Homomorphisms

Patrik Jansson

Example:

$$
\begin{aligned}
&exp : \mathbb{R} && \to \mathbb{R}_{>0} \\
&exp \;\; 0 && = 1 && \text{-- } e^0 = 1 \\
&exp \;\; (a + b) && = exp\ a * exp\ b && \text{-- } e^{a+b} = e^a e^b
\end{aligned}
$$

From: $\qquad\qquad\qquad\qquad\qquad$ To:

$$
\begin{tikzcd}
(\mathbb{R}, (+), 0) \arrow[r, bend left, "exp"] & (\mathbb{R}_{>0}, (*), 1) \\
(A, \oplus, z) \arrow[r, "h"] & (B, \otimes, u)
\end{tikzcd}
$$

$$
\begin{aligned}
\forall x, y : A.\quad & h\ (x \oplus y) = (h\ x) \otimes (h\ y) \\
& h \quad z \quad = \quad u
\end{aligned}
$$

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Homomorphisms

Types

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Example:

$$
\begin{aligned}
&exp : \mathbb{R} && \to \mathbb{R}_{>0} && \\
&exp\ \ 0 && = 1 && \text{-- } e^0 = 1 \\
&exp\ \ (a+b) && = exp\ a * exp\ b && \text{-- } e^{a+b} = e^a e^b
\end{aligned}
$$

$$(A, \oplus, z) \xrightarrow{h} (B, \otimes, u)$$

$$\forall x, y : A.\ h(x \oplus y) = (h\,x) \otimes (h\,y)$$

$$h\ \ z = u$$

$x : A$, $y : A$

$x \oplus y : A$

$z : A$

$\longmapsto$

$h\,x : B$, $h\,y : B$

$h(x \oplus y) : B$

$h\,z : B$

$(h\,x) \otimes (h\,y) : B$

$u : B$

Homomorphisms

Types

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Example:

$$
\begin{aligned}
&exp : \mathbb{R} &&\to \mathbb{R}_{>0} \\
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\end{aligned}
$$

$$(A, \oplus, z) \xrightarrow{h} (B, \otimes, u)$$

$$H_2 : (A \to B) \times (A \to A \to A) \times (B \to B \to B) \to Prop$$

$$H_2(h, \oplus, \otimes) \stackrel{def.}{=} \forall x, y : A.\ h(x \oplus y) == (h\,x) \otimes (h\,y)$$

We know $H_2(exp, +, *)$ & $H_0(exp, 0, 1)$

$$H_0(h, z, u) = \quad h\,z == u$$

Homomorphisms

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Example:

$$
\begin{aligned}
&exp : \mathbb{R} && \rightarrow \mathbb{R}_{>0} && \\
&exp\ \ 0 && = 1 && \text{-- } e^0 = 1 \\
&exp\ \ (a+b) && = exp\ a * exp\ b && \text{-- } e^{a+b} = e^a e^b
\end{aligned}
$$

From

$$
\begin{tikzcd}
(\mathbb{R}, (+), 0) \arrow[r, bend left, "exp"] & (\mathbb{R}_{>0}, (*), 1) \arrow[l, bend left, "log"]
\end{tikzcd}
$$

$$
\begin{aligned}
&log : \mathbb{R}_{>0} && \rightarrow \mathbb{R} && \\
&log\ \ 1 && = 0 && \text{-- } \log 1 = 0 \\
&log\ \ (a * b) && = log\ a + log\ b && \text{-- } \log(ab) = \log a + \log b
\end{aligned}
$$

Homomorphisms

Types

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$$H_2 : (A \to B) \times (A \to A \to A) \times (B \to B \to B) \to Prop$$

$$H_2(h, \oplus, \otimes) \stackrel{\text{def.}}{=} \forall x, y : A.\ h\,(x \oplus y) == (h\,x) \otimes (h\,y)$$

$$H_1 : (A \to B) \times (A \to A) \times (B \to B) \to Prop$$

$$H_1(h, fa, fb) = \forall x : A.\ h\,(fa\ x) == fb\,(h\ x)$$

$$H_0 : (A \to B) \times A \times B \to Prop$$

$$H_0(h, a, b) = h\ a == b$$

Homomorphisms Predicates

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$H_2(h, opa, opb) = \forall x{:}A.\ \forall y{:}A.\ h\ (opa\ x\ y) == opb\ (h\ x)\ (h\ y)$

$H_1(h, fa, fb) = \forall x{:}A.\ h\ (fa\ x) == fb\ (h\ x)$

$H_0(h, a, b) = h\ a == b$

$h : A \to B$

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Homomorphisms

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$\exists \otimes.\ H_2(odd, +, \otimes)$?

$odd : A \to B$

$\mathbb{Z} \to \mathbb{B}$

$(+) : \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}$

$(\otimes) : \mathbb{B} \to \mathbb{B} \to \mathbb{B}$

Homomorphisms

Patrik Jansson

$\exists \otimes.\ H_2(odd, +, \otimes)$?

$odd : \mathbb{N} \to \mathbb{B}$

$+ : \mathbb{N} \to \mathbb{N} \to \mathbb{N}$

$\otimes : \mathbb{B} \to \mathbb{B} \to \mathbb{B}$

Homomorphisms

Patrik Jansson

$\exists \otimes.\ H_2(odd, +, \otimes)$

$odd : \mathbb{N} \to \mathbb{B}$

$+ : \mathbb{N} \to \mathbb{N} \to \mathbb{N}$

$\otimes : \mathbb{B} \to \mathbb{B} \to \mathbb{B}$

$\exists \otimes.\ \forall x, y : \mathbb{N}.\ odd(x+y) == odd\,x \otimes odd\,y$

?

$\otimes$		$y=0$: F	$y=1$: T
$x=0$	F	F	T
$x=1$	T	T	F

$0+1=1$

$0+0$

$\downarrow$

$odd(0) = F$

$odd(1+1) = F$

Homomorphisms

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$$\exists \otimes.\ H_2(odd, +, \otimes)$$

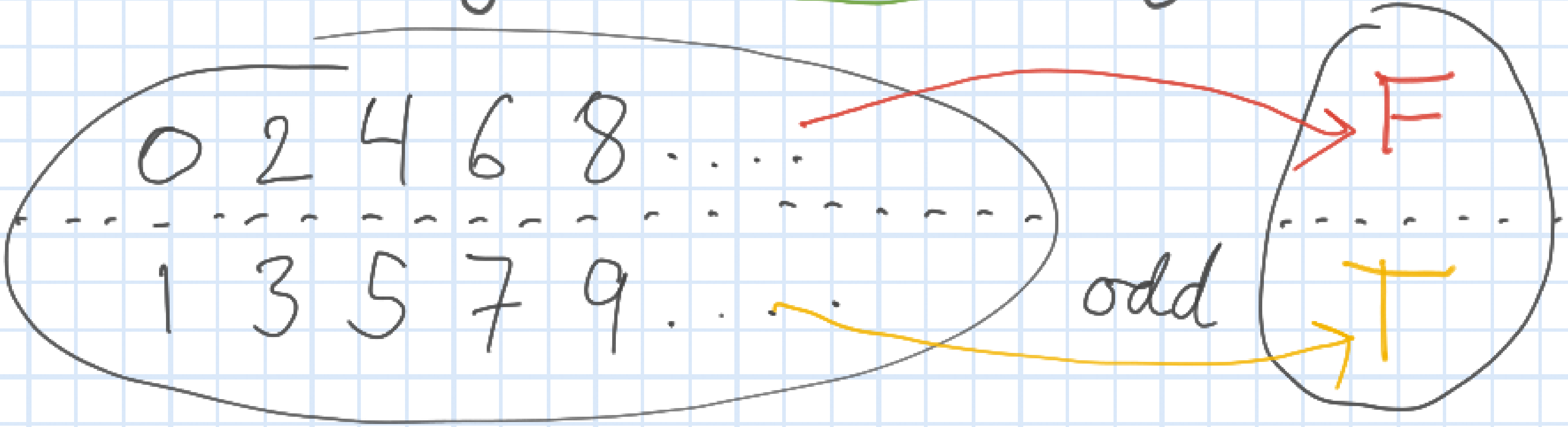
$odd : \mathbb{N} \to \mathbb{B}$

$+ : \mathbb{N} \to \mathbb{N} \to \mathbb{N}$

$\otimes : \mathbb{B} \to \mathbb{B} \to \mathbb{B}$

$$\exists \otimes.\ \forall x, y : \mathbb{N}.\ \boxed{odd(x+y)} == odd\ x \otimes odd\ y$$

	? $\otimes$	F ($y=2$)	T ($y=3$)
$x=2$	F	F	T
$x=3$	T	T	F



Homomorphisms

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$$\exists \otimes.\ H_2(odd, +, \otimes)$$

$odd : \mathbb{N} \to \mathbb{B}$

$+ : \mathbb{N} \to \mathbb{N} \to \mathbb{N}$

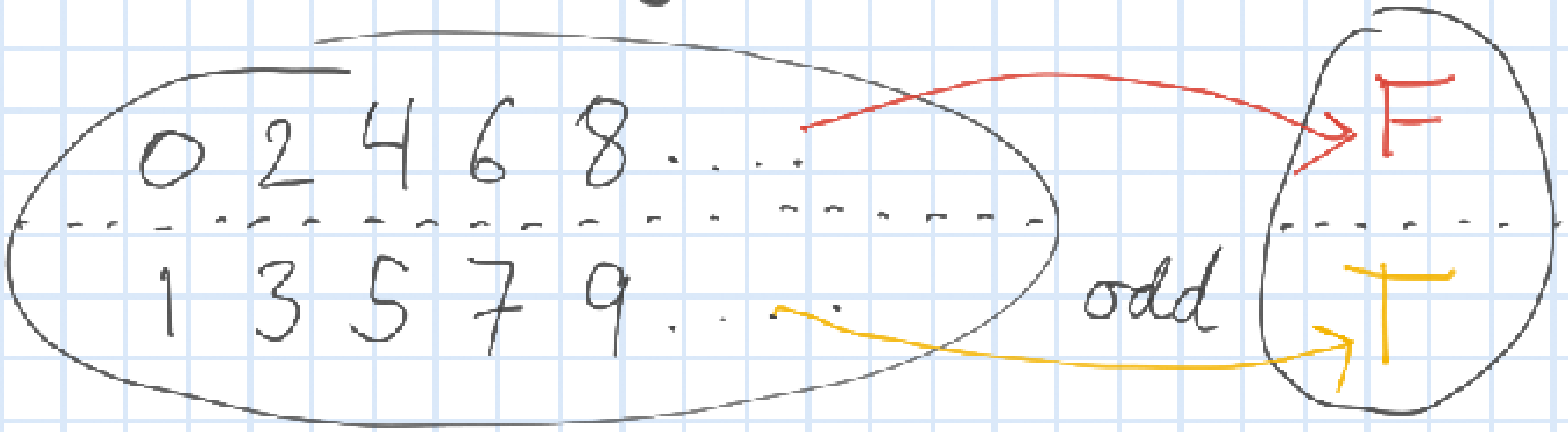
$\otimes : \mathbb{B} \to \mathbb{B} \to \mathbb{B}$

?

xor

$\otimes$	F	T
F	F	T
T	T	F

$$\exists \otimes.\ \forall x, y : \mathbb{N}.\ \boxed{odd(x+y)} == odd\ x \otimes odd\ y$$



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Homomorphisms

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Proof

$$\mathrm{odd}(x+y) = \mathrm{odd}\,x \otimes \mathrm{odd}\,y$$

$$\exists \otimes.\ H_2(\mathrm{odd}, +, \otimes)$$

Proof: We know $\forall n:\mathbb{N}.\ \exists k:\mathbb{N}.\ \exists b:\mathbb{Z}_2.\ n = 2\cdot k + b$

Thus $x = 2\cdot k_x + b_x \quad \wedge \quad y = 2\cdot k_y + b_y$

$\mathrm{odd}\,x = \mathrm{odd}\,b_x \quad \wedge \quad \mathrm{odd}\,y = \mathrm{odd}\,b_y$

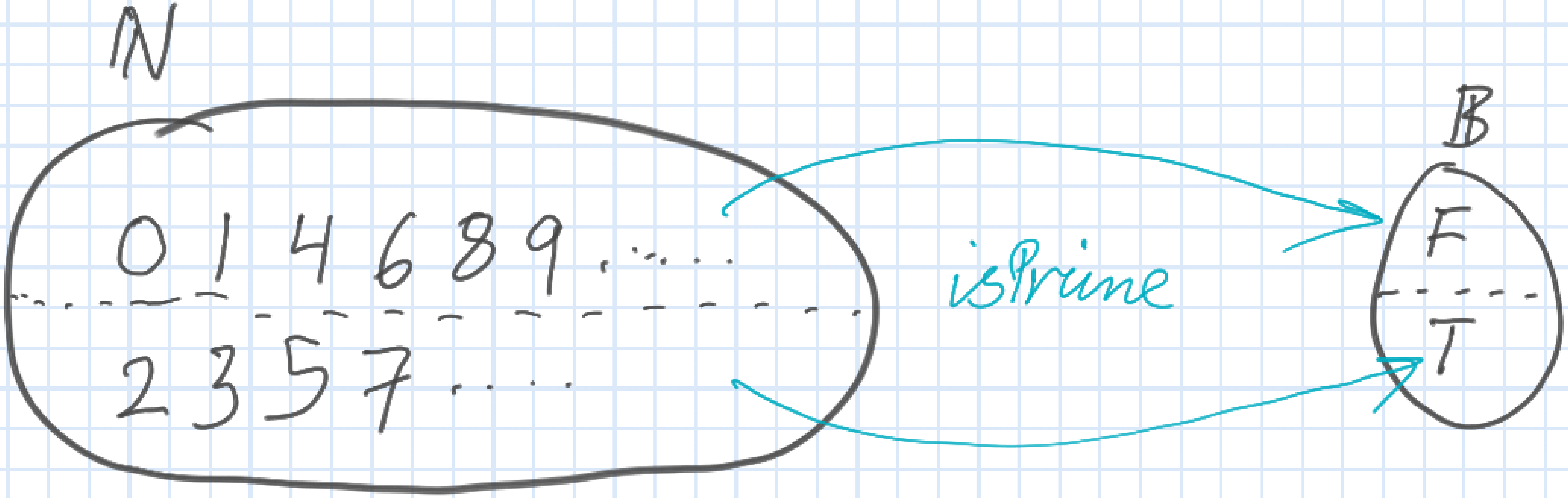
$x + y = 2\cdot(k_x + k_y) + (b_x + b_y)$

$\mathrm{odd}(x+y) = \mathrm{odd}(b_x + b_y)$

Check the four cases:

b_x	$b_y = 0$ or	$b_y = 1$
0	F == F	T == T
1	T == T	F == F

$\exists \otimes.\ H_2(\mathit{isPrime}, +, \otimes)$?



$\exists \otimes.\ H_2(\text{isPrime}, +, \otimes)$?

Proof by contradiction: Assume $\otimes : \mathbb{B} \to \mathbb{B} \to \mathbb{B}$ exists.

$$\forall x, y : \mathbb{N}.\ \text{isP}(x+y) == \text{isP}\,x \otimes \text{isP}\,y$$

$x = 2,\ y = 2 \Rightarrow x + y = 4$

$\text{isP}(4) = F$

$\text{isP}\,x = T$

$\text{isP}\,y = T$

$x = 2 \mapsto T$

$y = 3 \mapsto T$

$\text{isP}(2+3) = \text{isP}\,5 = T$

$\otimes$		T
T		$F = T$

$\exists \otimes.\ H_2(\mathrm{isPrime}, +, \otimes)$?

Proof by contradiction: Assume $\otimes : \mathbb{B} \to \mathbb{B} \to \mathbb{B}$ exists.

$\mathrm{isPrime}\ 2 = T$ $\qquad 2+2=4 \mapsto T \otimes T = F$

$\mathrm{isPrime}\ 3 = T$ $\qquad 2+3=5 \mapsto T \otimes T = T$

$\mathrm{isPrime}\ 4 = F$

$\mathrm{isPrime}\ 5 = T$

contradiction

Thus no such $\otimes$ can exist.