Model Predictive Control

Lecture 6: State Estimation and Prediction

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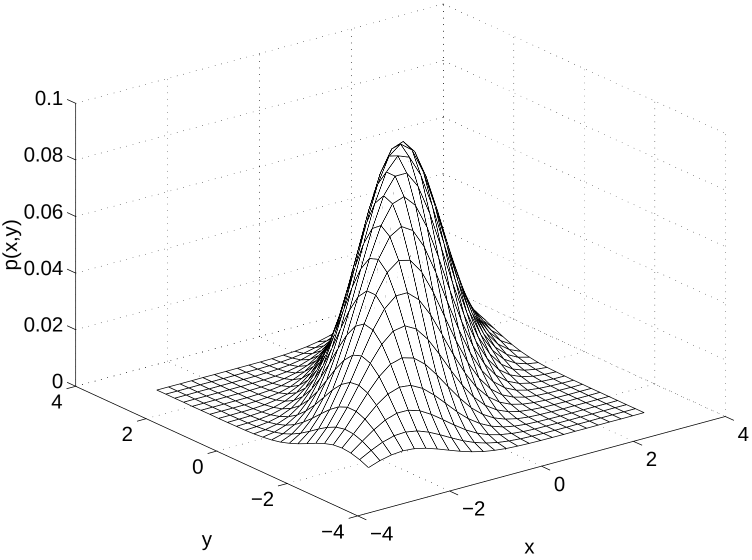
02619 Model Predictive Control

Learning Objectives

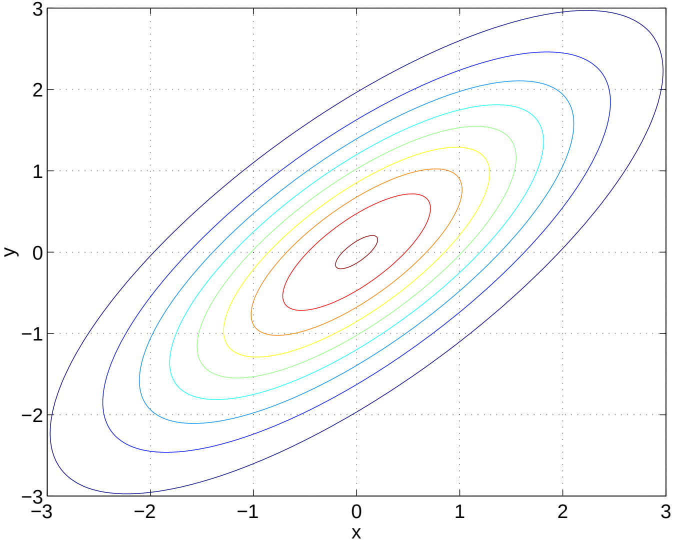
1. Kalman filtering

Outline

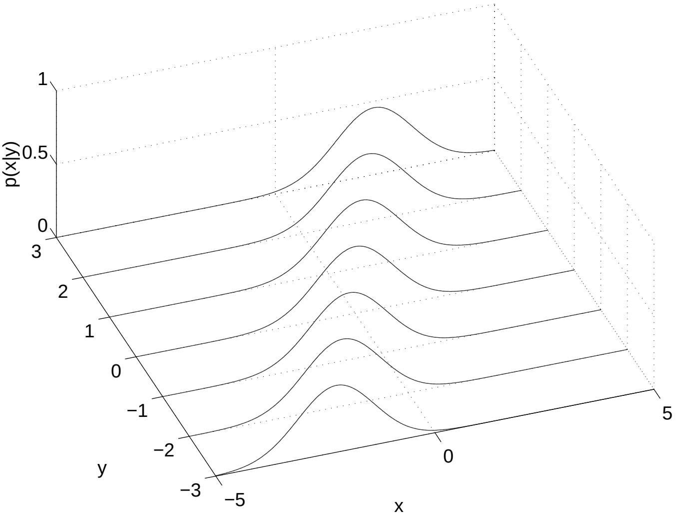
2D Normal Distribution - Joint Probability Density Function



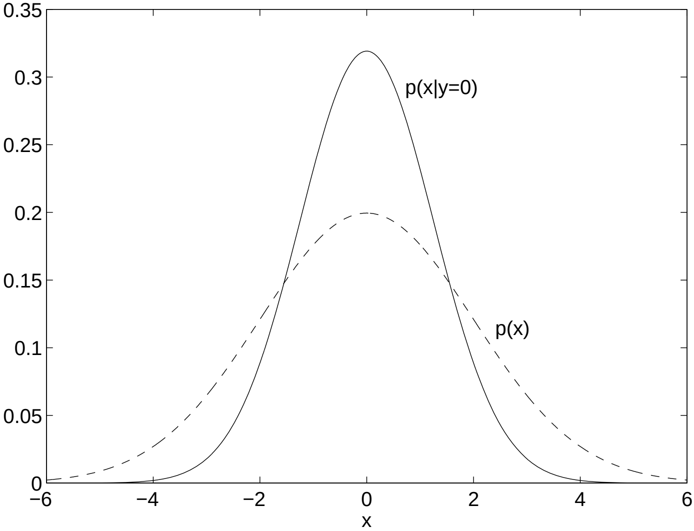
2D Normal Distribution - Confidence Region



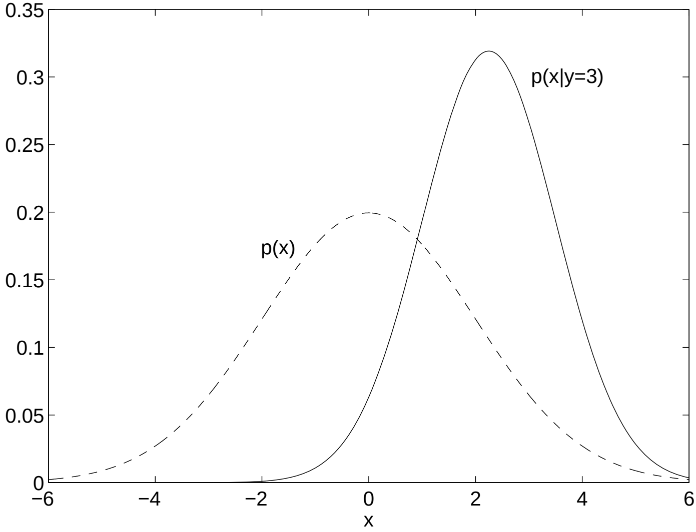
Conditional Probability Density Functions



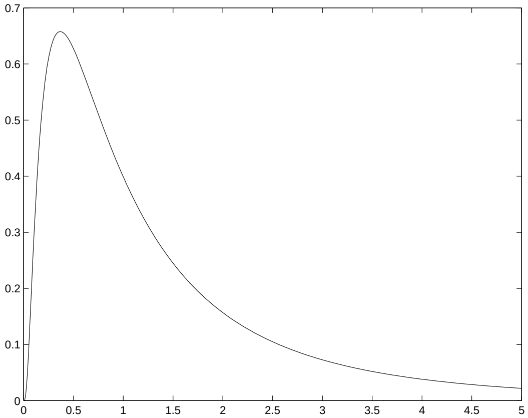
Conditional Probability Density Function



Conditional Probability Density Function



Estimation Principles



Estimator

- Mean
- Median
- Maximum Likelihood

Normally Distributed Variables

Uni-variate

$$\mathbf{x} \sim N(\mu, \sigma^2) \qquad \mu \in \mathbb{R},\ \sigma^2 \in \mathbb{R}_{++}$$

$$p_x(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

Multi-variate

$$\mathbf{x} \sim N(\mu, \Sigma) \qquad \mu \in \mathbb{R}^n,\ \Sigma \in \mathbb{R}^{n\times n}_{++}$$

$$p_x(x) = \frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}} \exp\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\right)$$

Conditional Distribution

Consider the normal distribution

$$\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} \sim N\left(\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix}, \begin{bmatrix} R_{xx} & R_{xy} \\ R_{yx} & R_{yy} \end{bmatrix} \right)$$

The estimate of $\boldsymbol{x}$ conditioned on $\boldsymbol{y}$ is denoted

$$\hat{\boldsymbol{x}} = \boldsymbol{x} | (\boldsymbol{y} = y)$$

and is normally distributed

$$\hat{\boldsymbol{x}} \sim N(\hat{x}, R_{\hat{x}})$$

with the mean

$$\hat{x} = \bar{x} + R_{xy} R_{yy}^{-1} (y - \bar{y})$$

and the variance

$$R_{\hat{x}} = R_{xx} - R_{xy} R_{yy}^{-1} R_{yx}$$

This is a key result in deriving the Kalman filter and predictor

Linear System

$$\boldsymbol{x}_{k+1} = A\boldsymbol{x}_k + G\boldsymbol{w}_k$$
$$\boldsymbol{y}_k = C\boldsymbol{x}_k + \boldsymbol{v}_k$$

$$\boldsymbol{x}_0 \sim N(\bar{x}_0, P_0)$$
$$\boldsymbol{w}_k \sim N_{iid}(0, Q)$$
$$\boldsymbol{v}_k \sim N_{iid}(0, R)$$

Linear System

$$\boldsymbol{x}_{k+1} = A\boldsymbol{x}_k + G\boldsymbol{w}_k \qquad \boldsymbol{x}_k \sim N(\bar{x}_k, P_k) \qquad \boldsymbol{w}_k \sim N_{iid}(0, Q)$$

$$\boldsymbol{x}_{k+1} \sim N(\bar{x}_{k+1}, P_{k+1})$$

Mean

$$\begin{aligned}\bar{x}_{k+1} &= E\left\{\boldsymbol{x}_{k+1}\right\} = E\left\{A\boldsymbol{x}_k + G\boldsymbol{w}_k\right\} \\ &= AE\left\{\boldsymbol{x}_k\right\} + GE\left\{\boldsymbol{w}_k\right\} = A\bar{x}_k\end{aligned}$$

Covariance

$$\begin{aligned}P_{k+1} &= \langle \boldsymbol{x}_{k+1}, \boldsymbol{x}_{k+1} \rangle = \langle A\boldsymbol{x}_k + G\boldsymbol{w}_k, A\boldsymbol{x}_k + G\boldsymbol{w}_k \rangle \\ &= \langle A\boldsymbol{x}_k, A\boldsymbol{x}_k \rangle + \langle A\boldsymbol{x}_k, G\boldsymbol{w}_k \rangle + \langle G\boldsymbol{w}_k, A\boldsymbol{x}_k \rangle + \langle G\boldsymbol{w}_k, G\boldsymbol{w}_k \rangle \\ &= A\langle \boldsymbol{x}_k, \boldsymbol{x}_k \rangle A' + A\langle \boldsymbol{x}_k, \boldsymbol{w}_k \rangle G' + G\langle \boldsymbol{w}_k, \boldsymbol{x}_k \rangle A' + G\langle \boldsymbol{w}_k, \boldsymbol{w}_k \rangle G' \\ &= AP_kA' + GQG'\end{aligned}$$

Discrete Lyapunov Equation

$$P_{k+1} = AP_kA' + GQG'$$

Assume $P_k \to P$ for $k \to \infty$. Then

$$P = APA' + GQG'$$

This is the **Discrete Lyapunov Equation**.
MATLAB:

```
P = dlyap(A,G*Q*G')
```

Linear System

$$\boldsymbol{y}_k = C\boldsymbol{x}_k + \boldsymbol{v}_k \quad \boldsymbol{x}_k \sim N(\bar{x}_k, P_k) \quad \boldsymbol{v}_k \sim N(0, R)$$

$$\begin{bmatrix} \boldsymbol{x}_k \\ \boldsymbol{y}_k \end{bmatrix} \sim N(??, ??)$$

Derivation

$$\bar{y}_k = E\left\{\boldsymbol{y}_k\right\} = E\left\{C\boldsymbol{x}_k + \boldsymbol{v}_k\right\} = C\bar{x}_k$$

$$\begin{aligned}
\langle \boldsymbol{y}_k, \boldsymbol{y}_k \rangle &= \langle C\boldsymbol{x}_k + \boldsymbol{v}_k, C\boldsymbol{x}_k + \boldsymbol{v}_k \rangle \\
&= C\overbrace{\langle \boldsymbol{x}_k, \boldsymbol{x}_k \rangle}^{=P_k} C' + C\overbrace{\langle \boldsymbol{x}_k, \boldsymbol{v}_k \rangle}^{=0} + \overbrace{\langle \boldsymbol{v}_k, \boldsymbol{x}_k \rangle}^{=0} C' + \overbrace{\langle \boldsymbol{v}_k, \boldsymbol{v}_k \rangle}^{=R} \\
&= CP_kC' + R
\end{aligned}$$

$$\begin{aligned}
\langle \boldsymbol{y}_k, \boldsymbol{x}_k \rangle &= \langle C\boldsymbol{x}_k + \boldsymbol{v}_k, \boldsymbol{x}_k \rangle \\
&= C\langle \boldsymbol{x}_k, \boldsymbol{x}_k \rangle + \langle \boldsymbol{v}_k, \boldsymbol{x}_k \rangle = CP_k
\end{aligned}$$

$$\begin{aligned}
\langle \boldsymbol{x}_k, \boldsymbol{y}_k \rangle &= \langle \boldsymbol{x}_k, C\boldsymbol{x}_k + \boldsymbol{v}_k \rangle \\
&= \langle \boldsymbol{x}_k, \boldsymbol{x}_k \rangle C' + \langle \boldsymbol{x}_k, \boldsymbol{v}_k \rangle = P_kC'
\end{aligned}$$

Linear System

$$\boldsymbol{y}_k = C\boldsymbol{x}_k + \boldsymbol{v}_k \quad \boldsymbol{x}_k \sim N(\bar{x}_k, P_k) \quad \boldsymbol{v}_k \sim N(0, R)$$

$$\begin{bmatrix} \boldsymbol{x}_k \\ \boldsymbol{y}_k \end{bmatrix} \sim N\left(\begin{bmatrix} \bar{x}_k \\ C\bar{x}_k \end{bmatrix}, \begin{bmatrix} P_k & P_k C' \\ C P_k & C P_k C' + R \end{bmatrix} \right)$$

Linear System

$$\boldsymbol{y}_k = C\boldsymbol{x}_k + \boldsymbol{v}_k \quad \boldsymbol{x}_k \sim N(\bar{x}_k, P_k) \quad \boldsymbol{v}_k \sim N(0, R)$$

$$\begin{bmatrix} \boldsymbol{x}_k \\ \boldsymbol{y}_k \end{bmatrix} \sim N\left(\begin{bmatrix} \bar{x}_k \\ C\bar{x}_k \end{bmatrix}, \begin{bmatrix} P_k & P_k C' \\ CP_k & CP_kC' + R \end{bmatrix} \right)$$

Conditional State Estimate

$$\hat{\boldsymbol{x}}_k = (\boldsymbol{x}_k | \boldsymbol{y}_k = y_k) \sim N(\hat{x}_{k|k}, P_{k|k})$$

$$\hat{x}_{k|k} = \bar{x}_k + \overbrace{P_k C'}^{R_{xy}} \overbrace{(CP_kC' + R)^{-1}}^{R_{yy}^{-1}} (y_k - \overbrace{C\bar{x}_k}^{\bar{y}_k})$$

$$P_{k|k} = \overbrace{P_k}^{R_{xx}} - \overbrace{P_k C'}^{R_{xy}} \overbrace{(CP_kC' + R)^{-1}}^{R_{yy}^{-1}} \overbrace{CP_k}^{R_{yx}}$$

Discrete Time Kalman Filter - Time Variant Case

$$\begin{aligned}\boldsymbol{x}_{k+1} &= A_k\boldsymbol{x}_k + G_k\boldsymbol{w}_k\\ \boldsymbol{z}_k &= C_{z,k}\boldsymbol{x}_k\\ \boldsymbol{y}_k &= C_k\boldsymbol{x}_k + \boldsymbol{v}_k\end{aligned}$$

$$\boldsymbol{x}_0 \sim N(\bar{x}_0, P_0)$$

$$\begin{bmatrix}\boldsymbol{w}_k\\ \boldsymbol{v}_k\end{bmatrix} \sim N_{iid}\left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}Q_k & S_k\\ S_k' & R_k\end{bmatrix}\right)$$

Discrete Time Kalman Filter - Time Variant Case

Measurement Update

$$\begin{aligned} R_{e,k} &= C_k P_{k|k-1} C_k' + R_k \\ K_{fx,k} &= P_{k|k-1} C_k' R_{e,k}^{-1} \\ K_{fw,k} &= S_k R_{e,k}^{-1} \end{aligned}$$

$$\begin{aligned} e_k &= y_k - C_k \hat{x}_{k|k-1} & & \\ \hat{x}_{k|k} &= \hat{x}_{k|k-1} + K_{fx,k} e_k & \qquad P_{k|k} &= P_{k|k-1} - K_{fx,k} R_{e,k} K_{fx,k}' \\ \hat{w}_{k|k} &= K_{fw,k} e_k & \qquad Q_{k|k} &= Q_k - K_{fw,k} R_{e,k} K_{fw,k}' \end{aligned}$$

Discrete Time Kalman Filter - Time Variant Case

One-Step Prediction (Time Update)

$$\hat{x}_{k+1|k} = A_k \hat{x}_{k|k} + G_k \hat{w}_{k|k}$$
$$P_{k+1|k} = A_k P_{k|k} A_k' + G_k Q_{k|k} G_k' - A_k K_{fx,k} S_k' G_k' - G_k S_k K_{fx,k}' A_k'$$

Prediction of the process noise and the states

1. Process noise

$$\hat{w}_{k+j|k} = 0 \qquad\qquad j = 1, 2, \ldots$$
$$Q_{k+j|k} = Q_{k+j}$$

2. States

$$\hat{x}_{k+1+j|k} = A_{k+j} \hat{x}_{k+j|k} + G_{k+j} \hat{w}_{k+j|k} = A_{k+j} \hat{x}_{k+j|k} \qquad j = 1, 2, \ldots$$
$$\begin{aligned} P_{k+1+j|k} &= A_{k+j} P_{k+j|k} A_{k+j}' + G_{k+j} Q_{k+j|k} G_{k+j}' \\ &= A_{k+j} P_{k+j|k} A_{k+j}' + G_{k+j} Q_{k+j} G_{k+j}' \end{aligned}$$

Discrete Time Kalman Filter - Time Variant Case

Prediction of the outputs and measurements

1. Outputs

$$
\begin{aligned}
\hat{z}_{k+j|k} &= C_{z,k+j}\hat{x}_{k+j|k} && j = 0, 1, 2, \ldots \\
R_{z,k+j|k} &= C_{z,k+j}P_{k+j|k}C'_{z,k+j}
\end{aligned}
$$

2. Measurement Noise

$$
\begin{aligned}
\hat{v}_{k+j|k} &= 0 && j = 1, 2, \ldots \\
R_{v,k+j|k} &= R_{k+j}
\end{aligned}
$$

3. Measurement

$$
\begin{aligned}
\hat{y}_{k+j|k} &= C_{k+j}\hat{x}_{k+j|k} && j = 1, 2, \ldots \\
R_{y,k+j|k} &= C_{k+j}P_{k+j|k}C'_{k+j} + R_{k+j}
\end{aligned}
$$

Application in an Output Predictor

Require: $\hat{x}_{k|k-1}$, $P_{k|k-1}$, y_k (and the model and covariances)

Compute the innovation

$$\hat{y}_{k|k-1} = C_k \hat{x}_{k|k-1} \qquad e_k = y_k - \hat{y}_{k|k-1} \qquad R_{e,k} = C_k P_{k|k-1} C_k' + R_k$$

Compute the filtered state and the filtered process noise

$$\begin{aligned} K_{fx,k} &= P_{k|k-1} C_k' R_{e,k}^{-1} & K_{fw,k} &= S_k R_{e,k}^{-1} \\ \hat{x}_{k|k} &= \hat{x}_{k|k-1} + K_{fx,k} e_k & P_{k|k} &= P_{k|k-1} - K_{fx,k} R_{e,k} K_{fx,k}' \\ \hat{w}_{k|k} &= K_{fw,k} e_k & Q_{k|k} &= Q_k - K_{fw,k} R_{e,k} K_{fw,k}' \end{aligned}$$

State Predictions

$$\begin{aligned} \hat{x}_{k+1|k} &= A_k \hat{x}_{k|k} + G_k \hat{w}_{k|k} \\ P_{k+1|k} &= A_k P_{k|k} A_k' + G_k Q_{k|k} G_k' - A_k K_{fx,k} S_k' G_k' - G_k S_k K_{fx,k}' A_k' \end{aligned}$$

evt + B_kU_k men NW

$j = 1, 2, \ldots, N-1$

$$\hat{x}_{k+1+j|k} = A_{k+j} \hat{x}_{k+j|k} \qquad P_{k+1+j|k} = A_{k+j} P_{k+j|k} A_{k+j}' + G_{k+j} Q_{k+j} G_{k+j}'$$

Output Predictions (and filtered output)

$$\hat{z}_{k+j|k} = C_{z,k+j} \hat{x}_{k+j|k} \qquad R_{z,k+j} = C_{z,k+j} P_{k+j|k} C_{z,k+j}' \qquad j = 0, 1, \ldots, N$$

return $\left\{ \hat{z}_{k+j|k}, R_{z,k+j|k} \right\}_{j=0}^{N}$ and $\left\{ \hat{x}_{k+1|k}, P_{k+1|k} \right\}$

Discrete Time Kalman Filter - Time Invariant Case

$$\begin{aligned} \boldsymbol{x}_{k+1} &= A\boldsymbol{x}_k + G\boldsymbol{w}_k \\ \boldsymbol{z}_k &= C_z\boldsymbol{x}_k \\ \boldsymbol{y}_k &= C\boldsymbol{x}_k + \boldsymbol{v}_k \end{aligned}$$

$$\boldsymbol{x}_0 \sim N(\bar{x}_0, P_0)$$

$$\begin{bmatrix} \boldsymbol{w}_k \\ \boldsymbol{v}_k \end{bmatrix} \sim N_{iid}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} Q & S \\ S & R \end{bmatrix} \right)$$

Discrete Time Kalman Filter - Time Invariant Case

Measurement Update

$$
\begin{aligned}
R_{e,k} &= C P_{k|k-1} C' + R \\
K_{fx,k} &= P_{k|k-1} C' R_{e,k}^{-1} \\
K_{fw,k} &= S R_{e,k}^{-1}
\end{aligned}
$$

$$
\begin{aligned}
e_k &= y_k - C \hat{x}_{k|k-1} \\
\hat{x}_{k|k} &= \hat{x}_{k|k-1} + K_{fx,k} e_k \qquad & P_{k|k} &= P_{k|k-1} - K_{fx,k} R_{e,k} K'_{fx,k} \\
\hat{w}_{k|k} &= K_{fw,k} e_k & Q_{k|k} &= Q - K_{fw,k} R_{e,k} K'_{fw,k}
\end{aligned}
$$

Discrete Time Kalman Filter - Time Invariant Case

One-Step Prediction (Time Update)

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k} + G\hat{w}_{k|k}$$
$$P_{k+1|k} = AP_{k|k}A' + GQ_{k|k}G' - AK_{fx,k}S'G' - GSK'_{fx,k}A'$$

Prediction of the process noise and the states

1. Process noise

$$\hat{w}_{k+j|k} = 0 \qquad\qquad j = 1, 2, \ldots$$
$$Q_{k+j|k} = Q$$

2. States

$$\hat{x}_{k+1+j|k} = A\hat{x}_{k+j|k} + G\hat{w}_{k+j|k} = A\hat{x}_{k+j|k} \qquad j = 1, 2, \ldots$$
$$\begin{aligned} P_{k+1+j|k} &= AP_{k+j|k}A' + GQ_{k+j|k}G' \\ &= AP_{k+j|k}A' + GQG' \end{aligned}$$

Discrete Time Kalman Filter - Time Invariant Case

Prediction of the outputs and measurements

1. Outputs

$$\hat{z}_{k+j|k} = C_z \hat{x}_{k+j|k} \qquad j = 0, 1, 2, \dots$$
$$R_{z,k+j|k} = C_z P_{k+j|k} C_z'$$

2. Measurement Noise

$$\hat{v}_{k+j|k} = 0 \qquad j = 1, 2, \dots$$
$$R_{v,k+j|k} = R$$

3. Measurement

$$\hat{y}_{k+j|k} = C \hat{x}_{k+j|k} \qquad j = 1, 2, \dots$$
$$R_{y,k+j|k} = C P_{k+j|k} C' + R$$

Discrete Time Kalman Filter - Time Invariant Case

Combining

$$\begin{aligned} R_{e,k} &= CP_{k|k-1}C' + R \\ P_{k|k} &= P_{k|k-1} - K_{fx,k}R_{e,k}K'_{fx,k} \qquad & K_{fx,k} &= P_{k|k-1}C'R_{e,k}^{-1} \\ Q_{k|k} &= Q - K_{fw,k}R_{e,k}K'_{fw,k} \qquad & K_{fw,k} &= SR_{e,k}^{-1} \end{aligned}$$

and

$$P_{k+1|k} = AP_{k|k}A' + GQ_{k|k}G' - AK_{fx,k}S'G' - GSK'_{fx,k}A'$$

yields

$$\begin{aligned} P_{k+1|k} = {} & AP_{k|k-1}A' + GQG' \\ & - (AP_{k|k-1}C' + GS)(CP_{k|k-1}C' + R)^{-1}(AP_{k|k-1}C' + GS)' \end{aligned}$$

Discrete Riccati Equation

$$P_{k+1|k} = AP_{k|k-1}A' + GQG' - (AP_{k|k-1}C' + GS)(CP_{k|k-1}C' + R)^{-1}(AP_{k|k-1}C' + GS)'$$

Assume $P_{k+1|k} \to P$ for $k \to \infty$. Then

$$P = APA' + GQG' - (APC' + GS)(CPC' + R)^{-1}(APC' + GS)'$$

This equation is called the **Discrete Algebraic Riccati Equation**.

Solution methods:

1. In MATLAB the command `dare` is used to solve this equation
   ```
   P = dare(A',C',G*Q*G',R,G*S)
   ```
2. An eigenvalue method for the corresponding stencil (Hamiltonian equation).
3. Alternatively, the solution may be computed by fixed-point iterations in the recursion defining the Discrete Algebraic Riccati Equation.

Discrete Time Kalman Filter - LTI Stationary Case

In the stationary case $P_{k|k-1} = P$ and

$$P = APA' + GQG' - (APC' + GS)(CPC' + R)^{-1}(APC' + GS)'$$

The matrices defining the stationary Kalman filter are

$$\begin{aligned}
R_e &= CPC' + R & R_e &= \lim_{k\to\infty} R_{e,k} \\
K_{fx} &= PC'R_e^{-1} & K_{fx} &= \lim_{k\to\infty} K_{fx,k} \\
K_{fw} &= SR_e^{-1} & K_{fw} &= \lim_{k\to\infty} K_{fw,k} \\
K_{px} &= AK_{fx} + GK_{fw} & K_{px} &= \lim_{k\to\infty} K_{px,k} \\
P_f &= P - K_{fx}R_eK'_{fx} & P_f &= \lim_{k\to\infty} P_{k|k} \\
Q_f &= Q - K_{fw}R_eK'_{fw} & Q_f &= \lim_{k\to\infty} Q_{k|k}
\end{aligned}$$

The equations in Kalman filter and one-step predictor are

$$\begin{aligned}
\hat{y}_{k|k-1} &= C\hat{x}_{k|k-1} \\
e_k &= y_k - \hat{y}_{k|k-1} \\
\hat{x}_{k|k} &= \hat{x}_{k|k-1} + K_{fx}e_k \\
\hat{w}_{k|k} &= K_{fw}e_k \\
\hat{x}_{k+1|k} &= A\hat{x}_{k|k} + G\hat{w}_{k|k} = A\hat{x}_{k|k-1} + K_{px}e_k
\end{aligned}$$

Discrete Time Kalman Filter - LTI Stationary Case

Innovation

$$e_k = y_k - \hat{y}_{k|k-1} = y_k - C\hat{x}_{k|k-1} \qquad R_e = CPC' + R$$

Filtered state and process noise

$$\begin{aligned} \hat{x}_{k|k} &= \hat{x}_{k|k-1} + K_{fx}e_k & \qquad P_f &= P - K_{fx}R_eK_{fx}' \\ \hat{w}_{k|k} &= K_{fw}e_k & Q_f &= Q - K_{fw}R_eK_{fw}' \end{aligned}$$

State Prediction

$$\begin{aligned} \hat{x}_{k+1|k} &= A\hat{x}_{k|k} + G\hat{w}_{k|k} & \qquad P_{x,1} &= P \\ &= A\hat{x}_{k|k-1} + K_{px}e_k & & \\ \hat{x}_{k+1+j|k} &= A\hat{x}_{k+j|k} & P_{x,1+j} &= AP_{x,j}A' + GQG' \qquad j = 1, 2, \ldots \end{aligned}$$

Discrete Time Kalman Filter - LTI Stationary Case

State Prediction

$$\begin{aligned}
\hat{x}_{k+1|k} &= A\hat{x}_{k|k} + G\hat{w}_{k|k} && P_{x,1} = P \\
&= A\hat{x}_{k|k-1} + K_{px}e_k && \\
\hat{x}_{k+1+j|k} &= A\hat{x}_{k+j|k} && P_{x,1+j} = AP_{x,j}A' + GQG' \qquad j = 1, 2, \ldots
\end{aligned}$$

Output Prediction

$$\begin{aligned}
\hat{z}_{k|k} &= C_z\hat{x}_{k|k} && R_{z,0} = C_zP_fC_z' \\
\hat{z}_{k+j|k} &= C_z\hat{x}_{k+j|k} && R_{z,j} = C_zP_{x,j}C_z' \qquad j = 1, 2, \ldots
\end{aligned}$$

Prediction of Measurements

$$\hat{y}_{k+j|k} = C\hat{x}_{k+j|k} \qquad R_{y,j} = CP_{x,j}C' + R \qquad j = 1, 2, \ldots$$

Notice that the covariances are independent of the measured data. Consequently, they may be computed in advance

Application in an Output Predictor

Require: $\hat{x}_{k|k-1}$, y_k (and the model (A, G, C, C_z) and the gains (K_{fx}, K_{fw}))

Compute the one-step prediction of the measurement and the innovation

$$\begin{aligned}\hat{y}_{k|k-1} &= C\hat{x}_{k|k-1}\\ e_k &= y_k - \hat{y}_{k|k-1}\end{aligned}$$

Compute the filtered state and the filtered process noise

$$\begin{aligned}\hat{x}_{k|k} &= \hat{x}_{k|k-1} + K_{fx}e_k\\ \hat{w}_{k|k} &= K_{fw}e_k\end{aligned}$$

State Predictions

$$\begin{aligned}\hat{x}_{k+1|k} &= A\hat{x}_{k|k} + G\hat{w}_{k|k}\\ \hat{x}_{k+j+1|k} &= A\hat{x}_{k+j|k} \qquad j = 1, 2, \dots, N-1\end{aligned}$$

Output Predictions (and filtered output)

$$\hat{z}_{k+j|k} = C_z\hat{x}_{k+j|k} \qquad j = 0, 1, \dots, N$$

return $\left\{\hat{z}_{k+j|k}\right\}_{j=0}^{N}$ and $\hat{x}_{k+1|k}$

The covariances may be computed in advance

Continuous-Discrete Time Kalman Filter

The system evolves in continuous time

$$\begin{aligned} d\boldsymbol{x}(t) &= A_c\boldsymbol{x}(t)dt + G_c d\boldsymbol{\omega}(t) \\ \boldsymbol{z}(t) &= C_z\boldsymbol{x}(t) \end{aligned}$$

while the measurements are at discrete times

$$\boldsymbol{y}(t_k) = C_k\boldsymbol{x}(t_k) + \boldsymbol{v}(t_k)$$

The stochastic specifications are

$$\begin{aligned} &\boldsymbol{x}(t_0) \sim N(\bar{x}_0, P_0) \\ &d\boldsymbol{\omega}(t) \sim N_{iid}(0, I dt) \\ &\boldsymbol{v}(t_k) \sim N_{iid}(0, R_k) \end{aligned}$$

Continuous-Discrete Time Kalman Filter

The equivalent discrete time system is

$$\begin{aligned} \boldsymbol{x}_{k+1} &= A\boldsymbol{x}_k + G\boldsymbol{w}_k \\ \boldsymbol{z}_k &= C_z\boldsymbol{x}_k \\ \boldsymbol{y}_k &= C_k\boldsymbol{x}_k + \boldsymbol{v}_k \end{aligned}$$

and

$$\boldsymbol{x}_0 \sim N(\bar{x}_0, P_0) \qquad \begin{bmatrix} \boldsymbol{w}_k \\ \boldsymbol{v}_k \end{bmatrix} \sim N_{iid}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} Q & S \\ S' & R_k \end{bmatrix} \right)$$

with

$$\begin{aligned} A &= e^{A_c T_s} \\ G &= I \qquad Q = \int_0^{T_s} e^{A_c \tau} G_c G_c' e^{A_c' \tau} d\tau \\ S &= 0 \end{aligned}$$

We may design a discrete time Kalman filter for this system. This is a Kalman filter for the continuous-discrete time system.

Continuous-Discrete Time Kalman Filter

Measurement Update

1. One-step prediction of measurement

$$\hat{y}_{k|k-1} = C_k \hat{x}_{k|k-1}$$

2. Innovation

$$e_k = y_k - \hat{y}_{k|k-1} \qquad R_{e,k} = C_k P_{k|k-1} C_k' + R_k$$

3. Filter constant

$$K_{fx,k} = P_{k|k-1} C_k' R_{e,k}^{-1}$$

4. Filtered state

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_{fx,k} e_k \qquad P_{k|k} = P_{k|k-1} - K_{fx,k} R_{e,k} K_{fx,k}'$$

Continuous-Discrete Time Kalman Filter

Time Update

1. Initial conditions

$$\hat{x}_k(t_k) = \hat{x}_{k|k}$$
$$P_k(t_k) = P_{k|k}$$

2. Solve the differential equations

$$\begin{aligned} \frac{d\hat{x}_k}{dt}(t) &= A_c \hat{x}_k(t) & t \in [t_k \; t_{k+1}] \\ \frac{dP_k}{dt}(t) &= A_c P_k(t) + P_k(t) A_c' + G_c G_c' & t \in [t_k \; t_{k+1}] \end{aligned}$$

3. One-step prediction of the states and associated covariance

$$\hat{x}_{k+1|k} = \hat{x}_k(t_{k+1})$$
$$P_{k+1|k} = P_k(t_{k+1})$$

Systems in Innovation Form

The system in innovation form

$$\begin{aligned} \boldsymbol{x}_{k+1} &= A\boldsymbol{x}_k + K\boldsymbol{e}_k \\ \boldsymbol{y}_k &= C\boldsymbol{x}_k + \boldsymbol{e}_k \end{aligned}$$

with

$$\boldsymbol{x}_0 \sim N(\bar{x}_0, P_0 = 0) \qquad \boldsymbol{e}_k \sim N_{iid}(0, R_e)$$

may be represented as

$$\begin{aligned} \boldsymbol{x}_{k+1} &= A\boldsymbol{x}_k + G\boldsymbol{w}_k \\ \boldsymbol{y}_k &= C\boldsymbol{x}_k + \boldsymbol{v}_k \end{aligned}$$

with

$$\boldsymbol{x}_0 \sim N(\bar{x}_0, P_0 = 0) \quad \begin{bmatrix} \boldsymbol{w}_k \\ \boldsymbol{v}_k \end{bmatrix} \sim N_{iid}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} Q & S \\ S' & R \end{bmatrix}\right)$$

$$G = K \qquad Q = S = R = R_e$$

Discrete Time Extended Kalman Filter

Plant and sensors

$$\begin{aligned} \boldsymbol{x}_{k+1} &= F_k(\boldsymbol{x}_k, \boldsymbol{w}_k) \\ \boldsymbol{z}_k &= g_z(\boldsymbol{x}_k) \\ \boldsymbol{y}_k &= g_k(\boldsymbol{x}_k) + \boldsymbol{v}_k \end{aligned}$$

Distributions

$$\begin{aligned} \boldsymbol{x}_0 &\sim N(\bar{x}_0, P_0) \\ \begin{bmatrix} \boldsymbol{w}_k \\ \boldsymbol{v}_k \end{bmatrix} &\sim N_{iid}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} Q_k & S_k \\ S_k' & R_k \end{bmatrix} \right) \end{aligned}$$

Discrete Time Extended Kalman Filter

Measurement Update (Estimation of filtered state and process noise)

1. One-step prediction of measurement

$$\hat{y}_{k|k-1} = g_k(\hat{x}_{k|k-1})$$
$$C_{k|k-1} = \frac{\partial g_k}{\partial x}(\hat{x}_{k|k-1})$$

2. Innovation

$$e_k = y_k - \hat{y}_{k|k-1} \qquad R_{e,k} = C_{k|k-1} P_{k|k-1} C'_{k|k-1} + R_k$$

3. Filter constants

$$K_{fx,k} = P_{k|k-1} C'_{k|k-1} R_{e,k}^{-1}$$
$$K_{fw,k} = S_k R_{e,k}^{-1}$$

4. Filtered state and filter process noise

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_{fx,k} e_k \qquad P_{k|k} = P_{k|k-1} - K_{fx,k} R_{e,k} K'_{fx,k}$$
$$\hat{w}_{k|k} = K_{fw,k} e_k \qquad Q_{k|k} = Q_k - K_{fw,k} R_{e,k} K'_{fw,k}$$

Discrete Time Extended Kalman Filter

Time Update (one-step prediction of the states)

1. State prediction

$$\hat{x}_{k+1|k} = F_k(\hat{x}_{k|k}, \hat{w}_{k|k})$$

2. Derivatives

$$A_{k|k} = \frac{\partial F_k}{\partial x}(\hat{x}_{k|k}, \hat{w}_{k|k})$$
$$G_{k|k} = \frac{\partial F_k}{\partial w}(\hat{x}_{k|k}, \hat{w}_{k|k})$$

3. Covariance of the state prediction

$$K_{p,k} = A_{k|k}K_{fx,k} + G_{k|k}K_{fw,k} = (A_{k|k}P_{k|k-1}C'_{k|k-1} + G_{k|k}S_k)R_{e,k}^{-1}$$
$$P_{k+1|k} = A_{k|k}P_{k|k}A'_{k|k} + G_{k|k}Q_{k|k}G'_{k|k} - K_{p,k}R_{e,k}K'_{p,k}$$

Discrete Time Extended Kalman Filter

Prediction of the states and the process noise

1. Process noise

$$\hat{w}_{k+j|k} = 0 \qquad Q_{k+j|k} = Q_{k+j} \qquad j = 1, 2, \ldots$$

2. State prediction

$$\hat{x}_{k+j+1|k} = F_{k+j}(\hat{x}_{k+j|k}, \hat{w}_{k+j|k}) = F_{k+j}(\hat{x}_{k+j|k}, 0) \qquad j = 1, 2, \ldots$$

3. Linearization (compute derivatives) for $j = 1, 2, \ldots$

$$\begin{aligned} A_{k+j|k} &= \frac{\partial F_{k+j}}{\partial x}(\hat{x}_{k+j|k}, \hat{w}_{k+j|k}) = \frac{\partial F_{k+j}}{\partial x}(\hat{x}_{k+j|k}, 0) \\ G_{k+j|k} &= \frac{\partial F_{k+j}}{\partial w}(\hat{x}_{k+j|k}, \hat{w}_{k+j|k}) = \frac{\partial F_{k+j}}{\partial w}(\hat{x}_{k+j|k}, 0) \end{aligned}$$

4. Covariances of the state prediction for $j = 1, 2, \ldots$

$$\begin{aligned} P_{k+j+1|k} &= A_{k+j|k} P_{k+j|k} A'_{k+j|k} + G_{k+j|k} Q_{k+j|k} G'_{k+j|k} \\ &= A_{k+j|k} P_{k+j|k} A'_{k+j|k} + G_{k+j|k} Q_{k+j} G'_{k+j|k} \end{aligned}$$

Discrete Time Extended Kalman Filter

Prediction of the outputs

$$
\begin{aligned}
\hat{z}_{k+j|k} &= g_z(\hat{x}_{k+j|k}) & j = 0, 1, 2, \ldots \\
C_{z,k+j|k} &= \frac{\partial g_z}{\partial x}(\hat{x}_{k+j|k}) \\
R_{z,k+j|k} &= C_{z,k+j|k} P_{k+j|k} C'_{z,k+j|k}
\end{aligned}
$$

Prediction of the measurements

$$
\begin{aligned}
\hat{y}_{k+j|k} &= g_{k+j}(\hat{x}_{k+j|k}) & j = 1, 2, \ldots \\
C_{k+j|k} &= \frac{\partial g_{k+j}}{\partial x}(\hat{x}_{k+j|k}) \\
R_{y,k+j|k} &= C_{k+j|k} P_{k+j|k} C'_{k+j|k} + R_{k+j}
\end{aligned}
$$

Continuous-Discrete Time Extended Kalman Filter

$$
\begin{aligned}
d\boldsymbol{x}(t) &= f(\boldsymbol{x}(t))dt + \sigma(t)d\boldsymbol{\omega}(t) \\
\boldsymbol{z}(t) &= g_z(\boldsymbol{x}(t)) \\
\boldsymbol{y}(t_k) &= g_k(\boldsymbol{x}(t_k)) + \boldsymbol{v}(t_k)
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{x}(t_0) &\sim N(\bar{x}_0, P_0) \\
d\boldsymbol{\omega}(t) &\sim N_{iid}(0, Idt) \\
\boldsymbol{v}(t_k) &\sim N_{iid}(0, R_k)
\end{aligned}
$$

Continuous-Discrete Time Extended Kalman Filter

Measurement Update

1. One-step prediction of measurement

$$\hat{y}_{k|k-1} = g_k(\hat{x}_{k|k-1})$$
$$C_{k|k-1} = \frac{\partial g_k}{\partial x}(\hat{x}_{k|k-1})$$

2. Innovation

$$e_k = y_k - \hat{y}_{k|k-1} \qquad R_{e,k} = C_{k|k-1} P_{k|k-1} C'_{k|k-1} + R_k$$

3. Filter constant

$$K_{fx,k} = P_{k|k-1} C'_{k|k-1} R_{e,k}^{-1}$$

4. Filtered state

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_{fx,k} e_k \qquad P_{k|k} = P_{k|k-1} - K_{fx,k} R_{e,k} K'_{fx,k}$$

Continuous-Discrete Time Extended Kalman Filter

Time Update

1. Initial conditions

$$\hat{x}_k(t_k) = \hat{x}_{k|k}$$
$$P_k(t_k) = P_{k|k}$$

2. Solve the differential equations

$$\frac{d\hat{x}_k}{dt}(t) = f(\hat{x}_k(t)) \qquad t \in [t_k \; t_{k+1}]$$
$$A_k(t) = \frac{\partial f}{\partial x}(\hat{x}_k(t)) \qquad t \in [t_k \; t_{k+1}]$$
$$\frac{dP_k}{dt}(t) = A_k(t)P_k(t) + P_k(t)A_k(t)' + \sigma(t)\sigma(t)' \qquad t \in [t_k \; t_{k+1}]$$

3. One-step prediction of the states and associated covariance

$$\hat{x}_{k+1|k} = \hat{x}_k(t_{k+1})$$
$$P_{k+1|k} = P_k(t_{k+1})$$

Continuous-Discrete Time Extended Kalman Filter

State Predictions

1. Initial conditions

$$\hat{x}_k(t_k) = \hat{x}_{k|k}$$
$$P_k(t_k) = P_{k|k}$$

2. Solve the differential equations

$$\frac{d\hat{x}_k}{dt}(t) = f(\hat{x}_k(t)) \qquad t \in [t_k \; t_{k+N}]$$
$$A_k(t) = \frac{\partial f}{\partial x}(\hat{x}_k(t)) \qquad t \in [t_k \; t_{k+N}]$$
$$\frac{dP_k}{dt}(t) = A_k(t)P_k(t) + P_k(t)A_k(t)' + \sigma(t)\sigma(t)' \quad t \in [t_k \; t_{k+N}]$$

3. One-step prediction of the states and associated covariance

$$\hat{x}_{k+j|k} = \hat{x}_k(t_{k+j}) \qquad j = 1, 2, \ldots, N$$
$$P_{k+j|k} = P_k(t_{k+j}) \qquad j = 1, 2, \ldots, N$$

Continuous-Discrete Time Extended Kalman Filter

Prediction of the outputs (at discrete time points)

$$\begin{aligned}
\hat{z}_{k+j|k} &= g_z(\hat{x}_{k+j|k}) && j = 0, 1, 2, \ldots, N \\
C_{z,k+j|k} &= \frac{\partial g_z}{\partial x}(\hat{x}_{k+j|k}) \\
R_{z,k+j|k} &= C_{z,k+j|k} P_{k+j|k} C'_{z,k+j|k}
\end{aligned}$$

Prediction of the measurements

$$\begin{aligned}
\hat{y}_{k+j|k} &= g_{k+j}(\hat{x}_{k+j|k}) && j = 1, 2, \ldots, N \\
C_{k+j|k} &= \frac{\partial g_{k+j}}{\partial x}(\hat{x}_{k+j|k}) \\
R_{y,k+j|k} &= C_{k+j|k} P_{k+j|k} C'_{k+j|k} + R_{k+j}
\end{aligned}$$

Continuous-Discrete Time Extended Kalman Filter

Prediction of the outputs (continuously)

$$
\begin{aligned}
\hat{z}_k(t) &= g_z(\hat{x}_k(t)) \qquad\qquad t \in [t_k \, t_{k+N}] \\
C_{z,k}(t) &= \frac{\partial g_z}{\partial x}(\hat{x}_k(t)) \\
R_{z,k}(t) &= C_{z,k}(t) P_k(t) C_{z,k}(t)'
\end{aligned}
$$

Questions and Comments

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