Model Predictive Control

Lecture 4: Modeling of Reactive and Distributed Systems

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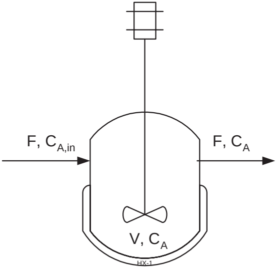
02619 Model Predictive Control

Learning Objectives

After this lecture you should be able to

1. Model systems with chemical reaction
2. Model flows in pipes (time delay systems, distributed systems)
3. Model chemical reaction and flow in pipes
4. Describe why we need time-delays for modeling

Chemical Reaction in a Tank

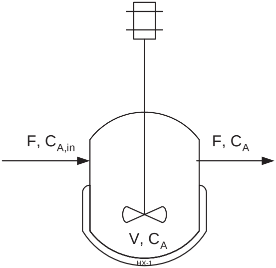


$$A \rightarrow P \qquad r = kC_A$$

The production rate of A is

$$R_A = -r$$

Chemical Reaction in a Tank



$$Accumulated = VC_A(t+\Delta t) - VC_A(t)$$
$$Influx = FC_{A,in}(t)\Delta t$$
$$Outflux = FC_A(t)\Delta t$$
$$Generated = R_A V \Delta t \qquad R_A = R_A(C_A(t))$$

$$A \to P \qquad r = kC_A \qquad R_A = -r$$

$$Accumulated = Influx - Outflux + \overbrace{Produced - Consumed}^{Generated}$$

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$$\begin{aligned} Accumulated &= VC_A(t+\Delta t) - VC_A(t) \\ Influx &= FC_{A,in}(t)\Delta t \\ Outflux &= FC_A(t)\Delta t \\ Generated &= R_A V \Delta t \qquad R_A = R_A(C_A(t)) \end{aligned}$$

1.

$$VC_A(t+\Delta t) - VC_A(t) = FC_{A,in}(t)\Delta t - FC_A(t)\Delta t + R_A(C_A(t))V\Delta t$$

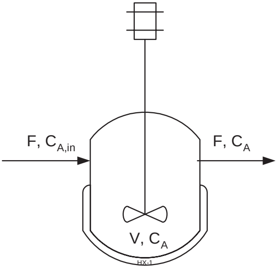
2.

$$\frac{C_A(t+\Delta t) - C_A(t)}{\Delta t} = \frac{F}{V}\left(C_{A,in}(t) - C_A(t)\right) + R_A(C_A(t))$$

3. $\Delta t \to 0$

$$\frac{dC_A}{dt} = \frac{F}{V}\left(C_{A,in}(t) - C_A(t)\right) + R_A(C_A(t))$$

Chemical Reaction in a Tank



$$\frac{dC_A}{dt}(t) = D\left(C_{A,in}(t) - C_A(t)\right) + R_A$$

$$D = \frac{F}{V}$$

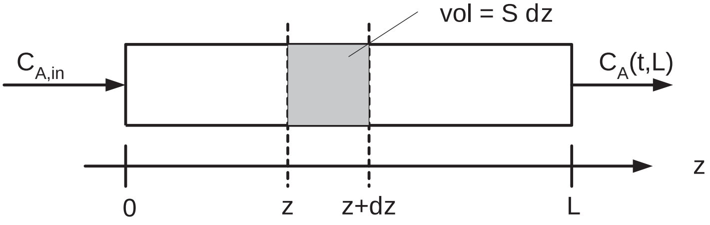
$$A \to P \qquad r = kC_A \qquad R_A = -r$$

Let $y(t) = C_A(t)$ and $u(t) = C_{A,in}(t)$. Then the corresponding LaPlace transform of this system can be written in the form

$$Y(s) = \frac{K}{\tau s + 1} U(s)$$

Question: What is K and τ?

Flow in a Pipe



$$Accumulated = \left[C_A(t+\Delta t, z) - C_A(t,z)\right] S\Delta z$$
$$Influx = N_A(t,z) S\Delta t$$
$$Outflux = N_A(t, z+\Delta z) S\Delta t$$

Flow in a Pipe

$$
\begin{aligned}
Accumulated &= \left[C_A(t+\Delta t, z) - C_A(t,z)\right] S\Delta z \\
Influx &= N_A(t,z)S\Delta t \\
Outflux &= N_A(t, z+\Delta z)S\Delta t
\end{aligned}
$$

1.

$$
\overbrace{\left[C_A(t+\Delta t, z) - C_A(t,z)\right] S\Delta z}^{Accumulated} = \overbrace{N_A(t,z)S\Delta t}^{Influx} - \overbrace{N_A(t, z+\Delta z)S\Delta t}^{Outflux}
$$

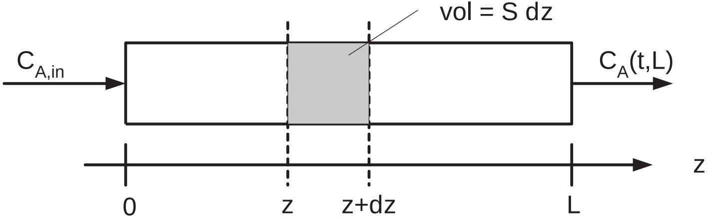
2.

$$
\frac{C_A(t+\Delta t, z) - C_A(t,z)}{\Delta t} = -\frac{N_A(t, z+\Delta z) - N_A(t,z)}{\Delta z}
$$

3. $\Delta \to 0$ and $\Delta z \to 0$

$$
\frac{\partial C_A}{\partial t}(t,z) = -\frac{\partial N_A}{\partial z}(t,z)
$$

Differential Equation Model



$$\frac{\partial C_A}{\partial t}(t,z) = -\frac{\partial N_A}{\partial z}(t,z)$$

Initial condition

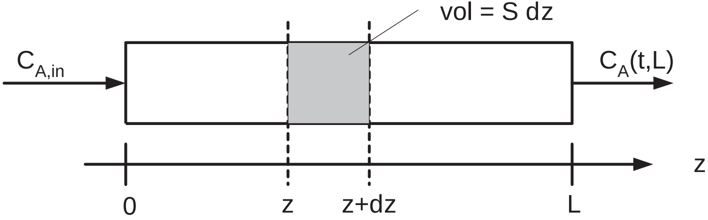
$$C_A(0,z) = C_{A0}(z) \qquad 0 \leq z \leq L$$

Boundary condition

$$C_A(t,0) = C_{A,in}(t) \qquad t \geq 0$$

Flux (convective flow)

$$N_A(t,z) = vC_A(t,z)$$



Solution

$$C_A(t, L) = C_{A,in}(t - \tau) \qquad \tau = \frac{L}{v}$$

Let

$$y(t) = C_A(t, L)$$
$$u(t) = C_{A,in}(t)$$

then

$$y(t) = u(t - \tau)$$

or

$$Y(s) = e^{-\tau s} U(s)$$

LaPlace Transform of a Time Delay

Consider a system with the solution

$$y(t) = u(t-\tau)$$

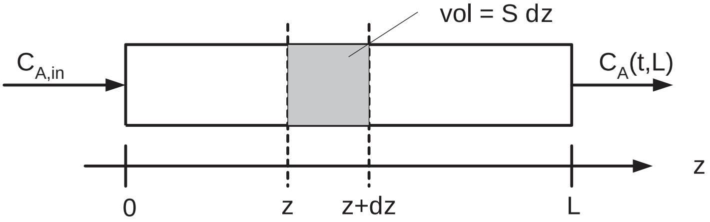
Let $Y(s)$ and $U(s)$ be the LaPlace transform of $y(t)$ and $u(t)$

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t) dt$$
$$U(s) = \mathcal{L}\{u(t)\} = \int_0^\infty e^{-st} u(t) dt$$

Assume $u(t) = 0$ for $t < 0$. Then

$$\begin{aligned} Y(s) &= \int_0^\infty e^{-st} y(t) dt = \int_0^\infty e^{-st} u(t-\tau) dt \\ &= \underbrace{e^{-s\tau} e^{s\tau}}_{=1} \int_0^\infty e^{-st} u(t-\tau) dt = e^{-s\tau} \int_0^\infty e^{-s(t-\tau)} u(t-\tau) dt \\ &= e^{-\tau s} U(s) \end{aligned}$$

Flow and Chemical Reaction in a Pipe



Chemical Reaction: $A \rightarrow P \qquad r = kC_A \qquad R_A = -r$

Flux for convective flow: $N_A = vC_A$

$$Accumulated = \left[C_A(t+\Delta t, z) - C_A(t,z)\right] S\Delta z$$
$$Influx = N_A(t,z)S\Delta t$$
$$Outflux = N_A(t, z+\Delta z)S\Delta t$$
$$Generated = R_A S\Delta z\Delta t$$

Flow and Chemical Reaction in a Pipe

$$Accumulated = Influx - Outflux + Generated$$

$$\begin{aligned} Accumulated &= [C_A(t+\Delta t, z) - C_A(t, z)]\, S\Delta z \\ Influx &= N_A(t, z) S \Delta t \\ Outflux &= N_A(t, z+\Delta z) S \Delta t \\ Generated &= R_A S \Delta z \Delta t \end{aligned}$$

leads to the partial differential equation

$$\frac{\partial C_A}{\partial t}(t, z) = -\frac{\partial N_A}{\partial z}(t, z) + R_A$$

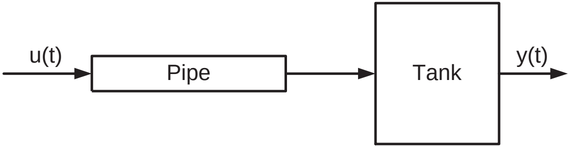
with the boundary equations

$$\begin{aligned} C_A(0, z) &= C_{A0}(z) \qquad 0 \le z \le L \\ C_A(t, 0) &= C_{A,in}(t) \qquad t \ge 0 \end{aligned}$$

The constitutive equations are

$$\begin{aligned} N_A &= v C_A \\ R_A &= -r \qquad r = k C_A \end{aligned}$$

Linear Systems with Delay



$$Y(s) = G(s)U(s) \qquad G(s) = \frac{B(s)}{A(s)} e^{-\tau_d s}$$

Examples

$$G(s) = \frac{K}{\tau s + 1}\, e^{-\tau_d s}$$
$$G(s) = \frac{K(\beta s + 1)}{(\tau_1 s + 1)(\tau_2 s + 1)} e^{-\tau_d s}$$

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Questions and Comments

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