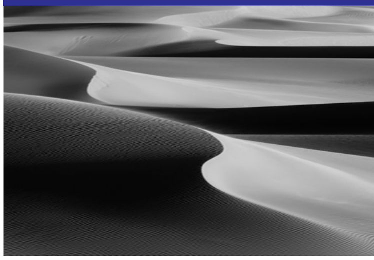
APPENDIX I



Complex Numbers

> Old Macdonald had a farm,
> Minus E-squared O

a mathematically simplified children's song

Many of the problems to which mathematics is applied involve the solution of equations. Over the centuries the number system had to be expanded many times to provide solutions for more and more kinds of equations. The natural numbers

$$\mathbb{N} = \{1,\ 2,\ 3,\ 4,\ \ldots\}$$

are inadequate for the solutions of equations of the form

$$x + n = m, \qquad (m,\ n \in \mathbb{N}).$$

Zero and negative numbers can be added to create the integers

$$\mathbb{Z} = \{\ldots,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ \ldots\}$$

in which that equation has the solution $x = m - n$ even if $m < n$. (Historically, this extension of the number system came much later than some of those mentioned below.) Some equations of the form

$$nx = m, \qquad (m,\ n \in \mathbb{Z}, \quad n \neq 0),$$

cannot be solved in the integers. Another extension is made to include numbers of the form m/n, thus producing the set of rational numbers

$$\mathbb{Q} = \left\{ \frac{m}{n} \ :\ m,\ n \in \mathbb{Z}, \quad n \neq 0 \right\}.$$

Every linear equation

$$ax = b, \qquad (a,\ b \in \mathbb{Q}, \quad a \neq 0),$$

has a solution $x = b/a$ in $\mathbb{Q}$, but the quadratic equation

$$x^2 = 2$$

has no solution in $\mathbb{Q}$, as was shown in Section P.1. Another extension enriches the rational numbers to the real numbers $\mathbb{R}$ in which some equations like $x^2 = 2$ have solutions. However, other quadratic equations, for instance,

$$x^2 = -1$$

do not have solutions, even in the real numbers, so the extension process is not complete. In order to be able to solve any quadratic equation, we need to extend the real number system to a larger set, which we call the **complex number system**. In this appendix we will define complex numbers and develop some of their basic properties.

Definition of Complex Numbers

We begin by defining the symbol i, called **the imaginary unit,**[1] to have the property

$$i^2 = -1.$$

Thus, we could also call i the **square root of** -1 and denote it $\sqrt{-1}$. Of course, i is not a real number; no real number has a negative square.

DEFINITION 1

A **complex number** is an expression of the form

$$a + bi \qquad \text{or} \qquad a + ib,$$

where a and b are *real numbers*, and i is the imaginary unit.

For example, $3+2i$, $\frac{7}{2}-\frac{2}{3}i$, $i\pi = 0+i\pi$, and $-3 = -3+0i$ are all complex numbers. The last of these examples shows that every real number can be regarded as a complex number. (We will normally use $a + bi$ unless b is a complicated expression, in which case we will write $a + ib$ instead. Either form is acceptable.)

It is often convenient to represent a complex number by a single letter; w and z are frequently used for this purpose. If a, b, x, and y are real numbers, and

$$w = a + bi \qquad \text{and} \qquad z = x + yi,$$

then we can refer to the complex numbers w and z. Note that $w = z$ if and only if $a = x$ and $b = y$. Of special importance are the complex numbers

$$0 = 0 + 0i, \qquad 1 = 1 + 0i, \qquad \text{and} \qquad i = 0 + 1i.$$

DEFINITION 2

If $z = x + yi$ is a complex number (where x and y are real), we call x the **real part** of z and denote it $\mathrm{Re}(z)$. We call y the **imaginary part** of z and denote it $\mathrm{Im}(z)$:

$$\mathrm{Re}(z) = \mathrm{Re}(x + yi) = x, \qquad \mathrm{Im}(z) = \mathrm{Im}(x + yi) = y.$$

Note that both the real and imaginary parts of a complex number are real numbers:

$$\begin{aligned} &\mathrm{Re}(3-5i) = 3 && \mathrm{Im}(3-5i) = -5 \\ &\mathrm{Re}(2i) = \mathrm{Re}(0+2i) = 0 && \mathrm{Im}(2i) = \mathrm{Im}(0+2i) = 2 \\ &\mathrm{Re}(-7) = \mathrm{Re}(-7+0i) = -7 && \mathrm{Im}(-7) = \mathrm{Im}(-7+0i) = 0. \end{aligned}$$

Graphical Representation of Complex Numbers

Since complex numbers are constructed from pairs of real numbers (their real and imaginary parts), it is natural to represent complex numbers graphically as points in a Cartesian plane. We use the point with coordinates (a, b) to represent the complex number $w = a + ib$. In particular, the origin $(0, 0)$ represents the complex number 0, the point $(1, 0)$ represents the complex number $1 = 1 + 0i$, and the point $(0, 1)$ represents the point $i = 0 + 1i$. (See Figure I.1.)

[1] In some fields, for example, electrical engineering, the imaginary unit is denoted j instead of i. Like "negative," "surd," and "irrational," the term "imaginary" suggests the distrust that greeted the new kinds of numbers when they were first introduced.