# DSLs of Mathematics: limit of functions

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# Course goal and focus

## Goal

Encourage students to approach mathematical domains
from a functional programming perspective.

## Course focus

- Make functions and types explicit
- Explicit distinction between syntax and semantics
- Types as carriers of semantic information
- Organize the types and functions in DSLs

Now Make variable binding and scope explicit

Lecture notes and more available at: `https://github.com/DSLsofMath/DSLsofMath`

# Example: The limit of a function

*We say that $f(x)$ **approaches the limit** $L$ as $x$ **approaches** $a$, and we write*

$$\lim_{x \to a} f(x) = L,$$

*if the following condition is satisfied:*
*for every number $\varepsilon > 0$ there exists a number $\delta > 0$, possibly depending on $\varepsilon$, such that if $0 < |x - a| < \delta$, then $x$ belongs to the domain of $f$ and*

$$|f(x) - L| < \varepsilon$$

- Adams & Essex, Calculus - A Complete Course

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*if*

$$\forall \varepsilon > 0$$

$$\exists \delta > 0$$

*such that if*

$$0 < |x - a| < \delta,$$

*then*

$$x \in \mathit{Dom}\, f \wedge |f(x) - L| < \varepsilon$$

# Limit of a function – continued

First attempt at translation:

$$
\begin{aligned}
&\mathit{lim}\ a\ f\ L = \forall\ \epsilon > 0.\ \ \exists\ \delta > 0.\ \ P\ \epsilon\ \delta \\
&\quad \text{where } P\ \epsilon\ \delta = (0 < |x - a| < \delta) \Rightarrow \\
&\qquad\qquad\qquad (x \in \mathit{Dom}\ f \wedge |f\ x - L| < \epsilon)
\end{aligned}
$$

Finally (after adding a binding for $x$):

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Lesson learned: be careful with scope and binding (of $x$ in this case).

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[We will now assume limits exist and use *lim* as a function from $a$ and $f$ to $L$.]

# Example 2: derivative

*The **derivative** of a function $f$ is another function $f'$ defined by*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

*at all points $x$ for which the limit exists (i.e., is a finite real number). If $f'(x)$ exists, we say that $f$ is **differentiable** at $x$.*

We can write

$D\ f\ x = \mathit{lim}\ 0\ g$ where $g\ h = \frac{f\ (x+h) - f\ x}{h}$

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$$D\ f \quad = \mathit{lim}\ 0 \circ \psi\ f\ \text{ where } \psi\ f\ \ x\ \ h = \frac{f\ (x+h) - f\ x}{h}$$

$$\begin{tikzcd}
& \mathbb{R} \arrow[dl, "D\ f"'] \arrow[d, "\psi\ f"] \\
\mathbb{R} & (\mathbb{R} \to \mathbb{R}) \arrow[l, "\mathit{lim}\ 0"]
\end{tikzcd}$$

# Derivatives, cont.

Examples:

$$
\begin{aligned}
&D : (\mathbb{R} \to \mathbb{R}) \to (\mathbb{R} \to \mathbb{R}) \\
&sq\ x \quad\ \ = x\hat{}2 \\
&double\ x = 2 * x \\
&c_2\ x \quad\ \ = 2 \\
&sq' \ \texttt{==}\ D\ sq \ \ \texttt{==}\ D\ (\lambda x \to x\hat{}2)\ \texttt{==}\ D\ (\hat{}2)\ \texttt{==}\ (2*)\ \texttt{==}\ double \\
&sq'' \ \texttt{==}\ D\ sq' \ \texttt{==}\ D\ double\ \texttt{==}\ c_2\ \texttt{==}\ const\ 2
\end{aligned}
$$

Note: we cannot *implement* $D$ (of this type) in Haskell.
Given only $f : \mathbb{R} \to \mathbb{R}$ as a "black box" we cannot compute the actual derivative $f' : \mathbb{R} \to \mathbb{R}$.
We need the "source code" of $f$ to apply rules from calculus.