Problem 1

x-axis : $[0, L_x]$ in $m-1$ subintervals of length h , $0 = x_1 < x_2 < \dots < x_{m-1} < x_m = L_x$

y-axis : $[0, L_y]$ in $n-1$ subintervals of length k , $0 = y_1 < y_2 < \dots < y_{n-1} < y_n = L_y$

Robin boundary method

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{2H}{K\delta} u \quad , \quad f(x,y) = \frac{2H}{K\delta} u$$

① Discretize the equation

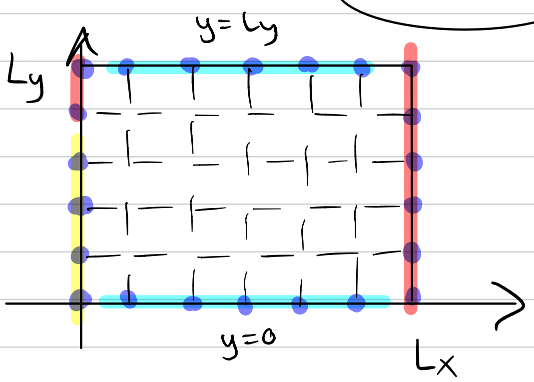
$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u(x+h,y) - 2u(x,y) + u(x-h,y)}{h^2} \longrightarrow \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

$$\frac{\partial^2 u}{\partial y^2} \approx \frac{u(x,y+h) - 2u(x,y) + u(x,y+h)}{k^2} \longrightarrow \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2}$$

m × n matrix

Eq : $$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = \frac{2H}{K\delta} u_{i,j}$$

works on the inner points $\boxed{\begin{matrix} 2 \le i \le m-1 \\ 2 \le j \le n-1 \end{matrix}}$



Depends on L

$$\frac{u_{i+1,j}}{h^2} - \frac{2u_{i,j}}{h^2} + \frac{u_{i-1,j}}{h^2} + \frac{u_{i,j+1}}{k^2} - \frac{2u_{i,j}}{k^2} + \frac{u_{i,j-1}}{k^2} - \frac{2H u_{i,j}}{K\delta} = 0$$

$$\frac{1}{h^2} u_{i+1,j} - \frac{2}{h^2} u_{i,j} + \frac{1}{h^2} u_{i-1,j} + \frac{1}{k^2} u_{i,j+1} - \frac{2}{k^2} u_{i,j} + \frac{1}{k^2} u_{i,j-1} - \frac{2H}{K\delta} u_{i,j} = 0$$

$$\frac{1}{h^2} u_{i+1,j} + \frac{1}{h^2} u_{i-1,j} + \frac{1}{k^2} u_{i,j+1} + \frac{1}{k^2} u_{i,j-1} - \left(\frac{2}{h^2} + \frac{2}{k^2} + \frac{2H}{K\delta}\right) u_{i,j} = 0$$

$$\boxed{\frac{u_{i+1,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} + u_{i,j-1}}{k^2} - \left(\frac{2}{h^2} + \frac{2}{k^2} + \frac{2H}{K\delta}\right) u_{i,j} = 0}$$

works on the inner points $\begin{matrix} 2 \le i \le m-1 \\ 2 \le j \le n-1 \end{matrix}$

② Robin boundary conditions $\boxed{\frac{\partial u}{\partial n} = \frac{H}{K} u}$

Bottom : $\frac{\partial u}{\partial n} = -\frac{\partial u}{\partial y}$
Top : $\frac{\partial u}{\partial n} = \frac{\partial u}{\partial y}$
Left : $\frac{\partial u}{\partial n} = -\frac{\partial u}{\partial x}$
Right : $\frac{\partial u}{\partial n} = \frac{\partial u}{\partial x}$
Heat BC (left) : $\frac{\partial u}{\partial n} = -\frac{\partial u}{\partial x} = \frac{P}{L\delta K}$

Bottom : $-\frac{H}{K} u_{ij} \approx \frac{-3u_{i,j} + 4u_{i,j+1} - u_{i,j+2}}{2k}$

$$-\frac{2kH}{K} u_{ij} = -3u_{i,j} + 4u_{i,j+1} - u_{i,j+2}$$

$$0 = \left(\frac{2kH}{K} - 3\right) u_{i,j} + 4u_{i,j+1} - u_{i,j+2}$$

$$\boxed{0 = \left(\frac{2kH}{K} - 3\right) u_{i,1} + 4u_{i,2} - u_{i,3}}$$ for $j=1$, $2 \le i \le m-1$

Top : $\frac{H}{K} u_{ij} \approx \frac{-3u_{i,j} + 4u_{i,j-1} - u_{i,j-2}}{-2k}$

$$-\frac{2kH}{K} u_{ij} = -3u_{i,j} + 4u_{i,j-1} - u_{i,j-2}$$

$$0 = \left(\frac{2kH}{K} - 3\right) u_{i,j} + 4u_{i,j-1} - u_{i,j-2}$$

$$\boxed{0 = \left(\frac{2kH}{K} - 3\right) u_{i,n} + 4u_{i,n-1} - u_{i,n-2}}$$ for $j=n$ $2 \le i \le m-1$

Left : $-\frac{H}{K} u_{ij} \approx \frac{-3u_{i,j} + 4u_{i+1,j} - u_{i+2,j}}{2h}$

$$-\frac{2hH}{K} u_{ij} = -3u_{i,j} + 4u_{i+1,j} - u_{i+2,j}$$

$$0 = \left(\frac{2hH}{K} - 3\right) u_{i,j} + 4u_{i+1,j} - u_{i+2,j}$$

$$\boxed{0 = \left(\frac{2hH}{K} - 3\right) u_{1,j} + 4u_{2,j} - u_{3,j}}$$ for $i=1$, $L < j \le n$

Right : $$\frac{H}{K} u_{ij} \approx \frac{-3u_{i,j} + 4u_{i-1,j} - u_{i-2,j}}{-2h}$$

$$-\frac{2hH}{K} u_{i,j} = -3u_{i,j} + 4u_{i-1,j} - u_{i-2,j}$$

$$0 = \left(\frac{2hH}{K} - 3\right) u_{i,j} + 4u_{i-1,j} - u_{i-2,j}$$

$$\boxed{0 = \left(\frac{2hH}{K} - 3\right) u_{m,j} + 4u_{m-1,j} - u_{m-2,j}}$$ for $i = m$, $1 \leq j \leq n$

Heat left : $$-\frac{P}{L\delta K} \approx \frac{-3u_{i,j} + 4u_{i+1,j} - u_{i+2,j}}{2h}$$

$$-\frac{2hP}{L\delta K} = -3u_{i,j} + 4u_{i+1,j} - u_{i+2,j}$$

$$\boxed{-\frac{2hP}{L\delta K} = -3u_{1,j} + 4u_{2,j} - u_{3,j}}$$ for $i = 1$, $1 \leq j \leq L$