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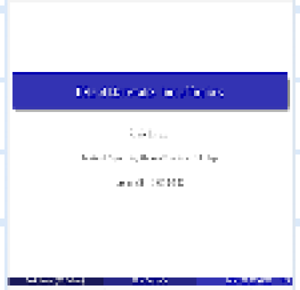

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DSLs of Mathematics: limit of functions

Patrik Jansson

Functional Programming, Chalmers University of Technology

Lecture 3.1, 2021-02-02

Course goal and focus

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Goal

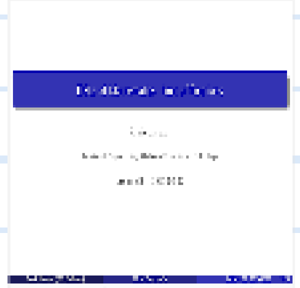
Encourage students to approach mathematical domains from a functional programming perspective.

Course focus

- Make functions and types explicit
- Explicit distinction between syntax and semantics
- Types as carriers of semantic information
- Organize the types and functions in DSLs

Now Make variable binding and scope explicit

Lecture notes and more available at: `https://github.com/DSLsofMath/DSLsofMath`



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Example: The limit of a function

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*We say that $f(x)$ **approaches the limit** L as x **approaches** a, and we write*

$$\lim_{x \to a} f(x) = L, \qquad lim(f, a, L)$$

if the following condition is satisfied:
for every number $\varepsilon > 0$ there exists a number $\delta > 0$, possibly depending on ε, such that if $0 < |x - a| < \delta$, then x belongs to the domain of f and

$$|f(x) - L| < \varepsilon$$

- Adams & Essex, Calculus - A Complete Course

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Limit of a function – continued

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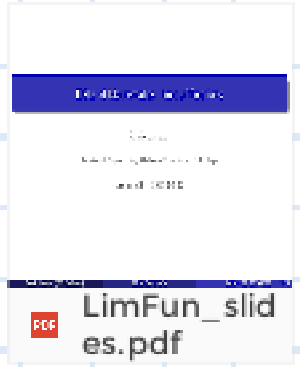

$$\lim_{x \to a} f(x) = L,$$

if

$$\forall \varepsilon > 0$$

$$\exists \delta > 0$$

such that if

$$0 < |x - a| < \delta,$$

then

$$x \in \mathit{Dom}\, f \wedge |f(x) - L| < \varepsilon$$

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First attempt at translation:

$$\begin{aligned} lim\ a\ f\ L &= \forall\ \epsilon > 0.\ \ \exists\ \delta > 0.\ \ P\ \epsilon\ \delta \\ \text{where } P\ \epsilon\ \delta &= (0 < |x - a| < \delta) \Rightarrow \\ &\quad (x \in Dom\ f \wedge |f\ x - L| < \epsilon) \end{aligned}$$

Scope check:

a, f, L — parameters to lim

ε — $\forall$-bound

δ — $\exists$-bound

x — ?

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Limit of a function – continued

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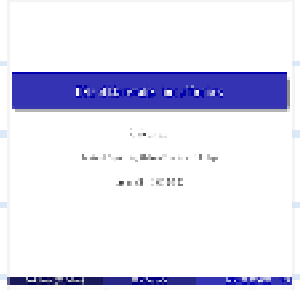
Finally (after adding a binding for x):

$$
\begin{aligned}
&\mathit{lim}\ a\ f\ L = \forall\ \epsilon > 0.\ \ \exists\ \delta > 0.\ \ P\ \epsilon\ \delta \\
&\quad \text{where } P\ \epsilon\ \delta = \quad \forall\ x.\ \ Q\ \epsilon\ \delta\ x \\
&\qquad\qquad Q\ \epsilon\ \delta\ x = (0 < |x - a| < \delta) \Rightarrow \\
&\qquad\qquad\qquad\qquad (x \in \mathit{Dom}\ f \wedge |f\ x - L| < \epsilon)
\end{aligned}
$$



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Limit of a function – continued

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Finally (after adding a binding for x):

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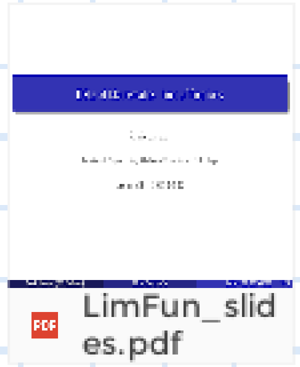
\end{aligned}
$$

Lesson learned: be careful with scope and binding (of x in this case).



lim properties

$$\lim\ a\ f\ L_1$$
$$\land\ \lim\ a\ f\ L_2$$
$$\Rightarrow\ L_1 = L_2$$

Thus lim can be used as a partial function from a and f.

lim typing

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$$\lim : X \to (X \to Y) \to Y \to Prop$$

or

$$\lim : X \to (X \to Y) \to Maybe\ Y$$

or

$$\lim : X \to (X \to Y) \to Y \quad \boxed{X \subseteq \mathbb{R},\ Y \subseteq \mathbb{R}}$$

lim a is linear:

$$\lim\ a\ (f \oplus g) = \lim\ a\ f + \lim\ a\ g$$
$$\lim\ a\ (c \odot f) = c \cdot (\lim\ a\ f)$$
$$\oplus : (X \to Y) \to (X \to Y) \to (X \to Y)$$
$$f \oplus g = \lambda x.\ f\,x + g\,x$$

Example 2: derivative

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*The **derivative** of a function f is another function f' defined by*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

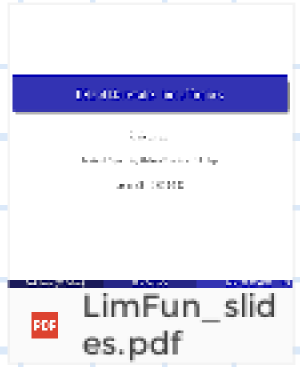
*at all points x for which the limit exists (i.e., is a finite real number). If $f'(x)$ exists, we say that f is **differentiable** at x.*

We can write

$$D\ f\ x = lim\ 0\ g \qquad \text{where} \qquad g\ h = \frac{f\ (x+h) - f\ x}{h}$$

$h \neq 0$

$D : (X \to Y) \to (X \to Y)$



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Example 2: derivative

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The ***derivative*** *of a function* f *is another function* f' *defined by*

$D = .'$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

at all points x *for which the limit exists (i.e., is a finite real number). If* $f'(x)$ *exists, we say that* f *is* ***differentiable*** *at* x.

$g : \mathbb{R}^{+} \to \mathbb{R}$

$g = \varphi\, x$

We can write

$D\ f\ x = lim\ 0\ g \quad$ where $\quad g\ h = \frac{f\ (x+h) - f\ x}{h}$

$D\ f\ x = lim\ 0\ (\varphi\ x)$ where $\quad \varphi\ x\ h = \frac{f\ (x+h) - f\ x}{h}$

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Example 2: derivative

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We can write

$$D\ f\ x = lim\ 0\ g \qquad \text{where} \qquad g\ h = \frac{f\ (x+h) - f\ x}{h}$$

$$g = \varphi\ x$$

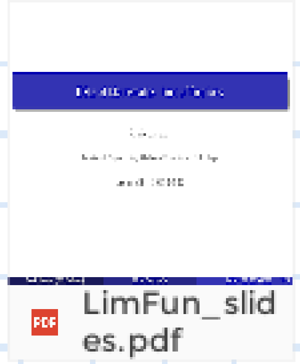
$$\varphi = \psi\ f$$

$$D\ f\ x = lim\ 0\ (\varphi\ x) \text{ where } \quad \varphi\ x\ h = \frac{f\ (x+h) - f\ x}{h}$$

$$= \underline{lim\ 0}\ (\underline{\psi\ f}\ x)$$

$$f' = \boxed{D\ f \quad = \underline{lim\ 0} \circ \underline{\psi\ f}} \text{ where } \psi\ f\ x\ h = \frac{f\ (x+h) - f\ x}{h}$$

$$\begin{tikzcd} & \mathbb{R} \ni x \arrow[dl, "D\ f"'] \arrow[d, "\psi\ f"] \\ \mathbb{R} & (\mathbb{R} \to \mathbb{R}) \ni \psi\ f\ x = g \arrow[l, "lim\ 0"] \end{tikzcd}$$



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Examples:

$(X \to Y) \to (X \to Y)$

$D : (\mathbb{R} \to \mathbb{R}) \to (\mathbb{R} \to \mathbb{R})$

$sq\ x = x\hat{}2$

$double\ x = 2 * x$

$c_2\ x = 2$

$sq' == D\ sq == D\ (\lambda x \to x\hat{}2) == D\ (\hat{}2) == (2*) == double$

$sq'' == D\ sq' == D\ double == c_2 == const\ 2$

Note: we cannot *implement* D (of this type) in Haskell.
Given only $f : \mathbb{R} \to \mathbb{R}$ as a "black box" we cannot compute the actual derivative $f' : \mathbb{R} \to \mathbb{R}$.
We need the "source code" of f to apply rules from calculus.

$D(f \oplus g) = Df \oplus Dg$

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$$D\ sq = \lim\ 0 \circ \Psi\ sq$$

$$\Psi : (\mathbb{R}\to\mathbb{R}) \to (\mathbb{R}\to\mathbb{R}\to\mathbb{R})$$

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$$\Psi\ sq\ x\ h = \frac{1}{h}\left((x+h)^2 - x^2\right) = \frac{1}{h}\left(\cancel{x^2} + 2xh + h^2 - \cancel{x^2}\right) =$$

$$= \frac{1}{h}\left(2xh + h^2\right) = 2 \cdot x + h$$

$$\Psi\ sq\ x = (2 \cdot x +)$$

$$D\ sq\ x = \lim\ 0\ (\Psi\ sq\ x) = \lim\ 0\ (2 \cdot x +) = 2 \cdot x + 0 = 2 \cdot x$$

$$D\ ({}^\wedge 2) = (2 \cdot)$$

$$D\ sq = (2 \cdot)$$

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$$D\ sq = lim\ 0 \circ \Psi\ sq$$

$$\Psi\ sq\ x\ h = \frac{(x+h)^2 - x^2}{h} = \frac{x^2 + 2xh + h^2 - x^2}{h} = \frac{2xh + h^2}{h} =$$

$$= 2x + h$$

$$\Psi\ sq\ x = (2 \cdot x +)$$

$$D\ sq\ x = lim\ 0\ (\Psi\ sq\ x) = lim\ 0\ (2 \cdot x +) = 2x + 0 = 2 \cdot x$$

$$D\ sq = (2 \cdot)$$

$$D\ ({}^\wedge 2) = (2 \cdot)$$

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$$D(f \oplus g) = \lim 0 \circ \Psi (f \oplus g) \qquad D({}^\wedge 2) = (2\cdot)$$

$$\Psi(f \oplus g)\, x\, h = \frac{1}{h} \cdot \big((f \oplus g)(x+h) - (f \oplus g)(x)\big) =$$

$$= \frac{1}{h}\big(f(x+h) + g(x+h) - (f x + g x)\big) =$$

$$= \frac{1}{h}\big((f(x+h) - f x) + (g(x+h) - g x)\big)$$

$$= \underbrace{\Psi f x}\, h + \underbrace{\Psi g x}\, h$$

$$\Psi(f \oplus g)\, x = \Psi f x \oplus \Psi g x$$

$$D(f \oplus g)\, x = \lim 0 \,(\qquad) = \lim 0\,(\Psi f x) + \lim 0\,(\Psi g x) =$$

$$= D f x + D g x$$

$$\boxed{D(f \oplus g) = D f \oplus D g}$$

$$D(f \oplus g) = \lim 0 \circ \Psi(f \oplus g)$$

$$D({}^\wedge 2) = (2 \cdot)$$

$$\Psi(f \oplus g)\, x\, h = \frac{(f \oplus g)(x+h) - (f \oplus g)\, x}{h} =$$

$$= \frac{1}{h}\big(\big(f(x+h) + g(x+h)\big) - (f x + g x)\big) =$$

$$= \frac{1}{h}\big(\big(f(x+h) - f x\big) + \big(g(x+h) - g x\big)\big) =$$

$$= \Psi f\, x\, h \quad + \quad \Psi g\, x\, h$$

$$\Psi(f \oplus g)\, x = \Psi f\, x \oplus \Psi g\, x$$

$$D(f \oplus g)\, x = \lim 0\, (\Psi f\, x \oplus \Psi g\, x) =$$

$$= \lim 0\, (\Psi f\, x) + \lim 0\, (\Psi g\, x) =$$

$$= D f\, x \quad + \quad D g\, x$$

$$D(f \oplus g) = Df \oplus Dg$$