Computer Arithmetic

David Marchant
Based on slides by Troels Henriksen

2022-09-13

Inspired by slides by Randal E. Bryant and David R. O'Hallaron.
Some material by Michael Kirkedal Tomsen.

Agenda

Biased number representation

For *biased numbers*, the raw bits are interpreted as unsigned, and then a constant *bias* is subtracted.

Unsigned

$$B2U(X) = \sum_{i=0}^{w-1} x_i \cdot 2^i$$

Two's complement

$$B2S(X) = -x_{w-1} \cdot 2^{w-1} + \sum_{i=0}^{w-2} x_i \cdot 2^i$$

Biased

$$B2I(X) = \left(\sum_{i=0}^{w-1} x_i \cdot 2^i \right) - \text{Bias}$$

- Typically

$$Bias = 2^{w-1} - 1$$

- **Examples for** $w = 8$, **Bias** $= 127$

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11111111_2	255_{10}	-1_{10}	128_{10}

Integral binary numbers

We have seen that

$$10010101_2$$

is basically interpreted like

$$149_{10}$$

"Structure" is the same, just with 2 instead of 10.

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Can we do the same thing for fractional numbers?

$$1011.101_2$$

Fractional numbers

$$123.456 = 1 \cdot 10^2 + 2 \cdot 10^1 + 3 \cdot 10^0 + 4 \cdot 10^{-1} + 5 \cdot 10^{-2} + 6 \cdot 10^{-3}$$

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Generally

$$a_{m-1} \cdots a_0 . a_{-1} \cdots a_{-n} = \sum_{i=-n}^{m-1} a_i \cdot 10^i$$

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Even more generally, for radix r

$$a_{m-1} \cdots a_0 . a_{-1} \cdots a_{-n} = \sum_{i=-n}^{m-1} a_i \cdot r^i$$

Fractional binary numbers

Weight	2^{m-1}	2^{m-2}	$\cdots$	4	2	1	1/2	1/4	1/8	$\cdots$	2^{-n}
Digits	b_{m-1}	b_{m-2}	$\cdots$	b_2	b_1	b_0	b_{-1}	b_{-2}	b_{-3}	$\cdots$	b_{-n}

Representation

- Bits to the right of "binary point" represents fractional powers of 2.
- Represents rational number.

$$b_{m-1} \cdots b_0 . b_{-1} \cdots b_{-n} = \sum_{i=-n}^{m-1} b_i \cdot 2^i$$

Examples of fractional binary numbers

Value	Representation
$5\frac{3}{4}$	101.11_2
$2\frac{7}{8}$	10.111_2
$1\frac{7}{16}$	1.0111_2

Observations

- Divide by 2 by logical shifting right.
- Multiply by 2 by shifting left.
- Numbers of form $0.111\ldots$ are just below 1.0.
  - $1/2 + 1/4 + 1/8 + \cdots 1/2^n + \cdots \sim 1.0$.
  - Use notation $1.0 - \epsilon$.

Representable numbers

Limitation #1

- Can only represent fractional part of form $x/2^k$
- Other rational numbers have repeating bit representation

Value	**Representation**
$\frac{1}{3}$	$0.0101010101[01]\cdots_2$
$\frac{1}{5}$	$0.001100110011[0011]\cdots_2$
$\frac{1}{10}$	$0.0001100110011[0011]\cdots_2$

Limitation #2

- Just one setting of binary point within the w bits.
  - Limited range of numbers—very small values? Very large?

The fixed-point dilemma

Consider $w = 8$

1 bit for fraction

- Largest number: $1111111.1_2 = 127.5_{10}$
- Increment: $0000000.1_2 = 0.5_{10}$

7 bits for fraction

- Largest number: $1.1111111_2 = 1.9921875_{10}$
- Increment: $0.0000001_2 = 0.0078125_{10}$

4 bits for fraction

- Largest number: $1111.1111_2 = 15.9375_{10}$
- Increment: $0000.0001_2 = 0.0625_{10}$

Fixed-point has same absolute precision everywhere, but this means relative precision is worse for numbers close to 0!

IEEE Floating Point

IEEE Standard 754

- Established in 1985 as uniform standard for floating point.
  - Many idiosyncratic formats before then.
- Supported by all major CPUs, GPUs, and most other processors.

Driven by numerical concerns

- Nice standards for rounding, overflow, underflow.
- Hard to make fast in hardware.
  - Numerical analysts predominated over hardware designers in defining standard.
  - ... but (later) Turing Award winner William Kahan secretly knew that Intel had figured out how to make accurate computation *fast*.
  - **Beware the wrath of Kahan!**
    - `http://people.eecs.berkeley.edu/~wkahan/`

Essentially scientific notation

$$3.5 \times 10^2 = 350$$

- **Significand** is 3.5
  - Conventionally a number in range $[1, 10)$, with sign.
- **Exponent** is 2.
  - Can also be negative.

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To keep significand in range, *adjust exponent*:

$$\begin{array}{lcl} 35 \times 10^1 & \Rightarrow & 3.5 \times 10^2 \\ 0.35 \times 10^3 & \Rightarrow & 3.5 \times 10^2 \end{array}$$

IEEE 754 uses bits instead of digits, and specifies a fixed-size encoding, but idea is the same.

Floating Point Representation

Numerical form

$$(-1)^S \cdot M \cdot 2^E$$

- **Sign bit** S determines whether number negative or positive.
- **Significand** M normally a fractional value in range $[1, 2)$.
- **Exponent** E weights value by power of two.

Encoding

- Most significant bit is sign bit.
- Exp field encodes E (but is not equal to E).
- Frac field encodes M (but is not equal to M).

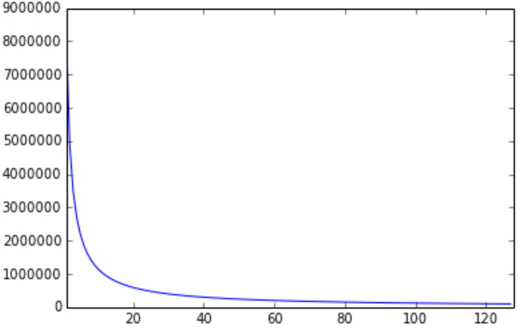
S	Exp	Frac

Why such a weird format?

The point is floating

- No fixed number of bits allocated to “fraction”.
- More bits close to 0, fewer bits for numbers with large magnitude.
- Symmetric around 0.

Density of floats



https://stackoverflow.com/a/24179424/6131552

Precision options

32-bit single precision: `float`

S	Exp	Frac
1 bit	8 bits	23 bits

64-bit double precision: `double`

S	Exp	Frac
1 bit	11 bits	52 bits

80-bit Extended precision (Intel only, never use): `long double`

S	Exp	Frac
1 bit	15 bits	64 bits

Newer standards contain more variants (16 bits, decimal floats) that we will not cover.

Main problem: not all numbers are representable

Format trouble

For example the number

$$0.1_{10}$$

cannot be represented on the form

$$(-1)^S \cdot M \cdot 2^E$$

Precision trouble

- A fixed number of bits cannot represent all numbers.
- Integer types represent an interval of natural numbers.
- *Any nontrivial interval* of rational numbers contains infinity elements.
- “Neighbouring” floats are separated by a “step size” 2^E (we’ll see).

Consequence

- **Rounding**.

Rounding of floating point numbers

- Floating point arithmetic returns the floating point number *closest* to the mathematically correct result.

Example

Mathematically,

$$1/10 = 0.1$$

But since 0.1 cannot be represented in binary floating point, we instead get the number that is closest:

$$1/10 = 0.1000000000000000055511151231257827021181583404541015625 0$$

(With 64-bit floats).

We will return to this, but writing algorithms that are robust to roundoff errors is a *big topic* that is outside the scope of this course.

Normalised values when Exp $\neq 0 \cdots 0$ and Exp $\neq 1 \cdots 1$

$$v = (-1)^S \cdot M \cdot 2^E$$

S	Exp	Frac

- **Exponent encoded as *biased* value**

$$E = \text{Exp} - \text{Bias}$$

  - Exp: unsigned value of Exp field.
  - Bias $= 2^{k-1} - 1$, where k is number of Exp bits.
    - Single precision: 127 (Exp $\in [1, 254]$, $E \in [-126, 127]$).
    - Double precision: 1023 (Exp $\in [1, 2046]$, $E \in [-1022, 1023]$).
- **Significand coded with implied leading 1:**

$$M = 1.xxx \cdots x_2$$

  - $xxx \cdots x$: bits of Frac field.
  - Minimum when Frac $= 0000 \cdots 0$ ($M = 1$).
  - Maximum when Frac $= 1111 \cdots 1$ ($M = 2 - \epsilon$).
  - Get extra leading bit for free.

Normalised encoding example

$$\boxed{v = (-1)^S \cdot M \cdot 2^E} \qquad \boxed{E = \text{Exp} - \text{Bias}}$$

Value: `float F = 15213.0`

$$\begin{aligned} 15213_{10} &= 11101101101101_2 \\ &= 1.1101101101101_2 \cdot 2^{13} \end{aligned}$$

Significand

$$\begin{aligned} M &= 1.1101101101101_2 \\ \text{Frac} &= \quad 11011011011010000000000_2 \end{aligned}$$

Exponent

$$\begin{aligned} E &= 13_{10} \\ \text{Bias} &= 127_{10} \\ \text{Exp} &= E + \text{Bias} = 140_{10} = 10001100_2 \end{aligned}$$

Result

0	10001100	11011011011010000000000

Denormal values

$$\boxed{v = (-1)^S \cdot M \cdot 2^E} \quad \boxed{E = 1 - \text{Bias}}$$

Occur when Exp $= 000\cdots0_2$.

- **Exponent encoded as**
$$E = 1 - \text{Bias}$$
- **Significand coded with implied leading 0:**
$$M = 0.xxx\cdots x_2$$
- **Cases**
  - $\text{Exp} = 000\cdots0_2, \text{Frac} = 000\cdots0_2$
    - Represents zero value.
    - Note distinct values $-0, +0$ — when might that be useful?
  - $\text{Exp} = 000\cdots0_2, \text{Frac} \neq 000\cdots0_2$
    - Numbers closest to 0.0.
    - Called **subnormal numbers.**
    - Ensure that $x \neq y \Rightarrow x - y \neq 0$, i.e. avoid underflow.

Special values

Occur when Exp $= 111\cdots1_2$.

When Exp $= 111\cdots1_2$, Frac $= 000\cdots0_2$

- Represents $\pm\infty$.
- Typically the result of *overflow*.
  - Overflow can be negative!
  - *Underflow* is when the result becomes zero due to rounding.
- Both positive and negative.
- Examples:

$$\frac{1}{0} = \frac{-1}{-0} = \infty \qquad \frac{1}{-0} = -\infty$$

When Exp $= 111\cdots1_2$, Frac $\neq 000\cdots0_2$

- Not A Number (NaN).
- Represents case when no numeric value can be determined.
- Examples:

$$\texttt{sqrt}(-1) \qquad \infty - \infty \qquad \infty \cdot 0$$

The floating point number line

← very positive E   very negative E →   ← very negative E   very positive E →

$-\infty$	-Normal	-Denorm	-0	$+0$	+Denorm	+Normal	$+\infty$

NaN		NaN

Note that NaNs are unordered:

- NaN is different from everything *even other NaNs*!
  - `NaN == NaN` is false.
  - Floating-point equality is not reflexive!
- `NaN > x` and `NaN < x` is false for all `x`.

Play the game

https://topps.diku.dk/compsys/floating-point.html

Tiny 8-bit floating point example

s	Exp	Frac
1b	4b	3b

8-bit floating point representation

- Sign bit is the most significant bit (leftmost).
- The next four bits are Exp with a bias of 7.
- The last three bits are Frac.

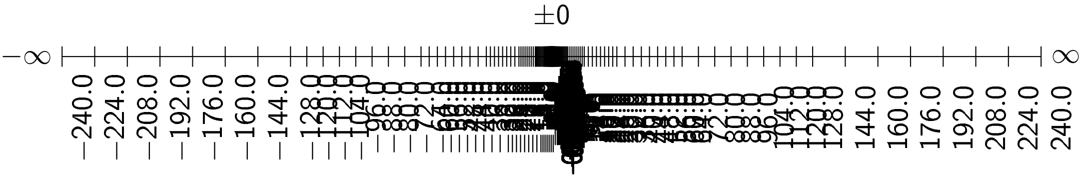
Same general form as IEEE Format

- Normalised, denormalised.
- Representation of 0, NaN, both infinities.

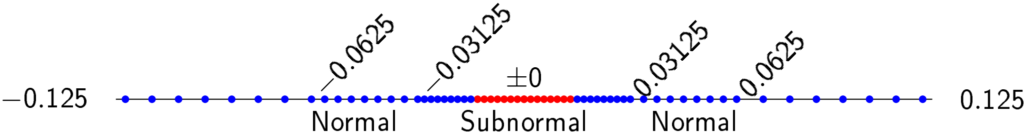
Let's look at their dynamic range.

	Sign	Exp	Frac	E	Value	
Denormalised	0	0000	000	−6	$0/8 \cdot 1/64 = 0/512$	zero
	0	0000	001	−6	$1/8 \cdot 1/64 = 1/512$	closest to zero
	0	0000	010	−6	$2/8 \cdot 1/64 = 2/512$	
	...					
	0	0000	111	−6	$7/8 \cdot 1/64 = 7/512$	largest denorm
Normalised	0	0001	000	−6	$8/8 \cdot 1/64 = 8/512$	smallest norm
	0	0001	001	−6	$9/8 \cdot 1/64 = 9/512$	
	...					
	0	0110	110	−1	$14/8 \cdot 1/2 = 14/16$	
	0	0110	111	−1	$15/8 \cdot 1/2 = 15/16$	closest to 1
	0	0111	000	0	$8/8 \cdot 1 = 8/8$	1
	0	0111	001	0	$9/8 \cdot 1 = 9/8$	closest to 1
	0	0111	010	0	$10/8 \cdot 1 = 10/8$	
	...					
	0	1110	110	7	$14/8 \cdot 128 = 224$	
	0	1110	111	7	$15/8 \cdot 128 = 240$	
	0	1111	000	N/A	∞	
	0	1111	001	N/A	NaN	
	...				NaN	

Distribution of values



Distribution of values (zooming in)



- Note how the distribution gets denser towards zero.
- Note the big gap there would be around 0 if we did not have subnormals.
- Each of the spans with same distance between neighbors corresponds to numbers with same Exp.

s	Exp	Frac
1b	4b	3b

Useful properties of the IEEE encoding

S	Exp	Frac

- **Floating-point zero same as integer zero**
  - All bits 0.
  - ...but negative zero is different.
- **Can almost compare floats with unsigned integer comparisons**
  - Must first compare sign bit.
  - Must consider $-0 = 0$.
  - NaNs problematic:
    - Greater than any other value (because $\text{Exp} = 111\cdots1_2$).
    - What should comparison yield?
  - Otherwise OK:
    - Normalised and denormalised compare as expected.
    - Infinities ordered properly relative to finities.

Basic idea behind floating point operations

$$x +_f y = \text{Round}(x + y)$$

$$x \times_f y = \text{Round}(x \times y)$$

- **Basic idea**
  - First *compute exact result*!
  - Then round it to fit into desired precision.
    - Overflow if exponent too large.
    - *Round to fit* into Frac.

Rounding and rounding modes

- There's more than one way to round a number, here to an integer.

	1.40	1.60	1.50	2.50	−1.50
Towards zero					

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Towards ∞	2	2	2	3	-1
Nearest even					

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	1.40	1.60	1.50	2.50	-1.50
Towards zero	1	1	1	2	-1
Towards $-\infty$	1	1	1	2	-2
Towards ∞	2	2	2	3	-1
Nearest even ∞	1	2	2	2	-2

- "Round to nearest, ties to even" is the default rounding mode.

Closer look at *nearest even*

- **Default rounding mode**
  - But can be changed dynamically.
    - `https://www.gnu.org/software/libc/manual/html_node/Rounding.html`
    - Never do this.
  - All others are statistically biased.
    - Biased: Sum of set of positive numbers will consistently be over- or under-estimated.
- **Applying to other decimal places / bit positions**
  - When exactly halfway between two possible values:
    - Round so that least significant digit is even.
  - E.g. rounding to nearest hundredth:
    - 7.8949999:

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    - 7.8990001:

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    - 7.8990001: 7.90
    - 7.8950000:

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    - Round so that least significant digit is even.
  - E.g. rounding to nearest hundredth:
    - 7.8949999: 7.89
    - 7.8990001: 7.90
    - 7.8950000: 7.90
    - 7.8850000:

Closer look at *nearest even*

- **Default rounding mode**
  - But can be changed dynamically.
    - `https://www.gnu.org/software/libc/manual/html_node/Rounding.html`
    - Never do this.
  - All others are statistically biased.
    - Biased: Sum of set of positive numbers will consistently be over- or under-estimated.
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Rounding binary numbers

- **Binary fractional numbers**
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  - Round to nearest $1/4$ (2 bits right of binary point).

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Floating point multiplication (assuming operands are numbers)

$$((-1)^{S_3} \cdot M_3 \cdot 2^{E_3}) = ((-1)^{S_1} \cdot M_1 \cdot 2^{E_1}) \cdot ((-1)^{S_2} \cdot M_2 \cdot 2^{E_2})$$

- **Exact result**

$$\begin{aligned} S_3 &= S_1 \oplus S_2 \\ M_3 &= M_1 \cdot M_2 \\ E_3 &= E_1 + E_2 \end{aligned}$$

  where $\oplus$ is exclusive-or.
- **Fixing**
  - If $M_3 \geq 2$, shift M_3 right and increment E_e.
  - If E_3 out of range, overflow to ∞.
  - Round M_3 to fit Frac precision.
- **Implementation**
  - Biggest chore is multiplying significands.

Floating point addition (assuming operands are numbers)

$$((-1)^{S_3} \cdot M_3 \cdot 2^{E_3}) = ((-1)^{S_1} \cdot M_1 \cdot 2^{E_1}) + ((-1)^{S_2} \cdot M_2 \cdot 2^{E_2})$$

- **Approach**
  - Assume without loss of generality that $E_1 \geq E_2$.
  - Rewrite smaller number such that its exponent matches E_1:

$$((-1)^{S_3} \cdot M_3 \cdot 2^{E_3}) = ((-1)^{S_1} \cdot M_1 \cdot 2^{E_1}) + ((-1)^{S_2} \cdot M_2' \cdot 2^{E_1})$$

- **Exact result**
  - Sign S_3, significant M_3:
    - Result of signed addition.
- **Fixing**
  - If $M_3 \geq 2$, shift M_3 right and increment E_3.
  - If $M_3 < 1$, shift M left k positions and decrement E_3 by k.
  - If E_3 out of range, overflow to ∞.
  - Round M to fit Frac precision.

$\leftarrow E_1 - E_2 \rightarrow$

$\boxed{-1^{S_1} \cdot M_1}$

$+ \quad \boxed{-1^{S_2} \cdot M_2}$

$\boxed{-1^{S_3} \cdot M_3}$

Example of floating-point addition with a 2-bit significand

$$
\begin{array}{rlr}
 & (-1.01 \cdot 2^2) + (1.1 \cdot 2^4) & \\
= & (-1.01 \cdot 2^2) + (110.0 \cdot 2^2) & \text{Align exponents} \\
= & (-1.01 + 110.0) \cdot 2^2 & \text{Distributivity} \\
= & 100.11 \cdot 2^2 & \text{Add significands} \\
= & 1.0011 \cdot 2^4 & \text{Normalise} \\
= & 1.01 \cdot 2^4 & \text{Perform rounding}
\end{array}
$$

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  - Closed under addition?

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    - Due to overflow and inexactness of rounding.
    - `(3.14 + 1e10)-1e10 = 0`
    - `3.14 + (1e10-1e10) = 3.14`
  - 0 is additive identity?

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  - $a \geq b \Rightarrow a + c \geq b + c$?

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- **Monotonicity**
  - $a \geq b \Rightarrow a + c \geq b + c$? **Almost**
    - Infinities and NaNs are the exception.

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    - `(1e20*1e20)*1e-20=`∞
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    - `1e20*(1e20*1e-20)= 1e20`
  - 1 is multiplicative identity? **Yes**
  - Multiplication distributes over addition? **No**
    - Overflow and rounding again.
    - `1e20*(1e20-1e20) = 0.0`
    - `1e20*1e20 - 1e20*1e20 = NaN`

Floating point arithmetic

- Biased numbers
- Background: Fractional binary numbers
- IEEE floating point standard
- Examples and properties
- Rounding, addition, and multiplication
- Floating point in C

Summary

Floating point in C

- **C guarantees two types**
  - `float`: 32-bit single precision.
  - `double`: 64-bit single precision.
- **Conversions/casting**
  - Casting between `int`, `float`, and `double` changes bit represensation.
  - `double`/`float` to `int`
    - Truncates fractional part.
    - Like rounding toward zero.
    - Not defined when out of range or NaN: generally sets to SMin.
  - `int` to `double`
    - Exact conversion as long as `int` fits in 53 bits.
  - `int` to `float`
    - Will round according to rounding mode.

Floating point is exciting!

First “flight” of the Ariane 5 in 1996.

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First “flight” of the Ariane 5 in 1996.

- A `double` storing horizontal velocity of the rocket was converted to a 16-bit signed integer.
- The number was larger than 32767 so the conversion failed, causing an exception, crashing the guidance module.

Floating point puzzles

For each of the following C expressions, either

- Argue that it is true for all argument values.
- Explain why it's not.

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int    x = ...;
float  f = ...;
double d = ...;
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Assume neither `d` nor `t` is NaN.

Assume `int` is 32 bits.

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For each of the following C expressions, either

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- `d * d >= 0.0`
- `(d+f)-d == f`

Summary

- **IEEE floating point has clear properties.**
  - But they may not match your intuition.
- **Represents numbers of the form $M \cdot 2^E$.**
- One can reason about operations independent of implementation.
  - Computed with perfect precision and then rounded.
  - But rounded after *every* "primitive" operation (e.g. addition, multiplication).
- **Not the same as $\mathbb{Q}/\mathbb{R}$ arithmetic.**
  - Violates associativity and distributivity, mostly due to rounding.
  - Sometimes makes life difficult for heavy-duty numerical programming.
  - But carefully designed such that "naive" use mostly does what one expects.

Also try this tool: `https://evanw.github.io/float-toy/`
And read this: `https://moyix.blogspot.com/2022/09/someones-been-messing-with-my-subnormals.html`