Nonlinear State Estimation

Model Predictive Control - Lecture 07A

John Bagterp Jørgensen

Department of Applied Mathematics and Computer Science
Technical University of Denmark

02619 Model Predictive Control

Major Methods for Nonlinear State Estimation

- Extended Kalman Filter (EKF)
- Unscented Kalman Filter (UKF)
- Ensemble Kalman Filter (EnKF)
- Particle Filtering (PF)
- Moving Horizon Estimation (MHE)
  - optimization based estimation
- Fokker-Planck Equation
  - also called Kolmogorow's forward equation

Model Classes Considered

- Discrete time process - discrete time measurement
  - Additive process noise

$$\begin{aligned} x_{k+1} &= F_k(x_k) + w_k \qquad & w_k &\sim N_{iid}(0, Q_k) \\ y_k &= h_k(x_k) + v_k \qquad & v_k &\sim N_{iid}(0, R_k) \end{aligned}$$

  - General process noise

$$\begin{aligned} x_{k+1} &= F_k(x_k, w_k) \qquad & w_k &\sim N_{iid}(0, Q_k) \\ y_k &= h_k(x_k) + v_k \qquad & v_k &\sim N_{iid}(0, R_k) \end{aligned}$$

- Continuous time process - discrete time measurement

$$\begin{aligned} dx(t) &= f(x(t))dt + G(x(t))d\omega(t) \qquad & d\omega(t) &\sim N_{iid}(0, Idt) \\ y(t_k) &= h_k(x(t_k)) + v_k \qquad & v_k &\sim N_{iid}(0, R_k) \end{aligned}$$

Extented Kalman Filter

(EKF)

EKF - Discrete Time Process - Additive Process Noise

- **Model:**

$$\begin{aligned} x_{k+1} &= F_k(x_k) + w_k \qquad & w_k \sim N_{iid}(0, Q_k) \\ y_k &= h_k(x_k) + v_k \qquad & v_k \sim N_{iid}(0, R_k) \end{aligned}$$

- **Filtering:** 1) Given y_k, $\hat{x}_{k|k-1}$ and $P_{k|k-1}$; 2) Compute $\hat{x}_{k|k}$ and $P_{k|k}$:

$$\begin{aligned} \hat{y}_{k|k-1} &= h_k(\hat{x}_{k|k-1}) & C_k &= \frac{\partial h_k}{\partial x}(\hat{x}_{k|k-1}) \\ e_k &= y_k - \hat{y}_{k|k-1} & R_{e,k} &= \langle e_k, e_k \rangle = C_k P_{k|k-1} C_k' + R_k \\ & & K_{fx,k} &= \langle x_k, e_k \rangle \langle e_k, e_k \rangle^{-1} = P_{k|k-1} C_k' R_{e,k}^{-1} \\ \hat{x}_{k|k} &= \hat{x}_{k|k-1} + K_{fx,k} e_k & P_{k|k} &= P_{k|k-1} - K_{fx,k} R_{e,k} K_{fx,k}' \end{aligned}$$

- **One-step prediction:** 1) Given $\hat{x}_{k|k}$ and $P_{k|k}$; 2) Compute $\hat{x}_{k+1|k}$ and $P_{k+1|k}$:

$$\hat{x}_{k+1|k} = F_k(\hat{x}_{k|k}) \quad A_k = \frac{\partial F_k}{\partial x}(\hat{x}_{k|k}) \quad P_{k+1|k} = A_k P_{k|k} A_k' + Q_k$$

- **j-step prediction:**

$$\begin{aligned} \hat{x}_{k+j+1|k} &= F_{k+j}(\hat{x}_{k+j|k}) \quad A_{k+j} = \frac{\partial F_{k+j}}{\partial x}(\hat{x}_{k+j|k}) \quad P_{k+j+1|k} = A_{k+j} P_{k+j|k} A_{k+j}' + Q_{k+j} \\ \hat{y}_{k+j|k} &= h_{k+j}(\hat{x}_{k+j|k}) \quad C_{k+j} = \frac{\partial h_{k+j}}{\partial x}(\hat{x}_{k+j|k}) \quad R_{k+j|k} = C_{k+j} P_{k+j|k} C_{k+j}' + R_{k+j} \end{aligned}$$

EKF - Discrete Time Process - General Process Noise

- **Model:**

$$x_{k+1} = F_k(x_k, w_k) \qquad w_k \sim N_{iid}(0, Q_k)$$
$$y_k = h_k(x_k) + v_k \qquad v_k \sim N_{iid}(0, R_k)$$

- **Filtering:** 1) Given y_k, $\hat{x}_{k|k-1}$ and $P_{k|k-1}$; 2) Compute $\hat{x}_{k|k}$ and $P_{k|k}$:

$$\begin{aligned}
\hat{y}_{k|k-1} &= h_k(\hat{x}_{k|k-1}) & C_k &= \frac{\partial h_k}{\partial x}(\hat{x}_{k|k-1}) \\
e_k &= y_k - \hat{y}_{k|k-1} & R_{e,k} &= \langle e_k, e_k \rangle = C_k P_{k|k-1} C_k' + R_k \\
& & K_{fx,k} &= \langle x_k, e_k \rangle \langle e_k, e_k \rangle^{-1} = P_{k|k-1} C_k' R_{e,k}^{-1} \\
\hat{x}_{k|k} &= \hat{x}_{k|k-1} + K_{fx,k} e_k & P_{k|k} &= P_{k|k-1} - K_{fx,k} R_{e,k} K_{fx,k}'
\end{aligned}$$

- **One-step prediction:** 1) Given $\hat{x}_{k|k}$ and $P_{k|k}$; 2) Compute $\hat{x}_{k+1|k}$ and $P_{k+1|k}$:

$$\hat{x}_{k+1|k} = F_k(\hat{x}_{k|k}, 0) \quad A_k = \frac{\partial F_k}{\partial x}(\hat{x}_{k|k}, 0) \quad G_k = \frac{\partial F_k}{\partial w}(\hat{x}_{k|k}, 0)$$
$$P_{k+1|k} = A_k P_{k|k} A_k' + G_k Q_k G_k'$$

- **j-step prediction:**

$$\hat{x}_{k+j+1|k} = F_{k+j}(\hat{x}_{k+j|k}, 0) \quad A_{k+j} = \frac{\partial F_{k+j}}{\partial x}(\hat{x}_{k+j|k}, 0) \quad G_{k+j} = \frac{\partial F_{k+j}}{\partial w}(\hat{x}_{k+j|k}, 0)$$
$$P_{k+j+1|k} = A_{k+j} P_{k+j|k} A_{k+j}' + G_{k+j} Q_{k+j} G_{k+j}'$$

EKF - Continuous Time Process

- **Model:**

$$\begin{aligned} dx(t) &= f(x(t))dt + G(x(t))d\omega(t) \qquad & d\omega(t) &\sim N_{iid}(0, Idt) \\ y(t_k) &= h_k(x(t_k)) + v_k & v_k &\sim N_{iid}(0, R_k) \end{aligned}$$

- **Filtering:** 1) Given y_k, $\hat{x}_{k|k-1}$ and $P_{k|k-1}$; 2) Compute $\hat{x}_{k|k}$ and $P_{k|k}$:

$$\begin{aligned} \hat{y}_{k|k-1} &= h_k(\hat{x}_{k|k-1}) & C_k &= \frac{\partial h_k}{\partial x}(\hat{x}_{k|k-1}) \\ e_k &= y_k - \hat{y}_{k|k-1} & R_{e,k} &= \langle e_k, e_k \rangle = C_k P_{k|k-1} C_k' + R_k \\ & & K_{fx,k} &= \langle x_k, e_k \rangle \langle e_k, e_k \rangle^{-1} = P_{k|k-1} C_k' R_{e,k}^{-1} \\ \hat{x}_{k|k} &= \hat{x}_{k|k-1} + K_{fx,k} e_k & P_{k|k} &= P_{k|k-1} - K_{fx,k} R_{e,k} K_{fx,k}' \end{aligned}$$

- **Prediction:** 1) Given $\hat{x}_{k|k}$ and $P_{k|k}$; 2) Solve

$$\begin{aligned} \frac{d\hat{x}_k(t)}{dt} &= f(\hat{x}_k(t)) & \hat{x}_k(t_k) &= \hat{x}_{k|k} \\ \frac{dP_k(t)}{dt} &= A_k(t)P_k(t) + P_k(t)A_k(t)' + G(\hat{x}_k(t))G(\hat{x}_k(t))' & P_k(t_k) &= P_{k|k} \\ A_k(t) &= \frac{\partial f}{\partial x}(\hat{x}_k(t)) \end{aligned}$$

- **One-step prediction:** $\hat{x}_{k+1|k} = \hat{x}_k(t_{k+1})$ and $P_{k+1|k} = P_k(t_{k+1})$
- **j-step prediction:** $\hat{x}_{k+j|k} = \hat{x}_k(t_{k+j})$ and $P_{k+j|k} = P_k(t_{k+j})$

Unscented Kalman Filter

(UKF)

- **Output and measurement model:**

$$\begin{aligned} z_k &= h_k(x_k) \\ y_k &= z_k + v_k \qquad v_k \sim N_{iid}(0, R_k) \end{aligned}$$

- **Filtering:**
  - Given $\hat{x}_{k|k-1}$ and $P_{k|k-1}$ - compute the sigma-points:

$$\hat{x}^{(0)}_{k|k-1} = \hat{x}_{k|k-1}$$
$$\hat{x}^{(i)}_{k|k-1} = \hat{x}_{k|k-1} + \sqrt{c}\left(\sqrt{P_{k|k-1}}\right)_i \quad i = 1, 2, \ldots, n_x$$
$$\hat{x}^{(i+n_x)}_{k|k-1} = \hat{x}_{k|k-1} - \sqrt{c}\left(\sqrt{P_{k|k-1}}\right)_i \quad i = 1, 2, \ldots, n_x$$

  - Compute output sigma points and mean

$$\hat{z}^{(i)}_{k|k-1} = h_k(\hat{x}^{(i)}_{k|k-1}) \qquad i = 0, 1, \ldots, 2n_x$$
$$\hat{y}_{k|k-1} = \hat{z}_{k|k-1} = \sum_{i=0}^{2m_x} W_m^{(i)} \hat{z}^{(i)}_{k|k-1}$$
$$e_k = y_k - \hat{y}_{k|k-1}$$

  - Compute covariances and gain

$$R_{zz,k} = \langle z_k, z_k \rangle = \sum_{i=0}^{2n_x} W_c^{(i)} \left(\hat{z}^{(i)}_{k|k-1} - \hat{z}_{k|k-1}\right)\left(\hat{z}^{(i)}_{k|k-1} - \hat{z}_{k|k-1}\right)'$$
$$R_{e,k} = R_{yy,k} = \langle y_k, y_k \rangle = R_{zz,k} + R_k$$
$$R_{xy,k} = \langle x_k, y_k \rangle = \langle x_k, z_k \rangle = \sum_{i=0}^{2n_x} W_c^{(i)} \left(\hat{x}^{(i)}_{k|k-1} - \hat{x}_{k|k-1}\right)\left(\hat{z}^{(i)}_{k|k-1} - \hat{z}_{k|k-1}\right)'$$
$$K_{fx,k} = \langle x_k, y_k \rangle \langle y_k, y_k \rangle^{-1} = R_{xy,k} R^{-1}_{yy,k} = R_{xy,k} R^{-1}_{e,k}$$

  - Compute: $\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_{fx,k} e_k \qquad P_{k|k} = P_{k|k-1} - K_{fx,k} R_{e,k} K'_{fx,k}$

- **One-step prediction:**
  - Given $\hat{x}_{k|k}$ and $P_{k|k}$ - compute the sigma-points:

$$
\begin{aligned}
\hat{x}_{k|k}^{(0)} &= \hat{x}_{k|k} \\
\hat{x}_{k|k}^{(i)} &= \hat{x}_{k|k} + \sqrt{c}\left(\sqrt{P_{k|k}}\right)_i \quad i = 1, 2, \ldots, n_x \\
\hat{x}_{k|k}^{(i+n_x)} &= \hat{x}_{k|k} - \sqrt{c}\left(\sqrt{P_{k|k}}\right)_i \quad i = 1, 2, \ldots, n_x
\end{aligned}
$$

  - Compute output sigma points and the mean

$$
\begin{aligned}
\hat{s}_{k|k}^{(i)} &= F_k(\hat{x}_{k|k}^{(i)}) \qquad i = 0, 1, \ldots, 2n_x \\
\hat{x}_{k+1|k} = \hat{s}_{k|k} &= \sum_{i=0}^{2m_x} W_m^{(i)} \hat{s}_{k|k}^{(i)}
\end{aligned}
$$

  - Compute the covariance

$$
\begin{aligned}
R_{ss,k} &= \langle s_k, s_k \rangle = \sum_{i=0}^{2n_x} W_c^{(i)} \left(\hat{s}_{k|k}^{(i)} - \hat{s}_{k|k}\right)\left(\hat{s}_{k|k}^{(i)} - \hat{s}_{k|k}\right)' \\
P_{k+1|k} &= R_{ss,k} + Q_k
\end{aligned}
$$

  - Result: $\hat{x}_{k+1|k}$ and $P_{k+1|k}$

- **One-step prediction:**
  - Given $\hat{x}_{k|k}$ and $P_{k|k}$ - compute the sigma-points:

$$
\begin{aligned}
\hat{x}_{k|k}^{(0)} &= \hat{x}_{k|k} \\
\hat{x}_{k|k}^{(i)} &= \hat{x}_{k|k} + \sqrt{c}\left(\sqrt{P_{k|k}}\right)_i \quad i = 1, 2, \ldots, n_x \\
\hat{x}_{k|k}^{(i+n_x)} &= \hat{x}_{k|k} - \sqrt{c}\left(\sqrt{P_{k|k}}\right)_i \quad i = 1, 2, \ldots, n_x
\end{aligned}
$$

  - Compute output sigma points and the mean

$$
\begin{aligned}
\hat{s}_{k|k}^{(0,0)} &= F_k(\hat{x}_{k|k}^{(0)}, \hat{w}_{k|k}^{(0)}) \\
\hat{s}_{k|k}^{(i,0)} &= F_k(\hat{x}_{k|k}^{(i)}, w_{k|k}^{(0)}) && i = 1, \ldots, 2n_x \\
\hat{s}_{k|k}^{(0,j)} &= F_k(\hat{x}_{k|k}^{(0)}, w_{k|k}^{(j)}) && j = 1, \ldots, 2n_w
\end{aligned}
$$

$$
\hat{x}_{k+1|k} = \hat{s}_{k|k} = W_m^{(0,0)} \hat{s}_{k|k}^{(0,0)} + \sum_{i=1}^{2n_x} W_m^{(i,0)} \hat{s}_{k|k}^{(i,0)} + \sum_{j=1}^{2n_w} W_m^{(0,j)} \hat{s}_{k|k}^{(0,j)}
$$

  - Compute the covariance

$$
P_{k+1|k} = R_{ss,k} = \langle s_k, s_k \rangle = \sum_{i=0}^{2n_x} W_c^{(i)} \left(\hat{s}_{k|k}^{(i)} - \hat{s}_{k|k}\right)\left(\hat{s}_{k|k}^{(i)} - \hat{s}_{k|k}\right)'
$$

  - Result: $\hat{x}_{k+1|k}$ and $P_{k+1|k}$

Ensemble Kalman Filter

(EnKF)

- **Continuous-discrete nonlinear model:**

$$
\begin{aligned}
dx(t) &= f(x(t))dt + G(x(t))d\omega(t), & \qquad d\omega(t) &\sim N_{iid}(0, Idt), \\
y(t_k) &= h_k(x(t_k)) + v_k, & \qquad v_k &\sim N_{iid}(0, R_k).
\end{aligned}
$$

- **Filtering:**
  - Given state ensemble, $\{\hat{x}^{(i)}_{k|k-1}\}_{i=1}^{N_p}$
  - Compute ensemble measurement prediction $\{\hat{y}^{(i)}_{k|k-1}\}_{i=1}^{N_p}$

$$\hat{y}^{(i)}_{k|k-1} = h_k(\hat{x}^{(i)}_{k|k-1}), \qquad i \in \{1, 2, \ldots, N_p\}$$

  - Compute innovations

$$e^{(i)}_k = y^{(i)}_k - \hat{y}^{(i)}_{k|k-1}, \qquad y^{(i)}_k = y_k + v^{(i)}_k, \qquad i \in \{1, 2, \ldots, N_p\},$$

    where $y^{(i)}_k$ are measurement perturbation, where $v^{(i)}_k$ are samples from $v_k \sim \mathcal{N}(0, R_k)$.
  - Compute Kalman gain

$$K_{fx,k} = R_{xy,k|k-1} R^{-1}_{yy,k|k-1},$$

    where the mean and covariances are approximated using the ensemble

$$\hat{x}_{k|k-1} = \frac{1}{N_p} \sum_{i=1}^{N_p} \hat{x}^{(i)}_{k|k-1},$$

$$R_{yy,k|k-1} = \frac{1}{N_p - 1} \sum_{i=1}^{N_p} \left(\hat{y}^{(i)}_{k|k-1} - \hat{y}_{k|k-1}\right) \left(\hat{y}^{(i)}_{k|k-1} - \hat{y}_{k|k-1}\right)^T,$$

$$R_{xy,k|k-1} = \frac{1}{N_p - 1} \sum_{i=1}^{N_p} \left(\hat{x}^{(i)}_{k|k-1} - \hat{x}_{k|k-1}\right) \left(\hat{y}^{(i)}_{k|k-1} - \hat{y}_{k|k-1}\right)^T$$

  - Compute filtering ensemble $\{\hat{x}^{(i)}_{k|k}\}_{i=1}^{N_p}$

$$\hat{x}^{(i)}_{k|k} = \hat{x}^{(i)}_{k|k-1} + K_{fx,k} e^{(i)}_k.$$

- **One-step prediction:**
  - Given state ensemble, $\{\hat{x}_{k|k}^{(i)}\}_{i=1}^{N_p}$
  - Compute state ensemble prediction as solution to

$$\begin{aligned} \hat{x}_k^{(i)}(t_k) &= \hat{x}_{k|k}^{(i)}, & i &= \{1, 2, \dots, N_p\}, \\ d\hat{\boldsymbol{x}}_k^{(i)}(t) &= f(\hat{\boldsymbol{x}}_k^{(i)}(t))dt + G(\hat{\boldsymbol{x}}_k^{(i)}(t))d\boldsymbol{\omega}_k^{(i)}(t), & i &= \{1, 2, \dots, N_p\}, \end{aligned}$$

  - Compute state mean and covariance (optional)

$$\hat{x}_{k|k} = \frac{1}{N_p} \sum_{i=1}^{N_p} \hat{x}_{k|k}^{(i)},$$

$$P_{k|k-1} = R_{xx,k|k-1} = \frac{1}{N_p - 1} \sum_{i=1}^{N_p} \left(\hat{x}_{k|k}^{(i)} - \hat{x}_{k|k} \right) \left(\hat{x}_{k|k}^{(i)} - \hat{x}_{k|k} \right)^T.$$

Particle Filter

(PF)

- **Continuous-discrete nonlinear model:**

$$\begin{aligned}
dx(t) &= f(x(t))dt + G(x(t))d\omega(t), && d\omega(t) \sim N_{iid}(0, Idt),\\
y(t_k) &= h_k(x(t_k)) + v_k, && v_k \sim N_{iid}(0, R_k).
\end{aligned}$$

- **Filtering:**
  - Given set of sampled particles with associated weights, $\{\hat{x}^{(i)}_{k|k-1}, w^{(i)}_k\}_{i=1}^{N_p}$
  - Compute innovations

$$e^{(i)}_k = y_k - \hat{y}^{(i)}_{k|k-1}, \qquad \hat{y}^{(i)}_{k|k-1} = h_k(\hat{x}^{(i)}_{k|k-1}), \qquad i \in \{1, 2, \ldots, N_p\}.$$

  - Compute new weights from likelihood function $p(y_k|x_k)$

$$w^{(i)}_k = \frac{1}{\sqrt{2\pi^{n_y}|R_k(\theta)|}} \exp\left(-\frac{1}{2}\left(e^{(i)}_k\right)^T R_k^{-1} e^{(i)}_k\right).$$

  - Resample particles according to new weights and assign new equal weights to all resampled particles
    1. Given weights $\{w^{(i)}_k\}$.
    2. Generate uniformly distributed sample $u_1 \sim \mathcal{U}\,[0, 1]$.
    3. Generate ordered resampling points

$$U = \frac{\{0, 1, \ldots, N_p - 1\} + u_1}{N_p}.$$

    4. Compute cumulative sum of weights, W_k.
    5. Resample m_i of each particle according to cumulative sum of weights, W_k, and ordered resampling points, U.
  - Compute mean and covariance (optional)

$$\hat{x}_{k|k} = \frac{1}{N_p}\sum_{i=1}^{N_p} \hat{x}^{(i)}_{k|k},$$

$$P_{k|k-1} = R_{xx,k|k-1} = \frac{1}{N_p - 1}\sum_{i=1}^{N_p}\left(\hat{x}^{(i)}_{k|k} - \hat{x}_{k|k}\right)\left(\hat{x}^{(i)}_{k|k} - \hat{x}_{k|k}\right)^T.$$

- **One-step prediction:**
  - Given set of sampled particles with associated weights, $\{\hat{x}_{k|k}^{(i)}, w_k^{(i)}\}_{i=1}^{N_p}$
  - Compute state particle prediction as solution to

$$\begin{aligned} \hat{x}_k^{(i)}(t_k) &= \hat{x}_{k|k}^{(i)}, & i &= \{1, 2, \ldots, N_p\}, \\ d\hat{\boldsymbol{x}}_k^{(i)}(t) &= f(\hat{\boldsymbol{x}}_k^{(i)}(t))dt + G(\hat{\boldsymbol{x}}_k^{(i)}(t))d\boldsymbol{\omega}_k^{(i)}(t), & i &= \{1, 2, \ldots, N_p\}, \end{aligned}$$

  - Compute state mean and covariance (optional)

$$\hat{x}_{k|k} = \frac{1}{N_p} \sum_{i=1}^{N_p} \hat{x}_{k|k}^{(i)},$$

$$P_{k|k-1} = R_{xx,k|k-1} = \frac{1}{N_p - 1} \sum_{i=1}^{N_p} \left(\hat{x}_{k|k}^{(i)} - \hat{x}_{k|k}\right) \left(\hat{x}_{k|k}^{(i)} - \hat{x}_{k|k}\right)^T.$$