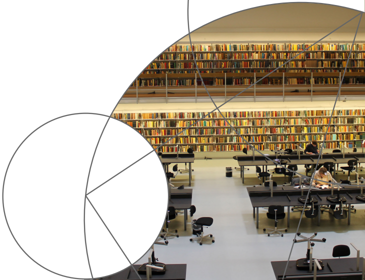
UNIVERSITY OF COPENHAGEN

DMA: Logic

Laura Mančinska
Institut for Matematiske Fag



Plan for today

- Logical propositions (statements)
- Logical operations: AND, OR, NOT, IMPLIES
- Logically equivalent propositions
- Truth tables

Reading for today: KBR 2.1–2.2

Logical propositions

A proposition or statement is a declarative sentence that is either true or false.

Examples:

- 25 is divisible by 5 (true)
- $\gcd(24, 18) = 4$ (false)
- Someone in this class has a birthday today (?)

Nonexamples:

- Math is beautiful (subjective)
- Is it your birthday today? (question)
- Have some cake! (not a declaration)

We use propositional variables, for example, p, q, r, to represent propositions.

Forming new (compound) statements

- We combine algebraic expressions using arithmetic operations like $+, \times, -$ to form new expressions:

$$(5x + 5) \times (3 - 5y)$$

- Similarly, we can combine propositions using logical operations ($\wedge, \vee, \sim, \Rightarrow, \Leftrightarrow$ etc.) to get new ones:

$$(q \vee p) \wedge (\sim q) \wedge (r \vee r)$$

Logical operations: AND, OR, NOT

- Conjunction: $p \wedge q$
  Pronounced: p and q
  True if p and q are both true. Otherwise false.
- Disjunction: $p \vee q$
  Pronounced: p or q
  True if at least one of p, q is true. Otherwise false
- Negation: $\sim p$
  Pronounced: not p
  True if p is false. False if p is true.

Note: In contrast to logic, in every day speech "or" is sometimes used in an exclusive manner:

"You may have chicken or you may have fish."

- Exclusive OR: $p \; \mathtt{xor} \; q$
  True if exactly one of p, q is true. Otherwise false.

Example

- p: Bob is younger than Alice
- q: Bob is Alice's brother

Which of the following English sentences correctly describe

$$(\sim p) \wedge q$$

1. Bob is not younger than Alice; also Bob is Alice's brother.
2. Bob is older than his sister Alice.
3. Bob is at least as old as his sister Alice.

Can you spot a mistake?

Wim van Dam @Wim_van_Dam · Aug 16

Civics Question#54 for the US Naturalization Test: "How old do citizens have to be to vote for President?"

Official USCIS answer: "eighteen (18) and older".

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Truth tables

A truth table gives the truth value (T/F) of a (compound) statement for all possible values of propositional variables.

p	q	$p \wedge q$	$p \vee q$	$\sim p$
F	F	F	F	T
F	T	F	T	T
T	F	F	T	F
T	T	T	T	F

Lets find the truth table for the proposition $\sim p \wedge (p \vee q)$

p	q	$\sim p$	$p \vee q$	$\sim p \wedge (p \vee q)$
F	F	T	F	F
F	T	T	T	T
T	F	F	T	F
T	T	F	T	F

Logical operations. Implication: $p \Rightarrow q$

Pronounced: "p implies q" / "if p then q"

p	q	$p \Rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T

Notes:

- p is called hypothesis and q is called conclusion.
- Examples: If pigs can fly, then $2+2=5$.
  If pigs can fly, then $2+2=4$.
  If it rains tomorrow, then 2 is the smallest prime.

  What are the truth values of the above propositions?

Contrapositive and converse

Def. Consider implication $p \Rightarrow q$. Its converse is $q \Rightarrow p$ and its contrapositive is $(\sim q) \Rightarrow (\sim p)$.

Example.
Implication: “If I'm hungry, then I'm grumpy.”
Converse:“If I'm grumpy, then I'm hungry.”
Contrapositive:“If I'm not grumpy, then I'm not hungry.”

Question. Consider implication $p \Rightarrow q$.

- Does its converse, $q \Rightarrow p$, express the same thing?
- How about its converse, $(\sim q) \Rightarrow (\sim p)$?

Logical operations: $p \Leftrightarrow q$

Pronounced: “p is equivalent to q” / “p if and only if q”
True if p and q have the same truth value. Otherwise false.

p	q	$p \Leftrightarrow q$
F	F	T
F	T	F
T	F	F
T	T	T

Notes:

- If $p \Leftrightarrow q$ is a true statement, then we say that p and q are (logically) equivalent and write $p \equiv q$.
- What is the difference between

$$p \Leftrightarrow q \quad \text{and} \quad p \equiv q?$$

Example: Logical equivalence

- Lets find the truth table for the contrapositive of $p \Rightarrow q$
- The contrapositive statement is $(\sim q) \Rightarrow (\sim p)$

p	q	$\sim q$	$\sim p$	$(\sim q) \Rightarrow (\sim p)$	$p \Rightarrow q$
F	F	T	T	T	T
F	T	F	T	T	T
T	F	T	F	F	F
T	T	F	F	T	T

Conclusion: $(p \Rightarrow q) \equiv ((\sim q) \Rightarrow (\sim p))$
In words: Truth values of an implication and its contrapositive are always the same.

Q: Are the truth values of an implication and its **converse** also always the same?

Tautology

Def. A tautology is a (compound) statement that is always true.

Example.
Let's find the truth table for $(p \Rightarrow q) \Leftrightarrow ((\sim p) \vee q)$

p	q	$\sim p$	$(\sim p) \vee q$	$p \Rightarrow q$	$(p \Rightarrow q) \Leftrightarrow ((\sim p) \vee q)$
F	F	T	T	T	T
F	T	T	T	T	T
T	F	F	F	F	T
T	T	F	T	T	T

Note: $(p \Rightarrow q) \equiv ((\sim p) \vee q)$

In general: $r \equiv s$ precisely when $r \Leftrightarrow s$ is a tautology.

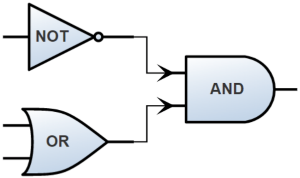
Absurdity

Def. An absurdity is a (compound) statement that is always false.

Example: $p \wedge (\sim p)$ is an absurdity (check!)

Exercise (electronic circuits)

p	q	$p \odot q$
F	F	T
F	T	T
T	F	T
T	T	F



Alternatively, we can define $\odot$ as

$$p \odot q \equiv \sim(p \wedge q)$$

Task: Find an equivalent expression for $\Rightarrow$(implies) using only $\odot$, $\sim$(not).

You should be able to:

- Rewrite simple declarative sentences using logical expressions.
- Translate logical expressions into everyday language.
- Calculate the truth table of a compound proposition (logical formula).
- Recognize logically equivalent propositions.

Exercises

Are Alice and Bob saying the same thing?

Alice: If it is raining, then I'll go home.

Bob: If I don't go home, then it is not raining.

Yes, since one statement is the contrapositive of the other and $(p \Rightarrow q) \equiv ((\sim p) \Rightarrow (\sim q))$.

Are Alice and Bob saying the same thing?

Alice: If you pass all of the weekly assignments, then you pass DMA.

Bob: If you don't pass all of the weekly assignments, then you don't pass DMA.

No! In general, $(p \Rightarrow q) \not\equiv ((\sim p) \Rightarrow (\sim q))$. Check!

Properties of AND, OR, NOT [Thm 2.2.1]

(1) $p \vee q \equiv q \vee p$

(2) $p \wedge q \equiv q \wedge p$

(3) $p \vee (q \vee r) \equiv (p \vee q) \vee r$

(4) $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$

(5) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

(6) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

(7) $\sim(\sim p) \equiv p$

(8) $\sim(p \vee q) \equiv (\sim p) \wedge (\sim q)$

(9) $\sim(p \wedge q) \equiv (\sim p) \vee (\sim q)$

We can prove the above by comparing the truth tables of LHS and RHS.

Properties invloving $\Rightarrow$ and $\Leftrightarrow$ [Thm. 2.2.2]

(a) $(p \Rightarrow q) \equiv ((\sim p) \vee q)$

(b) $(p \Rightarrow q) \equiv ((\sim q) \Rightarrow (\sim p))$

(c) $(p \Leftrightarrow q) \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$

(d) $\sim(p \Rightarrow q) \equiv (p \wedge (\sim q))$

(e) $\sim(p \Leftrightarrow q) \equiv (p \wedge (\sim q)) \vee (q \wedge (\sim p))$

We have already proven (a) and (b).
Items (d) and (e) can be proven using statements from 2.2.1.

Thm 2.2.4 gives more properties