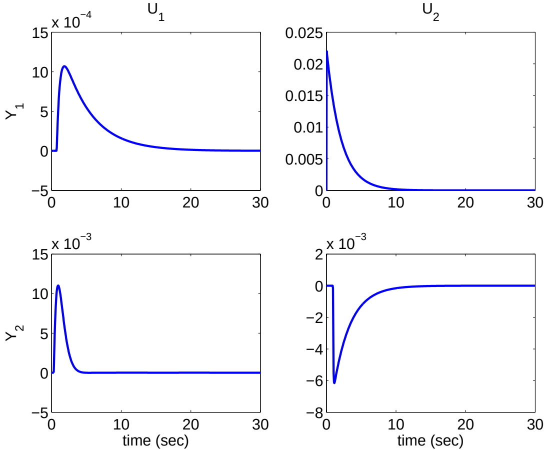
Compressor - Step Responses

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} \frac{0.1133e^{-.715s}}{1.783s^2+4.48s+1} & \frac{0.9222}{2.071s+1} \\ \frac{0.3378e^{-0.299s}}{0.361s^2+1.09s+1} & \frac{-0.321e^{-0.94s}}{0.104s^2+2.463s+1} \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix}$$



Compressor - Impulse Responses

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} \frac{0.1133e^{-.715s}}{1.783s^2+4.48s+1} & \frac{0.9222}{2.071s+1} \\ \frac{0.3378e^{-0.299s}}{0.361s^2+1.09s+1} & \frac{-0.321e^{-0.94s}}{0.104s^2+2.463s+1} \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix}$$



Learning Objectives

After this lecture you must be able to

Generate a step response by simulation of a linear or nonlinear system (state space representation)

Convert a transfer function (SISO and MIMO systems) to a discrete-time linear state space model

Manually fit a transfer function to a step response

Questions and Comments

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