DSLs of Mathematics: limit of functions

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Course goal and focus

Goal

Encourage students to approach mathematical domains from a functional programming perspective.

Course focus

- Make functions and types explicit
- Explicit distinction between syntax and semantics
- Types as carriers of semantic information
- Organize the types and functions in DSLs

Now Make variable binding and scope explicit

Lecture notes and more available at:
`https://github.com/DSLsofMath/DSLsofMath`

Example: The limit of a function

*We say that $f(x)$ **approaches the limit** L as x **approaches** a, and we write*

$$\lim_{x \to a} f(x) = L,$$

if the following condition is satisfied:
for every number $\varepsilon > 0$ there exists a number $\delta > 0$, possibly depending on ε, such that if $0 < |x - a| < \delta$, then x belongs to the domain of f and

$$|f(x) - L| < \varepsilon$$

- Adams & Essex, Calculus - A Complete Course

Limit of a function – continued

$$\lim_{x \to a} f(x) = L,$$

if

$$\forall \varepsilon > 0$$

$$\exists \delta > 0$$

such that if

$$0 < |x - a| < \delta,$$

then

$$x \in \mathit{Dom}\, f \wedge |f(x) - L| < \varepsilon$$

Limit of a function – continued

First attempt at translation:

$$\begin{aligned}
&\mathit{lim}\ a\ f\ L = \forall\ \epsilon > 0.\ \ \exists\ \delta > 0.\ \ P\ \epsilon\ \delta \\
&\quad \text{where } P\ \epsilon\ \delta = (0 < |x - a| < \delta) \Rightarrow \\
&\quad\quad\quad\quad\quad\quad\quad (x \in \mathit{Dom}\ f \wedge |f\ x - L| < \epsilon)
\end{aligned}$$

Limit of a function – continued

Finally (after adding a binding for x):

$$
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&\mathit{lim}\ a\ f\ L = \forall\ \epsilon > 0.\ \ \exists\ \delta > 0.\ \ P\ \epsilon\ \delta \\
&\quad \text{where } P\ \epsilon\ \delta = \quad \forall\ x.\ \ Q\ \epsilon\ \delta\ x \\
&\qquad\qquad Q\ \epsilon\ \delta\ x = (0 < |x - a| < \delta) \Rightarrow \\
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Limit of a function – continued

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Lesson learned: be careful with scope and binding (of x in this case).

Limit of a function – continued

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[We will now assume limits exist and use *lim* as a function from a and f to L.]

Example 2: derivative

*The **derivative** of a function f is another function f' defined by*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

*at all points x for which the limit exists (i.e., is a finite real number). If $f'(x)$ exists, we say that f is **differentiable** at x.*

We can write

$$D\ f\ x = lim\ 0\ g \qquad \text{where} \qquad g\ h = \frac{f\ (x+h) - f\ x}{h}$$

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$$D\ f \quad = lim\ 0 \circ \psi\ f \text{ where } \psi\ f\ \ x\ h = \frac{f\ (x+h) - f\ x}{h}$$

$$\begin{tikzcd}
 & \mathbb{R} \arrow[dl, "D\ f"'] \arrow[d, "\psi\ f"] \\
\mathbb{R} & (\mathbb{R} \to \mathbb{R}) \arrow[l, "lim\ 0"]
\end{tikzcd}$$

Derivatives, cont.

Examples:

$$
\begin{aligned}
&D : (\mathbb{R} \to \mathbb{R}) \to (\mathbb{R} \to \mathbb{R}) \\
&sq\ x \quad\ \ = x\hat{}2 \\
&double\ x = 2 * x \\
&c_2\ x \quad\ \ = 2 \\
&sq' \ \texttt{==}\ D\ sq\ \ \texttt{==}\ D\ (\lambda x \to x\hat{}2)\ \texttt{==}\ D\ (\hat{}2)\ \texttt{==}\ (2*)\ \texttt{==}\ double \\
&sq''\ \texttt{==}\ D\ sq'\ \texttt{==}\ D\ double\ \texttt{==}\ c_2\ \texttt{==}\ const\ 2
\end{aligned}
$$

Note: we cannot *implement* D (of this type) in Haskell.
Given only $f : \mathbb{R} \to \mathbb{R}$ as a "black box" we cannot compute the actual derivative $f' : \mathbb{R} \to \mathbb{R}$.
We need the "source code" of f to apply rules from calculus.