An Exact Method for Nonlinear Network Flow Interdiction Problems

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Abstract. We study network flow interdiction problems with nonlinear and nonconvex flow models. The resulting model is a max-min bilevel optimization problem in which the follower's problem is nonlinear and nonconvex. In this game, the leader attacks a limited number of arcs with the goal to maximize the load shed and the follower aims at minimizing the load shed by solving a transport problem in the interdicted network. We develop an exact algorithm consisting of lower and upper bounding schemes that computes an optimal interdiction under the assumption that the interdicted network remains weakly connected. The main challenge consists of computing valid upper bounds for the maximal load shed, whereas lower bounds can directly be derived from the follower's problem. To compute an upper bound, we propose solving a specific bilevel problem, which is derived from restricting the flexibility of the follower when adjusting the load flow. This bilevel problem still has a nonlinear and nonconvex follower's problem, for which we then prove necessary and sufficient optimality conditions. Consequently, we obtain equivalent single-level reformulations of the specific bilevel model to compute upper bounds. Our numerical results show the applicability of this exact approach using the example of gas networks.

1. Introduction

Bilevel optimization is a rapidly developing field in mathematical programming. We consider interdiction problems that are specific max-min bilevel problems, in which the leader takes interdiction actions to optimize the objective of the follower in the opposite direction; see e.g., [17, 28]. A special class of interdiction games are network interdiction problems in which the leader attacks a network and the follower solves a network-based optimization problem such as the shortest path, the max-flow, or the clique problem; see [28] for a recent overview.

In this paper, we study network flow interdiction problems with nonlinear and nonconvex flow models. These problems aim to identify the worst-case attack on a network. More precisely, the attacker, i.e., the leader, seeks to attack a limited number of arcs of the network with the goal to maximize the load shed. Contrarily, the defender, i.e., the follower, aims at minimizing the load shed by solving a nonlinear transport problem in the interdicted network. Here, load shed corresponds to the amount of load by which the original load flow is reduced so that the transport model of the follower admits a feasible point in the interdicted network.

To model the transport problem of the follower, we focus on nonlinear and nonconvex potential-based flows without controllable elements; see [12]. These potential-based flows extend classic network flows that are typically considered for network interdiction problems; see, e.g., [9, 28]. One of the main advantages of using potential-based flows consists of the broad applicability of these flows, which allow to model hydrogen, water, or natural gas networks while appropriately representing

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the underlying physics; see, e.g., [12]. Lossless DC power flow networks can also be modeled by linear potential-based flows. However, we focus on network flow interdiction problems with nonlinear and nonconvex potential-based flows in the following, which have not been addressed appropriately in the literature so far.

Prior works address variants of the considered network flow interdiction problem in lossless DC power flow networks. As one of the first, the authors of [24] introduce a bilevel model to identify the most critical attacks in terms of load shed. In several subsequent works, duality theory or the Karush–Kuhn–Tucker (KKT) conditions are applied to the follower's problem to obtain a single-level reformulation, which is then solved directly or by Benders decomposition [2, 4, 20, 26]. The authors of [3] empirically show that the duality-based reformulation outperforms the KKT approach for the considered network flow interdiction problem. However, as discussed in [16], the approaches exploiting duality theory require strong bounds on the duals of the follower's DC optimal power flow problem, which are not known in all cases. The authors of [16] dismiss the constraints regarding the potentials and apply duality theory with valid bounds for the duals to compute a lower bound for the maximal load shed.

However, we cannot apply approaches that are based on duality or other compact optimality certificates such as the KKT conditions to derive an equivalent single-level reformulation since the considered follower's problem is nonconvex. Further, we cannot exploit most of the common intuitions for network flow interdiction problems with capacitated linear flows in our potential-based setting due to the coupling of potentials and flows; see Section 3, where we discuss this in detail.

To the best of our knowledge, the only prior work [1] regarding nonlinear potential-based flows deals with interdicting gas networks. A cutting-plane approach is developed and convex relaxations for gas transport are exploited. However, if the potential-based flow model is not relaxed, then the developed cutting-plane approach is only valid under additional assumptions, e.g., the highly restrictive assumption that interdicting a component of the network does not increase the total load shed by more than the corresponding flow through the interdicted component. In general, this assumption is not satisfied in potential-based networks as noted in [25], where a generalized Benders decomposition method is applied under this assumption in lossless DC power flow networks. Further, we explicitly show in Section 3 that even interdicting a single arc in a potential-based network may lead to the failure of this assumption. Note that our approach is not based on this assumption and does not simplify the considered potential-based flow model.

Let us further remark that there is a large branch of research that deals with solving extended or more complicated models in the context of AC power flow networks, which then also leads to a nonlinear and nonconvex follower's problem; see e.g., [7, 11, 21]. However, AC power flow networks cannot be modeled by potential-based flows and, thus, these approaches cannot be directly transferred to the considered potential networks.

In this paper, we present an exact algorithm to solve the nonlinear network flow interdiction problem for potential-based flows. Since we focus on nonlinear and nonconvex potential-based flows, we further assume that the interdicted network remains weakly connected. As discussed in [6, p. 60], "it would be inaccurate to assume that contingencies that maintain connectivity are simple or uninteresting–quite the contrary." The reason is that a positive load shed in an interdicted, but still weakly connected, network is a dangerous contingency since it is based on the combination of physics and the network's structure and does not result from a lack of supply; see [6]. Thus, the made assumption does not render the considered network flow interdiction problem irrelevant at all.

Our exact approach is an iterative upper and lower bounding scheme, which solves the nonlinear network flow interdiction problem to global optimality. For a fixed interdiction decision, solving the follower's problem always leads to a valid lower bound. Thus, the main challenge consists of providing a valid and non-trivial upper bound. To compute such an upper bound, we restrict the flexibility of the follower when adjusting the load flow in the interdicted network. This leads to a specific bilevel problem that still has a nonlinear and nonconvex follower's problem. Using properties of potential-based flows and the special structure of the obtained bilevel problem, we then prove necessary and sufficient optimality conditions for the nonlinear and nonconvex follower's problem. Exploiting these optimality conditions, we derive equivalent single-level reformulations to compute a valid upper bound for the maximal load shed. We finally demonstrate the applicability of our algorithm by solving the network interdiction problem for gas networks. In the computational experiments, our approach significantly outperforms the enumeration approach. We highlight that in the developed approach, we do not simplify the considered nonlinear and nonconvex potential-based flows, e.g., to obtain a convex follower's problem. This is important since approximating the model of the underlying physics can lead to a wrong evaluation of an interdiction decision and of the vulnerability of the network as discussed in the recent work [11].

This paper is organized as follows. In Section 2, we introduce the network flow interdiction problem for potential-based flows. In Section 3, we present an example to illustrate the main challenges of potential-based flows and important differences compared to capacitated linear flows when considering the network flow interdiction problem. We then derive a valid upper bound that consists of solving a simplified but still nonlinear and nonconvex bilevel problem and provide different and equivalent single-level reformulations in Section 4. Afterward, in Section 5, we present an algorithm to compute an optimal interdiction decision that exploits the derived bounds. We then demonstrate the applicability of our approaches in a small computational study on the example of gas networks in Section 6 and discuss possibilities for future research in Section 7.

2. Problem Statement and Preliminary Results

We introduce the considered nonlinear potential-based flow model. Afterward, we specify the network interdiction problem, which is a specific bilevel problem in which the attacker at the upper level seeks to find the worst-case attack on the arcs of a network, which is used by the follower.

In a nutshell, potential-based flows consist of classic flow variables and additional nodal potential variables. The flow on an arc is determined by the difference of the potentials at the incident nodes. To make this more formal, we consider a directed graph $G = (V, A)$ with node set V and arc set A. The set of nodes V is further partitioned into entry nodes V_+, at which flow can be injected, exit nodes V_-, at which flow can be withdrawn, and inner nodes, at which flows can neither be injected nor withdrawn. For each arc $a \in A$, we introduce a flow variable q_a and for each node $u \in V$, we introduce a potential variable π_u that has to satisfy lower and upper bounds $0 < \pi_u^- \leq \pi_u^+$. In addition to [12], for every arc $a \in A$, we also consider lower and upper arc flow bounds $q_a^- \leq q_a^+$. Choosing directed graphs is a modeling choice that allows to track the direction of flow. For an arc $(u, v) \in A$, the flow q_a is positive if it flows from u to v and otherwise, the flow is negative.

For every arc $(u, v) \in A$, the arc flow q_a is determined by the difference of the potentials at the incident nodes, i.e.,

$$\pi_u - \pi_v = \Lambda_a \varphi(q_a), \tag{1}$$

where $\Lambda_a > 0$ is an arc-specific parameter and $\varphi\colon \mathbb{R} \to \mathbb{R}$ is a so-called potential function that is continuous, strictly increasing, and odd, i.e., $\varphi(-q_a) = -\varphi(q_a)$. For a given load flow vector that determines the amount of flow that is injected and withdrawn at the nodes, the flows additionally satisfy mass conservation. We specify this in our network interdiction problem later on.

In the following, we make some mild assumptions on the potential and flow bounds, which ensure that the zero load flow is feasible and that the potential level at a node is not fixed by the given bounds.

Assumption 1. *The intersection of the potential bound intervals has a nonempty interior, i.e.,* $\emptyset \neq \operatorname{int}\left(\bigcap_{u\in V} [\pi_u^-, \pi_u^+]\right) = (\underline{\pi}, \bar{\pi})$ *with* $\underline{\pi} < \bar{\pi}$*. For each arc* $a \in A$*, the lower and upper arc flow bounds satisfy* $q_a^- < 0 < q_a^+$.

For an arc $a \in A$, the choice of the potential function being nonlinear is made to accurately represent the underlying physics of the network flow. In [12], explicit choices of the potential functions for stationary gas φ^{G}, water φ^{W}, or lossless DC power networks φ^{DC} are derived by approximating physical laws. These potential functions are given by

$$\varphi^{\mathrm{G}}(q_a) = q_a\,|q_a|\,, \quad \varphi^{\mathrm{W}}(q_a) = \operatorname{sgn}(q_a)\,|q_a|^{\,1.852}, \quad \varphi^{\mathrm{DC}}(q_a) = q_a. \tag{2}$$

Let us now describe the studied network interdiction game with potential-based flows as a bilevel problem. We start with a given and balanced load flow $\ell \in \mathbb{R}^V_{\geq 0}$, i.e., $\sum_{u\in V_+} \ell_u = \sum_{u\in V_-} \ell_u$ holds, that represents the injections and withdrawals at every node of the network. The leader then interdicts a limited number of arcs with the goal to maximize the load shed. Here, load shed equals the amount of load by which the given load flow has to be reduced so that it can be transported through the interdicted network. After such an attack by the leader, the follower then solves a transportation problem in the interdicted network with the goal to minimize the load shed. Here, the interdicted network is the network without the arcs that have been interdicted by the leader. Formally, the considered network interdiction problem is given

$$\max_x \quad \sum_{u\in V_-} \lambda_u \ell_u \quad \text{s.t.} \quad x \in X \subseteq \{0,1\}^A, \ (\lambda, q, \pi) \in S(x), \tag{3}$$

where $S(x)$ denotes the set of optimal solutions of the x-parameterized lower-level problem

$$\min_{\lambda,q,\pi} \quad \sum_{u\in V_-} \lambda_u \ell_u \tag{4a}$$

$$\text{s.t.} \quad \sum_{a\in\delta^{\mathrm{out}}(v)} q_a - \sum_{a\in\delta^{\mathrm{in}}(v)} q_a = \begin{cases} (1-\lambda_v)\ell_v, & \text{if } v \in V_+,\\ -(1-\lambda_v)\ell_v, & \text{if } v \in V_-,\\ 0, & \text{else,} \end{cases} \quad v \in V, \tag{4b}$$

$$(1-x_a)q_a^- \leq q_a \leq (1-x_a)q_a^+, \quad a \in A, \tag{4c}$$

$$x_a M_a^- \leq \pi_u - \pi_v - \Lambda_a \varphi(q_a) \leq x_a M_a^+, \quad a = (u,v) \in A, \tag{4d}$$

$$\pi_u^- \leq \pi_u \leq \pi_u^+, \quad u \in V, \tag{4e}$$

$$0 \leq \lambda_u \leq 1, \ u \in V_-, \quad \lambda_u^- \leq \lambda_u \leq 1, \ u \in V_+. \tag{4f}$$

In the upper-level problem (3), the leader determines an interdiction decision $x \in X \subseteq \{0,1\}^A$ that maximizes the load shed. The interdiction decision has to satisfy additional constraints, which are modeled using the set X. For example, these constraints can represent a limited interdiction budget or geographic correlations between the attacks [29]. Our approach presented in Section 4 and 5 does not depend on the choice of X and, thus, allows to address different variants of the considered network interdiction problem.

In contrast to the attacker, the follower minimizes the load shed by solving the transport problem (4). Constraints (4b) represent flow conservation at every node of the network. These constraints further imply that for every feasible point (λ, q, π) of Problem (4), the load flow vector is still balanced after the load shed, i.e., $\sum_{u \in V_+}(1-\lambda_u)\ell_u = \sum_{u \in V_-}(1-\lambda_u)\ell_u$ holds. Consequently, it is sufficient to only consider the load shed of the exit nodes in the objective function (4a). Constraints (4c) ensure that if an arc is interdicted, then the flow through this arc is 0. Furthermore, the constraints require that if an arc is not interdicted, then the flow through this arc satisfies specific lower and upper flow bounds. Analogously, Constraints (4d) ensure that for every non-interdicted arc the dependency (1) between arc flow and potentials at the incident nodes is satisfied. However, if an arc is interdicted, then the corresponding arc flow and potentials at the incident nodes are decoupled. This is also ensured by Constraints (4d) together with a valid choice of the big-M values M_a^- and M_a^+, which we discuss below. Due to technical restrictions, the potentials have to satisfy lower and upper potential bounds given by Constraints (4e). Constraints (4f) state that we can decrease the load of exits. For an exit node $u \in V_-$, the value $\lambda_u = 1$ corresponds to a complete load shed of this exit, which means that no flow is withdrawn at this node. Contrarily, $\lambda_u = 0$ states that the load of the corresponding exit is unchanged. For entry nodes $u \in V_+$, Constraints (4f) allow to increase as well as decrease the load of entries within certain limits in order to minimize the load shed. Here, $\lambda_u^- \leq \lambda_u \leq 0$ represents an increase of the load of an entry, whereas $\lambda_u \in [0,1]$ again represents a decrease of the corresponding load. We note that allowing to decrease the load of exits and to flexibly adjust the load of entries is considered in different network interdiction problems of the literature [1, 7, 16].

We now briefly discuss that for each arc $(u,v) \in V$ the big-M values

$$M_a^- = \pi_u^- - \pi_v^+, \quad M_a^+ = \pi_u^+ - \pi_v^-, \tag{5}$$

are valid for Problem (4). Here, valid means that for every interdiction $x \in X$ and $a \in A$ with $x_a = 1$, the corresponding constraint (4d) is redundant. To this end, consider an upper-level decision $x \in X$, flows $q \in \mathbb{R}^A$ that satisfy Constraints (4c), and potentials $\pi \in \mathbb{R}^V$ that satisfy Constraints (4e). Then, for any $a \in A$ with $x_a = 1$, the inequalities $\pi_u^- - \pi_v^+ \leq \pi_u - \pi_v - \varphi(0) = \pi_u - \pi_v \leq \pi_u^+ - \pi_v^-$ are satisfied due to Constraints (4c) and (4e). Consequently, choosing the big-M values M_a^- and M_a^+ according to (5) ensures that for an interdicted arc, the arc flow and the potentials at the incident nodes are completely decoupled.

If no flow bounds are available, then $\sum_{V_- \in V} \ell_u$ can be used as an upper bound for the absolute flow through an arc since there cannot be any cyclic flow in the considered potential networks; see e.g., [14].

Remark 1. *For a linear potential function φ, the lower-level problem (4) is a linear problem. Consequently, the lower level can be reformulated using the strong-duality theorem or the KKT approach; see, e.g., [2, 3, 20, 26] for the case of network interdiction problems for lossless DC power flow networks.*

In the light of this remark, we now focus on nonlinear and nonconvex potential constraints. Thus, we obtain a nonlinear and nonconvex lower-level problem and approaches that are based on duality or other compact optimality certificates such as KKT conditions cannot be used to derive a single-level reformulation of (3). The approaches discussed in Section 4 and 5 are valid for general potential functions but are especially useful for nonlinear and nonconvex potential functions, which have not been addressed appropriately in the literature so far.

From Assumption 1 it follows that for every feasible leader's decision $x \in X$, the follower's problem (4) is feasible since $\lambda_u = 1$, $u \in V$, is always feasible. In addition, the variables $q \in \mathbb{R}^A$, $\lambda \in \mathbb{R}^V$, and $\pi \in \mathbb{R}^V$ are bounded due to Constraints (4b)–(4f). Thus, we obtain the following result by the theorem of Weierstraß.

Lemma 2. *For every feasible interdiction decision $x \in X$, there is an optimal solution (λ, q, π) of the lower-level problem* (4).

Consequently, the bilevel problem (3) is solvable because there is only a finite number of feasible interdiction decision $x \in X \subseteq \{0,1\}^A$.

To conclude this section, we state a uniqueness result for the considered potential-based flows from the literature that we apply to the network interdiction problem in the following. To this end, we dismiss flow and potential bounds. In the early works [10, 18] together with [23], it is proven that for a given balanced load flow ℓ, the corresponding flows are uniquely determined by flow conservation and the dependencies of flows and potentials in Constraints (1). Moreover, the corresponding potential differences are unique and, thus, the potentials themselves are unique up to a constant shift. This result can be applied to every connected component of a network if the load flow is balanced w.r.t. every connected component. Since we later focus on weakly connected graphs, we only present the result for this case.

We now apply the uniqueness result of the literature to the follower's problem (4). For a leader's interdiction decision $x \in X$, we denote the interdicted graph by $G(x) = (V, A(x))$, i.e., $G(x)$ only contains the non-interdicted arcs of graph G. Since we dismiss the potential and flow bounds, we further replace the big-M values M_a^- and M_a^+, given in (5), by

$$M_a^+ = \sum_{a \in A} \Lambda_a \varphi(Q) = -M_a^-, \quad a \in A, \quad \text{with } Q = \sum_{u \in V_-} \ell_u. \tag{6}$$

Lemma 3. *Let a leader's decision $x \in X$, a balanced load flow $\ell \in \mathbb{R}^V$, and $Q = \sum_{u \in V_-} \ell_u$ be given. Further, let the interdicted graph $G(x)$ be weakly connected and for all $a \in A$, we choose M_a^- and M_a^+ according to* (6). *Then, there exist potentials $\pi \in \mathbb{R}^V$ and unique flows $q \in \mathbb{R}^A$ so that the set of points that satisfy Constraints* (4d) *and*

$$\sum_{a \in \delta^{\text{out}}(v)} q_a - \sum_{a \in \delta^{\text{in}}(v)} q_a = \begin{cases} \ell_v, & \text{if } v \in V_+, \\ -\ell_v, & \text{if } v \in V_-, \\ 0, & \text{else}, \end{cases} \quad v \in V, \tag{7}$$

$$-(1 - x_a)Q \le q_a \le (1 - x_a)Q, \quad a \in A, \tag{8}$$

is nonempty and given by

$$\{(q, \tilde{\pi}) : \tilde{\pi} = \pi + \tau \mathbb{1},\ \tau \in \mathbb{R}\}, \tag{9}$$

where $\mathbb{1} = (1, \dots, 1)^\top \in \mathbb{R}^V$ is the vector of ones.

Proof. We apply the known uniqueness result from the literature, see, e.g., Theorem 7.1 of [15], to the interdicted network $G(x)$. Consequently, we obtain unique flows q and a set of points given by (9) that satisfy (4d) and (7) w.r.t. the interdicted graph $G(x)$. Since $Q = \sum_{u \in V_-} \ell_u$ is an upper bound on the absolute flow through an arc, the obtained points also satisfy (8) for $a \in A(x)$, i.e., for those arcs with $x_a = 0$. We now extend the set in (9) by setting $q_a = 0$ for all $a \in A \setminus A(x)$. The points in the extended set directly satisfy (7) and (8) w.r.t. $G = (V, A)$.

We now show that for $a \in A \setminus A(x)$, Constraints (4d) are also satisfied by the extended points of (9). To this end, we consider nodes $u, v \in V$ and a path $P(u, v)$ from u to v, which exists since the graph $G(x)$ is weakly connected. For an arc $a \in A$, the function $\sigma_a(P(u, v))$ evaluates to 1 if arc a is a forward arc of the path P, it evaluates to -1 if a is an backward arc of P, and otherwise, it is zero. From Constraints (4d), which are satisfied for $a \in A(x)$, and (7), it then follows that any extended point (q, π) of (9) satisfies

$$|\pi_u - \pi_v| = \left| \sum_{a \in P(u,v)} \sigma_a(P(u,v)) \Lambda_a \varphi(q_a) \right| \leq \sum_{a \in A} \Lambda_a \varphi(Q),$$

because the potential function φ is strictly increasing and $|q_a| \leq Q$ holds. This implies that Constraints (4d) are also satisfied by the extended points of (9) due to the choice of M_a^- and M_a^+ according to (6). □

We note that the flow bound Q can be replaced by any other bound that does not restrict the flow through any of the arcs.

From the uniqueness result it follows that for a feasible upper-level decision $x \in X$ and after fixing the load shedding variables $\lambda \in \mathbb{R}^V$ such that $(1 - \lambda) \circ \ell$ is balanced, the corresponding flows are already uniquely determined in the follower's problems. Here, $a \circ b$ denotes the Hadamard product of two vectors a and b. Further, the potentials are already determined up to a constant shift, which we will later use in our solution approach.

3. Why Potential-Based Flows are Difficult

We now briefly discuss that going from capacitated linear to potential-based flows leads to additional challenges when considering network interdiction problems, since most of the common intuitions for the capacitated linear case cannot be transferred to the potential-based setting.

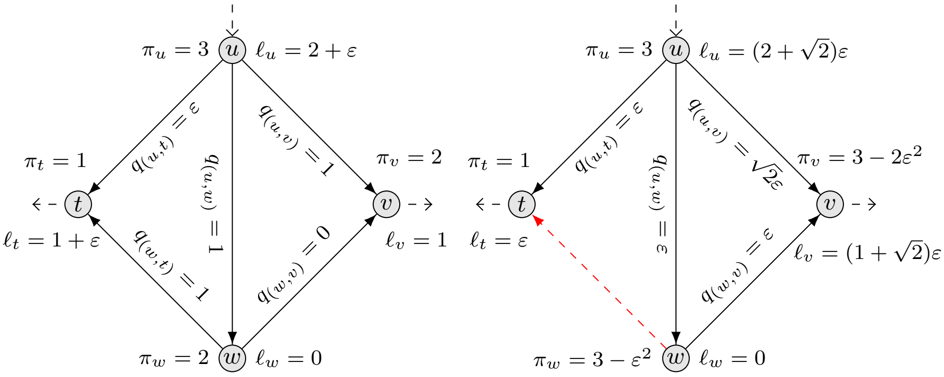
One of the core challenges of potential-based flows consists of their "global" nature, which means that changing the load at specific nodes generally affects the potentials at every node and the arc flow of every arc of the network. For network interdiction problems with potential flows, this global nature implies that interdicting a single arc $a \in A$ with load q_a can cause a load shed, i.e., $\sum_{u \in V_-} \lambda_u \ell_u$, that significantly exceeds the value q_a or even exceeds the maximum flow capacity, i.e., $\max\{|q_a^-|, q_a^+\}$, of the interdicted arc. This is in contrast to network interdiction problems with capacitated linear flows, where the total capacity of an arc represents a valid upper bound for the total load shed that is necessary if the arc is interdicted.

We now consider the potential function $\varphi(q) = \varphi^{\mathrm{G}} = q|q|$; see (2). For $\varepsilon \in (0, 0.4]$, we give an example in Figure 1 in which interdicting the arc (w, t) with a flow capacity of 1 leads to a total load shed of $2 - \varepsilon(1 + \sqrt{2}) > 1$. Hence, the load shed exceeds the total capacity of the interdicted arc.

More precisely, after interdicting arc (w, t), we can observe two different effects. First, we have to reduce the load of node t from $1 + \varepsilon$ to ε since this exit node can only be supplied through the arc (u, t) with arc capacity ε. We note that considering capacitated linear instead of potential-based flows leads to the same reduction of the load ℓ_t. However, the next observation only applies to potential networks. From the coupling of flow and potentials by Constraints (4d) and mass flow conservation (4b), it follows that

$$\pi_u - \pi_v = q_{(u,v)} \left| q_{(u,v)} \right| = q_{(u,w)} \left| q_{(u,w)} \right| + q_{(w,v)} \left| q_{(w,v)} \right|, \quad q_{(u,w)} = q_{(w,v)}, \qquad (10)$$

has to be satisfied for the interdicted network. Consequently, these equations and flow conservation at node w enforce that for any positive load $\ell_v > 0$, we now have positive flow through arc (w, v) in the interdicted network. Since the arc (w, v) has a small flow capacity, it is necessary to reduce the load of node v from 1 to $(1 + \sqrt{2})\varepsilon < 1$. We note that the latter reduction of load ℓ_v is not necessary



$$\varphi(q) = q|q|,\ \pi_i^+ = \infty,\ \pi_i^- = -\infty,\ i \in V, \quad \Lambda_a = 1, a \in A \setminus \{(u,t)\},\ \Lambda_{(u,t)} = 2/\varepsilon^2,$$
$$q_a^- = -1,\ q_a^+ = 1,\ a \in A \setminus \{(u,t),(w,v)\},\ q_{(u,t)}^- = q_{(w,v)}^- - \varepsilon,\ q_{(u,t)}^+ = q_{w,v}^+ = \varepsilon.$$

FIGURE 1. Example in which for fixed $\varepsilon \in (0, 0.4]$, interdicting the single arc (w,t) causes a load shed of $2 - \varepsilon(1 + \sqrt{2}) > 1$ that exceeds the absolute flow capacity of 1 of the interdicted arc.

if a capacitated linear flow model is considered since then there is no coupling of potentials and arc flows according to (10) and we can set $q_{(u,v)} = 1$.

Let us remark that we can adjust the lower and upper potential bounds in the example of Figure 1 so that we can neglect the flow bounds but we still observe the effect that interdicting a single arc can cause a load shed that exceeds the absolute flow on this arc; see Appendix A. Thus, the observed effect does not necessarily depend on the flow bounds but is driven by the coupling of potentials and flows.

As a further modification, we can remove the arc (u,t) in both graphs of the example in Figure 1 and adjust ℓ_t to $\ell_t = 1$. Then, we still observe that interdicting an arc leads to load shed larger than the corresponding flow capacity. This demonstrates that even for interdicted arcs that are not part of a cycle, the intuition of network interdiction problems with capacitated linear flows is wrong for the case of potential networks.

To conclude this section, we highlight the consequences of the presented example if applying specific cutting-plane approaches or a generalized Benders decomposition to network interdiction problems in potential networks; see, e.g., [1, 25]. Typically, these approaches exploit linear cuts, which for a given interdiction decision bound from above the total possible load shed. For a given interdiction decision $\tilde{x} \in X$, the linear cuts are typically of the form $\psi(x) \le \psi(\tilde{x}) + \sum_{a \in A} \delta_a(\tilde{x}) x_a$, where the function ψ represents the optimal objective function of (3) w.r.t. the considered interdiction decision. For potential networks, the example of Figure 1 now implies that this cut is not valid in general if the coefficients $\delta_a(\tilde{x})$ equal the absolute flow capacity of the corresponding arc, i.e., $\delta_a(\tilde{x}) = \max\{|q_a^-|, q_a^+\}$ for every $a \in A$. Consequently, the latter cut is only valid under additional assumptions or relaxations of the potential-based flow model, e.g., if it is assumed that interdicting an arc with flow q_a only causes a maximum load shed of q_a as stated in [25]. However, this assumption is not satisfied in general potential-based networks as demonstrated in the example above and we do not exploit this assumption in the following.

4. A Bilevel Approach for Computing Valid Upper Bounds

In general, the bilevel problem (3) is very challenging because the lower level is a nonlinear and nonconvex problem. In the following, we develop a bilevel model that computes a valid upper bound for (3). Although the obtained bilevel model preserves the nonlinear and nonconvex structure of the follower, it has a special structure that can be exploited to derive a single-level reformulation. For this, the uniqueness result of Lemma 3 plays an important role. We then develop an approach that exploits the derived upper bounds to compute the optimal solution of (3) in Section 5.

The main idea consists of restricting the follower so that he can only evenly reduce the given load flow ℓ, i.e., we replace the $|V|$-many load shed variables $\lambda \in \mathbb{R}^V$ by a single variable. Moreover, we make the mild assumption that the potential function φ is positively homogeneous of order $r > 0$.

Assumption 2. *The potential function $\varphi\colon \mathbb{R} \to \mathbb{R}$ is positive homogeneous of order $r > 0$, i.e., $\varphi(\lambda q_a) = \lambda^r \varphi(q_a)$ for all $\lambda > 0$.*

As stated in [12], homogeneity is motivated by physical laws and the restriction to $r > 0$ still allows to model hydrogen, natural gas, water, or lossless DC power flow networks; see, e.g., (2). Moreover, the positive homogeneity implies that φ is given by $\varphi(q_a) = \alpha \operatorname{sgn}(q_a)|q_a|^r$ with $\alpha = \varphi(1) > 0$; see [12]. The positive homogeneity allows to transfer Observation 3.1 of [12] to network interdiction games with nonlinear potential-based flows, which leads to the result that we can scale feasible points of our transport problem as follows.

Lemma 4. *Let a feasible decision $x \in X$ of the leader, a balanced load flow $\ell \in \mathbb{R}^V$, and a point (q, π) that satisfies the Constraints (4d), (7), and (8) w.r.t. the load $\ell \in \mathbb{R}^V$ be given. Further, let the interdicted graph $G(x)$ be weakly connected and for all $a \in A$, we choose M_a^- and M_a^+ according to (6). Then, for any $\lambda \geq 0$ the point $(\lambda q, \lambda^r \pi)$ satisfies the Constraints (4d), (7), and (8) w.r.t. the load $\lambda \ell$.*

We now algorithmically exploit this structural property to derive an upper bound of the original interdiction problem (3). To this end, we assume that the follower is only allowed to evenly reduce the given balanced load flow $\ell \in \mathbb{R}^V$, i.e., for every pair of nodes $(u, v) \in V^2$ the load shed variables are the same, i.e., $\lambda_u = \lambda_v = \lambda$. Consequently, we only consider a scalar $\lambda \in [0, 1]$ instead of a vector $\lambda \in \mathbb{R}^V$ in our modified interdiction problem that is then given by

$$\max_{x \in X} \min_{\lambda, q, \pi} \quad \sum_{u \in V_-} \lambda \ell_u \tag{11a}$$

$$\text{s.t.} \quad \sum_{a \in \delta^{\text{out}}(v)} q_a - \sum_{a \in \delta^{\text{in}}(v)} q_a = \begin{cases} (1-\lambda)\ell_v, & \text{if } v \in V_+, \\ -(1-\lambda)\ell_v, & \text{if } v \in V_-, \\ 0, & \text{else}, \end{cases} \quad v \in V, \tag{11b}$$

$$\text{(4c)–(4e)}, \ 0 \leq \lambda \leq 1. \tag{11c}$$

We choose the big-M values M_a^- and M_a^+ according to (5). For every feasible upper-level decision $x \in X$, the follower's problem of (11) is still feasible since $\lambda = 1$ leads to the zero load flow that is feasible for every interdicted network. Consequently, from Lemma 2 it follows that solving Problem (11) leads to an upper bound for the objective value of the original interdiction problem.

Lemma 5. *Let (x, λ, q, π) be an optimal solution of the bilevel problem (11). Then, $\sum_{u \in V_-} \lambda \ell_u \geq \sum_{u \in V_-} \tilde{\lambda}_u \tilde{\ell}_u$ holds for every bilevel feasible point $(\tilde{x}, \tilde{\lambda}, \tilde{q}, \tilde{\pi})$ of the original interdiction problem (3).*

This lemma shows that restricting the flexibility of the follower leads to an upper bound for the original network interdiction problem (3). However, the simplifications made in the lower-level problem still maintain the nonconvex and nonlinear structure of the follower's problem and do not simplify the considered potential-based flow model. Thus, Problem (11) is still a challenging bilevel problem.

Fortunately, Problem (11) now provides some additional structure, which consists of the uniqueness result of Lemma 3 and the positive homogeneity of the potential-based flows of Lemma 4. More precisely, after the interdiction decision $x \in X$ is made by the leader, the unique flows $q \in \mathbb{R}^A$ corresponding to the original load flow $\ell \in \mathbb{R}^V$ as well as the potentials $\pi \in \mathbb{R}^V$, which are unique up to a constant shift, are directly determined by Constraints (7), (8), and (4d) due to Lemma 3. Moreover, after determining the flows q and potentials π, every feasible point of the follower's problem is of the form $(\lambda, (1-\lambda)q, (1-\lambda)^r\pi + \mathbb{1}\tau)$ for a given constant $\tau \in \mathbb{R}$, which follows from Lemma 3 and 4. Consequently, the structural properties of the bilevel problem (11) allow us to characterize the feasible points of the follower depending on the flows q and potentials π that correspond to the original load flow ℓ.

We now exploit these structural properties to move the nonlinear and nonconvex computations of the flows and potentials from the lower to the upper level. This is possible since for a given and balanced load flow the computation of the flows is unique. Afterward, the follower exploits the positive homogeneity of the potential-based flows to scale the corresponding flows and potentials such that the flow and potential bounds are satisfied. This leads to the following bilevel problem with a nonconvex and nonlinear follower problem that only contains $(1-\lambda)^r$ as a nonlinear term:

$$\max_{x,q,\pi} \quad \sum_{u \in V_-} \lambda \ell_u \tag{12a}$$

$$\text{s.t.} \quad \sum_{a \in \delta^{\text{out}}(v)} q_a - \sum_{a \in \delta^{\text{in}}(v)} q_a = \begin{cases} \ell_v, & \text{if } v \in V_+, \\ -\ell_v, & \text{if } v \in V_-, \\ 0, & \text{else}, \end{cases} \quad v \in V, \tag{12b}$$

$$x_a M_a^- \le \pi_u - \pi_v - \Lambda_a \varphi(q_a) \le x_a M_a^+, \quad a = (u,v) \in A, \tag{12c}$$

$$-(1-x_a)Q \le q_a \le (1-x_a)Q, \quad a \in A, \tag{12d}$$

$$x \in X, \ (\lambda, \tau) \in S(x, q, \pi). \tag{12e}$$

Here, $Q = \sum_{u \in V_-} \ell_u$ is an upper bound on the absolute flow through an arc and we choose M_a^- as well as M_a^+ according to (6). Further, the set $S(x, q, \pi)$ consists of all optimal solutions of the x-parameterized problem

$$\min_{\lambda,\tau} \quad \sum_{u \in V_-} \lambda \ell_u \tag{13a}$$

$$\text{s.t.} \quad \pi_u^- \le (1-\lambda)^r \pi_u + \tau \le \pi_u^+, \quad u \in V, \tag{13b}$$

$$(1-x_a)q_a^- \le (1-\lambda)q_a \le (1-x_a)q_a^+, \quad a \in A, \tag{13c}$$

$$0 \le \lambda \le 1, \ \tau \in \mathbb{R}. \tag{13d}$$

In the upper-level problem, the leader now chooses a feasible interdiction decision $x \in X$. For the original load flow $\ell \in \mathbb{R}^V$, the leader then computes the unique flows q and potentials π. This computation of flows and potentials is determined by Constraints (12b)–(12d). In doing so, the leader dismisses flow and potential bounds and the computed flows and potentials do not necessarily satisfy these bounds. Thus, the follower now exploits the positive homogeneity and uniqueness of the flows as well as the uniqueness of the potentials up to a constant shift. More precisely, the follower scales the flows as well as potentials down and shifts the potentials, if necessary, so that the corresponding bounds are satisfied. In doing so, the follower minimizes the load shed variable $\lambda \in [0,1]$, i.e., the follower scales the flows and potentials down as little as necessary to satisfy the bounds.

We note that a similar approach is applied in [22] to a bilevel model of the European entry-exit gas market system. In [22], moving the nonlinearities of the potential-based flows from the lower to the upper level leads to a linear lower-level problem. In contrast to this, we obtain a lower level that is still nonlinear but contains less challenging nonlinear constraints than the original formulation. The lower level now consists of the task to determine scalars $\lambda \in [0,1]$ and $\tau \in \mathbb{R}$ such that $(\lambda, (1-\lambda)q, (1-\lambda)^r\pi + \mathbb{1}\tau)$ satisfies the potential and flow bounds in the follower's problem of (11) while minimizing the load shed variable $\lambda \in [0,1]$.

We now prove that the reformulated bilevel problem (12) and the bilevel problem (11) admit the same optimal value. In addition, the set of optimal interdiction decisions is the same, i.e., for each interdiction decision that is part of a bilevel optimal solution of (12), there is also a bilevel optimal solution of (11) with the same interdiction decision and vice versa. To this end, we have to assume that the interdicted network $G(x)$ is weakly connected.

Assumption 3. *For every interdiction decision $x \in X$, the interdicted graph $G(x)$ is weakly connected.*

As discussed in [6, p. 60], this assumption does not simplify the problem. Further, it guarantees that for each $x \in X$, Constraints (12b) can always be satisfied, which is not necessarily the case if an interdiction decomposes the interdicted graph into multiple disconnected components. For what follows, we need the following technical lemma.

Lemma 6. *Let Assumptions 1–3 be satisfied and $x \in X$. Further, let the point (λ, q, π) satisfy (4d) with M_a^- and M_a^+ chosen according to (5) or (6), (11b), and for each arc $a \in A$ with $x_a = 1$, $q_a = 0$ holds. Then, for every $a = (u,v) \in A$ with $x_a = 1$, the inequalities*

$$-(1-\lambda)^r\varphi\left(\sum_{u\in V_-}\ell_u\right) \le \pi_u - \pi_v \le (1-\lambda)^r\varphi\left(\sum_{u\in V_-}\ell_u\right) \tag{14}$$

are satisfied.

Proof. For $a' = (u,v) \in A$ with $x_{a'} = 1$, there exists a path $P(u,v)$ between u and v in the interdicted network $G(x)$ due to Assumption 3. Thus, each arc $a \in P(u,v)$ is not interdicted, i.e., $x_a = 0$ holds. For an arc $a \in A$, the function $\sigma_a(P(u,v))$ evaluates to 1 if arc a is a forward arc of the path P, it evaluates to -1 if a is an backward arc of P, and otherwise, it is zero. Consequently, from Constraints (4d) it then follows

$$\begin{aligned}|\pi_u - \pi_v| &= \left|\sum_{a\in P(u,v)}\sigma_a(P(u,v))\Lambda_a\varphi(q_a)\right| \le \sum_{a\in A}\Lambda_a\varphi\left(\sum_{u\in V_-}(1-\lambda)\ell_u\right)\\ &= (1-\lambda)^r\sum_{a\in A}\varphi\left(\sum_{u\in V_-}\ell_u\right),\end{aligned}$$

since the potential function φ is positive homogeneous and strictly increasing. Further, $|q_a| \le (1-\lambda)\sum_{u\in V_-}\ell_u$ holds since (λ, q) satisfy Constraints (11b) and there cannot be any cyclic flow in the considered potential network. □

Theorem 7. *Let Assumptions 1–3 be satisfied. Then, Problem (11) and (12) admit the same optimal value and the same sets of optimal interdiction decisions.*

Proof. We start by proving that the optimal objective value of (12) is an upper bound for the optimal objective value of (11). Let $(x, q, \pi, \lambda, \tau)$ be a bilevel feasible point of (12). For fixed interdiction x, we now prove that $(\lambda, (1-\lambda)q, (1-\lambda)^r\pi + \tau)$ is a feasible point of the lower-level problem of (11). The feasibility of the Constraints (11b) and (4c) directly follows from (12b) and (13c). Further, the feasibility of Constraints (4e) and $\lambda \in [0,1]$ follows from (13b) and (13d). For $a \in A$ with $x_a = 0$, Constraints (4d) are directly satisfied due to (12c). However, for $a \in A$ with $x_a = 1$, we have to take into account that M_a^- and M_a^+ are defined by (5) in (11) and not by (6) as in (12). For $a \in A$ with $x_a = 1$, we obtain $q_a = 0$ due to (13c). In this case, Constraints (4d) reduce to

$$\pi_u^- - \pi_v^+ \leq (1-\lambda)^r(\pi_u - \pi_v) = (1-\lambda)^r(\pi_u - \pi_v) + \tau - \tau \leq \pi_u^+ - \pi_v^-,$$

which is satisfied due to Constraints (13b). Consequently, $(\lambda, (1-\lambda)q, (1-\lambda)^r\pi+\tau)$ is feasible for the lower-level problem of Problem (11) and note further that all lower-level variables are bounded. Thus, there exists a bilevel feasible point (x, λ', q', π') with the same interdiction decision x for Problem (11) and each of these points satisfies $\sum_{u \in V_-} \lambda \ell_u \geq \sum_{u \in V_-} \lambda' \ell_u$. Thus, the optimal objective of (12) is an upper bound for the optimal objective value of (11) since for every $x \in X$, there exists a corresponding bilevel feasible point of (12) due to Lemma 3.

Next, we prove that the optimal objective value of (12) is a lower bound for the optimal objective value of (11). Let (x, λ, q, π) be a bilevel feasible point of Problem (11). If $\lambda = 1$ holds, then the complete load is shed, which is an upper bound for the objective value of (12). We now show that for $\lambda \in [0, 1)$ and fixed interdiction x, the point $(x, (1/(1-\lambda))q, ((1/(1-\lambda))^r\pi))$ satisfies the constraints of the upper-level problem (12). The feasibility of Constraints (12b) directly follows from (11b). For $a \in A$ with $x_a = 0$, Constraints (12c) are satisfied due to (4d). If $x_a = 1$ holds, Constraints (4c) imply that $q_a = 0$. In this case, the feasibility of Constraints (12c) follows from applying Lemma 6 and dividing Constraints (14) by $(1-\lambda)^r$. Finally, $|q_a| \leq (1-\lambda)\sum_{u \in V_-} \ell_u$ holds since (λ, q) satisfies Constraints (11b) and because there cannot be any cyclic flow in the considered potential network. From this and Constraints (4c), it follows that Constraints (12d) are valid. Consequently, the point $(x, 1/(1-\lambda)q, (1/(1-\lambda))^r\pi)$ satisfies the constraints of the upper-level problem (12). Moreover, the point $(\lambda, 1/(1-\lambda)q, (1-\lambda)^r\pi, 0)$ is a feasible point of the lower-level problem (13) due to (4c) and (4e). Further, all lower-level variables that affect the objective function are bounded. Consequently, there exists a bilevel feasible point $(x, q', \pi', \lambda', \tau')$ of (12) with the same interdiction x and each of these points satisfies $\sum_{u \in V_-} \lambda \ell_u \geq \sum_{u \in V_-} \lambda' \ell_u$. Thus, the optimal objective of (12) is a lower bound for the optimal objective value of (11) since for every $x \in X$, there exists a corresponding bilevel feasible point of (11).

From the previous cases, it follows that the optimal objective values of (12) and (11) are the same. Further, for an optimal solution of one of the considered bilevel problems, there is always an bilevel optimal solution of the other bilevel problem with the same interdiction. Thus, the set of optimal interdiction decisions are the same. □

The follower's problem (13) is still nonconvex and nonlinear. Thus, a single-level reformulation cannot be obtained by applying strong duality or the KKT conditions. However, Problem (13) contains a special structure that can be exploited to provide necessary and sufficient optimality conditions.

In a nutshell, these conditions mainly express that in an optimal solution of Problem (13) at least one lower as well as one upper potential bound is tight or at least one arc flow bound is tight. The intuition behind these conditions is that if for a feasible upper-level point (x, q, π) and a corresponding feasible lower-level point (λ, τ) neither potential nor flow bounds are tight in (13), then we can decrease the load shed variable $\lambda \in [0,1]$ and if necessary adjust $\tau \in \mathbb{R}$ to obtain a feasible point of the follower with a smaller objective value.

Theorem 8. *Let Assumptions 1–3 be satisfied. For any feasible upper-level decision (x, q, π) of Problem (12), the point (λ, τ) is an optimal solution of the follower's problem (13) if and only if (λ, τ) is feasible for the follower's problem (13) and the point $(x, q, \pi, \lambda, \tau)$ satisfies at least one of the following conditions:*

(i) There are nodes $u, v \in V$ such that the corresponding lower and upper potential bound are tight, i.e.,

$$(1-\lambda)^r \pi_u + \tau = \pi_u^+, \quad (1-\lambda)^r \pi_v + \tau = \pi_v^-. \tag{15}$$

(ii) There is an arc $a \in A$ such that the lower or upper flow bound is tight, i.e.,

$$(1-\lambda) q_a = q_a^- \quad \textit{or} \quad (1-\lambda) q_a = q_a^+. \tag{16}$$

(iii) There is no load shed, i.e.,

$$\lambda = 0. \tag{17}$$

Proof. We first prove that every optimal solution of the follower's problem (13) satisfies at least one of the Conditions (15)–(17). To this end, we now contrarily assume that the bilevel feasible point $(x, q, \pi, \lambda, \tau)$ of (12) does not satisfy any of the Conditions (15)–(17). Consequently, we can replace τ by $\tilde{\tau}$ such that $(x, q, \pi, \lambda, \tilde{\tau})$ is a point that satisfies all constraints of the bilevel problem (12) and for every node $u \in V$ and every arc $a \in A$, we have the strict inequalities

$$\pi_u^- < (1-\lambda)^r \pi_u + \tilde{\tau} < \pi_u^+, \quad q_a^- < (1-\lambda) q < q_a^+, \quad \lambda > 0.$$

Consequently, none of the flow and potential bounds is tight and we have a positive load shed for the considered bilevel feasible point. Our modification of τ does not affect the objective value and only affects the lower-level constraints. Consequently, $(x, q, \pi, \lambda, \tilde{\tau})$ is still a bilevel feasible point of (12). However, now none of the lower-level constraints is tight, i.e., we only have strict inequalities, and, thus, we can decrease $\lambda > 0$, which also decreases the objective value of the follower. Consequently, (λ, τ) is not the optimal response of the follower for the given interdiction decision x with flows q and potentials π. This is a contradiction to the bilevel feasibility of the considered point $(x, q, \pi, \lambda, \tau)$.

Next, we prove that for any feasible upper-level decision (x, q, π) of Problem (12), any feasible point (λ, τ) of the follower's problem (13) that satisfies at least one of the Conditions (15)–(17) is optimal. To this end, we consider the three following cases. First, if Condition (17) is satisfied by the point (λ, τ), then $\lambda = 0$ holds. Consequently, there is no load shed and the objective value equals zero, which is optimal. Second, we assume that for at least a single arc $a \in A$, Condition (16) is satisfied. If $\lambda = 0$ holds, we apply the first case. Otherwise, it follows from Condition (16) that we cannot decrease $\lambda > 0$ without violating a lower q_a^- or an upper flow bound q_a^+ since $q_a^- < 0 < q_a^+$ holds. This is independent from the choice of τ. Third, we assume that there are nodes $u, v \in V$ so that Conditions (15) are satisfied by (λ, τ). If $\lambda = 0$ holds, we apply the first case. Otherwise, every feasible point $(\tilde{\lambda}, \tilde{\tau})$ of the follower's problem (13) satisfies the inequality

$$(1-\tilde{\lambda})^r (\pi_u - \pi_v) = (1-\tilde{\lambda})^r \pi_u + \tilde{\tau} - (1-\tilde{\lambda})^r \pi_v - \tilde{\tau} \le \pi_u^+ - \pi_v^- \tag{18}$$

due to Constraints (13b). The inequality is independent of $\tilde{\tau}$ and $\pi_u^+ - \pi_v^- > 0$ holds due to Assumption 1. Moreover, from Condition (15) it follows that the point (λ, τ) satisfies the inequality in (18) with equality. Consequently, $(\tilde{\lambda}, \tilde{\tau})$ with $0 \le \tilde{\lambda} < \lambda$ cannot be a feasible solution of the follower's problem (13) w.r.t. (x, q, π) since it violates Inequality (18). Thus, (λ, τ) is an optimal solution of the follower's problem (13). □

Corollary 1. *Let Assumptions 1–3 be satisfied. For any feasible upper-level decision (x, q, π) of Problem (12), every optimal solution (λ, τ) of the lower-level problem (13) satisfies $\lambda \in [0, 1)$.*

Proof. If $\lambda = 1$ holds, then none of the Conditions (15)–(17) can be satisfied due to $\pi_u^- < \pi_u^+$ for all nodes $u \in V$ and $q_a^- < 0 < q_a^+$ for all arcs $a \in A$; see Assumption 1. Consequently, from Theorem 8, it follows that every optimal solution (λ, τ) of Problem (13) satisfies $\lambda \in [0, 1)$. □

We now exploit the necessary and sufficient optimality conditions for the follower's problem of Theorem 8 to derive a single-level reformulation. To this end, we represent the necessary and sufficient optimality conditions (15)–(17) using additional binary variables and linear constraints. We further exploit that for a balanced load flow $\ell \in \mathbb{R}^V$, the corresponding flows are unique and the potentials are unique up to a constant shift. This leads to the single-level reformulation

Single level ref

$$
\begin{aligned}
\max_{\substack{x,\lambda,q,\pi,\tilde{\varepsilon}^+,\tilde{\varepsilon}^-,\varepsilon^+,\\ \varepsilon^-,\bar{y},\underline{y},\tilde{y},y'}} \quad & \sum_{u \in V_-} \lambda \ell_u && \text{(19a)}\\
\text{s.t.} \quad & \text{(12b)–(12d)}, \ \lambda \in [0,1], && \text{(19b)}\\
& (1-\lambda)^r \pi_u + \tilde{\varepsilon}_u^+ = \pi_u^+, \quad u \in V, && \text{(19c)}\\
& (1-\lambda)^r \pi_u - \tilde{\varepsilon}_u^- = \pi_u^-, \quad u \in V, && \text{(19d)}\\
& (1-\lambda) q_a + \varepsilon_a^+ = q_a^+, \quad a \in A, && \text{(19e)}\\
& (1-\lambda) q_a - \varepsilon_a^- = q_a^-, \quad a \in A, && \text{(19f)}\\
& \tilde{\varepsilon}^+, \tilde{\varepsilon}^-, \varepsilon^+, \varepsilon^- \geq 0, && \text{(19g)}\\
& \min_{u \in V}\{\tilde{\varepsilon}_u^+\} \leq \tilde{M}\bar{y}, \ \min_{u \in V}\{\tilde{\varepsilon}_u^-\} \leq \tilde{M}\underline{y}, && \text{(19h)}\\
& \min_{a \in A}\{\varepsilon_a^+\} \leq \tilde{Q}\tilde{y}, \ \min_{a \in A}\{\varepsilon_a^-\} \leq \tilde{Q}y', && \text{(19i)}\\
& (\bar{y} + \underline{y})\tilde{y}y'\lambda = 0, && \text{(19j)}\\
& \bar{y}, \underline{y}, \tilde{y}, y' \in \{0,1\}. && \text{(19k)}
\end{aligned}
$$

Constraints (12b)–(12d) are the upper-level constraints of Problem (12), which determine the flows q and potentials π corresponding to the load flow ℓ in the interdicted network. Constraints (19c)–(19g) and $\lambda \in [0, 1]$ represent the lower-level constraints of Problem (13). In comparison to Problem (13), we additionally introduce nonnegative slack variables $\tilde{\varepsilon}^+, \tilde{\varepsilon}^- \in \mathbb{R}^V$ and $\varepsilon^+, \varepsilon^- \in \mathbb{R}^A$ to reformulate the inequalities regarding the flow and potential bounds as equalities. These slack variables enable us to model the necessary and sufficient optimality conditions of Theorem 8 by Constraints (19h)–(19k). Note that

$$
\tilde{M} := \max_{u \in V}\{\pi_u^+ - \pi_u^-\}, \quad \tilde{Q} := \max_{a \in A}\{q_a^+ - q_a^-\} \tag{20}
$$

are valid upper bounds for the slack variables $\tilde{\varepsilon}^+, \tilde{\varepsilon}^-, \varepsilon^+$, and ε^- due to Constraints (19c)–(19g). Additionally, for $a \in A$, we choose M_a^- and M_a^+ according to (6). Constraints (19c), (19d), (19g), and (19h) ensure that the binary variables $\bar{y}$ and $\underline{y}$ can be set to zero if and only if Condition (15) is satisfied. Analogously, Constraints (19e)–(19g) and (19i) ensure that at least one of the binary variables $\tilde{y}$ or y' can be set to zero if and only if Condition (16) holds. Further, Constraint (19j) models that at least one of the necessary and sufficient optimality conditions of Theorem 8 is satisfied. We note that we can linearize the minima in Constraints (19h) and (19i) as well as the left-hand side of Constraint (19j) using additional binary variables since the corresponding variables are bounded. We explicitly discuss these reformulations at the end of this section.

We now formally prove that the single-level reformulation (19) and the bilevel Problem (12) admit the same optimal objective values. Furthermore, the set of optimal interdiction decisions is the same, i.e., for each interdiction decision belonging to an optimal solution of (19), there is a bilevel optimal solution of (12) with the same interdiction decision and vice versa. We now abbreviate a feasible point of the single-level problem (19) by

$$s := (x, \lambda, q, \pi, \tilde{\varepsilon}^+, \tilde{\varepsilon}^-, \varepsilon^+, \varepsilon^-, \bar{y}, \underline{y}, \tilde{y}, y')$$

to obtain a more compact representation of the results.

Theorem 9. *Let Assumptions 1–3 be satisfied. The single-level problem* (19) *and the bilevel problem* (12) *admit the same optimal objective value and the set of optimal interdiction decisions is the same as well.*

Proof. Let $(\tilde{x}, \tilde{q}, \tilde{\pi}, \tilde{\lambda}, \tau)$ be a bilevel feasible point of (12). From Corollary 1 it follows that $\tilde{\lambda} < 1$ holds. We now consider the point $z := (\tilde{x}, \tilde{\lambda}, \tilde{q}, \tilde{\pi} + \mathbb{1}\tau/(1-\tilde{\lambda})^r, \tilde{\varepsilon}^+, \tilde{\varepsilon}^-, \varepsilon^+, \varepsilon^-, \bar{y}, \underline{y}, \tilde{y}, y')$, in which the slack variables $\tilde{\varepsilon}^+, \tilde{\varepsilon}^-, \varepsilon^+, \varepsilon^-$ are uniquely determined by the Constraints (19c)–(19g). Moreover, we choose the binary variables $\bar{y}, \underline{y}, \tilde{y}, y'$ as small as possible. The constructed point z satisfies Constraints (19b)–(19i) and (19k) since $(\tilde{x}, \tilde{\lambda}, \tilde{q}, \tilde{\pi}, \tau)$ is a bilevel feasible point of (12). If $\tilde{\lambda} = 0$ holds, then it directly follows that Constraint (19j) is satisfied by z. If $\tilde{\lambda} > 0$ holds, then the bilevel feasible point $(\tilde{x}, \tilde{q}, \tilde{\pi}, \tilde{\lambda}, \tau)$ satisfies Conditions (15) or (16) due to Theorem 8. Consequently, it follows that $(\bar{y} + \underline{y}) = 0$ or $\tilde{y}y' = 0$ holds and, thus, Constraint (19j) is satisfied by the point z. In total, z is a feasible point of Problem (19) with the same interdiction decision $x \in X$ and load shed $\lambda \in [0, 1)$.

We now consider an optimal solution s of the single-level problem (19) and prove that the point $z := (x, q, \pi, \lambda, 0)$ is a bilevel feasible point of Problem (12). Since the point s is feasible for Problem (19), it follows that z is feasible for the upper-level constraints (12b)–(12d) and also satisfies the lower-level constraints (13b)–(13d). Further, from Constraint (19j), it follows that $\bar{y} = \underline{y} = 0$ or at least one of the variables $\tilde{y}, y', \lambda$ is equal to zero. However, this directly implies that the point z satisfies at least one of the Conditions (15)–(17) due to the following three cases. First, if $\lambda = 0$ holds, then the point z satisfies Condition (17). Second, if $\tilde{y} = 0$ or $y' = 0$ holds, then the point z satisfies Condition (16). Third and finally, if $\bar{y} = 0$ and $\underline{y} = 0$ is satisfied, then the point z satisfies Condition (15). Consequently, from Theorem 8, it follows that for the upper-level decisions (x, q, π), the point $(\lambda, 0)$ is an optimal solution of the follower's problem. Thus, z is a bilevel feasible point with the same interdiction decision and load shed as the single-level optimal solution s. □

Remark 10. *Let Assumptions 1–3 be satisfied. Every interdiction decision $x \in X$ can be extended to a feasible solution of Problem* (19) *as it is done in the proof of Theorem 9. However, we can reduce the set of feasible solutions by replacing Constraint* (19j) *with*

$$(\bar{y} + \underline{y})\tilde{y}y' = 0. \tag{21}$$

Consequently, all solutions that only satisfy Condition (17)*, i.e., $\lambda = 0$, but do neither satisfy Condition* (15) *nor* (16)*, are not feasible anymore. However, the set of optimal solution stays the same as long as there is an interdiction decision $x \in X$ that causes any load shed. We further note that this adaption of Problem* (19) *turns out to be beneficial in our computational study.*

We further note that we can model the min-operators involved in the single-level reformulation (19) by applying following standard linearization technique.

Remark 11 (Linearization of min-operators)**.** *To model the min-operators involved in Problem* (19)*, we apply the following classic technique. For a finite set of* *indices I, we model $\min_{i\in I}\{f_i\}$ by using additional binary variables t_i, $i \in I$, and the continuous variable g. Let further $L, U \in \mathbb{R}$ be chosen so that the inequalities $L \leq f_i \leq U$ hold for every $i \in I$. Then, $g = \min_{i\in I}\{f_i\}$ holds if and only if*

$$f_i - (U - L)(1 - t_i) \leq g \leq f_i,\ i \in I, \quad \sum_{i\in I} t_i = 1, \quad t_i \in \{0, 1\},\ i \in I. \tag{22}$$

We can further linearize Constraint (19j) using additional binary variables and additional linear constraints with the help of McCormick inequalities [19].

By applying the previously discussed linearization techniques, we can reformulate the single-level reformulation (19) as a mixed-integer nonlinear optimization problem (MINLP) consisting of finitely many variables and constraints. This problem then can be solved using state-of-the-art MINLP solvers to compute optimal interdiction decisions. However, the single-level reformulation (19) contains many nonlinear terms in Constraints (12c) as well as in Constraints (19c)–(19f), which makes it very challenging to solve. Hence, in the next two sections, we present two equivalent reformulations of Problem (19) that significantly reduce the number of nonlinear terms within constraints (19c)–(19f).

4.1. **First Reduced Single-Level Reformulation.** In (19), we compute the flows regarding the given load flow $\ell \in \mathbb{R}^V$. Afterward, we then scale the potentials and flows down to satisfy the flow and potential bounds in Constraints (19c)–(19g), which leads to a large number of nonlinear terms.

We now propose to compute the flows directly for the scaled load flow $(1 - \lambda)\ell$ in Constraint (R1.2) below. Since the considered potential-based flows are positively homogeneous and the flows are unique for a given nomination, we then do not have to scale the corresponding flows and potentials down anymore, which eliminates all the nonlinearities in the Constraints (19c)–(19g). The obtained model reads as

$$\max_{s} \quad \sum_{u\in V_-} \lambda \ell_u \tag{R1.1}$$

$$\text{s.t.} \quad \sum_{a\in\delta^{\text{out}}(v)} q_a - \sum_{a\in\delta^{\text{in}}(v)} q_a = \begin{cases} (1-\lambda)\ell_v, & \text{if } v \in V_+, \\ -(1-\lambda)\ell_v, & \text{if } v \in V_-, \\ 0, & \text{else}, \end{cases} \quad v \in V, \tag{R1.2}$$

$$\text{(12c)(12d)},\ \lambda \in [0, 1], \tag{R1.3}$$

$$\pi_u + \tilde{\varepsilon}_u^+ = \pi_u^+,\ \pi_u - \tilde{\varepsilon}_u^- = \pi_u^-, \quad u \in V, \tag{R1.4}$$

$$q_a + \varepsilon_a^+ = q_a^+,\ q_a - \varepsilon_a^- = q_a^-, \quad a \in A, \tag{R1.5}$$

$$\tilde{\varepsilon}^+, \tilde{\varepsilon}^-, \varepsilon^+, \varepsilon^- \geq 0,\ \text{(19h)–(19k)}. \tag{R1.6}$$

We now prove that there is a 1-to-1 correspondence between the optimal points of (19) and (R1) so that the set of feasible interdictions and load sheds is the same.

Lemma 12. *Let Assumptions 1–3 be satisfied. A point $s := (x, \lambda, q, \pi, \tilde{\varepsilon}^+, \tilde{\varepsilon}^-, \varepsilon^+, \varepsilon^-, \bar{y}, \underline{y}, \tilde{y}, y')$ is optimal for Problem (19) if and only if the point $z := (x, \lambda, (1-\lambda)q, (1-\lambda)^r\pi, \tilde{\varepsilon}^+, \tilde{\varepsilon}^-, \varepsilon^+, \varepsilon^-, \bar{y}, \underline{y}, \tilde{y}, y')$ is optimal for Problem (R1).*

Proof. Let s be an optimal point of Problem (19). Corollary 1 and Theorem 9 imply that $\lambda \in [0, 1)$ holds. We now prove that the point z is feasible for (R1). Multiplying Constraints (12b) by $(1-\ell) \in [0, 1)$ shows the feasibility of (R1.2). The feasibility of Constraints (12c) w.r.t. $(1-\lambda)^r\pi$ follows from their feasibility w.r.t. π as well as from $(1 - \lambda) \in [0, 1)$ and the positive homogeneity of the potential function. Analogously, Constraints (12d) are satisfied for $(1 - \lambda)q$. Constraints (R1.4) and (R1.5) are valid for z due to the feasibility of Constraints (19c)–(19g) w.r.t. s. Thus, the point z is feasible for Problem (R1).

Let now z be an optimal solution of Problem (R1). Further, $\lambda \in [0,1)$ holds since, otherwise, for each arc $a \in A$, $q_a = 0$ is satisfied and Constraint (19j) cannot be valid due to Assumption 1. The feasibility of Constraints (12b) follows from Constraints (R1.2). Further, Constraints (19c)–(19k) are valid for s since Constraints (R1.3)–(R1.6) are satisfied by z. For an arc $a \in A$ with $x_a = 0$, the feasibility of Constraints (12c) w.r.t. q and π follows from their feasibility for $(1-\lambda)q$ and $(1-\lambda)^r\pi$. For arc $a \in A$ with $x_a = 1$, we obtain $q_a = 0$ due to (12d). Thus, the feasibility of Constraints (12c) w.r.t. q and π follows from Lemma 6 and dividing Constraints (14) by $(1-\lambda)^r$. Finally, $(1-\lambda)|q_a| \le (1-\lambda)\sum_{u \in V_-} \ell_u$ holds since $(\lambda, (1-\lambda)q)$ satisfies Constraints (R1.2) and there cannot be any cyclic flow. Consequently, Constraints (12d) are valid for s. Thus, point s is feasible for Problem (19). □

Model (R1) does not contain nonlinear terms in (R1.4) and (R1.5) to decide if flow or potential bounds are tight, which is in contrast to Model (19) and Constraints (19c)–(19f). In addition, we do not add any additional nonlinearities in Model (R1).

4.2. **Second Reduced Single-Level Reformulation.** We now derive a different reformulation of the single-level problem (19) that again contains less nonlinear terms. To this end, we exploit that $\lambda \in [0,1)$ holds due to Corollary 1, which enables us to apply a specific variable transformation. More precisely, we derive the second reduced single-level reformulation by multiplying the Constraints (19c) and (19d) with $(1/(1-\lambda))^r$. Further, we multiply Constraints (19e) and (19f) by $(1/(1-\lambda))$. In addition, we can consider $\tilde{\varepsilon}^+$ instead of $1/(1-\lambda)^r\tilde{\varepsilon}^+$ since $1/(1-\lambda)^r > 0$ and $\tilde{\varepsilon}^+ \ge 0$ hold. Analogously, we proceed with $\tilde{\varepsilon}^-, \varepsilon^+$, and ε^-. We then apply the variable substitution $\lambda^* = 1/(1-\lambda)$ for which $\lambda^* \ge 1$ holds. After applying the variable transformation, we can additionally reformulate the objective function $\max\ (1 - 1/\lambda^*) \sum_{u \in V_-} \ell_u$ as the linear objective function $\max\ \lambda^* \sum_{u \in V_-} \ell_u$ and still obtain the same optimal solutions after reversing the variable substitution. The second single-level reformulation is thus given by

$$\max_{\substack{x,\lambda^*,q,\pi,\tilde{\varepsilon}^+,\tilde{\varepsilon}^-,\varepsilon^+,\\ \varepsilon^-,\bar{y},\underline{y},\tilde{y},y'}} \quad \lambda^* \sum_{u \in V_-} \ell_u \tag{R2.1}$$

$$\text{s.t.} \quad (12b)\text{–}(12d), \tag{R2.2}$$

$$\pi_u + \tilde{\varepsilon}_u^+ = (\lambda^*)^r \pi_u^+, \ \pi_u - \tilde{\varepsilon}_u^- = (\lambda^*)^r \pi_u^-, \quad u \in V, \tag{R2.3}$$

$$q_a + \varepsilon_u^+ = \lambda^* q_a^+, \ q_a - \varepsilon_u^- = \lambda^* q_a^-, \quad a \in A, \tag{R2.4}$$

$$\tilde{\varepsilon}^+, \tilde{\varepsilon}^-, \varepsilon^+, \varepsilon^- \ge 0, \quad (19h)\text{–}(19k), \quad 1 \le \lambda^*. \tag{R2.5}$$

Since we only applied a variable transformation to Problem (19) to derive the second reduced problem (R2), it directly follows that the set of optimal interdiction decisions of both problems is the same.

Let us remark that after applying the variable transformation, the big-M values $\tilde{M}$ and $\tilde{Q}$ of (20), which are necessary to linearize the min-operators in Constraints (19h)–(19k), now depend on the variable λ^*, i.e., $\tilde{M} = (\lambda^*)^r \max_{u \in V}\{\pi_u^+ - \pi_u^-\}$ and $\tilde{Q} = \lambda^* \max_{a \in A}\{q_a^+ - q_a^-\}$.

For fixed λ^*, these big-M values are valid since the potentials π satisfy Constraints (R2.3) and the flows q satisfy Constraints (R2.4). Consequently, it is necessary to prove a finite upper bound for $\lambda^* = (1/(1-\lambda))$ to provide valid big-M values $\tilde{M}$ and $\tilde{Q}$.

We now prove that under Assumptions 1–3, there is a finite upper bound of λ^*. This means that, for every interdiction, the follower can still transport a very small amount of flow through the network. This is in line with realistic attacks on electricity, gas, or water networks, for which it is unlikely that an interdiction can shed the entire load in the network.

Lemma 13. *Let Assumptions 1–3 be satisfied, the potential function be given by* $\varphi(q_a) = \operatorname{sgn}(q_a)|q_a|^r$*, and* $Q = \sum_{u \in V_-} \ell_u$*. Then, for every upper-level feasible point* (x, q, π) *of Problem (12), there is a feasible point* (λ, τ) *of the lower level (13) with*

$$\lambda = \max\left\{0, 1 - \frac{\min_{a \in A}\{|q_a^-|, q_a^+\}}{Q}, 1 - \left(\frac{\bar{\pi} - \underline{\pi}}{\sum_{a \in A} \Lambda_a \varphi(Q)}\right)^{1/r}\right\} =: \lambda^+ < 1. \quad (25)$$

Proof. Let the point (x, q, π) be feasible for the upper-level constraints of (12). For each arc $a \in A$, the absolute flow $|q_a|$ is bounded by Q since there is no cyclic flow. If the flow bounds satisfy $q_a^- \le -Q \le Q \le q_a^+$, then these flow bounds never cause load shedding and can be neglected. Thus, we focus on the case that at least one arc $a \in A$ satisfies $q_a^- > -Q$ or $q_a^+ < Q$. Consequently, choosing $\lambda \in [1 - \min_{a \in A}\{|q_a^-|, q_a^+\}/Q, 1)$ implies that $(1 - \lambda)q$ satisfies (13c) since for $a \in A$ with $x_a = 1$, Constraints (12d) imply $q_a = 0$ and otherwise, we obtain

$$q_a^- \le (1 - \lambda)(-Q) \le (1 - \lambda)q_a \le (1 - \lambda)Q \le q_a^+.$$

From Assumption 1, it follows $\emptyset \ne \operatorname{int}\left(\bigcap_{u \in V}[\pi_u^-, \pi_u^+]\right) = (\underline{\pi}, \bar{\pi})$, i.e., $\underline{\pi} < \bar{\pi}$. Additionally, for the point $(0, q, \pi)$, we can apply Lemma 6 and, thus, for $u, v \in V$, we obtain $|\pi_u - \pi_v| \le \sum_{a \in A} \Lambda_a \varphi(Q)$. If $\bar{\pi} - \underline{\pi} \ge \sum_{a \in A} \Lambda_a \varphi(Q)$, then the potential bounds never cause load shedding and can be neglected. Thus, we focus on the case $\bar{\pi} - \underline{\pi} < \sum_{a \in A} \Lambda_a \varphi(Q)$. We now choose $\lambda \in [1 - ((\bar{\pi} - \underline{\pi})/\sum_{a \in A} \Lambda_a \varphi(Q))^{1/r}, 1)$. Due to Lemma 3, we can further choose η such that for each $u \in V$, the inequality $(1 - \lambda)^r \pi_u + \eta \ge 0$ holds and such that there exists at least one node w that satisfies $(1 - \lambda)^r \pi_w + \eta = 0$. For each node $u \in V$, we obtain

$$\pi_u^- \le \underline{\pi} \le (1 - \lambda)^r \pi_u + \eta + \underline{\pi} \le (1 - \lambda)^r (\pi_u - \pi_w) + \eta - \eta + \underline{\pi} \le \bar{\pi} \le \pi_u^+.$$

Thus, the point $(\lambda, \eta + \underline{\pi})$ with $\lambda < 1$ satisfying Condition (25) is a feasible point of the lower level (13). □

This lemma implies that every feasible point of the bilevel problem (12) satisfies $0 \le \lambda < 1$. The same also holds for every feasible point of (19), (R1), and (R2) due to Theorem 9 and Lemma 12. Thus, we can bound the variable λ^* by $1 \le \lambda^* = 1/(1 - \lambda) \le 1/(1 - \lambda^+)$.

Let us finally remark that for each arc $a \in A$ we can also choose the big-M values M_a^- and M_a^+ in Models (19) and (R1) according to (5) instead of (6). This is based on the fact that in (19) and (R1), we do not dismiss any potential bounds and it follows in analogy to the given explanation after (5) in Section 2. Again, for Model (R2) these big-M values are given by

$$M_a^- = (\lambda^*)^r (\pi_u^- - \pi_v^+), \quad M_a^+ = (\lambda^*)^r (\pi_u^+ - \pi_v^-), \quad (26)$$

in which we again bound λ^* by $1 \le \lambda^* = 1/(1 - \lambda) \le 1/(1 - \lambda^+)$.

5. An Exact Method for Potential-Based Flow Interdiction

We now present an exact algorithm that computes an optimal interdiction decision by exploiting the derived single-level reformulation (R1), respectively (R2), which compute upper bounds for the interdiction problem (3).

The algorithm works as follows. We set the lower bound $\phi_{\text{LB}} = 0$ and the upper bound $\phi_{\text{UB}} = \infty$ for the total load shed, i.e., for the objective value of the interdiction Problem (3). Additionally, we initialize the load vector $\ell^* = \ell$ to the original load vector $\ell \in \mathbb{R}^V$. We then solve one of the reduced-single level reformulations, i.e., (R1) or (R2), w.r.t. ℓ^* to obtain an upper bound $\phi^{\mathrm{L}}_{\mathrm{UB}}$ (where L stands for "leader"), on the load shed and a corresponding interdiction decision $x \in X$. If the obtained upper bound $\phi^{\mathrm{L}}_{\mathrm{UB}}$ is smaller than the best current upper bound ϕ_{UB}, we update the incumbent. We then solve the follower's problem w.r.t. ℓ^* and the interdiction decision x to obtain a lower bound $\phi^{\mathrm{F}}_{\mathrm{LB}}$ (where F stands for "follower"), on the load shed. We denote the corresponding solution by (λ, q, π). If the lower bound $\phi^{\mathrm{F}}_{\mathrm{LB}}$ is larger than the best current lower bound ϕ_{LB}, then we update the latter. In addition, we update the load flow to the best response of the follower, i.e., $\ell^* = (1-\lambda) \circ \ell$, and store the computed interdiction x, which now represents the best known interdiction decision. Afterward, we add the no-good cut

$$\sum_{a \in A: x_a = 1} (1 - x) + \sum_{a \in A: x_a = 0} x \geq 1 \tag{27}$$

to the considered reduced single-level reformulation (R1) or (R2) to cut off the previously computed interdiction. Then, the procedure is repeated. We terminate either if we satisfy a relative tolerance w.r.t. the gap given by the lower and upper bound or if there is no feasible interdiction decision left.

The method is given in Algorithm 1. Note that as long as the best lower bound ϕ_{LB} equals zero, we replace the termination condition in Line 2 by $\phi_{\mathrm{UB}} - \phi_{\mathrm{LB}} \geq \varepsilon$.

Algorithm 1: Solving potential-based network flow interdiction problems

Input: load flow $\ell \in \mathbb{R}^V$ and optimality tolerance $\varepsilon \geq 0$.
1 Initialize $\phi_{\mathrm{LB}} \leftarrow 0$, $\phi_{\mathrm{UB}} \leftarrow \infty$, $\ell^* \leftarrow \ell$, and $x^* \leftarrow 0$.
2 **while** $(\phi_{\mathrm{UB}} - \phi_{\mathrm{LB}})/\phi_{\mathrm{LB}} \geq \varepsilon$ **do**
3 Solve Problem (R1) or (R2) w.r.t. ℓ^* and obtain the interdiction decision $x \in X$ and the objective value $\phi^{\mathrm{L}}_{\mathrm{UB}}$.
4 **if** *Problem* (R1) *or* (R2) *w.r.t.* ℓ^* *is infeasible* **then**
5 Update $\phi_{\mathrm{UB}} \leftarrow \phi_{\mathrm{LB}}$ and **return** interdiction $x^* \in X$.
6 **if** $\phi_{\mathrm{UB}} > \phi^{\mathrm{L}}_{\mathrm{UB}} + \phi_{\mathrm{LB}}$ **then**
7 Update $\phi_{\mathrm{UB}} \leftarrow \phi^{\mathrm{L}}_{\mathrm{UB}} + \phi_{\mathrm{LB}}$.
8 Solve the lower-level problem (4) w.r.t. x as well as ℓ and obtain the solution (λ, q, π) with objective value $\phi^{\mathrm{F}}_{\mathrm{LB}}$.
9 **if** $\phi^{\mathrm{F}}_{\mathrm{LB}} > \phi_{\mathrm{LB}}$ **then**
10 Update $\phi_{\mathrm{LB}} \leftarrow \phi^{\mathrm{F}}_{\mathrm{LB}}$, $\ell^* \leftarrow (1-\lambda) \circ \ell$, and set $x^* = x$.
11 Add the no-good cut (27) to X to cut off the current interdiction x.
12 **return** interdiction $x^* \in X$

In Line 3, we solve the single-level reformulation w.r.t. the load flow ℓ^*. This load flow ℓ^* represents the best response of the follower w.r.t. the best known interdiction x^*, i.e., ℓ^* is obtained by applying the smallest load shed to the original load flow ℓ so that ℓ^* can be transported in the interdicted network $G(x^*)$. Thus, in each iteration of Algorithm 1, we seek to find an interdiction that violates the best response ℓ^* of the follower w.r.t. the current most effective interdiction x^*. Consequently, the update of the upper bound in Line 7 consists of ϕ_{LB}, which represents the load shed necessary to obtain the best responds ℓ^*, plus $\phi^{\mathrm{L}}_{\mathrm{UB}}$, which represent the load shed caused by the interdiction x regarding ℓ^* in the current iteration.

Finite termination of Algorithm 1 directly follows from the fact that the set X only contains a finite number of feasible interdiction decisions and, in each iteration, we cut off an interdiction decision in Line 11. We further note that Algorithm 1 always returns at least an ε-optimal interdiction decision. However, if $\phi_{\text{UB}} \leq \phi_{\text{LB}}$ holds, the computed interdiction is optimal since there is no further interdiction decision that can cause more load shed then the current best interdiction x^*. We can explicitly enforce this by setting $\varepsilon = 0$.

Theorem 14. *Let Assumptions 1–3 be satisfied. For a balanced load flow $\ell \in \mathbb{R}^V$ and an optimality tolerance $\varepsilon \geq 0$, Algorithm 1 terminates after a finite number of iterations. Let further (x^*, λ, q, π) be the bilevel feasible point that we obtain by solving the follower's problem (4) w.r.t. x^* and let ϕ be the corresponding objective value. The value ϕ^{opt} denotes the optimal objective value of the original interdiction problem (3). Then, one of the two cases occurs. If $\phi_{\text{LB}} > 0$, then $0 \leq (\phi^{\text{opt}} - \phi)/\phi^{\text{opt}} \leq \varepsilon$ holds. Otherwise, if $\phi_{\text{LB}} = 0$, then $0 \leq \phi^{\text{opt}} - \phi \leq \varepsilon$ holds. In addition, if $\phi_{\text{UB}} \leq \phi_{\text{LB}}$, then $\phi = \phi^{\text{opt}}$ is satisfied and (x^*, λ, q, π) is an optimal solution of the original bilevel problem (3).*

Proof. Let (x, ℓ', q', π') be an optimal solution of the original interdiction problem (3). From the construction of Algorithm 1, it directly follows $\phi^{\text{opt}} \geq \varphi = \phi_{\text{LB}}$. Further, it holds $\phi_{\text{LB}} \leq \phi_{\text{UB}}$. We now prove $\phi^{\text{opt}} \leq \phi_{\text{UB}}$. To this end, we consider a distinction of different cases.

If x is cut off by (27) in any iteration, then it directly follows $\phi^{\text{opt}} \leq \phi_{\text{LB}}$. Thus, we now consider that x is not cut off during the solution process. If algorithm terminates in Line 4, then again $\phi^{\text{opt}} \leq \phi_{\text{LB}} = \phi_{\text{UB}}$ holds since the interdiction x cannot cause any load shed for ℓ^* and ϕ_{LB} equals the load shed necessary to obtain ℓ^*. Thus, we now consider the case that the algorithm terminates due to the termination condition in Line 2. Let ℓ' be the load flow for which Problem (R1), respectively (R2), is solved in the algorithm to obtain the final upper bound ϕ_{UB}. This upper bound is the sum of two values. First, the load shed necessary to obtain ℓ' from the original load flow ℓ. Second, the maximal load shed necessary to make ℓ' feasible by scaling down for any feasible interdiction that has not been cut off in Problem (R1), respectively (R2). Consequently, for the optimal interdiction x, we can always scale down ℓ' to a feasible point of the follower's problem (4) with a load shed of ϕ_{UB}. Thus, $\phi_{\text{UB}} \geq \phi^{\text{opt}} \geq \varphi = \phi_{\text{LB}} \geq 0$ holds and the claim follows from the termination condition in Line 2. □

As explained in Remark 10, we can consider Constraint (21) instead of (19j) in the reduced single-level reformulation (R1) or (R2), which can significantly reduce the set of feasible points. This modification leads to the result that if there is no feasible interdiction that can cause any positive load shed w.r.t. ℓ^*, then Problem (R1) and (R2) becomes infeasible.

Remark 15 (Adaption of Algorithm 1)**.** *The motivation for the following variant of Algorithm 1 is that we only have to solve the single-level reformulations to optimality a few times to obtain the best response of the follower ℓ^* w.r.t. the best known interdiction x^*. In the remaining iterations, we are mainly interested in cutting off interdiction decisions that have a positive load shed regarding the best response ℓ^*.*

We now assume that the single-level reformulations are modeled according to Remark 10. We then solve the single-level reformulations to optimality only at the first and in every i-th iteration of Algorithm 1. In the remaining iterations, we only compute a feasible point for the single-level reformulation (R1) or (R2). In every iteration in which we obtain a feasible but not optimal point for (R1) or (R2), we skip the update of the upper bound in Line 7. As observed in our computational study, the proposed variant of Algorithm 1 significantly increases the performance of the algorithm.

Since the single-level reformulation is modeled according to Remark 10, almost every feasible point leads to an interdiction that causes a positive load shed w.r.t. the considered load flow. Here, "almost" means that there can be feasible points with load shed zero that at the same time satisfy Conditions (15)–(17), which is rather rare in practice. We further highlight that the adapted algorithm is only meaningful if Remark 10 is applied. Otherwise, every interdiction $x \in X$ leads to a feasible solution for the single-level models and, consequently, the adapted algorithm would only enumerate all interdiction decisions.

Finally note that Theorem 14 is still valid for the presented modification since we cut off a feasible interdiction in every iteration and only update the upper bound if (R1) or (R2) are solved to optimality.

6. Numerical Results

We now present a computational comparison of Algorithm 1 w.r.t. the single-level reformulations (R1) and (R2). To this end, we focus on gas networks, i.e., $\varphi(q) = q|q|$. Since there are no other approaches to deal with the considered network interdiction games with nonlinear and nonconvex potential-based flows, we compare our results with a simple enumeration approach.

We discuss our computational setup, the considered instances, and the implementation of our models in Sections 6.1 and 6.2. Afterward, we present our numerical results regarding the running times and number of iterations in Section 6.3. Finally, we compare the convergence of the lower and upper bounds of the considered approaches in Section 6.4.

6.1. **Implementation Details.** We briefly specify some modeling aspects of the single-level reformulations and the enumeration approach. For all models, the set of feasible interdictions $X \subseteq \{0,1\}^A$ contains a so-called budget constraint $\sum_{a \in A} x_a \le K$, which restricts the number of interdicted arcs by a given value $K \in \mathbb{N}$. We model Assumption 3, i.e., the weak-connectivity of the interdicted network, by using a standard flow problem that sends from an arbitrary node a single unit of artificial flow to each other node in the interdicted network. As a consequence of our preliminary computational tests, we implemented the single-level reformulations (R1) and (R2) according to Remark 10 and use the big-M values given in (5), respectively (26). Further, we set the optimality tolerance of Algorithm 1 to $\varepsilon = 10^{-4}$.

In pipe-only gas networks, there are typically no flow bounds since the flow through an arc is implicitly bounded by the potential bounds at the incident nodes. Thus, the given flow bounds of the tested instances are redundantly large, i.e., the flow bounds are so large such that for a given load flow, Condition (16) is never satisfied. Consequently, we remove these flow bounds, i.e., Constraint (R1.5), respectively (R2.4), and Conditions (16) modeled by (19i). Due to this and Remark 10, Constraint (19j) simplifies to $\bar{y} + \underline{y} = 0$. Further, in Model (R2) we can replace λ^* by λ^{*r} in the objective function and then apply a variable substitution $\lambda' = \lambda^{*r}$ to reduce the number of nonlinear terms.

In pipe-only potential networks, the highest potential level is always attained at a source node, i.e., $\max_{u \in V} \pi_u = \max_{u \in V_+} \pi_u$. In the considered gas networks, all upper potential bounds are equal, i.e., $\pi_u^+ = \pi_v^+$ holds for all $u, v \in V$. Due to this and Lemma 3, there always exists an optimal solution in which the maximum potential level is attained at a source node and that it equals the upper bound. We exploit this relation by adding a corresponding valid inequality to our single-level reformulations.

If we solve the considered single-level reformulation to optimality and if the corresponding interdiction decision does not lead to an improved lower bound, we solve the single-level reformulation including the additional no-good cut (27) in the next iteration. In this case, we use the optimal objective value of the previous iteration as an upper bound for the objective function of the single-level reformulation in the next iteration.

We finally discuss our implementation of the enumeration approach. In this iterative approach, we first compute a feasible interdiction $x \in X$ that satisfies Assumption 3 using a mixed-integer linear problem. Afterward, we then solve the follower's problem for fixed interdiction x and store the interdiction decision if it leads to a larger load shed then the best interdiction known so far. Then, we add the no-good cut (27) to the set X and repeat the procedure until no feasible interdiction remains.

6.2. **Computational Setup and Instances.** The models are implemented in Python 3.7 using Pyomo 6.4.2.dev0; see [8]. We solve the models using SCIP 8.0.0 [5] with Gurobi 9.0.3 [13] as LP solver and Ipopt 3.14.4 [30] as NLP solver. Additionally, we disabled PAPILO in SCIP. We note that we also tested different state-of-the-art solvers using GAMS such as BARON and ANTIGONE. However, SCIP generally performed better on the considered types of problems. The computations are carried out on a single core of a single node of a server[1] with XEON_SP_6126 CPUs. Further, we set a limit for the RAM of 16 GB and a total time limit of 24 h.

For the computations, we consider two different sets of SCIP parameters. We call the first set SCIP-default-options, which consists of the default SCIP parameters and we additionally set the time limit to 500 s as well as the relative optimality gap to 10^{-4}. The second set of parameters is called SCIP-feasibility-options and adds the predefined set of parameters given by SCIP settings "set emphasize feasibility" to the first set SCIP-default-options.

Based on preliminary tests, we use SCIP-feasibility-options for the single-level reformulation (R1) and SCIP-default-options for (R2). If we find a feasible point within the time limit of 500 s but do not prove its optimality, then we proceed as in the adapted algorithm, see Remark 15, i.e., we proceed without updating the upper bound. If no feasible solution can be found within the time limit, then we solve the problem with the other set of SCIP options. If this still does not lead to a feasible solution, we double the time limit and repeat the procedure. In practice, restarting the solution process with a different set of SCIP parameters has been more efficient than directly increasing the time limit. This is based on the observation that the solver can get stuck at specific steps in the solution process, which in some cases can be circumvented by considering different SCIP parameters. If we use the adapted algorithm 1 described in Remark 15, then we set the time limit of each iteration that we solve to optimality directly to 1000 s.

For solving the original follower's problem (4), we use SCIP-default-options and set the time limit to the remaining time of the total time limit of 24 h.

In case that we only search for a feasible point in our approaches, we stop the solution process after obtaining the first feasible point and always start the above described procedure using SCIP-feasibility-options.

For our computational study, we consider the GasLib-40 network [27]. The GasLib-40 consists of 40 nodes, out of which 3 are sources and 29 are sinks. Moreover, the network contains 39 pipes and we bypass the 6 compressors in the network since we consider potential-based flows without controllable elements. We further consider two different load flows $\ell \in \mathbb{R}^V$ that represent the injections and withdrawals in the non-interdicted network at a single point in time. First, we consider the load flow, given in [27], and slightly scale each of the injections and withdrawals by 0.9991. The latter scaling is necessary to obtain a feasible load flow in the non-interdicted network since we bypass the compressors. We call this instance GasLib-40-scaled. As the

[1] https://elwe.rhrk.uni-kl.de/elwetritsch/hardware.shtml

second load flow, we perturb the scaled load flow by randomly sampling the demand of the sinks. For every sink, we sample an integer demand in $[0, 200] \times 1000\,\mathrm{m}^3\,\mathrm{h}^{-1}$ using a fixed seed of zero. Further, the demand of the sources stays unchanged. To guarantee that the load flow is balanced, we scale each of these demands of the exits by the total amount of injections divided by the total amount of the sampled exits. We call this instance GasLib-40-scaled-randomExits.

6.3. **Running Times and Number of Iterations.** We now compare Algorithm 1 w.r.t. both single-level reformulations (R1) and (R2) and the enumeration approach. Afterward, we analyze the adaption of Algorithm 1 given in Remark 15, which significantly improves the running times.

In Table 1, we summarize the running times and number of iterations for both considered instances and different interdiction budgets $K \in \{1, \ldots, 5\}$. To this end, we solve in each iteration the corresponding single-level reformulation (R1), respectively (R2), to optimality. From the results, it follows that for all instances Algorithm 1 needs significantly less iterations compared the enumeration approach. However, in each of these iterations, we have to solve an MINLP within Algorithm 1, whereas the enumeration approach solves an NLP in each iteration. We note that current state-of-the-art solvers are really challenged by solving the MINLPs (R1) and (R2) to global optimality and often struggle to prove optimality in reasonable time. This is the reason why for some of the instances the approaches based on Algorithm 1 are outperformed regarding the running times by the enumeration approach despite the significantly less number of iterations.

However, as stated in Remark 15, it is not necessary to always solve the single-level reformulations (R1), respectively (R2), to optimality in each iteration of Algorithm 1 to obtain an optimal interdiction decision. Thus, we also applied Algorithm 1 with the adaption that we only solve the corresponding MINLP to optimality in the first and in each 50th iteration. In the remaining iterations, we only compute a feasible point. This adaption drastically improves the performance of Algorithm 1 as the running times in Table 2 show. Regarding the running times, the adapted Algorithm 1 clearly outperforms the enumeration approach except for the smallest interdiction budget $K = 1$. Further, for $K = 5$, the approaches based on the adapted Algorithm 1 solve all instances in the time limit, which is not the case for the enumeration approach.

As discussed previously, the number of iterations is again significantly lower for the adapted algorithm compared to the enumeration approach. We note that the difference of the number of iterations between the Algorithm 1 and its adapted version can generally be traced back to the fact that if we do not solve the MINLPs to optimality, we obtain different interdiction decisions. Consequently, we may update the best response of the follower, i.e., ℓ^* in Algorithm 1, at a different iteration, which affects the feasible region of the considered MINLPs. In general, it is theoretically not guaranteed that solving the MINLPs to optimality always leads to a smaller number of iterations. However, we observe this effect in practice.

We finally note that using the single-level reformulation (R2) tends to be faster over all instances compared to using (R1) except for two instances. However, we also observe that solving the MINLP (R2) is more prone to numerical difficulties. This leads to the result that in some iterations of Algorithm 1, we cannot compute a feasible point and, thus, we reach the time limit; see, e.g., Table 1 for $K = 4$ and instance GasLib-40-scaled-randomExits. These numerical difficulties also explain the small difference in the number of iterations for (R1) and (R2) in Table 1 with $K = 4$ and instance GasLib-40-scaled. The numerical difficulties most likely arise since the used variable substitution in Model (R2) significantly increases specific big-M values as discussed in Section 4.2. In general, the single-level reformulation (R1) seems to

Table 1. Running times (in seconds) and number of iterations of Algorithm 1 w.r.t. (R1), respectively (R2), and the enumeration approach. The single-level reformulation (R1), respectively (R2), is solved to optimality in each iteration of Algorithm 1.

	$K = 1$		$K = 2$		$K = 3$		$K = 4$		$K = 5$	
	time	#iter	time	#iter	time	#iter	time	#iter	time	#iter
enum.	33.2	26	268.1	270	1760.1	1619	14 237.1	5893	—	—
(R1)	57.0	2	172.9	3	1167.8	11	18 202.0	152	—	—
(R2)	31.2	2	107.1	3	701.8	11	17 583.6	155	72 505.0	725
GasLib-40-scaled-randomExits										
enum.	16.4	26	195.2	270	1673.8	1619	13 883.7	5893	86 125.6	13 221
(R1)	87.6	3	86.7	2	3013.0	24	24 299.6	185	—	—
(R2)	46.6	3	108.9	2	1375.3	24	—	—	—	—

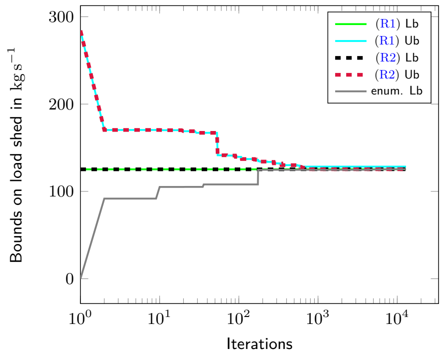
Table 2. Running times (in seconds) and number of iterations of the adapted Algorithm 1 w.r.t. (R1), respectively (R2), and the enumeration approach. The single-level reformulation (R1), respectively (R2) is solved to optimality in the first and in each 50th iteration of Algorithm 1.

	$K = 1$		$K = 2$		$K = 3$		$K = 4$		$K = 5$	
	time	#iter	time	#iter	time	#iter	time	#iter	time	#iter
enum.	33.2	26	268.1	270	1760.1	1619	14 237.1	5893	—	—
(R1)	53.5	2	122.4	3	533.2	11	4938.1	162	36 468.7	784
(R2)	37.8	2	89.4	3	365.3	12	3976.1	161	21 256.1	912
GasLib-40-scaled-randomExits										
enum.	16.4	26	195.2	270	1673.8	1619	13 883.7	5893	86 125.6	13 221
(R1)	76.3	3	96.3	2	946.5	24	7922.4	194	46 487.9	855
(R2)	42.6	3	79.3	2	760.7	24	5703.4	194	37 273.9	876

be numerically more stable, but the solver needs more time to find a feasible point or an optimal solution.

6.4. **Convergence of Lower and Upper Bounds.** We now compare the convergence of the lower and upper bound for the load shed if applying the approaches based on Algorithm 1. In addition, we consider the development of the corresponding lower bound of the enumeration approach. However, we stress that the convergence of this lower bound of the enumeration approach can be subject to strong fluctuations since there is no criterion on how to choose the next feasible interdiction. We further do not obtain an upper bound for the load shed if applying the enumeration approach.

In Figure 2, we compare the lower and upper bounds for the load shed of the instance GasLib-40-scaled with interdiction budget $K = 5$. We observe that both variants of Algorithm 1 compute a nearly optimal interdiction already in the first iteration. The optimal interdiction decision is found in later iterations, but the difference regarding the load shed compared to the first iteration is smaller than $0.1\,\mathrm{kg\,s^{-1}}$. Both approaches tighten the upper bound to a relative small gap in a small number of iterations. As expected, the upper bound obtained by the adapted algorithm decreases much slower compared to the original version of the algorithm. This is based on the fact that in each iteration, in which we only compute a feasible point, we do not obtain a valid upper bound and, thus, cannot update the best known upper bound for the load shed. However, the adapted algorithm is still superior in terms of total running times. Optimality for the considered instance can only be formally proven in the approaches based on the adapted algorithm and in



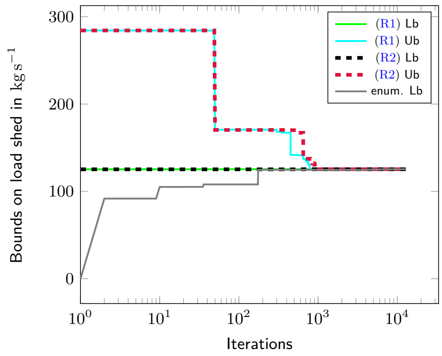


FIGURE 2. GasLib-40-scaled for budget $K = 5$. Left: Single-level reformulations solved to optimality in each iteration. Right: Problems solved to optimality in the first and in each 50th iteration.

Algorithm 1 w.r.t. (R2). The enumeration approach fails to prove the optimality in this case. For the remaining instance, we can observe a similar development of the lower and upper bounds as in Figure 2.

Over all considered instances, we observe that applying Algorithm 1 yields to a convergence of the lower and upper bound for the load shed within a relatively small number of iterations. Moreover, the load shed corresponding to the best interdiction decision of the very first iterations is almost always close to the optimal one.

Beside the faster running times and smaller number of iterations, two major benefits of our approaches based on Algorithm 1 are the following. First, these approaches provide a lower and upper bound for the maximum load shed in each iteration. Consequently, we obtain an optimality gap, which is not the case for the enumeration approach. Second, we can control the trade-off between fast total running times and obtaining tight upper bounds within a small number of iterations in the adapted algorithm by choosing which iterations should be solved to optimality. This makes Algorithm 1 especially useful to compute a good, but not necessarily optimal, interdiction decision within a small time limit.

7. CONCLUSION

We studied a network flow interdiction problem with nonlinear and nonconvex potential-based flows. We developed an exact algorithm that exploits upper and lower bounding schemes to compute optimal interdictions. Finally, the applicability of the developed approach is demonstrated using the example of gas networks.

This paper paves the way for different directions of future research. First, extending the considered potential-based flow model to include controllable elements, e.g., compressors in gas networks or pumps in water networks, can be a next step for future research. However, introducing these controllable elements adds challenging integer variables yielding MINLPs in the lower level. Second, it is of interest to develop approaches that can also handle the case that the interdicted network is not connected. We believe that it is possible to extend the presented approach to this case at the cost of further binary variables in the upper level to model the connected components. Third and finally, moving forward to trilevel defender-attacker-defender games subject to potential-based flows is a challenging, but very interesting, topic for future research.

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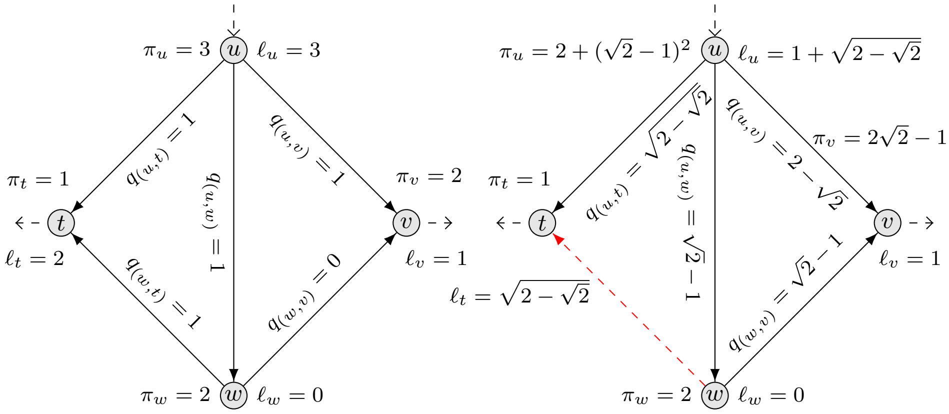
Appendix A. Why Potential-Based Flows are Difficult—Even without Flow Bounds

We now modify the example of Section 3 by dismissing arc flow bounds. Again, we demonstrate that interdicting an arc can cause a load shed that exceeds the absolute flow through this arc in the non-interdicted network.

The example in Figure 3 illustrates that interdicting arc (w, t) with absolute flow of 1 leads to a total load shed of $2 - \sqrt{2 - \sqrt{2}} > 1$. Consequently, the load shed exceeds the absolute flow trough the interdicted arc in the non-interdicted network. More precisely, after interdicting arc (w, t), from flow conservation (4b) and Constraints (4d), we obtain

$$q_{(u,w)} = q_{(w,v)}, \quad q^2_{(u,w)} + q^2_{(w,v)} = q^2_{(u,v)}, \quad q_{(u,v)} = \ell_v - q_{(u,w)}.$$

From these equations and $\ell_v \in [0, 1]$, it follows that for $q_{(u,w)} = \sqrt{2} - 1 = q_{(w,v)}$, $q_{(u,v)} = 2-\sqrt{2}$, and $\pi_u = 2+(\sqrt{2}-1)^2$, it is possible to avoid any load shed at node v, i.e., $\ell_v = 1$. We note that the requirement $\pi_w \leq 2$ and Constraints (4d) imply that $\pi_u = 2 + (\sqrt{2} - 1)^2$ is the maximal potential level at node u. However, then it is necessary to shed load at node t to satisfy $\pi_u - 2q^2_{(u,t)} \geq 1 = \pi^-_t$, which leads to $q_{(u,t)} \leq \sqrt{2 - \sqrt{2}} = \ell_t$. Let us further remark, that any other solution that causes



$$\varphi(q) = q|q|,\ \pi_i^+ = 3,\ \pi_i^- = 1,\ i \in V \setminus \{w\},\ \pi_w^+ = 2,\ \pi_w^- = 1,$$
$$q^- = -\infty,\ q_a^+ = \infty,\ a \in A, \quad \Lambda_a = 1, a \in A \setminus \{(u,t)\},\ \Lambda_{(u,t)} = 2.$$

FIGURE 3. An example where interdicting the single arc (w,t) causes a load shed larger than the original absolute flow through arc (w,t).

a load shed at node v, i.e., $\ell_v < 1$, also decreases the pressure level $\pi_u = 2+(\sqrt{2}-1)^2$, which then leads to an even larger load shed at node t. Thus, the presented solution is optimal, i.e., the load shed is minimal.

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