Reading Group: Spatial Dynamics

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Spatial Dynamics

I. Introduction

What is spatial dynamics? → Viewing one unbound space variable as evolution variable to gain insight into solution profiles

Example: travelling waves in 1D

$$U_t = U_{xx} + f(U) \quad (x \in \mathbb{R})$$

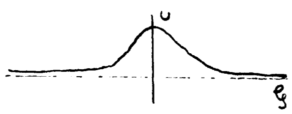
$$U(x,t) = u(x-ct) \rightarrow$$

$$-cu_\xi = u_{\xi\xi} + f(u) \quad (\xi \in \mathbb{R})$$

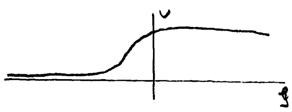
$$\mathcal{F}(u,c) = u_{\xi\xi} + cu_\xi + f(u) = 0$$

$$\mathcal{F}: C^2(\mathbb{R}) \times \mathbb{R} \rightarrow C^0(\mathbb{R})$$

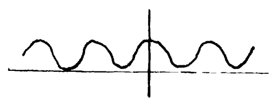
zeros of functions



pulse



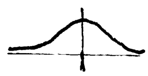
front



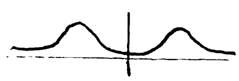
wave train

→ implicit-function theorems

⊖ multi-pulses



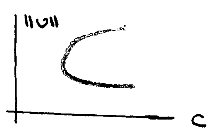
1-pulse



2-pulse

not close in function space

⊕

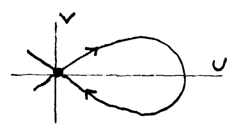


saddle-node

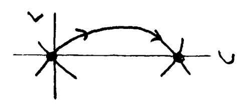
IFT works ...

$$\begin{pmatrix} u \\ v \end{pmatrix}_\xi = \begin{pmatrix} v \\ -[cv + f(u)] \end{pmatrix}$$

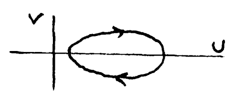
dynamical-system in ξ



homoclinic orbit



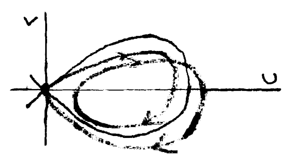
heteroclinic orbit



periodic orbit

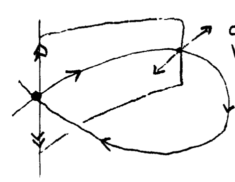
→ dynamical-systems techniques

⊕



pointwise close

⊖



does not break linearly in speed c → not easy to analyse

Spatial dynamics more than dimensions :

- stationary solutions to PDEs posed on $\mathbb{R}^n \times \Omega$, $\Omega \subset \mathbb{R}^k$ bdd, $n+k \geq 2$
- time-periodic solutions to PDEs on $\mathbb{R}^n \times \Omega$, $n \geq 1$

Example : $\begin{cases} u_t = \Delta u + (1+\mu) u + u^2 + \varepsilon \cos(\omega x) & (x,y) \in \mathbb{R} \times (0,\pi), \ t>0 \\ u(x,0,t) = u(x,\pi,t) = 0 \quad \forall (x,t) & \text{Dirichlet} \end{cases}$

stationary solutions $\rightarrow$

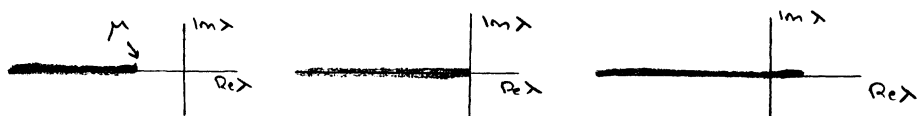
(1) $\begin{cases} \Delta u + (1+\mu) u + u^2 + \varepsilon \cos(\omega x) = 0 & (x,y) \in \mathbb{R} \times (0,\pi) \\ u(x,0) = u(x,\pi) = 0 \quad \forall x \end{cases}$

- $\varepsilon = 0$ $\rightarrow$ $u = 0$ is a solution :

  linearization about $u=0$: $\begin{cases} \Delta v + (1+\mu) v = \lambda v \\ v(x,0) = v(x,\pi) = 0 \end{cases}$ temporal eigenvalue

  $v(x,y) = e^{i\ell x} \sin(ky)$, with $\ell \in \mathbb{R}$, $k \in \mathbb{N}$ $(k \neq 0)$ $\rightarrow$

  spectrum $= \{ \lambda \mid \lambda = -\ell^2 - k^2 + 1 + \mu, \ \ell \in \mathbb{R}, \ k \geq 1 \}$



$\mu < 0$ $\qquad$ $\mu = 0$ $\qquad$ $\mu > 0$

- Fix $\mu < 0$, then there is an $\varepsilon_0 > 0$ so that (1) has a unique small bounded solution $u(\varepsilon)$ near $u=0$ for each ε with $|\varepsilon| < \varepsilon_0$.

- what happens near $\mu = 0$?

$$u_{xx} + u_{yy} + (1+\mu) u + u^2 + \varepsilon \cos(\omega x) = 0$$

(2) $$\begin{pmatrix} u \\ v \end{pmatrix}_x = \begin{pmatrix} 0 & 1 \\ -\partial_{yy} - 1 - \mu & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} 0 \\ -u^2 + \varepsilon \cos \omega x \end{pmatrix}$$

where $(u,v)(x) \in Y =$ appropriate space of functions in y on $(0,\pi)$ with $u(0) = u(\pi) = 0$ (Dirichlet in y-variable)

We write (1) as

(3) $\quad U_x = A(\mu) U + N(U;\varepsilon)$ with $U \in Y$

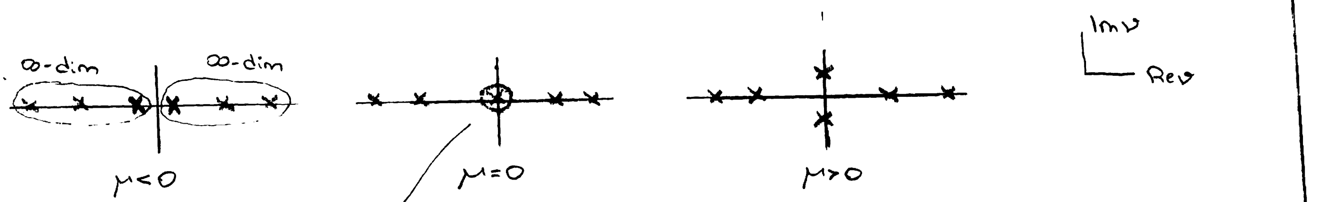
Spectrum of $A(\mu)$:

$$\begin{pmatrix} 0 & 1 \\ -\partial_{yy} - 1 - \mu & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \nu \begin{pmatrix} u \\ v \end{pmatrix}$$ spatial eigenvalue

$\rightarrow$ $\begin{cases} -u_{yy} - (1+\mu) u = \nu^2 u \\ u(0) = u(1) = 0 \end{cases}$

$\rightarrow$ $u = \sin(ky)$ with $k \geq 1$ and we get

$\nu_k^{\pm} = \pm \sqrt{k^2 - 1 - \mu}$ $\quad k \geq 1$



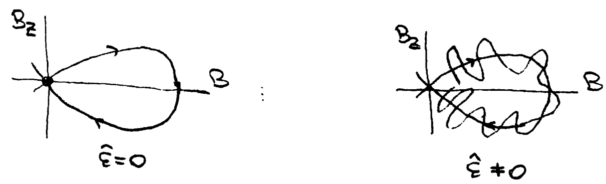
two-dimensional center manifold near $U=0$ for $(\mu,\varepsilon)\approx 0$
tangent to center eigenspace, smooth in (U,μ,ε).
[Kirchgässner '81], [Fischer '84], [Mielke '85, ...]

1) $U = \begin{pmatrix} A \\ A_x \end{pmatrix} \sin y + \underbrace{O\big((|A|+|\varepsilon|+|\mu|)^2\big)}_{\text{contains only } \sin(ky) \text{ with } k\neq 1}$

$\rightarrow \quad A_{xx} + \mu A + \frac{2}{3\pi} A^2 + \frac{4\varepsilon}{\pi}\cos(\omega x) + O\big((|A|+|\varepsilon|+|\mu|)^3\big) = 0$

rescaling: $z = \sigma x$, $\mu = -\sigma^2$, $A = \sigma^2 B$, $\varepsilon = \sigma^4 \hat{\varepsilon}$, $\hat{\omega} = \frac{\omega}{\sigma}$ gives

$$\underbrace{B_{zz} - B + \frac{2}{3\pi}B^2 + \frac{4\hat{\varepsilon}}{\pi}\cos(\hat{\omega} z) + O(\sigma) = 0}$$ [assume $\omega = \hat{\omega}\,\varepsilon^{1/4}$]



many small bounded solutions for $-1 \ll \mu < 0$

2) why is it nontrivial to obtain a center manifold for (3)?

$U(x) = e^{\nu_k^{\pm} x}\begin{pmatrix} 1 \\ \nu_k^{\pm}\end{pmatrix}$ satisfy $U_x = \mathcal{A}(\mu)\, U$ for all $k \geq 1$

general solution for $\mu=0$:

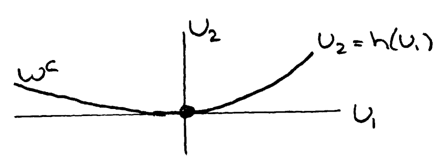
$$U(x) = \underbrace{\sum_{k=2}^{\infty} a_k e^{\sqrt{k^2-1}\,x}\overbrace{\begin{pmatrix}\frac{1}{\sqrt{k^2-1}} \\ 1\end{pmatrix}}^{\sin ky}}_{\text{no convergence for } x>0} + \underbrace{\sum_{k=2}^{\infty} b_k e^{-\sqrt{k^2-1}\,x}\overbrace{\begin{pmatrix}\frac{-1}{\sqrt{k^2-1}} \\ 1\end{pmatrix}}^{\sin ky}}_{\text{no convergence for } x<0} + \begin{pmatrix}1 & x \\ 0 & 1\end{pmatrix}\begin{pmatrix}a_0 \\ a_1\end{pmatrix}\sin y$$

(2) or (3) are not well-posed as initial-value problems / dynamical systems.

idea for proof of existence of center manifolds:

$$\begin{cases} U_1 \in E^c = \text{center eigenspace} =: \operatorname{span}\{U \sin y;\ U \in \mathbb{R}^2\} \\ U_2 \in E^h = \text{hyperbolic eigenspace} =: \operatorname{span}\{U \sin ky;\ U\in\mathbb{R}^2,\ k\geq 2\} \end{cases}$$

$$(4) \quad \begin{cases} U_1' = \mathcal{A}^c(\mu) + \mathcal{N}^c(U_1+U_2;\varepsilon) & U_1(0) = U_1^0 \\ U_2' = \mathcal{A}^h(\mu) + \mathcal{N}^h(U_1+U_2;\varepsilon) \end{cases}$$



We think of $\mathcal{N}^c, \mathcal{N}^h$ as being truncated so that both functions are small independently of (U_1, U_2).

(this can be achieved by multiplying with appropriate cutoff functions).

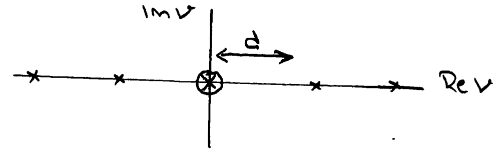
Any bounded or mildly growing solution to (4) must satisfy the integral equation

$$U_1(t) = e^{A^c(\mu)t}\, U_1^0 + \int_0^t e^{A^c(\mu)(t-s)}\, \mathcal{N}_1(U_1(s)+U_2(s))\, ds$$

$$U_2(t) = \int_{-\infty}^t e^{A^s(\mu)(t-s)}\, \mathcal{N}_2(U_1(s)+U_2(s))\, ds + \int_{\infty}^t e^{A^u(\mu)(t-s)}\, \mathcal{N}_2(U_1(s),U_2(s))\, ds$$

$$= \int_{-\infty}^{\infty} G(t-s)\, \mathcal{N}_2(U_1(s),U_2(s))\, ds$$

where $G(t) = \begin{cases} e^{A^s(\mu)t} & t<0 \\ -e^{A^u(\mu)t} & t>0 \end{cases}$ with $\|G(t)\| \leq K e^{-d|t|}$ $t \in \mathbb{R}$



$\rightarrow$ apply contraction-mapping principle to get unique fixed-point which depend Lipschitz continuously on U_1^0.

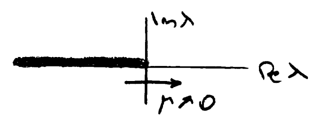
Summary: $\begin{cases} \Delta u + (1+\mu)u + u^2 + \varepsilon\cos(\omega x) = 0 & (x,y) \in \mathbb{R}\times(0,\pi) \\ u(x,0) = 0 = u(x,\pi) \end{cases}$

linearized problem: $\Delta u + (1+\mu)u = \lambda u$
$u(x,0) = 0 = u(x,\pi)$

seek solutions $u(x,y) = e^{\nu x}\sin ky \quad (k \geq 1)$

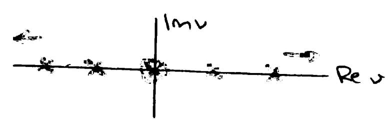
$\rightarrow \quad \lambda = \nu^2 - k^2 + 1 + \mu \quad$ to, $k \geq 1$

λ temporal: $\nu = i\ell \rightarrow \lambda = -\ell^2 - k^2 + 1 + \mu \qquad k \geq 1,\ \ell \in \mathbb{R}$



$u_t = \Delta u + (1+\mu)u$

ν = spatial: $\lambda = 0 \rightarrow \nu = \pm\sqrt{k^2 - 1 - \mu} \qquad k \geq 1$



$U_x = \begin{pmatrix} 0 & 1 \\ -\partial_{yy} - 1 - \mu & 0 \end{pmatrix} U$

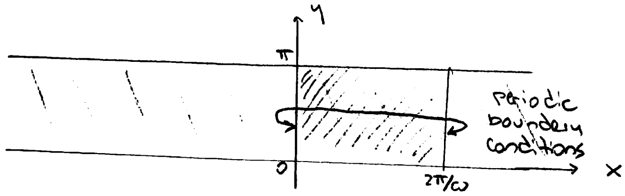
$\lambda = 0$ in the essential spectrum $\Leftrightarrow$ $\nu \in i\mathbb{R}$ is in spectrum of $\begin{pmatrix} 0 & 1 \\ -\partial_{yy} - 1 - \mu & 0 \end{pmatrix}$

a) $u_t = \Delta u + (1+\mu)u + u^2 + \varepsilon\cos\omega x$: derive amplitude equation

b) $\begin{pmatrix} u \\ v \end{pmatrix}_x = \begin{pmatrix} v \\ -u_{yy} - (1+\mu)u - u^2 - \varepsilon\cos\omega x \end{pmatrix}$: derive vector field on centre manifold

$\rightarrow$ stationary amplitude equation from a) $=$ scaled vector field from b)

Remark



$$\begin{cases} \Delta u + (1+\mu) u + u^2 + \varepsilon \cos \omega x = 0 \\ u(x,0) = 0 = u(x,\pi) \end{cases}$$

restrict to $\frac{2\pi}{\omega}$ - periodic functions in x

$\rightarrow \quad u(x,y) = e^{i\ell\omega x} \sin ky \qquad \ell \in \mathbb{Z},\ k \geq 1$

„discretizes" spectrum: $\quad \lambda = -\omega^2\ell^2 - k^2 + 1 + \mu \quad$ since $\ell \in \mathbb{Z},\ k \geq 1$.

$\rightarrow$ can use IFT

The center manifold theorem: [Vanderbauwhede Dynamics Reported 2, 1989]

① Consider

(1) $$\dot{U} = F(U,\varepsilon) \qquad U \in \mathbb{R}^N,\ \varepsilon \in \mathbb{R}^p$$

where $F(0,0) = 0$ and $\operatorname{spec}(D_U F(0,0)) \cap i\mathbb{R} \neq \emptyset$. We then reformulate (1) as

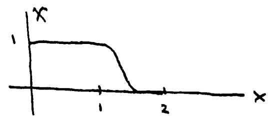
(2) $$\begin{pmatrix} \dot{U} \\ \dot{\varepsilon} \end{pmatrix} = \begin{pmatrix} F(U,\varepsilon) \\ 0 \end{pmatrix} \qquad \text{or} \qquad \dot{u} = f(u) \qquad u = (U,\varepsilon) \in \mathbb{R}^N \times \mathbb{R}^p = \mathbb{R}^n$$

Note that $$D_u f(0,0) = \begin{pmatrix} D_U F(0,0) & D_\varepsilon F(0,0) \\ 0 & 0 \end{pmatrix}$$

It therefore suffices to construct center manifolds for

(3) $$\dot{u} = f(u) \qquad u \in \mathbb{R}^n \qquad \text{with} \quad f(0) = 0, \quad \operatorname{spec}(Df(0)) \cap i\mathbb{R} \neq \emptyset.$$

We need to use a cutoff function χ which we choose according to



Replacing (3) by

(4) $$\dot{u} = Df(0)u + [f(u) - f(0) - Df(0)u]\ \chi\left(\frac{|u|}{\delta}\right)$$

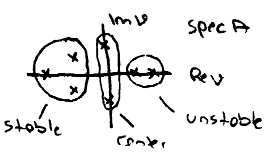
we observe that (3) and (4) coincide for $|u| < \delta$.

<u>Lemma</u> If $G \in C^k$ with $G(0) = DG(0) = 0$, then $g(u) := G(u)\,\chi\left(\frac{|u|}{\delta}\right)$ satisfies: $g \in C^k_b$ and $|g|_1 \to 0$ as $\delta \to 0$.

② Therefore, we consider from now on the system

(*) $$\dot{u} = Au + g(u) \qquad u \in \mathbb{R}^n \qquad g(0) = Dg(0) = 0, \quad \operatorname{spec}(A) \cap i\mathbb{R} \neq \emptyset$$

with $|g|_{C^1_b}$ sufficiently small



$$A = \begin{pmatrix} A^c & & \\ & A^s & \\ & & A^u \end{pmatrix}, \quad A^c = \begin{pmatrix} A^c & & \\ & 0 & \\ & & 0 \end{pmatrix}, \quad A^s = \begin{pmatrix} 0 & & \\ & A^s & \\ & & 0 \end{pmatrix}, \quad A^u = \begin{pmatrix} 0 & & \\ & 0 & \\ & & A^u \end{pmatrix}$$

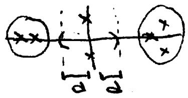
with projections P^c, P^s, P^u onto E^c, E^s, E^u, respectively.

<u>Theorem</u> Let A be given, then there is a $\rho_0 > 0$ such that the following is true for each $g \in C^{1,1}_b$ with $|g|_1 \le \rho_0$:

$$W^c = \{u_0 \,/\, \sup_{t \in \mathbb{R}} |P^h u(t)| < \infty \text{ where } u(0) = u_0 \text{ and } u(t) \text{ satisfies } (*)\}$$
$$= \operatorname{graph} \pi^c = \{u = u^c + \pi^c(u^c)\,;\ u^c \in E^c\}$$

for an appropriate Lipschitz-continuous function $\pi^c: E^c \to E^h$. Furthermore, W^c is invariant under (*). If $g \in C^1_b \cap C^{k,\alpha}$ for some $k \ge 1$, $0 < \alpha \le 1$, then $\pi^c \in C^{k,\alpha}$.
(Note: The theorem is also true for $\alpha = 0$).

③ Outline of the proof:



Spectral gap $> d$

$$|e^{A^c t} P^c| \le K(\eta)\, e^{\eta|t|} \qquad t \in \mathbb{R},\ \eta > 0$$
$$|e^{A^s t} P^s| + |e^{-A^u t} P^u| \le K_0 e^{-dt} \qquad t \ge 0,$$

$$X_\eta = \{ v \in C^0(\mathbb{R}, \mathbb{R}^n) \mid |v|_\eta = \sup_{t\in\mathbb{R}} e^{-\eta|t|}\, |v(t)| < \infty \}$$

$v(0)$ lies in W^c iff

$$(5) \quad v(t) = e^{A^c t} P^c v_0^c + \int_0^t e^{A^c(t-s)} P^c g(v(s))\,ds + \int_{-\infty}^t e^{A^s(t-s)} P^u g(v(s))\,ds + \int_{\infty}^t e^{A^u(t-s)} P^u g(v(s))\,ds$$

where $v_0^c = P^c v(0)$ $\left(\begin{array}{l}\text{Since the integral eqn. differs from variation-of-constant formula by} \\ \text{terms } e^{A^s t} P^s v_0^s \text{ and } e^{A^u t} P^u v_0^u\end{array}\right)$

It therefore suffices to find solutions of (5) for $v_0^c \in E^c$. We write

$$B(t) = \begin{cases} e^{A^s t} P^s & t > 0 \\ -e^{A^u t} P^u & t < 0 \end{cases} \qquad |B(t)| \le K_0\, e^{-d|t|} \qquad t \in \mathbb{R}$$

so that (5) becomes

$$(**) \quad v(t) = e^{A^c t} P_0^c v_0^c + \int_0^t e^{A^c(t-s)} P^c g(v(s))\,ds + \int_{-\infty}^{\infty} B(t-s) P^h g(v(s))\,ds$$

We write $(**)$ as

$$(***) \quad v = \mathcal{G}(v, v_0^c) \qquad v_0^c \in E^c, \quad v \in X_\eta$$

Lemma 1 $\quad \mathcal{B} : X_\eta \to X_\eta, \quad v \mapsto \mathcal{B}v$ with $[\mathcal{B}v](t) = \int_0^t e^{A^c(t-s)} P^c v(s)\,ds + \int_{-\infty}^{\infty} B(t-s) P^h v(s)\,ds$

is linear and bounded with $\|\mathcal{B}\| \le \dfrac{K(\eta/2)}{\eta} + \dfrac{2K_0}{d-\eta}$

Proof
$$|\mathcal{B}v|_\eta = \sup_{t\in\mathbb{R}} e^{-\eta|t|} \left| \int_0^t e^{A^c(t-s)} P^c v(s)\,ds + \int_{-\infty}^{\infty} B(t-s) P^h v(s)\,ds \right|$$
$$\le \sup_{t\in\mathbb{R}} e^{-\eta|t|} \left(\int_0^t K(\tfrac{\eta}{2})\, e^{\frac{\eta}{2}|t-s|} e^{\eta|s|} |v|_\eta\,ds + \int_{-\infty}^{\infty} K_0\, e^{-d|t-s|} e^{\eta|s|} |v|_\eta\,ds \right)$$
$$\le |v|_\eta \sup_{t\in\mathbb{R}} \left(\int_0^t K(\tfrac{\eta}{2})\, e^{-\frac{\eta}{2}|t-s|}\,ds + \int_{-\infty}^{\infty} K_0\, e^{-(d-\eta)|t-s|}\,ds \right)$$
$$\le \left(\frac{K(\eta/2)}{\eta} + \frac{2K_0}{d-\eta} \right) |v|_\eta .$$

Lemma 2 $\quad H : E^c \to X_\eta, \quad v_0^c \mapsto H(v_0^c)$ with $[H(v_0^c)](t) = e^{A^c t} P^c v_0^c$ is linear and bounded by $K(\eta)$.

Lemma 3 $\quad$ Let $g \in C_b^1$, then $G : X_\eta \to X_\eta$, $v \mapsto G(v)$ with $[G(v)](t) = g(v(t))$ is well defined and Lipschitz continuous with $|G(v_1) - G(v_2)|_\eta \le |g|_1\, |v_1 - v_2|_\eta$

Proof $\quad$ Firstly, $|G(v)|_\eta = \sup_{t\in\mathbb{R}} e^{-\eta|t|} |g(v(t))| \le |g|_1 \quad \forall v \in X_\eta$
and $g(v(t))$ is continuous, so G is well defined.

$$|G(v_1) - G(v_2)|_\eta = \sup_{t\in\mathbb{R}} e^{-\eta|t|}\, |g(v_1(t)) - g(v_2(t))|$$
$$= \sup_{t\in\mathbb{R}} e^{-\eta|t|} \left| \int_0^1 g'(v_2(t) + \tau(v_1(t) - v_2(t)))\,d\tau\, (v_1(t) - v_2(t)) \right|$$
$$\le \sup_{t\in\mathbb{R}} e^{-\eta|t|}\, |g|_1\, |v_1(t) - v_2(t)| \le |g|_1\, |v_1 - v_2|_\eta .$$

In summary,
$$\mathcal{G}(u,u_0^c) = H(u_0^c) + (\mathcal{B}\circ G)(u)$$

$\mathcal{G} : E_0^c \times X_\eta \to X_\eta$ well-defined, continuous, and Lipschitz continuous in u with

$$|\mathcal{G}(u_1,u_0^c) - \mathcal{G}(u_2,u_0^c)|_\eta \le \|\mathcal{B}\|_\eta \, |G(u_1) - G(u_2)|_\eta$$
$$\le \left(\frac{K(\eta/2)}{\eta} + \frac{2K_0}{d-\eta}\right) |g|_1 \; |u_1 - u_2|_\eta$$

uniformly in $u_0^c \in E^c$

Lemma 4 Assume $|g|_1 \le \frac{1}{2}\left(\frac{K(\eta/2)}{\eta} + \frac{2K_0}{d-\eta}\right)^{-1}$, then $u = \mathcal{G}(u,u_0^c)$ has a unique fixed point $\Phi(u_0^c) \in X_\eta$ for each $u_0^c \in E^c$ and $\Phi : E_0^c \to X_\eta$ is Lipschitz continuous in u_0^c.

Proof Apply uniform contraction mapping principle: $|\mathcal{G}(u_1,u_0^c) - \mathcal{G}(u_2,u_0^c)| \le \Theta |u_1 - u_2|$ $(0 \le \Theta < 1)$

1) Fixed points are unique: $u_j = \mathcal{G}(u_j,\mu)$ $\Rightarrow$ $|u_1 - u_2| = |\mathcal{G}(u_1,\mu) - \mathcal{G}(u_2,\mu)| \le \Theta |u_1 - u_2|$

2) Consider $u_n = \mathcal{G}^n(u_0,\mu)$ then $|u_{n+1} - u_n| \le \Theta^n \, |\mathcal{G}(u_0,\mu) - u_0|$

$$|u_n - u_m| \le \left(\sum_{k=n}^{m} \Theta^k\right) |\mathcal{G}(u_0,\mu) - u_0| \le \frac{\Theta^{\min(n,m)}}{1-\Theta} |\mathcal{G}(u_0,\mu) - u_0|$$

$\to \{u_n\}_{n\in\mathbb{N}}$ is Cauchy, and limit is fixed point by continuity of $\mathcal{G}$

3) Let $\Phi(\mu)$ denote the fixed point, then

$$\Phi(\mu_1) - \Phi(\mu_2) = \mathcal{G}(\Phi(\mu_1),\mu_1) - \mathcal{G}(\Phi(\mu_2),\mu_2) = \mathcal{G}(\Phi(\mu_1),\mu_1) - \mathcal{G}(\Phi(\mu_2),\mu_1) + \mathcal{G}(\Phi(\mu_2),\mu_1) - \mathcal{G}(\Phi(\mu_2),\mu_2)$$
$$|\Phi(\mu_1) - \Phi(\mu_2)| \le |\mathcal{G}(\Phi(\mu_1),\mu_1) - \mathcal{G}(\Phi(\mu_2),\mu_1)| + |\mathcal{G}(\Phi(\mu_2),\mu_1) - \mathcal{G}(\Phi(\mu_2),\mu_2)|$$
$$\le \Theta |\Phi(\mu_1) - \Phi(\mu_2)| + \mathrm{Lip}(\mathcal{G},\mu) \, |\mu_1 - \mu_2|$$
$$|\Phi(\mu_1) - \Phi(\mu_2)| \le \frac{\mathrm{Lip}(\mathcal{G},\mu)}{1-\Theta} |\mu_1 - \mu_2|$$

Lemma 5 The map $\pi^c(u_0^c) = [\Phi(u_0^c)](0)$ gives the center manifold

Proof By construction (invariance follows from uniqueness).

Note $G : X_\eta \to X_\eta$, $u \mapsto g(u)$ is not C^k if $g \in C_b^1 \cap C^k$.

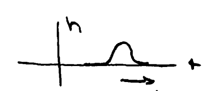
Lemma 6 The map $G : X_\eta \to X_{\eta+\varepsilon}$, $u \mapsto g(u)$ is C^k if $g \in C_b^1 \cap C^k$ for any $\varepsilon > 0$.

Proof
$$|G(u+h) - G(u) - G'(u)h|_{\eta+\varepsilon} = \sup_{t\in\mathbb{R}} e^{-(\eta+\varepsilon)|t|} |g(u(t)+h(t)) - g(u(t)) - g'(u(t))h(t)|$$
$$\le \sup e^{-(\eta+\varepsilon)|t|} \int_0^1 |g'(u(t) + \tau h(t)) - g'(u(t))| \, d\tau \; |h(t)|$$
$$\le |h|_\eta \underbrace{\sup_{t\in\mathbb{R}} e^{-\varepsilon|t|} \int_0^1 |g'(u(t) + \tau h(t)) - g'(u(t))| \, d\tau}_{\to 0 \text{ as } |h|_\eta \to 0}$$

provided $\varepsilon > 0$.

(Pick $\kappa > 0$, choose $T > 0$ so that $2|g|_1 e^{-\varepsilon T} \le \frac{\kappa}{2}$
Next choose $\delta > 0$ so small that $|g'(u(t)+v) - g'(u(t))| \le \frac{\kappa}{2}$
for all $|t| \le T$ and $|v| < \delta e^{\varepsilon T}$ $\to$ $\underbrace{\cdots}_{<\kappa}$ for $|h|_\eta < \delta$.)

($\varepsilon = 0$



move to ∞
then $|h|_\eta \to 0$
but $\int_0^1 |\ldots| \ge \delta > 0$
for $u \equiv$ const.)

smoothness of π^c: [Henry: Springer 1981 (§6)]

$Y = \{ \sigma : E^c \to E^h \, ; \; |\sigma|_0 \le 1 \, , \; \mathrm{Lip}\, \sigma \le 1 \}$

$G : Y \to Y \, , \; \sigma \mapsto G(\sigma)$ so that $\pi^c = G(\pi^c)$.

$B = \{ \sigma \in Y ; \; \sigma \in C^{k,\alpha}$ with $|\sigma|_{k,\alpha} \le b \}$ for $k \ge 1$, $0 < \alpha \le 1$

(i) show that $G : B \to B$ for an appropriate $b > 0$

(ii) show that B is closed in Y

Construction of G:

$\sigma \in Y \;\to\; \exists!$ soln $u^c = u^c(u_0^c)$ of $u^c(t) = e^{A^c t} P^c u_0^c + \int_0^t e^{A^c(t-s)} P^c g(u^c(s) + \sigma(u^c(s)))\, ds$ in X_η

$\to \; [G(\sigma)](u_0^c) = \int_{-\infty}^{0} B(-s) P^h g(u^c(s) + \sigma(u^c(s)))\, ds$

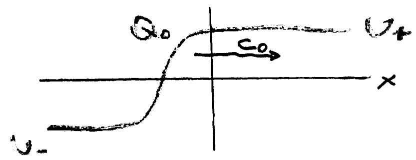
1) $G : Y \to Y$ is well defined

2) $G : Y \to Y$ is a contraction

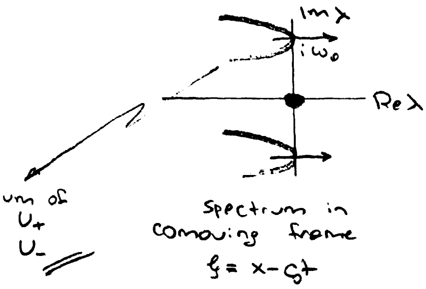
3) $G(B) \subset B$ for an appropriate b

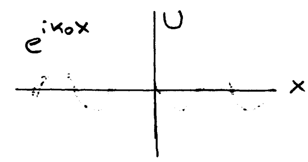
II. Spatial dynamics for large-amplitude structures

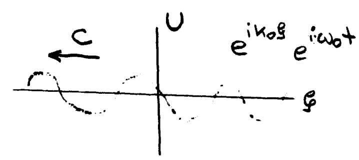
Fronts in reaction-diffusion systems that undergo essential instabilities:



front ($c_0 > 0$)







$\rightsquigarrow$ resembles a Hopf bifurcation, but now with a continuum of modes crossing

(1) $\quad U_t = DU_{\xi\xi} + cU_\xi + F(U;\mu) \qquad \xi \in \mathbb{R},\ U \in \mathbb{R}^N \qquad$ reaction-diffusion system

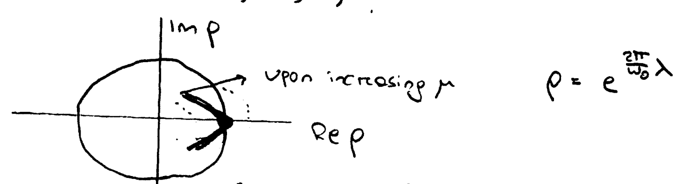
critical temporal frequency $= \omega_0 \quad \rightsquigarrow \quad$ temporal period $\approx \frac{2\pi}{\omega_0} = T_0$

1) temporal dynamics:

semiflow of (1) $= \Phi_t(U; c, \mu)$ (say on $C^0_{unif}(\mathbb{R}, \mathbb{R}^N)$)

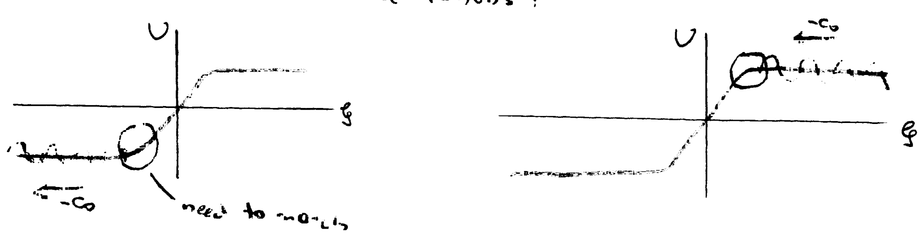
$\Phi_t(Q_0; c_0, 0) = Q \quad \forall t$

$D_U \Phi_{T_0}(Q_0; c_0, 0)$:



spectrum of $D_U\Phi_{T_0}(Q_0; c_0, 0)$

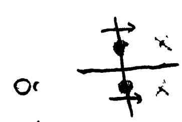
Can use amplitude equations (complex Ginzburg-Landau equation) to describe the evolution of small-amplitude patterns:



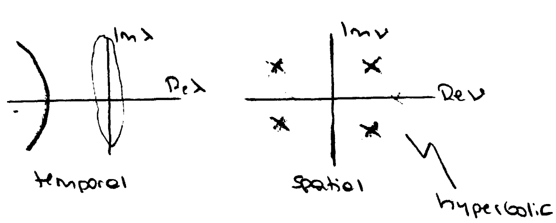
2) spatial dynamics:

want to find solutions $U(\xi, t)$ of (1) with temporal period $\frac{2\pi}{\omega_0}$ (or period close to $\frac{2\pi}{\omega_0}$) which are close to $Q_0(\xi)$ $\forall \xi$:

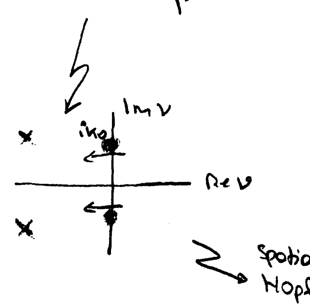
$$\begin{pmatrix} U \\ V \end{pmatrix}_\xi = \begin{pmatrix} V \\ D^{-1}[U_t - cV - F(U;\mu)] \end{pmatrix} \qquad u = (U, V)$$



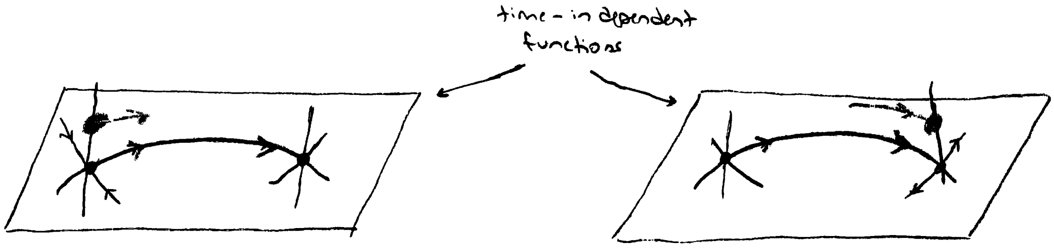
with $u(\xi)$: $\frac{2\pi}{\omega_0}$-periodic function in t for each fixed ξ.



stable rest state



destabilizing rest state



bifurcation behind front

bifurcation ahead of front

Setting

(1) $U_t = DU_{xx} + F(U;\mu) \qquad x \in \mathbb{R},\ U \in \mathbb{R}^N,\ \mu \in \mathbb{R}$

We are interested in travelling waves, $U(x,t) = U_*(x-ct)$, and therefore consider eqn. (1) in the comoving frame $\xi = x - ct$ in which (1) becomes

(2) $U_t = DU_{\xi\xi} + cU_\xi + F(U;\mu) \qquad \xi \in \mathbb{R}$.

(H1) Assume that $U(\xi,t) = Q(\xi)$ is a stationary solution of (2) for $c = c_0 > 0$ at $\mu = 0$.
We also assume that
(i) $Q(\xi) \to U^0_\pm$ as $\xi \to \pm\infty$
(ii) $\det F_U(U^0_\pm, 0) \neq 0$.

In particular, the homogeneous rest states $U(x,t) = U^0_\pm$ of (1) (or (2)) at $\mu = 0$ persist for all μ close to zero as they satisfy $F(U;\mu) = 0$ which we can solve (locally uniquely) for $U = U_\pm(\mu)$ with $U_\pm(0) = U^0_\pm$.

Consider the linearization

(3) $U_t = DU_{xx} + F_U(U^\pm(\mu), \mu)\, U$

of (1) about $U^\pm$. We seek solutions of the form $e^{\lambda t + ikx} U_0$ with $U_0 \neq 0$ which exist iff

(4) $d^0_\pm(\lambda, ik, \mu) = \det\left(-Dk^2 + F_U(U^\pm(\mu),\mu) - \lambda\right) = 0$.

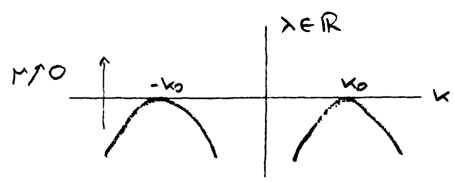
(H2) [bifurcation ahead] We assume that there are $\varepsilon, \delta > 0$ so that the following is true for $\mu \approx 0$:
(i) $d^0_-(\lambda, ik, \mu) = 0$ for some $k \in \mathbb{R}$, $\lambda \in \mathbb{C}$ implies $\operatorname{Re}\lambda < -\delta$.
(ii) $\exists!\ k_0 > 0$ such that

- $d^0_+(\lambda, ik, \mu) = 0$ for some $k \in \mathbb{R} \setminus U_\varepsilon(\pm k_0)$, $\lambda \in \mathbb{C}$ implies $\operatorname{Re}\lambda < -\delta$
- $d^0_+(\lambda, ik, \mu) = c_1\left[-d(k-k_0)^2 + \mu - \lambda\right] + \mathcal{O}\left(|\mu|^2 + |\lambda|^2 + |k-k_0||\mu| + (|\mu| + |\lambda| + |k-k_0|)^3\right)$
  for some $c_1 \neq 0$, $d > 0$, for $k \in U_\varepsilon(k_0)$.

In particular, we have

(5) $\lambda = \lambda^0_*(ik) = -d(k-k_0)^2 + \mu + \text{h.o.t.}$

satisfies $d^0_+(\lambda, ik, \mu) = 0$.



These assumptions on $U_\pm$ in the laboratory frame carry over immediately to the comoving frame: Consider

$$U_t = DU_{\xi\xi} + cU_\xi + F_U(U_\pm(\mu),\mu)\, U$$

then nontrivial solutions of the form $e^{\lambda t + ik\xi} U_0$ exist iff

$$d^c_\pm(\lambda, ik, \mu) = \det\left(-Dk^2 + ikc + F_U(U_\pm(\mu),\mu) - \lambda\right) = 0 \quad \to \quad \lambda - ikc = \lambda^0$$

Thus, (λ^0, ik, μ) satisfies $d^0(\lambda^0, ik, \mu) = 0$ iff $(\lambda, ik, \mu) = (\lambda^0 + ikc, ik, \mu)$ satisfies $d^c_\pm(\lambda, ik, \mu) = 0$.
Hence, (5) becomes

(6) $\lambda = \lambda^c_*(ik) = \mu + ikc - d(k-k_0)^2 + \text{h.o.t.} \qquad (k \text{ near } k_0)$

in the frame moving with speed c.

Remark $\lambda^c(ik) = \mu + ikc - dk^2 = \mu + ikc + d(ik)^2$

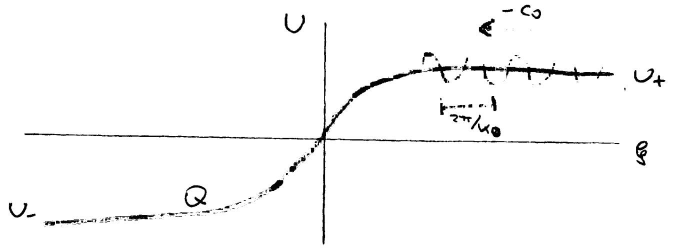
corresponds, via $\lambda \to \partial_t$, $ik \to \partial_x$ (Fourier transform), to

$$v_t = d v_{xx} + c v_x + \mu v$$

$\uparrow$ $c>0$: transport to the left.

Group velocity: $c_g = -\frac{d\lambda}{d\nu}\Big|_{\nu = ik_0} = -c < 0$ direction of transport.

Remark $\lambda^{c_0}(ik_0) = ik_0c_0 =: i\omega_0 \neq 0$: anticipated temporal frequency of bifurcating patterns



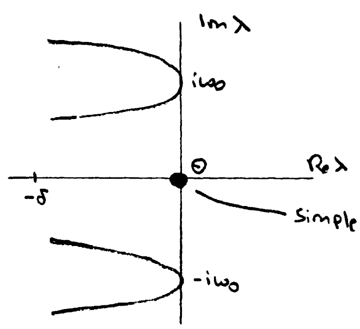
$\omega_0 = k_0c_0$ frequency.

Return to front Q at $\mu=0$. Linearizing (2) about Q at $\mu=0$ with $c=c_0$, we get

$$\mathcal{L}_0 = D\partial_{\xi\xi} + c\partial_\xi + F_U(Q(\xi),0)$$

(H3) We assume that $\mathcal{L}_0$ posed on $L^2(\mathbb{R},\mathbb{R}^N)$ has the eigenvalue $\lambda=0$ (with eigenfunction Q') is simple and that any other isolated eigenvalue λ has $\mathrm{Re}\,\lambda < -\delta$.

This implies



spec$(\mathcal{L}_0)$ on $L^2(\mathbb{R},\mathbb{R}^N)$

We need to exclude that $\lambda = \pm i\omega_0$ is an embedded eigenvalue but will state this assumption later.

Spatial dynamics

We take $\omega_0 = k_0c_0$ and seek solutions of (2),

$$U_t = DU_{\xi\xi} + cU_\xi + F(U,\mu)$$

that have temporal period $\frac{2\pi}{\omega_0}$ for $c \approx c_0$, $\mu \approx \mu_0$ and are close, in an appropriate sense, to $Q(\xi)$.

Thus, we consider

(7) $$\begin{pmatrix} U \\ V \end{pmatrix}_\xi = \begin{pmatrix} V \\ D^{-1}[U_t - F(U,\mu)] - cD^{-1}V \end{pmatrix} \quad \text{on } Y = H^1(S^1) \times H^{1/2}(S^1), \quad S^1 = [0, \tfrac{2\pi}{\omega_0}]/\sim$$

where

$$H^k(S^1) = \Big\{ u(t) = \sum_{n\in\mathbb{Z}} a_n e^{int};\ \sum_{n\in\mathbb{Z}} (1+|n|^{2k})|a_n|^2 < \infty \Big\}$$

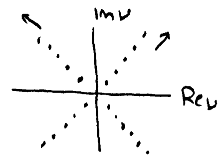
with norm $|u|_k := \sqrt{\sum_{n\in\mathbb{Z}} (1+|n|^{2k})|a_n|^2}$

note: $H^1(S^1) \hookrightarrow C^0(S^1)$ and $F(U,\mu)$ makes sense for $U \in H^1(S^1)$.

Remarks

- Eqn. (7) is ill-posed since

$$v_\xi = \begin{pmatrix} 0 & 1 \\ \partial_t & 0 \end{pmatrix} v \quad \text{has spectrum} \quad \nu = \pm\sqrt{ik}$$



- Eqn. (7) is equivariant under the action

$$T_\theta : Y \to Y, \quad v(\cdot) \mapsto v(\cdot - \theta) \quad : \quad \text{shifts in time}$$

- Define $Y_0 = \{ v \in Y;\ v(t) = v_0 \in \mathbb{R}^n \} = \{ v \in Y;\ v \text{ does not depend on time} \}$ then (7) leaves Y_0 invariant and reduces to the travelling-wave ODE

(8) $$\begin{pmatrix} U \\ V \end{pmatrix}_\xi = \begin{pmatrix} V \\ -D^{-1}[F(U;\mu) + cV] \end{pmatrix}$$

We begin by studying (8) with $\mu = 0$ and $c = c_0$:

(9) $$\begin{pmatrix} U \\ V \end{pmatrix}_\xi = \begin{pmatrix} V \\ -D^{-1}[F(U;0) + c_0 V] \end{pmatrix} \qquad (U,V) \in \mathbb{R}^N \times \mathbb{R}^N = Y_0$$

By assumption, this equation has the equilibria $v_\pm = (U^0_\pm, 0)$ and a heteroclinic orbit q $q(\xi) = (Q(\xi), Q_\xi(\xi))$ that connects v_- at $\xi = -\infty$ to v_+ at $\xi = \infty$. Since we assumed that $\det F_U(U^0_\pm;0) \neq 0$, we know that $v_\pm$ are both hyperbolic since the linearization of (9) about $v_\pm$ is given by

$$A^0_\pm = \begin{pmatrix} 0 & 1 \\ -D^{-1} F_U(U^0_\pm;0) & -c_0 D^{-1} \end{pmatrix} \quad \text{in } Y_0$$

Lemma 1 $A^0_\pm$ has N eigenvalues ν with $\mathrm{Re}\,\nu > 0$ and N eigenvalues ν with $\mathrm{Re}\,\nu < 0$.

Before proving this lemma, we write Hypothesis (H2) using the dispersion relation

$$d^c_\pm(\lambda, \nu, \mu) = \det\left(D\nu^2 + \nu c + F_U(U_\pm(\mu);\mu) - \lambda \right)$$

which gives

(i) $d^c_-(\lambda, ik, \mu) = 0$ for some $k \in \mathbb{R}$ implies $\mathrm{Re}\,\lambda < -\delta$

(ii) $d^c_+(\lambda, ik, \mu) = 0$ for some $k \in \mathbb{R}$ implies either $\mathrm{Re}\,\lambda < -\delta$ or else $k \in U_\delta(\pm k_0)$ and
$\lambda = \lambda^c_\pm(k) = \mu + ikc - d\,(k \pm k_0)^2 + \text{h.o.t.}$

Proof of Lemma 1

Consider $d^{c_0}_\pm(\lambda, \nu, 0) = \det\left(D\nu^2 + \nu c + F_U(U^0_\pm;0) - \lambda \right) = 0$ for $\lambda \gg 1$.

For fixed λ, $d^{c_0}_\pm(\lambda, \nu, 0)$ is a polynomial in ν of degree $2N$ which therefore has precisely $2N$ roots. For $\lambda \gg 1$, we set $\nu = \sqrt{\lambda}\, \check{\nu}$ to get

$$\det\left(D\nu^2 + \nu c_0 + F_U(U^0_\pm;0) - \lambda\right) = \det\left(D\lambda\check{\nu}^2 + \sqrt{\lambda}\,\check{\nu}c_0 + F_U(U^0_\pm;0) - \lambda\right)$$

$$= \lambda^N \det\left(D\check{\nu}^2 + \frac{\check{\nu}c_0}{\sqrt{\lambda}} + \frac{1}{\lambda}F_U(U^0_\pm;0) - 1\right)$$

which vanishes if, and only if, $\det\left(D\check{\nu}^2 + \frac{\check{\nu}c_0}{\sqrt{\lambda}} + \frac{1}{\lambda}F_U(U^0_\pm;0) - 1\right) = 0.$

Setting $\varepsilon = \frac{1}{\sqrt{\lambda}}$ with $\varepsilon > 0$ close to zero, we obtain

(10) $$\det\left(D\check{\nu}^2 + \varepsilon\check{\nu}c_0 + \varepsilon^2 F_U(U^0_\pm;0) - 1\right) = (d_1\check{\nu}^2-1)\cdot\ldots\cdot(d_N\check{\nu}^2-1) + \underbrace{O(\varepsilon)}_{} \overset{!}{=} 0$$

where $D = \mathrm{diag}(d_j) > 0$. For $\varepsilon = 0$, we obtain $2N$ roots

$$\check{\nu}_j^\pm = \pm\sqrt{d_j} \qquad j=1,\ldots,N \qquad \text{(counted with multiplicity)}$$

polynomial of degree $2N-1$ in $\check{\nu}$ with coefficients that are smooth in ε and vanish at $\varepsilon = 0$.

Rouché's theorem (see any „complex variables" textbook) implies that (10) has $2N$ solutions which are close to $\check{\nu}_j^\pm$ for all $\varepsilon > 0$ sufficiently small. Hence, we see that

$d^{c_0}_\pm(\lambda,\nu,0) = 0$ has precisely N solution ν with $\mathrm{Re}\,\nu > 0$ and N solutions with $\mathrm{Re}\,\nu < 0$ for all $\lambda \gg 1$. We now move λ along the positive axis from $\lambda \gg 1$ towards $\lambda = 0$, while monitoring the $2N$ roots of $d^{c_0}_\pm(\lambda,\nu,0) = 0$. Rouché's theorem implies that the number of roots with positive or negative real part can only change if one of these roots crosses the imaginary axis: Thus, if the roots ν of $d^{c_0}_\pm(0,\nu,0)$ do not split $N:N$, then there is a $\lambda \geq 0$ and a $k \in \mathbb{R}$ with $d^{c_0}_\pm(\lambda, ik, 0) = 0$: this, however, cannot happen due to (i) and (ii) on the previous page. Thus, the roots of $\det\left(D\nu^2 + \nu c_0 + F_U(U^0_\pm;0) - \lambda\right) = 0$ split $N:N$ at $\lambda = 0$. For $\lambda = 0$, this is the characteristic polynomial of $A^0_\pm$, and the lemma is proved. $\square$

The proof of Lemma 1 is very useful: It shows that spatial roots ν of $d^c_\pm(\lambda,\nu,\mu)$ are very rigid for $\lambda \gg 1$ and that they depend only on the leading-order term $U_t = DU_{xx}$ of the underlying PDE. We will use this property again later.

As a consequence of Lemma 1, the stable and unstable manifolds of the hyperbolic equilibria $u_\pm$ of (9) have dimension N. We also know that

$$q(\xi) \in W^u(u_-) \cap W^s(u_+),$$

and therefore

$$q_\xi(\xi) \in T_{q(\xi)}W^u(u_-) \cap T_{q(\xi)}W^s(u_+).$$

I claim that $\dim\left[T_{q(0)} W^u(u_-) \cap T_{q(0)} W^s(u_+)\right] = 1$

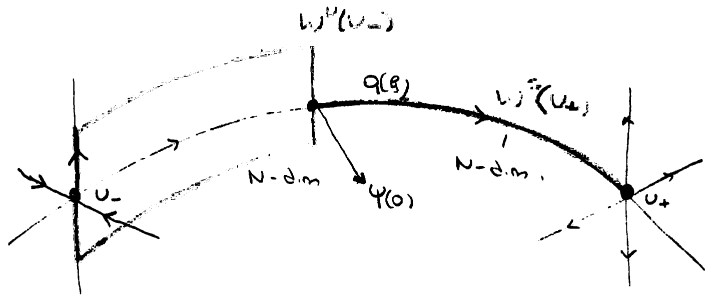
which corresponds to the statement that $v(\xi) = q_\xi(\xi)$ is the only bounded solution (up to scalar multiples) of the variational equation

$$v_\xi = \begin{pmatrix} 0 & 1 \\ -D^{-1} F_U(Q(\xi),0) & -c_0 D^{-1} \end{pmatrix} v \qquad v \in Y_0 \tag{11}$$

of the travelling-wave ODE (9) associated with the heteroclinic orbit $q(\xi)$. Any solution $v(\xi) = (U,V)(\xi)$ of (11) corresponds to a solution U of

$$\mathcal{L}_0 U = D U_{\xi\xi} + c_0 U_\xi + F_U(Q(\xi);0)\, U = 0.$$

and vice versa. If $v(\xi)$ is a bounded solution of (11), then $v(\xi)$ decays exponentially as $\xi \to \pm\infty$ and corresponds therefore to an element $U(\xi)$ in the null space of $\mathcal{L}_0$ in L^2. We assumed in (H3) that this null space contains only multiples of $Q_\xi(\xi)$ which proves that $\dim\left[T_{q(0)} W^u(u_-) \cap T_{q(0)} W^s(u_+)\right] = 1$ as claimed (more precisely, the intersection is one-dimensional since the <u>geometric</u> multiplicity of $\lambda = 0$ was assumed to be one, which gives $\dim N(\mathcal{L}_0) = 1$).
null space



in $Y_0 = \mathbb{R}^N \times \mathbb{R}^N$

$$\psi(0) \perp \left(T_{q(0)} W^u(u_-) + T_{q(0)} W^s(u_+)\right)$$

Next, we need to understand how $W^u(u_-)$ and $W^s(u_+)$ behave when c is varied near c_0. We expect that these manifolds no longer intersect when $c \neq c_0$ but need to make this expectation more precise.

Exponential dichotomies

Consider

(12) $$u_x = A(x)u \qquad u \in \mathbb{R}^n$$

where

$$|A(x) - A_\pm| \le K e^{-\alpha|x|} \qquad x \to \pm\infty$$

for appropriate hyperbolic matrices $A_\pm$. The solution operator $\Phi(x,y)$ of (12) maps $u(y)$ to $u(x)$ for any solution u of (12). We have

$$\Phi(x,x) = 1, \qquad \Phi(x,y)\,\Phi(y,z) = \Phi(x,z) \qquad \forall x,y,z \in \mathbb{R}.$$

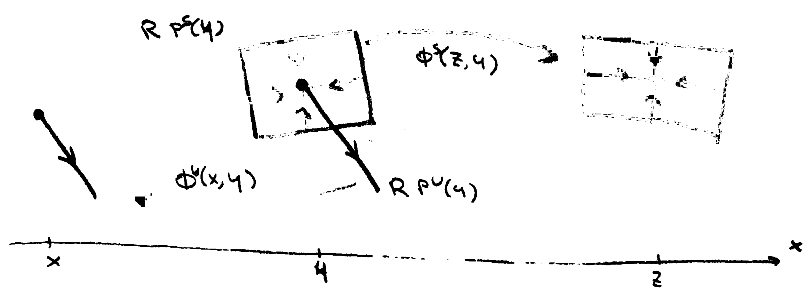
We need to separate initial data of solutions that decay as x increases from data whose solutions decay as x decreases.

Definition Let $J \subset \mathbb{R}$ be an unbounded interval (ie $J = \mathbb{R}, \mathbb{R}^+$ or $\mathbb{R}^-$). Eqn. (12) is said to have an exponential dichotomy on J if there are operators $\Phi^s(x,y)$ and $\Phi^u(x,y)$ defined for $x,y \in J$ with $x \ge y$ and $x \le y$, respectively, and numbers $\eta, K > 0$ so that

(i) $\Phi^s(x,y)v_0$ and $\Phi^u(x,y)v_0$ are solutions of (12) for $x,y \in J$ with

$$\begin{cases} |\Phi^s(x,y)v_0| \le K e^{-\eta|x-y|}\,|v_0| & x \ge y \\ |\Phi^u(x,y)v_0| \le K e^{-\eta|x-y|}\,|v_0| & x \le y \end{cases}$$

(ii)
$$\begin{cases} \Phi^s(x,y)\,\Phi^s(y,z) = \Phi^s(x,z) & x \ge y \ge z \\ \Phi^u(x,y)\,\Phi^u(y,z) = \Phi^u(x,z) & x \le y \le z \\ \Phi^s(x,y)\,\Phi^u(y,z) = 0 & x \ge y,\ z \ge y \\ \Phi^u(x,y)\,\Phi^s(y,z) = 0 & x \le y,\ z \le y \\ \Phi^s(x,x) + \Phi^u(x,x) = 1 & \forall x \end{cases}$$



$$\left.\begin{aligned} P^s(x) &= \Phi^s(x,x) \\ P^u(x) &= \Phi^u(x,x) \end{aligned}\right\} \text{Projections}$$

Proposition 1 Assume that $A(x) \to A_\pm$ as $x \to \pm\infty$ for appropriate hyperbolic matrices $A_\pm$, then

1) Eqn. (12) has exponential dichotomies $\Phi^s_+(x,y), \Phi^u_+(x,y)$ on $\mathbb{R}^+$ and $\Phi^s_-(x,y), \Phi^u_-(x,y)$ on $\mathbb{R}^-$.

2) Eqn. (12) has an exponential dichotomy on $\mathbb{R}$ iff $R\,\Phi^s_+(0,0) \oplus R\,\Phi^u_-(0,0) = \mathbb{R}^n$.

Remark • $R\,\Phi^s_+(0,0)$ is unique, but $N\,\Phi^s_+(0,0) = R\,\Phi^u_+(0,0)$ is not

Proof of Proposition 1 $\longrightarrow$ following typed notes. $\square$

B

Integral formulation for exponential dichotomies

We write

$$u_t = [A + B(t)]u$$

where A is hyperbolic and $|B(t)| \to 0$ as $t \to \infty$. If we denote the spectral projections of A onto the set of eigenvalues with positive and negative real part by $P_0^{\rm s}$ and $P_0^{\rm s}$, respectively, then there are positive constants $\eta > 0$ and $K > 0$ with

$$|\mathrm{e}^{At} P_0^{\rm s}| + |\mathrm{e}^{-At} P_0^{\rm u}| \le K \mathrm{e}^{-\eta t}$$

for $t \ge 0$. Pick $0 < \gamma < \eta$ and choose $t_0 \gg 1$ so that

$$\sup_{t \ge t_0} |B(t)| \le \frac{\eta - \gamma}{6K}.$$

Consider the fixed-point equation

$$\begin{aligned}
\Phi^{\rm s}(t,s) &= \mathrm{e}^{A(t-s)} P_0^{\rm s} + \int_s^t \mathrm{e}^{A(t-\tau)} P_0^{\rm s} B(\tau)\,\Phi^{\rm s}(\tau,s)\,\mathrm{d}\tau + \int_\infty^t \mathrm{e}^{A(t-\tau)} P_0^{\rm u} B(\tau)\,\Phi^{\rm s}(\tau,s)\,\mathrm{d}\tau \\
&\quad - \int_{t_0}^s \mathrm{e}^{A(t-\tau)} P_0^{\rm s} B(\tau)\,\Phi^{\rm u}(\tau,s)\,\mathrm{d}\tau, \qquad (t \ge s \ge t_0) \\
\Phi^{\rm u}(t,s) &= \mathrm{e}^{A(t-s)} P_0^{\rm u} + \int_s^t \mathrm{e}^{A(t-\tau)} P_0^{\rm u} B(\tau)\,\Phi^{\rm u}(\tau,s)\,\mathrm{d}\tau + \int_{t_0}^t \mathrm{e}^{A(t-\tau)} P_0^{\rm s} B(\tau)\,\Phi^{\rm u}(\tau,s)\,\mathrm{d}\tau \\
&\quad + \int_s^\infty \mathrm{e}^{A(t-\tau)} P_0^{\rm u} B(\tau)\,\Phi^{\rm s}(\tau,s)\,\mathrm{d}\tau, \qquad (s \ge t \ge t_0)
\end{aligned}$$

Define the spaces

$$\begin{aligned}
\mathcal{X}^{\rm s} &= \Big\{\varphi^{\rm s};\ \varphi^{\rm s}(t,s) \in \mathbb{R}^{n\times n} \text{ defined and continuous for } t \ge s \ge t_0 \text{ with } \|\varphi^{\rm s}\|_{\rm s} = \sup_{t \ge s \ge t_0} \mathrm{e}^{\gamma(t-s)} |\varphi^{\rm s}(t,s)|\Big\} \\
\mathcal{X}^{\rm u} &= \Big\{\varphi^{\rm u};\ \varphi^{\rm u}(t,s) \in \mathbb{R}^{n\times n} \text{ defined and continuous for } s \ge t \ge t_0 \text{ with } \|\varphi^{\rm u}\|_{\rm u} = \sup_{s \ge t \ge t_0} \mathrm{e}^{\gamma(s-t)} |\varphi^{\rm u}(t,s)|\Big\}
\end{aligned}$$

Define $R = (R^{\rm s}, R^{\rm u}) \in \mathcal{X}^{\rm s} \oplus \mathcal{X}^{\rm u}$ by

$$\begin{aligned}
R^{\rm s}(t,s) &= \mathrm{e}^{A(t-s)} P_0^{\rm s}, \qquad (t \ge s \ge t_0) \\
R^{\rm u}(t,s) &= \mathrm{e}^{A(t-s)} P_0^{\rm u}, \qquad (s \ge t \ge t_0)
\end{aligned}$$

Define the linear map $\mathcal{T} : \mathcal{X}^{\rm s} \oplus \mathcal{X}^{\rm u} \longmapsto \mathcal{X}^{\rm s} \oplus \mathcal{X}^{\rm u}$ by

$$\begin{aligned}
[\mathcal{T}^{\rm s}(\varphi^{\rm s},\varphi^{\rm u})](t,s) &= \int_s^t \mathrm{e}^{A(t-\tau)} P_0^{\rm s} B(\tau)\,\varphi^{\rm s}(\tau,s)\,\mathrm{d}\tau + \int_\infty^t \mathrm{e}^{A(t-\tau)} P_0^{\rm u} B(\tau)\,\varphi^{\rm s}(\tau,s)\,\mathrm{d}\tau \\
&\quad - \int_{t_0}^s \mathrm{e}^{A(t-\tau)} P_0^{\rm s} B(\tau)\,\varphi^{\rm u}(\tau,s)\,\mathrm{d}\tau, \qquad (t \ge s \ge t_0) \\
[\mathcal{T}^{\rm u}(\varphi^{\rm s},\varphi^{\rm u})](t,s) &= \int_s^t \mathrm{e}^{A(t-\tau)} P_0^{\rm u} B(\tau)\,\varphi^{\rm u}(\tau,s)\,\mathrm{d}\tau + \int_{t_0}^t \mathrm{e}^{A(t-\tau)} P_0^{\rm s} B(\tau)\,\varphi^{\rm u}(\tau,s)\,\mathrm{d}\tau \\
&\quad + \int_s^\infty \mathrm{e}^{A(t-\tau)} P_0^{\rm u} B(\tau)\,\varphi^{\rm s}(\tau,s)\,\mathrm{d}\tau, \qquad (s \ge t \ge t_0)
\end{aligned}$$

The fixed-point equation can then be written as

$$(\mathrm{id} - \mathcal{T})(\varphi^{\rm s},\varphi^{\rm u}) = R \tag{1}$$

where $(\varphi^{\rm s},\varphi^{\rm u}) \in \mathcal{X}^{\rm s} \oplus \mathcal{X}^{\rm u}$. It follows that

$$\|\mathcal{T}\| \le \frac{3K}{\eta-\gamma} \sup_{t \ge t_0} |B(t)| \le \frac{1}{2}.$$

Denote the unique solution of (1) by $(\Phi^{\rm s}(t,s), \Phi^{\rm u}(t,s))$. Fix $s \ge \sigma \ge t_0$ and define

$$\begin{aligned}
\varphi^{\rm s}(t) &= \Phi^{\rm s}(t,s)\Phi^{\rm s}(s,\sigma), \qquad (t \ge s) \\
\varphi^{\rm u}(t) &= \Phi^{\rm u}(t,s)\Phi^{\rm s}(s,\sigma), \qquad (t \le s)
\end{aligned}$$

We then have from the fixed-point equation that

$$\begin{aligned}
\varphi^{\rm s}(t) &= \mathrm{e}^{A(t-s)} P_0^{\rm s}\,\Phi^{\rm s}(s,\sigma) + \int_s^t \mathrm{e}^{A(t-\tau)} P_0^{\rm s} B(\tau)\,\varphi^{\rm s}(\tau)\,\mathrm{d}\tau + \int_\infty^t \mathrm{e}^{A(t-\tau)} P_0^{\rm u} B(\tau)\,\varphi^{\rm s}(\tau)\,\mathrm{d}\tau \\
&\quad - \int_{t_0}^s \mathrm{e}^{A(t-\tau)} P_0^{\rm s} B(\tau)\,\varphi^{\rm u}(\tau)\,\mathrm{d}\tau, \qquad (t \ge s) \\
\varphi^{\rm u}(t) &= \mathrm{e}^{A(t-s)} P_0^{\rm u}\,\Phi^{\rm s}(s,\sigma) + \int_s^t \mathrm{e}^{A(t-\tau)} P_0^{\rm u} B(\tau)\,\varphi^{\rm u}(\tau)\,\mathrm{d}\tau + \int_{t_0}^t \mathrm{e}^{A(t-\tau)} P_0^{\rm s} B(\tau)\,\varphi^{\rm u}(\tau)\,\mathrm{d}\tau \\
&\quad + \int_s^\infty \mathrm{e}^{A(t-\tau)} P_0^{\rm u} B(\tau)\,\varphi^{\rm s}(\tau)\,\mathrm{d}\tau, \qquad (t \le s)
\end{aligned}$$

Regarding $(\varphi^{\rm s},\varphi^{\rm u})$ as unknowns, it follows as before that this equation has a unique solution that is therefore given by the above expression for $(\varphi^{\rm s},\varphi^{\rm u})$. Since

$$\begin{aligned}
\Phi^{\rm s}(s,\sigma) &= \mathrm{e}^{A(s-\sigma)} P_0^{\rm s} + \int_\sigma^s \mathrm{e}^{A(s-\tau)} P_0^{\rm s} B(\tau)\,\Phi^{\rm s}(\tau,\sigma)\,\mathrm{d}\tau + \int_\infty^s \mathrm{e}^{A(s-\tau)} P_0^{\rm u} B(\tau)\,\Phi^{\rm s}(\tau,\sigma)\,\mathrm{d}\tau \\
&\quad - \int_{t_0}^\sigma \mathrm{e}^{A(s-\tau)} P_0^{\rm s} B(\tau)\,\Phi^{\rm u}(\tau,\sigma)\,\mathrm{d}\tau,
\end{aligned}$$

we obtain

$$\begin{aligned}
P_0^{\rm s}\,\Phi^{\rm s}(s,\sigma) &= \mathrm{e}^{A(s-\sigma)} P_0^{\rm s} + \int_\sigma^s \mathrm{e}^{A(s-\tau)} P_0^{\rm s} B(\tau)\,\Phi^{\rm s}(\tau,\sigma)\,\mathrm{d}\tau - \int_{t_0}^\sigma \mathrm{e}^{A(s-\tau)} P_0^{\rm s} B(\tau)\,\Phi^{\rm u}(\tau,\sigma)\,\mathrm{d}\tau, \\
P_0^{\rm u}\,\Phi^{\rm s}(s,\sigma) &= \int_\infty^s \mathrm{e}^{A(s-\tau)} P_0^{\rm u} B(\tau)\,\Phi^{\rm s}(\tau,\sigma)\,\mathrm{d}\tau
\end{aligned}$$

and therefore

$$\begin{aligned}
\varphi^{\rm s}(t) &= \mathrm{e}^{A(t-s)} P_0^{\rm s}\Big(\mathrm{e}^{A(s-\sigma)} P_0^{\rm s} + \int_\sigma^s \mathrm{e}^{A(s-\tau)} P_0^{\rm s} B(\tau)\,\Phi^{\rm s}(\tau,\sigma)\,\mathrm{d}\tau - \int_{t_0}^\sigma \mathrm{e}^{A(s-\tau)} P_0^{\rm s} B(\tau)\,\Phi^{\rm u}(\tau,\sigma)\,\mathrm{d}\tau\Big) \\
&\quad + \int_s^t \mathrm{e}^{A(t-\tau)} P_0^{\rm s} B(\tau)\,\varphi^{\rm s}(\tau)\,\mathrm{d}\tau + \int_\infty^t \mathrm{e}^{A(t-\tau)} P_0^{\rm u} B(\tau)\,\varphi^{\rm s}(\tau)\,\mathrm{d}\tau \\
&\quad - \int_{t_0}^s \mathrm{e}^{A(t-\tau)} P_0^{\rm s} B(\tau)\,\varphi^{\rm u}(\tau)\,\mathrm{d}\tau, \qquad (t \ge s) \\
\varphi^{\rm u}(t) &= \mathrm{e}^{A(t-s)} P_0^{\rm u} \int_\infty^s \mathrm{e}^{A(s-\tau)} P_0^{\rm u} B(\tau)\,\Phi^{\rm s}(\tau,\sigma)\,\mathrm{d}\tau + \int_s^t \mathrm{e}^{A(t-\tau)} P_0^{\rm u} B(\tau)\,\varphi^{\rm u}(\tau)\,\mathrm{d}\tau \\
&\quad + \int_{t_0}^t \mathrm{e}^{A(t-\tau)} P_0^{\rm s} B(\tau)\,\varphi^{\rm u}(\tau)\,\mathrm{d}\tau + \int_s^\infty \mathrm{e}^{A(t-\tau)} P_0^{\rm u} B(\tau)\,\varphi^{\rm s}(\tau)\,\mathrm{d}\tau, \qquad (t \le s)
\end{aligned}$$

Simplifying these expressions, we obtain

$$\begin{aligned}
\varphi^{\rm s}(t) &= \mathrm{e}^{A(t-\sigma)} P_0^{\rm s} + \int_\sigma^s \mathrm{e}^{A(t-\tau)} P_0^{\rm s} B(\tau)\,\Phi^{\rm s}(\tau,\sigma)\,\mathrm{d}\tau - \int_{t_0}^\sigma \mathrm{e}^{A(t-\tau)} P_0^{\rm s} B(\tau)\,\Phi^{\rm u}(\tau,\sigma)\,\mathrm{d}\tau \\
&\quad + \int_s^t \mathrm{e}^{A(t-\tau)} P_0^{\rm s} B(\tau)\,\varphi^{\rm s}(\tau)\,\mathrm{d}\tau + \int_\infty^t \mathrm{e}^{A(t-\tau)} P_0^{\rm u} B(\tau)\,\varphi^{\rm s}(\tau)\,\mathrm{d}\tau
\end{aligned}$$

$$
\begin{aligned}
&\quad - \int_{t_0}^{s} \mathrm{e}^{A(t-\tau)} P_0^{\mathrm{s}} \, B(\tau) \, \varphi^{\mathrm{u}}(\tau) \, \mathrm{d}\tau, && (t \geq s) \\
\varphi^{\mathrm{u}}(t) &= \int_{\infty}^{s} \mathrm{e}^{A(t-\tau)} P_0^{\mathrm{u}} \, B(\tau) \, \Phi^{\mathrm{s}}(\tau,\sigma) \, \mathrm{d}\tau + \int_{s}^{t} \mathrm{e}^{A(t-\tau)} P_0^{\mathrm{u}} \, B(\tau) \, \varphi^{\mathrm{u}}(\tau) \, \mathrm{d}\tau \\
&\quad + \int_{t_0}^{t} \mathrm{e}^{A(t-\tau)} P_0^{\mathrm{s}} \, B(\tau) \, \varphi^{\mathrm{u}}(\tau) \, \mathrm{d}\tau + \int_{s}^{\infty} \mathrm{e}^{A(t-\tau)} P_0^{\mathrm{u}} \, B(\tau) \, \varphi^{\mathrm{s}}(\tau) \, \mathrm{d}\tau, && (t \leq s)
\end{aligned}
$$

We had shown that this equation has the unique solution

$$
\begin{aligned}
\varphi^{\mathrm{s}}(t) &= \Phi^{\mathrm{s}}(t,s)\Phi^{\mathrm{s}}(s,\sigma), && (t \geq s) \\
\varphi^{\mathrm{u}}(t) &= \Phi^{\mathrm{u}}(t,s)\Phi^{\mathrm{s}}(s,\sigma), && (t \leq s)
\end{aligned}
$$

On the other hand, we can substitute

$$
\begin{aligned}
\varphi^{\mathrm{s}}(t) &= \Phi^{\mathrm{s}}(t,\sigma), && (t \geq s) \\
\varphi^{\mathrm{u}}(t) &= 0 && (t \leq s)
\end{aligned}
$$

and obtain

$$
\begin{aligned}
\Phi^{\mathrm{s}}(t,\sigma) &= \mathrm{e}^{A(t-\sigma)} P_0^{\mathrm{s}} + \int_{\sigma}^{s} \mathrm{e}^{A(t-\tau)} P_0^{\mathrm{s}} \, B(\tau) \, \Phi^{\mathrm{s}}(\tau,\sigma) \, \mathrm{d}\tau - \int_{t_0}^{\sigma} \mathrm{e}^{A(t-\tau)} P_0^{\mathrm{s}} \, B(\tau) \, \Phi^{\mathrm{u}}(\tau,\sigma) \, \mathrm{d}\tau \\
&\quad + \int_{s}^{t} \mathrm{e}^{A(t-\tau)} P_0^{\mathrm{s}} \, B(\tau) \, \Phi^{\mathrm{s}}(\tau,\sigma) \, \mathrm{d}\tau + \int_{\infty}^{t} \mathrm{e}^{A(t-\tau)} P_0^{\mathrm{u}} \, B(\tau) \, \Phi^{\mathrm{s}}(\tau,\sigma) \, \mathrm{d}\tau, && (t \geq s) \\
0 &= \int_{\infty}^{s} \mathrm{e}^{A(t-\tau)} P_0^{\mathrm{u}} \, B(\tau) \, \Phi^{\mathrm{s}}(\tau,\sigma) \, \mathrm{d}\tau + \int_{s}^{\infty} \mathrm{e}^{A(t-\tau)} P_0^{\mathrm{u}} \, B(\tau) \, \Phi^{\mathrm{s}}(\tau,\sigma) \, \mathrm{d}\tau, && (t \leq s)
\end{aligned}
$$

The second equation is obviously satisfied, while the first equation can be simplified to

$$
\begin{aligned}
\Phi^{\mathrm{s}}(t,\sigma) &= \mathrm{e}^{A(t-\sigma)} P_0^{\mathrm{s}} + \int_{\sigma}^{t} \mathrm{e}^{A(t-\tau)} P_0^{\mathrm{s}} \, B(\tau) \, \Phi^{\mathrm{s}}(\tau,\sigma) \, \mathrm{d}\tau - \int_{t_0}^{\sigma} \mathrm{e}^{A(t-\tau)} P_0^{\mathrm{s}} \, B(\tau) \, \Phi^{\mathrm{u}}(\tau,\sigma) \, \mathrm{d}\tau \\
&\quad + \int_{\infty}^{t} \mathrm{e}^{A(t-\tau)} P_0^{\mathrm{u}} \, B(\tau) \, \Phi^{\mathrm{s}}(\tau,\sigma) \, \mathrm{d}\tau, && (t \geq s)
\end{aligned}
$$

This equation, however, is also met: it is the first equation in the fixed-point equation that is satisfied by $(\Phi^{\mathrm{s}}, \Phi^{\mathrm{u}})$. We conclude that

$$
\begin{aligned}
\Phi^{\mathrm{s}}(t,s)\Phi^{\mathrm{s}}(s,\sigma) &= \Phi^{\mathrm{s}}(t,\sigma) \\
\Phi^{\mathrm{u}}(t,s)\Phi^{\mathrm{s}}(s,\sigma) &= 0
\end{aligned}
$$

Consider now the full linearization about the equilibria $U_\pm^0$ in Y:

$$A_\pm = \begin{pmatrix} 0 & 1 \\ D^{-1}[\partial_t - F_U(U_\pm^0,0)] & -cD^{-1} \end{pmatrix} \qquad Y = H^1(S^1)\times H^{1/2}(S^1) \qquad S^1 = [0, \tfrac{2\pi}{\omega_0}]/\sim$$

Since $U_\pm^0 \in \mathbb{R}^N$ does not depend on t, the operator decouples on each Fourier space:

$$Y_\ell = \left\{ e^{i\omega_0 \ell t}\begin{pmatrix} U_\ell \\ V_\ell \end{pmatrix};\ \begin{pmatrix} U_\ell \\ V_\ell \end{pmatrix} \in \mathbb{R}^N \times \mathbb{R}^N \right\} \qquad \ell \in \mathbb{Z}$$

$$A_\pm e^{i\omega_0 \ell t}\begin{pmatrix} U_\ell \\ V_\ell \end{pmatrix} = e^{i\omega_0 \ell t}\begin{pmatrix} 0 & 1 \\ D^{-1}[i\omega_0 \ell - F_U(U_\pm^0,0)] & -cD^{-1} \end{pmatrix}\begin{pmatrix} U_\ell \\ V_\ell \end{pmatrix} = \nu e^{i\omega_0 \ell t}\begin{pmatrix} U_\ell \\ V_\ell \end{pmatrix}$$

i.e.

$$\begin{pmatrix} 0 & 1 \\ D^{-1}[i\omega_0 \ell - F_U(U_0^\pm,0)] & -cD^{-1} \end{pmatrix}\begin{pmatrix} U_\ell \\ V_\ell \end{pmatrix} = \nu \begin{pmatrix} U_\ell \\ V_\ell \end{pmatrix}$$

has nontrivial solutions $\begin{pmatrix} U_\ell \\ V_\ell \end{pmatrix}$ iff

$$\begin{aligned} 0 &= \det\left[\nu(\nu + cD^{-1}) - D^{-1}(i\omega_0 \ell - F_U(U_\pm^0,0)) \right] \\ &= (\det D^{-1}) \det\left[D\nu^2 + c\nu + F_U(U_\pm^0,0) - i\omega_0 \ell \right] \\ &= (\det D^{-1})\ d_\pm^c(i\omega_0 \ell, \nu, 0) \qquad \ell \in \mathbb{Z} \end{aligned}$$

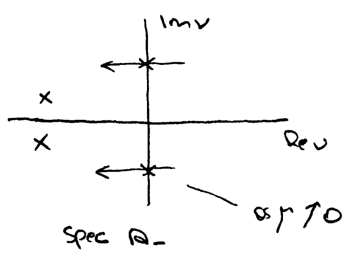
We therefore conclude that

$\ell \neq \pm 1$: $A_\pm|_{Y_\ell}$ is hyperbolic with an $N{:}N$ splitting of its eigenvalues ν

$\ell = \pm 1$: $A_+|_{Y_{\pm 1}}$ is hyperbolic with an $N{:}N$ splitting

$\ell = \pm 1$: $A_-|_{Y_{\pm 1}}$ is hyperbolic except for a simple pair of spatial eigenvalues ν_H given by $\nu_H = \dfrac{\pm i\omega_0 - \mu}{c} + O(\mu^2)$ for $\mu \approx 0$.

Furthermore, $A_-|_{Y_{\pm 1}}$ is hyperbolic for $\mu < 0$ with an $N{:}N$ splitting.



$\ell = \pm 1$
$c > 0$

$$\nu^H = \frac{\pm i\omega}{c} - \frac{\mu}{c} + O(\mu^2).$$

as $\mu \nearrow 0$ since we have an $N{:}N$ splitting for $\mu < 0$

Proof Homotopy from $\lambda = i\omega_0 \ell + \infty$ to $\lambda = i\omega_0 \ell$ and use (i)-(ii) on p. 6

The spatial Hopf eigenvalues ν_H arise for A_- at $\ell = \pm 1$ by solving

$$\lambda = i\omega_0 = \mu + ikc - d(k-k_0)^2 + \text{h.o.t.}$$

for k and setting $\nu = ik$. □

We need to solve the variational equation

$$(13) \qquad \begin{pmatrix} U \\ V \end{pmatrix}_\xi = \begin{pmatrix} 0 & 1 \\ D^{-1}(\partial_t - F_U(Q(\xi),0)) & -cD^{-1} \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} \qquad \text{in } Y$$

Recall

$$Y_\ell = \left\{ e^{i\omega_0 \ell t} \begin{pmatrix} U_\ell \\ V_\ell \end{pmatrix} ; \begin{pmatrix} U_\ell \\ V_\ell \end{pmatrix} \in \mathbb{R}^N \times \mathbb{R}^N \right\}$$

For $\ell \neq 0$, the norm on Y_ℓ is

$$\left| e^{i\omega_0 \ell t} \begin{pmatrix} U_\ell \\ V_\ell \end{pmatrix} \right|^2_{H^1 \times H^{1/2}} = |\ell|^2 |U_\ell|^2 + |\ell| \, |V_\ell|^2$$

so that $\sum_{\ell \in \mathbb{Z}} \left(|\ell|^2 |U_\ell|^2 + |\ell| \, |V_\ell|^2 \right) < \infty$.

Thus, we set $\begin{pmatrix} U_\ell \\ V_\ell \end{pmatrix} = \begin{pmatrix} \hat{U}_\ell / |\ell| \\ \hat{V}_\ell / |\ell|^{1/2} \end{pmatrix}$ with the ordinary ℓ^2-norm for $\begin{pmatrix} \hat{U} \\ \hat{V} \end{pmatrix}$.

In these variables, equation (13) becomes

$$\begin{pmatrix} \hat{U}_\ell \\ \hat{V}_\ell \end{pmatrix}_\xi = \begin{pmatrix} 0 & |\ell|^{1/2} \\ D^{-1}(i\omega_0 \ell - F_U(Q(\xi),0)) \frac{1}{|\ell|^{1/2}} & -cD^{-1} \end{pmatrix} \begin{pmatrix} \hat{U}_\ell \\ \hat{V}_\ell \end{pmatrix}$$

$$= |\ell|^{1/2} \begin{pmatrix} 0 & 1 \\ D^{-1}\left(i\omega_0 \operatorname{sign}\ell - \frac{1}{|\ell|} F_U(Q(\xi),0)\right) & -\frac{c}{|\ell|^{1/2}} D^{-1} \end{pmatrix} \begin{pmatrix} \hat{U}_\ell \\ \hat{V}_\ell \end{pmatrix}$$

Rescaling $\zeta = |\ell|^{1/2} \xi$, we obtain

$$\begin{pmatrix} \hat{U}_\ell \\ \hat{V}_\ell \end{pmatrix}_\zeta = \underbrace{\begin{pmatrix} 0 & 1 \\ D^{-1}\left(\pm i\omega_0 - \frac{1}{|\ell|} F_U(Q(\zeta/|\ell|^{1/2}); 0)\right) & -\frac{c}{|\ell|^{1/2}} D^{-1} \end{pmatrix}}_{\text{matrices in } \mathbb{C}^N \times \mathbb{C}^N} \begin{pmatrix} \hat{U}_\ell \\ \hat{V}_\ell \end{pmatrix}$$

For $\ell = \infty$, we obtain

$$(*) \qquad u_\zeta = \begin{pmatrix} 0 & 1 \\ \pm D^{-1} i\omega_0 & 0 \end{pmatrix} u \qquad u \in \mathbb{C}^N \times \mathbb{C}^N$$

while, for $\ell \gg 1$, we get

$$(**) \qquad u_\zeta = \left[\begin{pmatrix} 0 & 1 \\ \pm D^{-1} i\omega_0 & 0 \end{pmatrix} + \underbrace{\frac{1}{|\ell|^{1/2}} \begin{pmatrix} 0 & 0 \\ -\frac{1}{|\ell|^{1/2}} D^{-1} F_U \, Q(\zeta/|\ell|^{1/2}); 0) & -cD^{-1} \end{pmatrix}}_{\substack{\to 0 \text{ in norm as } |\ell| \to \infty \\ \text{uniformly in } \zeta \in \mathbb{R}}} \right] u$$

Equation $(*)$ has an exponential dichotomy on $\mathbb{R}$. Thus, $(**)$ has exponential dichotomies on $\mathbb{R}^\pm$ for all $\ell \in \mathbb{Z}$ with $|\ell| \geq \ell_0$ for some $\ell_0 \in \mathbb{N}$, with exponential rates and constants that do not depend on ℓ.

We study the spatial dynamical system

$$\begin{pmatrix} U \\ V \end{pmatrix}_\xi = \begin{pmatrix} V \\ D^{-1}(U_t - F(U,\mu) - cV) \end{pmatrix} \qquad \begin{pmatrix} U \\ V \end{pmatrix}(\xi) \in H^1(S^1) \times H^{1/2}(S^1) =: Y$$

The linearization

$$\begin{pmatrix} U \\ V \end{pmatrix}_\xi = \begin{pmatrix} 0 & 1 \\ D^{-1}[\partial_t - F_U(U_*,\mu)] & -cD^{-1} \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}$$

about any time-independent solution $(U_*, V_*)(\xi) \in Y_0$ decouples on each Fourier subspace

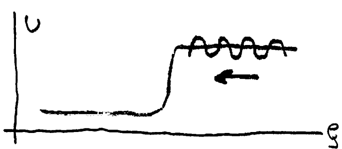
$$Y_\ell = \left\{ e^{i\omega_0 \ell t} \begin{pmatrix} U_\ell \\ V_\ell \end{pmatrix} ; \begin{pmatrix} U_\ell \\ V_\ell \end{pmatrix} \in \mathbb{C}^{2N} \right\} \cong \mathbb{C}^{2N} \qquad \ell \in \mathbb{Z}$$

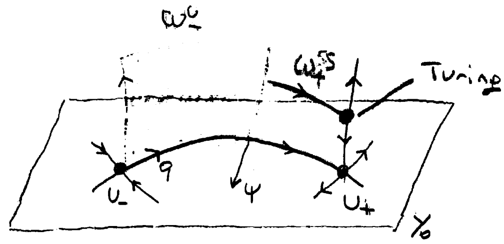
where it becomes

(1) $$\begin{pmatrix} U_\ell \\ V_\ell \end{pmatrix}_\xi = \begin{pmatrix} 0 & 1 \\ D^{-1}[i\omega_0 \ell - F_U(U_*,\mu)] & -cD^{-1} \end{pmatrix} \begin{pmatrix} U_\ell \\ V_\ell \end{pmatrix} \qquad 2N\text{-dim. ODE}$$

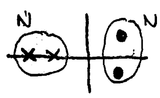
The two geometric situations we are interested in are:

1) Turing bifurcation ahead of fronts:

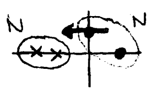




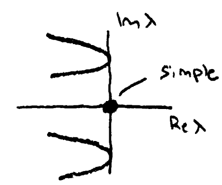
supercritical: $\mu > 0$



Spectrum of (1) about U_- $\forall \ell$, U_+ $\forall \ell \neq \pm 1$



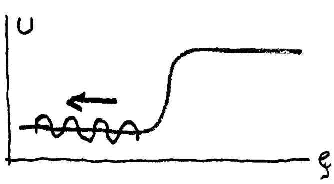
spectrum of (1) about U_+ at $\ell = \pm 1$

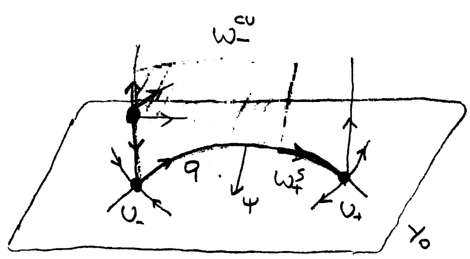


Spec $\mathcal{L}$

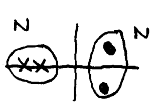
$\mathcal{L} = D\partial_{xx} + c\partial_x + F_U(Q,0)$

2) Turing bifurcation behind fronts:

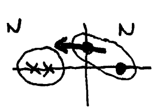




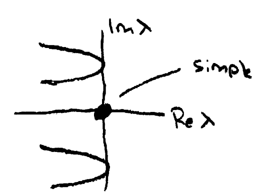
supercritical: $\mu > 0$



Spectrum of (1) about U_- $\forall \ell \neq \pm 1$, U_+ $\forall \ell$



spectrum of (1) about U_- at $\ell = \pm 1$

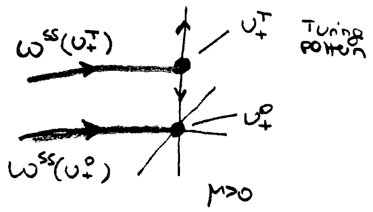


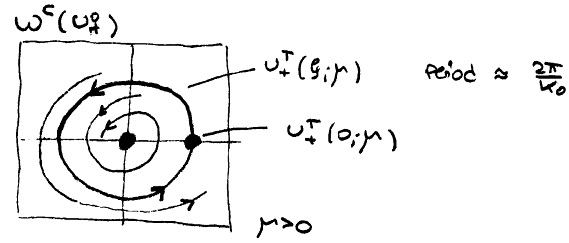
Spec $\mathcal{L}$

$\mathcal{L} = D\partial_{xx} + c\partial_x + F_U(Q,0)$

1) We concentrate on Turing bifurcation ahead of fronts :

- Center manifold





The bifurcating solution $U_+^T(\xi;\mu)$ is continuous in μ, and $U_+^T(\xi;0) = U_+^0 \ \forall \xi$.

We consider the (strong) stable fibre $W^{ss}(U_+^T(0;\mu))$ which consists of all initial data $\tilde{U}(0)$ such that $|\tilde{U}(\xi) - U_+^T(\xi;\mu)| \le Ke^{-\alpha\xi}$ as $\xi \to \infty$ for some $\alpha>0$.

$W^{ss}(U_+^T(0;\mu))$ is a smooth manifold that depends continuously on μ in the C^1-topology with $W^{ss}(U_+^T(0;0)) = W^{ss}(U_+^0)$: (strong) stable manifold of U_+^0.

(Note that these manifolds are constructed as solutions to appropriate integral equations which can be used to deduce the above properties — see below for more details)

- We set $\mu=0$ and discuss the geometry of $W^u(U_-^0)$ and $W^{ss}(U_+^0)$:
Since $Q(\xi) \in Y_0$, the linearization of the spatial dynamical system decouples on each Y_ℓ.
We then obtain that

$$E^u := T_{q(0)} W^u(U_-^0) = \bigoplus_{\ell\in\mathbb{Z}} E_\ell^u$$
$$E^s := T_{q(0)} W^{ss}(U_+^0) = \bigoplus_{\ell\in\mathbb{Z}} E_\ell^s$$

$\Big\}$ E_ℓ^u, E_ℓ^s are N-dimensional $\forall \ell$.

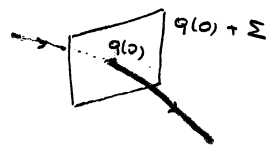
and

$$E^u \cap E^s = \mathbb{R} q_\xi(0) \subset Y_0$$

(Since we assumed that $\mathcal{L}_T\, e^{i\ell p}$ has no null space in L^2 unless $\ell=0$ where the null space is one-dimensional)

which implies $(E^u + E^s)^\perp = \mathbb{R}\psi_0 \in Y_0$ for some $\psi_0 \in Y_0$ with $\psi_0 \ne 0$.

Let $\Sigma := \{\mathbb{R} q_\xi(0)\}^\perp$, then



$$\tilde{E}^u := T_{q(0)} W^u(U_-^0) \cap \Sigma$$
$$\tilde{E}^s := T_{q(0)} W^{s}(U_+^0) \cap \Sigma$$

satisfy
$$\tilde{E}^s \cap \tilde{E}^u = \{0\}$$
$$(\tilde{E}^s + \tilde{E}^u)^\perp = \mathbb{R}\psi_0$$
$$\Sigma = \tilde{E}^s \times \tilde{E}^u \times \mathbb{R}\psi_0$$

- Therefore, we can parametrize $W^u(U_-(\mu))$ and $W^{ss}(U_+^T(0;\mu))$ for $c \approx c_0$ by

$$W^u(U_-(\mu)) \cap q(0)+\Sigma = q(0) + \{ (h^u(w^u,c,\mu),\ w^u,\ h_\psi^u(w^u,c,\mu)) ;\ w^u \in \tilde{E}^u \}$$
$$W^{ss}(U_+^T(0;\mu)) \cap q(0)+\Sigma = q(0) + \{ (w^s,\ h^s(w^s,c,\mu),\ h_\psi^s(w^s,c,\mu)) ;\ w^s \in \tilde{E}^s \}$$

where

$$D_w h^j,\ D_w h_\psi^j = 0 \quad \text{at } (0,c_0,0) \quad \text{for } j=s,u$$

$$h_\psi^u(0,c,0) - h_\psi^s(0,c,0) = M(c-c_0) + O((c-c_0)^2) \qquad \text{for some } M \ne 0$$

h^j, h_ψ^j is smooth in (w,c) for each fixed μ, and in h^j, h_ψ^j and its derivatives with respect to (w,c) are continuous in μ

I had argued previously that $M \neq 0$ is equivalent to the assumption that $\lambda = 0$ is algebraically simple as an eigenvalue of $\mathcal{L}_*$

Intersections of $W^u(u_-(\mu))$ and $W^{ss}(u_+^T(0;\mu))$ in $q(0)+\Sigma$ are therefore found as solutions of the system

$$\mathcal{F}: \quad \tilde{E}^s \times \tilde{E}^{uu} \times \mathbb{R} \times \mathbb{R} \longrightarrow \tilde{E}^s \times \tilde{E}^u \times \mathbb{R}\psi_0$$
$$(w^s, w^u, c, \mu) \longmapsto \left(h^u(w^u,c,\mu) - w^s,\ w^u - h^s(w^s,c,\mu),\ h^u_\psi(w^u,c,\mu) - h^s_\psi(w^s,c,\mu)\right)$$

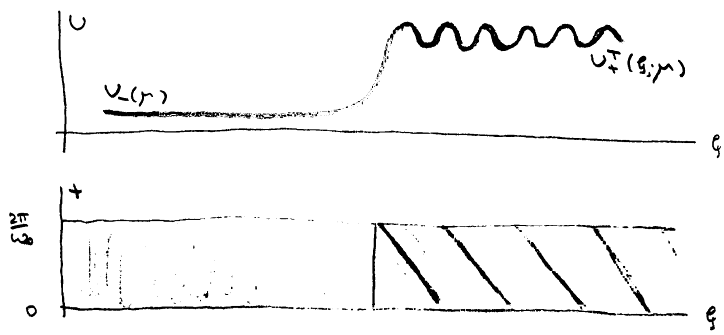
We have that

$\mathcal{F}$ is smooth in (w^s, w^u, c) for fixed μ and $\mathcal{F}$, $D_{(w^s,w^u,c)}\mathcal{F}$ are C^0 in μ

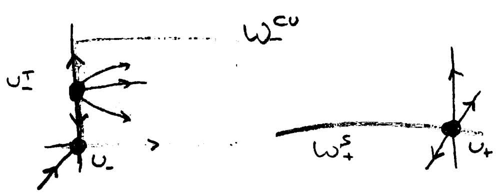
$\mathcal{F}(0,0,c_0,0) = 0$

$$D_{(w^s,w^u,c)}\mathcal{F}(0,0,c_0,0) = \begin{pmatrix} -1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & M \end{pmatrix} \quad \text{invertible}$$

$\Rightarrow$ The implicit function theorem shows the existence of a unique solution of $\mathcal{F}(w^s, w^u, c, \mu) = 0$ near $(0,0,c_0,0)$ for each fixed $\mu \geq 0$, and this solution is continuous in μ



2) For Turing bifurcations behind fronts, the following manifolds are considered:



Any pattern that converges to $u_-^T(\xi;\mu)$ as $\xi \to -\infty$ and to $u_+(\mu)$ as $\xi \to \infty$ lies in the intersection of $W^{cu}(u_-(\mu))$ and $W^s(u_+(\mu))$

Note that $W^u(u_-(\mu))$ for $\mu<0$ becomes $W^{cu}(u_-^0)$ at $\mu=0$, and extends smoothly in μ into $\mu>0$ as $W^{cu}(u_-(\mu))$. The reason is that eigenvalues cross from right (unstable) into left (stable) half plane.

Proceeding as in 1), we find that $W^{cu}(u_-(\mu))$ and $W^s(u_+(\mu))$ intersect in $q(0)+\Sigma$ for a unique $c \approx c_0$ and at a unique intersection point for each μ near zero. However, the same construction works also for the restriction of the spatial dynamical system to the invariant subspace Y_0, where it gives the continuation of the front $Q(\xi)$ for $\mu \approx 0$. Uniqueness in either case implies that the unique intersection points of $W^{cu}(u_-(\mu))$ and $W^s(u_+(\mu))$ in Y actually lies in Y_0 and is given by the persisting front.

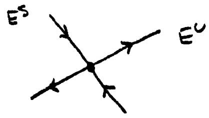
$\Rightarrow$ Genuinely time-periodic solutions near the front do <u>not</u> bifurcate.

Exponential dichotomies for general spatial dynamical systems:

(1) $\quad u_x = [A + B(x)]\, u \qquad u \in X \qquad$ (X reflexive Banach space)

We need the following assumptions:

1) $u_x = Au$ has an exponential dichotomy with rate η_* on $\mathbb{R}$:



$$A = A^s + A^u, \quad E^u \oplus E^s = Y$$

$$\begin{cases} e^{A^s x} : \text{ semigroup on } E^s \\ e^{-A^u x} : \text{ semigroup on } E^u \end{cases}$$

2) $B \in C^{0,\Theta}(\mathbb{R}^+, L(X^\alpha, X))$ for some $\Theta > 0$ and $\alpha \in [0,1)$. Furthermore, $\exists x_* \in \mathbb{R}^+$: $B(x) = S(x) + K(x)$ $\forall x \in \mathbb{R}^+$ with $\|S(x)\|_{L(X^\alpha, X)} \leq \varepsilon$ $\forall x$ and $K(x) = 0$ $\forall x \geq x_*$

3) Compactness: A^{-1} is compact as operator on X

4) Backward uniqueness: If u satisfies (1) or its adjoint on $\mathbb{R}^+$ with $u(0) = 0$, then $u \equiv 0$.

Theorem: Assume 1. is met, then for each η with $0 < \eta < \eta_*$ $\exists\, \varepsilon > 0$ and $K \geq 1$ so that the following is true when 2.-4. are satisfied: Equation (1) has an exponential dichotomy on $\mathbb{R}^+$ with rate η and constant K.

Outline of the proof

$$(2)\begin{cases} e^{A^s(x-y)} P^s u_0 = \Phi^s(x,y) + e^{A^s x} P^s \Phi^u(0,y) + \int_x^\infty e^{A^u(x-z)} P^u B(z)\, \Phi^s(z,y)\, dz \\ \qquad - \int_y^x e^{A^s(x-z)} P^s B(z)\, \Phi^s(z,y)\, dz + \int_0^y e^{A^s(x-z)} P^s B(z)\, \Phi^u(z,y)\, dz & (x \geq y) \\ e^{A^u(x-y)} P^u u_0 = \Phi^u(x,y) - e^{A^s x} P^s \Phi^u(0,y) - \int_y^\infty e^{A^u(x-z)} P^u B(z)\, \Phi^s(z,y)\, dz \\ \qquad - \int_y^x e^{A^u(x-z)} P^u B(z)\, \Phi^u(z,y)\, dz + \int_x^0 e^{A^s(x-z)} P^s B(z)\, \Phi^u(z,y)\, dz & (x \leq y) \end{cases}$$

(i) Find the stable subspace at $y=0$, i.e. all initial data leading to exp. decaying solutions of (1).
$\longrightarrow$ set $y = 0$ and $\Phi^u(0,0) = 0$ in (2) to get

$$\begin{cases} e^{A^s x} P^s u_0 = \Phi^s(x,0) + \int_x^\infty e^{A^u(x-z)} P^u B(z)\, \Phi^s(z,0)\, dz - \int_0^x e^{A^s(x-z)} P^s B(z)\, \Phi^s(z,0)\, dz \\ P^u u_0 = -\int_0^\infty e^{A^u(x-z)} P^u B(z)\, \Phi^s(z,0)\, dz \end{cases}$$

which we write as $h u_0 = T_0^s \Phi^s$ which h given.

Lemma: T_0^s is Fredholm with index zero

Proof: $T_0^s = \text{id} + \underbrace{\text{small}}_{S(x)} + \underbrace{\text{compact}}_{\substack{A^{-1} \text{ compact} \\ K(x) \text{ acts on compact interval } [0, x_*]}} \longrightarrow$ Fredholm index 0. $\square$

$\rightarrow \quad E^s = \{ ((T_0^s)^{-1} Rh)(x=0) \}$.

(ii) Fix $y \geq 0$ and consider again (2).
We choose a closed complement E^u of E^s in X^α (note: the existence of closed complements requires a proof!)

$$T : \{(\phi^s, \phi^u) \mid \phi^u(0,y) \in E^u\} \to \{(\phi^s, \phi^u) \mid \phi^u(0,y) \in RP^u\}$$

<u>Lemma</u> T is an isomorphism

<u>Proof</u>
- T is Fredholm with index for $B \geq 0$.
- T has trivial null space. □