

# DSLs of Mathematics: limit of functions

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# Course goal and focus

## Goal

Encourage students to approach mathematical domains from a functional programming perspective.

## Course focus

- Make functions and types explicit
- Explicit distinction between syntax and semantics
- Types as carriers of semantic information
- Organize the types and functions in DSLs

Now Make variable binding and scope explicit

Lecture notes and more available at:

<https://github.com/DSLsofMath/DSLsofMath>

## Example: The limit of a function

We say that  $f(x)$  *approaches the limit*  $L$  as  $x$  *approaches*  $a$ , and we write

$$\lim_{x \rightarrow a} f(x) = L,$$

if the following condition is satisfied:

for every number  $\varepsilon > 0$  there exists a number  $\delta > 0$ , possibly depending on  $\varepsilon$ , such that if  $0 < |x - a| < \delta$ , then  $x$  belongs to the domain of  $f$  and

$$|f(x) - L| < \varepsilon$$

- Adams & Essex, Calculus - A Complete Course

# Limit of a function – continued

$$\lim_{x \rightarrow a} f(x) = L,$$

*if*

$$\forall \varepsilon > 0$$

$$\exists \delta > 0$$

*such that if*

$$0 < |x - a| < \delta,$$

*then*

$$x \in \text{Dom } f \wedge |f(x) - L| < \varepsilon$$

First attempt at translation:

$$\begin{aligned} \lim_{x \rightarrow a} f(x) = L &= \forall \epsilon > 0. \exists \delta > 0. P \in \delta \\ \text{where } P \in \delta &= (0 < |x - a| < \delta) \Rightarrow \\ &\quad (x \in \text{Dom } f \wedge |f(x) - L| < \epsilon) \end{aligned}$$

Finally (after adding a binding for  $x$ ):

$$\lim a f L = \forall \epsilon > 0. \exists \delta > 0. P \epsilon \delta$$

$$\text{where } P \epsilon \delta = \forall x. Q \epsilon \delta x$$

$$Q \epsilon \delta x = (0 < |x - a| < \delta) \Rightarrow \\ (x \in \text{Dom } f \wedge |f x - L| < \epsilon)$$

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Lesson learned: be careful with scope and binding (of  $x$  in this case).

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[We will now assume limits exist and use *lim* as a function from  $a$  and  $f$  to  $L$ .]



## Example 2: derivative

The **derivative** of a function  $f$  is another function  $f'$  defined by

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

at all points  $x$  for which the limit exists (i.e., is a finite real number).  
If  $f'(x)$  exists, we say that  $f$  is **differentiable** at  $x$ .

We can write

$$Df_x = \lim_{h \rightarrow 0} g \quad \text{where} \quad g(h) = \frac{f(x+h) - f(x)}{h}$$

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$$D f x = \lim_{h \rightarrow 0} g \quad \text{where} \quad g h = \frac{f(x+h) - f x}{h}$$

$$D f x = \lim_{h \rightarrow 0} (\varphi x) \quad \text{where} \quad \varphi x h = \frac{f(x+h) - f x}{h}$$

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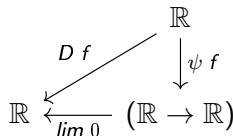
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We can write

$$D f \ x = \lim 0 \ g \quad \text{where} \quad g \ h = \frac{f(x+h) - f \ x}{h}$$

$$D f \ x = \lim 0 \ (\varphi \ x) \quad \text{where} \quad \varphi \ x \ h = \frac{f(x+h) - f \ x}{h}$$

$$D f \ = \lim 0 \circ \psi \ f \quad \text{where} \quad \psi \ f \ x \ h = \frac{f(x+h) - f \ x}{h}$$



Examples:

$$D : (\mathbb{R} \rightarrow \mathbb{R}) \rightarrow (\mathbb{R} \rightarrow \mathbb{R})$$

$$sq\ x = x^2$$

$$double\ x = 2 * x$$

$$c_2\ x = 2$$

$$sq' == D\ sq == D\ (\lambda x \rightarrow x^2) == D\ (^2) == (2*) == double$$

$$sq'' == D\ sq' == D\ double == c_2 == const\ 2$$

Note: we cannot *implement*  $D$  (of this type) in Haskell.

Given only  $f : \mathbb{R} \rightarrow \mathbb{R}$  as a “black box” we cannot compute the actual derivative  $f' : \mathbb{R} \rightarrow \mathbb{R}$ .

We need the “source code” of  $f$  to apply rules from calculus.