Nonlinear State Estimation

02619 Model Predictive Control - Lecture 07A

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Major Methods for Nonlinear State Estimation

- Extended Kalman Filter (EKF)
- Unscented Kalman Filter (UKF)
- Ensemble Kalman Filter (EnKF)
- Particle Filtering (PF)
- Moving Horizon Estimation (MHE)
  - optimization based estimation
- Fokker-Planck Equation
  - also called Kolmogorow's forward equation

Model Classes

- Discrete-Discrete Systems
  - Additive Process Noise

$$
\begin{aligned}
\boldsymbol{x}_{k+1} &= F_k(\boldsymbol{x}_k) + \boldsymbol{w}_k, \qquad & \boldsymbol{w}_k &\sim N_{iid}(0, Q_k), \\
\boldsymbol{y}_k &= h_k(\boldsymbol{x}_k) + \boldsymbol{v}_k, \qquad & \boldsymbol{v}_k &\sim N_{iid}(0, R_k).
\end{aligned}
$$

  - General Process Noise

$$
\begin{aligned}
\boldsymbol{x}_{k+1} &= F_k(\boldsymbol{x}_k, \boldsymbol{w}_k), \qquad & \boldsymbol{w}_k &\sim N_{iid}(0, Q_k), \\
\boldsymbol{y}_k &= h_k(\boldsymbol{x}_k) + \boldsymbol{v}_k, \qquad & \boldsymbol{v}_k &\sim N_{iid}(0, R_k).
\end{aligned}
$$

- Continuous-Discrete Systems

$$
\begin{aligned}
d\boldsymbol{x}(t) &= f(\boldsymbol{x}(t))dt + \sigma(\boldsymbol{x}(t))d\boldsymbol{\omega}(t), \quad & d\boldsymbol{\omega}(t) &\sim N_{iid}(0, I dt), \\
\boldsymbol{y}(t_k) &= h_k(\boldsymbol{x}(t_k)) + \boldsymbol{v}_k, & \boldsymbol{v}_k &\sim N_{iid}(0, R_k).
\end{aligned}
$$

Extented Kalman Filter

(EKF)

EKF: References

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- **Discrete-Discrete System - Additive Process Noise:**

$$
\begin{aligned}
\boldsymbol{x}_{k+1} &= F_k(\boldsymbol{x}_k) + \boldsymbol{w}_k, & \boldsymbol{w}_k &\sim N_{iid}(0, Q_k), \\
\boldsymbol{y}_k &= h_k(\boldsymbol{x}_k) + \boldsymbol{v}_k, & \boldsymbol{v}_k &\sim N_{iid}(0, R_k).
\end{aligned}
$$

EKF: Discrete-Discrete System - Additive Process Noise

- **Filtering:**
  - Given y_k, $\hat{x}_{k|k-1}$ and $P_{k|k-1}$
  - Compute innovation and covariance

$$e_k = y_k - \hat{y}_{k|k-1}, \qquad \hat{y}_{k|k-1} = h_k(\hat{x}_{k|k-1}),$$

$$R_{e,k} = \langle \boldsymbol{e}_k, \boldsymbol{e}_k \rangle = C_k P_{k|k-1} C_k^T + R_k, \qquad C_k = \frac{\partial h_k}{\partial x}(\hat{x}_{k|k-1}).$$

  - Compute Kalman gain

$$K_{fx,k} = \langle \boldsymbol{x}_k, \boldsymbol{e}_k \rangle \langle \boldsymbol{e}_k, \boldsymbol{e}_k \rangle^{-1} = P_{k|k-1} C_k^T R_{e,k}^{-1}.$$

  - Compute filtered state mean and covariance

$$\begin{aligned} \hat{x}_{k|k} &= \hat{x}_{k|k-1} + K_{fx,k} e_k, \\ P_{k|k} &= P_{k|k-1} - K_{fx,k} R_{e,k} K_{fx,k}^T \\ &= \left(I - K_{fx,k} C_k\right) P_{k|k-1} \left(I - K_{fx,k} C_k\right)^T + K_{fx,k} R_k K_{fx,k}^T. \end{aligned}$$

EKF: Discrete-Discrete System - Additive Process Noise

- **One-step prediction:**
  - Given $\hat{x}_{k|k}$ and $P_{k|k}$
  - Compute prediction of state mean

$$\hat{x}_{k+1|k} = F_k(\hat{x}_{k|k}).$$

  - Compute prediction of state covariance

$$P_{k+1|k} = A_k P_{k|k} A_k^T + Q_k, \qquad A_k = \frac{\partial F_k}{\partial x}(\hat{x}_{k|k}).$$

- **j-step prediction:**
  - Given $\hat{x}_{k+j-1|k}$ and $P_{k+j-1|k}$
  - Compute predicton of state mean

$$\hat{x}_{k+j|k} = F_{k+j-1}(\hat{x}_{k+j-1|k}).$$

  - Compute prediction of state covariance

$$\begin{aligned} P_{k+j|k} &= A_{k+j-1} P_{k+j-1|k} A_{k+j-1}^T + Q_{k+j-1}, \\ A_{k+j-1} &= \frac{\partial F_{k+j-1}}{\partial x}(\hat{x}_{k+j-1|k}). \end{aligned}$$

- **Discrete-Discrete System - General Process Noise:**

$$
\begin{aligned}
\boldsymbol{x}_{k+1} &= F_k(\boldsymbol{x}_k, \boldsymbol{w}_k), & \boldsymbol{w}_k &\sim N_{iid}(0, Q_k), \\
\boldsymbol{y}_k &= h_k(\boldsymbol{x}_k) + \boldsymbol{v}_k, & \boldsymbol{v}_k &\sim N_{iid}(0, R_k).
\end{aligned}
$$

EKF: Discrete-Discrete System - General Process Noise

- **Filtering:**
  - Given y_k, $\hat{x}_{k|k-1}$ and $P_{k|k-1}$
  - Compute innovation and covariance

$$
\begin{aligned}
e_k &= y_k - \hat{y}_{k|k-1}, & \hat{y}_{k|k-1} &= h_k(\hat{x}_{k|k-1}), \\
R_{e,k} &= \langle \boldsymbol{e}_k, \boldsymbol{e}_k \rangle = C_k P_{k|k-1} C_k^T + R_k, & C_k &= \frac{\partial h_k}{\partial x}(\hat{x}_{k|k-1}).
\end{aligned}
$$

  - Compute Kalman gain

$$
K_{fx,k} = \langle \boldsymbol{x}_k, \boldsymbol{e}_k \rangle \langle \boldsymbol{e}_k, \boldsymbol{e}_k \rangle^{-1} = P_{k|k-1} C_k^T R_{e,k}^{-1}.
$$

  - Compute filtered state mean and covariance

$$
\begin{aligned}
\hat{x}_{k|k} &= \hat{x}_{k|k-1} + K_{fx,k} e_k, \\
P_{k|k} &= P_{k|k-1} - K_{fx,k} R_{e,k} K_{fx,k}^T.
\end{aligned}
$$

EKF: Discrete-Discrete System - General Process Noise

- **One-step prediction:**
  - Given $\hat{x}_{k|k}$ and $P_{k|k}$
  - Compute prediction of state mean

$$\hat{x}_{k+1|k} = F_k(\hat{x}_{k|k}, 0).$$

  - Compute prediction of state covariance

$$P_{k+1|k} = A_k P_{k|k} A_k^T + Q_k, \qquad A_k = \frac{\partial F_k}{\partial x}(\hat{x}_{k|k}, 0).$$

- **j-step prediction:**
  - Given $\hat{x}_{k+j-1|k}$ and $P_{k+j-1|k}$
  - Compute predicton of state mean

$$\hat{x}_{k+j|k} = F_{k+j-1}(\hat{x}_{k+j-1|k}, 0).$$

  - Compute prediction of state covariance

$$P_{k+j|k} = A_{k+j-1} P_{k+j-1|k} A_{k+j-1}^T + G_{k+j-1} Q_{k+j-1} G_{k+j-1}^T,$$

$$A_{k+j-1} = \frac{\partial F_{k+j-1}}{\partial x}(\hat{x}_{k+j-1|k}, 0), \quad G_{k+j-1} = \frac{\partial F_{k+j-1}}{\partial w}(\hat{x}_{k+j-1|k}, 0).$$

- **Contiunous-Discrete System:**

$$
\begin{aligned}
d\boldsymbol{x}(t) &= f(\boldsymbol{x}(t))dt + \sigma(\boldsymbol{x}(t))d\boldsymbol{\omega}(t), && d\boldsymbol{\omega}(t) \sim N_{iid}(0, Idt),\\
\boldsymbol{y}(t_k) &= h_k(\boldsymbol{x}(t_k)) + \boldsymbol{v}_k, && \boldsymbol{v}_k \sim N_{iid}(0, R_k).
\end{aligned}
$$

EKF: Continuous-Discrete System

- **Filtering:**
  - Given y_k, $\hat{x}_{k|k-1}$ and $P_{k|k-1}$
  - Compute innovation and covariance

$$
\begin{aligned}
e_k &= y_k - \hat{y}_{k|k-1}, & \hat{y}_{k|k-1} &= h_k(\hat{x}_{k|k-1}), \\
R_{e,k} &= \langle \boldsymbol{e}_k, \boldsymbol{e}_k \rangle = C_k P_{k|k-1} C_k^T + R_k, & C_k &= \frac{\partial h_k}{\partial x}(\hat{x}_{k|k-1}).
\end{aligned}
$$

  - Compute Kalman gain

$$
K_{fx,k} = \langle \boldsymbol{x}_k, \boldsymbol{e}_k \rangle \langle \boldsymbol{e}_k, \boldsymbol{e}_k \rangle^{-1} = P_{k|k-1} C_k^T R_{e,k}^{-1}.
$$

  - Compute filtered state mean and covariance

$$
\begin{aligned}
\hat{x}_{k|k} &= \hat{x}_{k|k-1} + K_{fx,k} e_k, \\
P_{k|k} &= P_{k|k-1} - K_{fx,k} R_{e,k} K_{fx,k}^T.
\end{aligned}
$$

EKF: Continuous-Discrete System

- **Prediction:**
  - Given $\hat{x}_{k|k}$ and $P_{k|k}$
  - Compute prediction of state mean as solution to

$$\frac{\hat{x}_k(t)}{dt} = f(\hat{x}_k(t)), \qquad \hat{x}_k(t_k) = \hat{x}_{k|k},$$

    for $t \in [t_k, \infty[$.
  - Compute prediction of state covariance as solution to

$$\frac{dP_k(t)}{dt} = A_k(t)P_k(t) + P_k(t)A_k(t)^T + \sigma_k(t)\sigma_k(t)^T, \quad P_k(t_k) = P_{k|k},$$

$$A_k(t) = \frac{\partial f}{\partial x}(\hat{x}_k(t)), \qquad \sigma_k(t) = \sigma(\hat{x}_k(t)),$$

    for $t \in [t_k, \infty[$.
- **One-step prediction:**
  - Compute state mean and covariance predictions as

$$\hat{x}_{k+1|k} = \hat{x}_k(t_{k+1}), \qquad P_{k+1|k} = P_k(t_{k+1}).$$

- **j-step prediction:**
  - Compute state mean and covariance predictions as

$$\hat{x}_{k+j|k} = \hat{x}_k(t_{k+j}), \qquad P_{k+j|k} = P_k(t_{k+j}).$$

Unscented Kalman Filter

(UKF)

UKF: References

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[36] Renato Zanetti and Kyle J DeMars. "Joseph formulation of unscented and quadrature filters with application to consider states". In: *Journal of Guidance, Control, and Dynamics* 36.6 (2013), pages 1860–1864.

- **Discrete-Discrete System - Additive Process Noise:**

$$
\begin{aligned}
\boldsymbol{x}_{k+1} &= F_k(\boldsymbol{x}_k) + \boldsymbol{w}_k, && \boldsymbol{w}_k \sim N_{iid}(0, Q_k), \\
\boldsymbol{z}_k &= h_k(\boldsymbol{x}_k), && \\
\boldsymbol{y}_k &= \boldsymbol{z}_k + \boldsymbol{v}_k, && \boldsymbol{v}_k \sim N_{iid}(0, R_k).
\end{aligned}
$$

UKF: Discrete-Discrete System - Additive Process Noise

- **Filtering:**
  - Pre-compute parameters

$$
\begin{aligned}
c &= \alpha^2 \left(n_x + \kappa\right), \\
\lambda &= \alpha^2 \left(n_x + \kappa\right) - n_x,
\end{aligned}
$$

    where $\alpha \in \left]0, 1\right]$ and $\kappa = 0$.
  - Pre-compute weights

$$
\begin{aligned}
W_m^{(0)} &= \frac{\lambda}{n_x + \lambda}, \\
W_c^{(0)} &= \frac{\lambda}{n_x + \lambda} + \left(1 - \alpha^2 + \beta\right), \\
W_m^{(i)} = W_c^{(i)} &= \frac{1}{2\left(n_x + \lambda\right)}, \qquad\qquad i \in \{1, 2, \ldots, 2n_x\},
\end{aligned}
$$

    where $\beta = 2$ is optimal for normal distributions.

UKF: Discrete-Discrete System - Additive Process Noise

- **Filtering:**
  - Given $\hat{x}_{k|k-1}$ and $P_{k|k-1}$
  - Compute the sigma-points:

$$
\begin{aligned}
x^{(0)}_{k|k-1} &= \hat{x}_{k|k-1}, \\
x^{(i)}_{k|k-1} &= \hat{x}_{k|k-1} + \sqrt{c}\left(\sqrt{P_{k|k-1}}\right)_i, && i \in \{1, 2, \dots, n_x\}, \\
x^{(n_x+i)}_{k|k-1} &= \hat{x}_{k|k-1} - \sqrt{c}\left(\sqrt{P_{k|k-1}}\right)_i, && i \in \{1, 2, \dots, n_x\},
\end{aligned}
$$

    where the vector $\left(\sqrt{P_{k|k-1}}\right)_i$ denotes the i'th column of the Cholesky factorisation of the covariance matrix $P_{k|k-1}$.
  - Compute innovation

$$e_k = y_k - \hat{y}_{k|k-1},$$

    where the mean measurement prediction is computed from the sigma-point measurement predictions

$$\hat{y}_{k|k-1} = \hat{z}_{k|k-1} = \sum_{i=0}^{2n_x} W_m^{(i)} z^{(i)}_{k|k-1}, \qquad z^{(i)}_{k|k-1} = h_k(x^{(i)}_{k|k-1}),$$

    for $i \in \{1, 2, \dots, 2n_x\}$.

UKF: Discrete-Discrete System - Additive Process Noise

- Compute covariances from sigma-points

$$
\begin{aligned}
R_{zz,k} &= \langle \boldsymbol{z}_k, \boldsymbol{z}_k \rangle = \sum_{i=0}^{2n_x} W_c^{(i)} \left(\hat{z}_{k|k-1}^{(i)} - \hat{z}_{k|k-1} \right) \left(\hat{z}_{k|k-1}^{(i)} - \hat{z}_{k|k-1} \right)^T , \\
R_{e,k} &= R_{yy,k} = \langle \boldsymbol{y}_k, \boldsymbol{y}_k \rangle = R_{zz,k} + R_k , \\
R_{xy,k} &= \langle \boldsymbol{x}_k, \boldsymbol{y}_k \rangle = \langle \boldsymbol{x}_k, \boldsymbol{z}_k \rangle = \sum_{i=0}^{2n_x} W_c^{(i)} \left(\hat{x}_{k|k-1}^{(i)} - \hat{x}_{k|k-1} \right) \left(\hat{z}_{k|k-1}^{(i)} - \hat{z}_{k|k-1} \right)^T .
\end{aligned}
$$

- Compute Kalman gain

$$
K_{fx,k} = \langle \boldsymbol{x}_k, \boldsymbol{y}_k \rangle \langle \boldsymbol{y}_k, \boldsymbol{y}_k \rangle^{-1} = R_{xy,k} R_{yy,k}^{-1} = R_{xy,k} R_{e,k}^{-1}.
$$

- Compute filtered state mean and covariance

$$
\begin{aligned}
\hat{x}_{k|k} &= \hat{x}_{k|k-1} + K_{fx,k} e_k , \\
P_{k|k} &= P_{k|k-1} - K_{fx,k} R_{e,k} K_{fx,k}^T .
\end{aligned}
$$

UKF: Discrete-Discrete System - Additive Process Noise

- **One-step prediction:**
  - Pre-compute parameters

$$
\begin{aligned}
c &= \alpha^2 \left(n_x + \kappa\right), \\
\lambda &= \alpha^2 \left(n_x + \kappa\right) - n_x,
\end{aligned}
$$

    where $\alpha \in]0, 1]$ and $\kappa = 0$.
  - Pre-compute weights

$$
\begin{aligned}
W_m^{(0)} &= \frac{\lambda}{n_x + \lambda}, \\
W_c^{(0)} &= \frac{\lambda}{n_x + \lambda} + \left(1 - \alpha^2 + \beta\right), \\
W_m^{(i)} = W_c^{(i)} &= \frac{1}{2\left(n_x + \lambda\right)}, \qquad i \in \{1, 2, \ldots, 2n_x\},
\end{aligned}
$$

    where $\beta = 2$ is optimal for normal distributions.

UKF: Discrete-Discrete System - Additive Process Noise

- Given $\hat{x}_{k|k}$ and $P_{k|k}$
- Compute the sigma-points:

$$
\begin{aligned}
x_{k|k}^{(0)} &= \hat{x}_{k|k}, \\
x_{k|k}^{(i)} &= \hat{x}_{k|k} + \sqrt{\bar{c}}\left(\sqrt{P_{k|k}}\right)_i, && i \in \{1, 2, \ldots, n_x\}, \\
x_{k|k}^{(n_x+i)} &= \hat{x}_{k|k} - \sqrt{\bar{c}}\left(\sqrt{P_{k|k}}\right)_i, && i \in \{1, 2, \ldots, n_x\},
\end{aligned}
$$

  where the vector $\left(\sqrt{P_{k|k}}\right)_i$ denotes the i'th column of the Cholesky factorisation of the covariance matrix $P_{k|k}$.
- Compute sigma-point predictions

$$
s_{k+1|k}^{(i)} = F_k(x_{k|k}^{(i)}), \qquad i \in \{0, 1, \ldots, 2n_x\}.
$$

- Compute state prediction mean

$$
\hat{x}_{k+1|k} = \hat{s}_{k+1|k} = \sum_{i=0}^{2n_x} W_m^{(i)} s_{k+1|k}^{(i)}.
$$

- Compute state prediction covariance

$$
\begin{aligned}
R_{ss,k+1} &= \langle \boldsymbol{s}_{k+1}, \boldsymbol{s}_{k+1} \rangle = \sum_{i=0}^{2n_x} W_c^{(i)} \left(s_{k+1|k}^{(i)} - \hat{s}_{k+1|k}\right)\left(s_{k+1|k}^{(i)} - \hat{s}_{k+1|k}\right)^T, \\
P_{k+1|k} &= R_{ss,k+1} + Q_k.
\end{aligned}
$$

- **Discrete-Discrete System - General Process Noise:**

$$\begin{aligned} \boldsymbol{x}_{k+1} &= F_k(\boldsymbol{x}_k, \boldsymbol{w}_k), & \boldsymbol{w}_k &\sim N_{iid}(0, Q_k), \\ \boldsymbol{y}_k &= h_k(\boldsymbol{x}_k) + \boldsymbol{v}_k, & \boldsymbol{v}_k &\sim N_{iid}(0, R_k). \end{aligned}$$

UKF: Discrete-Discrete System - General Process Noise

- **Filtering:**
  - Pre-compute parameters

$$
\begin{aligned}
c &= \alpha^2 \left(n_x + \kappa\right), \\
\lambda &= \alpha^2 \left(n_x + \kappa\right) - n_x,
\end{aligned}
$$

    where $\alpha \in \left]0, 1\right]$ and $\kappa = 0$.
  - Pre-compute weights

$$
\begin{aligned}
W_m^{(0)} &= \frac{\lambda}{n_x + \lambda}, \\
W_c^{(0)} &= \frac{\lambda}{n_x + \lambda} + \left(1 - \alpha^2 + \beta\right), \\
W_m^{(i)} = W_c^{(i)} &= \frac{1}{2\left(n_x + \lambda\right)}, \qquad\qquad i \in \{1, 2, \ldots, 2n_x\},
\end{aligned}
$$

    where $\beta = 2$ is optimal for normal distributions.

UKF: Discrete-Discrete System - General Process Noise

- Given $\hat{x}_{k|k-1}$ and $P_{k|k-1}$
- Compute the sigma-points:

$$
\begin{aligned}
x_{k|k-1}^{(0)} &= \hat{x}_{k|k-1}, \\
x_{k|k-1}^{(i)} &= \hat{x}_{k|k-1} + \sqrt{c}\left(\sqrt{P_{k|k-1}}\right)_i, & i \in \{1, 2, \ldots, n_x\}, \\
x_{k|k-1}^{(n_x+i)} &= \hat{x}_{k|k-1} - \sqrt{c}\left(\sqrt{P_{k|k-1}}\right)_i, & i \in \{1, 2, \ldots, n_x\},
\end{aligned}
$$

  where the vector $\left(\sqrt{P_{k|k-1}}\right)_i$ denotes the i'th column of the Cholesky factorisation of the covariance matrix $P_{k|k-1}$.
- Compute innovation

$$
e_k = y_k - \hat{y}_{k|k-1},
$$

  where the mean measurement prediction is computed from the sigma-point measurement predictions

$$
\hat{y}_{k|k-1} = \hat{z}_{k|k-1} = \sum_{i=0}^{2n_x} W_m^{(i)} z_{k|k-1}^{(i)}, \qquad z_{k|k-1}^{(i)} = h_k(x_{k|k-1}^{(i)}),
$$

  for $i \in \{1, 2, \ldots, 2n_x\}$.

UKF: Discrete-Discrete System - General Process Noise

- Compute covariances from sigma-points

$$
\begin{aligned}
R_{zz,k} &= \langle \boldsymbol{z}_k, \boldsymbol{z}_k \rangle = \sum_{i=0}^{2n_x} W_c^{(i)} \left(\hat{z}_{k|k-1}^{(i)} - \hat{z}_{k|k-1} \right) \left(\hat{z}_{k|k-1}^{(i)} - \hat{z}_{k|k-1} \right)^T , \\
R_{e,k} &= R_{yy,k} = \langle \boldsymbol{y}_k, \boldsymbol{y}_k \rangle = R_{zz,k} + R_k , \\
R_{xy,k} &= \langle \boldsymbol{x}_k, \boldsymbol{y}_k \rangle = \langle \boldsymbol{x}_k, \boldsymbol{z}_k \rangle = \sum_{i=0}^{2n_x} W_c^{(i)} \left(\hat{x}_{k|k-1}^{(i)} - \hat{x}_{k|k-1} \right) \left(\hat{z}_{k|k-1}^{(i)} - \hat{z}_{k|k-1} \right)^T .
\end{aligned}
$$

- Compute Kalman gain

$$
K_{fx,k} = \langle \boldsymbol{x}_k, \boldsymbol{y}_k \rangle \langle \boldsymbol{y}_k, \boldsymbol{y}_k \rangle^{-1} = R_{xy,k} R_{yy,k}^{-1} = R_{xy,k} R_{e,k}^{-1} .
$$

- Compute filtered state mean and covariance

$$
\begin{aligned}
\hat{x}_{k|k} &= \hat{x}_{k|k-1} + K_{fx,k} e_k , \\
P_{k|k} &= P_{k|k-1} - K_{fx,k} R_{e,k} K_{fx,k}^T .
\end{aligned}
$$

UKF: Discrete-Discrete System - General Process Noise

- **One-step prediction:**
  - Pre-compute parameters

$$
\begin{aligned}
\bar{c} &= \alpha^2 \left(\bar{n} + \kappa\right), \\
\bar{\lambda} &= \alpha^2 \left(\bar{n} + \kappa\right) - \bar{n},
\end{aligned}
$$

    where $\alpha \in \left]0, 1\right]$, $\kappa = 0$, and $\bar{n} = n_x + n_w$.
  - Pre-compute weights

$$
\begin{aligned}
\bar{W}_m^{(0)} &= \frac{\bar{\lambda}}{\bar{n} + \bar{\lambda}}, \\
\bar{W}_c^{(0)} &= \frac{\bar{\lambda}}{\bar{n} + \bar{\lambda}} + \left(1 - \alpha^2 + \beta\right), \\
\bar{W}_m^{(i)} = \bar{W}_c^{(i)} &= \frac{1}{2\left(\bar{n} + \bar{\lambda}\right)}, \qquad\qquad i \in \{1, 2, \dots, 2\bar{n}\},
\end{aligned}
$$

    where $\beta = 2$ is optimal for normal distributions.

UKF: Discrete-Discrete System - General Process Noise

- Given $\hat{x}_{k|k}$ and $P_{k|k}$
- Compute the sigma-points:

$$
\begin{aligned}
x^{(i)}_{k|k} &= \hat{x}_{k|k}, & i &\in \{0, 2n_x+1, 2n_x+2, \ldots, 2n_x+2n_w\},\\
x^{(i)}_{k|k} &= \hat{x}_{k|k} + \sqrt{\bar{c}}\left(\sqrt{P_{k|k}}\right)_i, & i &\in \{1, 2, \ldots, n_x\},\\
x^{(n_x+i)}_{k|k} &= \hat{x}_{k|k} - \sqrt{\bar{c}}\left(\sqrt{P_{k|k}}\right)_i, & i &\in \{1, 2, \ldots, n_x\},
\end{aligned}
$$

where the vector $\left(\sqrt{P_{k|k}}\right)_i$ denotes the i'th column of the Cholesky factorisation of the covariance matrix $P_{k|k}$.

- Compute noise

$$
\begin{aligned}
w^{(i)}_{k|k} &= 0, & i &\in \{0, 1, \ldots, 2n_x\},\\
w^{(i)}_{k|k} &= \sqrt{\bar{c}}\left(\sqrt{Q_k}\right)_i, & i &\in \{2n_x+1, 2n_x+2, \ldots, 2n_x+n_w\},\\
w^{(i)}_{k|k} &= -\sqrt{\bar{c}}\left(\sqrt{Q_k}\right)_i, & i &\in \{2n_x+n_w+1, 2n_x+n_w+2, \ldots, 2n_x+2n_w\},
\end{aligned}
$$

where the vector $\left(\sqrt{Q_k}\right)_i$ denotes the i'th column of the Cholesky factorisation of the covariance matrix Q_k.

UKF: Discrete-Discrete System - General Process Noise

- Compute sigma-point predictions

$$x_{k+1|k}^{(i)} = F_k(x_{k|k}^{(i)}, w_{k|k}^{(i)}), \qquad\qquad i \in \{0, 1, \ldots, 2\bar{n}\}.$$

- Compute state prediction mean

$$\hat{x}_{k+1|k} = \sum_{i=0}^{2\bar{n}} \bar{W}_m^{(i)} x_{k+1|k}^{(i)}.$$

- Compute state prediction covariance

$$P_{k+1|k} = \langle \boldsymbol{x}_{k+1}, \boldsymbol{x}_{k+1} \rangle = \sum_{i=0}^{2\bar{n}} \bar{W}_c^{(i)} \left(x_{k+1|k}^{(i)} - \hat{x}_{k+1|k} \right) \left(x_{k+1|k}^{(i)} - \hat{x}_{k+1|k} \right)^T.$$

- **Contiunous-Discrete System:**

$$\begin{aligned} d\boldsymbol{x}(t) &= f(\boldsymbol{x}(t))dt + \sigma(\boldsymbol{x}(t))d\boldsymbol{\omega}(t), & \qquad d\boldsymbol{\omega}(t) &\sim N_{iid}(0, Idt), \\ \boldsymbol{y}(t_k) &= h_k(\boldsymbol{x}(t_k)) + \boldsymbol{v}_k, & \qquad \boldsymbol{v}_k &\sim N_{iid}(0, R_k). \end{aligned}$$

UKF: Continuous-Discrete System

- **Filering:**
  - Pre-compute parameters

$$
\begin{aligned}
c &= \alpha^2 \left(n_x + \kappa\right), \\
\lambda &= \alpha^2 \left(n_x + \kappa\right) - n_x,
\end{aligned}
$$

    where $\alpha \in]0, 1]$ and $\kappa = 0$.
  - Pre-compute weights

$$
\begin{aligned}
W_m^{(0)} &= \frac{\lambda}{n_x + \lambda}, \\
W_c^{(0)} &= \frac{\lambda}{n_x + \lambda} + \left(1 - \alpha^2 + \beta\right), \\
W_m^{(i)} = W_c^{(i)} &= \frac{1}{2\left(n_x + \lambda\right)}, \qquad\qquad i \in \{1, 2, \ldots, 2n_x\},
\end{aligned}
$$

    where $\beta = 2$ is optimal for normal distributions.

UKF: Continuous-Discrete System

- Given $\hat{x}_{k|k-1}$ and $P_{k|k-1}$
- Compute the sigma-points:

$$\begin{aligned}
x_{k|k-1}^{(0)} &= \hat{x}_{k|k-1}, \\
x_{k|k-1}^{(i)} &= \hat{x}_{k|k-1} + \sqrt{c}\left(\sqrt{P_{k|k-1}}\right)_i, \qquad & i \in \{1, 2, \ldots, n_x\}, \\
x_{k|k-1}^{(n_x+i)} &= \hat{x}_{k|k-1} - \sqrt{c}\left(\sqrt{P_{k|k-1}}\right)_i, \qquad & i \in \{1, 2, \ldots, n_x\},
\end{aligned}$$

  where the vector $\left(\sqrt{P_{k|k-1}}\right)_i$ denotes the i'th column of the Cholesky factorisation of the covariance matrix $P_{k|k-1}$.
- Compute innovation

$$e_k = y_k - \hat{y}_{k|k-1},$$

  where the mean measurement prediction is computed from the sigma-point measurement predictions

$$\hat{y}_{k|k-1} = \hat{z}_{k|k-1} = \sum_{i=0}^{2n_x} W_m^{(i)} z_{k|k-1}^{(i)}, \qquad z_{k|k-1}^{(i)} = h_k(x_{k|k-1}^{(i)}),$$

  for $i \in \{1, 2, \ldots, 2n_x\}$.

UKF: Continuous-Discrete System

- Compute covariances from sigma-points

$$
\begin{aligned}
R_{zz,k} &= \langle \boldsymbol{z}_k, \boldsymbol{z}_k \rangle = \sum_{i=0}^{2n_x} W_c^{(i)} \left(\hat{z}_{k|k-1}^{(i)} - \hat{z}_{k|k-1} \right) \left(\hat{z}_{k|k-1}^{(i)} - \hat{z}_{k|k-1} \right)^T, \\
R_{e,k} &= R_{yy,k} = \langle \boldsymbol{y}_k, \boldsymbol{y}_k \rangle = R_{zz,k} + R_k, \\
R_{xy,k} &= \langle \boldsymbol{x}_k, \boldsymbol{y}_k \rangle = \langle \boldsymbol{x}_k, \boldsymbol{z}_k \rangle = \sum_{i=0}^{2n_x} W_c^{(i)} \left(\hat{x}_{k|k-1}^{(i)} - \hat{x}_{k|k-1} \right) \left(\hat{z}_{k|k-1}^{(i)} - \hat{z}_{k|k-1} \right)^T.
\end{aligned}
$$

- Compute Kalman gain

$$
K_{fx,k} = \langle \boldsymbol{x}_k, \boldsymbol{y}_k \rangle \langle \boldsymbol{y}_k, \boldsymbol{y}_k \rangle^{-1} = R_{xy,k} R_{yy,k}^{-1} = R_{xy,k} R_{e,k}^{-1}.
$$

- Compute filtered state mean and covariance

$$
\begin{aligned}
\hat{x}_{k|k} &= \hat{x}_{k|k-1} + K_{fx,k} e_k, \\
P_{k|k} &= P_{k|k-1} - K_{fx,k} R_{e,k} K_{fx,k}^T.
\end{aligned}
$$

UKF: Continuous-Discrete System

- **One-step prediction:**
  - Pre-compute parameters

$$\bar{c} = \alpha^2 \left(\bar{n} + \kappa\right),$$
$$\bar{\lambda} = \alpha^2 \left(\bar{n} + \kappa\right) - \bar{n},$$

    where $\alpha \in \left]0, 1\right]$, $\kappa = 0$, and $\bar{n} = n_x + n_w$.
  - Pre-compute weights

$$\bar{W}_m^{(0)} = \frac{\bar{\lambda}}{\bar{n} + \bar{\lambda}},$$
$$\bar{W}_c^{(0)} = \frac{\bar{\lambda}}{\bar{n} + \bar{\lambda}} + \left(1 - \alpha^2 + \beta\right),$$
$$\bar{W}_m^{(i)} = \bar{W}_c^{(i)} = \frac{1}{2\left(\bar{n} + \bar{\lambda}\right)}, \qquad i \in \{1, 2, \ldots, 2\bar{n}\},$$

    where $\beta = 2$ is optimal for normal distributions.

UKF: Continuous-Discrete System

- Given $\hat{x}_{k|k}$ and $P_{k|k}$
- Compute the sigma-points:

$$
\begin{aligned}
x_{k|k}^{(i)} &= \hat{x}_{k|k}, && i \in \{0, 2n_x+1, 2n_x+2, \ldots, 2n_x+2n_w\}, \\
x_{k|k}^{(i)} &= \hat{x}_{k|k} + \sqrt{\bar{c}}\left(\sqrt{P_{k|k}}\right)_i, && i \in \{1, 2, \ldots, n_x\}, \\
x_{k|k}^{(n_x+i)} &= \hat{x}_{k|k} - \sqrt{\bar{c}}\left(\sqrt{P_{k|k}}\right)_i, && i \in \{1, 2, \ldots, n_x\},
\end{aligned}
$$

  where the vector $\left(\sqrt{P_{k|k}}\right)_i$ denotes the i'th column of the Cholesky factorisation of the covariance matrix $P_{k|k}$.
- Compute noise

$$
\begin{aligned}
dw_{k|k}^{(i)} &= 0, && i \in \{0, 1, \ldots, 2n_x\}, \\
dw_{k|k}^{(i)} &= \sqrt{\bar{c}dt}\,(I)_i\,, && i \in \{2n_x+1, 2n_x+2, \ldots, 2n_x+n_w\}, \\
dw_{k|k}^{(i)} &= -\sqrt{\bar{c}dt}\,(I)_i\,, && i \in \{2n_x+n_w+1, 2n_x+n_w+2, \ldots, 2n_x+2n_w\},
\end{aligned}
$$

  where the vector $(I)_i$ denotes the i'th column of the Identity matrix I of size n_w.

UKF: Continuous-Discrete System

- Compute sigma-point predictions as solution to

$$\left.\begin{aligned} x_k^{(i)}(t_k) &= x_{k|k}^{(i)} \\ dx_k^{(i)}(t) &= f(x_k^{(i)}(t))dt + \sigma(x_k^{(i)}(t))dw_{k|k}^{(i)} \end{aligned}\right., \qquad i \in \{0, 1, \ldots, 2\bar{n}\},$$

  for $t \in \left[t_k, t_{k+1}\right]$, and where the one-step predictions are $x_{k+1|k}^{(i)} = x_k^{(i)}(t_{k+1})$.
- Compute state prediction mean

$$\hat{x}_{k+1|k} = \sum_{i=0}^{2\bar{n}} \bar{W}_m^{(i)} x_{k+1|k}^{(i)}.$$

- Compute state prediction covariance

$$P_{k+1|k} = \langle \boldsymbol{x}_{k+1}, \boldsymbol{x}_{k+1} \rangle = \sum_{i=0}^{2\bar{n}} \bar{W}_c^{(i)} \left(x_{k+1|k}^{(i)} - \hat{x}_{k+1|k}\right) \left(x_{k+1|k}^{(i)} - \hat{x}_{k+1|k}\right)^T.$$

Ensemble Kalman Filter

(EnKF)

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- **Discrete-Discrete System - Additive Process Noise:**

$$\begin{aligned} \boldsymbol{x}_{k+1} &= F_k(\boldsymbol{x}_k) + \boldsymbol{w}_k, & \boldsymbol{w}_k &\sim N_{iid}(0, Q_k), \\ \boldsymbol{y}_k &= h_k(\boldsymbol{x}_k) + \boldsymbol{v}_k, & \boldsymbol{v}_k &\sim N_{iid}(0, R_k). \end{aligned}$$

EnKF: Discrete-Discrete System - Additive Process Noise

- **Filtering:**
  - Given state ensemble, $\{x_{k|k-1}^{(i)}\}_{i=1}^{N_p}$
  - Compute ensemble measurement prediction $\{y_{k|k-1}^{(i)}\}_{i=1}^{N_p}$

$$y_{k|k-1}^{(i)} = h_k(x_{k|k-1}^{(i)}), \qquad i \in \{1, 2, \ldots, N_p\}.$$

  - Compute innovations

$$e_k^{(i)} = y_k - \left(y_{k|k-1}^{(i)} + v_k^{(i)} \right), \qquad i \in \{1, 2, \ldots, N_p\},$$

    where $v_k^{(i)}$ are samples $v_k \sim \mathcal{N}(0, R_k)$, i.e., the measurement noise distribution.
  - Compute state and measurement prediction means

$$\hat{x}_{k|k-1} = \frac{1}{N_p} \sum_{i=1}^{N_p} x_{k|k-1}^{(i)}, \qquad \hat{y}_{k|k-1} = \frac{1}{N_p} \sum_{i=1}^{N_p} y_{k|k-1}^{(i)}.$$

EnKF: Discrete-Discrete System - Additive Process Noise

- Compute state prediction covariance and cross-covariance

$$R_{yy,k|k-1} = \frac{1}{N_p - 1} \sum_{i=1}^{N_p} \left(y_{k|k-1}^{(i)} - \hat{y}_{k|k-1}\right) \left(y_{k|k-1}^{(i)} - \hat{y}_{k|k-1}\right)^T + R_k,$$

$$R_{xy,k|k-1} = \frac{1}{N_p - 1} \sum_{i=1}^{N_p} \left(x_{k|k-1}^{(i)} - \hat{x}_{k|k-1}\right) \left(y_{k|k-1}^{(i)} - \hat{y}_{k|k-1}\right)^T .$$

- Compute Kalman gain

$$K_{fx,k} = R_{xy,k|k-1} R_{yy,k|k-1}^{-1},$$

- Compute filtering ensemble, $\{x_{k|k}^{(i)}\}_{i=1}^{N_p}$

$$x_{k|k}^{(i)} = x_{k|k-1}^{(i)} + K_{fx,k} e_k^{(i)}, \qquad i \in \{1, 2, \ldots, N_p\}.$$

- Compute filtered state mean

$$\hat{x}_{k|k} = \frac{1}{N_p} \sum_{i=1}^{N_p} x_{k|k}^{(i)}.$$

- Compute filtered state covariance

$$P_{k|k} = \frac{1}{N_p - 1} \sum_{i=1}^{N_p} \left(x_{k|k}^{(i)} - \hat{x}_{k|k}\right) \left(x_{k|k}^{(i)} - \hat{x}_{k|k}\right)^T .$$

EnKF: Discrete-Discrete System - Additive Process Noise

- **One-step prediction:**
  - Given state ensemble, $\{x_{k|k}^{(i)}\}_{i=1}^{N_p}$
  - Compute state ensemble prediction

$$x_{k+1|k}^{(i)} = F_k(x_{k|k}^{(i)}) + w_k^{(i)}, \qquad i = \{1, 2, \ldots, N_p\},$$

    where $w_k^{(i)}$ are samples from $\boldsymbol{w}_k \sim \mathcal{N}(0, Q_k)$, i.e., the process noise distribution.
  - Compute state mean

$$\hat{x}_{k+1|k} = \frac{1}{N_p} \sum_{i=1}^{N_p} x_{k+1|k}^{(i)}.$$

  - Compute covariance

$$\begin{aligned} P_{k+1|k} &= \langle \boldsymbol{x}_{k+1|k}, \boldsymbol{x}_{k+1|k} \rangle \\ &= \frac{1}{N_p - 1} \sum_{i=1}^{N_p} \left(x_{k+1|k}^{(i)} - \hat{x}_{k+1|k} \right) \left(x_{k+1|k}^{(i)} - \hat{x}_{k+1|k} \right)^T. \end{aligned}$$

- **Discrete-Discrete System - General Process Noise:**

$$\begin{aligned} \boldsymbol{x}_{k+1} &= F_k(\boldsymbol{x}_k, \boldsymbol{w}_k), & \boldsymbol{w}_k &\sim N_{iid}(0, Q_k), \\ \boldsymbol{y}_k &= h_k(\boldsymbol{x}_k) + \boldsymbol{v}_k, & \boldsymbol{v}_k &\sim N_{iid}(0, R_k). \end{aligned}$$

EnKF: Discrete-Discrete System - General Process Noise

- **Filtering:**
  - Given state ensemble, $\{x_{k|k-1}^{(i)}\}_{i=1}^{N_p}$
  - Compute ensemble measurement prediction $\{y_{k|k-1}^{(i)}\}_{i=1}^{N_p}$

$$y_{k|k-1}^{(i)} = h_k(x_{k|k-1}^{(i)}), \qquad i \in \{1, 2, \dots, N_p\}.$$

  - Compute innovations

$$e_k^{(i)} = y_k - \left(y_{k|k-1}^{(i)} + v_k^{(i)} \right), \qquad i \in \{1, 2, \dots, N_p\},$$

    where $v_k^{(i)}$ are samples $v_k \sim \mathcal{N}(0, R_k)$, i.e., the measurement noise distribution.
  - Compute state and measurement prediction means

$$\hat{x}_{k|k-1} = \frac{1}{N_p} \sum_{i=1}^{N_p} x_{k|k-1}^{(i)}, \qquad \hat{y}_{k|k-1} = \frac{1}{N_p} \sum_{i=1}^{N_p} y_{k|k-1}^{(i)}.$$

EnKF: Discrete-Discrete System - General Process Noise

- Compute state prediction covariance and cross-covariance

$$R_{yy,k|k-1} = \frac{1}{N_p - 1} \sum_{i=1}^{N_p} \left(y_{k|k-1}^{(i)} - \hat{y}_{k|k-1} \right) \left(y_{k|k-1}^{(i)} - \hat{y}_{k|k-1} \right)^T + R_k,$$

$$R_{xy,k|k-1} = \frac{1}{N_p - 1} \sum_{i=1}^{N_p} \left(x_{k|k-1}^{(i)} - \hat{x}_{k|k-1} \right) \left(y_{k|k-1}^{(i)} - \hat{y}_{k|k-1} \right)^T .$$

- Compute Kalman gain

$$K_{fx,k} = R_{xy,k|k-1} R_{yy,k|k-1}^{-1},$$

- Compute filtering ensemble, $\{x_{k|k}^{(i)}\}_{i=1}^{N_p}$

$$x_{k|k}^{(i)} = x_{k|k-1}^{(i)} + K_{fx,k} e_k^{(i)}, \qquad\qquad i \in \{1, 2, \ldots, N_p\}.$$

- Compute filtered state mean

$$\hat{x}_{k|k} = \frac{1}{N_p} \sum_{i=1}^{N_p} x_{k|k}^{(i)}.$$

- Compute filtered state covariance

$$P_{k|k} = \frac{1}{N_p - 1} \sum_{i=1}^{N_p} \left(x_{k|k}^{(i)} - \hat{x}_{k|k} \right) \left(x_{k|k}^{(i)} - \hat{x}_{k|k} \right)^T .$$

EnKF: Discrete-Discrete System - General Process Noise

- **One-step prediction:**
  - Given state ensemble, $\{x_{k|k}^{(i)}\}_{i=1}^{N_p}$
  - Compute state ensemble prediction

$$x_{k+1|k}^{(i)} = F_k(x_{k|k}^{(i)}, w_k^{(i)}), \qquad i = \{1, 2, \ldots, N_p\},$$

    where $w_k^{(i)}$ are samples from $\boldsymbol{w}_k \sim \mathcal{N}(0, Q_k)$, i.e., the process noise distribution.
  - Compute state mean

$$\hat{x}_{k+1|k} = \frac{1}{N_p} \sum_{i=1}^{N_p} x_{k+1|k}^{(i)}.$$

  - Compute covariance

$$\begin{aligned} P_{k+1|k} &= \langle \boldsymbol{x}_{k+1|k}, \boldsymbol{x}_{k+1|k} \rangle \\ &= \frac{1}{N_p - 1} \sum_{i=1}^{N_p} \left(x_{k+1|k}^{(i)} - \hat{x}_{k+1|k} \right) \left(x_{k+1|k}^{(i)} - \hat{x}_{k+1|k} \right)^T. \end{aligned}$$

- **Contiunous-Discrete System:**

$$\begin{aligned}
d\boldsymbol{x}(t) &= f(\boldsymbol{x}(t))dt + \sigma(\boldsymbol{x}(t))d\boldsymbol{\omega}(t), \qquad && d\boldsymbol{\omega}(t) \sim N_{iid}(0, Idt), \\
\boldsymbol{y}(t_k) &= h_k(\boldsymbol{x}(t_k)) + \boldsymbol{v}_k, && \boldsymbol{v}_k \sim N_{iid}(0, R_k).
\end{aligned}$$

EnKF: Continuous-Discrete System

- **Filtering:**
  - Given state ensemble, $\{x_{k|k-1}^{(i)}\}_{i=1}^{N_p}$
  - Compute ensemble measurement prediction $\{y_{k|k-1}^{(i)}\}_{i=1}^{N_p}$

$$y_{k|k-1}^{(i)} = h_k(x_{k|k-1}^{(i)}), \qquad\qquad i \in \{1, 2, \dots, N_p\}.$$

  - Compute innovations

$$e_k^{(i)} = y_k^{(i)} - y_{k|k-1}^{(i)}, \qquad y_k^{(i)} = y_k + v_k^{(i)}, \qquad i \in \{1, 2, \dots, N_p\},$$

    where $y_k^{(i)}$ are measurement perturbation, where $v_k^{(i)}$ are samples from $v_k \sim \mathcal{N}(0, R_k)$.
  - Compute state and measurement prediction means

$$\hat{x}_{k|k-1} = \frac{1}{N_p}\sum_{i=1}^{N_p} x_{k|k-1}^{(i)}, \qquad\qquad \hat{y}_{k|k-1} = \frac{1}{N_p}\sum_{i=1}^{N_p} y_{k|k-1}^{(i)}.$$

EnKF: Continuous-Discrete System

- Compute state prediction covariance and cross-covariance

$$R_{yy,k|k-1} = \frac{1}{N_p - 1} \sum_{i=1}^{N_p} \left(y_{k|k-1}^{(i)} - \hat{y}_{k|k-1} \right) \left(y_{k|k-1}^{(i)} - \hat{y}_{k|k-1} \right)^T + R_k,$$

$$R_{xy,k|k-1} = \frac{1}{N_p - 1} \sum_{i=1}^{N_p} \left(x_{k|k-1}^{(i)} - \hat{x}_{k|k-1} \right) \left(y_{k|k-1}^{(i)} - \hat{y}_{k|k-1} \right)^T .$$

- Compute Kalman gain

$$K_{fx,k} = R_{xy,k|k-1} R_{yy,k|k-1}^{-1},$$

- Compute filtering ensemble, $\{x_{k|k}^{(i)}\}_{i=1}^{N_p}$

$$x_{k|k}^{(i)} = \hat{x}_{k|k-1}^{(i)} + K_{fx,k} e_k^{(i)}, \qquad\qquad i \in \{1, 2, \ldots, N_p\}.$$

EnKF: Continuous-Discrete System

- **One-step prediction:**
  - Given state ensemble, $\{x_{k|k}^{(i)}\}_{i=1}^{N_p}$
  - Compute state ensemble prediction as solution to

$$
\begin{aligned}
x_k^{(i)}(t_k) &= x_{k|k}^{(i)}, && i = \{1, 2, \dots, N_p\}, \\
d\boldsymbol{x}_k^{(i)}(t) &= f(\boldsymbol{x}_k^{(i)}(t))dt + \sigma(\boldsymbol{x}_k^{(i)}(t))d\boldsymbol{\omega}_k(t), && i = \{1, 2, \dots, N_p\},
\end{aligned}
$$

    for $t \in [t_k, t_{k+1}]$, and where the predictions $x_{k+1|k}^{(i)} = x_k^{(i)}(t_{k+1})$.
  - Compute state mean and covariance (optional)

$$
\hat{x}_{k+1|k} = \frac{1}{N_p} \sum_{i=1}^{N_p} x_{k+1|k}^{(i)},
$$

$$
P_{k+1|k} = R_{xx,k+1|k} = \frac{1}{N_p - 1} \sum_{i=1}^{N_p} \left(x_{k+1|k}^{(i)} - \hat{x}_{k+1|k} \right) \left(x_{k+1|k}^{(i)} - \hat{x}_{k+1|k} \right)^T .
$$

Particle Filter

(PF)

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- **Discrete-Discrete System - Additive Process Noise:**

$$\begin{aligned} \boldsymbol{x}_{k+1} &= F_k(\boldsymbol{x}_k) + \boldsymbol{w}_k, & \boldsymbol{w}_k &\sim N_{iid}(0, Q_k), \\ \boldsymbol{y}_k &= h_k(\boldsymbol{x}_k) + \boldsymbol{v}_k, & \boldsymbol{v}_k &\sim N_{iid}(0, R_k). \end{aligned}$$

PF: Discrete-Discrete System - Additive Process Noise

- **Filtering:**
  - Given set of sampled particles, $\{x_{k|k-1}^{(i)}\}_{i=1}^{N_p}$
  - Compute measurement predictions

$$y_{k|k-1}^{(i)} = h_k(x_{k|k-1}^{(i)}), \qquad i \in \{1, 2, \ldots, N_p\}.$$

  - Compute innovations

$$e_k^{(i)} = y_k - y_{k|k-1}^{(i)}, \qquad i \in \{1, 2, \ldots, N_p\}.$$

  - Compute updated weights from likelihood function $p(y_k|x_k)$

$$w_k^{(i)} = \frac{1}{\sqrt{2\pi^{n_y}|R_k|}} \exp\left(-\frac{1}{2}\left(e_k^{(i)}\right)^T R_k^{-1} e_k^{(i)}\right), \quad i \in \{1, 2, \ldots, N_p\}.$$

  and normalise

$$\tilde{w}_k^{(i)} = \frac{w_k^{(i)}}{\sum_{j=1}^{N_p} w_k^{(j)}}, \qquad i \in \{1, 2, \ldots, N_p\}.$$

PF: Discrete-Discrete System - Additive Process Noise

- Resample particles according to new weights and assign new equal weights to all resampled particles
  1. Given normalised likelihoods $\{\tilde{w}_k^{(i)}\}$.
  2. Generate uniformly distributed sample $q_1 \sim \mathcal{U}[0,1]$.
  3. Compute ordered resampling points

$$q_k^{(i)} = \frac{(i-1)+q_1}{N_p}, \qquad\qquad i \in \{1,2,\ldots,N_p\}.$$

  4. Resample by producing $m^{(i)}$ copies of particle, $\hat{x}_{k|k-1}^{(i)}$, for which $m^{(i)}$ is the number of indices, l, where $q_k^{(l)} \in \left]s^{(i-1)}, s^{(i)}\right]$ where $s^{(i)} = \sum_{j=1}^{i} \tilde{w}_k^{(j)}$.
  5. Resampled set is $\{x_{k|k}^{(i)}\}_{i=1}^{N_p}$.
- Compute mean and covariance (optional)

$$\hat{x}_{k|k} = \frac{1}{N_p}\sum_{i=1}^{N_p} x_{k|k}^{(i)},$$

$$P_{k|k} = R_{xx,k|k} = \frac{1}{N_p-1}\sum_{i=1}^{N_p}\left(\hat{x}_{k|k}^{(i)} - \hat{x}_{k|k}\right)\left(\hat{x}_{k|k}^{(i)} - \hat{x}_{k|k}\right)^T.$$

PF: Discrete-Discrete System - Additive Process Noise

- **One-step prediction:**
  - Given set of sampled particles, $\{x_{k|k}^{(i)}\}_{i=1}^{N_p}$
  - Compute state particle prediction

$$x_{k+1|k}^{(i)} = F_k(x_{k|k}^{(i)}) + w_k^{(i)}, \qquad\qquad i = \{1, 2, \dots, N_p\},$$

    where $w_k^{(i)}$ are samples from $\boldsymbol{w}_k \sim \mathcal{N}(0, Q_k)$, i.e., the process noise distribution.
  - Compute state mean

$$\hat{x}_{k+1|k} = \frac{1}{N_p} \sum_{i=1}^{N_p} x_{k+1|k}^{(i)}.$$

  - Compute covariance

$$\begin{aligned} P_{k+1|k} &= \langle \boldsymbol{x}_{k+1|k}, \boldsymbol{x}_{k+1|k} \rangle \\ &= \frac{1}{N_p - 1} \sum_{i=1}^{N_p} \left(x_{k+1|k}^{(i)} - \hat{x}_{k+1|k} \right) \left(x_{k+1|k}^{(i)} - \hat{x}_{k+1|k} \right)^T. \end{aligned}$$

- **Discrete-Discrete System - General Process Noise:**

$$
\begin{aligned}
\boldsymbol{x}_{k+1} &= F_k(\boldsymbol{x}_k, \boldsymbol{w}_k), & \boldsymbol{w}_k &\sim N_{iid}(0, Q_k), \\
\boldsymbol{y}_k &= h_k(\boldsymbol{x}_k) + \boldsymbol{v}_k, & \boldsymbol{v}_k &\sim N_{iid}(0, R_k).
\end{aligned}
$$

PF: Discrete-Discrete System - General Process Noise

- **Filtering:**
  - Given set of sampled particles, $\{x_{k|k-1}^{(i)}\}_{i=1}^{N_p}$
  - Compute measurement predictions

$$y_{k|k-1}^{(i)} = h_k(x_{k|k-1}^{(i)}), \qquad i \in \{1, 2, \ldots, N_p\}.$$

  - Compute innovations

$$e_k^{(i)} = y_k - y_{k|k-1}^{(i)}, \qquad i \in \{1, 2, \ldots, N_p\}.$$

  - Compute updated weights from likelihood function $p(y_k|x_k)$

$$w_k^{(i)} = \frac{1}{\sqrt{2\pi^{n_y}|R_k|}} \exp\left(-\frac{1}{2}\left(e_k^{(i)}\right)^T R_k^{-1} e_k^{(i)}\right), \quad i \in \{1, 2, \ldots, N_p\}.$$

    and normalise

$$\tilde{w}_k^{(i)} = \frac{w_k^{(i)}}{\sum_{j=1}^{N_p} w_k^{(j)}}, \qquad i \in \{1, 2, \ldots, N_p\}.$$

PF: Discrete-Discrete System - General Process Noise

- Resample particles according to new weights and assign new equal weights to all resampled particles
  1. Given normalised likelihoods $\{\tilde{w}_k^{(i)}\}$.
  2. Generate uniformly distributed sample $q_1 \sim \mathcal{U}[0,1]$.
  3. Compute ordered resampling points

$$q_k^{(i)} = \frac{(i-1)+q_1}{N_p}, \qquad\qquad i \in \{1, 2, \ldots, N_p\}.$$

  4. Resample by producing $m^{(i)}$ copies of particle, $\hat{x}_{k|k-1}^{(i)}$, for which $m^{(i)}$ is the number of indices, l, where $q_k^{(l)} \in \left]s^{(i-1)}, s^{(i)}\right]$ where $s^{(i)} = \sum_{j=1}^{i} \tilde{w}_k^{(j)}$.
  5. Resampled set is $\{x_{k|k}^{(i)}\}_{i=1}^{N_p}$.
- Compute mean and covariance (optional)

$$\hat{x}_{k|k} = \frac{1}{N_p} \sum_{i=1}^{N_p} x_{k|k}^{(i)},$$

$$P_{k|k} = R_{xx,k|k} = \frac{1}{N_p - 1} \sum_{i=1}^{N_p} \left(\hat{x}_{k|k}^{(i)} - \hat{x}_{k|k}\right)\left(\hat{x}_{k|k}^{(i)} - \hat{x}_{k|k}\right)^T.$$

PF: Discrete-Discrete System - General Process Noise

- **One-step prediction:**
  - Given state ensemble, $\{x_{k|k}^{(i)}\}_{i=1}^{N_p}$
  - Compute state ensemble prediction

$$x_{k+1|k}^{(i)} = F_k(x_{k|k}^{(i)}, w_k^{(i)}), \qquad\qquad i = \{1, 2, \ldots, N_p\},$$

    where $w_k^{(i)}$ are samples drawn from the process noise distribution $\boldsymbol{w}_k \sim \mathcal{N}(0, Q_k)$.
  - Compute state mean

$$\hat{x}_{k+1|k} = \frac{1}{N_p} \sum_{i=1}^{N_p} x_{k+1|k}^{(i)}.$$

  - Compute covariance

$$P_{k+1|k} = \frac{1}{N_p - 1} \sum_{i=1}^{N_p} \left(x_{k+1|k}^{(i)} - \hat{x}_{k+1|k} \right) \left(x_{k+1|k}^{(i)} - \hat{x}_{k+1|k} \right)^T.$$

- **Contiunous-Discrete System:**

$$\begin{aligned} d\boldsymbol{x}(t) &= f(\boldsymbol{x}(t))dt + \sigma(\boldsymbol{x}(t))d\boldsymbol{\omega}(t), & \quad d\boldsymbol{\omega}(t) &\sim N_{iid}(0, Idt), \\ \boldsymbol{y}(t_k) &= h_k(\boldsymbol{x}(t_k)) + \boldsymbol{v}_k, & \quad \boldsymbol{v}_k &\sim N_{iid}(0, R_k). \end{aligned}$$

PF: Continuous-Discrete System

- **Filtering:**
  - Given set of sampled particles, $\{x_{k|k-1}^{(i)}\}_{i=1}^{N_p}$
  - Compute innovations

$$
\begin{aligned}
e_k^{(i)} &= y_k - y_{k|k-1}^{(i)}, & i \in \{1, 2, \ldots, N_p\}, \\
y_{k|k-1}^{(i)} &= h_k(x_{k|k-1}^{(i)}), & i \in \{1, 2, \ldots, N_p\}.
\end{aligned}
$$

  - Compute updated weights from likelihood function $p(y_k|x_k)$

$$
w_k^{(i)} = \frac{1}{\sqrt{2\pi^{n_y}|R_k|}} \exp\left(-\frac{1}{2}\left(e_k^{(i)}\right)^T R_k^{-1} e_k^{(i)}\right), \quad i \in \{1, 2, \ldots, N_p\}.
$$

and normalise

$$
\tilde{w}_k^{(i)} = \frac{w_k^{(i)}}{\sum_{j=1}^{N_p} w_k^{(j)}}, \qquad i \in \{1, 2, \ldots, N_p\}
$$

PF: Continuous-Discrete System

- Resample particles according to new weights and assign new equal weights to all resampled particles
  1. Given normalised likelihoods $\{\tilde{w}_k^{(i)}\}$.
  2. Generate uniformly distributed sample $q_1 \sim \mathcal{U}\,[0,1]$.
  3. Compute ordered resampling points

$$q_k^{(i)} = \frac{(i-1)+q_1}{N_p}, \qquad\qquad i \in \{1, 2, \ldots, N_p\}.$$

  4. Resample by producing $m^{(i)}$ copies of particle, $\hat{x}_{k|k-1}^{(i)}$, for which $m^{(i)}$ is the number of indices, l, where $q_k^{(l)} \in \left]s^{(i-1)}, s^{(i)}\right]$ where $s^{(i)} = \sum_{j=1}^{i} \tilde{w}_k^{(j)}$.
  5. Resampled set is $\{x_{k|k}^{(i)}\}_{i=1}^{N_p}$.
- Compute mean and covariance (optional)

$$\hat{x}_{k|k} = \frac{1}{N_p} \sum_{i=1}^{N_p} x_{k|k}^{(i)},$$

$$P_{k|k} = R_{xx,k|k} = \frac{1}{N_p - 1} \sum_{i=1}^{N_p} \left(\hat{x}_{k|k}^{(i)} - \hat{x}_{k|k}\right) \left(\hat{x}_{k|k}^{(i)} - \hat{x}_{k|k}\right)^T.$$

PF: Continuous-Discrete System

- **One-step prediction:**
  - Given state ensemble, $\{x_{k|k}^{(i)}\}_{i=1}^{N_p}$
  - Compute state ensemble prediction as solution to

$$
\begin{aligned}
x_k^{(i)}(t_k) &= x_{k|k}^{(i)}, && i = \{1, 2, \ldots, N_p\}, \\
d\boldsymbol{x}_k^{(i)}(t) &= f(\boldsymbol{x}_k^{(i)}(t))dt + \sigma(\boldsymbol{x}_k^{(i)}(t))d\boldsymbol{\omega}_k(t), && i = \{1, 2, \ldots, N_p\},
\end{aligned}
$$

    for $t \in [t_k, t_{k+1}]$, and where the predictions $x_{k+1|k}^{(i)} = x_k^{(i)}(t_{k+1})$.
  - Compute state mean and covariance (optional)

$$
\begin{aligned}
\hat{x}_{k|k} &= \frac{1}{N_p} \sum_{i=1}^{N_p} x_{k|k}^{(i)}, \\
P_{k|k} = R_{xx,k|k} &= \frac{1}{N_p - 1} \sum_{i=1}^{N_p} \left(x_{k|k}^{(i)} - \hat{x}_{k|k} \right) \left(x_{k|k}^{(i)} - \hat{x}_{k|k} \right)^T.
\end{aligned}
$$

Moving Horizon Estimation

(MHE)

Fokker-Planck Equation

(Kolmogorow's forward equation)