Group 9

Smaller project in third week

$y''(x) - y(x) = 0 \qquad [0,1] \qquad y_{general}(x) = c_1 e^{x} + c_2 e^{-x}$

Problem 1 & 2

$x_1 = 0 < x_2 < \ldots < x_{n-1} < x_n = 1$

$$\begin{cases} y''(x) = y(x) \\ y(0) = 1 \\ y(1) = -1 \end{cases}$$

$$f''(x) \simeq \frac{\underbrace{f(x-h)}_{y_{i-1}} - 2\underbrace{f(x)}_{y_i} + \underbrace{f(x+h)}_{y_{i+1}}}{h^2}$$

$$y''(x_i) = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$$

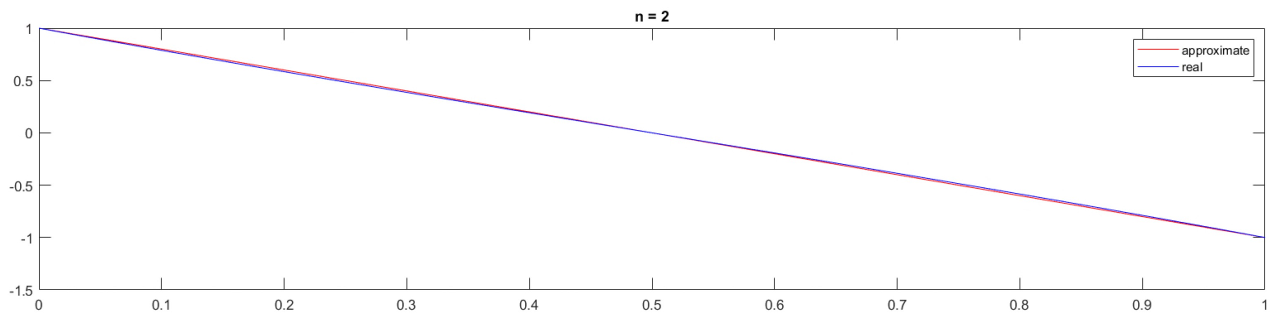
$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = y_i \qquad \text{for} \quad 2 \le i \le n-1$$

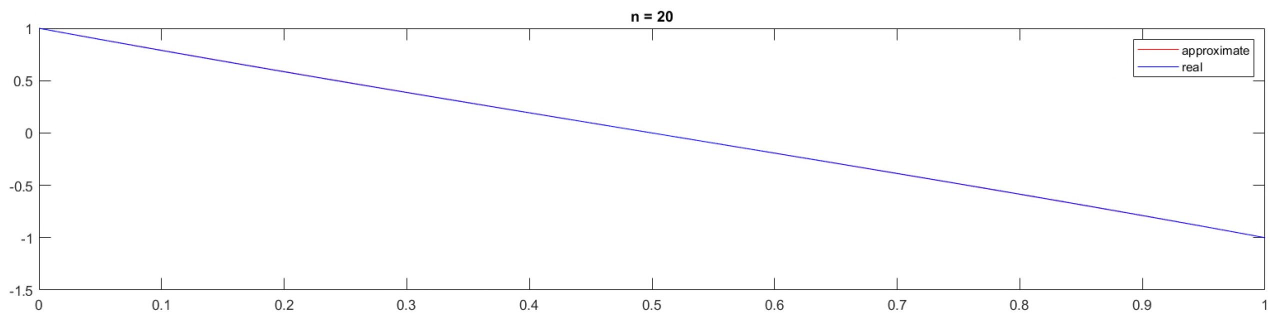
Rewrite as : $A\bar{y} = b$ where $\bar{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

$$y_{i+1} - (2-h^2)y_i + y_{i-1} = 0 \qquad \text{for} \quad 2 \le i \le n-1$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 1 & -(2-h^2) & 1 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 1 & -(2-h^2) & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & -(2-h^2) & 1 & \cdots & 0 & 0 \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -(2-h^2) & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix} \bar{y} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

Because the solution is almost linear, the difference by using n=2 is relatively small between the approximation and the real function.





Problem 3 - Neumann boundary equations

$y'(0) = 0 \qquad y'(1) = 1$

$$f'(x) \simeq \frac{-3f(x) + 4f(x+h) - f(x+2h)}{2h} \qquad \text{at } x = 0$$

$$f'(x) \simeq \frac{-3f(x) + 4f(x-h) - f(x-2h)}{2h} \qquad \text{at } x = 1$$

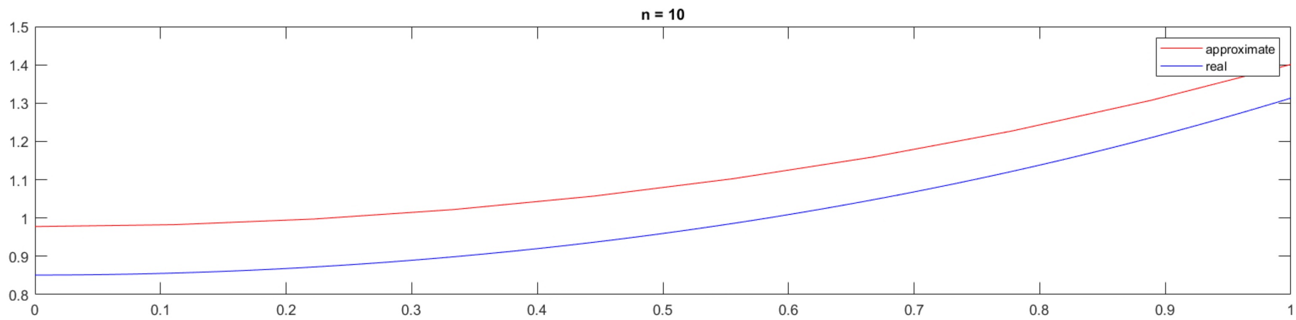
$$0 = \frac{-3y_i + 4y_{i+1} - y_{i+2}}{2h} \implies 0 = -3y_i + 4y_{i+1} - y_{i+2} \qquad \boxed{i=1}$$ - because of the left side boundary

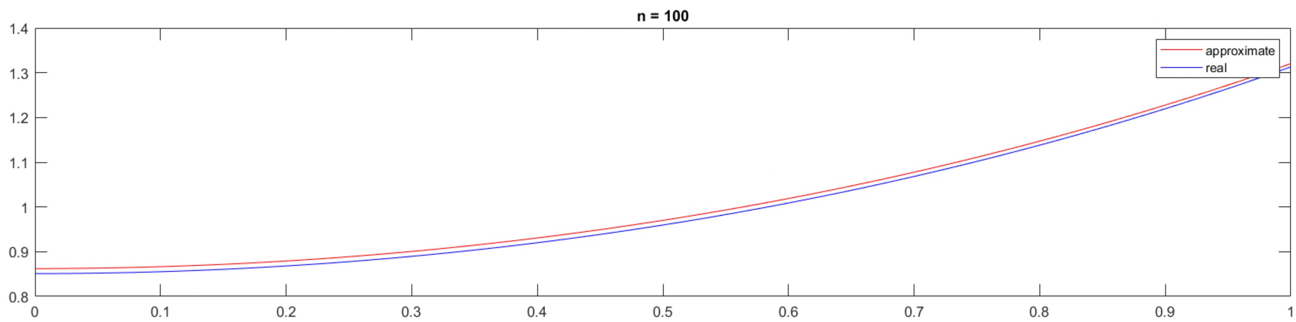
$$1 = \frac{-3y_n + 4y_{n-1} - y_{n-2}}{-2h} \implies -2h = -y_{n-2} + 4y_{n-1} - 3y_n \qquad \boxed{n}$$ - because of the right side boundary

$A\bar{y} = b \qquad n\times n$ matrix

$$\begin{bmatrix} -3 & 4 & -1 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 1 & -(2-h^2) & 1 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 1 & -(2-h^2) & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & -(2-h^2) & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -(2-h^2) & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 4 & -3 \end{bmatrix} \qquad \bar{y} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ -2h \end{bmatrix}$$

In the following we show the difference to the approximation by using n=10 and n=100 as seen in the graph.





Problem 4 – Robin boundary conditions

$y'(0) = y(0)+1 \qquad y'(1) = -y(1)$

$f'(x) \simeq \dfrac{-3f(x)+4f(x+h)-f(x+2h)}{2h}$ at $x=0$

$f'(x) \simeq \dfrac{-3f(x)+4f(x-h)-f(x-2h)}{-2h}$ at $x=1$

$\boxed{i=1}$

$$y(0)+1 = \frac{-3y_i+4y_{i+1}-y_{i+2}}{2h} \Rightarrow (y_i+1)\cdot 2h = -3y_i+4y_{i+1}-y_{i+2}$$

$$2hy_i+2h = -3y_i+4y_{i+1}-y_{i+2}$$

$$2h = -y_i(2h+3)+4y_{i+1}-y_{i+2}$$

$$-y(1) = \frac{-3y_n+4_{n-1}-y_{n-2}}{-2h} \Rightarrow 2hy_n = -3y_n+4_{n-1}-y_{n-2}$$

$$0 = -(2h+3)y_n+4_{n-1}-y_{n-2}$$

$A\bar{y}=b$ nxn matrix

$$\begin{bmatrix} -(2h+3) & 4 & -1 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 1 & -(2-h^2) & 1 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 1 & -(2-h^2) & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & -(2-h^2) & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -(2-h^2) & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 4 & -(2h+3) \end{bmatrix} \qquad \bar{y} = \begin{bmatrix} 2h \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

In the following we show the difference to the approximation by using n=10 and n=100 as seen in the graph.