Problem 1

x-axis : $[0, L_x]$ in m-1 subintervals of length h , $\quad 0 = x_1 < x_2 < \dots < x_{m-1} < x_m = L_x$

y-axis : $[0, L_y]$ in n-1 subintervals of length k , $\quad 0 = y_1 < y_2 < \dots < y_{n-1} < y_n = L_y$

Robin boundary method

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{2H}{K\delta} u \quad , \quad f(x,y) = \frac{2H}{K\delta} u$$
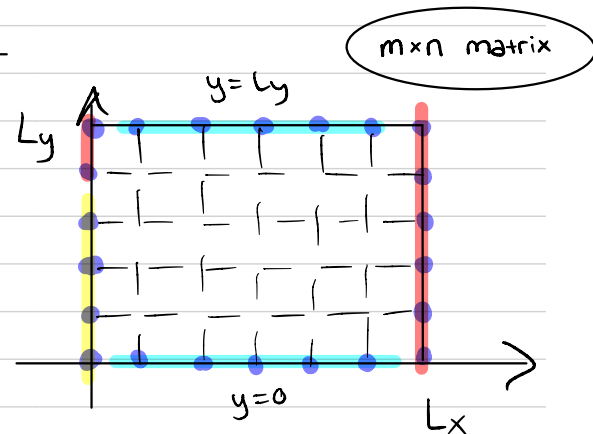
① Discretize the equation

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u(x+h,y) - 2u(x,y) + u(x-h,y)}{h^2} \longrightarrow \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

$$\frac{\partial^2 u}{\partial y^2} \approx \frac{u(x, y+h) - 2u(x,y) + u(x, y+h)}{k^2} \longrightarrow \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2}$$

$m \times n$ matrix

Eq : $\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = \frac{2H}{K\delta} u_{i,j}$

works on the inner points $\boxed{\begin{array}{l} 2 \le i \le m-1 \\ 2 \le j \le n-1 \end{array}}$

Depends on L

$$\frac{u_{i+1,j}}{h^2} - \frac{2u_{i,j}}{h^2} + \frac{u_{i-1,j}}{h^2} + \frac{u_{i,j+1}}{k^2} - \frac{2u_{i,j}}{k^2} + \frac{u_{i,j-1}}{k^2} - \frac{2H u_{i,j}}{K\delta} = 0$$

$$\frac{1}{h^2} u_{i+1,j} - \frac{2}{h^2} u_{i,j} + \frac{1}{h^2} u_{i-1,j} + \frac{1}{k^2} u_{i,j+1} - \frac{2}{k^2} u_{i,j} + \frac{1}{k^2} u_{i,j-1} - \frac{2H}{K\delta} u_{i,j} = 0$$

$$\frac{1}{h^2} u_{i+1,j} - \frac{1}{h^2} u_{i-1,j} + \frac{1}{k^2} u_{i,j+1} + \frac{1}{k^2} u_{i,j-1} - \left(\frac{2}{h^2} + \frac{2}{k^2} + \frac{2H}{K\delta} \right) u_{i,j} = 0$$

$$\frac{u_{i+1,j} - u_{i-1,j}}{h^2} + \frac{u_{i,j+1} + u_{i,j-1}}{k^2} - \left(\frac{2}{h^2} + \frac{2}{k^2} + \frac{2H}{K\delta} \right) u_{i,j} = 0$$

works on the inner points $\begin{array}{l} 2 \le i \le m-1 \\ 2 \le j \le n-1 \end{array}$

② Robin boundary conditions $\boxed{\dfrac{\partial u}{\partial n} = \dfrac{H}{K} u}$

Bottom : $\dfrac{\partial u}{\partial n} = -\dfrac{\partial u}{\partial y}$
Top : $\dfrac{\partial u}{\partial n} = \dfrac{\partial u}{\partial y}$
Left : $\dfrac{\partial u}{\partial n} = -\dfrac{\partial u}{\partial x}$
Right : $\dfrac{\partial u}{\partial n} = \dfrac{\partial u}{\partial x}$
Heat BC (left): $\dfrac{\partial u}{\partial n} = -\dfrac{\partial u}{\partial x} = \dfrac{P}{L\delta K}$

Bottom : $-\dfrac{H}{K} u_{i,j} \approx \dfrac{-3u_{i,j} + 4u_{i,j+1} - u_{i,j+2}}{2k}$

$-\dfrac{2kH}{K} u_{i,j} = -3u_{i,j} + 4u_{i,j+1} - u_{i,j+2}$

$0 = \left(\dfrac{2kH}{K} - 3\right) u_{i,j} + 4u_{i,j+1} - u_{i,j+2}$

$\boxed{0 = \left(\dfrac{2kH}{K} - 3\right) u_{i,1} + 4u_{i,2} - u_{i,3}}$ for $j=1$, $2 \le i \le m-1$

Top : $\dfrac{H}{K} u_{i,j} \approx \dfrac{-3u_{i,j} + 4u_{i,j-1} - u_{i,j-2}}{-2k}$

$-\dfrac{2kH}{K} u_{i,j} = -3u_{i,j} + 4u_{i,j-1} - u_{i,j-2}$

$0 = \left(\dfrac{2kH}{K} - 3\right) u_{i,j} + 4u_{i,j-1} - u_{i,j-2}$

$\boxed{0 = \left(\dfrac{2kH}{K} - 3\right) u_{i,n} + 4u_{i,n-1} - u_{i,n-2}}$ for $j=n$ $2 \le i \le m-1$

Left : $-\dfrac{H}{K} u_{i,j} \approx \dfrac{-3u_{i,j} + 4u_{i+1,j} - u_{i+2,j}}{2h}$

$-\dfrac{2hH}{K} u_{i,j} = -3u_{i,j} + 4u_{i+1,j} - u_{i+2,j}$

$0 = \left(\dfrac{2hH}{K} - 3\right) u_{i,j} + 4u_{i+1,j} - u_{i+2,j}$

$\boxed{0 = \left(\dfrac{2hH}{K} - 3\right) u_{1,j} + 4u_{2,j} - u_{3,j}}$ for $i=1$, $L < j \le n$

Right : $\dfrac{H}{K} u_{i,j} \approx \dfrac{-3u_{i,j} + 4u_{i-1,j} - u_{i-2,j}}{-2h}$

$\dfrac{-2hH}{K} u_{i,j} = -3u_{i,j} + 4u_{i-1,j} - u_{i-2,j}$

$0 = \left(\dfrac{2hH}{K} - 3\right) u_{i,j} + 4u_{i-1,j} - u_{i-2,j}$

$\boxed{0 = \left(\dfrac{2hH}{K} - 3\right) u_{m,j} + 4u_{m-1,j} - u_{m-2,j}}$ $\qquad$ for $i = m$, $\quad 1 \le j \le n$

Heat left : $-\dfrac{P}{L\delta K} \approx \dfrac{-3u_{i,j} + 4u_{i+1,j} - u_{i+2,j}}{2h}$

$-\dfrac{2hP}{L\delta K} = -3u_{i,j} + 4u_{i+1,j} - u_{i+2,j}$

$\boxed{-\dfrac{2hP}{L\delta K} = -3u_{1,j} + 4u_{2,j} - u_{3,j}}$ $\qquad$ for $i = 1$, $\quad 1 \le j \le L$