- All electric and magnetic phenomena in nature can be attributed to the existence of electrical charge and charged particle motions. In classical descriptions, charge carriers having charge q and mass m are treated as **"point particles"** (or "test charges") which obey Newton's 2nd law of motion ($\mathbf{F} = m\ d\mathbf{v}/dt$). In the presence of an electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ (which are related to distant charge carriers as described by Maxwell's equations), such a point particle will not affect the fields in its vicinity, yet it will experience a force $\mathbf{F}$ (and thus an acceleration) as it moves with a velocity $\mathbf{v}$ through the fields as described by the **Lorentz force law**: $\mathbf{F} = q\ (\mathbf{E} + \mathbf{v} \times \mathbf{B})$.

- Here, $\mathbf{F}$, $\mathbf{E}$, $\mathbf{B}$ and $\mathbf{v}$ are all vector fields which can be expressed in Cartesian coordinates in terms of mutually orthogonal unit vectors $\hat{x}$, $\hat{y}$ and $\hat{z}$, which are related in terms of the **right hand rule**. In general, each component of a vector field can itself be position and time dependent: i.e, $\mathbf{E}(\mathbf{r}, t) = \hat{x}E_x(x, y, z, t) + \hat{y}E_y(x, y, z, t) + \hat{z}E_z(x, y, z, t)$. Vector fields obey the principle of superposition and the vector operations of dot product (yielding a scalar) and cross product (yielding another vector).

- Stationary distributions of charge carriers generate static electric fields $\mathbf{E}$ [V/m]. The $\mathbf{E}$ generated in a vacuum by a single stationary point charge having charge of Q [C] is radially symmetric around Q and decreases inversely as the square of the distance from the charge (**Coulomb's Law:** $\mathbf{E} = Q/(4\pi\epsilon_0 r^2)\ \hat{\mathbf{r}}$ [V/m]), where $\epsilon_0 \approx 10^{-9}/(36\pi)$ [F/m] is the **permittivity of free space**.

- **Gauss's Law for E in integral form** states that the flux of the electric **displacement vector** $\mathbf{D} \equiv \epsilon_0\mathbf{E}$ [C/m^2] through any closed surface S embedded in a vacuum equals the total amount of charge enclosed inside the volume V bounded by S:

$$\oint_S \mathbf{D} \cdot \mathbf{dS} = \int_V \rho\ dV = Q_{encl}$$

  Note that $\mathbf{dS}$ always points towards the outside of the volume. When using Gauss' Law to derive $\mathbf{E}$ generated by symmetric charge distributions, it is wise to construct the Gaussian surface to exploit the symmetry in order to simplify the evaluation of the surface integral in the LHS of the equation. For example, by considering a single point charge inside a spherically symmetric Gaussian surface around the charge, Coulomb's Law can be derived. You should be familiar with using Gauss' Law to derive the electric field generated by simple, stationary charge distributions, including point charges (in terms of Q [C]), line charges (in terms of ρ_l [C/m]), surface charges (in terms of ρ_S [C/m^2]), and symmetric volumetric charges (in terms of ρ [C/m^3]).

- Since vector fields obey superposition, $\mathbf{E}$ generated by more complicated charge distributions can be found straightforwardly by appropriately adding the simple results above. For example, two infinite sheets holding equal and algebraically opposite charge density $\pm\rho_S$ [C/m^2] produce $\mathbf{E} = \hat{\mathbf{e}}\ \rho_s/\epsilon_0$ V/m between the sheets (where $\hat{\mathbf{e}}$ is the unit vector normal to the two sheets pointing away from the positive charge density) and $\mathbf{E} = 0$ on either side of them.

- Stationary charge distributions in nature commonly exist in two forms: (1) as "free" charges distributed on the surface of conducting materials, or (2) as "bound" volumetric charge distributed within a dielectric solid. Different material substances are distinguished by their response to applied fields, which in turn depends on the proportions of free or bound charges they contain. These two materials, and the behavior of electric fields within them, are discussed in more detail below:

- Electric fields inside ***conducting materials***:
  - In highly conducting materials, many of the charge carriers contained within move easily in response the Lorentz force they experience in the presence of applied fields. Collisions with the lattice of atomic nuclei in the material, which occur at a frequency ν [s^{-1}], cause them to lose momentum. In response to these two forces (electrostatic Lorentz force and opposing friction term), the charge carriers experience a net drift velocity $\mathbf{v}$ which can be determined from Newton's Force Law.
  - The drift of electrons inside a conducting material constitutes a **free current density** $\mathbf{J} = qN\mathbf{v}$ [$\mathrm{A/m^2}$], where q, N, and $\mathbf{v}$ are the charge, volumetric number density, and velocity of the charge carriers. The current density can also be expressed by Ohm's Law in terms of the applied $\mathbf{E}$ and **electrical conductivity** σ [S/m] as $\mathbf{J} = \sigma\mathbf{E}$.
  - When a **perfect conductor** (having $\sigma = \infty$ and no bound charges) is placed in a static electric field $\mathbf{E_0}$, the conducting electrons inside will move freely and accumulate at the surface of the material until the resulting surface charge distribution generates a secondary field $\mathbf{E}'$ that completely cancels the applied $\mathbf{E}$ inside the conductor. As a result, $\mathbf{E} = \mathbf{E_0} + \mathbf{E}' = 0$ inside a perfect conductor instantaneously, and $\mathbf{E} = 0$ inside any conductor in steady-state equilibrium.
- Electric fields inside ***dielectric materials***:
  - Dielectrics are materials having a large number of charged particles in bound form, which are not free to move far away from the nucleus that they orbit in response to an applied field $\mathbf{E_0}$. Instead, a negatively charged cloud of bound (orbiting) electrons (of total charge $-Q$) becomes slightly displaced (by d) from the positively charged atomic nucleus (of charge Q) and produces a charge dipole having a **dipole moment** $\mathbf{p} \equiv Q\mathbf{d}$ [Cm] and thus an induced electric field in the opposite direction as the applied field. Some molecules have permanent dipole moments, which can change orientation/alignment in response to an applied $\mathbf{E}$. A **perfect dielectric** contains no free charges (thus $\sigma = 0$) and all charges are bound.
  - In a dielectric material which has N_d dipoles (stretched atoms and/or re-aligned molecules with intrinsic $\mathbf{p}$) per unit volume, the material itself has a dipole moment per unit volume $\mathbf{P} = N_d\,\mathbf{p}$ [$\mathrm{C/m^2}$], also known as the **polarization field**. Superposing all of the individual dipole moments inside a perfect dielectric having a uniform polarization $\mathbf{P}$ yields a macroscopic induced electric field $\mathbf{E}' = -\mathbf{P}/\epsilon_0$. Superposing $\mathbf{E}'$ with an externally applied electric field $\mathbf{E_0}$ yields a total field $\mathbf{E} = \mathbf{E}_0 - \mathbf{P}/\epsilon_0$. The displacement vector is generalized to account for these induced fields by being **REDEFINED** as $\mathbf{D} \equiv \epsilon_0\mathbf{E} + \mathbf{P}$ [$\mathrm{C/m^2}$]. Since $\mathbf{P} = 0$ in a vacuum, $\mathbf{D}$ reverts to its original free space expression.
  - In most dielectrics, the polarization induced by an applied electric field $\mathbf{E_0}$ is proportional to the total field $\mathbf{E}$, such that $\mathbf{P} = \epsilon_0\chi_e\mathbf{E}$, where χ_e is the (dimensionless) **susceptibility**. Thus, the displacement vector is $\mathbf{D} \equiv \epsilon_0\mathbf{E} + \mathbf{P} = \epsilon_0(1 + \chi_e)\mathbf{E} \equiv \epsilon_0\epsilon_r\mathbf{E} \equiv \epsilon\mathbf{E}$ in terms of the **relative permittivity** ϵ [F/m], which is always greater than the permittivity of free space ϵ_0. Note that, in a perfect dielectric (having no free charges such that $\nabla \cdot \mathbf{D} = 0$), the displacement field is constant and does not change from its free space value.
- Vector fields $\mathbf{D}$ and $\mathbf{E}$ will be continuous in space except across boundaries carrying non-zero surface charge density ρ_S [$\mathrm{C/m^2}$]. **Boundary conditions** relating the fields on one side of the boundary to fields on the other are given by:

$$\hat{n} \times (\mathbf{E}^+ - \mathbf{E}^-) = 0$$
$$\hat{n} \cdot (\mathbf{D}^+ - \mathbf{D}^-) = \rho_S$$

Here, $\hat{n}$ is defined as the unit vector pointing perpendicularly away from the boundary, and region + is defined to be on the side towards which $\hat{n}$ is pointing. Note that the fields on both sides must be evaluated at the location of the boundary. These equations imply that tangential components of $\mathbf{E}$ are continuous while the normal components of $\mathbf{E}$ can be discontinuous if a surface charge is present on the boundary AND/OR if the electric permitivitties are different in the two materials.

  - If the material on one side of the boundary is a conductor, $\mathbf{E}$ must be set to zero there (since steady-state EM fields cannot exist inside a conductor). If external fields exist on the other side of the boundary, free charges inside the conductor must have moved to the surface to shield the interior of the conductor – thus $\rho_s \neq 0$.
  - If the materials on both sides of the boundary are perfect dielectrics, $\rho_s = 0$ since free charges do not exist within either medium.

- In order to relate the spatial variations of $\mathbf{E}$ at a given point to the volumetric charge density at that point, we can rewrite Gauss' Law using the **divergence theorem:** $\oint_S \mathbf{A} \cdot \mathbf{dS} = \int_V (\nabla \cdot \mathbf{A})\, dV$, where the divergence $\nabla \cdot \mathbf{A}$ of a given vector field $\mathbf{A}$ is the scalar flux density of $\mathbf{A}$, expressed in Cartesian coordinates as:

$$\nabla \cdot \mathbf{A} = \frac{\delta A_x}{\delta x} + \frac{\delta A_y}{\delta y} + \frac{\delta A_z}{\delta z}$$

  A non-zero divergence corresponds to the existence of spatial variations in the vector field in the direction of the field itself. A positive divergence of $\mathbf{A}$ implies that a source of $\mathbf{A}$ exists at that point in space, while a negative divergence corresponds to a sink. Thus, **Gauss' Law for E in differential form** is $\nabla \cdot \mathbf{D} = \rho$.

- According to the Helmholtz Theorem, both the divergence and curl of a vector field must be known in order to specify the field uniquely. The curl of a vector field $\mathbf{A}$ is the vector circulation density of $\mathbf{A}$, expressed in Cartesian coordinates as:

$$\nabla \times \mathbf{A} = \hat{x}(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}) + \hat{y}(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}) + \hat{z}(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y})$$

  A non-zero curl corresponds to the existence of spatial variations in the vector field in a direction perpendicular to the field itself. The curl of vector field $\mathbf{A}$ is related to its circulation (i.e., its line integral around a closed contour C) via the **Stokes theorem:** $\oint_C \mathbf{A} \cdot \mathbf{dl} = \int_S (\nabla \times \mathbf{A}) \cdot \mathbf{dS}$.

- Note that:
  - $\nabla \cdot \nabla \times \mathbf{A} = 0$ for any vector field $\mathbf{A}$
  - $\nabla \times \nabla \phi = 0$ for any scalar field ϕ
  - $\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$ for any $\mathbf{A}$

- All static electric fields are curl-free, i.e., $\nabla \times \mathbf{E} = 0$. Correspondingly, the circulation of static electric fields must be zero: $\oint_C \mathbf{E} \cdot \mathbf{dl} = 0$. Fields obeying these conditions are known as "conservative fields".

- Based on the second vector identity above, all static electric fields can be expressed in terms of a scalar function V, where $\mathbf{E} = -\nabla V$. Here, V is known as the **electrostatic potential** in [V]. Note that $\mathbf{E}$ points in the direction from high to low potential and is perpendicular to contours of constant potential ("equipotential surfaces"). Given $\mathbf{E}$, the potential drop $V_A - V_B$ between two points A and B can be found by taking a line integral of $\mathbf{E}$ along ***any path*** from A to B:

$$V_A - V_B = \int_B^A dV = \int_B^A \nabla V \cdot \mathbf{dl} = -\int_B^A \mathbf{E} \cdot \mathbf{dl} = \int_A^B \mathbf{E} \cdot \mathbf{dl}$$

Note that the integral $\int_A^B \mathbf{E} \cdot \mathbf{dl}$ represents the ***work done by the field* E *per unit charge*** moved from A to B. Thus, for conservation of energy to hold, $\int_A^B \mathbf{E} \cdot \mathbf{dl} = -\int_B^A \mathbf{E} \cdot \mathbf{dl}$, so that $\oint \mathbf{E} \cdot \mathbf{dl} = 0$, as required for conservative fields.

- In static situations (i.e., when ρ is time-independent) and in situations when the electric permittivity ϵ is homogeneous (not spatially varying), Gauss' Law for $\mathbf{E}$ reduces to the **Poisson's equation**: $\nabla^2 V = -\rho/\epsilon$. In the absence of static free charges, Poisson's equation reduces to **Laplace's equation**: $\nabla^2 V = 0$. Note that the solution to Laplace's equation in Cartesian coordinates must be a function V that is linearly dependent on x, y, or z, and the corresponding electric field must be constant.

- Electrostatics allows us to develop concepts such as capacitance, conductance, and resistance which are useful to describe both distributed and lumped circuits.
  - The **capacitance** C is equal to the net amount of charge Q moved between two uncharged conductors divided by the resulting potential drop V ($C = Q/V$ [F]).
  - If a current I [A/m] flows between the two conductors, the **conductance** G is defined as $G = I/V$ [S].
  - The **resistance** is defined as the reciprocal of the conductance: $R \equiv 1/G$ [Ω].

  For distributed circuits, we will be most concerned with the above parameters *per unit length along the line*, which are denoted by $\mathcal{C}$, $\mathcal{G}$, and $\mathcal{R}$, respectively.
  - An idealized **parallel plate transmission line** (T.L.) consists of a pair of perfectly conducting plates having width W separated by a perfect dielectric of thickness d (fringing effects at the edges of the plates can be neglected if $W \gg d$). With this geometry, $\mathcal{C} = \epsilon W/d$ and $\mathcal{G} = \sigma W/d$.
  - For a co-axial cable geometry with inner and outer cylindrical radii a and b, respectively, $\mathcal{C} = \epsilon(2\pi/\ln(b/a))$ and $\mathcal{G} = \sigma(2\pi/\ln(b/a))$
  - In the above configuration, W/d and $2\pi/\ln(b/a)$ are the **geometric factor** GF specific to that geometry. In general (for arbitrary geometries with different geometric factors), $\mathcal{C} = \epsilon\ (GF)$ and $\mathcal{G} = \sigma\ (GF)$.

- A charge carrier moving at a constant velocity (i.e., **a steady current**) generates magnetic flux density $\mathbf{B}$ (in units of [Wb/m^2] or [T]). The (infinitesimal) magnetic flux density $\mathbf{dB}$ generated at a radius r by an infinitesimal current element $I\mathbf{dl}$ in a vacuum is along the direction $I\mathbf{dl} \times \hat{r}$ and is given by the ***Biot-Savart Law:*** $\mathbf{dB} = \frac{\mu_0 (I\mathbf{dl} \times \hat{r})}{4\pi r^2}$, where $\mu_0 \approx 4\pi \times 10^{-7}$ [H/m] is the **permeability of free space**. The total field generated by a steady current distribution can be found by using the Biot-Savart Law in superposition for each infinitesimal current element in the distribution. Note that the Biot-Savart Law is analogous to Coulomb's Law.

- The ***static* form of Ampere's Law** restates the Biot-Savart Law by relating the magnetic intensity vector $\mathbf{H} \equiv \mathbf{B}/\mu_0$ [A/m] (in a vacuum) to volumetric current density $\mathbf{J}$ [A/m^2]. In integral form, Ampere's Law states that the magnetomotive force around a closed contour C equals the flux of current density (i.e., total current) crossing any (open) surface S bounded by C due to the motion of free charges through the surface: $\oint_C \mathbf{H} \cdot \mathbf{dl} = \int_S \mathbf{J} \cdot \mathbf{dS}$ (static case). Note that the direction of the contour C and the unit surface vector $\mathbf{dS}$ must adhere to the right hand rule. It is wise to construct the Amperian loop C to exploit the symmetry of the current distribution in order to simplify the evaluation of the surface integral in the LHS of the equation. In differential form, Ampere's Law states that $\nabla \times \mathbf{H} = \mathbf{J}$ (static case). You should be familiar with using Ampere's Law to derive the magnetic field generated by simple, steady current distributions, including line currents (in terms of I [A]), surface currents (in terms of $\mathbf{J_s}$ [A/m]) and symmetric volumetric currents.

- **Gauss' Law for B** expresses the experimental finding that magnetic fields have no point sources (unlike electric fields). In integral form, the law states that the magnetic flux through a closed surface S equals zero: $\oint \mathbf{B} \cdot \mathbf{dS} = 0$. In differential form, the law states that the divergence of $\mathbf{B}$ must be zero: $\nabla \cdot \mathbf{B} = 0$.

- Magnetic fields inside **magnetic materials**:
  - Magnetic materials are characterized by the presence of bound, looping currents, which are divergence free and do not contribute to charge accumulation. Such a looping current can arise, for example, from the orbital motions of an electron around the atomic nucleus or from the quantum mechanical electron spin. The **magnetic dipole moment m** of such a looping current is given as $\mathbf{m} = IA\,\hat{n}$ in terms of the effective current I and area A of the loop. For a magnetic material having N_d magnetic moments per unit volume, the **magnetization M** is the volume density of magnetic dipole moments, $\mathbf{M} = N_d\ \mathbf{m}$ [A/m]. The bound current density is expressed as $\mathbf{J}_{bound} = \nabla \times \mathbf{M}$.
  - Each magnetic moment produces its own $\mathbf{B}$. Superposing all the magnetic moments inside a magnetic material having uniform magnetization $\mathbf{M}$ yields a macroscopic secondary magnetic field $\mathbf{B_m}$, which when superposed with an external (applied) magnetic field $\mathbf{B_0}$ yields a total field $\mathbf{B} = \mathbf{B_m} + \mu_0\mathbf{M}$. The magnetic intensity vector is **REDEFINED** as $\mathbf{H} \equiv (1/\mu_0)\mathbf{B} - \mathbf{M}$.
  - In many magnetic materials, the magnetization $\mathbf{M}$ induced by an applied magnetic field $\mathbf{B_0}$ is proportional to the total $\mathbf{B}$, such that $\mathbf{M} = \chi_m\mathbf{H}$, where χ_m is the **magnetic susceptibility**. As a result, the magnetic flux density becomes $\mathbf{B} = \mu_0(1 + \chi_m)\mathbf{H} \equiv \mu_0\mu_r\mathbf{H} \equiv \mu\mathbf{H}$. In most materials, $\chi_m << 1$ (it can be positive or negative), though in ferromagnetic materials, it can be larger though always postive.

- Magnetostatics allows us to develop the concept of **inductance**. The (self) inductance L of a current loop is the ratio of the magnetic flux $\Phi = \int_S \mathbf{B} \cdot \mathbf{dS}$ through the surface bounded by the loop to the current I. In general, the inductance *per unit length* $\mathcal{L}$ of an arbitrary configuration of current-carrying loops is related to the magnetic permeability μ of the medium in terms of a geometric factor GF, such that $\mathcal{L} = \mu/GF$. Note the similarity of this relation with that of capacitance per unit length $\mathcal{C}$ as described above.

- **Faraday's Law** in integral form states that the induced voltage or electromotive force (EMF or $\mathcal{E}$) around a loop C (i.e., $\mathcal{E} = \oint_C \mathbf{E} \cdot \mathbf{dl}$) equals the negative time rate of change of the magnetic flux Ψ through any (open) surface S bounded by C. In differential form, Faraday's Law is stated as $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$. According to **Lenz's Law**, if a conducting wire were placed along C, current would flow in a direction that would *oppose the change* in the magnetic flux. Several situations give rise to magnetically induced EMF, such as: (1) a static contour within a time-varying $\mathbf{B}$, (2) a deforming loop C (and thus surface area S) within a static $\mathbf{B}$, (3) a rigid loop moving through a spatially varying $\mathbf{B}$, or (4) a rotating loop.

- Maxwell noted an asymmetry between Faraday's and Ampere's Laws and deduced that Ampere's Law as written above is incomplete under time-varying situations. In addition to currents due to free charges, magnetomotive force around a closed loop C also can be induced by time-varying $\mathbf{D}$ through any (open) surface S bounded by C. Thus, in time-varying situations, **Ampere's Law in integral form** becomes: $\int_C \mathbf{H} \cdot \mathbf{dl} = \int_S \mathbf{J} \cdot \mathbf{dS} + \frac{d}{dt}(\int_S \mathbf{D} \cdot \mathbf{dS})$, where $\mathbf{D} = \epsilon\mathbf{E}$ [C/m^2] is the displacement vector field. Note that the surfaces S for the integration of current due to free charges and displacement flux must be the same. In differential form, Ampere's Law becomes $\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$.

- Vector fields $\mathbf{B}$ and $\mathbf{H}$ will be continuous in space except across boundaries carrying non-zero surface current density $\mathbf{J_s}$. **Boundary conditions** relating the fields on one side of the boundary ($\mathbf{B_1}$ and $\mathbf{H_1}$) to fields on the other ($\mathbf{B_2}$ and $\mathbf{H_2}$) are given by:

$$\hat{n} \times (\mathbf{H_1} - \mathbf{H_2}) = \mathbf{J_s}$$
$$\hat{n} \cdot (\mathbf{B_1} - \mathbf{B_2}) = 0$$

  As with the boundary conditions for $\mathbf{E}$ and $\mathbf{D}$ (see above), $\hat{n}$ is defined as the unit vector pointing perpendicularly away from the boundary, and region 1 is defined to be on the side towards which $\hat{n}$ is pointing. Note that the fields on both sides must be evaluated at the location of the boundary. These equations imply that normal components of $\mathbf{B}$ are continuous while the tangential components of $\mathbf{B}$ can be discontinuous if a surface current is present on the boundary AND/OR if the magnetic permeabilities are different in the two materials.

  – If the material on one side of the boundary is a perfect conductor, $\mathbf{B}$ must be set to zero there (since EM fields cannot exist inside a perfect conductor) and thus no normal component can exist on the other side. If a tangential component exists on the other side, free charges inside the conductor must have moved to the surface to shield the interior of the conductor − thus $\mathbf{J_s} \neq 0$.

  – If the materials on both sides of the boundary are perfect dielectrics, $\mathbf{J_s} = 0$ since free charges do not exist within either medium.

- The law of **charge conservation** is an important corrolary to Maxwell's equations which can be derived from Gauss' Law for $\mathbf{E}$ together with Ampere's Law. This law states that the net current due to a flow of free charges through a closed surface S equals the negative time rate of change of the charge enclosed in the volume V bounded by S. Restating charge continuity in differential form yields $\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$.

- Together, Gauss's Laws, Faraday's Law, and Ampere's Law comprise ***Maxwell's equations***, which relate the field vectors to their source charge and current densities. To summarize, Maxwell's equations are written as follows for a linear material medium, where we distinguish between free and bound charges in the formulation of charge and current densities ρ and $\mathbf{J}$:

$$\nabla \cdot \mathbf{D} = \rho_{free}$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{H} = \mathbf{J}_{free} + \frac{\partial \mathbf{D}}{\partial t}$$

Here, the displacement vector is defined as $\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon \mathbf{E}$ in terms of the uniform **polarization** $\mathbf{P} = -\nabla \cdot \rho_{bound}$ and the magnetic intensity vector is defined as $\mathbf{H} \equiv (1/\mu_0)\mathbf{B} - \mathbf{M} = (1/\mu)\mathbf{B}$ in terms of the uniform **magnetization** $\mathbf{M}$, where $\mathbf{J}_{bound} = \partial \mathbf{P}/\partial t + \nabla \times \mathbf{M}$. In a vacuum, $\mathbf{P} = \mathbf{M} = 0$. We typically drop the subscript *free* from ρ and $\mathbf{J}$ in the equations above, with the understanding that bound charge and current descriptions are contained in the definitions for $\mathbf{D}$ and $\mathbf{H}$.

- Under static conditions (i.e, $\frac{\partial}{\partial t} = 0$), Maxwell's equations decouple into two sets of two equations governing electrostatics (stationary charges) and magnetostatics (steady currents), respectively. The interdependence of $\mathbf{E}$ and $\mathbf{H}$ described by the time-dependent Maxwell's equations implies the existence of electromagnetic (EM) waves which propagate away from their current source. Maxwell's equations must be solved simultaneously to yield solutions for the components of $\mathbf{E}$ and $\mathbf{B}$ via the electromagnetic **wave equation**, which is obtained by taking the curl of Faraday's law and substituting from Ampere's Law. ***Assuming that the waves are propagating in a homogeneous*** (constant μ and ϵ), ***perfect dielectric*** ($\rho = \mathbf{J} = \sigma = 0$), and that the wavefield is transverse and 1D (plane) with $\mathbf{E} = \hat{x}\, E_x(z,t)$ generated by a current source on the $z = 0$ plane, the 3D wave equation reduces to its $\hat{x}$ component only:

$$\frac{\partial^2 E_x}{\partial z^2} = \mu\epsilon \frac{\partial^2 E_x}{\partial t^2}.$$

  - The solution to the 1D plane wave equation above is the linear superposition of two waveforms $f(t)$ and $g(t)$ traveling in the $\pm\hat{z}$ directions, respectively (i.e., away from the source at $z = 0$ in both directions), which are weighted by constant coefficients A and B, such that $E_x(z,t) = Af(t - z/v_p) + Bg(t + z/v_p)$. In free, unbounded space, the phase speed v_p of the propagating EM wave is $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \approx 3 \times 10^8$ m/s, while in a perfect dielectric medium, the propagation speed is $v_p = \frac{1}{\sqrt{\mu\epsilon}} \leq c$.
  - For arbitrary plane wave orientations, Maxwell's equations require that $\hat{e} \times \hat{h}$ is always along the direction of wave propagation. For the electric field given above ($\mathbf{E} = \hat{x}\, E_x(z,t)$) the associated magnetic field must be given by $\mathbf{H} = \pm\hat{y}\, H_y(z,t)$ since in this case $\hat{x} \times \pm\hat{y} = \pm\hat{z}$. Maxwell's equations also imply that $|\mathbf{E}|/|\mathbf{B}| = v_p$ and that $|\mathbf{E}|/|\mathbf{H}| = \eta$ where $\eta = \sqrt{\frac{\mu}{\epsilon}}$ is the **intrinsic impedence** of the material medium in $[\Omega]$. In free space, $\eta = \eta_0 \approx 120\pi$. The constant of integration which results from solving for $\mathbf{B}$ or $\mathbf{H}$ from $\mathbf{E}$ via Faraday's Law represents a DC (background) magnetic field which is set to zero when no DC magnetic field is present.
  - The functional forms of $f(t)$ and $g(t)$, together with the coefficients A and B, are governed by the magnitude and time dependence of the surface current source $J_s(t)$. Maxwell's equations in integral form show that $Af(t) = Bg(t) = \frac{\eta}{2} J_s(t)$. For a current sheet at $z = 0$ with arbitrary time varying current density $J_s(t)$ pointing in the $-\hat{x}$ direction, the solution to the 1D plane wave equation is:

$$\mathbf{E}(z,t) = \begin{cases} \hat{x}\ \frac{\eta}{2} J_s(t - \frac{z}{v_p}) & z > 0 \\ \hat{x}\ \frac{\eta}{2} J_s(t + \frac{z}{v_p}) & z < 0 \end{cases} \tag{1}$$

$$\mathbf{H}(z,t) = \begin{cases} \hat{y}\ \frac{1}{2}J_s(t - \frac{z}{v_p}) & z > 0 \\ -\hat{y}\ \frac{1}{2}J_s(t + \frac{z}{v_p}) & z < 0 \end{cases} \quad (2)$$

Note that in general, $\mathbf{E}(z,t)$ is anti-parallel to $\mathbf{J_s}$ and has even symmetry around the sheet, while $\mathbf{H}(z,t)$ is perpendicular to $\mathbf{E}$ (such that $\hat{e} \times \hat{h} = \hat{v_p}$) and has odd symmetry around the sheet.

– For help visualizing plane EM wave propagation from various current sheet sources, click the link on the course website to view some time-lapse animations.

– Cosinusoidal surface current variation $J_s(t) = J_0 \cos(\omega t)$ generates cosinusoidal EM waves which have the same **angular frequency** ω [rad/s]. Cosinusoidal traveling waves are characterized by a length scale (i.e., **wavelength**) λ [m] and timescale (i.e., period) T [s], where $\omega = 2\pi/T$. The traveling waveform $\cos(\omega(t \mp \frac{z}{v_p}))$ is typically written in terms of the **wave number** $\beta \equiv 2\pi/\lambda$ [rad/m] as $\cos(\omega t \mp \beta z)$, where $v_p \equiv \omega/\beta$. Note that the linear frequency f can be expressed in terms of the period or the angular frequency: $f = 1/T = \omega/2\pi$ [1/s] or [Hz]. Thus, ω, T, and f describe the time variation of the wave, while λ and β describe its spatial variation. Note that $\lambda f = \omega/\beta = v_p$.

  * **Phasor notation** can be a convenient way to express cosinusoidally varying fields. For example, the phasor $\tilde{\mathbf{E}}$ of wave field $\mathbf{E} = \hat{e}\ E_0 \cos(\omega t \mp \beta z + \theta) = \mathrm{Re}\{\hat{e}\ E_0\ e^{j(\omega t \mp \beta z + \theta)}\} = \mathrm{Re}\{\tilde{\mathbf{E}}\ e^{j\omega t}\}$ is given as $\tilde{\mathbf{E}} = \hat{e}\ E_0\ e^{j(\mp\beta z + \theta)} = \hat{e}\ E_0\ e^{\mp j\beta z}e^{j\theta}$. Note that phasors have a direction and magnitude and only depend on spatial coordinates.

– Plane EM waves transport energy, momentum, and angular momentum along their direction of propagation. The transported energy per unit time (i.e. power) per unit area in the wave plane (i.e., power density) is known as the **Poynting flux S**, where $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ [W/m^2]. Note that $\mathbf{S}$ is in the direction of wave propagation.

– Energy is also stored in the fields themselves, at a volumetric density of $\frac{1}{2}\mathbf{D}\cdot\mathbf{E}$ [J/m^3] and $\frac{1}{2}\mathbf{B}\cdot\mathbf{H}$ [J/m^3] for the electric and magnetic fields, respectively. Stored power density is thus the time rate of change of the stored energy density.

– Electromagnetic energy can also be injected or dissipated locally (through the work required to accelerate conduction electrons, for example). The rate of energy injection/dissipation per unit volume is given as $\mathbf{J} \cdot \mathbf{E}$ [W/m^3].

– Together, the power density stored in the fields, the power density dissipated (or produced) locally, and the flux of transported power density must equal zero, such that electromagnetic energy is conserved. The equation describing this conservation of energy is known as the **Poynting theorem**. Locally (i.e., applying to a point in space), this equation is given by:

$$\nabla \cdot \mathbf{S} + \frac{\partial}{\partial t}(\tfrac{1}{2}\mathbf{D} \cdot \mathbf{E} + \tfrac{1}{2}\mathbf{B} \cdot \mathbf{H}) + \mathbf{J} \cdot \mathbf{E} = 0 \quad [\mathrm{W/m^3}]$$

For a volumetric region in space bounded by a closed surface, the Poynting theorem becomes (via the divergence theorem):

$$\int_S \mathbf{S} \cdot \mathbf{dS} + \frac{d}{dt}\int_V (\tfrac{1}{2}\mathbf{D} \cdot \mathbf{E} + \tfrac{1}{2}\mathbf{B} \cdot \mathbf{H})\, dV + \int_V \mathbf{J} \cdot \mathbf{E}\, dV = 0 \quad [\mathrm{W}]$$

– The Poynting flux of time-varying EM waves will also be time-varying. The time-averaged Poynting flux $\langle\mathbf{S}\rangle$ is defined as $\frac{1}{T}\int_0^T \mathbf{S}\ dt$. Using phasor notation, the time-averaged Poynting vector is $\langle\mathbf{S}\rangle = \frac{1}{2}\mathrm{Re}\{\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^*\}$ in terms of the complex conjugate of $\tilde{\mathbf{H}}$, denoted $\tilde{\mathbf{H}}^*$. Note that the time-average of the stored power density (time-rate change of energy density) is zero.

– Maxwell's equations as formulated above can be used to derive the 1D (plane) wave equation for an arbitrary material media having a finite, non-zero conductivity σ. In this case (unlike the case of wave propagation in a perfect dielectric described above), the conduction current term in Ampere's Law cannot be neglected. For linear plane waves given by $\mathbf{E} = \hat{x}E_x(z,t)$ and $\mathbf{H} = \pm\hat{y}H_y(z,t)$, the wave equation becomes

$$\frac{\partial^2 E_x}{\partial z^2} = \mu\sigma\frac{\partial E_x}{\partial t} + \mu\epsilon\frac{\partial^2 E_x}{\partial t^2}$$

In phasor notation (for cosinusoidal current sources), the plane wave equation is given as:

$$\frac{\partial^2 \tilde{E}_x}{\partial z^2} = (j\omega)^2\mu(\epsilon - j\frac{\sigma}{\omega})\tilde{E}_x$$

The solution to the phasor wave equation is a superposition of damped co-sinusoidal waves. In phasor notation, the general solution is $\tilde{E_x} = E_0\, e^{\mp\gamma z}e^{j\phi}$, where $\gamma = j\omega\sqrt{\mu(\epsilon - j\sigma/\omega)}$ is known as the **propagation constant** and ϕ is the (constant) phase of the field at the initial condition $t = z = 0$. Defining $\gamma \equiv \alpha + j\beta$ in terms of the **attenuation constant** α [1/m] and **phase constant** β [rad/m], the phasor electric field becomes $\tilde{\mathbf{E}} = \hat{x}E_0 e^{\mp\alpha z}e^{\mp j\beta z}e^{j\phi}$.

* The propagation speed of the wave is $v_p = \omega/\beta$ (same as for waves in a perfect dielectric). However, in conducting materials β is generally not linearly proportional to ω, such that v_p is a function of ω. As a result, waves of different frequency travel at different speeds, a property known as **dispersion**.

* The amplitude of the wave, $E_0\, e^{\mp\alpha z}$, is attenuated exponentially as it propagates through a material medium. The **penetration depth** δ is defined as the distance the wave must travel for its amplitude to decay by a factor of $1/e$. Thus, $\delta = 1/\alpha$ [m]. As with waves through any medium, the wavelength is $\lambda = 2\pi/\beta$; however, note that in the case where the wave amplitude attenuates, the wavelength now corresponds to the distance between two alternate zero crossings and not between two consecutive maxima.

* The corresponding magnetic field in a conducting material ($\sigma \neq 0$) is determined by the same two criteria as with TEM waves in a perfect dielectric (where $\sigma = 0$). These are: (1) $\hat{h} = \hat{v_p} \times \hat{e}$ ($= \mp\hat{z} \times \hat{x} = \pm\hat{y}$ in this case) and (2) $|\mathbf{E}|/|\mathbf{H}| = \eta$. However, in a conducting medium, the intrinsic impedance is redefined to be: $\eta = \sqrt{\mu/(\epsilon - j\frac{\sigma}{\omega})}$ [Ω]. Thus, for $\mathbf{H} = \pm\hat{y}H_y$, $\tilde{H_y} = \pm(E_0/\eta)\; e^{\mp\alpha z}e^{\mp j\beta z}e^{j\phi}$. Since η is in general a complex number, it can be written as $\eta = |\eta|e^{j\tau}$, such that $\tilde{H_y} = (E_0/|\eta|)\; e^{\mp\alpha z}e^{\mp j\beta z}e^{j\phi}e^{-j\tau}$. Note that in a material medium having a non-zero conductivity and thus complex impedence η, the magnetic field is *out of phase* with respect to the electric field by an amount τ. In general, $\tau = (1/2)\tan^{-1}(\sigma/\omega\epsilon)$. The quantity $\sigma/\omega\epsilon$ is known as the **loss tangent** and is used to determine whether the material is a good or poor conductor:

  · In a **perfect dielectric**, $\sigma = 0$, such that $\gamma = j\omega\sqrt{\mu\epsilon} = j\beta$ and $\eta = \sqrt{\mu/\epsilon}$. Note that EM waves do not attenuate as they propagate though a perfect dielectric (since $\alpha = 0$) and $\mathbf{E}$ and $\mathbf{B}$ are in phase (since η is purely real, $\tau = 0$).

  · If $\sigma/\omega\epsilon \ll 1$, then the material is a poor conductor, also known as an **imperfect dielectric**. In that case, $\gamma \approx (\sigma/2)\sqrt{\mu/\epsilon} + j\omega\sqrt{\mu\epsilon}$ and $\eta \approx \sqrt{\mu/\epsilon}(1 + j\sigma/2\omega\epsilon)$. Note that β and $\mathrm{Re}\{\eta\}$ are the same as for a perfect dielectric.

  · If $\sigma/\omega\epsilon \gg 1$, then the material is a **good conductor** and $\gamma \approx (1+j)\sqrt{\omega\mu\sigma/2}$ and $\eta \approx \sqrt{j\omega\mu/\sigma} = \sqrt{\omega\mu/\sigma}e^{j\pi/4}$. Note that in a good conductor, $\alpha = \beta$ and $\tau = \pi/4$.

    * The properties of the material medium (σ, μ, and ϵ) can be determined from knowledge of the propagation parameters of an EM wave traveling through the material, where $\sigma = \text{Re}\{\gamma/\eta\}$, $\mu = \gamma\eta/j\omega$ and $\epsilon = (1/\omega)\text{Im}\{\gamma/\eta\}$.

- **EM wave polarization** describes how the "tip" of the electric field vector at a given point in space changes position as a function of time.
  - Co-sinusiodal plane waves $\mathbf{E_1} = \hat{x}\ \cos(\omega t \mp \beta z + \theta_1)$ and $\mathbf{E_2} = \hat{y}\ \cos(\omega t \mp \beta z + \theta_2)$ are linearly polarized in the $\hat{x}$ and $\hat{y}$ directions, respectively.
  - The weighted superposition $\mathbf{E} = a\mathbf{E_1} + b\mathbf{E_2} = \hat{x}E_x + \hat{y}E_y$ is also linearly polarized if $\theta_1 - \theta_2 = 0$ or π (that is, if E_x and E_y are in phase or phase opposition).
  - For $a = b$ **AND** $\theta_1 - \theta_2 = \pm\pi/2$, then $\mathbf{E}$ is circularly polarized, such that at a given location in space, $\mathbf{E}$ rotates in a circle with increasing time. If the rotation is counterclockwise when looking FROM the direction of propagation (i.e., the direction of rotation and the direction of propagation satisfy the right-hand rule), then $\mathbf{E}$ is right-handed circularly polarized. If the rotation is clockwise, $\mathbf{E}$ is left-handed circularly polarized.
  - If none of the above criteria are fulfilled, then $\mathbf{E}$ is elliptically polarized.Note that elliptically polarized waves also rotate, such that the elliptical polarization can be right or left handed following the same convention as circularly polarized waves.

- When a TEM wave propagating through an arbitrary material medium (characterized by μ_1, ϵ_1, and σ_1) encounters a region of differing intrinsic impedance (η_2), the wave in general will both reflect from and transmit through the boundary. Maxwell's boundary conditions are used to determine the amplitude of reflected and transmitted electric wavefields ($\mathbf{E_r}$ and $\mathbf{E_t}$, respectively) relative to that of the incident electric wavefield $\mathbf{E_i}$.

  For normal incidence of linearly polarized TEM waves, the ratio $\mathbf{E_r}/\mathbf{E_i}$ *evaluated at the boundary* equals $\Gamma \equiv \frac{\eta_2-\eta_1}{\eta_2+\eta_1}$, a dimensionless scaling factor known as the **reflection coefficient**. The ratio $\mathbf{E_t}/\mathbf{E_i}$ *evaluated at the boundary* equals $\tau \equiv \frac{2\eta_2}{\eta_2+\eta_1}$, a dimensionless scaling factor known as the **transmission coefficient**. Note that $1 + \Gamma = \tau$.

  - If the wave is incident on a perfect conductor (such that $\eta_2 = 0$), the wavefield will reflect completely: $\Gamma = -1$ and $\tau = 0$. Total reflection of an incident cosinusoidal wavefield off of a normal planar boundary yields a **standing wave** which carries no net energy.
  - If the wave is incident on a region having $\eta_2 = \eta_1$ (i.e., **matched impedance**), the wavefield will transmit completely: $\Gamma = 0$ and $\tau = 1$.

- **Guided wavefields** confined to a region between two parallel conducting plates (possible if plate width W is much larger than their separation d) will generate charge and current variations on the interior surfaces of the plates facing one another in accordance with Maxwell's boundary condition equations. Alternatively, applying a time-varying surface current or charge density at some location on the plates will in turn generate propagating guided waves.

  Assuming that the TEM field between the plates is linearly polarized with $\mathbf{E} = E_x(z,t)\hat{x}$ and $\mathbf{B} = B_y(z,t)\hat{y}$, we can express the fields in terms of voltage between the plates $V \equiv E_x d$ and surface current on the bottom plate $I \equiv H_y W$. Neglecting loss terms (i.e., assuming that the conductor and dielectrics are perfect materials), Ampere's and Faraday's laws can be rewritten in terms of the circuit parameters ($\mathcal{L}$ and $\mathcal{C}$) to obtain geometry-independent equations known as **telegrapher's equations**:

$$-\frac{\partial V}{\partial z} = \mathcal{L}\frac{\partial I}{\partial t} \text{ and } -\frac{\partial I}{\partial z} = \mathcal{C}\frac{\partial V}{\partial t}$$

The telegrapher's equations can be combined to obtain a wave equation:

$$\frac{\partial^2 V}{\partial z^2} - \mathcal{LC}\frac{\partial^2 V}{\partial t^2} = 0$$

with superposed $\pm\hat{z}$-direction propagating wave solutions: $V(z,t) = f(t \mp z/v_p)$ and $I(z,t) = \pm(1/Z_0)f(t \mp z/v_p)$. The phase speed of the waves is $v_p \equiv 1/\sqrt{\mathcal{LC}} = 1/\sqrt{\mu\epsilon}$ (such that voltage and current waveforms travel at the same speed as the associated fields inside the perfect dielectric between the plates). The ratio of the voltage and current of each waveform (i.e., V^+/I^+ or V^-/I^-), is known as the **characteristic impedance** of the T.L. and denoted $Z_0 \equiv \sqrt{\mathcal{L}/\mathcal{C}} = \eta\frac{1}{\text{GF}}$, in terms of the geometric factor GF (see p.4). Note that the telegrapher's equations are valid in general for other types of two-wire T.L.s provided that appropriate (geometry-dependent) expressions for $\mathcal{L}$ and $\mathcal{C}$ are used.

- The total power transported in the T.L. waveguide is given by the Poynting vector $\mathbf{E} \times \mathbf{H}$ times the cross section of the waveguide. For the parallel plate geometry described above, power transported in the $\hat{z}$ direction reduces to $P(z,t) = V(z,t)I(z,t)$. Hence, the familiar circuit theory formula for **time-averaged power** $\langle P \rangle = (1/2)\text{Re}\{VI^*\}$ (in terms of voltage and current phasors V and I) will also hold in sinusoidal steady-state problems.

- For a T.L. having characteristic impedance Z_0, phase speed v_p, and length l, the voltage and current variations (i.e., general, unbounded solutions to the telegrapher's equations) are given as

$$V(z,t) = V^+(t - \frac{z}{v_p}) + V^-(t + \frac{z}{v_p})$$
$$I(z,t) = \frac{1}{Z_0}\left[V^+(t - \frac{z}{v_p}) - V^-(t + \frac{z}{v_p})\right]$$

where $V^+(t)$ and $V^-(t)$ are forward propagating (in the $+\hat{z}$-direction towards the load) and backwards propagating (in the $-\hat{z}$-direction towards the generator) voltage waveforms.

- When the above general T.L. is **terminated by a resistive load R_L**:
  - Voltage and current waveforms are initially scaled by a factor denoted τ_g, known as the **injection coefficient**. They will propagate down the line towards the load end, where they will be scaled (by a factor Γ_L) and reflected off of the load if its impedance R_L does not match the characteristic impedance Z_0 of the T.L. Waveforms propagating down the line towards the source end will similarly be scaled (by Γ_g) and reflected at the source if its internal resistance does not equal Z_0.
  - Familiar lumped circuit boundary conditions at the load and source ends are used to determine the injection and reflection coefficients:
    * The boundary condition at the load end requires that $V(l,t)/I(l,t) = R_L$. This condition implies that
      $\Gamma_L \equiv \dfrac{R_L - Z_0}{R_L + Z_0}$ is the **load reflection coefficient**.
      Note that *at the source* $V^-(t)$ is also a time-shifted (by delay $2l/v_p$) replica of $V^+(t)$ after scaling by Γ_L.

    * At the source end ($z = 0$) (having a source voltage $f_i(t)$ and internal resistance R_g), the boundary condition requires that $V(0,t) = f(t) - R_g\ I(0,t)$. This condition implies that $\Gamma_g \equiv \dfrac{R_g - Z_0}{R_g + Z_0}$ is the **source reflection coefficient**
      and $\tau_g = \dfrac{Z_0}{R_g + Z_0}$ is the **injection coefficient**.
  - A **short-circuit** has $R_L = 0$, such that $\Gamma_L = -1$ and the voltage reflected at the load end is the negative of the incident (forward-propagating) voltage (thus voltage across the short circuit is zero).
  - An **open circuit** has $R_L = \infty$, such that $\Gamma_L = 1$ and the reflected voltage is equal to the incident voltage (thus current through the open circuit is zero).
  - A **matched load** has $R_L = Z_0$, such that $\Gamma_L = 0$ and no reflection at the load occurs. Similarly, when $R_g = Z_0$, the internal resistance at the source end is matched to Z_0 and no reflection (of the backwards propagating wave) occurs at the source.

- The **zero-state impulse response** of the T.L. can by found by solving for $V^+(z,t)$ and $V^-(z,t)$ when the source voltage is a delta function: $f_i(t) = \delta(t)$. For the above T.L. with resistive load (and thus Γ_L defined as above), the steady-state impulse response is found to be an infinite summation of amplitude-scaled and time-shifted replicas of the delta function input:

$$V^+(z,t) = \tau_g \sum (\Gamma_L \Gamma_g)^n \delta(t - \frac{z}{v} - n\frac{2l}{v_p})$$
$$V^-(z,t) = \tau_g \Gamma_L \sum (\Gamma_L \Gamma_g)^n \delta(t + \frac{z}{v} - (n+1)\frac{2l}{v_p})$$

  (the summation above is over the number n of reflections off of the source, ranging from 0 to ∞). Note that the total voltage at any location z on the line at a given time t is the superposition of both waveforms given above. When $f_i(t)$ is an arbitrary waveform instead of $\delta(t)$, the voltage waveforms are found by convolving the impulse response functions with $f_i(t)$, which yields the **general zero-state voltage solution**. *Since the impulse response is a sum of delta functions, the convolution results in a sum of appropriately delayed $f_i(t)$.*

- **Bounce diagrams** are a useful time-domain analysis tool for calculating scaling factors and time delays following successive reflections from the load and source ends of a T.L. Note that similar (but different!) bounce diagrams can be constructed for current as well as voltage, where the coefficients of current amplitude $I_n^\pm$ are the same as those for voltage ($V_n^\pm$), but scaled by an additional factor of $\pm\frac{1}{Z_0}$.

  Bounce diagrams are also useful for analyzing multiple T.L. circuits; coefficients for voltage waveform reflection and transmission between T.L. segments connected in series, which have characteristic impedances Z_j and Z_k, respectively, are given by $\Gamma_{jk} = \dfrac{Z_k - Z_j}{Z_k + Z_j}$ and $\tau_{jk} = 1 + \Gamma_{jk}$.

- For co-sinusoidal sources $f(t) = Re\{\tilde{F}e^{j\omega t}\}$, the steady-state T.L. responses $V(z,t)$ and $I(z,t)$ are also co-sinusoidal and best handled using phasor techniques. It is conventional in sinusoidal steady-state T.L. analysis to define a new spatial coordinate $d = -z$, where $d = 0$ at the load end and $d = l$ at the generator end, where l is the length of the T.L. Phasors $\tilde{V}$ and $\tilde{I}$ *(note that the phasor tilde notation is dropped from here on)* have steady-state solutions expressed as the superpositions of forward and reverse propagating waves.

$$V(d) = V^+e^{j\beta d} + V^-e^{-j\beta d} \quad \text{and} \quad I(d) = \frac{V^+e^{j\beta d}}{Z_0} - \frac{V^-e^{-j\beta d}}{Z_0}$$

where $\beta = \omega/v_p = \omega\sqrt{\mathcal{LC}}$ and $V^\pm/I^\pm = \pm Z_0 = \pm\sqrt{\mathcal{L}/\mathcal{C}}$.

- At the load end ($d = 0$) having arbitrary (complex) impedance Z_L, the reflected voltage waveform is $V^- = \Gamma_L V^+$, where $\Gamma_L \equiv \frac{Z_L - Z_0}{Z_L + Z_0}$ is the **load reflection coefficient**.
- $V(d)$ and $I(d)$ are related in terms of the **line impedance**:

$$Z(d) \equiv V(d)/I(d) = Z_0 \frac{1 + \Gamma_L e^{-j2\beta d}}{1 - \Gamma_L e^{-j2\beta d}} = Z_0 \frac{1 + \Gamma(d)}{1 - \Gamma(d)} \; [\Omega]$$

  where $\Gamma(d) = \Gamma_L e^{-j2\beta d}$ is known as the **generalized reflection coefficient**. The impedance at the input terminal of the line (i.e., $Z(d = l)$) is known as the **input impedance** and frequently denoted Z_{in}. Note that **line admittance** is the reciprocal of impedance $Y(d) = 1/Z(d)$ and that $Y(d) = Z(d + \frac{\lambda}{4})$.
- At the generator end of the T.L. ($d = l$), the boundary condition that $V(l) = F\frac{Z(l)}{Z_s + Z(l)}$ and $I(l) = V(l)/Z(l)$ (where F is the source voltage phasor and $Z(l)$ is the input impedance of the line) can be used to solve for V^+ in terms of F.

- A **short-circuited T.L.** (having $Z_L = 0$ such that $\Gamma_L = -1$) is characterized by:
  - Phasor voltage $V(d) = V^+(e^{j\beta d} - e^{-j\beta d}) = j2V^+\sin(\beta d)$. Note that $V_L \equiv V(d = 0) = 0$.
  - Phasor current $I(d) = Y_0 2V^+\cos(\beta d)$
  - Phasors $V(d)$ and $I(d)$ describe **complete standing waves** on a short-circuited T.L., where the amplitude of cosinusoidal time-variation of line current and voltage itself varies cosinusoidally along the line. Voltage nulls occur at the load and at $\frac{\lambda}{2}$ intervals from the end. Current nulls occur at $d = \frac{\lambda}{4}$ from the load and at $\frac{\lambda}{2}$ intervals from the first minimum along the line.
  - Line impedance $Z(d) = jZ_0\tan(\beta d)$ is purely reactive and has a $\frac{\lambda}{2}$ periodicity. Note that at $d = \frac{\lambda}{4}$ (or $\beta d = \frac{\pi}{2}$), $Z(d) = \infty$, such that the line impedance of a shorted line has become that of an open circuit. In other words, a translation in distance along the line by $d = \frac{\lambda}{4}$ transforms the short into an open. Another translation by $d = \frac{\lambda}{4}$ will transform it back into a short.
- An **open-circuited T.L** (having $Z_L = \infty$ such that $\Gamma_L = 1$) is characterized by:
  - Phasor voltage $V(d) = V^+(e^{j\beta d} + e^{-j\beta d}) = 2V^+\cos(\beta d)$
  - Phasor current $I(d) = jY_0 2V^+\sin(\beta d)$, Note that $I_L \equiv I(d = 0) = 0$.
  - Like the short-circuited line, phasors $V(d)$ and $I(d)$ describe **complete standing waves** along the line with $\frac{\lambda}{2}$ periodicity and $d = \frac{\lambda}{4}$ spacing between voltage and current nulls.
  - The line admittance of an open line is $Y(d) = I(d)/V(d) = jY_0\tan(\beta d)$.
- A lossless T.L. of length l having open/short terminations on both ends can sustain unforced voltage and current standing waves at a set of discrete resonant frequencies.
  - Resonance for $Z_{in} = Z(l) = 0$ or $Y(l) = \infty$ is known as **series resonance** (regardless of the load).
  - Resonance for $Z_{in} = Z(l) = \infty$ or $Y(l) = 0$ is known as **parallel resonance** (regardless of the load).

Series or parallel resonance frequencies can also be sustained when the open or shorted stub T.L.s are used in series or parallel networks with other circuit elements, though the resonance frequencies may be shifted.

- T.L. segments of half- and quarter-wave electrical lengths have unique properties regarding the transformation between voltage, current, and impedance.
  - The **half-wave transformer** inverts the algebraic sign of the voltage and current inputs at the load end (and vice versa): $V_{in} = -V_L$ and $I_{in} = -I_L$, such that the line impedance $Z(d)$ is identical at both ends.
  - The **quarter-wave transformer** advances the voltage and current such that $V_{in} = jI_L Z_0$ and $I_{in} = jV_L/Z_0$. The line impedances at each end are related by $Z_{in}Z_L = (Z_0)^2$. Since $Z_{in} = V_{in}/I_{in}$, we can express the current at the load end simply in terms of the input voltage and characteristic impedance:

$$I_L = -j\frac{V_{in}}{Z_0}$$

    Note that the current at the load end is independent of load impedance Z_L, a property known as current forcing. This is the principle behind corporate ladder networks.

- **GENERAL CASE:** For a T.L terminated by an **arbitrary load** impedance Z_L, the load reflection coefficient $\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$ is arbitrary and generally complex, so that it can be written in terms of its magnitude and phase: $\Gamma_L = |\Gamma_L|e^{j\theta_L}$. In that case:
  - Line voltage becomes $V(d) = V^+e^{j\beta d}(1 + \Gamma(d)) = V^+e^{j\beta d}(1 + |\Gamma_L|e^{j(\theta_L - 2\beta d)})$.
  - Voltage maxima occur where $|V(d)|_{max} = |V^+|(1 + |\Gamma_L|)$ and voltage minima occur where $|V(d)|_{min} = |V^+|(1 - |\Gamma_L|)$. Note that the phase angle of $\Gamma(d)$ is $\theta = \theta_L - 2\beta d$.
  - The **standing wave ratio (VSWR)** relates the maximum and minimum voltages on the line as VSWR $= \dfrac{|V(d)|_{max}}{|V(d)|_{min}} = \dfrac{1 + |\Gamma_L|}{1 - |\Gamma_L|}$. The VSWR ranges from 1 (for $\Gamma_L = 0$, i.e., a matched line), to ∞ (for $|\Gamma_L| = 1$, i.e., the load is purely reactive).
  - Note that $|\Gamma_L| = \frac{\text{VSWR}-1}{\text{VSWR}+1}$.
  - **Normalized line impedance** is $z(d) = \frac{Z(d)}{Z_0} = \dfrac{1 + \Gamma(d)}{1 - \Gamma(d)} \equiv r(d) + jx(d)$. Note that the normalized impedance at the load is $z(0) \equiv z_L = \dfrac{1 + \Gamma_L}{1 - \Gamma_L}$.

- The parameters $\Gamma(d)$ and $z(d) = r + jx$ (and/or $y(d) = 1/z(d)$) are related graphically in a **Smith Chart**, which represents the complex plane of $\Gamma(d)$, with the $\text{Re}\{\Gamma(d)\}$ axis along the horizontal. The numerical value for $\Gamma(d)$ at a point along a T.L. segment can be represented in complex polar coordinates on the S.C. in terms of its magnitude $|\Gamma(d)| = |\Gamma_L|$ and phase angle $\theta = \theta_L - 2\beta d$.
  - The center of the chart corresponds to $|\Gamma_L| = 0$, thus VSWR=1, and thus matched conditions.
  - The perimeter of the chart is the unit circle on the complex $\Gamma(d)$ plane and corresponds to: $|\Gamma_L| = 1$, VSWR $= \infty$, and purely reactive loads.
  - For arbitrary loads (on lossless lines), $|\Gamma(d)|$ is a constant between 1 and 0, such that VSWR is a finite constant that is greater than 1. Thus, starting from the load end of the T.L. at an arbitrary location on the S.C. and moving toward the generator end, the generalized reflection coefficient progresses clockwise in a circle around the origin, since the phase angle θ decreases with increasing d while $|\Gamma(d)|$ stays constant. Counterclockwise rotation of $\Gamma(d)$ would thus correspond to decreasing distance from the load.

- The quantity $|1+\Gamma(d)|$ is the proportionality factor that relates $|V(d)|$ to $|V^+|$. Since this proportionality term is maximized at $\Gamma(d) = |\Gamma_L|$, voltage maxima occur on a S.C. at the intersection of the constant Γ circle and the positive $\mathrm{Re}\{\Gamma(d)\}$ axis. Similarly, voltage minima occur on a S.C. at the intersection of the constant Γ circle and negative $\mathrm{Re}\{\Gamma(d)\}$ axis.
- Every full rotation around the S.C. corresponds to a translation in electrical length of $\lambda/2$ (thus a half rotation corresponds to $\lambda/4$). Electrical length along the transmission line is indicated as a function of phase angle along a perimeter axis on the S.C. Note that voltage minima and maxima occur every $\lambda/4$ from each other.

Curved grid lines of constant reactance r and resistance x are superimposed on the complex $\Gamma(d)$ plane of the Smith Chart. The horizontal axis corresponds to purely real (resistive) impedance, while the unit circle perimeter corresponds to purely imaginary (reactive) impedance. The normalized line impedance $z(d)$ at a point along the T.L. (along the constant Γ circle on the S.C.) can be determined from the unique intersection of r and x contours at that location on the S.C. This mapping makes the following calculations much easier to solve graphically using a S.C. than algebraically:

- Relate d_{min} (the electrical length between the load and the nearest voltage minimum) to θ_L and thus the load reflection coefficient Γ_L
- Derive VSWR $= z(d_{max})$, the (purely real) line impedance at voltage maxima.
- Calculate normalized line admittance $y(d)$ from the intersection of contour lines at a point 180° from $z(d)$ around the constant Γ circle, since $y(d) = z(d + \lambda/4)$.

• Time-average power transported on a T.L. from generator to load is written equivalently as

$$P = \tfrac{1}{2}\mathrm{Re}\{V(d)I^*(d)\} = \frac{|V^+|^2}{2Z_0} - \frac{|V^-|^2}{2Z_0}$$

Because the T.L. is lossless, the power at the input must equal the power transported along the line which must equal the power delivered to the load, so that:

$$P = \tfrac{1}{2}\mathrm{Re}\{V(0)I^*(0)\} = \tfrac{1}{2}\mathrm{Re}\{V(d)I^*(d)\} = \tfrac{1}{2}\mathrm{Re}\{V(l)I^*(l)\}$$

• Typically, generators are designed to operate into loads matching the generator internal resistance Z_g. While it is generally easy to select T.L. having $Z_0 = Z_g$, frequently the load impedance $Z_L \neq Z_0$. **Impedance matching** can be achieved by inserting an impedance matching network in front of the load in order to achieve a VSWR=1 on the main line segment between the generator and the network.

- **Quarter wave matching of resistive loads** involves inserting a quarter-wavelength T.L. in series between the end of the T.L. and the load. A match will be achieved if the characteristic impedance of the QWT $Z_T = \sqrt{Z_0 Z_L}$.
- **Quarter wave matching of reactive loads** involves inserting a quarter-wavelength T.L. in series at some distance d_q from the load end of the T.L. A match will be achieved when the characteristic impedance of the QWT is $Z_T = \sqrt{Z_0 Z(d_q)}$. Since we require Z_T to be a purely real (resistive) impedance, d_q is chosen so that $Z(d_q)$ is purely resistive (occurs at voltage maxima/minima). Purely real impedances lie along the horizontal axis of the S.C., so finding d_q is thus a matter of finding the electrical length (or phase difference) between the load and a voltage maximum/minimum.
- **Single stub matching** involves inserting a shorted or open T.L. of length l_s in parallel at a distance d_s from the load end of the main line. A match will be achieved when $\Gamma(d_s) = 0$, or

when the input admittance at $d = d_s$ (i.e., treating the main line w/ load and stub in parallel) is $y(d_s) = 1 + j0$. Since $y(d_s)$ is the sum of the admittances of the two sections in parallel, it can be written (assuming both the T.L. and the stub have the same characteristic impedance Z_0) as $y(d_s) = y'(d_s) + y_s(l_s)$, where $y'(d_s) = g'(d_s) + jb'(d_s)$ is the input admittance of the main T.L. segment with its arbitrary load Z_L and $y_s(l_s) = jb_s(l_s)$ is the purely susceptive (imaginary) admittance of the stub evaluated at its input end. Thus a match will be achieved when $g'(d_s) = 1$ and $b_s(l_s) = -b'(d_s)$.

The steps to solving a single stub matching problem using a Smith Chart are as follows:

* Find z_L on the Smith Chart and draw the constant Γ circle around the origin through z_L.
* Find y_L, located 180° around the constant Γ circle from z_L.
* Transform y_L in a CLOCKWISE direction to a location where $g = 1$ (where the constant Γ circle intersects the unit conductance circle). The admittance at this point is $y'(d_s) = 1 + jb'(d_s)$. Note that there will be two solutions (one for each of the two intersections).
* Find the electrical length d_s as the distance between y_L and the desired intersection using the WTG axis on the outer perimeter of the S.C.
* Note the susceptance $b'(d_s)$ on the main line at the intersection, and find the negative of that susceptance on the constant Γ circle of the stub section (i.e., the unit circle). This admittance here is $y_s(l_s) = jb_s(l_s)$.
* Transform $y_s(l_s)$ to the load admittance y_L of the stub. If the stub is shorted, $y_L = \infty$ whereas if the stub is open $y_L = 0$. Note that this translation corresponds to COUNTER-CLOCKWISE rotation.
* Find the length of the stub l_s as the electrical length between $y_s(l_s)$ and y_L using the WTL axis on the perimeter of the S.C..