Finite Horizon MPC for Systems in Innovation Form

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Introduction

- Model Predictive Control is one of the most successful advanced control technologies
- The major part of industrial MPC commissioning is generation of data and identification of models
- Successful implementation of MPC requires a good model for the prediction
- MPC and System Identification have developed as two separate disciplines

In this contribution we will provide one of the overlooked missing links needed for bridging the gap between MPC and system identification.

We provide the correct finite horizon LQG control law for **state space models in innovation form** and demonstrate how this law can be used in the tuning of constrained finite horizon predictive control systems.

Systems in Innovation Form

Models identified by system identification methods

- ARX

$$A(q^{-1})\boldsymbol{y}_k = B(q^{-1})u_k + \boldsymbol{e}_k$$

- ARMAX

$$A(q^{-1})\boldsymbol{y}_k = B(q^{-1})u_k + C(q^{-1})\boldsymbol{e}_k$$

- Box-Jenkins

$$\boldsymbol{y}_k = \frac{B(q^{-1})}{A(q^{-1})}u_k + \frac{C(q^{-1})}{D(q^{-1})}\boldsymbol{e}_k$$

- Subspace methods

may be realized as models in innovation form

$$\begin{aligned}\boldsymbol{x}_{k+1} &= A\boldsymbol{x}_k + Bu_k + K\boldsymbol{e}_k \\ \boldsymbol{y}_k &= C\boldsymbol{x}_k + \boldsymbol{e}_k\end{aligned}$$

These models have correlated process and measurement noise

State Space Realization of ARMAX Models - MIMO

$$A(q^{-1})\boldsymbol{y}(t) = B(q^{-1})\boldsymbol{u}(t) + C(q^{-1})\boldsymbol{e}(t)$$

with

$$\begin{aligned} A(q^{-1}) &= I + A_1 q^{-1} + A_2 q^{-2} + \ldots + A_n q^{-n} \\ B(q^{-1}) &= B_1 q^{-1} + B_2 q^{-2} + \ldots + B_n q^{-n} \\ C(q^{-1}) &= I + C_1 q^{-1} + C_2 q^{-2} + \ldots + C_n q^{-n} \end{aligned}$$

may be realized as a state space model in innovation form

$$\begin{aligned} \boldsymbol{x}_{k+1} &= A\boldsymbol{x}_k + Bu_k + K\boldsymbol{e}_k \\ \boldsymbol{y}_k &= C\boldsymbol{x}_k + \boldsymbol{e}_k \end{aligned}$$

$$A = \left[\begin{array}{c|cccc} -A_1 & I & 0 & \ldots & 0 \\ -A_2 & 0 & I & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -A_{n-1} & 0 & 0 & \ldots & I \\ \hline -A_n & 0 & 0 & \ldots & 0 \end{array}\right] B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_{n-1} \\ B_n \end{bmatrix} K = \begin{bmatrix} C_1 - A_1 \\ C_2 - A_2 \\ \vdots \\ C_{n-1} - A_{n-1} \\ C_n - A_n \end{bmatrix}$$

$$C = \begin{bmatrix} I & 0 & 0 & \ldots & 0 \end{bmatrix}$$

Offset Free Control

Model:

$$A(q^{-1})\boldsymbol{y}_k = B(q^{-1})u_k + C(q^{-1})\boldsymbol{e}_k$$

$$A(q^{-1}) = I + A_1q^{-1} + A_2q^{-2} + \ldots + A_nq^{-n}$$
$$B(q^{-1}) = B_1q^{-1} + B_2q^{-2} + \ldots + B_nq^{-n}$$
$$C(q^{-1}) = I + C_1q^{-1} + C_2q^{-n} + \ldots + C_nq^{-n}$$

Disturbance Model:

$$F(q^{-1})\boldsymbol{e}_k = G(q^{-1})\boldsymbol{\varepsilon}_k \quad \boldsymbol{\varepsilon}_k \sim N(0, R_\varepsilon)$$

$$F(q^{-1}) = I + F_1q^{-1} + F_2q^{-2} + \ldots + F_mq^{-m}$$
$$G(q^{-1}) = I + G_1q^{-1} + G_2q^{-2} + \ldots + G_mq^{-m}$$

Combined Model (SISO case):

$$A(q^{-1})y_k = B(q^{-1})u_k + \frac{T(q^{-1})}{1 - q^{-1}}\varepsilon_k$$

$F(q^{-1}) = 1 - q^{-1}$, $T(q^{-1}) = C(q^{-1})G(q^{-1})$

Industrial Model Predictive Control

Qin and Badgwell (2003)

Table 4
Comparison of linear MPC control technology

Company	Aspen Tech	Honeywell Hi-Spec	Adersa	Adersa	Invensys	SGS
Product	DMC-plus	RMPCT	HIECON	PFC	Connois.	SMOC
Linear Model	FSR	ARX, TF	FIR	LSS,TF,ARX	ARX,FIR	LSS
Forms[a]	L,S,I,U	L,S,I,U	L,S,I	L,N,S,I,U	L,S,I,U	L,S,I,U
Feedback[b]	CD, ID	CD, ID	CD, ID	CD, ID	CD, ID	KF
Rem Ill-cond[c]	IMS	SVT	—	—	IMS	IMS
SS Opt Obj[d]	L/Q[I, O], …, R	Q[I,O]	—	Q[I,O]	L[I,O]	Q[I,O],R
SS Opt Const[e]	IH,OS,R	IH,OH	—	IH,OH	IH,OH	IH,OS
Dyn Opt Obj[f]	Q[I,O,M],S	Q[I,O]	Q[O],Q[I]	Q[I,O],S	Q[I,O,M]	Q[I,O]
Dyn Opt Const[g]	IH	IH,OS	IH,OH,OS,R	IA,OH,OS,R	IH,OS,R	IH,OS
Output Traj[h]	S,Z	S,Z,F	S,Z,RT	S,Z,RT	S,Z	S,Z,RTB,F
Output Horiz[i]	FH	FH	FH	CP	FH	FH
Input Param[j]	MMB	MM	SM	BF	MMB	MMB
Sol. Method[k]	SLS	ASQP	ASQP	LS	ASQP	ASQP
References	Cutler and Ramaker (1979) and DMC Corp., (1994)	Honeywell Inc., (1995)	Richalet (1993)	Richalet (1993)		Marquis and Broustail (1998)

[a] Model form: Finite impulse response (FIR), finite step response (FSR), Laplace transfer function (TF), linear state-space (LSS), auto-regressive with exogenous input (ARX), linear (L), nonlinear (N), stable (S), integrating (I), unstable (U).

[b] Feedback: Constant output disturbance (CD), integrating output disturbance (ID), Kalman filter (KF).

[c] Removal of Ill-conditioning: Singular value thresholding (SVT), input move suppression (IMS).

[d] Steady-state optimization objective: linear (L), quadratic (Q), inputs (I), outputs (O), multiple sequential objectives (…), outputs ranked in order of priority (R).

[e] Steady-state optimization constraints: Input hard maximum, minimum, and rate of change constraints (IH), output hard maximum and minimum constraints (OH), constraints ranked in order of priority (R).

[f] Dynamic optimization objective: Quadratic (Q), inputs (I), Outputs (O), input moves (M), sub-optimal solution (S).

[g] Dynamic optimization constraints: Input hard maximum, minimum and rate of change constraints, IH with input acceleration constraints (IA), output hard maximum and minimum constraints (OH), output soft maximum and minimum constraints (OS), constraints ranked in order of priority (R).

[h] Output trajectory: Setpoint (S), zone (Z), reference trajectory (RT), RT bounds (RTB), funnel (F).

[i] Output horizon: Finite horizon (FH), coincidence points (CP).

[j] Input parameterization: Single move (SM), multiple move (MM), MM with blocking (MMB), basis functions (BF).

[k] Solution method: Least squares (LS), sequential LS (SLS), active set quadratic program (ASQP).

Filtering and Prediction

State Space System with Correlated Process and Measurement Noise

$$
\begin{aligned}
\boldsymbol{x}_{k+1} &= A\boldsymbol{x}_k + Bu_k + G\boldsymbol{w}_k \\
\boldsymbol{y}_k &= C\boldsymbol{x}_k + \boldsymbol{v}_k
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{x}_0 &\sim N(\hat{x}_{0|-1}, P_{0|-1}) \\
\begin{bmatrix} \boldsymbol{w}_k \\ \boldsymbol{v}_k \end{bmatrix} &\sim N_{iid}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} R_{ww} & R_{wv} \\ R_{vw} & R_{vv} \end{bmatrix} \right)
\end{aligned}
$$

1. State space models in innovation form is a special case of models with correlated noise
2. FIR, ARX, ARMAX, Box-Jenkins models may be realized as state space models in innovation form
3. Subspace methods identify state space models in innovation form

The state space model in innovation form

$$\begin{aligned} \boldsymbol{x}_{k+1} &= A\boldsymbol{x}_k + Bu_k + K\boldsymbol{e}_k \\ \boldsymbol{y}_k &= C\boldsymbol{x}_k + \boldsymbol{e}_k \end{aligned}$$

corresponds to

$$\begin{aligned} \boldsymbol{x}_{k+1} &= A\boldsymbol{x}_k + Bu_k + G\boldsymbol{w}_k \\ \boldsymbol{y}_k &= C\boldsymbol{x}_k + \boldsymbol{v}_k \end{aligned}$$

$$\begin{aligned} \boldsymbol{x}_0 &\sim N(\hat{x}_{0|-1}, P_{0|-1}) \\ \begin{bmatrix} \boldsymbol{w}_k \\ \boldsymbol{v}_k \end{bmatrix} &\sim N_{iid}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} R_{ww} & R_{wv} \\ R_{vw} & R_{vv} \end{bmatrix}\right) \end{aligned}$$

with

$$\begin{bmatrix} \boldsymbol{w}_k \\ \boldsymbol{v}_k \end{bmatrix} = \begin{bmatrix} \boldsymbol{e}_k \\ \boldsymbol{e}_k \end{bmatrix} \sim N_{iid}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} R_e & R_e \\ R_e & R_e \end{bmatrix}\right) \quad G = K$$

Stationary Kalman Filter

Discrete Algebraic Riccati Equation (DARE)

$$P = APA' + GR_{ww}G' - (APC' + GR_{wv})(R_{vv} + CPC')^{-1}(APC' + GR_{wv})'$$

Gains

$$\begin{aligned} R_{fe} &= CPC' + R_{vv} \\ K_{fx} &= PC'R_{fe}^{-1} \\ K_{fw} &= R_{wv}R_{fe}^{-1} \end{aligned}$$

Filtered estimates:

$$\begin{aligned} e_k &= y_k - C\hat{x}_{k|k-1} \\ \hat{x}_{k|k} &= \hat{x}_{k|k-1} + K_{fx}e_k \\ \hat{w}_{k|k} &= K_{fw}e_k \end{aligned}$$

Predicted estimates:

$$\begin{aligned} \hat{x}_{k+1|k} &= A\hat{x}_{k|k} + Bu_k + G\hat{w}_{k|k} \\ \hat{x}_{k+1+j|k} &= A\hat{x}_{k+j|k} + Bu_{k+j} & j &= 1, 2, \ldots, N-1 \\ \hat{y}_{k+j|k} &= C\hat{x}_{k+j|k} & j &= 1, 2, \ldots, N \end{aligned}$$

Regulation

Regulation - Unconstrained

The finite horizon **unconstrained** linear quadratic regulation problem

$$\min_{\{u_{k+j|k}\}_{j=0}^{N-1}} \phi = \frac{1}{2}\sum_{j=0}^{N-1} \left\| \hat{y}_{k+1+j|k} - r_{k+1+j|k} \right\|_Q^2 + \left\| \Delta u_{k+j|k} \right\|_S^2$$

$$\begin{aligned} s.t. \qquad & \hat{x}_{k+1|k} = A\hat{x}_{k|k} + Bu_{k|k} + G\hat{w}_{k|k} & \\ & \hat{x}_{k+1+j|k} = A\hat{x}_{k+j|k} + Bu_{k+j|k} & j = 1, ..., N-1 \\ & \hat{y}_{k+j|k} = C\hat{x}_{k+j|k} & j = 1, \dots, N \end{aligned}$$

has the solution

$$u_{k|k} = L_x \hat{x}_{k|k} + L_w \hat{w}_{k|k} + L_R R_k + L_u u_{k-1|k}$$

Regulation - Constrained

The finite horizon **constrained** linear quadratic regulation problem

$$\min_{\{u_{k+j|k}\}_{j=0}^{N-1}} \phi = \frac{1}{2} \sum_{j=0}^{N-1} \left\| \hat{y}_{k+1+j|k} - r_{k+1+j|k} \right\|_Q^2 + \left\| \Delta u_{k+j|k} \right\|_S^2$$

$$\begin{aligned}
s.t. \quad & \hat{x}_{k+1|k} = A\hat{x}_{k|k} + Bu_{k|k} + G\hat{w}_{k|k} \\
& \hat{x}_{k+1+j|k} = A\hat{x}_{k+j|k} + Bu_{k+j|k} && j = 1, ..., N-1 \\
& \hat{y}_{k+j|k} = C\hat{x}_{k+j|k} && j = 1, \ldots, N \\
& u_{\min} \leq u_{k+j|k} \leq u_{\max} && j = 0, 1, \ldots, N-1 \\
& \Delta u_{\min} \leq \Delta u_{k+j|k} \leq \Delta u_{\max} && j = 0, 1, \ldots, N-1
\end{aligned}$$

has a solution that is computed by a convex QP algorithm and is denoted

$$u_{k|k} = \mu_k(\hat{x}_{k|k}, \hat{w}_{k|k}, \{r_{k+j|k}\}_{j=1}^{N}, u_{k-1|k})$$

MPC Controller Algorithm

Require: y_k, $\left\{r_{k+j|k}\right\}_{j=1}^{N}$, $\hat{x}_{k|k-1}$, $u_{k-1|k}$

Filter:

$e_k = y_k - C\hat{x}_{k|k-1}$

$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_{fx}e_k$

$\hat{w}_{k|k} = K_{fw}e_k$

Regulator:

$u_{k|k} = \mu(\hat{x}_{k|k}, \hat{w}_{k|k}, \left\{r_{k+j|k}\right\}_{j=1}^{N}, u_{k-1|k})$

One-step predictor:

$\hat{x}_{k+1|k} = A\hat{x}_{k|k} + Bu_{k|k} + G\hat{w}_{k|k}$

Return: $u_{k|k}$, $\hat{x}_{k+1|k}$

Unconstrained case:

$$\begin{aligned} u_{k|k} &= \mu_k(\hat{x}_{k|k}, \hat{w}_{k|k}, \left\{r_{k+j|k}\right\}_{j=1}^{N}, u_{k-1|k}) \\ &= L_x\hat{x}_{k|k} + L_w\hat{w}_{k|k} + L_R R_k + L_u u_{k-1|k} \end{aligned}$$

Closed-Loop Properties

System

$$\begin{aligned}\boldsymbol{x}_{k+1} &= A\boldsymbol{x}_k + Bu_k + G\boldsymbol{w}_k + Ed_k\\ \boldsymbol{z}_k &= C_z\boldsymbol{x}_k\\ \boldsymbol{y}_k &= C\boldsymbol{x}_k + \boldsymbol{v}_k\end{aligned}$$

The model used for the controller design

$$\begin{aligned}\bar{\boldsymbol{x}}_{k+1} &= \hat{A}\bar{\boldsymbol{x}}_k + \hat{B}u_k + \hat{G}\bar{\boldsymbol{w}}_k\\ \boldsymbol{y}_k &= \hat{C}\bar{\boldsymbol{x}}_k + \bar{\boldsymbol{v}}_k\end{aligned}$$

gives the output feedback LQ controller

$$\begin{aligned}e_k &= y_k - \hat{C}\hat{x}_{k|k-1}\\ \hat{x}_{k|k} &= \hat{x}_{k|k-1} + K_{fx}e_k\\ \hat{w}_{k|k} &= K_{fw}e_k\\ u_k &= L_x\hat{x}_{k|k} + L_w\hat{w}_{k|k} + L_R R_k + L_u u_{k-1}\\ \hat{x}_{k+1|k} &= \hat{A}\hat{x}_{k|k} + \hat{B}u_k + \hat{G}\hat{w}_{k|k}\end{aligned}$$

Closed-Loop State-Space System

Combining

1. the system model
2. the equations for the unconstrained LQ controller in state space form

gives the following state space representation of the closed loop system

$$\begin{bmatrix} x_{k+1} \\ \hat{x}_{k+1|k} \\ u_k \end{bmatrix} = \begin{bmatrix} A_{cl} & B(L_x - \Lambda\hat{C}) & BL_u \\ \hat{\Lambda}C & \hat{A} + \hat{B}L_x - \hat{\Lambda}\hat{C} & \hat{B}L_u \\ \Lambda C & L_x - \Lambda\hat{C} & L_u \end{bmatrix} \begin{bmatrix} x_k \\ \hat{x}_{k|k-1} \\ u_{k-1} \end{bmatrix} + \begin{bmatrix} G & B\Lambda & BL_R & E \\ 0 & \hat{\Lambda} & \hat{B}L_R & 0 \\ 0 & \Lambda & L_R & 0 \end{bmatrix} \begin{bmatrix} w_k \\ v_k \\ R_k \\ d_k \end{bmatrix}$$

$$\begin{bmatrix} z_k \\ y_k \\ u_k \end{bmatrix} = \begin{bmatrix} C_z & 0 & 0 \\ C & 0 & 0 \\ \Lambda C & L_x - \Lambda\hat{C} & L_u \end{bmatrix} \begin{bmatrix} x_k \\ \hat{x}_{k|k-1} \\ u_{k-1} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & \Lambda & L_R & 0 \end{bmatrix} \begin{bmatrix} w_k \\ v_k \\ R_k \\ d_k \end{bmatrix}$$

$$\begin{aligned} \Lambda &= L_x K_{fx} + L_w K_{fw} \\ \hat{\Lambda} &= \hat{A}K_{fx} + \hat{G}K_{fw} + \hat{B}\Lambda \\ A_{cl} &= A + B\Lambda C \end{aligned}$$

State Space Model, Frequency Domain, Variance

The state space model

$$x_{k+1} = Ax_k + Bu_k$$
$$y_k = Cx_k + Du_k$$

may be represented as the input-output model

$$Y(z) = G(z)U(z) \qquad G(z) = C(zI - A)^{-1}B + D$$

This IO-model is used to compute $|G(z)|$ for $z = e^{i\omega T_s}$ for various frequencies.

Assuming that A is stable, the stationary variance of the outputs, R_y, may be computed by

$$R_x = AR_xA' + BR_uB'$$
$$R_y = CR_xC' + DR_uD'$$

R_u is the variance of the input signal. R_x is computed by solving a discrete Lyapunov equation.

Simulated Furnace Example

Simulated Furnace Example - System

The system

$$\boldsymbol{Z}(s) = \left[\frac{20\, e^{-50s}}{(40s+1)(4s+1)}\right] U(s) + \left[\frac{-5\, e^{-10s}}{(5s+1)^2}\right] (D(s) + \boldsymbol{W}(s))$$
$$\boldsymbol{y}(t_k) = \boldsymbol{z}(t_k) + \boldsymbol{v}(t_k)$$

is realized as a discrete-time state space system

$$\begin{aligned} \boldsymbol{x}_{k+1} &= A\boldsymbol{x}_k + Bu_k + G\boldsymbol{w}_k + Ed_k \\ \boldsymbol{z}_k &= C_z\boldsymbol{x}_k \\ \boldsymbol{y}_k &= C\boldsymbol{x}_k + \boldsymbol{v}_k \end{aligned}$$

using a sample time of $T_2 = 2$

Simulated Furnace Example - Controller

The controller prediction model

$$A(q^{-1})\boldsymbol{y}_k = B(q^{-1})u_k + \frac{1-\alpha q^{-1}}{1-q^{-1}}\boldsymbol{e}_k$$

$$A(q^{-1}) = 1 - 1.5578q^{-1} + 0.5769q^{-2}$$
$$B(q^{-1}) = 0.2094q^{-26} + 0.1744q^{-27}$$

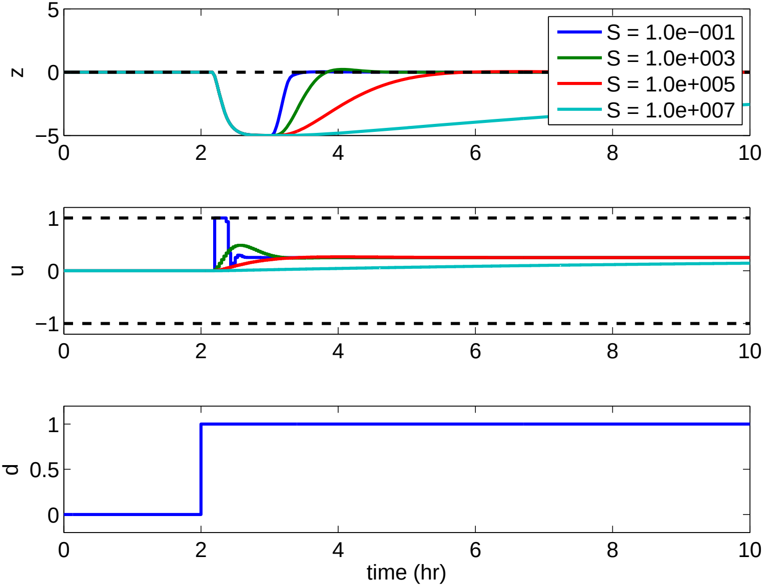
is realized as state space model in innovation form

$$\bar{\boldsymbol{x}}_{k+1} = \bar{A}\bar{\boldsymbol{x}}_k + \bar{B}u_k + \bar{K}\boldsymbol{e}_k$$
$$\boldsymbol{y}_k = \bar{C}\bar{\boldsymbol{x}}_k + \boldsymbol{e}_k$$

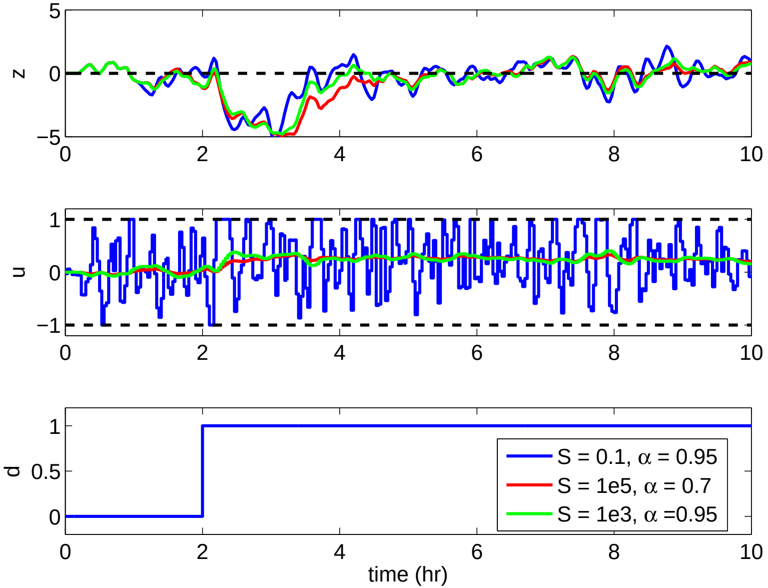
Controller objective and constraints ($N = 150$):

$$\phi = \sum_{j=0}^{N-1} \left\| \hat{y}_{k+j+1|k} - r_{k+j+1|k} \right\|_2^2 + \left\| \Delta u_{k+j|k} \right\|_S^2$$
$$-1 \leq u_{k+j|k} \leq 1 \quad j = 0, 1, \ldots, N-1$$

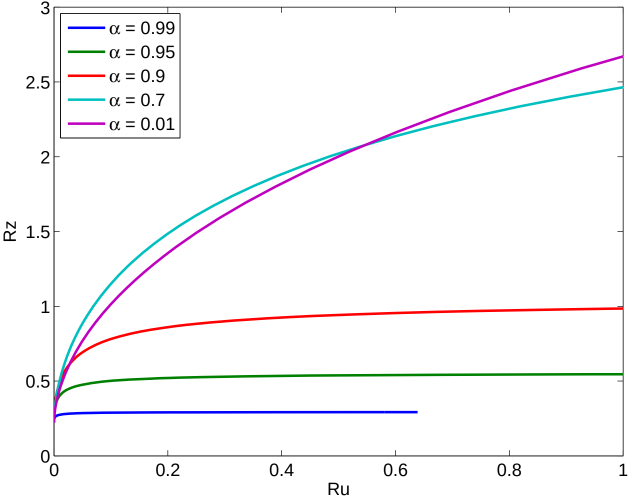
Deterministic Closed-Loop Simulation - $\alpha = 0.95$



Stochastic Closed-Loop Simulation

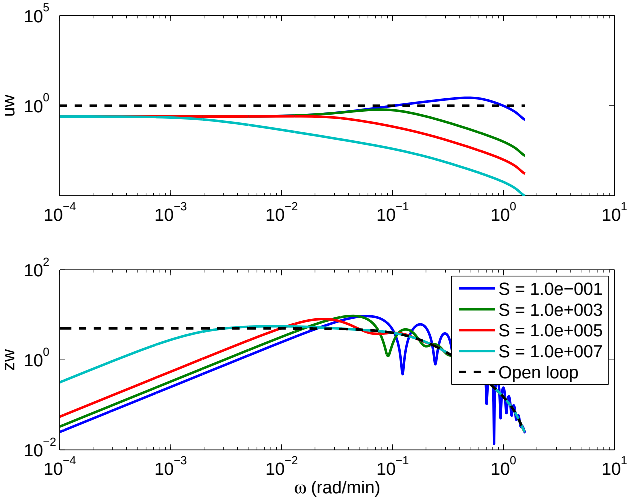


Closed-Loop Variance



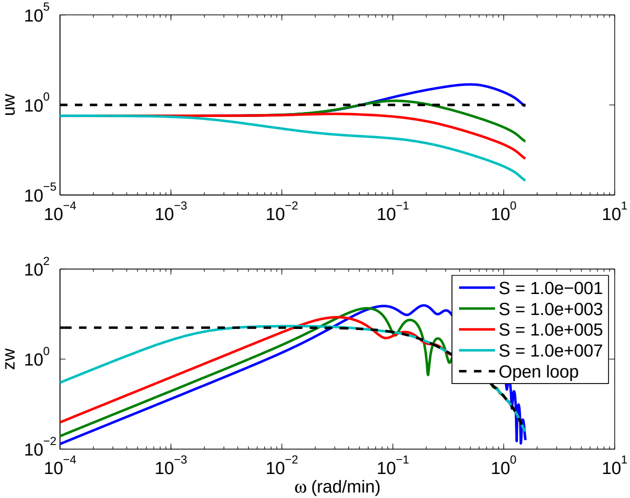
Variance of the output vs the input for different tunings of the controller

Closed-Loop Simulation - $\alpha = 0.95$



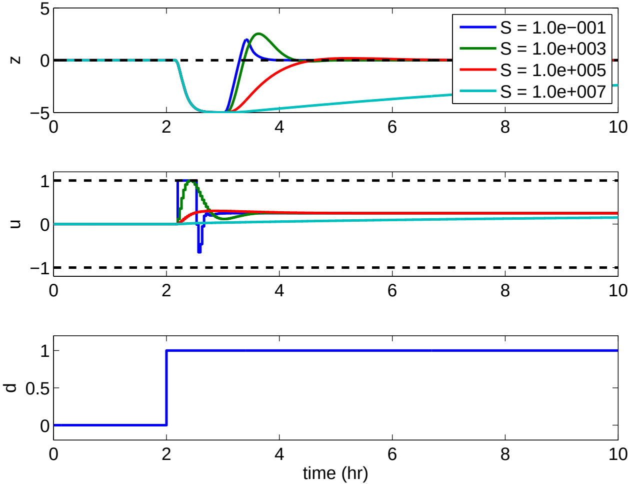
Frequency Response

Closed-Loop Simulation - $\alpha = 0.7$

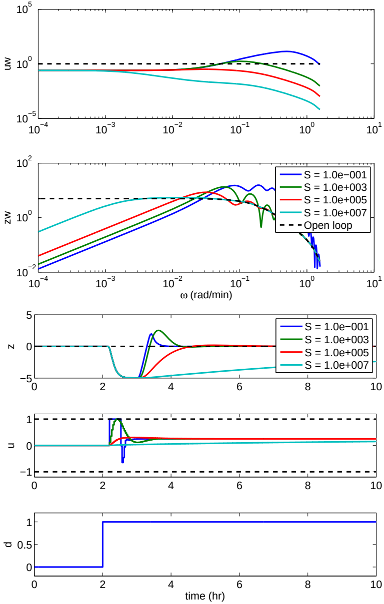


Frequency Response

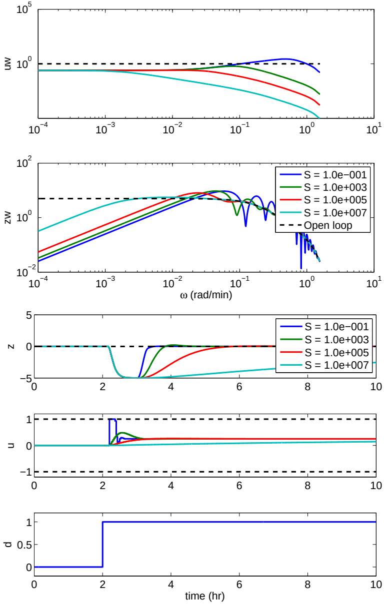
Closed-Loop Simulation - $\alpha = 0.7$



Closed-loop response of input constrained MPC.

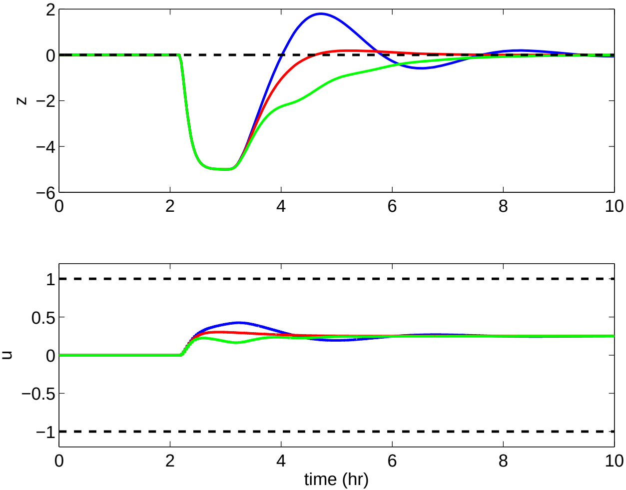


$\alpha = 0.7$



$\alpha = 0.95$

Robustness - Closed-Loop Simulation $(S, \alpha) = (10^5, 0.7)$



MPC with a model corresponding to the gains $K = 40$ (blue), $K = 20$ (red), and $K = 10$ (green). The plant has gain $K = 20$.

Conclusion

- We provide the correct control law for finite horizon linear quadratic systems with correlated process and measurement noise
- We demonstrate how this case is relevant for state space models in innovation form
- State space models in innovation form arise from FIR, ARX, ARMAX, BJ and subspace methods for system identification
- The correct control laws may be used to develop analytical closed loop expressions for the system with the unconstrained controller. These expressions can be used to guide the non-trivial tuning of Model Predictive Controllers

Questions and Comments

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