Absolute and essential spectra

The linear heat equation

$$u_t = u_{xx} + 2u_x + au$$

with transport provides a nice illustration of the various concepts involved in predicting absolute and convective instabilities. Consider this PDE on the interval $[0, L]$ with separated boundary conditions where we may, for instance, take Dirichlet boundary conditions $u(0) = 0 = u(L)$. The function $u(x,t) = \mathrm{e}^{\lambda t}\mathrm{e}^{\nu x}$ satisfies the heat equation precisely when the spatial and temporal exponents ν and λ satisfy the dispersion relation $\lambda = \nu^2 + 2\nu + a$. Upon solving for ν, we obtain $\nu_\pm = -1 \pm \sqrt{\lambda + 1 - a}$ which shows how the spatial wavenumbers $\nu_\pm$ depend on the temporal growth rate λ.

The stability of the background state $u = 0$ on the real line is determined by the essential spectrum given by

$$\Sigma_{\mathrm{ess}} = \{\lambda \in \mathbb{C};\ \lambda = -k^2 + 2\mathrm{i}k + a \text{ for some } k \in \mathbb{R}\}.$$

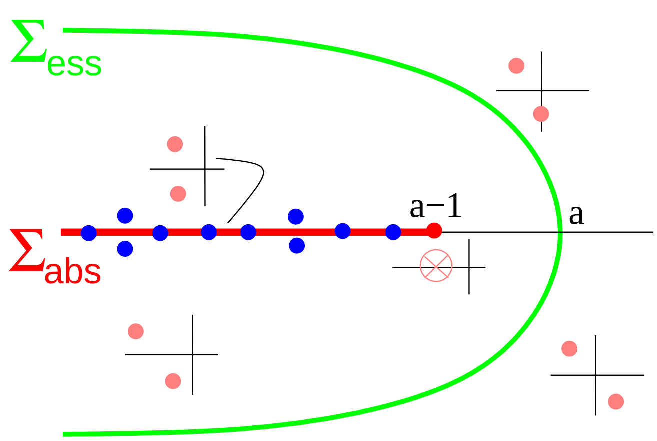
It corresponds to all values of λ for which one of the spatial eigenvalues $\nu_\pm$ is purely imaginary. The corresponding solution is $u(x,t) = \mathrm{e}^{\lambda(k)t}\sin kx$ with $\lambda(k) = -k^2 + 2\mathrm{i}k + a$.

The absolute spectrum

$$\Sigma_{\mathrm{abs}} = \{\lambda \in \mathbb{C};\ \mathrm{Re}\,\nu_+ = \mathrm{Re}\,\nu_-\} = \{\lambda \in \mathbb{R};\ \lambda \le a - 1\}.$$

consists of all points λ for which the corresponding spatial wavenumbers ν_+ and ν_- have the same real part. The absolute spectrum is not really spectrum; its significance, however, is that it provides the asymptotic location, as the domain size tends to infinity, of all but possibly a finite number of discrete eigenvalues of the heat equation on all large bounded intervals for any type of separated boundary conditions. The rightmost point of the absolute spectrum is the branchpoint $\lambda_{\mathrm{bp}} = a - 1$.

The following schematic picture illustrates the various spectra. The essential spectrum is plotted in green, while the absolute spectrum is the red line. The blue bullets symbolize the actual spectrum, consisting of point eigenvalues, of the heat equation on the interval $[0, L]$. Lastly, the insets consist of the spatial spectra, i.e., of the solutions $\nu_\pm$ of the dispersion relation where λ is fixed somewhere in the complex plane. In particular, the absolute spectrum consists of all points λ whose associated spatial wavenumbers ν_+ and ν_- have the same real part.



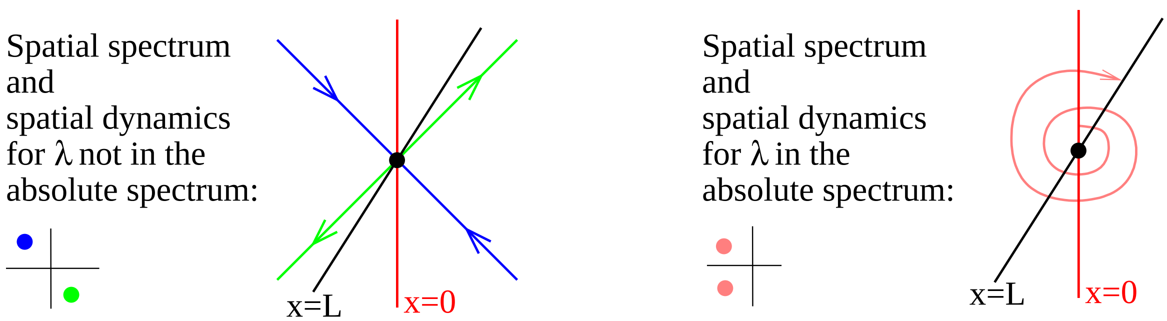
To see that the absolute spectrum really gives the asymptotic location of the discrete spectrum of the heat equation with separated boundary conditions, consider the eigenvalue equation

$$u_{xx} + 2u_x + (a - \lambda)u = 0.$$

We may seek eigenvalues on the interval $[0, L]$ by adopting a shooting approach. Consider the associated first-order system

$$\begin{pmatrix} u_x \\ v_x \end{pmatrix} = \begin{pmatrix} v \\ -2v + (\lambda - a)u \end{pmatrix}$$

which we solve starting at $x = 0$ by taking the line of all elements that satisfy the boundary conditions at $x = 0$ as initial condition. For Dirichlet boundary conditions, these initial conditions are given by the red line in the image below. We then solve until $x = L$ and see that λ is in the point spectrum provided the resulting line at $x = L$ satisfies the boundary conditions at $x = L$. These lines at $x = L$ are shown in black in the schematic picture below.



In the above picture, the small insets show the eigenvalues of the matrix

$$\begin{pmatrix} 0 & 1 \\ \lambda - a & -2 \end{pmatrix}$$

that appears in the above first-order system, while the large insets show the dynamics in of first-order system. Thus, the resulting black lines at $x = L$ can coincide with the red lines only if λ is in the absolute spectrum. Otherwise, the black lines converge to the green lines that denote the eigendirections of the spatial eigenvalue ν with the largest real part.