Modeling and Simulation of Distributed Systems

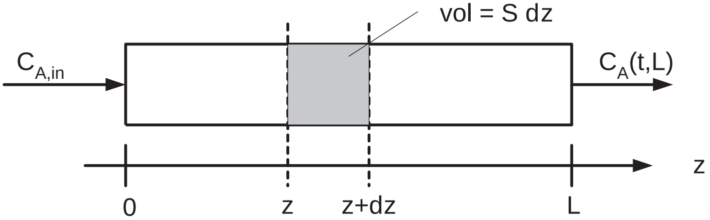
Partial Differential Equations

John Bagterp Jørgensen

Department of Applied Mathematics and Computer Science
Technical University of Denmark

02619 Model Predictive Control - Lecture 4

Flow and Chemical Reaction in a Pipe



Chemical Reaction: $A \to P \qquad r = kC_A \qquad R_A = -r$

Flux for convective and diffusive flow: $N_A = vC_A + J_A \qquad J_A = -D_A \dfrac{\partial C_A}{\partial z}$

$$\begin{aligned} Accumulated &= \left[C_A(t+\Delta t, z) - C_A(t,z)\right] S\Delta z \\ Influx &= N_A(t,z) S\Delta t \\ Outflux &= N_A(t, z+\Delta z) S\Delta t \\ Generated &= R_A S\Delta z \Delta t \end{aligned}$$

- Model (mass balance)

$$\frac{\partial C_A(t,z)}{\partial t} = -\frac{\partial N_A(t,z)}{\partial z} + R_A(t,z)$$

- Boundary conditions

$$\begin{aligned} z = 0: &\qquad N_A(t,0) = vC_{A,in} \\ z = L: &\qquad N_A(t,L) = vC_A(t,L) \end{aligned}$$

- Initial condition

$$t = 0: \qquad C_A(0,z) = C_{A0}(z)$$

- Flux

$$N_A(t,z) = \overbrace{vC_A(t,z)}^{\text{convection}} \overbrace{-D_A \frac{\partial C_A(t,z)}{\partial z}}^{\text{diffusion}}$$

- Stoichiometry and kinetics

$$A \rightarrow P \qquad r = kC_A$$

- Production rates

$$R_A = -r$$

- Equidistant spatial discretization

$$z_j = \left(j - \frac{1}{2}\right)\Delta z \qquad \Delta z = \frac{L}{N_z}$$

- Spatial discretization of partial differential equation (method of lines)

$$\frac{dC_{A,j}(t)}{dt} = -\frac{N_{A,j+1/2}(t) - N_{A,j-1/2}(t)}{\Delta z} + R_{A,j}(t) \qquad j = 1, 2, \dots, N_z$$

- Fluxes

$$\begin{aligned}
N_{A,j+1/2}(t) &= vC_{A,in}(t) && j = 0 \\
N_{A,j+1/2}(t) &= vC_{A,j}(t) - D_A \frac{C_{A,j+1}(t) - C_{A,j}(t)}{\Delta z} && j = 1, 2, \dots, N_z - 1 \\
N_{A,j+1/2}(t) &= vC_{A,j}(t) && j = N_z
\end{aligned}$$

- Reaction rates

$$r_j(t) = kC_{A,j}(t) \qquad j = 1, 2, \dots, N_z$$

- Production rates

$$R_{A,j}(t) = -r_j(t) \qquad j = 1, 2, \dots, N_z$$

The model

- Equidistant spatial discretization

$$z_j = \left(j - \frac{1}{2}\right) \Delta z \qquad \Delta z = \frac{L}{N_z}$$

- Spatial discretization of partial differential equation (method of lines)

$$\frac{dC_{A,j}(t)}{dt} = -\frac{N_{A,j+1/2}(t) - N_{A,j-1/2}(t)}{\Delta z} + R_{A,j}(t) \qquad j = 1, 2, \ldots, N_z$$

- Fluxes

$$\begin{aligned} N_{A,j+1/2}(t) &= vC_{A,in}(t) && j = 0 \\ N_{A,j+1/2}(t) &= vC_{A,j}(t) - D_A \frac{C_{A,j+1}(t) - C_{A,j}(t)}{\Delta z} && j = 1, 2, \ldots, N_z - 1 \\ N_{A,j+1/2}(t) &= vC_{A,j}(t) && j = N_z \end{aligned}$$

- Reaction rates

$$r_j(t) = kC_{A,j}(t) \qquad j = 1, 2, \ldots, N_z$$

- Production rates

$$R_{A,j}(t) = -r_j(t) \qquad j = 1, 2, \ldots, N_z$$

can be represented as

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \qquad x(t_0) = x_0 \\ y(t) &= g(x(t)) \end{aligned}$$

with $x = [C_{A,1};\, C_{A,2};\, \ldots, C_{A,N_z}]$, $u = C_{A,in}$, and $y = C_{A,out} = C_{A,N_z}$

$$\frac{dC_{A,1}}{dt} = \left(\frac{v}{\Delta z}\right) C_{A,in} + \left(-\frac{v + \frac{D_A}{\Delta z}}{\Delta z} - k\right) C_{A,1} + \left(\frac{\frac{D_A}{\Delta z}}{\Delta z}\right) C_{A,2}$$

$$\frac{dC_{A,j}}{dt} = \left(\frac{v + \frac{D_A}{\Delta z}}{\Delta z}\right) C_{A,j-1} + \left(-\frac{v + 2\frac{D_A}{\Delta z}}{\Delta z} - k\right) C_{A,j} + \left(\frac{\frac{D_A}{\Delta z}}{\Delta z}\right) C_{A,j+1} \qquad j = 2, \ldots, N_z - 1$$

$$\frac{dC_{A,N_z}}{dt} = \left(\frac{v + \frac{D_A}{\Delta z}}{\Delta z}\right) C_{A,N_z-1} + \left(-\frac{v + \frac{D_A}{\Delta z}}{\Delta z} - k\right) C_{A,N_z}$$

$$\dot{x}(t) = A_c x(t) + B_c u(t)$$
$$y(t) = C_c x(t)$$

$N_z = 5$: $x = [C_{A,1};\, C_{A,2};\, C_{A,3};\, C_{A,4};\, C_{A,5}]$, $u = C_{A,in}$, $y = C_{A,out} = C_{A,5}$

$$A_c = \begin{bmatrix} -\tilde{\alpha} & \gamma & 0 & 0 & 0 \\ \beta & -\alpha & \gamma & 0 & 0 \\ 0 & \beta & -\alpha & \gamma & 0 \\ 0 & 0 & \beta & -\alpha & \gamma \\ 0 & 0 & 0 & \beta & -\tilde{\alpha} \end{bmatrix} \qquad B_c = \begin{bmatrix} \delta \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad C_c = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

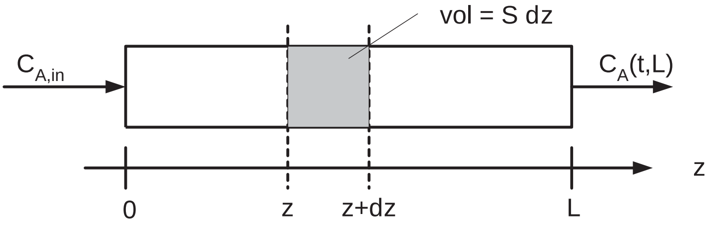
$$\alpha = \beta + \gamma + k \quad \tilde{\alpha} = \beta + k \quad \beta = \frac{v + \frac{D_A}{\Delta z}}{\Delta z} \quad \gamma = \frac{\frac{D_A}{\Delta z}}{\Delta z} \quad \delta = \frac{v}{\Delta z}$$

ZOH-discretization

$$x_{k+1} = A x_k + B u_k$$
$$y_k = C x_k$$

$$\begin{bmatrix} A & B \\ 0 & I \end{bmatrix} = \exp\left(\begin{bmatrix} A_c & B_c \\ 0 & 0 \end{bmatrix} T_s\right) \qquad C = C_c$$

Flow and Chemical Reaction in a Pipe



Chemical Reaction: $2A \to P \qquad r = kC_A^2 \qquad R_A = -2r$

Flux for convective and diffusive flow: $N_A = vC_A + J_A \qquad J_A = -D_A \dfrac{\partial C_A}{\partial z}$

$$\begin{aligned} Accumulated &= \left[C_A(t+\Delta t, z) - C_A(t,z)\right] S\Delta z \\ Influx &= N_A(t,z)S\Delta t \\ Outflux &= N_A(t, z+\Delta z)S\Delta t \\ Generated &= R_A S\Delta z\Delta t \end{aligned}$$

- Model (mass balance)

$$\frac{\partial C_A(t,z)}{\partial t} = -\frac{\partial N_A(t,z)}{\partial z} + R_A(t,z)$$

- Boundary conditions

$$\begin{aligned} z = 0: &\qquad N_A(t,0) = vC_{A,in} \\ z = L: &\qquad N_A(t,L) = vC_A(t,L) \end{aligned}$$

- Initial condition

$$t = 0: \qquad C_A(0,z) = C_{A0}(z)$$

- Flux

$$N_A(t,z) = \overbrace{vC_A(t,z)}^{\text{convection}} \overbrace{-D_A \frac{\partial C_A(t,z)}{\partial z}}^{\text{diffusion}}$$

- Stoichiometry and kinetics

$$2A \rightarrow P \qquad r = kC_A^2$$

- Production rates

$$R_A = -2r$$

- Equidistant spatial discretization

$$z_j = \left(j - \frac{1}{2}\right) \Delta z \qquad \Delta z = \frac{L}{N_z}$$

- Spatial discretization of partial differential equation (method of lines)

$$\frac{dC_{A,j}(t)}{dt} = -\frac{N_{A,j+1/2}(t) - N_{A,j-1/2}(t)}{\Delta z} + R_{A,j}(t) \qquad j = 1, 2, \ldots, N_z$$

- Fluxes

$$\begin{aligned}
N_{A,j+1/2}(t) &= vC_{A,in}(t) && j = 0 \\
N_{A,j+1/2}(t) &= vC_{A,j}(t) - D_A \frac{C_{A,j+1}(t) - C_{A,j-1}(t)}{\Delta z} && j = 1, 2, \ldots, N_z - 1 \\
N_{A,j+1/2}(t) &= vC_{A,j}(t) && j = N_z
\end{aligned}$$

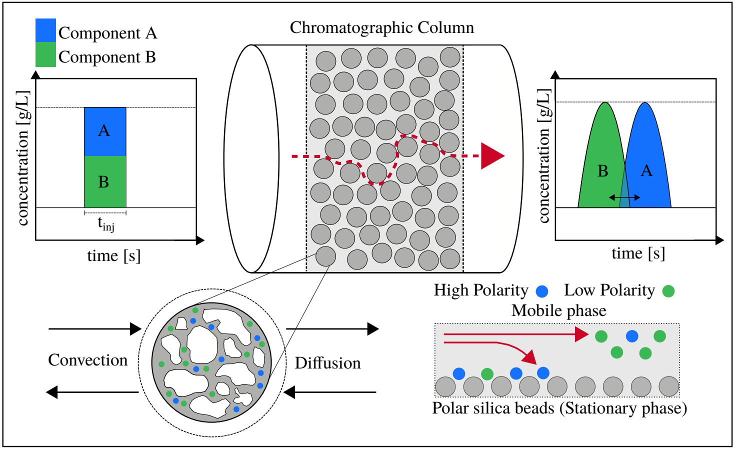
- Reaction rates

$$r_j(t) = k\left(C_{A,j}(t)\right)^2 \qquad j = 1, 2, \ldots, N_z$$

- Production rates

$$R_{A,j}(t) = -2r_j(t) \qquad j = 1, 2, \ldots, N_z$$

Chromatography



Mathematical Model - Linear Equilibrium

- Mass balance

$$\frac{\partial}{\partial t}\bar{c}_i(t,z) = -\frac{\partial}{\partial z}\bar{N}_i(t,z)$$

- Flux

$$\bar{N}_i = \epsilon N_i$$
$$N_i = v c_i + J_i$$

- Diffusion

$$J_i = -D_i \frac{\partial}{\partial z} c_i$$

- Mobile and immobile phase

$$V\bar{c}_i = \epsilon V c_i + (1-\epsilon) V q_i$$

  is equivalent to

$$\bar{c}_i = \epsilon c_i + (1-\epsilon) q_i$$

- Equilibrium (linear)

$$K_i = \frac{q_i}{c_i}$$

Mathematical Model - Linear Equilibrium

The equations

- Mass balance

$$\frac{\partial}{\partial t}\bar{c}_i(t,z) = -\frac{\partial}{\partial z}\bar{N}_i(t,z)$$

- Flux

$$\begin{aligned} \bar{N}_i &= \epsilon N_i \\ N_i &= v c_i + J_i \end{aligned}$$

- Diffusion

$$J_i = -D_i \frac{\partial}{\partial z} c_i$$

- Mobile and immobile phase

$$V\bar{c}_i = \epsilon V c_i + (1-\epsilon) V q_i$$

  is equivalent to

$$\bar{c}_i = \epsilon c_i + (1-\epsilon) q_i$$

- Equilibrium (linear)

$$K_i = \frac{q_i}{c_i}$$

correspond to

$$\begin{aligned} &\left(1 + \frac{1-\epsilon}{\epsilon} K_i\right) \frac{\partial}{\partial t} c_i = -v\frac{\partial}{\partial z} c_i + D_i \frac{\partial^2}{\partial z^2} c_i \\ &q_i = K_i c_i \\ &\bar{c}_i = \epsilon c_i + (1-\epsilon) q_i \end{aligned}$$

Mathematical Model - Linear Equilibrium

$$\left(1 + \frac{1-\epsilon}{\epsilon} K_i\right) \frac{\partial}{\partial t} c_i = -v \frac{\partial}{\partial z} c_i + D_i \frac{\partial^2}{\partial z^2} c_i$$

is a linear system of partial differential equations

$$\frac{\partial}{\partial t} c_i = a_i \frac{\partial}{\partial z} c_i + b_i \frac{\partial^2}{\partial z^2} c_i$$

with

$$a_i = \frac{-v}{1 + \frac{1-\epsilon}{\epsilon} K_i}$$
$$b_i = \frac{D_i}{1 + \frac{1-\epsilon}{\epsilon} K_i}$$

For short we write this as

$$\partial_t c_i = a_i \partial_z c_i + b_i \partial^2_{zz} c_i$$

Mathematical Model - Linear Equilibrium

- Mathematical model:

$$t > 0,\, 0 < z < L: \qquad \partial_t c_i = a_i \partial_z c_i + b_i \partial_{zz}^2 c_i$$

- Boundary conditions:

$$\begin{aligned} z = 0: &\qquad N_i = v c_i - D_i \partial_z c_i = v c_{i,in} \\ z = L: &\qquad \partial_z c_i = 0 \end{aligned}$$

- Initial value condition:

$$t = 0: \qquad c_i(0, z) = c_{i,init}(z)$$

- Linear continuous-time and discrete-time state space model

$$\begin{aligned} \dot{x}(t) &= A_c x(t) + B_c u(t) & \qquad x_{k+1} &= A x_k + B u_k \\ y(t) &= C x(t) & y_k &= C x_k \end{aligned}$$

- Mathematical model:

$$
\begin{aligned}
&t>0,\, 0<z<L: && \left(1+\frac{1-\epsilon}{\epsilon}K_i\right)\partial_t c_i = -\partial_z N_i, \quad N_i = vc_i - D_i\partial_z c_i \\
&z=0: && N_i = vc_i - D_i\partial_z c_i = vc_{i,in} \\
&z=L: && \partial_z c_i = 0
\end{aligned}
$$

- Spatial discretization

$$
\begin{aligned}
&\left(1+\frac{1-\epsilon}{\epsilon}K_i\right)\frac{d}{dt}c_{i,1} = -\frac{N_{i,3/2}-N_{i,1/2}}{\Delta z} \\
&\left(1+\frac{1-\epsilon}{\epsilon}K_i\right)\frac{d}{dt}c_{i,j} = -\frac{N_{i,j+1/2}-N_{i,j-1/2}}{\Delta z}, \qquad j=2,\dots,N_z-1 \\
&\left(1+\frac{1-\epsilon}{\epsilon}K_i\right)\frac{d}{dt}c_{i,N_z} = -\frac{N_{i,N_z+1/2}-N_{i,N_z-1/2}}{\Delta z}
\end{aligned}
$$

- Flux

$$
\begin{aligned}
&N_{i,1/2} = vc_{i,in} \\
&N_{i,j+1/2} = vc_{i,j} - D_i\frac{c_{i,j+1}-c_{i,j}}{\Delta z}, \quad j=1,2,\dots,N_z-1 \\
&N_{i,N_z+1/2} = vc_{i,N_z}
\end{aligned}
$$

- Discretized system

$$\left(1+\frac{1-\epsilon}{\epsilon}K_i\right)\frac{d}{dt}c_{i,1} = \left(\frac{v}{\Delta z}\right)c_{i,in} + \left(-\frac{v+D_i/\Delta z}{\Delta z}\right)c_{i,1} + \left(\frac{D_i/\Delta z}{\Delta z}\right)c_{i,2}$$
$$\left(1+\frac{1-\epsilon}{\epsilon}K_i\right)\frac{d}{dt}c_{i,j} = \left(\frac{v+D_i/\Delta z}{\Delta z}\right)c_{i,j-1} + \left(-\frac{v+2D_i/\Delta z}{\Delta z}\right)c_{i,j} + \left(\frac{D_i/\Delta z}{\Delta z}\right)c_{i,j+1}$$
$$\left(1+\frac{1-\epsilon}{\epsilon}K_i\right)\frac{d}{dt}c_{i,N_z} = \left(\frac{v+D_i/\Delta z}{\Delta z}\right)c_{i,N_z-1} + \left(-\frac{v+D_i/\Delta z}{\Delta z}\right)c_{i,N_z}$$

- State space form

$$\dot{x}_i(t) = A_{c,i}x_i(t) + B_{c,i}u_i(t)$$
$$y_i(t) = C_{c,i}x_i(t)$$

$N_z = 5$

$$x_i = \left[c_{i,1};\, c_{i,2}; c_{i,3}; c_{i,4}; c_{i,5}\right] \quad u_i = c_{i,in} \quad y_i = c_{i,N_z+1} = c_{i,N_z}$$

$$A_{c,i} = \begin{bmatrix} -\beta_i & \gamma_i & & & \\ \beta_i & -\alpha_i & \gamma_i & & \\ & \beta_i & -\alpha_i & \gamma_i & \\ & & \beta_i & -\alpha_i & \gamma_i \\ & & & \beta_i & -\beta_i \end{bmatrix} \quad B_{c,i} = \begin{bmatrix} \delta_i \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad C_{c,i} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\alpha_i = \beta_i + \gamma_i \quad \beta_i = \frac{\frac{v+D_i/\Delta z}{\Delta z}}{\kappa_i} \quad \delta_i = \frac{\frac{v}{\Delta z}}{\kappa_i} \quad \gamma_i = \frac{\frac{D_i/\Delta z}{\Delta z}}{\kappa_i} \quad \kappa_i = 1 + \frac{1-\epsilon}{\epsilon}K_i$$

- State-space model
  - Continuous time

$$\dot{x}_i(t) = A_{c,i}x_i(t) + B_{c,i}u_i(t), \quad x_{i,k} = x_i(t_k) \quad u_i(t) = u_{i,k} \text{ for } t_k \leq t < t_{k+1}$$
$$y_i(t) = C_c x_i(t)$$

  - Discrete time

$$x_{i,k+1} = A_i x_{i,k} + B_i u_{i,k}$$
$$y_{i,k} = C_i x_{i,k} \qquad\qquad C_i = C_{c,i}$$

- Expicit-Euler discretization

$$A_i = I + \Delta t A_{c,i} \qquad B_i = \Delta t B_{c,i}$$

- Implicit-Euler discretization

$$A_i = \left(I - \Delta t A_{c,i}\right)^{-1} \qquad B_i = \Delta t(I - \Delta t A_{c,i})^{-1} B_{c,i}$$

- Trapezoidal disretization

$$A_i = \left(I - \frac{1}{2}\Delta t A_{c,i}\right)^{-1}\left(I + \frac{1}{2}\Delta t A_{c,i}\right) \qquad B_i = \Delta t \left(I - \frac{1}{2}\Delta t A_{c,i}\right) B_{c,i}$$

- Exact discretization

$$A_i = \exp\left(A_{c,i}\Delta t\right) \qquad B_i = \int_0^{\Delta t} \exp\left(A_{ci}s\right) B_{ci}\, ds$$

$$\begin{bmatrix} A_i & B_i \\ 0 & I \end{bmatrix} = \exp\left(\begin{bmatrix} A_{c,i} & B_{c,i} \\ 0 & 0 \end{bmatrix} \Delta t\right)$$

Stochastic Discrete-Time Model and Kalman Filter

$$\begin{aligned} x_{i,k+1} &= A_i x_{i,k} + B_i(u_{i,k} + w_{i,k}) \qquad & w_{i,k} &\sim N_{iid}(0, Q_i) \\ y_{i,k} &= C_i x_{k,i} + v_{i,k} & v_{i,k} &\sim N_{iid}(0, R_i) \end{aligned}$$

Dynamic Kalman filter

$$\begin{aligned} \hat{y}_{i,k|k-1} &= C_i \hat{x}_{i,k|k-1} \\ e_{i,k} &= y_{i,k} - \hat{y}_{i,k|k-1} \\ R_{e,i,k} &= C_i P_{i,k|k-1} C_i' + R_i \\ K_{i,k} &= P_{i,k|k-1} C_i' R_{e,i,k}^{-1} \\ \hat{x}_{i,k|k} &= \hat{x}_{i,k|k-1} + K_{i,k} e_{i,k} \\ P_{i,k|k} &= P_{i,k|k-1} - K_{i,k} R_{e,i,k} K_{i,k}' = (I - K_{i,k} C_i) P_{i,k|k-1} (I - K_{i,k} C_i)' + K_{i,k} R_i K_{i,k}' \end{aligned}$$

One-step predictor

$$\begin{aligned} \hat{x}_{i,k+1|k} &= A_i \hat{x}_{i,k|k} + B_i \hat{u}_{i,k|k} \\ P_{i,k+1|k} &= A_i P_{i,k|k} A_i' + B_i Q_i B_i' \end{aligned}$$

j-step predictor

$$\begin{aligned} \hat{x}_{i,k+1+j|k} &= A_i \hat{x}_{i,k+j|k} + B_i \hat{u}_{i,k+j|k} \\ P_{i,k+1+j|k} &= A_i P_{i,k+j|k} A_i' + B_i Q_i B_i' \end{aligned}$$

- The linear ordinary differential equation system (deterministic)

$$dx_i(t) = \big(A_{c,i}x_i(t) + B_{c,i}u_i(t)\big)\,dt$$

can be expressed as

$$dx_i(t) = \big(A_{c,i}x_i(t) + B_{c,i}u_i(t)\big)\,dt$$

- Linear stochastic differential equation system (stochastic)

$$\begin{aligned} dx_i(t) &= \big(A_{c,i}x_i(t) + B_{c,i}u_i(t)\big)\,dt + B_{c,i}\sigma_i d\omega_i(t) \\ &= A_{c,i}x_i(t)dt + B_{c,i}\left(u_i(t)dt + \sigma_i d\omega_i(t)\right) \end{aligned}$$

$d\omega_i(t) \sim N_{iid}(0, dt)$ such that $\omega_i(t)$ is a standard Wiener process.

- Exact discretization for $u_i(t) = u_{i,k}$ when $t_k \leq t < t_{k+1}$:

$$x_{i,k+1} = A_i x_{i,k} + B_i u_{i,k} + w_{i,k} \qquad w_{i,k} \sim N_{iid}(0, Q_i)$$

$$A_i = \exp\left(A_{c,i}\Delta t\right) \quad B_i = \int_0^{\Delta t} \exp\left(A_{c,i}s\right) B_{c,i} ds$$

$$Q_i = \int_0^{\Delta t} \exp(A_{c,i}s) B_{c,i}\sigma_i\sigma_i' B_{c,i}' \exp(A_{c,i}' s) ds$$

$$\begin{bmatrix} A_i & B_i \\ 0 & I \end{bmatrix} = \exp\left(\begin{bmatrix} A_{c,i} & B_{c,i} \\ 0 & 0 \end{bmatrix}\Delta t\right)$$

$$\begin{bmatrix} \Phi_{i,11} & \Phi_{i,12} \\ 0 & \Phi_{i,22} \end{bmatrix} = \exp\left(\begin{bmatrix} -A_{c,i} & B_{c,i}\sigma_i\sigma_i' B_{c,i}' \\ 0 & A_{c,i}' \end{bmatrix}\Delta t\right)$$

$$Q_i = \Phi_{i,22}'\Phi_{i,12}$$

- Discrete measurement equation

$$y_{i,k} = C_i x_{i,k} + v_{i,k} \qquad v_{i,k} \sim N_{iid}(0, R_i)$$

- We can apply the discrete-time Kalman filter for the discrete-time system corresponding to the continuous-discrete system

Kalman filter for continuous-discrete systems

- Continous-discrete system

$$
\begin{aligned}
dx_i(t) &= A_{c,i}x_i(t)dt + B_{c,i}\left(u_i(t)dt + \sigma_i d\omega_i(t)\right) \qquad && d\omega_i(t) \sim N_{iid}(0, Idt) \\
y_i(t_k) &= C_i x_i(t_k) + v_i(t_k) && v_i(t_k) \sim N_{iid}(0, R_i)
\end{aligned}
$$

- Equivalent discrete-discrete system

$$
\begin{aligned}
x_{i,k+1} &= A_i x_{i,k} + B_i u_{i,k} + w_{i,k} \qquad && w_{i,k} \sim N_{iid}(0, Q_i) \\
y_{i,k} &= C_i x_{i,k} + v_{i,k} && v_{i,k} \sim N_{iid}(0, R_i)
\end{aligned}
$$

- Dynamic Kalman filter for the discrete-discrete system
  Filter

$$
\begin{aligned}
\hat{y}_{i,k|k-1} &= C_i \hat{x}_{i,k|k-1} \\
e_{i,k} &= y_{i,k} - \hat{y}_{i,k|k-1} \\
R_{e,i,k} &= C_i P_{i,k|k-1} C_i' + R_i \\
K_{i,k} &= P_{i,k|k-1} C_i' R_{e,i,k}^{-1} \\
\hat{x}_{i,k|k} &= \hat{x}_{i,k|k-1} + K_{i,k} e_{i,k} \\
P_{i,k|k} &= P_{i,k|k-1} - K_{i,k} R_{e,i,k} K_{i,k}' = (I - K_{i,k}C_i) P_{i,k|k-1} (I - K_{i,k}C_i)' + K_{i,k} R_i K_{i,k}'
\end{aligned}
$$

  One-step predictor

$$
\begin{aligned}
\hat{x}_{i,k+1|k} &= A_i \hat{x}_{i,k|k} + B_i \hat{u}_{i,k|k} \\
P_{i,k+1|k} &= A_i P_{i,k|k} A_i' + Q_i
\end{aligned}
$$

  j-step predictor

$$
\begin{aligned}
\hat{x}_{i,k+1+j|k} &= A_i \hat{x}_{i,k+j|k} + B_i \hat{u}_{i,k+j|k} \\
P_{i,k+1+j|k} &= A_i P_{i,k+j|k} A_i' + Q_i
\end{aligned}
$$