Numerical Integration via the Euler Method of the Equations of Motion for a 1-D Simple Harmonic Oscillator

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Introduction

For this project a one-dimensional simple harmonic oscillator (SHO) and its equations of motion were considered and calculated both analytically and computationally. The Hamiltonian for this system is as follows

$$H = \frac{p^2}{2m} + \frac{kx^2}{2}$$

where *p* is the momentum, *x* is the position, *m* is the mass, and *k* is the spring constant. The relevant partial derivatives for this Hamiltonian are

$$\frac{\partial H}{\partial t} = \frac{dx}{dt} = \frac{p}{m}$$

$$\frac{\partial H}{\partial x} = \frac{dp}{dt} = -kx$$

Analytically solving the coupled ordinary differential equations provides the following equations of motion for the system

$$x(t) = x_0 \cos(\omega t)$$

$$p(t) = -x_o m\omega \sin(\omega t)$$

where ω is equal to $\sqrt{\frac{k}{m}}$. Using the Euler method for solving ODEs, values for both the position and momentum were determined numerically from the coupled ODEs obtained from the Hamiltonian. The computational results were then compared to the analytical results in order to determine the efficiency of the Euler method algorithm.

Methods

The Euler method used in this simulation is a numerical algorithm for solving initial value ODEs. The algorithm considers the ODE as an equation for the slope of the tangent line to at a given point on a curve that solves the ODE. Given an initial value, the tangent line is computed and then a small time-step, Δt, is taken along the tangent line and a new tangent line is calculated. The recursion relation for this algorithm is as follows

$$x_{n+1} = x_n + \Delta t * \frac{dx}{dt}$$

The Euler method can be derived from the Taylor expansion of the function, and from the expansion it is seen that this method is a first-order approximation, therefore the error in this method is on the order of Δt^2 for each time-step.

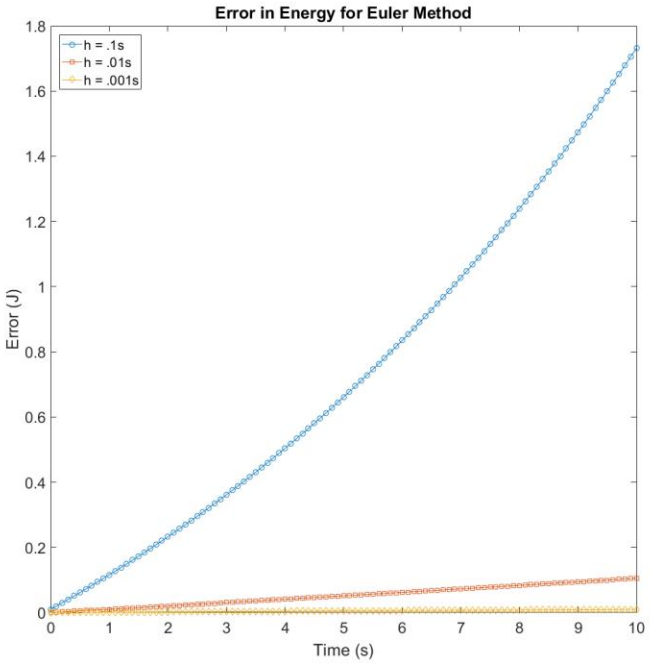

Figure 1

Results

For the results presented in this project, all data was computed using code written in C++, compiled using the GCC 6.3 compiler. Initial conditions and values for the system were chosen to be m = 1.0 kg, k = 1.0 kg/s^2, x_0 = 1.0 m, and p_0 = 0.0 kg*m/s. Time-step values, h, of 0.1 s, 0.01 s, and 0.001 s were used over a range of t_0 = 0.0 s to t_{max} = 10.0 s. Other values were utilized to test the robustness of the code and are noted otherwise.

Figure 1 shows a plot of the error for each of the three time steps in reference to the constant analytical energy of the system. The maximum error for time steps 0.1s, 0.01s and 0.001s are 1.7 J, 0.11 J and 0.010 J, respectively. It can be seen from the plot that as the time-step decreases the error between the Euler method and the analytic solution significantly decreases.

Figure 2 shows a plot of position versus time for each of the time-steps and analytic solution, and **Figure 3** is an error plot in reference to the analytic solution. The maximum error for time steps 0.1s, 0.01s and 0.001s are .60 m, .048 m and .005 m respectively.

Figures 4-6 show plots for position versus momentum for each of the three time steps.

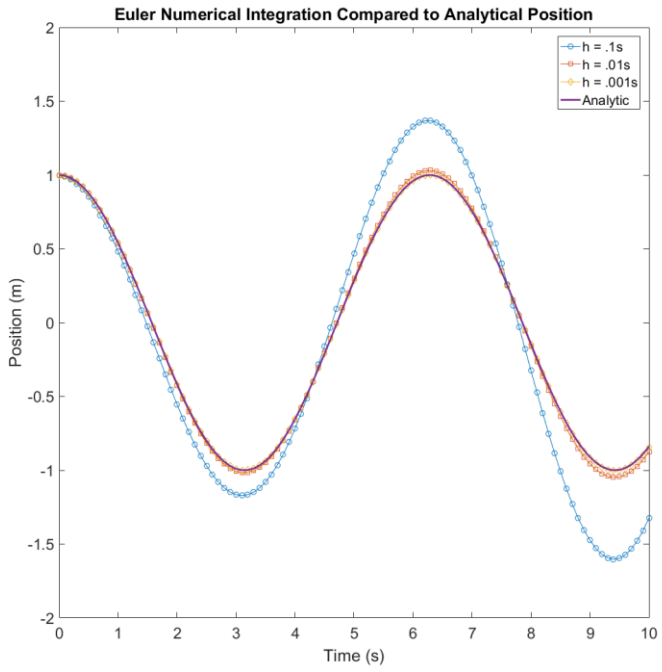

Figure 2

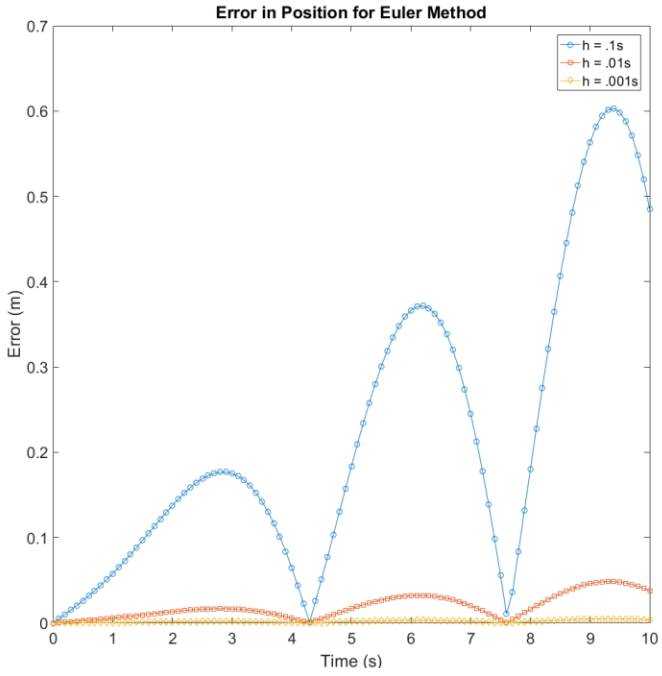

Figure 3

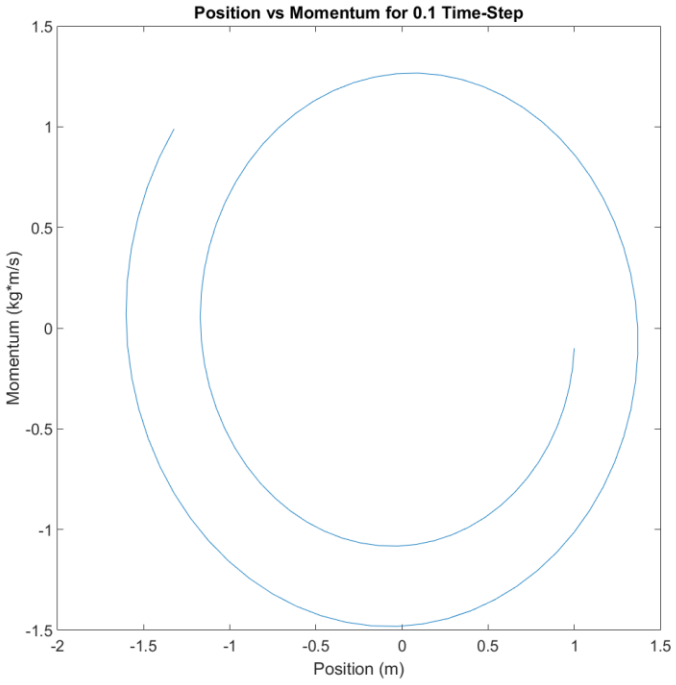

Figure 4

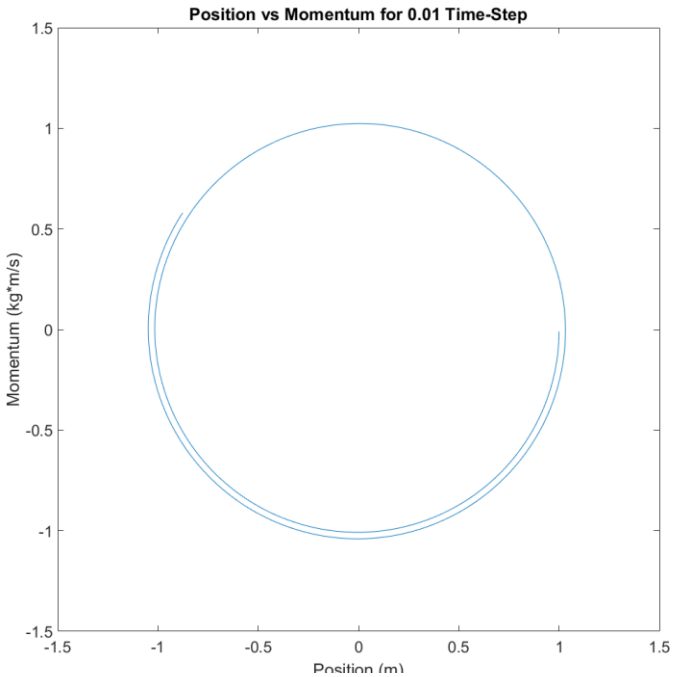

Figure 5

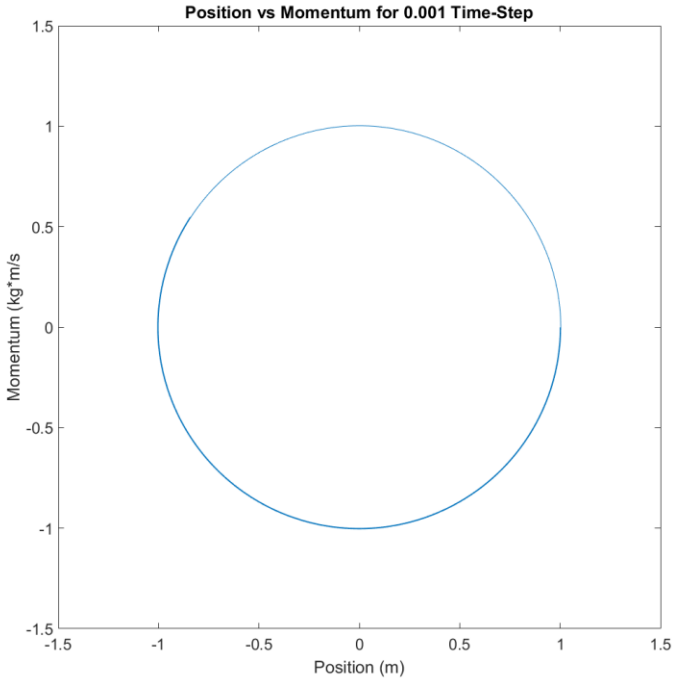

Figure 6

Discussion

The Euler method is a first-order method, meaning that the error is proportional to the square of the step-size. This is a systematic error that can be improved upon by reducing the step-size. The error plots **Figure 1** and **Figure 3** both demonstrate this reduction in error as they show results approaching the analytical solution for the system with each step-size reduction. Interestingly, for the error plot in **Figure 3** we see large peaks in the error. This is a result of the harmonic nature of the position equation, and these spikes occur in a cycle near the maximum amplitude of the SHO. The perceived reduction in error after every local maximum is a product of the nature of the system and therefore not a global reduction in error.

Considering the energy as constant, with mass and spring constant both equal to 1.0, the system's Hamiltonian can be arranged into an equation of a circle with a radius of 2*energy. Therefore, analytically the plots in **Figures 4-6** should prove to be circular with a radius of the value of the energy (which for the system is 1.0). For the time step 0.1 s we see that the plot shows a spiral instead of a circle due to the large errors present in the energy and position which can be seen in **Figure 1** and **Figure 3**. The plots for time steps 0.01 and 0.001 further tighten (**Figures 5 and 6**) to eventually approach a circle. This further shows that as the time-step decreases, so does the error in the Euler method.

In order to test the robustness of the code, a positive valued spring constant of 1.0 was used to see how this would affect the results. The values for both energy and position increased without bound. This result is expected since the model of a SHO requires a restoring force in order to provide harmonic motion.