# MPC Relevant Prediction-Error Identification

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# Outline

# Model Predictive Control

The MPC regulator objective function

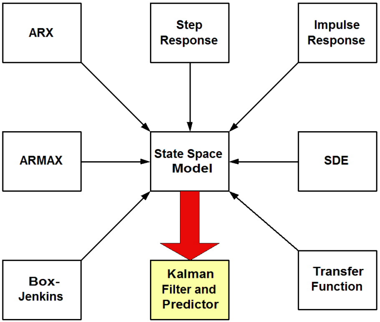
$$
\begin{aligned}
\phi_k = & \frac{1}{2}\sum_{j=1}^{N_p}(\hat{y}_{k+j|k} - r_{k+j|k})' Q (\hat{y}_{k+j|k} - r_{k+j|k}) \\
& + \frac{1}{2}\sum_{j=0}^{N_c} \Delta \hat{u}'_{k+j|k} S \Delta \hat{u}_{k+j|k}
\end{aligned}
$$

requires a multi-step $N_p$-step-ahead prediction at each time point.

**Contributions**

1. Use a system identification criterion that is consistent with this objective.
2. Parameterize the predictor using a continuous-discrete stochastic model.

# State Space Model Realization




1. Realize the chosen model parameterization as a stochastic discrete-time state space model.
2. The optimal filter and predictor for the state space model is the Kalman filter and predictor.

$$\begin{aligned} \mathbf{x}_{k+1} &= A(\theta)\mathbf{x}_k + B(\theta)u_k + \mathbf{w}_k \\ \mathbf{y}_k &= C(\theta)\mathbf{x}_k + \mathbf{v}_k \end{aligned}$$

with

$$\begin{bmatrix} \mathbf{w}_k \\ \mathbf{v}_k \end{bmatrix} \sim N_{iid}\left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} R_{ww}(\theta) & R_{wv}(\theta) \\ R_{wv}(\theta)' & R_{vv}(\theta) \end{bmatrix} \right)$$

$$\mathbf{x}_0 \sim N(\hat{x}_{0|-1}(\theta), P_{0|-1}(\theta))$$

# Kalman Filter

## Dynamic Kalman Filter

**Innovation and gains**

$$\begin{aligned}
\hat{y}_{k|k-1} &= C\hat{x}_{k|k-1} \\
e_k &= y_k - \hat{y}_{k|k-1} \\
R_{e,k} &= CP_{k|k-1}C' + R_{vv} \\
K_{fx,k} &= P_{k|k-1}C'R_{e,k}^{-1} \\
K_{fw,k} &= R_{wv}R_{e,k}^{-1}
\end{aligned}$$

**Filtered estimates**

$$\begin{aligned}
\hat{x}_{k|k} &= \hat{x}_{k|k-1} + K_{fx,k}e_k \\
\hat{w}_{k|k} &= K_{fw,k}e_k
\end{aligned}$$

**and covariances**

$$\begin{aligned}
P_{k|k} &= P_{k|k-1} - K_{fx,k}R_{e,k}K'_{fx,k} \\
Q_{k|k} &= R_{ww} - K_{fw,k}R_{e,k}K'_{fw,k}
\end{aligned}$$

## Static Kalman Filter

**Riccati equation ($P = \lim_{k\to\infty} P_{k|k-1}$)**

$$\begin{aligned}
P &= APA' + R_{ww} \\
&\quad - (APC' + R_{wv})(R_{vv} + CPC')^{-1}(APC' + R_{wv})'
\end{aligned}$$

**Gains**

$$\begin{aligned}
R_e &= CPC' + R_{vv} \\
K_{fx} &= PC'R_e^{-1} \\
K_{fw} &= R_{wv}R_e^{-1}
\end{aligned}$$

**Innovation**

$$\begin{aligned}
\hat{y}_{k|k-1} &= C\hat{x}_{k|k-1} \\
e_k &= y_k - \hat{y}_{k|k-1}
\end{aligned}$$

**Filtered estimates**

$$\begin{aligned}
\hat{x}_{k|k} &= \hat{x}_{k|k-1} + K_{fx}e_k \\
\hat{w}_{k|k} &= K_{fw}e_k
\end{aligned}$$

# Kalman Predictors

## One-Step Predictor

**Estimated predictions**

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k} + B\hat{u}_{k|k} + \hat{w}_{k|k}$$
$$\hat{y}_{k+1|k} = C\hat{x}_{k+1|k}$$

**and covariances**

$$\begin{aligned} P_{k+1|k} &= AP_{k|k}A' + Q_{k|k} \\ &\quad - AK_{fx,k}R'_{wv} - R_{wv}K'_{fx,k}A' \\ R_{k+1|k} &= CP_{k+1|k}C' + R_{vv} \end{aligned}$$

## j-Step Predictor

**Estimated predictions for $j \geq 2$**

$$\hat{x}_{k+j|k} = A\hat{x}_{k+j-1|k} + B\hat{u}_{k+j-1|k}$$
$$\hat{y}_{k+j|k} = C\hat{x}_{k+j|k}$$

**and covariances**

$$P_{k+j|k} = AP_{k+j-1|k}A' + R_{ww}$$
$$R_{k+j|k} = CP_{k+j|k}C' + R_{vv}$$

# One-Step Prediction Error Estimation

The innovations (prediction errors) are normally distributed

$$\mathbf{e}_k(\theta) \sim N(0, R_{e,k}(\theta))$$

and computed from the Kalman filter recursions

$$e_k(\theta) = y_k - \hat{y}_{k|k-1}(\theta)$$
$$R_{e,k}(\theta) = R_{vv}(\theta) + C(\theta)P_{k|k-1}(\theta)C(\theta)'$$

Estimation problem

$$\hat{\theta} = \arg\min_{\theta \in \Theta} V(\theta)$$

Estimation criteria

$$\text{LS}: \quad V(\theta) = \frac{1}{2}\sum_{k=1}^{N} \|e_k(\theta)\|_2^2$$

$$\text{ML}: \quad V(\theta) = \frac{1}{2}\sum_{k=1}^{N} \left[\ln\left(\det R_{e,k}(\theta)\right) + e_k(\theta)' R_{e,k}^{-1}(\theta) e_k(\theta)\right]$$

# Multi-Step Prediction Error Maximum-Likelihood Estimation

$$\hat{\theta} = \arg\min_{\theta \in \Theta} V_{ML}(\theta)$$

in which the likelihood function is

$$V_{ML}(\theta) = \frac{n_y f}{2} \ln(2\pi) + \frac{1}{2} \sum_{k=-1}^{N-2} \ln\left(\det R_k\right) + \epsilon_k R_k^{-1} \epsilon_k$$

$f = N_p[N - \frac{1}{2}(N_p - 1)]$, $\boldsymbol{\epsilon}_k = \mathbf{Y}_k - \hat{Y}_k(\theta)$, $R_k = \langle \boldsymbol{\epsilon}_k, \boldsymbol{\epsilon}_k \rangle$, and

$$\mathbf{Y}_k = \begin{bmatrix} \mathbf{y}_{k+1} \\ \mathbf{y}_{k+2} \\ \vdots \\ \mathbf{y}_{k+N_p} \end{bmatrix} \quad \hat{Y}_k(\theta) = \begin{bmatrix} \hat{y}_{k+1|k}(\theta) \\ \hat{y}_{k+2|k}(\theta) \\ \vdots \\ \hat{y}_{k+N_p|k}(\theta) \end{bmatrix}$$

Data used for SYSID: $\{(y_k, u_k)\}_{k=0}^{N-1}$

# Wood and Berry Distillation

## Nominal IO-model

$$Y(s) = G(s)U(s) + G_d(s)D(s)$$

with

$$G(s) = \begin{bmatrix} \frac{12.8e^{-s}}{16.7s+1} & \frac{-18.9e^{-3s}}{21.0s+1} \\ \frac{6.6e^{-7s}}{10.9s+1} & \frac{-19.4e^{-3s}}{14.4s+1} \end{bmatrix}$$

$$G_d(s) = \begin{bmatrix} \frac{3.8e^{-8.1s}}{14.9s+1} \\ \frac{4.9e^{-3.4s}}{13.2s+1} \end{bmatrix}$$

$Y_1$: Overhead methanol %
$Y_2$: Bottom methanol %
$U_1$: Reflux flow rate
$U_2$: Steam flow rate

$D$: Feed flow rate

## Stochastic Model

The following model is used for generating the IO-data

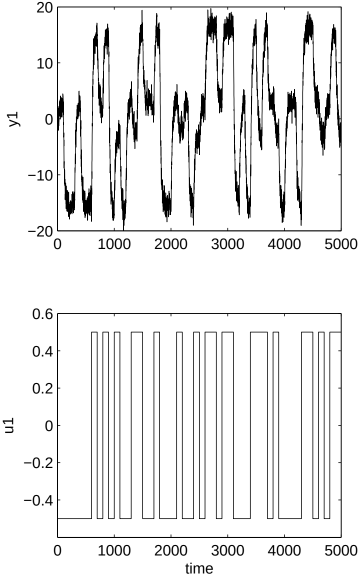
$$\begin{aligned} \mathbf{Z}(s) &= G(s)U(s) \\ &\quad + G_d(s)(D(s) + \sigma \mathbf{E}(s)) \\ \mathbf{y}(t_k) &= \mathbf{z}(t_k) + \mathbf{v}(t_k) \end{aligned}$$

$\mathbf{E}(s)$ is white noise and $\sigma = 1$

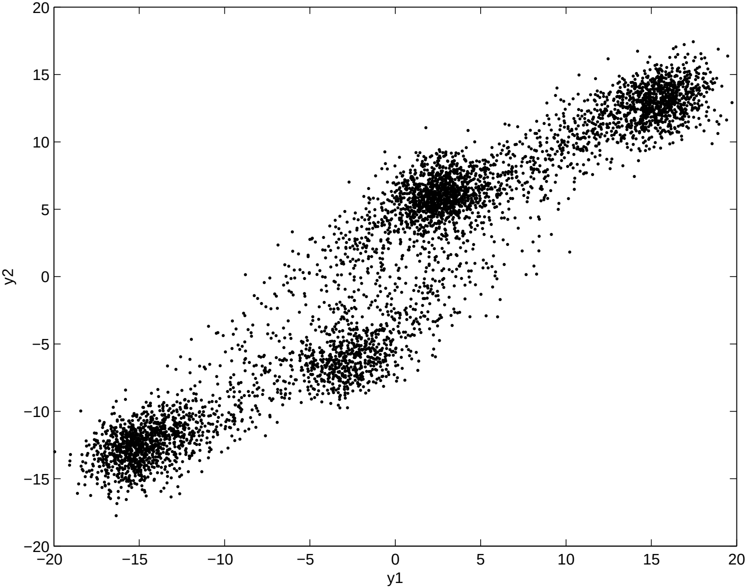
$$\mathbf{v}(t_k) \sim N_{iid}\left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} r_1^2 & 0 \\ 0 & r_2^2 \end{bmatrix} \right)$$

in which $r_1 = r_2 = 1.0$.

# Wood and Berry Distillation: Experiment for SYSID



# Wood and Berry Distillation: Measured Outputs



# Wood and Berry Distillation: System Identification

## Model Parametrization

**Coninuous-Discrete Stochastic Model**

$$\hat{\mathbf{Z}}(s) = \hat{G}(s)U(s) + \hat{H}(s)\hat{\mathbf{E}}(s)$$
$$\mathbf{y}(t_k) = \hat{\mathbf{z}}(t_k) + \hat{\mathbf{v}}(t_k)$$

**Transfer Function**

$$\hat{G}(s) = \begin{bmatrix} \hat{g}_{11}(s) & \hat{g}_{12}(s) \\ \hat{g}_{21}(s) & \hat{g}_{22}(s) \end{bmatrix}$$

$$\hat{g}_{ij}(s) = \frac{k_{ij}}{\tau_{ij}s + 1} e^{-d_{ij}s} \; i,j = \{1,2\}$$

**Disturbance Model**

$$\hat{H}(s) = \begin{bmatrix} \hat{h}_{11}(s) & 0 \\ 0 & \hat{h}_{22}(s) \end{bmatrix}$$

$$\hat{h}_{ii}(s) = \frac{1}{s} \frac{\sigma_{ii}}{\gamma_{ii}s + 1} \quad i = 1,2$$

## Identified Model

**Model and disturbance model**

$$\hat{G}(s) = \begin{bmatrix} \frac{13.21e^{-0.84s}}{17.20s+1} & \frac{-18.52e^{-3.34s}}{20.67s+1} \\ \frac{6.72e^{-7.69s}}{10.03s+1} & \frac{-19.28e^{-3.07s}}{14.77s+1} \end{bmatrix}$$

$$\hat{H}(s) = \begin{bmatrix} \frac{1}{s}\frac{0.18}{0.16s+1} & 0 \\ 0 & \frac{1}{s}\frac{0.27}{0.16s+1} \end{bmatrix}$$

**Estimated measurement noise**

$$\hat{R}_{vv} = \begin{bmatrix} 1.03^2 & 0 \\ 0 & 1.04^2 \end{bmatrix}$$

# Wood and Berry Distillation: True and Identified Model

## Nominal IO-model

**Model and Disturbance Model**

$$G(s) = \begin{bmatrix} \frac{12.8e^{-s}}{16.7s+1} & \frac{-18.9e^{-3s}}{21.0s+1} \\ \frac{6.6e^{-7s}}{10.9s+1} & \frac{-19.4e^{-3s}}{14.4s+1} \end{bmatrix}$$

$$G_d(s) = \begin{bmatrix} \frac{3.8e^{-8.1s}}{14.9s+1} \\ \frac{4.9e^{-3.4s}}{13.2s+1} \end{bmatrix}$$

**Measurement noise**

$$R_{vv} = \begin{bmatrix} 1.0^2 & 0 \\ 0 & 1.0^2 \end{bmatrix}$$

## Identified Model

**Model and disturbance model**

$$\hat{G}(s) = \begin{bmatrix} \frac{13.21e^{-0.84s}}{17.20s+1} & \frac{-18.52e^{-3.34s}}{20.67s+1} \\ \frac{6.72e^{-7.69s}}{10.03s+1} & \frac{-19.28e^{-3.07s}}{14.77s+1} \end{bmatrix}$$

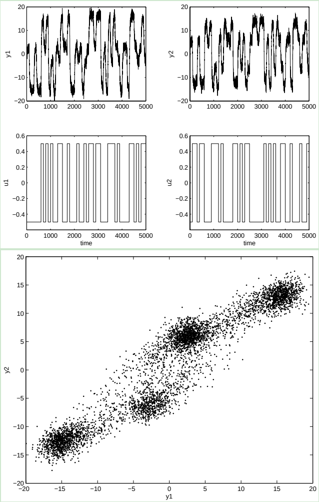
$$\hat{H}(s) = \begin{bmatrix} \frac{1}{s}\frac{0.18}{0.16s+1} & 0 \\ 0 & \frac{1}{s}\frac{0.27}{0.16s+1} \end{bmatrix}$$

**Estimated measurement noise**

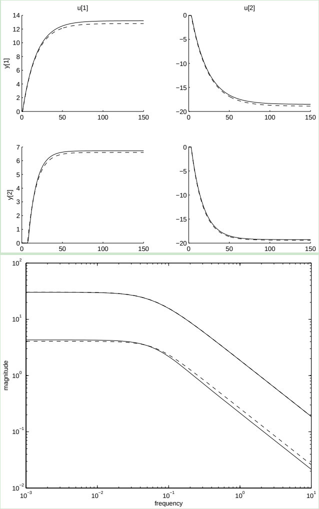
$$\hat{R}_{vv} = \begin{bmatrix} 1.03^2 & 0 \\ 0 & 1.04^2 \end{bmatrix}$$

# Wood and Berry Distillation: System Identification

## Example (IO-Data)

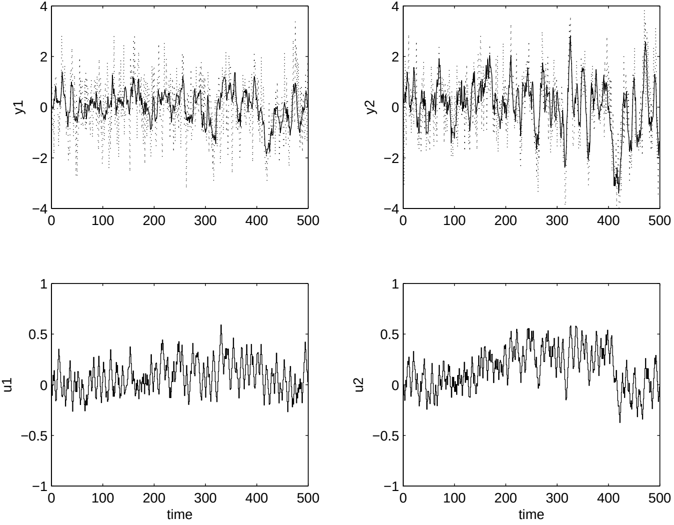



## Example (Identified Model)



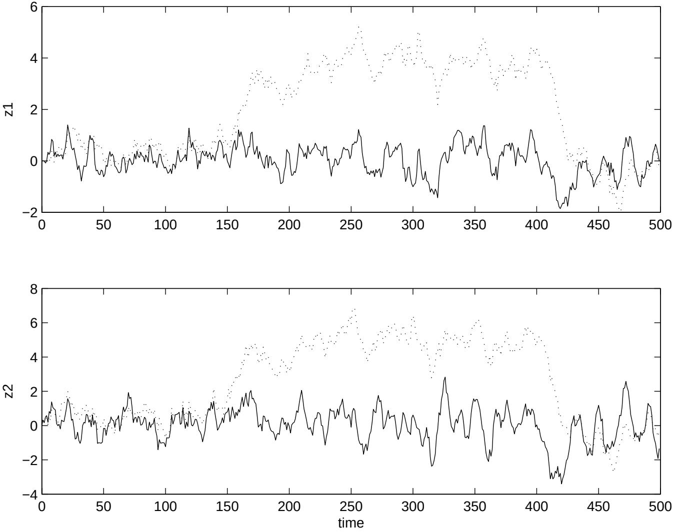
# Wood and Berry Disstillation: MPC Control

An unknown disturbance (D=1) enters at t=150 and disappears at t =400.



# Wood and Berry Distillation: MPC Performance

Comparison of OL and MPC controlled Wood and Berry distillation



# Conclusions

1. Method for estimation of parameters in MIMO continuous-discrete-time stochastic systems described by transfer functions with time delays.
2. Multi-Step Maximum Likelihood Prediction-Error-Method compatible with MPC objective function
3. Computational feasibility of the method is demonstrated for the Wood and Berry distillation column example.

# Future work:

Comparison of closed loop MPC performance based on predictive models obtained by

1. One-step Prediction-Error-Method
2. Multi-Step Maximum Likelihood Prediction-Error-Method
3. ARX parameterizations
4. Subspace identification (ARX parameterization)