The Evans function: An example

Consider the scalar partial differential equation (PDE)

$$u_t = u_{xx} - u + u^3, \qquad u \in \mathbb{R}, \qquad x \in \mathbb{R} \tag{1}$$

which has the stationary solution $u(x,t) = q(x) := \sqrt{2}\,\mathrm{sech}\,x$.

Linearizing of (1) about $q(x)$ gives the linear PDE

$$u_t = u_{xx} + (3q(x)^2 - 1)u$$

or

$$u_t = u_{xx} + (6\,\mathrm{sech}^2(x) - 1)u.$$

The essential spectrum is given by

$$\Sigma_{\mathrm{rm}} = \{\lambda \in \mathbb{C};\ \lambda = -k^2 - 1 \text{ for } k \in \mathbb{R}\}$$

The eigenvalue problem associated with the linearization about the pulse q is given by

$$\lambda u = u_{xx} + (6\,\mathrm{sech}^2(x) - 1)u, \tag{2}$$

which we shall also write as the linear first-order differential equation

$$\begin{pmatrix} u_x \\ v_x \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 + \lambda - 6\,\mathrm{sech}^2(x) & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \tag{3}$$

To construct the Evans function, we need to find solutions of (2) or (3) that decay to zero as $x \to \infty$ or as $x \to -\infty$.

Using hypergeometric series, these solutions are found to be

$$u_-(x;\lambda) = \mathrm{e}^{\sqrt{1+\lambda}x}\left[1 + \frac{\lambda}{3} - \sqrt{1+\lambda}\tanh(x) - \mathrm{sech}^2(x)\right],$$

which decays to 0 as $x \to -\infty$ for $\mathrm{Re}\,\lambda > -1$, and

$$u_+(x;\lambda) = \mathrm{e}^{-\sqrt{1+\lambda}x}\left[1 + \frac{\lambda}{3} + \sqrt{1+\lambda}\tanh(x) - \mathrm{sech}^2(x)\right]$$

which decays to 0 as $x \to \infty$ for $\mathrm{Re}\,\lambda > -1$.

The Evans function $E(\lambda)$ is defined to be the Wronskian

$$E(\lambda) = \det\begin{pmatrix} u_-(0;\lambda) & u_+(0;\lambda) \\ u_-'(0;\lambda) & u_+'(0;\lambda) \end{pmatrix} = -\frac{2}{9}\lambda(\lambda - 3)\sqrt{1+\lambda} \tag{4}$$

of the two solutions $u_\pm(x;\lambda)$.

By construction, a complex number λ is a root of the Evans function precisely when (2) and (3) have a bounded nonzero solution for that value of λ: indeed, the solutions $u_-(x;\lambda)$ and $u_+(x;\lambda)$ are then linearly dependent and generate a bounded nonzero solution of (2).

Thus, the spectrum is given by