DMA: Welcome & Sets and Sequences

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Welcome to the DM part of DMA!

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  - 5-th time teaching DMA
  - Undergrad: CS 🇱🇻
  - PhD: Quantum information, University of Waterloo 🇨🇦

  - Personal: Came to DK about 6 years ago.
  - Research: Quantum computing 🇸🇬 🇬🇧 🇩🇰
  - Use quantum mechanics for faster algorithms, more secure crypto etc.

Goals

- Introduce the basic mathematical concepts relevant in computer science
- Train your ability to solve problems, reason rigorously (write proofs).
  - It’s ok not to get thigs right at first. Rather aim to improve.

A recipe for doing well in this class

Actively engage with the covered content

- Passive listening won't lead to good outcomes!
- Take notes during the class.
- **Do** the small in-class **exercises**. Solve exercises during the exercise session.
- **Read** the assigned text before and/or after the lecture.
  Lectures do not cover all the material!
- **Look over the previous lecture**. Ask yourself
  - What were the main points?
  - Was there something I didn't understand?

Plan for today

- Sequences
  - Series
  - $\sum -$notation
  - Sum-formulas
- Sets

Reading for today

- Section 1 from Notes for Week 3
- KBR 1.1-1.3. Base-b notation from KBR 1.4

Reading for tomorrow (read <u>before</u> the class!)

- Sections 2 and 3 from Notes for Week 3
- Parts of chapter 3 (pages 49-57) from CLRS

Sequences

Sequences

Def. A **sequence** is an ordered collection of elements or terms.

Examples:

$$1, 1, 2, 3, 5, 8, 13, 21, \cdots \qquad (a_n)$$

$$0, 3, 8, 15, 24, 35, \cdots \qquad (b_n)$$

- Notation: (a_n) denotes the whole sequence
  a_n is the n-th term
- n usually ranges from 0 or 1 up to infinity
- We can explicitly define an infinite sequence (b_n) via a function f by letting

$$b_n = f(n)$$

Defining sequences explicitly: an example

Example: $0, 3, 8, 15, 24, 35, \cdots$ (b_n)

Can you find a function $f: \mathbb{Z}^+ \to \mathbb{R}$ such that

$$b_n = f(n) \text{ for } n \geq 1$$

Answer: $f(n) = n^2 - 1$

Let's check

n	1	2	3	4	5
$f(n)$	0	3	8	15	24

Recursively defined sequences

In a recursive definition of (a_n) we

1. Specify some finite number of initial elements $a_1, a_{2,} \dots, a_k$
2. Define elements a_n with $n > k$ in terms of the previous elements $a_1, a_{2,} \dots, a_{n-1}$.

Example: $\begin{cases} a_1 = 1 \\ a_n = n\ a_{n-1} \text{ for } n > 1 \end{cases}$

- Can you find an explicit expression for a_n?

  $a_n = 1 \cdot 2 \cdot 3\ \dots (n-1) \cdot n = n!$ (pronounced "n factorial")

- How about our 1st example: 1, 1, 2, 3, 5, 8, 13, 21, $\cdots$

Defining sequences recursively: an example

Example: 1, 1, 2, 3, 5, 8, 13, 21, $\cdots$ (f_n)

Can you find a recursive definition for sequence (f_n)?

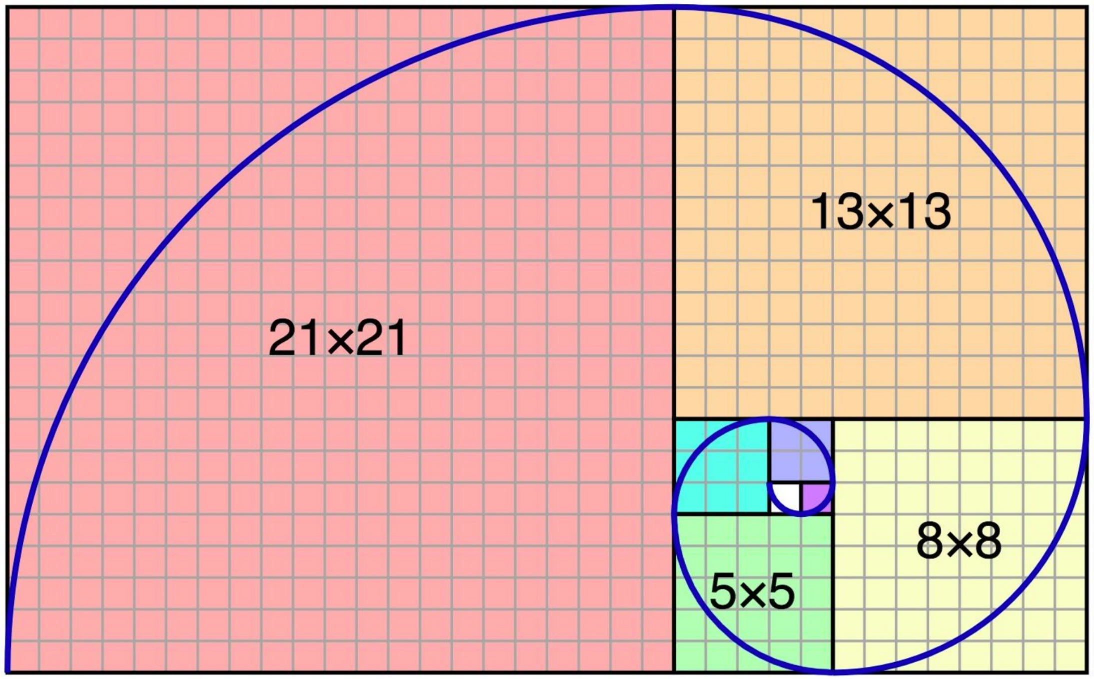
Answer: $\begin{cases} f_1 = 1\,,\ f_2 = 1 \\ f_n = f_{n-1} + f_{n-2} \text{ for } n \geq 3 \end{cases}$

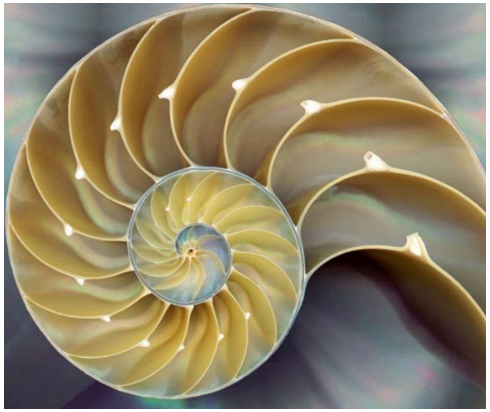
Let's check

n	3	4	5	6	5
$f_{n-1} + f_{n-2}$	1+1=2	1+2=3	2+3=5	3+5=8	5+8=13

This is called Fibonacci sequence.

Fibonacci spiral





Recursive definitions often appear as runtimes of recursive programs

- (T_n): worst-case runtime of binary search

```
BINÆRSØGNING(A,i,j,x):
    If j < i: Return False
    m = floor((i+j)/2)
    If A[m] == x:
        Return True
    Elseif A[m] < x:
        Return BINÆRSØGNING(A,m+1,j,x)
    Else: Return BINÆRSØGNING(A,i,m-1,x)
```

- Often easy to write down a recursive expression

$$\begin{cases} T_1 = const_1 & \\ T_n = T_{\frac{n}{2}} + const_2 & \text{for } n > 1 \end{cases}$$

Series

Before we discuss SERIES, let's get to know the **summation** symbol

$$\sum_{i=1}^{n} a_i$$

Making friends with the summation-symbol

Consider the sum

$$1+4+9+16+25+\cdots+n^2$$

Summation symbol lets us to succinctly specify it as

$$\sum_{i=1}^{n} i^2$$

- Pronounced "sum of i^2 from $i = 1$ to n".
- Summation index: variable that changes from one summand to the next
- Lower limit: the smallest value the index will take on
- Upper limit: the largest value the index will take on

Series

Suppose we have a sequence (a_n):

$$a_1, a_2, a_3, \ldots$$

Series is a sequence (s_k) whose k-th term is given by

$$s_k = a_1 + a_2 + \cdots + a_k$$

Series

Suppose we have a sequence (a_n):

$$a_1, a_2, a_3, \ldots$$

Series is a sequence (s_k) whose k-th term is given by

$$s_k = a_1 + a_2 + \cdots + a_k = \sum_{n=1}^{k} a_n$$

We can also define (s_k) from (a_n) recursively

$$\begin{cases} s_1 = a_1 & \\ s_k = s_{k-1} + a_k & \text{for } k > 1 \end{cases}$$

Explicit expressions for common series

* $1 + 2 + \cdots + n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2} = \Theta(n^2)$

* $1 + 4 + 9 + \cdots + n^2 = \sum_{k=1}^{n} k^2 = \frac{2n^3 + 3n^2 + n}{6} = \Theta(n^3)$

* $c + c^2 + c^3 + \cdots + c^n = \sum_{k=1}^{n} c^k = \frac{c^{n+1} - c}{c - 1}$ for $c \neq 1$

  $= \Theta(c^n)$ for $c > 1$.

Useful identities

* $\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

* $\sum_{k=1}^{n} c a_k = c \sum_{k=1}^{n} a_k$

* $\sum_{k=1}^{n} a_k = \sum_{k=1}^{m-1} a_k + \sum_{k=m}^{n} a_k$ where $m \leq n$

Summary (sequences)

Def. A sequence is an ordered list of numbers.

- Explicit definition (by defining the n-th term via some $f(n)$)

$$b_n = n^2 - 1$$

- Recursive definition (by specifying a recursive rule)

Fibonacci sequence

$$\begin{cases} a_1 = 1 \\ a_2 = 1 \\ a_n = a_{n-1} + a_{n-2} \text{ for } n > 2 \end{cases}$$

Sets

Sets

Def. A **set** is a collection of elements (with no duplicates).

Notation: $\mathrm{A} = \{1,2,3\}$

- "$x \in A$" means that x is an element of set
  - Ex: $2 \in A$, $5 \notin A$
- No duplicate elements and order doesn't matter:
  - $\{1,2,3\} = \{3,1,2\} = \{1,2,2,3\}$
- Sets can be finite or infinite

Def. **Cardinality** of a set A, denoted $|A|$, is the number of elements it contains.

- Ex: $|\{1,2,3\}| = 3$
  $|\mathbb{Z}| = \infty$

How to specify sets

1. By listing all the elements:
   - Ex. $\{a, b, c, d, e, f, g\}$
2. Set-builder notation: $\{x \mid x \text{ has property } P\}$
   - Ex: $\mathbb{Z} = \{x \mid x \text{ is an integer}\}$
     $\mathbb{Z}^+ = \{x \in \mathbb{Z} \mid x > 0\}$
     $A = \{x \mid x = y^2 \text{ for some } y \in \mathbb{Z}\}$

Other important sets you should know:

$$\mathbb{R}, \mathbb{R}^+, \mathbb{R}^-, \mathbb{N}, \mathbb{Q}$$

- The **empty set**, denoted ∅ or { }, is the set containing no elements.

Power set

Def. A is a **subset** of B, denoted $\boldsymbol{A \subseteq B}$, if every element of A is also contained in B.

Ex. List all the subsets of $\mathrm{A} = \{a, b, c\}$. How many are there?
Ans: $\emptyset$, $\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}$

Def. The **power set of A**, denoted $\mathrm{P}(A)$ or 2^A is the set consisting of all subsets of A.

Given ordering $a_1, a_2, \ldots, a_n$ of elements of A, we can identify subsets A' of A with n-bit strings

$$A' \leftrightarrow s \in \{0,1\}^n, \text{ where } s_i = \begin{cases} 0 \text{ if } a_i \notin A' \\ 1 \text{ if } a_i \in A' \end{cases}$$

Ex. Find the string corresponding to subset $\mathrm{A}' = \{a, c\}$ of $A = \{a, b, c\}$
Find the subset of $A = \{a, b, c\}$ corresponding to string 011

Operations on sets

Union: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

Intersection: $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

Complement (of A with respect to B): $A - B = \{x \mid x \in A \text{ and } x \notin B\}$

Def. Sets A and B are said to be disjoint if $A \cap B = \emptyset$