

# Problem 1

x-axis :  $[0, L_x]$  in  $m-1$  subintervals of length  $h$  ,  $0 = x_1 < x_2 < \dots < x_{m-1} < x_m = L_x$   
 y-axis :  $[0, L_y]$  in  $n-1$  subintervals of length  $k$  ,  $0 = y_1 < y_2 < \dots < y_{n-1} < y_n = L_y$   
 Robin boundary method

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{2H}{K\delta} u, \quad f(x, y) = \frac{2H}{K\delta} u$$

① Discretize the equation

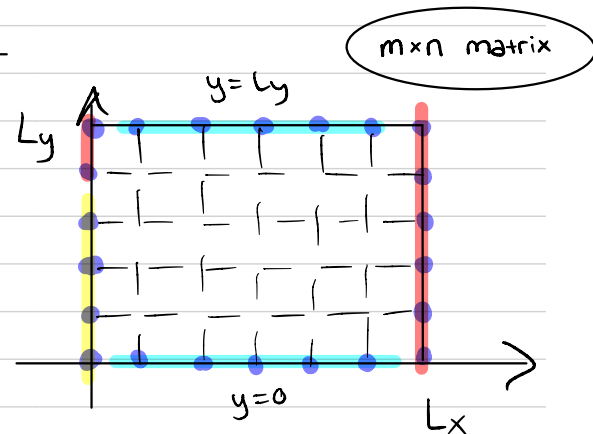
$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u(x+h, y) - 2u(x, y) + u(x-h, y)}{h^2} \rightarrow \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

$$\frac{\partial^2 u}{\partial y^2} \approx \frac{u(x, y+h) - 2u(x, y) + u(x, y-h)}{k^2} \rightarrow \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2}$$

$$\text{Eq: } \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = \frac{2H}{K\delta} u_{i,j}$$

works on the inner points

$$\begin{matrix} 2 \leq i \leq m-1 \\ 2 \leq j \leq n-1 \end{matrix}$$



Depends on  $L$

$$\frac{u_{i+1,j}}{h^2} - \frac{2u_{i,j}}{h^2} + \frac{u_{i-1,j}}{h^2} + \frac{u_{i,j+1}}{k^2} - \frac{2u_{i,j}}{k^2} + \frac{u_{i,j-1}}{k^2} - \frac{2H}{K\delta} u_{i,j} = 0$$

$$\frac{1}{h^2} u_{i+1,j} - \frac{2}{h^2} u_{i,j} + \frac{1}{h^2} u_{i-1,j} + \frac{1}{k^2} u_{i,j+1} - \frac{2}{k^2} u_{i,j} + \frac{1}{k^2} u_{i,j-1} - \frac{2H}{K\delta} u_{i,j} = 0$$

$$\frac{1}{h^2} u_{i+1,j} + \frac{1}{h^2} u_{i-1,j} + \frac{1}{k^2} u_{i,j+1} + \frac{1}{k^2} u_{i,j-1} - \left( \frac{2}{h^2} + \frac{2}{k^2} + \frac{2H}{K\delta} \right) u_{i,j} = 0$$

$$\frac{u_{i+1,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} + u_{i,j-1}}{k^2} - \left( \frac{2}{h^2} + \frac{2}{k^2} + \frac{2H}{K\delta} \right) u_{i,j} = 0$$

works on the inner points  $\begin{matrix} 2 \leq i \leq m-1 \\ 2 \leq j \leq n-1 \end{matrix}$

② Robin boundary conditions

$$\frac{\partial u}{\partial n} = \frac{H}{K} u$$

$$\text{Bottom: } \frac{\partial u}{\partial n} = -\frac{\partial u}{\partial y}$$

$$\text{Top: } \frac{\partial u}{\partial n} = \frac{\partial u}{\partial y}$$

$$\text{Left: } \frac{\partial u}{\partial n} = -\frac{\partial u}{\partial x}$$

$$\text{Right: } \frac{\partial u}{\partial n} = \frac{\partial u}{\partial x}$$

$$\text{Heat BC (left): } \frac{\partial u}{\partial n} = -\frac{\partial u}{\partial x} = \frac{P}{L\delta K}$$

$$\text{Bottom: } -\frac{H}{K} u_{ij} \approx \frac{-3u_{ij} + 4u_{i,j+1} - u_{i,j+2}}{2k}$$

$$-\frac{2kH}{K} u_{ij} = -3u_{ij} + 4u_{i,j+1} - u_{i,j+2}$$

$$0 = \left(\frac{2kH}{K} - 3\right) u_{i,j} + 4u_{i,j+1} - u_{i,j+2}$$

$$0 = \left(\frac{2kH}{K} - 3\right) u_{i,1} + 4u_{i,2} - u_{i,3}$$

for  $j=1$ ,  $2 \leq i \leq m-1$

$$\text{Top: } \frac{H}{K} u_{ij} \approx \frac{-3u_{ij} + 4u_{i,j-1} - u_{i,j-2}}{-2k}$$

$$-\frac{2kH}{K} u_{ij} = -3u_{ij} + 4u_{i,j-1} - u_{i,j-2}$$

$$0 = \left(\frac{2kH}{K} - 3\right) u_{i,j} + 4u_{i,j-1} - u_{i,j-2}$$

$$0 = \left(\frac{2kH}{K} - 3\right) u_{i,n} + 4u_{i,n-1} - u_{i,n-2}$$

for  $j=n$ ,  $2 \leq i \leq m-1$

$$\text{Left: } -\frac{H}{K} u_{ij} \approx \frac{-3u_{i,j} + 4u_{i+1,j} - u_{i+2,j}}{2h}$$

$$-\frac{2hH}{K} u_{ij} = -3u_{i,j} + 4u_{i+1,j} - u_{i+2,j}$$

$$0 = \left(\frac{2hH}{K} - 3\right) u_{i,j} + 4u_{i+1,j} - u_{i+2,j}$$

$$0 = \left(\frac{2hH}{K} - 3\right) u_{1,j} + 4u_{2,j} - u_{3,j}$$

for  $i=1$ ,  $1 \leq j \leq n$

Right :  $\frac{H}{K} u_{i,j} \approx \frac{-3u_{i,j} + 4u_{i-1,j} - u_{i-2,j}}{-2h}$

$$-\frac{2hH}{K} u_{i,j} = -3u_{i,j} + 4u_{i-1,j} - u_{i-2,j}$$

$$0 = \left(\frac{2hH}{K} - 3\right) u_{i,j} + 4u_{i-1,j} - u_{i-2,j}$$

$$0 = \left(\frac{2hH}{K} - 3\right) u_{m,j} + 4u_{m-1,j} - u_{m-2,j}$$

for  $i = m$ ,  $1 \leq j \leq n$

Heat left :  $-\frac{P}{L\delta K} \approx \frac{-3u_{i,j} + 4u_{i+1,j} - u_{i+2,j}}{2h}$

$$-\frac{2hP}{L\delta K} = -3u_{i,j} + 4u_{i+1,j} - u_{i+2,j}$$

$$-\frac{2hP}{L\delta K} = -3u_{1,j} + 4u_{2,j} - u_{3,j}$$

for  $i = 1$ ,  $1 \leq j \leq L$