ECE 329

Introduction to Electromagnetic Fields

Section E

Adapted from Prof. Cunningham's Notes

Introduction

Professor Lynford Goddard
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Nanotechnology Lab
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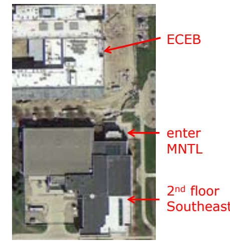
Office hrs: TH **10-11AM**
Room 2254 MNTL
Opportunity to practice solving problems and discuss concepts



Other administrative

- Course webpage: https://courses.engr.illinois.edu/ece329/
  - Syllabus
  - Course Calendar
  - Grading Policy
  - Homework assignments
  - Past Exams
  - Class Notes
  - Recorded Lectures
- Rao's book (7th ed. or 6th is good also)

How to get an A in ECE329

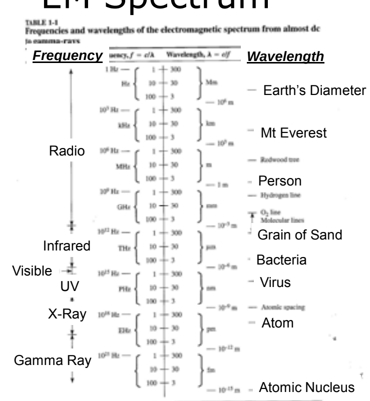
- ***Time Management.*** Allocate 10 hrs/wk of *regularly scheduled times* in the week outside of class for 329:
  - 30 min for **reading** of textbook **before** each class
  - 30 min for **studying** online notes **before** each class
  - 30 min for **studying** my notes **between** classes
  - 75 min for **practicing** problems at the tutorial session
  - 4-5 hrs/wk for HWs
    - In a semester, all lectures total only 32.5hrs, which is less than 1 week at a job! It's up to you to put in the time to learn
  - Get a 1" binder (organize lecture notes/HWs/exams)
  - Start assignments early. **Do all problems by yourself first.** If you get stuck, form study groups to work on problems together but **ALWAYS** write-up and submit **YOUR OWN** solutions. Do not blindly copy.
  - Ask questions and come to office hrs if you get stuck. Don't let confusion snowball.

How to get an A in ECE329

- ***Practice doing problems.*** Get comfortable with the math manipulations and associated physical meaning, and you will find exam problems to be easier
  - HW problems
  - Example problems worked in lecture
  - Example problems worked in online class notes
  - Old exam problems
  - Office hours
- ***Come to class!!***
  - *HW & Class Participation = 15% of your grade*
  - I will discuss topics to be emphasized on exams and give hints about how to approach the more difficult homework problems.

EM Spectrum

$\lambda f = c$



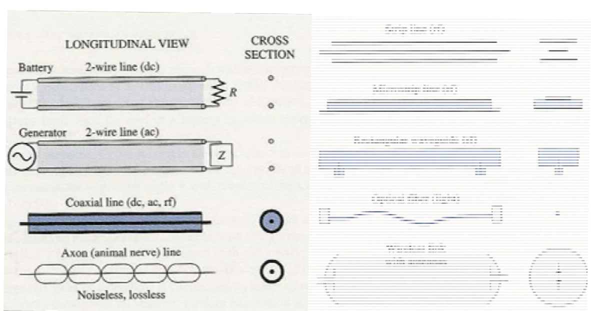
Speed of Light

Distance Travelled	**Travel Time**
Across a virus	100×10^{-18} sec
300 meters	1 microsecond
NY to LA	13 milliseconds
Around earth	0.133 sec
Earth to Moon	1.2 sec
Earth to Sun	8.3 minutes
Earth to Mars	3-21 minutes
Sun to nearest star	4 years
Diameter of our galaxy	100,000 years
Edge of known universe	15 billion years

c ≡ 299,792,458 m/s (used to define the meter!)

Sending/Receiving EM Waves

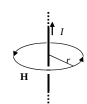
Guiding EM Waves



Brief History of EM

Charles Coulomb in 1785 demonstrated how electric charges repel one another

Andre Marie Ampere discovered that an electric current produces a magnetic field in 1820



Michael Faraday in 1831 showed that since an electric current could produce a magnetic field, a changing magnetic field can produce an electric current. "Principle of Induction" used for first electric generators.

James Clerk Maxwell (1831-1879)

- 1855-1868 - formulates field equations for electromagnetism. Predicts existence of EM wave propagation and the speed of light. Shows theoretical possibility of generating electromagnetic radiation
- 1873: publishes *Treatise on Electricity and Magnetism*

Maxwell's Equations

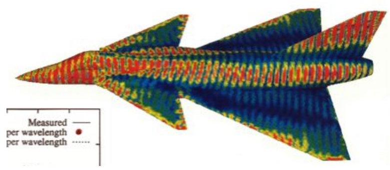
Integral form		Differential Form
$\oint_C \mathbf{E}\cdot d\mathbf{L} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S}$	Faraday	$\nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
$\oint_C \mathbf{H}\cdot d\mathbf{L} = \int_S \mathbf{J}\cdot d\mathbf{S} + \frac{d}{dt}\int_S \mathbf{D}\cdot d\mathbf{S}$	Ampere	$\nabla\times\mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$
$\oint_S \mathbf{D}\cdot d\mathbf{S} = \int_{Vol} \rho\, dv$	Gauss	$\nabla\cdot\mathbf{D} = \rho$
$\oint_S \mathbf{B}\cdot d\mathbf{S} = 0$	Gauss	$\nabla\cdot\mathbf{B} = 0$

E	Electric Field
H	Magnetic Field
D=ε**E**	"Displacement Flux Density"
B=μ**H**	"Magnetic Flux Density"
J	Current Density
ρ	Charge Density

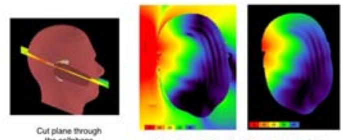
Computers use Differential Form for Complex Objects

From huge objects to the nanoscale, Maxwell's Equations always work!

VFY-218 Jet Fighter at 500 MHz



Cell Phone Interaction with Human Head

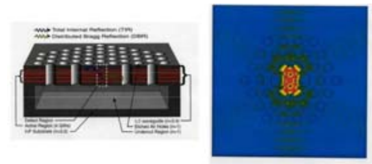


Cut plane through the cellphone

Maps of the E-field and SAR within the cut plane. Relative intensities are shown in dB.

Source: Remcom Inc. website: http://www.remcominc.com/html/index.html

Photonic Bandgap Defect Mode Lasers



Painter et al., *Science*, June 11, 1999.

Sections 1.1-1.2

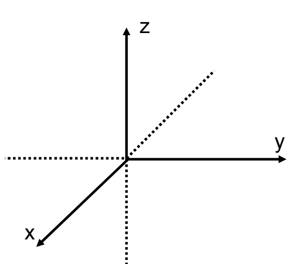
Vector Algebra
Cartesian Coordinates
Differential Length Vector
Differential Surface Vector

Adapted from Prof. Cunningham's Notes

Today's Topics

- Review
  - Scalars (numbers) and vectors
  - Unit vectors
  - Vector addition & subtraction
  - Magnitude
  - Dot product
  - Cross product
- New Topics
  - Differential length vector
  - Differential surface vector

Cartesian Coordinate System



Unit vectors

$\hat{a}_x \quad \hat{a}_y \quad \hat{a}_z$

Notation

$\vec{A} = A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z$

Alternate Notation

$\vec{A} = < A_x, A_y, A_z >$

$\vec{A} = A_1\hat{a}_1 + A_2\hat{a}_2 + A_3\hat{a}_3$

(Review at home)

Vector Math

$\vec{A} = A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z \qquad \vec{B} = B_x\hat{a}_x + B_y\hat{a}_y + B_z\hat{a}_z$

Magnitude of **A**

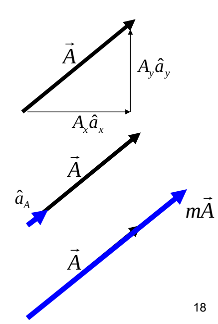
$|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$

Unit vector in direction of **A**

$\hat{a}_A = \frac{\vec{A}}{|\vec{A}|} = \frac{A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}}$

Multiplication of **A** by a scalar

$m\vec{A} = mA_x\hat{a}_x + mA_y\hat{a}_y + mA_z\hat{a}_z$

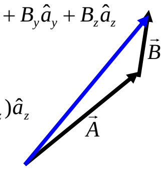


(Review at home) Vector Math

$\vec{A} = A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z$ $\vec{B} = B_x\hat{a}_x + B_y\hat{a}_y + B_z\hat{a}_z$

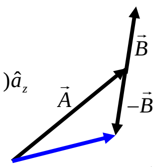
Addition of **A** and **B**

$$\vec{A}+\vec{B} = (A_x + B_x)\hat{a}_x + (A_y + B_y)\hat{a}_y + (A_z + B_z)\hat{a}_z$$



Subtraction of **A** and **B**

$$\vec{A}-\vec{B} = (A_x - B_x)\hat{a}_x + (A_y - B_y)\hat{a}_y + (A_z - B_z)\hat{a}_z$$



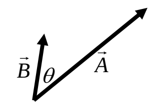
Dot Product

$\vec{A} = A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z$ $\vec{B} = B_x\hat{a}_x + B_y\hat{a}_y + B_z\hat{a}_z$

Dot Product of **A** and **B**

$\vec{A}\bullet\vec{B} = |\vec{A}||\vec{B}|\cos\theta$

Resulting quantity is a SCALAR

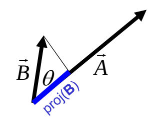


Physical Meaning

(Magnitude of **A**)*(Projection of **B** onto **A**)

or (Magnitude of **B**)*(Projection of **A** onto **B**)

$\vec{A}\bullet\vec{B} = \vec{B}\bullet\vec{A}$



Dot Product

$\vec{A} = A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z$ $\vec{B} = B_x\hat{a}_x + B_y\hat{a}_y + B_z\hat{a}_z$

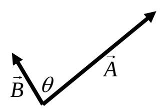
Dot Product of **A** and **B**

$\vec{A}\bullet\vec{B} = |\vec{A}||\vec{B}|\cos\theta$

$\vec{A}\perp\vec{B}$

$\cos\theta = 0$

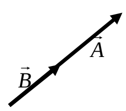
$\vec{A}\bullet\vec{B} = 0$ $\boxed{\vec{A}\perp\vec{B} \Leftrightarrow \vec{A}\bullet\vec{B}=0}$



$\vec{A}\|\vec{B}$

$\cos\theta = 1$

$\vec{A}\bullet\vec{B} = |\vec{A}||\vec{B}|$ $\boxed{\vec{A}\parallel\vec{B} \Leftrightarrow \vec{A}\bullet\vec{B}=|\vec{A}||\vec{B}|}$



Dot Product

$\vec{A} = A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z$ $\vec{B} = B_x\hat{a}_x + B_y\hat{a}_y + B_z\hat{a}_z$

$$\boxed{\begin{array}{l}\vec{A}\perp\vec{B} \Leftrightarrow \vec{A}\bullet\vec{B}=0\\ \vec{A}\parallel\vec{B} \Leftrightarrow \vec{A}\bullet\vec{B}=|\vec{A}||\vec{B}|\end{array}}$$

Dot Product of **Unit Vectors**

$$\begin{array}{lll}\hat{a}_x\bullet\hat{a}_x = 1 & \hat{a}_y\bullet\hat{a}_x = 0 & \hat{a}_z\bullet\hat{a}_x = 0\\ \hat{a}_x\bullet\hat{a}_y = 0 & \hat{a}_y\bullet\hat{a}_y = 1 & \hat{a}_z\bullet\hat{a}_y = 0\\ \hat{a}_x\bullet\hat{a}_z = 0 & \hat{a}_y\bullet\hat{a}_z = 0 & \hat{a}_z\bullet\hat{a}_z = 1\end{array}$$

$$\vec{A}\bullet\vec{B} = \left(A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z\right)\bullet\left(B_x\hat{a}_x + B_y\hat{a}_y + B_z\hat{a}_z\right)$$

$$\boxed{\vec{A}\bullet\vec{B} = A_xB_x + A_yB_y + A_zB_z}$$ Scalar

Discussion Problem

- Given **A**=<3,2,1>, **B**=<1,1,-1>, **C**=<1,2,3>, find:
  - a. |**A**+**B**-4**C**|
  - b. unit vector along (**A**+2**B**-**C**)
  - c. **A**·**C**

Problem taken from p11 in old book

Cross Product

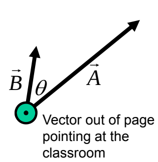
$\vec{A} = A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z$ $\vec{B} = B_x\hat{a}_x + B_y\hat{a}_y + B_z\hat{a}_z$

Cross Product of **A** and **B**

Resulting quantity is a VECTOR

Magnitude

$\vec{A}\times\vec{B} = |\vec{A}||\vec{B}|\sin\theta$



Direction

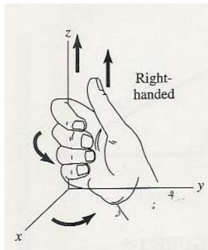
Perpendicular to BOTH **A** and **B**

Two possible vectors satisfy this condition

Determined using the RIGHT HAND RULE

Right Hand Rule

Bus Driver in Kauai Demonstrating the Right Hand Rule



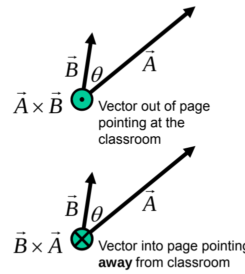
Cross Product

$\vec{A} = A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z$ $\qquad \vec{B} = B_x\hat{a}_x + B_y\hat{a}_y + B_z\hat{a}_z$

<u>Cross Product of **A** and **B**</u>

$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$

Same magnitude
Opposite direction



Cross Product

$\vec{A} = A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z$ $\qquad \vec{B} = B_x\hat{a}_x + B_y\hat{a}_y + B_z\hat{a}_z$

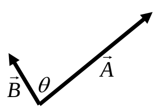
<u>Cross Product of **A** and **B**</u>

$\vec{A}\times\vec{B} = |\vec{A}||\vec{B}|\sin\theta\hat{a}_N$

$\vec{A}\perp\vec{B}$

$\sin\theta = 1$

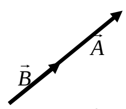
$\vec{A}\times\vec{B} = |\vec{A}||\vec{B}|\hat{a}_N$ $\qquad \boxed{\vec{A}\perp\vec{B} \Leftrightarrow \vec{A}\times\vec{B} = |\vec{A}||\vec{B}|\hat{a}_N}$



$\vec{A}\|\vec{B}$

$\sin\theta = 0$

$\vec{A}\times\vec{B} = 0$ $\qquad \boxed{\vec{A}\parallel\vec{B} \Leftrightarrow \vec{A}\times\vec{B} = 0}$



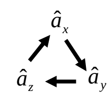
Cross Product

$\vec{A} = A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z$ $\qquad \vec{B} = B_x\hat{a}_x + B_y\hat{a}_y + B_z\hat{a}_z$

$$\boxed{\begin{array}{l}\vec{A}\perp\vec{B} \Leftrightarrow \vec{A}\times\vec{B} = |\vec{A}||\vec{B}|\hat{a}_N \\ \vec{A}\parallel\vec{B} \Leftrightarrow \vec{A}\times\vec{B} = 0\end{array}}$$

<u>Cross Product of **Unit Vectors**</u>

$\hat{a}_x\times\hat{a}_x = 0$	$\hat{a}_y\times\hat{a}_x = -\hat{a}_z$	$\hat{a}_z\times\hat{a}_x = \hat{a}_y$
$\hat{a}_x\times\hat{a}_y = \hat{a}_z$	$\hat{a}_y\times\hat{a}_y = 0$	$\hat{a}_z\times\hat{a}_y = -\hat{a}_x$
$\hat{a}_x\times\hat{a}_z = -\hat{a}_y$	$\hat{a}_y\times\hat{a}_z = \hat{a}_x$	$\hat{a}_z\times\hat{a}_z = 0$



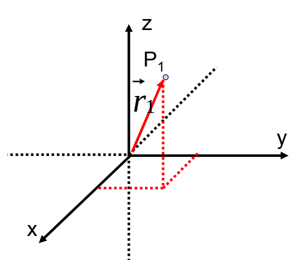
RH Coordinate System Sign Convention

$$\vec{A}\times\vec{B} = \begin{vmatrix}\hat{a}_x & \hat{a}_y & \hat{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z\end{vmatrix} = \underbrace{(A_yB_z - A_zB_y)\hat{a}_x + (A_zB_x - A_xB_z)\hat{a}_y + (A_xB_y - A_yB_x)\hat{a}_z}_{\text{Vector}}$$

Perpendicular to **A** & **B**
Right Hand Rule to Select Direction

(Review at home)

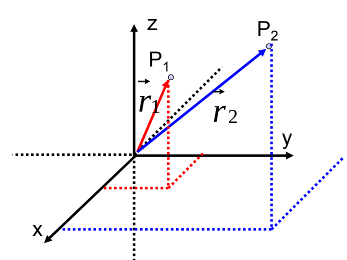
Vector From Origin to a Point



P_1: (x_1, y_1, z_1)

(Review at home)

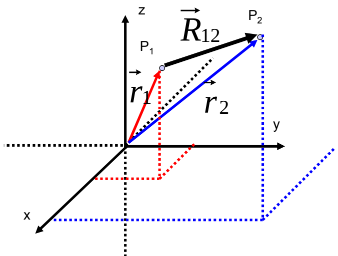
Vector Between Two Points



P_1: (x_1, y_1, z_1) $\qquad P_2$: (x_2, y_2, z_2)

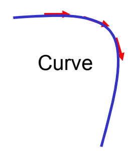
(Review at home)

Vector Between Two Points



$$\vec{R}_{12} = (FinalPosition) - (InitialPosition)$$
$$\vec{R}_{12} = (x_2 - x_1)\hat{a}_x + (y_2 - y_1)\hat{a}_y + (z_2 - z_1)\hat{a}_z$$

Differential Length Vector



Always tangent to a curve or surface

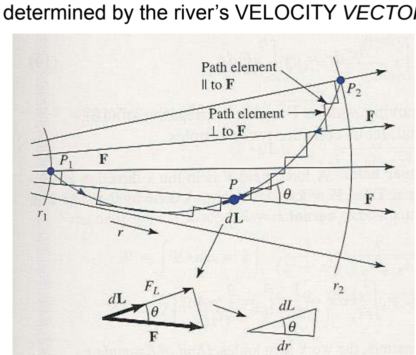
Exact vector is different at different places on the curve or surface

What we will plug into Faraday's law

$$d\vec{l} = dx\hat{a}_x + dy\hat{a}_y + dz\hat{a}_z$$

(in cartesian coordinates only)

Imagine you are walking in a river and the water pushes with a force, **F**, determined by the river's VELOCITY *VECTOR FIELD*.



How much work does the river do if you walk from P_1 to P_2 along the path shown?

Discussion Problem

- Given **A**=<3,2,1>, **B**=<1,1,-1>, **C**=<1,2,3>, find:
  a. **B** x **C**
  b. **A** · (**B** x **C**)
- For each line, find d**l** if the z-component of d**l** is dz:
  a. x=3, y=4
  b. x+y=0, y+z=1
  c. the line passing from (2,0,0) thru (0,0,1)

Problem taken from p11 and p20 in old book

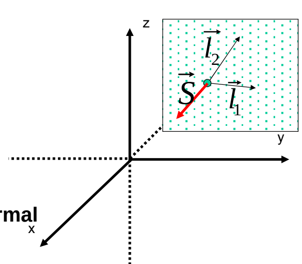
Surface Vector

Easy example surface:
Flat surface in the yz plane

Pick a point on the surface

Find two vectors at that point that are tangent to the surface

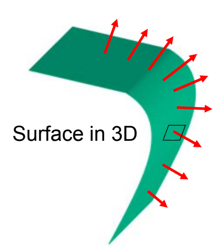
A **surface vector** is always **normal** (perpendicular) to the surface



You get **normal** vectors by performing a **cross product**

$$\vec{S} = \vec{l}_{\_} \times \vec{l}_{\_}$$

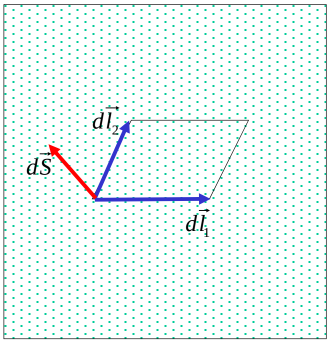
Surface Vectors



How do we come up with a surface vector when a surface is curved?

If we zoom into a small enough area, the surface will "look" flat

Differential Surface Vectors



Find two differential length vectors at the point on the surface.

Think of **dS** as a tiny parallelogram with sides bounded by **dl₁** and **dl₂** and direction perp to the surface.

$$d\vec{S} = d\vec{l}_1 \times d\vec{l}_2$$

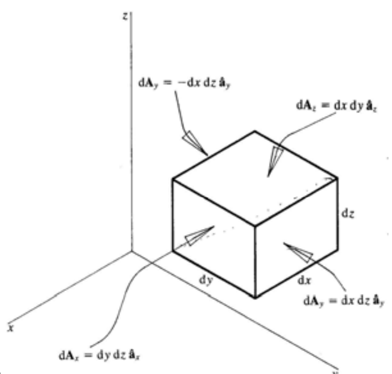
Cartesian Coordinates

(x,y,z)

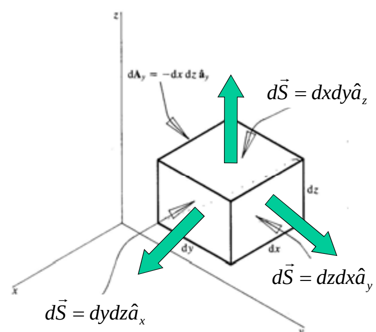
$$d\vec{l} = dx\hat{a}_x + dy\hat{a}_y + dz\hat{a}_z$$

$$d\vec{S} = \pm dydz\hat{a}_x$$
$$d\vec{S} = \pm dzdx\hat{a}_y$$
$$d\vec{S} = \pm dxdy\hat{a}_z$$

Differential area in cartesian coords



Differential surface vectors for unit vectors in cartesian coords



Lecture 1 Summary

- Dot Product $\vec{A} \bullet \vec{B} =$ _______ = ________
- Cross Product

$$\vec{A} \times \vec{B} = \begin{vmatrix} \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \end{vmatrix} = < \_\_\_\_\_\_, \_\_\_\_\_\_, \_\_\_\_\_\_ >$$

- Next two lectures:
  - Scalar and Vector Fields (1.3)
  - The Lorentz Force (1.6)
  - Coulomb's Law (1.4-1.6)
  - Surf. Integrals/Gauss' Law (2.2,2.5)

Upcoming Schedule

- Lectures 1-6 are a review of PHYS 212 and MATH 241 – covered quickly
  - Read Chapters 1 and 2 of Rao's text over the next 2 weeks
- I-clickers will be used for challenge problems beginning in Lecture 3
  - Bring them next time to test out

Lectures 2-3
Sections 1.3-1.6, 2.2, 2.5

Scalar and Vector Fields

Lorentz Force Equation

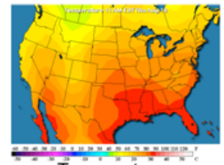
Coulomb's Law

Surface Integrals

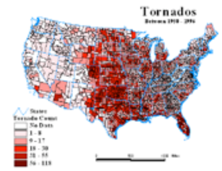
Connecting Coulomb's and Gauss' Law

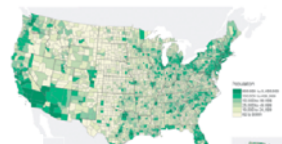
Adapted from Prof. Cunningham's Notes

Scalar Field



Temperature

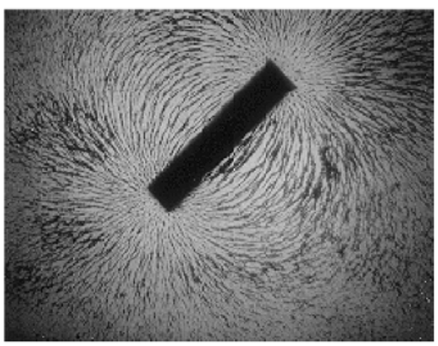




Population Density

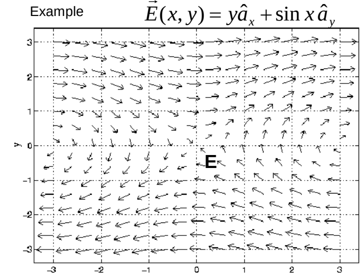
Elevation

Vector Field



Vector Field

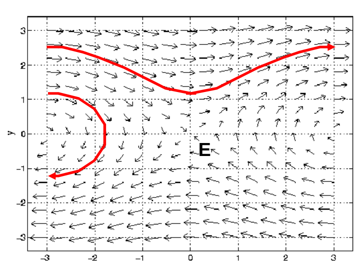
Example $\vec{E}(x,y) = y\hat{a}_x + \sin x\,\hat{a}_y$



Vector Field

Example $\vec{E}(x,y) = y\hat{a}_x + \sin x\,\hat{a}_y$

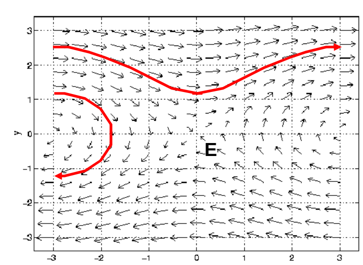
"Direction Line" or "stream line" or "flux line"



E vectors are tangent to the direction line (parallel to differential length vectors)
Thus, slope of direction line is m = dy/dx = E_y/E_x

Electric force

$$\vec{F}_E = q\vec{E}$$

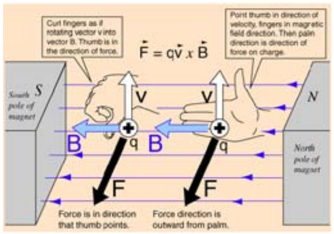


Force vectors $\mathbf{F}_E$ are parallel to **E** for a positive charge q

Magnetic force

$$\vec{F}_M = q\vec{v}\times\vec{B}$$

Direction: Perpendicular to velocity vector – does no work!
Perpendicular to B vector



Lorentz Force Equation

So if a region of space contains BOTH an **E** field and a **B** field, a moving charge will experience force from both at the same time…

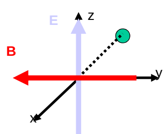
$$\vec{F}_{TOTAL} = \vec{F}_E + \vec{F}_M$$

$$\vec{F}_{TOTAL} = q\vec{E} + q\vec{v}\times\vec{B}$$

Application: Mass Spectrometers

- Part I: Velocity Selector
  - Particles with a specific velocity in crossed EM fields are undeflected

$$\vec{E} = E_0\hat{a}_z$$
$$\vec{B} = -B_0\hat{a}_y$$
$$\vec{v} = v_0\hat{a}_x$$

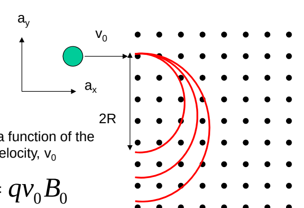


$$\vec{F}_{TOTAL} = q(E_0 - v_0B_0)\hat{a}_z = 0 \; iff \;\; v_0 = E_0/B_0$$

Application: Mass Spectrometers

- Part II: Mass Selector
  - Mass sets the radius of B-field orbit since particle velocity is the same

$$\vec{B} = B_0\hat{a}_z$$
$$\vec{v} = v_0\hat{a}_x$$



Problem: Solve for R as a function of the particle's mass, m, and velocity, v_0

$$F_c = mv_0^2/R = qv_0B_0$$

Current = Moving Charge

What is CURRENT? CHARGES IN MOTION!!

$I\,d\vec{l}$

$$I\,d\vec{l} = q\vec{v}$$

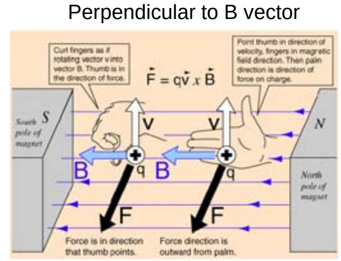
$$\frac{coul}{\sec}\cdot m = coul\cdot\frac{m}{\sec}$$

So the current in a wire, ***I***, flowing across a magnetic field will feel a force…

Magnetic force

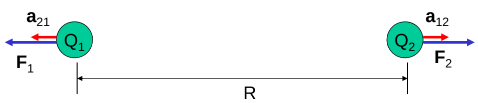
$$F_M = (I\,d\vec{l})\times\vec{B} = q\vec{v}\times\vec{B}$$

Direction: Perpendicular to velocity vector
Perpendicular to B vector



Electrostatic Force

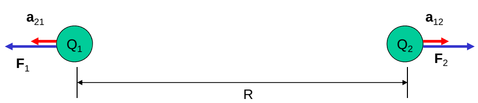
What is the force F_2 on a point charge Q_2 due to a single point charge Q_1 located a distance R away?



Coulomb's observations

- The magnitude of F is
  - proportional to the product of the charges
  - inversely proportional to the square of the distance
  - depends on the medium
- F points along the joining line
- Like charges repel; unlike charges attract

Coulomb's Law

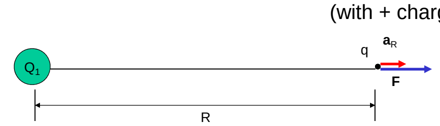


$$\vec{F}_1 = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2}\hat{a}_{21} \qquad \vec{F}_2 = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2}\hat{a}_{12}$$

Electric Field

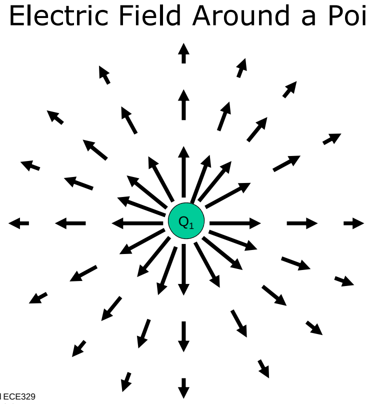
- The electric field **E** is the force per unit charge caused by the source charges

Tiny "test charge" (with + charge)



$$\vec{E} = \lim_{Q_2 \to 0} \frac{\vec{F}_2}{Q_2} = \frac{Q_1}{4\pi\varepsilon_0 R^2}\hat{a}_R$$

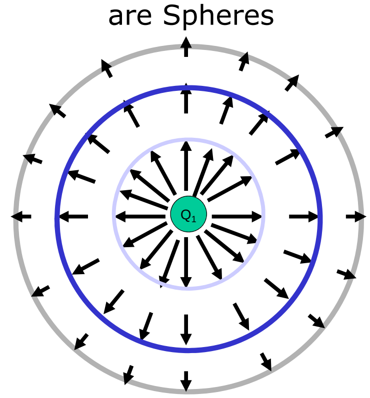
Electric Field Around a Point Charge



$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 R^2}\hat{a}_R$$

Field strength is proportional to the length of vectors

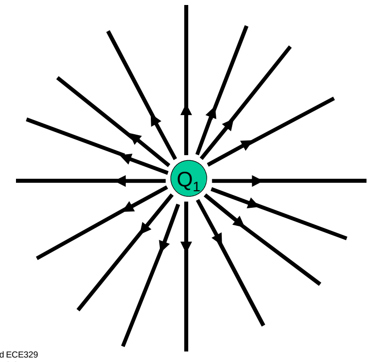
Surfaces of Constant E Magnitude are Spheres



Field Lines

- Another way to graphically represent vector fields
- The field strength is proportional to the **density** of field lines
- **E**-field lines begin on + charges, initially emanating uniformly in all directions, and end on – charges
  - Can't stop in midair but can extend to ∞
- They **never** intersect

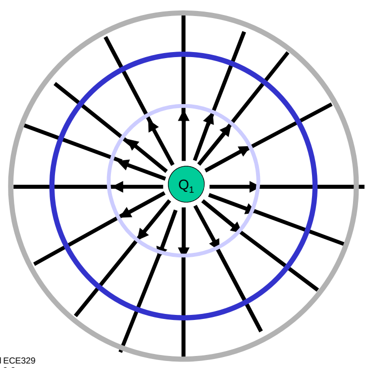
Electric Field Around a Point Charge



$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 R^2}\hat{a}_R$$

Field strength is proportional to the **density** of field lines

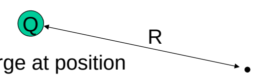
Electric Field Around a Point Charge



$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 R^2}\hat{a}_R$$

Gauss' Law
Number of field lines passing thru **any** surface that encloses Q_1 is constant

Calculating the Electric Field



Point Charge at position (x_1,y_1,z_1)

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 R^2}\hat{a}_R$$

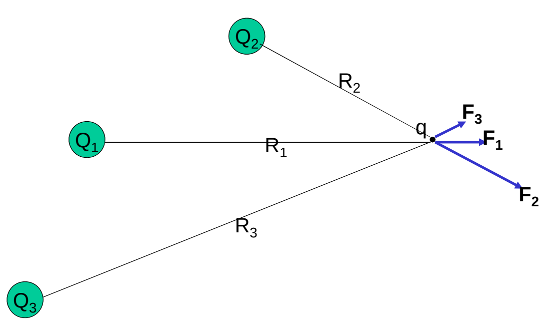
Position where we want to calculate electric field at Position (x_2,y_2,z_2)

$$R = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

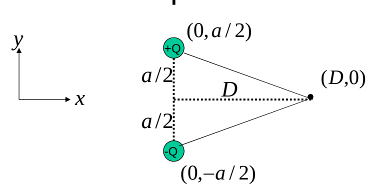
$$\hat{a}_R = \frac{(x_2-x_1)\hat{a}_x+(y_2-y_1)\hat{a}_y+(z_2-z_1)\hat{a}_z}{R}$$

Unit vector pointing along direction from Q to Point

Superposition of E Fields

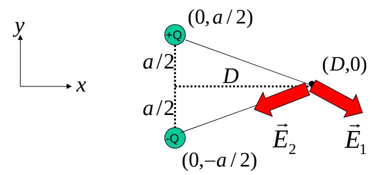


Example: Calculate the E-field of a Dipole at a Point



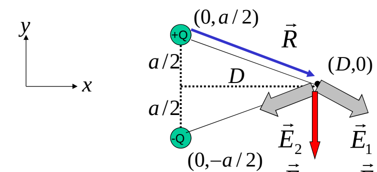
Tip: We only need to work in 2D. Why?

Example: E of Dipole



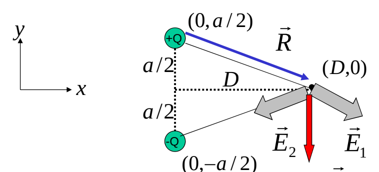
Tip: Use symmetry to eliminate components that cancel. Here, we only need to calculate the y-component and only for one of the charges (let's say for +Q). Why?

Example: E of Dipole



$$\vec{E}_{TOT} = \vec{E}_1 + \vec{E}_2 = 2E_{1y}\hat{a}_y$$

Example: E of Dipole

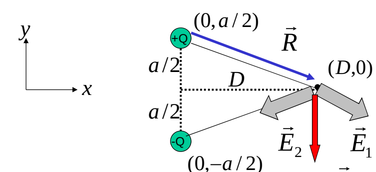


$$\vec{E}_{TOT} = 2E_{1y}\hat{a}_y$$

$$\vec{R}_{from\ +Q\ to\ P} = < x_2 - x_1, y_2 - y_1 > = < D, -a/2 >$$

$$\hat{a}_R = \frac{< D, -a/2 >}{\sqrt{D^2 + a^2/4}} \qquad \vec{E}_1 = \frac{Q}{4\pi\varepsilon_0 R^2}\hat{a}_R$$

Example: E of Dipole



$$\vec{E}_{TOT} = 2E_{1y}\hat{a}_y$$

$$E_{1y} = \left(\frac{Q}{4\pi\varepsilon_0 R^2}\hat{a}_R\right) \bullet \hat{a}_y$$

$$\vec{E}_{TOT} = 2\frac{Q}{4\pi\varepsilon_0 R^2}\left(\frac{-a/2}{R}\right)\hat{a}_y$$

UNITS
(Newtons per
Coulomb)

Patented 5-Step Program for Problem Solving

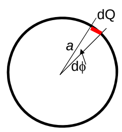
1. MAKE A **LARGE CLEAR** DRAWING
   a. Also draw cross-sections if the problem is in 3D
   b. Pick a coordinate system that is appropriate for the symmetry of the problem
2. Divide charge distributions into tiny pieces
3. Find d**E** of one tiny piece
4. Use SYMMETRY to eliminate any components that cancel (i.e. add to ZERO)
5. INTEGRATE to add contribution of ALL the tiny pieces

Example Charge Distributions

- Several discrete points of charge
- Line of charge
- Ring of charge
- Nonuniform lines or rings of charge
- An infinite sheet of charge
- Infinite box of charge
- Spherical surface of charge
- Cylindrical surface of charge
- Spherical volume of charge
- Cylindrical volume of charge

(Review at home)
Example 1: Find Total Charge of a Linear Charge Distribution

Linear Charge λ_0 (C/m) distribution
in a circular loop of radius = *a*

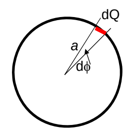


One little piece has a charge (in coulombs)

$$dQ = (\lambda_0)(ad\phi)$$

(Review at home)
Example 1: Find Total Charge of a Linear Charge Distribution

Linear Charge λ_0 (C/m) distribution
in a circular loop of radius = *a*



One little piece has a charge (in coulombs)

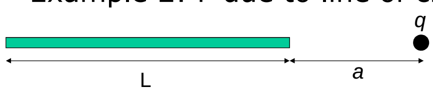
$$dQ = (\lambda_0)(ad\phi)$$

Integrate to get the entire charge of the loop:

$$Q = \int_{\phi=0}^{2\pi} (\lambda_0)(ad\phi) = 2\pi\lambda_0 a$$

Units: Coulombs

(Review at home)
Example 2: F due to line of charge

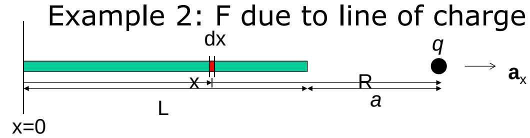


Rod has a TOTAL charge = Q (coul)

So the rod has a charge DENSITY $\rho_l = Q/L$ (coul/m)

Find the FORCE exerted by the whole charged rod on the charge q

(Review at home)
Example 2: F due to line of charge



What is the small amount of force, dF, applied by a small sliver of the rod?

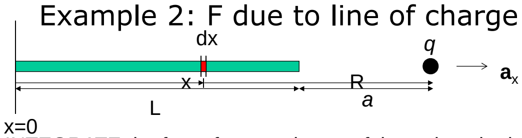
Differential force applied to q

$$d\vec{F} = \frac{\left[\frac{Q}{L}dx\right]}{4\pi\varepsilon_0 R^2} q\hat{a}_x$$

Differential charge in one small sliver (coul)

$$R = (L + a) - x$$

(Review at home)
Example 2: F due to line of charge

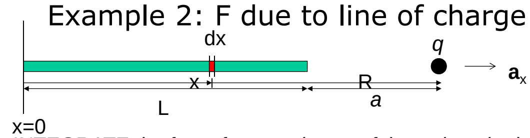


INTEGRATE the force from each part of the rod to obtain the force due to the whole thing:

$$\vec{F} = \int_0^L \frac{qQ}{4\pi\varepsilon_0 L} \frac{dx}{(L + a - x)^2} \hat{a}_x$$

Fortunately, a_x is constant and can be taken outside of the integral. Not so simple in cylindrical/spherical coordinates!

(Review at home)
Example 2: F due to line of charge



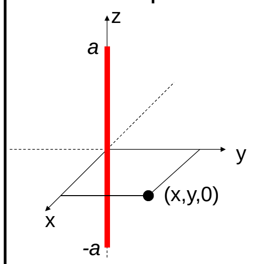
INTEGRATE the force from each part of the rod to obtain the force due to the whole thing:

$$\vec{F} = \frac{qQ}{4\pi\varepsilon_0 L}\hat{a}_x \int_0^L \frac{dx}{(L + a - x)^2} = \frac{qQ}{4\pi\varepsilon_0 L}\hat{a}_x \left.\frac{1}{L + a - x}\right|_0^L$$

$$\vec{F} = \frac{qQ}{4\pi\varepsilon_0 L}\hat{a}_x\left[\frac{1}{a} - \frac{1}{L + a}\right] = \frac{qQ}{4\pi\varepsilon_0 a(L + a)}\hat{a}_x$$

Units: N

Example 3: **E** due to line of charge

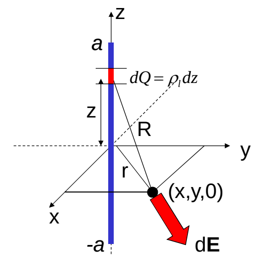


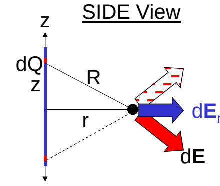
Linear charge distribution
ρ_l (coul/m) from for $-a<z<a$

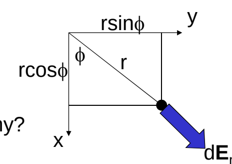
Find **E** at point on xy plane

Next, consider what happens when the line is infinitely long

Example 3: **E** due to line of charge

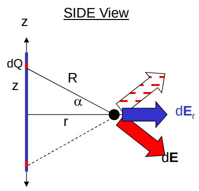






Cylindrical symmetry so use cylindrical coordinates (r,ϕ,z)
$\mathbf{E}_{total}$ will point in a_r direction, why?

Example 3: **E** due to line of charge



$$\overrightarrow{dE} = \frac{dQ}{4\pi\varepsilon_0 R^2}\hat{a}_R$$

$$dE_r = dE\cos(\alpha) = dE\frac{r}{R}$$

$$R = \sqrt{r^2+z^2}$$

$$E_r = \frac{\rho_l}{4\pi\varepsilon_0}\int_{z=-a}^{a}\frac{r}{\left(r^2+z^2\right)^{3/2}}dz(\hat{a}_r)$$

Slightly difficult to integrate directly

Example 3: **E** due to line of charge

$$E_r = \frac{\rho_l}{4\pi\varepsilon_0}\int_{z=-a}^{a}\frac{r}{\left(r^2+z^2\right)^{3/2}}dz(\hat{a}_r)$$

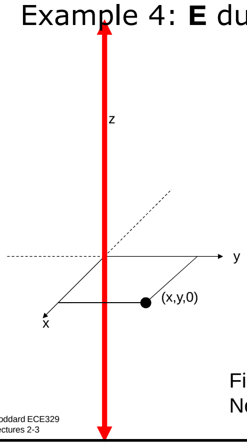
$$R = \frac{r}{\cos(\alpha)},\quad z = r\tan(\alpha),\ \ dz = r\sec^2(\alpha)d\alpha$$

$$E_r = \frac{\rho_l}{4\pi\varepsilon_0}\int_{z=-a}^{a} r\frac{\cos^3(\alpha)}{r^3}(r\sec^2(\alpha)d\alpha)(\hat{a}_r)$$

$$= \frac{\rho_l}{4\pi\varepsilon_0 r}(\hat{a}_r)\int_{\tan(\alpha)=-a/r}^{\tan(\alpha)=+a/r}\cos(\alpha)d\alpha = \frac{\rho_l}{4\pi\varepsilon_0 r}(\hat{a}_r)\sin(\alpha)$$

$$= \frac{\rho_l}{4\pi\varepsilon_0 r}(\hat{a}_r)2\frac{a}{\sqrt{a^2+r^2}}$$

Example 4: **E** due to ∞ line of charge



a >> r so we get

$$\lim_{a\to\infty}E_r = \frac{\rho_l}{4\pi\varepsilon_0 r}(\hat{a}_r)2\lim_{a\to\infty}\frac{a}{\sqrt{a^2+r^2}}$$

$$= \frac{\rho_l}{2\pi\varepsilon_0 r}(\hat{a}_r)$$

$$\boxed{E_r = \frac{\rho_l}{2\pi\varepsilon_0 r}(\hat{a}_r)}$$

Field strength now drops off as 1/r,
Not as $1/r^2$ like a point charge

Lecture 2 Summary

- The electric field **E** is the ______ per _____ _______ caused by the source charges. It points along the _______ line. For a point charge,

E =

- Next class:
  - Surface Integrals (2.2)
  - Connecting Coulomb's and Gauss' Law (2.5)

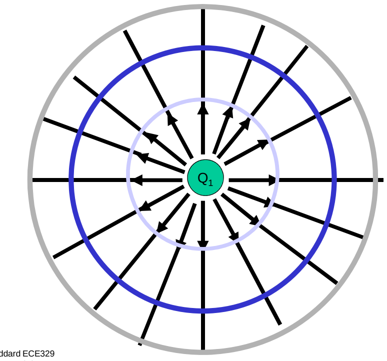
Lecture 3
Sections 2.2, 2.5

Surface Integrals

Connecting Coulomb's and Gauss' Law

Adapted from Prof. Cunningham's Notes

Electric Field Around a Point Charge

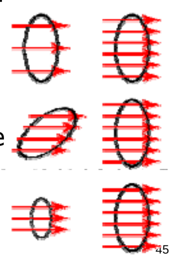


$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 R^2}\hat{a}_R$$

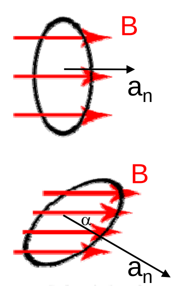
Gauss' Law
Number of field lines passing thru **any** surface that encloses Q_1 is constant

Surface Integrals

- Flux = # of arrows that pass thru a surface; it depends on:
  - The density of vectors
  - The angle of the surface
  - The area of the surface



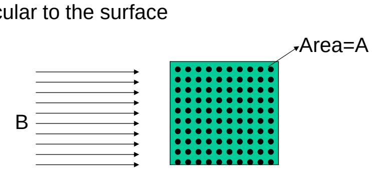
Surface Integral describes the Flux of a Vector Field



$$\begin{aligned} Flux &= \vec{B}\bullet\Delta\vec{S} \\ &= (\vec{B}\bullet\hat{a}_n)|\Delta S| \\ &= (B\cos\alpha)|\Delta S| \\ &= |B_n||\Delta S| \end{aligned}$$

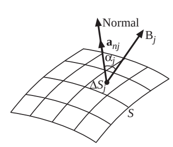
Trick that works sometimes

If the flux is
- Uniform (has equal magnitude across whole surface)
- Perpendicular to the surface



$$\psi = \iint_S \vec{B}\bullet d\vec{S} = B\cdot A$$

In the general case ...



$$\text{Flux} = \sum_{j=1}^{n}\Delta\psi_j$$

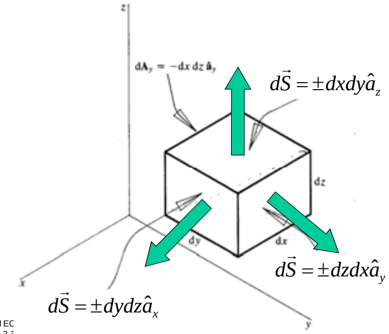
$$= \sum_{j=1}^{n}\mathbf{B}_j\bullet\Delta\mathbf{S}_j$$

In the limit $n \to \infty$,

$$\text{Flux,}\quad \boxed{\psi = \int_S \mathbf{B}\bullet d\mathbf{S}}$$

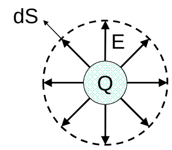
= Surface integral of B over S.

Differential surface vectors for unit vectors in cartesian coords



Electric Flux = Charge Enclosed

- Coulomb's Law for the Electric field of a point charge: $\vec{E} = \dfrac{Q}{4\pi\varepsilon_0 R^2}\hat{a}_R$



Define **D**=ε_0**E** to be the "displacement flux density"

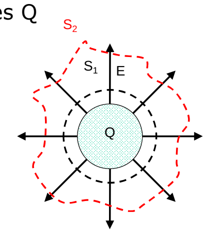
$$\psi_E = \oiint_S \vec{D} \bullet d\vec{S} = \oiint_S \varepsilon_0 \vec{E} \bullet d\vec{S}$$

$$= \varepsilon_0 E(Surf\ \ Area)$$

$$= \varepsilon_0 \frac{Q}{4\pi\varepsilon_0 R^2}(4\pi R^2) = Q$$

Same Flux Out of Any Surface

- Same # of field lines pass thru any surface that encloses Q

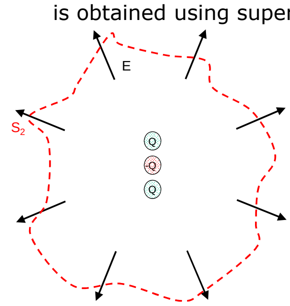


$$\psi_E = \oiint_S \vec{D} \bullet d\vec{S} = Q_{enclosed}$$

Our first Maxwell Equation!

Superposition

- The flux for an arbitrary distribution of charges is obtained using superposition

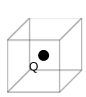


$$\psi_E = \oiint_S \vec{D} \bullet d\vec{S} = Q_{enclosed}$$

FLUX OUT = CHARGE ENCLOSED

Simple Example 1

6-sided cube with Q at the center:

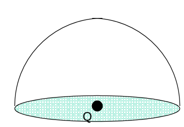


Flux out of entire box = Q
Flux out of one side = Q/6

What if Q is not at the center?

Simple Example 2

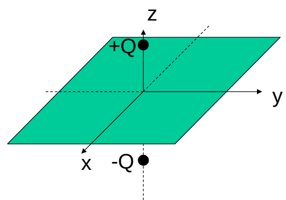
Flux out of hemisphere with Q at the center = Q/2



Challenge Question 1

- What is the flux across the xy plane $\mathbf{a}_n = \mathbf{a}_z$ for a dipole: +Q at (0,0,a/2) and -Q at (0,0,-a/2)?

(a) Q/2
(b) -Q/2
(c) -Q
(d) -3Q/2
(e) -2Q



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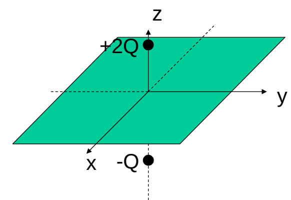
Challenge Question 2

- What is the flux across the xy plane $\mathbf{a}_n = \mathbf{a}_z$ for the charge distribution: +2Q at (0,0,a/2) and -Q at (0,0,-a/2)? (Hint: Use superposition.)

(a) Q/2
(b) -Q/2
(c) -Q
(d) -3Q/2
(e) -2Q



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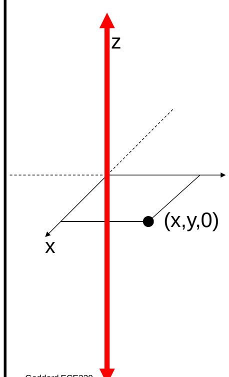
Example 3: **E** due to ∞ line of charge



Linear charge distribution
ρ (coul/m)

Find **E** at point on xy plane

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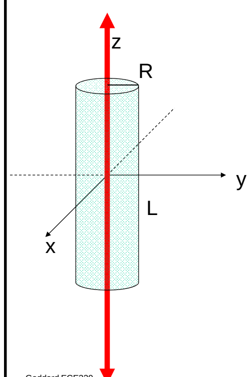
Example 3: **E** due to ∞ line of charge



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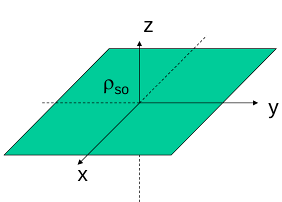
Example 4: **E** due to a surface of charge

What surface
shall we draw?



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The (Fictional) Yadaraf Bug: Flux and Surface Integrals

- The Yadaraf Bug
  - They live in the ground
  - They only come out at night to search for food
  - Very hard to see - REALLY small
  - After gathering food, they <u>always</u> return to their hole in the ground

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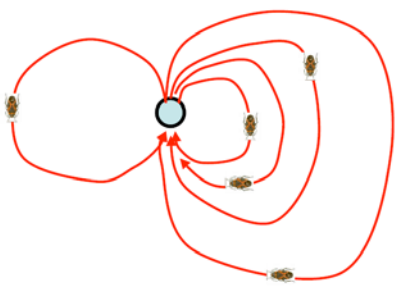
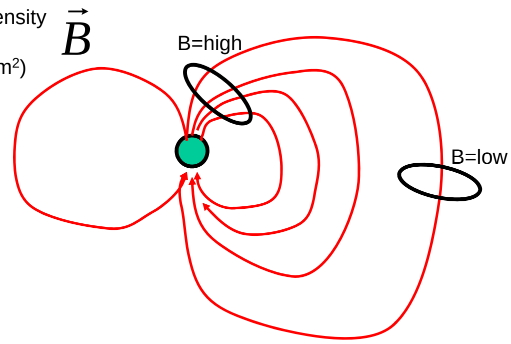
Yadaraf Bug Travel Paths

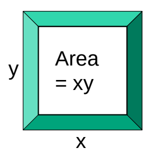


Yadaraf Bug Travel Paths

Bug Density Vector (Bugs/m^2) $\vec{B}$



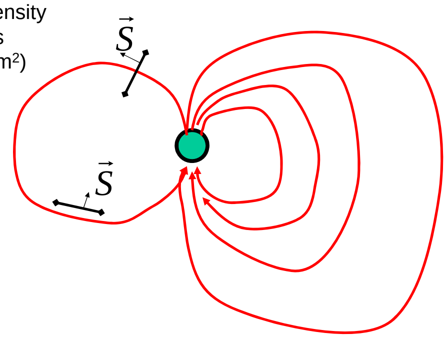
Yadaraf Bug Counter



- Register a +1 count for each bug going through in one direction
- Register a -1 cound for each bug going through in the opposite direction
- Has a known area for the bugs to pass through

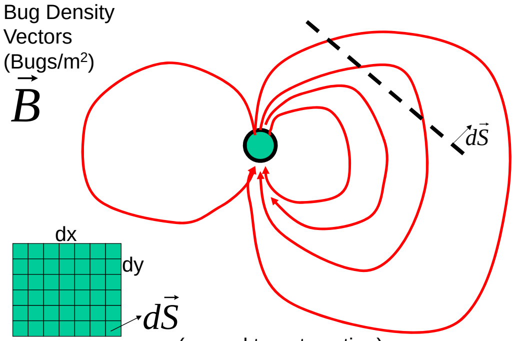
Yadaraf Bug Travel Paths

Bug Density Vectors (Bugs/m^2) $\vec{B}$



Bug Counting Net

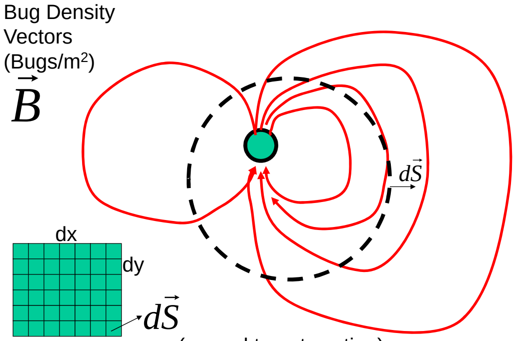
Bug Density Vectors (Bugs/m^2) $\vec{B}$



(normal to net section)

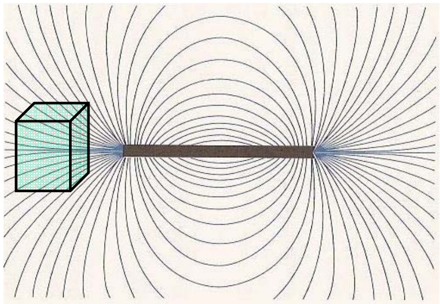
"Closed" Bug Counting Net

Bug Density Vectors (Bugs/m^2) $\vec{B}$



(normal to net section)

So magnetic flux, ψ_B, through a CLOSED surface is always zero



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Lectures 2-3

Magnetic field lines form closed paths (they do not begin or end)

Gauss' Law for B Fields

Net flux of magnetic field lines through any closed surface MUST be zero.

$$\oiint_S \vec{B} \bullet d\vec{S} = 0$$

Our Second Maxwell Equation!

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Lectures 2-3

Lecture 3 Summary

- Gauss' Law

$$\psi_E = \oiint_S \vec{D} \bullet d\vec{S} = Q_{enclosed}$$

FLUX OUT = CHARGE ENCLOSED

$$\oiint_S \vec{B} \bullet d\vec{S} = 0$$

MAGNETIC FLUX LINES DO NOT BEGIN/END

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Lectures 2-3

Lectures 4-5
Sections 3.1-3.3

Review of Vector Calculus

Curl and Divergence

Maxwell's Equations in Differential Form

Adapted from Prof. Cunningham's Notes

Fundamental Theorem of Single Variable Calculus

$$\int_a^b f'(x)dx = f(b) - f(a)$$

df=f'(x)dx is the infinitesimal change of f in going from x to x+dx

Thus, chopping up the interval (a,b) into pieces dx and adding up df gives the total change

Fundamental Theorem of Multi-Variable Calculus

$$\int_a^b \nabla f \bullet d\vec{l} = f(b) - f(a)$$

$$\nabla f = \frac{\partial f}{\partial x}\hat{a}_x + \frac{\partial f}{\partial y}\hat{a}_y + \frac{\partial f}{\partial z}\hat{a}_z$$

df= ∇f·d**l** is the infinitesimal change of f in going from (x,y,z) to (x+dx,y+dy,z+dz)

Thus, chopping up the path (a,b) into pieces d**l** and adding up df gives the total change

∇f is a conservative field

$$\int_a^b \nabla f \bullet d\vec{l} = f(b) - f(a)$$

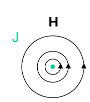
$$\nabla f = \frac{\partial f}{\partial x}\hat{a}_x + \frac{\partial f}{\partial y}\hat{a}_y + \frac{\partial f}{\partial z}\hat{a}_z$$

The right hand side doesn't depend on path so
- ∇f is conservative

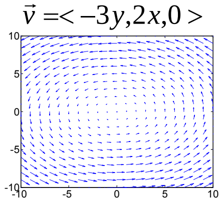
$$\oint_C \nabla f \bullet d\vec{l} = 0$$

- ∇f is curl-free

Curl



$$\vec{v} = <-3y, 2x, 0>$$

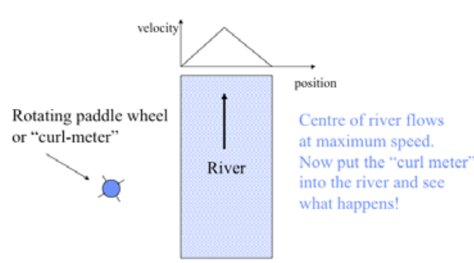


$$\nabla \times \vec{v} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} = 5\hat{a}_z$$

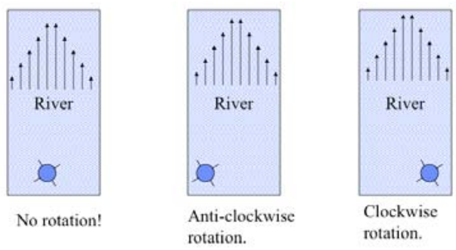
∇ x **v** is a measure of how much the vector field **v** circulates at a given point

A place of high curl is like a whirlpool
- Everywhere here is a whirlpool of strength 5

Physical Interpretation of Curl



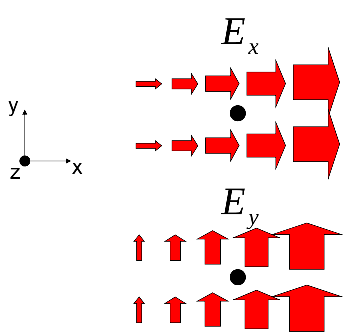
Physical Interpretation of Curl



Meaning of Curl

- The curl meter only spins if there is a non-uniformity in the vector field in a direction **perpendicular** to the field
  – Curl describes variation ACROSS the flow of the field
- Rotation rate is proportional to the degree of non-uniformity
- Rotation is described with a magnitude and a direction – so it's a VECTOR and it's given by the right hand rule

Curl



$\frac{dE_x}{dx} \neq 0$

NO CURL
Varies ALONG

$\frac{dE_y}{dx} \neq 0$

YES CURL
Varies ACROSS

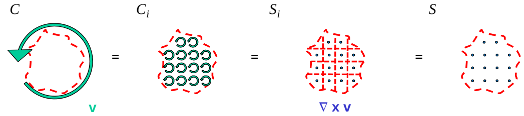
Example: Curl

- Find the curl of $\mathbf{A} = (x^2-4)\mathbf{a}_y$

From D3.7 (p 159) of old book

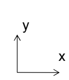
Stokes' Theorem

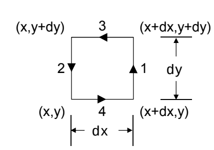
$$\oint_C \vec{v} \bullet d\vec{l} = \sum \oint_{C_i} \vec{v} \bullet d\vec{l} = \sum \iint_{S_i} (\nabla \times \vec{v}) \bullet d\vec{S} = \iint_S (\nabla \times \vec{v}) \bullet d\vec{S}$$



∇ x **v** is a measure of how much the vector field **v** circulates at a given point
A place of high curl is like a whirlpool
Thus, adding up the circulation from each whirlpool inside a region = the total circ. along the boundary

Optional: Derivation of Stokes' Theorem





$$\oint_C \vec{v} \bullet d\vec{l} = \left[v_y(x+dx, y+dy/2) - v_y(x, y+dy/2)\right] \cdot dy$$
$$-\left[v_x(x+dx/2, y+dy) - v_x(x+dx/2, y)\right] \cdot dx$$
$$= \left(\frac{dv_y}{dx} - \frac{dv_x}{dy}\right)_{x+dx/2,\, y+dy/2} \cdot dx \cdot dy$$
$$\oint_C \vec{v} \bullet d\vec{l} = \left(\vec{\nabla} \times \vec{v}\right) \bullet d\vec{S} \approx \iint_S \left(\vec{\nabla} \times \vec{v}\right) \bullet d\vec{S}$$

v is a conservative field iff ∇ x **v** = 0

$$\oint_C \vec{v} \bullet d\vec{l} = 0 = \iint_S (\nabla \times \vec{v}) \bullet d\vec{S}$$

The following are therefore equivalent:

v is conservative **v** is curl-free

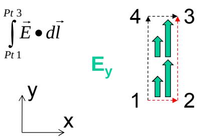
$\oint_C \vec{v} \bullet d\vec{l} = 0$ $\int_a^b \vec{v} \bullet d\vec{l}$ is path independent

$\vec{v} = -\nabla f$ for some scalar potential f

Curl at a Point

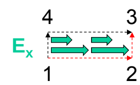
"E has CURL" means $EMF \neq 0$ around a tiny closed path at a particular point – line integral is path dependent

$\int_{Pt\,1}^{Pt\,3} \vec{E} \bullet d\vec{l}$



If there is any DIFFERENCE in E_y in the x direction $\frac{dE_y}{dx} \neq 0$

Path 123 ≠ Path 143



Or if there is any DIFFERENCE in E_x in the y direction $\frac{dE_x}{dy} \neq 0$

Path 123 ≠ Path 143

Maxwell's Equations in Differential Form

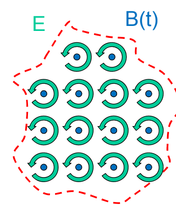
Faraday's Law $\oint_C \vec{E} \bullet d\vec{l} = -\frac{d}{dt} \iint_S \vec{B} \bullet d\vec{S}$

Stokes' Thm

$$\oint_C \vec{E} \bullet d\vec{l} = \iint_S (\nabla \times \vec{E}) \bullet d\vec{S} = -\frac{d}{dt} \iint_S \vec{B} \bullet d\vec{S}$$

$$\Rightarrow \nabla \times \vec{E} = -\frac{d\vec{B}}{dt}$$

Faraday's Law In Differential Form

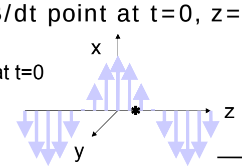


$$\nabla \times \vec{E} = -\frac{d\vec{B}}{dt}$$

Challenge Question

- For **E**=E$_0$cos(ωt-βz)**a**$_x$, which direction will d**B**/dt point at t=0, z=π/(4β)?

E-Field at t=0



Wave moves along +**a**$_z$

(a) **a**$_x$, (b) **a**$_y$, (c) -**a**$_y$, (d) **a**$_z$, (e) dB/dt=0

- Hint: find d**B**/dt directly or use the sketch

From D3.1 (p 141) of old book

Maxwell's Equations in Differential Form

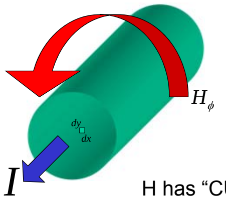
Ampere's Law $\oint_C \vec{H} \bullet d\vec{l} = \iint_S \vec{J} \bullet d\vec{S} + \frac{d}{dt} \iint_S \vec{D} \bullet d\vec{S}$

Stokes' Thm

$$\oint_C \vec{H} \bullet d\vec{l} = \iint_S (\nabla \times \vec{H}) \bullet d\vec{S} = \iint_S (\vec{J} + \frac{d\vec{D}}{dt}) \bullet d\vec{S}$$

$$\Rightarrow \nabla \times \vec{H} = \vec{J} + \frac{d\vec{D}}{dt}$$

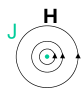
Ampere's Law In Differential Form



$MMF = \oint \vec{H} \bullet d\vec{l} \neq 0$

If there is any current going through a particular point

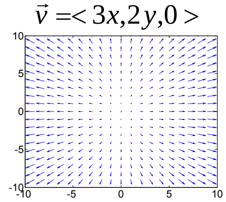
H has "CURL" at a point if there is current going through the point:

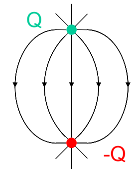


CONDUCTION current
DISPLACEMENT current

Divergence

$\vec{v} =< 3x, 2y, 0 >$





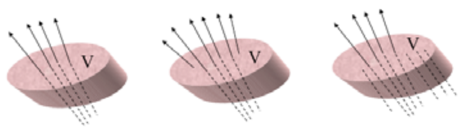
$$\nabla \bullet \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 5$$

$\nabla \cdot$ **v** is a measure of how many field lines for **v** are created at a given point

A place of high divergence is like a water faucet

- Everywhere here is a faucet of strength 5

Meaning of Divergence



Flux in = flux out so no sources or sinks inside V.

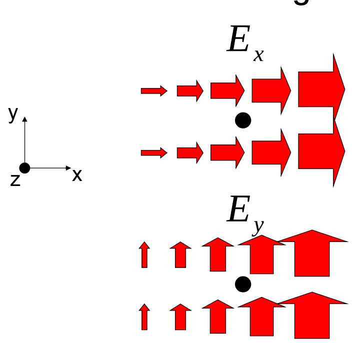
Flux out > flux in Positive divergence. Must be a source inside V.

Flux out < flux in Negative divergence. Must be a sink or drain inside V.

Meaning of Divergence

- There is divergence if there is a non-uniformity in the vector field in a direction **parallel** to the field
  - Divergence describes variation ALONG the flow of the field
- Divergence is proportional to the degree of non-uniformity
- Divergence is described only by the magnitude – so it's a SCALAR

Divergence



$\frac{dE_x}{dx} \neq 0$

YES DIVERGENCE Varies ALONG

$\frac{dE_y}{dx} \neq 0$

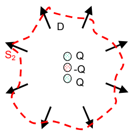
NO DIVERGENCE Varies ACROSS

Example: Divergence

- Find the divergence of $\mathbf{A} = (x-2)^2 \mathbf{a}_x$

From D3.8 (p 159) of old book

Divergence Theorem



$$\iiint_V \nabla \bullet \vec{v}\ dV = \oiint_{\partial V} \vec{v} \bullet d\vec{S}$$

$\nabla \cdot \mathbf{v}$ is a measure of how many field lines for **v** are created at a given point

Thus, adding up the net # lines created inside a volume = the flux of lines out its boundary

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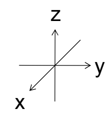
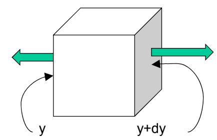
Lectures 4-5

Optional: Derivation of Divergence Theorem





$$\oiint_{\partial V} \vec{v} \bullet d\vec{S} = [v_x(x+dx) - v_x(x)] \cdot dy \cdot dz$$
$$+ [v_y(y+dy) - v_y(y)] \cdot dx \cdot dz$$
$$+ [v_z(z+dz) - v_z(z)] \cdot dx \cdot dy$$

$$\oiint_{\partial V} \vec{v} \bullet d\vec{S} = (\vec{\nabla} \bullet \vec{v}) dV \approx \iiint_V (\vec{\nabla} \bullet \vec{v}) dV$$

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Maxwell's Equations in Differential Form

Gauss' Law $\qquad \oiint_S \vec{B} \bullet d\vec{S} = 0$

Divergence Thm

$$\oiint_S \vec{B} \bullet d\vec{S} = \iiint_V \nabla \bullet \vec{B}\ dV = 0$$

$$\Rightarrow \nabla \bullet \vec{B} = 0$$

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Does **B** satisfy $\nabla \cdot \mathbf{B} = 0$

$$\vec{B} = \frac{\mu_0 I}{2\pi r} \hat{a}_\phi$$

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Lectures 4-5

Maxwell's Equations in Differential Form

Gauss' Law $\qquad \oiint_S \vec{D} \bullet d\vec{S} = \iiint_V \rho dV$

Divergence Thm

$$\oiint_S \vec{D} \bullet d\vec{S} = \iiint_V \nabla \bullet \vec{D}\ dV = \iiint_V \rho dV$$

$$\Rightarrow \nabla \bullet \vec{D} = \rho$$

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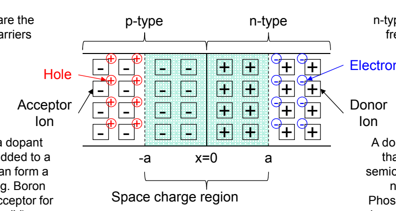
E-field of pn junction (ECE 340)

p-type: holes are the free charge carriers

n-type: electrons are the free charge carriers



An acceptor is a dopant atom that when added to a semiconductor can form a p-type region, e.g. Boron (group III) is an acceptor for Silicon (group IV)

A donor is a dopant atom that when added to a semiconductor can form an n-type region, e.g. Phosphorus (group V) is a donor for Silicon (group IV)

- Find the **E**-field for an evenly doped: $N_d = N_a$ pn junction: $\rho = \{ -\rho_0 = -qN_a$ for $-a < x < 0$,
  $\{ \ \rho_0 = qN_d$ for $0 < x < a$,
  $\{ \ 0$ for $|x| > a$

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From Example 3.5 (p 145) of old book

Blank space for work

Challenge Question

- Can $\mathbf{A}=y\mathbf{a}_x+x\mathbf{a}_y$ be an E or B field in a region of free space where J=0, ρ=0, and electrostatics applies? Workspace:

(a) Yes, but **A** can only be an E-field
(b) Yes, but **A** can only be a B-field
(c) Yes, **A** can be either an E or B field
(d) No, **A** cannot be either an E or B field

From Problem 3.11a (p 198) of old book

Continuity Equation in Differential Form

Continuity Eqn $$\oint\!\!\oint_S \vec{J}\bullet d\vec{S} = -\frac{d}{dt}\iiint_V \rho dV$$

Divergence Thm

$$\oint\!\!\oint_S \vec{J}\bullet d\vec{S} = \iiint_V \nabla\bullet\vec{J}\ dV = -\frac{d}{dt}\iiint_V \rho dV$$

$$\Rightarrow\ \nabla\bullet\vec{J} = -\frac{d\rho}{dt}$$

Useful Relationships

$\nabla\bullet(\nabla\times\vec{A})=0$

$\nabla\times(\nabla f)=0$ We already knew ∇f is conservative

$\nabla\times(\nabla\times\vec{A})=\nabla(\nabla\bullet\vec{A})-\nabla^2\vec{A}$ Must use cartesian coordinates!

$$\nabla^2\vec{A}\equiv(\nabla^2 A_x)\hat{a}_x+(\nabla^2 A_y)\hat{a}_y+(\nabla^2 A_z)\hat{a}_z$$

$$0=\nabla\bullet(\nabla\times\vec{H})=\nabla\bullet(\vec{J}+\frac{d\vec{D}}{dt})$$

$$=\nabla\bullet\vec{J}+\frac{d}{dt}(\nabla\bullet\vec{D})=\nabla\bullet\vec{J}+\frac{d\rho}{dt}$$

$$\therefore\ \nabla\bullet\vec{J}=-\frac{d\rho}{dt}$$

The continuity eqn. is contained in Maxwell's Eqns.

Example: Continuity Eqn.

- For $\mathbf{J}=J_0(x^2\mathbf{a}_x+y^2\mathbf{a}_y+z^2\mathbf{a}_z)$, find dρ/dt at the point (0.02,0.01,0.01)

From D3.6a (p 149) of old book

Summary of Maxwell's Equations

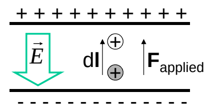
Faraday's Law	$\oint_C \vec{E}\bullet d\vec{l} = -\frac{d}{dt}\iint_S \vec{B}\bullet d\vec{S}$	$\nabla\times\vec{E}=-\frac{d\vec{B}}{dt}$
Ampere's Law	$\oint_C \vec{H}\bullet d\vec{l} = \iint_S \vec{J}\bullet d\vec{S}+\frac{d}{dt}\iint_S \vec{D}\bullet d\vec{S}$	$\nabla\times\vec{H}=\vec{J}+\frac{d\vec{D}}{dt}$
Gauss' Law	$\oint\!\!\oint_S \vec{B}\bullet d\vec{S}=0$	$\nabla\bullet\vec{B}=0$
Gauss' Law	$\oint\!\!\oint_S \vec{D}\bullet d\vec{S}=\iiint_V \rho dV$	$\nabla\bullet\vec{D}=\rho$
Continuity Eq.	$\oint\!\!\oint_S \vec{J}\bullet d\vec{S}=-\frac{d}{dt}\iiint_V \rho dV$	$\nabla\bullet\vec{J}=-\frac{d\rho}{dt}$

Lectures 6-7
Sections 6.1-6.2

Potential Functions for Static Fields
Poisson's and Laplace's Equation
PN Junction

Adapted from Prof. Cunningham's Notes

Electric Potential



How much potential energy does a charge have in an electric field?

Answer: It depends on the work that you had to do to get it to that spot. You have to work against the E-field.

$$\Delta PE = \vec{F}_{applied} \bullet d\vec{l} = -\vec{F}_e \bullet d\vec{l}$$
$$\vec{F}_e = q\vec{E}$$
$$\Delta PE = -q\vec{E} \bullet d\vec{l}$$

Answer depends on the amount of charge q
Can we define something determined by the field only

Electric Potential - Definition

Work you do per unit charge is the ELECTRIC POTENTIAL DIFFERENCE between the two points

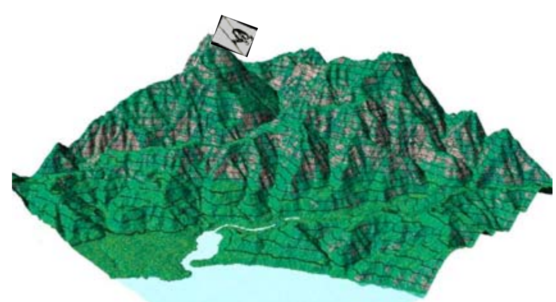
$$\Delta V = \frac{\Delta PE}{q} = -\vec{E} \bullet d\vec{l} = -E_x \Delta x \;\; if \;\; d\vec{l} = \Delta x \hat{a}_x$$

Units: $\frac{Nm}{C} = \frac{J}{C} = VOLTS$

$$E_x = -\frac{\Delta V}{\Delta x}, E_y = -\frac{\Delta V}{\Delta y}, E_z = -\frac{\Delta V}{\Delta z}$$ Units: $\frac{V}{m}$

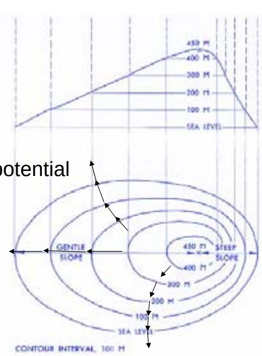
$$\boxed{\vec{E} = -\vec{\nabla} V}$$

Potential: Mount Electron



What is the steepest slope on the mountain?

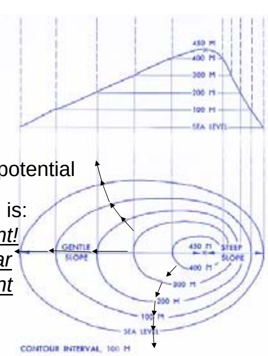
Potential



Surfaces of equal potential

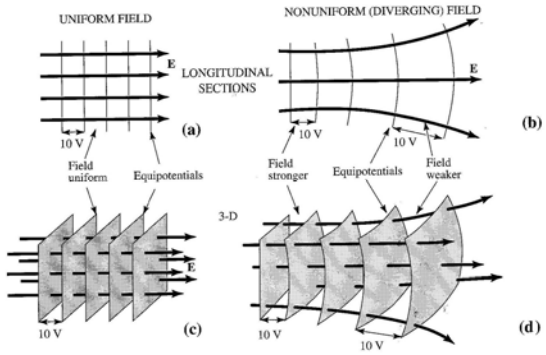
What direction is the steepest slope at a given point?

Potential



Surfaces of equal potential

Steepest downslope is:
Opposite the gradient!
Always perpendicular to surface of constant potential



Gradient Definition

$$\vec{E} = -\left(\frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z\right)$$

Most important definition of today:

$$\vec{E} = -\nabla V$$

$$\nabla = \left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right)$$

Gradient Operator (Cartesian Coordinates)

Gradient Operator

$$\vec{E} = -\nabla V$$

$$\nabla = \left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right)$$

V is a SCALAR FIELD (potential as a function of position)

∇V is a VECTOR FIELD
Has magnitude and direction -**E**
Is the DIRECTION with the FASTEST <u>INCREASE</u> in V
Fastest direction for an electron (negative charge) to decrease its potential

Scalar Potentials

- For: $\Phi_1(x, y, z) = x^2 + y^2 + z^2$
  $\Phi_2(x, y, z) = x + 2y + 2z$

Find the following quantities at (3, 4, 12):

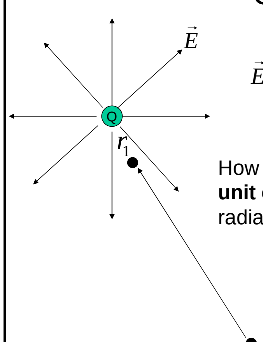
(a) the maximum <u>rate</u> of increase of Φ_1

(b) the maximum <u>rate</u> of increase of Φ_2

(c) the <u>rate</u> of increase of Φ_1 along the direction of the maximum rate of increase of Φ_2

From D5.2 (p 290) of old book

Example: Potentials for a Point Charge

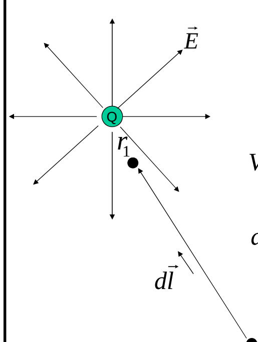


$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{a}_r$$

In spherical coords

How much work is required to move a **unit** charge from infinity to a radial distance = r_1

Example: Potentials for a Point Charge



$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{a}_r$$

In spherical coords

$$\vec{E} = -\nabla V$$

$$V(r) = -\int_{\infty}^{r_1} \vec{E} \bullet d\vec{l}$$

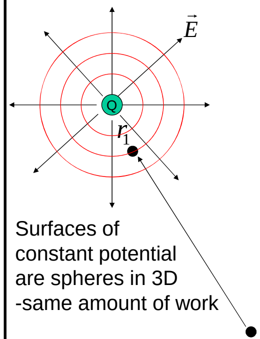
"Absolute" potential at r_1 using zero potential at r = ∞

$$d\vec{l} = -|dr|\,\hat{a}_r = dr\hat{a}_r$$

Since dr<0 going from r=∞

$$V(r) = -\int_{\infty}^{r_1} \frac{Q}{4\pi\varepsilon_0 r^2}dr$$

Example: Potentials for a Point Charge



$$V(r) = -\int_{\infty}^{r_1} \frac{Q}{4\pi\varepsilon_0 r^2}\,dr$$

$$= \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_1} - \frac{1}{\infty}\right)$$

$$\boxed{= \frac{Q}{4\pi\varepsilon_0 r_1}}$$

Challenge Question: Scalar Potentials

- If the scalar potential V can be defined, which of the following is true?

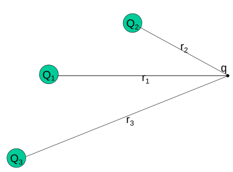
(a) The potential difference between two points is identical, independent of path
(b) The electric field is conservative
(c) The electric field lines will always be perpendicular to equipotential surfaces
(d) The greatest decrease in potential per unit length is along the electric field direction
(e) All of the above

LG's question

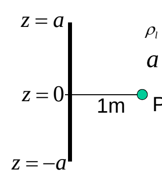
Potential Superposition

Potentials of more than one point charge are superimposed by addition

$$V_{\text{point}} = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q_1}{r_1} + \frac{Q_2}{r_2} + \frac{Q_3}{r_3} + \bullet\bullet\bullet\right)$$



Superposition Example: Potential of a Line Charge

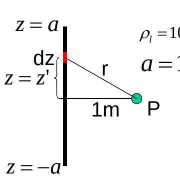


$\rho_l = 10^{-10}(C/m)$

$a = 1m$

Find Potential at point "P", 1 m away from the line

Superposition Example: Potential of a Line Charge



$\rho_l = 10^{-10}(C/m)$

$a = 1m$

Find Potential at point "P", 1 m away from the line

$$V_P = \frac{1}{4\pi\varepsilon_0}\int_{-a}^{a}\frac{dQ}{r}$$

$$V_P = \frac{1}{4\pi\varepsilon_0}\int_{-a}^{a}\frac{\rho_l dz'}{r}$$

$$V_P = \frac{1}{4\pi\varepsilon_0}\int_{-a}^{a}\frac{\rho_l dz'}{\sqrt{z'^2+1^2}}$$

$$V_P = \frac{\rho_l}{4\pi\varepsilon_0}\ln\left(z' + \sqrt{z'^2+1}\right)\Big|_{-a}^{a}$$

Scalar Potentials

- For: $\vec{E} = yz\hat{a}_x + (y+zx)\hat{a}_y + xy\hat{a}_z$

(a) Find the scalar potential if V(0,0,0)=0. (Hint: Use a direct line path. At home, try using 3 separate segments along x, y, and z directions.)
(b) Evaluate the potential difference V_A-V_B for:

1. A=(2,1,1) and B=(1,4,0.5)
2. A=(2,2,2) and B=(1,1,1)
3. A=(5,1,0.2) and B=(1,2,3)

Adapted from D 5.4 (p299) of old book

Useful facts about Potential

- Potential is always the DIFFERENCE between two points
- If one point is at infinity, then the potential is an absolute potential
- The gradient of V gives the vector -**E** at a particular point
- Potential between two points is identical, regardless of whether a straight or curved path is taken
  - **E** is a conservative field
- Electric field lines are always perpendicular to equipotential surfaces (constant voltage)
  - E-field lines are along the direction of the greatest decrease in potential

Gradient Function in Cylindrical and Spherical Coordinates

Φ = scalar field (like voltage)

$$\nabla\Phi = \begin{cases} \left(\frac{\partial\Phi}{\partial x}\hat{a}_x + \frac{\partial\Phi}{\partial y}\hat{a}_y + \frac{\partial\Phi}{\partial z}\hat{a}_z\right) & \text{Cartesian} \\ \left(\frac{\partial\Phi}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\phi}\hat{a}_\phi + \frac{\partial\Phi}{\partial z}\hat{a}_z\right) & \text{Cylindrical} \\ \left(\frac{\partial\Phi}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\hat{a}_\phi\right) & \text{Spherical} \end{cases}$$

Laplacian Operator

$$\boxed{\vec{\nabla}\bullet(\nabla\Phi) = \nabla^2\Phi}$$

The "Laplacian" of a scalar field. (also called "Del Squared")

$\Phi(x,y,z)$ Scalar Field

$$\nabla\Phi = \frac{\partial\Phi}{\partial x}\hat{a}_x + \frac{\partial\Phi}{\partial y}\hat{a}_y + \frac{\partial\Phi}{\partial z}\hat{a}_z$$ Vector

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2}$$ Scalar

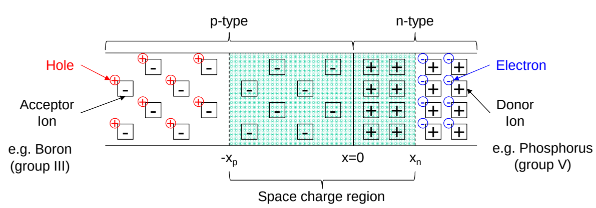
Poisson Equation for Potentials (static fields)

$$\boxed{\vec{\nabla}\bullet(\nabla\Phi) = -\frac{\rho}{\varepsilon}}$$

$$\boxed{\nabla^2\Phi = -\frac{\rho}{\varepsilon}}$$

Known as "Poisson's Equation"
It's just Gauss' Law in terms of Potentials for a Static Field
If ρ=0, it is known as "Laplace's Equation"

Potential of pn junction (ECE 340)



- Find the scalar potential V for a step pn junction in silicon:

$$\rho = \begin{cases} -qN_A & \text{for } -d_p < x < 0 \\ qN_D & \text{for } 0 < x < d_n \\ 0 & \text{otherwise} \end{cases}$$

From Example 5.5 (p 300) of old book

Maxwell's Eqns - Integral form

$$\oint_C \vec{E}\bullet d\vec{l} = -\frac{d}{dt}\iint_S \vec{B}\bullet d\vec{S}$$

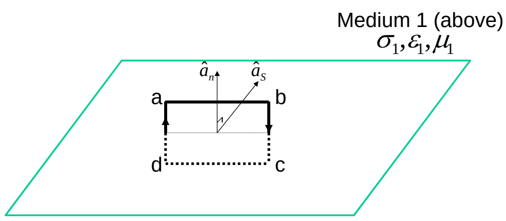
$$\oint_C \vec{H}\bullet d\vec{l} = \iint_S \vec{J}\bullet d\vec{S} + \frac{d}{dt}\iint_S \vec{D}\bullet d\vec{S}$$

$$\oiint_S \vec{B}\bullet d\vec{S} = 0$$

$$\oiint_S \vec{D}\bullet d\vec{S} = \iiint_V \rho dV$$

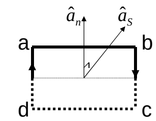
They are valid for ALL closed paths and closed surfaces, EVEN WHEN THEY SPAN A BOUNDARY BETWEEN TWO MATERIALS

Closed Path Through a Boundary



Closed path: abcd
Apply Faraday's Law and Ampere's Law to the closed path

Normal Vectors



Medium 1 (above) $\sigma_1, \varepsilon_1, \mu_1$

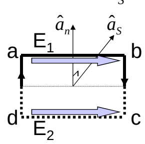
Medium 2 (below) $\sigma_2, \varepsilon_2, \mu_2$

$\hat{a}_n$ Vector NORMAL to the boundary. Points INTO medium 1

$\hat{a}_S$ Vector normal to the path, TANGENT to the interface. Use right hand rule for path to define direction

Faraday's Law at Boundary

$$\oint_C \vec{E} \bullet d\vec{l} = -\frac{d}{dt}\iint_S \vec{B} \bullet d\vec{S} = 0$$



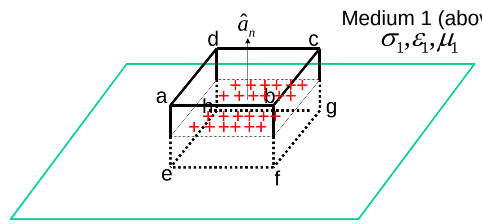
Medium 1 (above) $\sigma_1, \varepsilon_1, \mu_1$

Medium 2 (below) $\sigma_2, \varepsilon_2, \mu_2$

Take limit as *ad* and *bc* go to zero
Consider remaining E_1 and E_2 TANGENT TO SURFACE

$$E_1 = E_2 \;\; i.e. E_t \text{ is continuous}$$

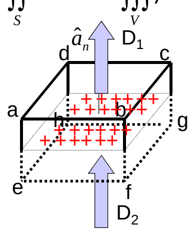
The closed volume can enclose surface charges



Example
1. Free charges on the surface of a conductor

Gauss' Law for D at Boundary

$$\oiint_S \vec{D} \bullet d\vec{S} = \iiint_V \rho dV$$

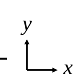


Take limit as *ae*, *bf*, *cg*, and *dh* go to zero
Consider D_1 and D_2 NORMAL TO SURFACE

$$D_1 - D_2 = \rho \;\; i.e. D_n \text{ is discontinuous because of } \rho_S$$

Challenge Question: Boundary Conditions

- What is $\mathbf{D_1}$? $\quad \varepsilon_1 = 1\varepsilon_0 \quad \rho = 3\ C/m^2$; $\quad \varepsilon_2 = 2\varepsilon_0 \quad \vec{D}_2 = 3\hat{x} + 2\hat{y}\ C/m^2$



(a) $3\hat{x} + 2\hat{y}\ C/m^2$
(b) $3\hat{x} + 5\hat{y}\ C/m^2$
(c) $3\hat{x} - 1\hat{y}\ C/m^2$
(d) $\frac{3}{2}\hat{x} + 5\hat{y}\ C/m^2$
(e) $\frac{3}{2}\hat{x} - 1\hat{y}\ C/m^2$

LG's question

Lecture 6-7 Summary

- Since div **D** = ρ and **D** = ε **E**,

$$\boxed{\vec{E} = -\nabla\Phi}$$

satisfy Gauss' Laws and is valid for static fields, if ε is constant and Φ satisfies Poisson's equation: $\boxed{\nabla^2\Phi = -\dfrac{\rho}{\varepsilon}}$

- At a boundary, $E_{1t}=E_{2t}$ but $D_{1n}-D_{2n}=\rho$

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Lectures 8-9
Section 5.1

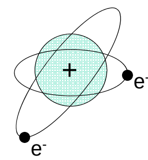
Conductors
Dielectrics

Adapted from Prof. Cunningham's Notes

1

Goddard ECE329
Lectures 8-9

Atomic Model of Conductivity

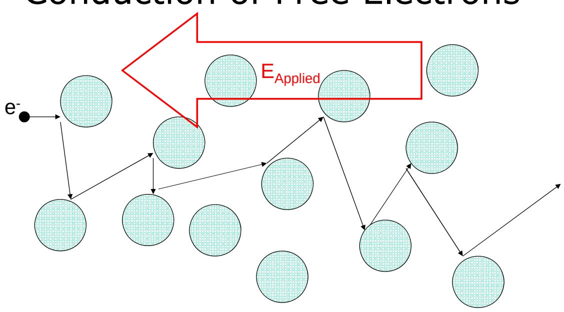


- Tightly bound inner orbitals
- Loosely bound outer orbitals
  - Free to escape the nucleus and move around inside the material

2

Goddard ECE329
Lectures 8-9

Conduction of Free Electrons



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Lectures 8-9

Conduction of free electrons

- The electron cannot travel in a straight path for very long
- It keeps running into the nuclei, getting deflected MANY times, meeting RESISTANCE
- Net motion is still in direction opposite of the applied **E** and drift velocity is:

$$\vec{v}_d = -\mu_e \vec{E} \quad \text{for electrons}$$

$\mu_e = \frac{|e|\tau}{m}$ is the electron mobility,

$\tau \approx 10^{-14}$ sec is the average collision time

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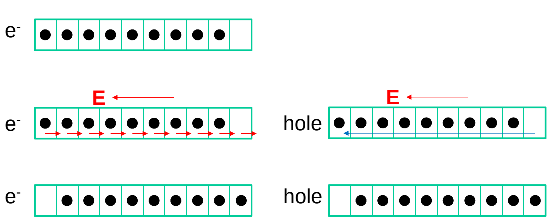
Holes are missing electrons



- Holes are missing electrons
  - Move in direction of **E** field

$$\vec{v}_d = +\mu_h \vec{E} \quad \text{for holes}$$

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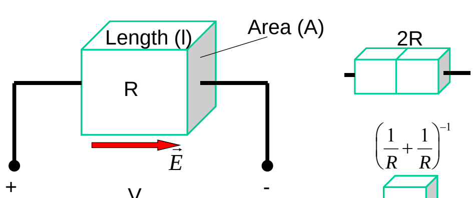
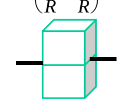
Lectures 8-9

Resistors



$$R = \frac{V}{I}$$

$$\Omega(Ohms) = \frac{volts}{amps}$$



R tells us how much total resistance is encountered, but does not tell us anything fundamental about the resisting material

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Lectures 8-9

Resistivity

$$\rho = \text{Resistivity, units of } (\Omega \cdot \text{m})$$

- Tells us how much resistance the *material* provides, factoring out the dimensions of the resistor

$$R = \rho \frac{l}{A}$$

Conductivity

$$\sigma = \frac{1}{\text{Resistivity}} = \frac{l}{R \cdot A} = \frac{1}{\Omega \cdot m}$$

- More common to describe materials in terms of conductivity rather than resistivity
- High conductivity = low resistivity
- Special units:

$$\sigma = \frac{1}{\Omega \cdot m} = \frac{Siemens}{m} = \frac{S}{m}$$

ABLE 2-1
onductivities†

	Material	Conductivity, ℧ m^{-1}		Material	Conductivity, ℧ m^{-1}
Insulators	Quartz, fused	$\sim 10^{-17}$	Conductors	Silicon	10^{3}
	Ceresin wax	$\sim 10^{-17}$		Carbon	$\sim 3 \times 10^{4}$
	Polystyrene	$\sim 10^{-16}$		Graphite	$\sim 10^{5}$
	Sulfur	$\sim 10^{-15}$		Cast iron	$\sim 10^{6}$
	Mica	$\sim 10^{-15}$		Mercury	10^{6}
	Paraffin	$\sim 10^{-15}$		Nichrome	10^{6}
	Rubber, hard	$\sim 10^{-15}$		Stainless steel	10^{6}
	Porcelain	$\sim 10^{-14}$		Constantan	2×10^{6}
	Glass	$\sim 10^{-12}$		Silicon steel	2×10^{6}
	Bakelite	$\sim 10^{-9}$		German silver	3×10^{6}
	Distilled water	$\sim 10^{-4}$		Lead	5×10^{6}
insulators	Dry, sandy soil	$\sim 10^{-3}$		Tin	9×10^{6}
	Marshy soil	$\sim 10^{-2}$		Phosphor bronze	10^{7}
	Fresh water	$\sim 10^{-2}$		Brass	1×10^{7}
	Animal fat‡	4×10^{-2}		Zinc	1.7×10^{7}
Poor conductors	Animal muscle (⊥ to fiber)‡	0.08		Tungsten	1.8×10^{7}
	Animal, body (ave)‡	0.2		Duralumin	3×10^{7}
	Animal muscle (∥ to fiber)‡	0.35		Aluminum, hard-drawn	3.5×10^{7}
	Animal blood	0.7		Gold	4.1×10^{7}
	Germanium (semiconductor)	~ 2		Copper	5.7×10^{7}
Conductors	Seawater	~ 4		Silver	6.1×10^{7}
	Ferrite	10^{2}	Superconductors	Hg (at <4.1 K)	∞
	Tellurium	$\sim 5 \times 10^{2}$		Nb (at <9.2 K)	∞
				Nb_3(Al-Ge) (at <21 K)	∞
				$YBa_2Cu_3O_7$ (at <80 K)	∞

†or low frequencies. At 20°C except where noted.

Conductivity

- Conductivity takes everything into account in one number
  - Number of free electrons
  - Frequency of collisions between electrons and the nuclei
  - Scattering properties of the nuclei
  - Motion for electrons and holes

$$\sigma = \mu_e N_e |e| + \mu_h N_h |e|$$

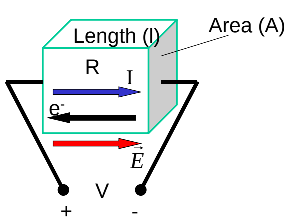
N_e, N_h = # of free electrons or holes per cm^3

Ohm's Law

$$|E| = \frac{V}{l} \quad \left(\frac{V}{m}\right)$$

$$|J| = \frac{I}{A} \quad \left(\frac{A}{m^2}\right)$$

$$R = \frac{l}{\sigma A} \quad (\Omega)$$



$$V = IR$$

$$E \cdot l = (J \cdot A)\left(\frac{l}{\sigma A}\right) = \frac{1}{\sigma} J \cdot l$$

$$\boxed{J = \sigma E}$$

Maxwell's Equations in a Conductor

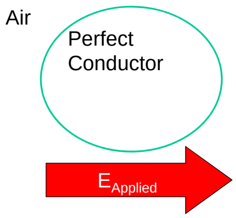
$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$ In free space

$$\boxed{\vec{\nabla} \times \vec{H} = \sigma \vec{E} + \frac{\partial \vec{D}}{\partial t}}$$ In a conducting material

Conduction current: Moving charge

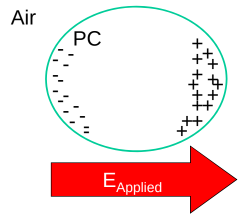
Displacement Current: Time varying E-field

Perfect Conductor (PC) in an Applied Electric Field



What Happens?

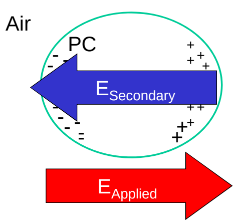
Perfect Conductor (PC) in an Applied Electric Field



<u>What Happens?</u>

-Free e^- move in direction opposite of applied E
- When e^- moves away from its nucleus, it leaves a + charge behind
- The material as a whole is still NEUTRALLY CHARGED but the charge has now been redistributed

E=0 inside perfect conductor

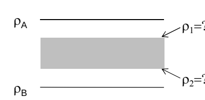


<u>What Happens Next?</u>

-The separated + and - charges create their own SECONDARY INTERNAL E field that <u>*EXACTLY CANCELS*</u> the applied E field
- The TOTAL E field inside the perfect conductor is ALWAYS ZERO
- If it weren't zero, free charge would continue drifting till it is!

Infinite plane conducting slab

- An infinite plane conducting slab lies between two infinite plane sheets of uniform charge density ρ_A and ρ_B. Find the surface charge densities on the two slab surfaces.



From D4.2 (p 217) of old book

If **E**=0 inside a PC, how about **H**?

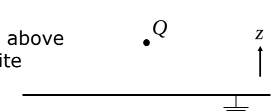
$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \Rightarrow \quad 0 = -\frac{\partial \vec{B}}{\partial t}$$

B and H must be STATIC

Technically, a non-zero static H-field can exist inside a PC, but how did it get there? It must have existed there for all time. But, then what created it in the first place. This is an ill-posed problem so in 329, we will assume H=0 inside a PC.

Challenge Question: Point charge above a sheet

- A positive charge is located above a perfectly conducting infinite ground plane at z=0



(a) **E**=0 above the plane

(b) **E**=0 below the plane

(c) A uniform charge density is induced on the plane

(d) The total induced charge on the plane is +Q

(e) The electric field lines have a radial component at the plane

LG's question

Lecture 8 Summary

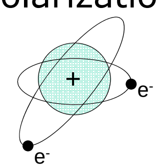
- Ohm's Law:
- Conductivity: $\sigma = \frac{1}{\text{Resistivity}} = \frac{l}{R \cdot A}$
- Inside a perfect conductor, **E**=
- Next class
  - Dielectrics

ECE 329
Lecture 9

Dielectrics

Adapted from Prof. Cunningham's Notes

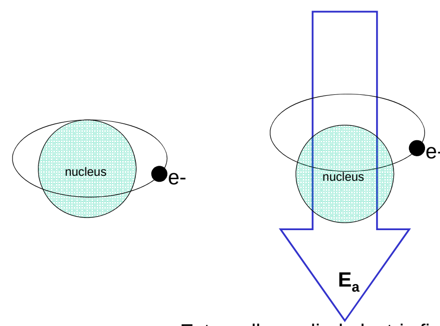
Atomic Model of Dielectric Polarization



Bound charge

- Tightly bound inner orbitals
- Not many loosely bound outer orbital electrons

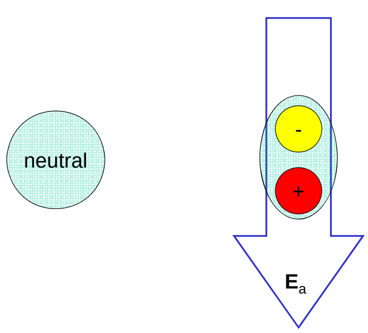
Polarization of an atom



Bound charge

Externally applied electric field separates electron and nucleus

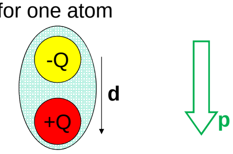
Polarization of an atom



External field distorts atom slightly
E inside is **reduced** but is non-zero

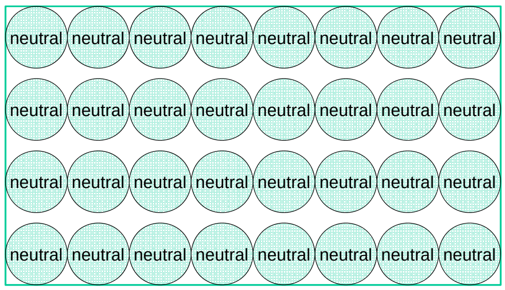
Polarization or Electric Dipole Moment per Unit Volume

"mini" dipole moment for one atom



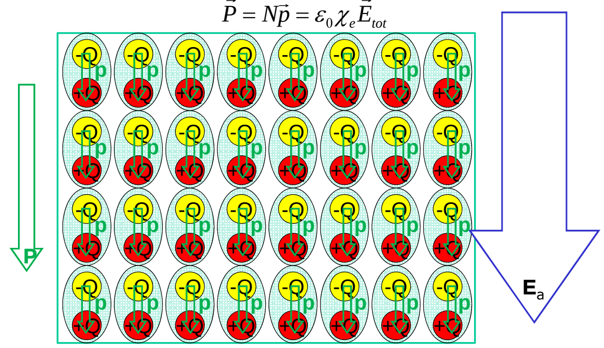
d is the distance VECTOR going from "-" to "+" side

p=Q**d** is the electric dipole moment



Apply **E** to Dielectric Material

$$\vec{P} = N\vec{p} = \varepsilon_0 \chi_e \vec{E}_{tot}$$

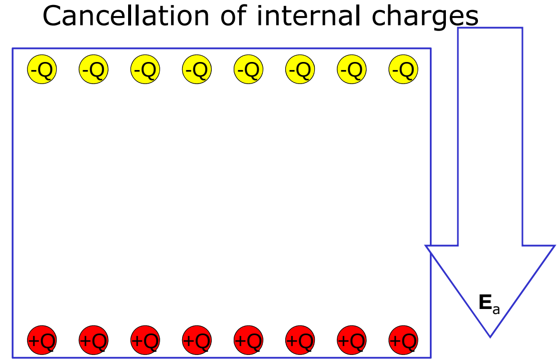


Dielectric Susceptibility χ_e

- Measures how easy it is to shift electrons from their centered orbit around the nuclei of a material to form internal dipoles

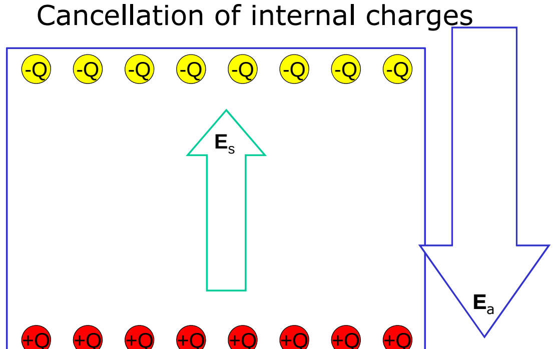
$$\vec{P} = \varepsilon_0 \chi_e \vec{E}_{tot}$$

Apply **E** to Dielectric Material
Cancellation of internal charges



"Polarization Surface Charges" Remain

Apply **E** to Dielectric Material
Cancellation of internal charges



Secondary Electric Field Produced
* *Does not cancel applied field inside dielectric*

$\mathbf{E}_{tot}$ is reduced but not zero

Unlike conductors where **E**=0, in the dielectric slab, the total field is **reduced (but NOT eliminated)** by the secondary field produced by the surface polarization charge

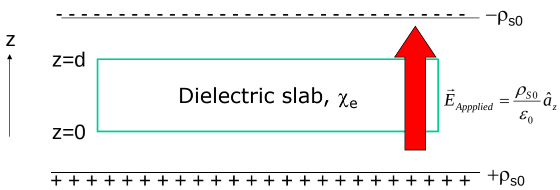
$$\vec{E}_{tot} = \vec{E}_a + \vec{E}_s$$

And the polarization vector, **P**, is the polarization of atoms in the dielectric due to the TOTAL field (after it has been reduced by $\mathbf{E}_s$)

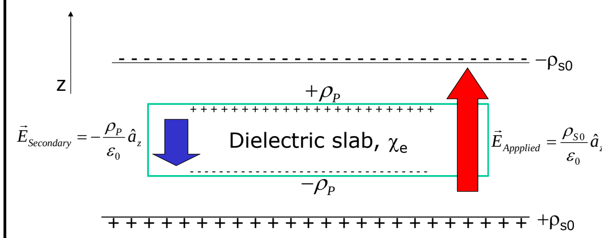
$$\vec{P} = \varepsilon_0 \chi_e \vec{E}_{tot}$$

Example

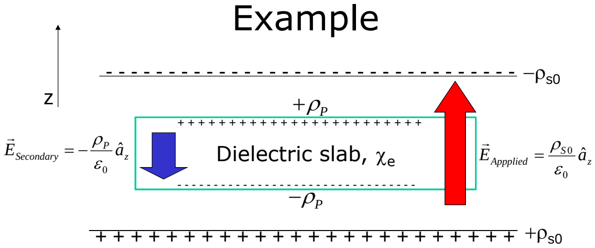
- Apply external electric field to a slab of dielectric material
- Apply field by placing the slab between two equal and opposite charge densities



Example



Example

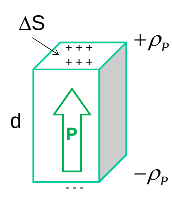


$$\vec{E}_{Total} = \vec{E}_{Appplied} + \vec{E}_{Secondary}$$

$$\vec{P} = \varepsilon_0 \chi_e \vec{E}_{total} = \chi_e(\rho_{S0} - \rho_P)\hat{a}_z$$

But how much charge ρ_P is generated by ρ_{S0}??

Example



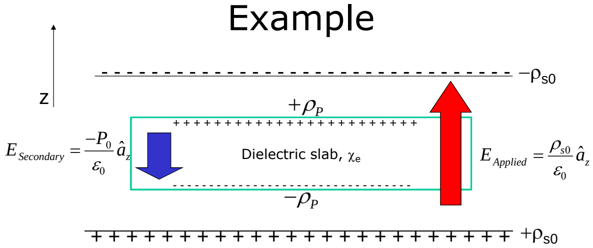
$$\vec{P} = P_0\hat{a}_z$$ where P_0 is the dipole moment per unit volume

$$\vec{P}V = P_0(d\Delta S)\hat{a}_z$$ the total dipole moment of the column

$$Q\vec{d} = (\rho_P \Delta S)d\hat{a}_z$$ also the total dipole moment of the column

$$\therefore\ P_0 = \rho_P$$ dipole moment per unit volume = surface charge density

Example



$$\vec{P} = \varepsilon_0 \chi_e \vec{E}_{total} = \chi_e(\rho_{S0} - \rho_P)\hat{a}_z$$

$$P_0\hat{a}_z = \chi_e(\rho_{S0} - \rho_P)\hat{a}_z = \chi_e(\rho_{S0} - P_0)\hat{a}_z$$

$$P_0 = \chi_e\rho_{S0} - \chi_e P_0 \qquad \boxed{\therefore\ \rho_P = P_0 = \frac{\chi_e \rho_{S0}}{1+\chi_e}}$$

Example

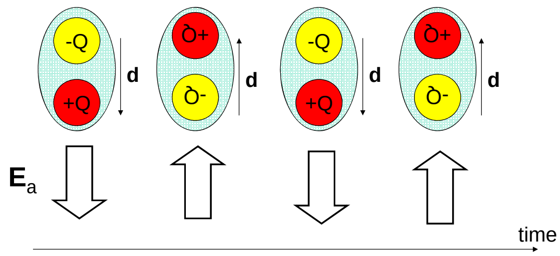
So finally, the “final answer” is:

$$\vec{E}_{Total} = \frac{P_0}{\varepsilon_0 \chi_e} = \frac{\rho_{S0}/\varepsilon_0}{1+\chi_e}$$

The total electric field strength inside the dielectric is reduced from its “free space” value by $(1+\chi_e)$

Free space: χ_e=0 Perfect Conductor: $\chi_e \to \infty$

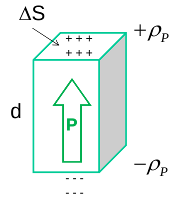
Dipole Atoms in Alternating **E** Field



Moving charges means a current
Definition of the "Polarization Current"

$$I_P = \frac{dQ}{dt}$$

Recall from the Example



$+\rho_P$

$-\rho_P$

$\vec{P} = P_0 \hat{a}_z$ where P_0 is the dipole moment per unit volume

$\vec{P}V = P_0(d\Delta S)\hat{a}_z$ the total dipole moment of the column

$Q\vec{d} = (\rho_P \Delta S)d\hat{a}_z$ also the total dipole moment of the column

$$P_0 \Delta S = Q$$

$$\boxed{\therefore \;\; J_p = \frac{I_p}{\Delta S} = \frac{dQ/dt}{\Delta S} = \frac{dP_0}{dt}}$$

Polarization Current

- Application of an alternating **E** field results in a polarization current due to motion of charge between the two surfaces of the dielectric material

$$\boxed{\vec{J}_P = \frac{\partial \vec{P}}{\partial t}}$$

Ampere's Law in a Dielectric

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

In a dielectric medium, we need to include J_P with the total current

$$\vec{\nabla} \times \vec{H} = \vec{J} + \vec{J}_P + \frac{\partial(\varepsilon_0 \vec{E}_{total})}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{P}}{\partial t} + \frac{\partial(\varepsilon_0 \vec{E}_{total})}{\partial t}$$

Ampere's Law in a Dielectric

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{P}}{\partial t} + \frac{\partial(\varepsilon_0 \vec{E}_{total})}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial}{\partial t}\underbrace{(\vec{P} + \varepsilon_0 \vec{E}_{total})}$$

So Ampere's law is the same and we modify the displacement vector's definition

$$\boxed{\vec{D} = \vec{P} + \varepsilon_0 \vec{E}_{total}}$$

Definition of Dielectric Constant

$$\vec{D} = \vec{P} + \varepsilon_0 \vec{E}_{total}$$

$$\vec{D} = \varepsilon_0 \chi_e \vec{E}_{total} + \varepsilon_0 \vec{E}_{total}$$

$$\vec{D} = \varepsilon_0 (1 + \chi_e) \vec{E}_{total}$$

$$\boxed{\vec{D} = \varepsilon_0 \varepsilon_r \vec{E}_{total}}$$

Definition of Dielectric Constant

$$\boxed{\vec{D} = \varepsilon_0 \varepsilon_r \vec{E}_{total}}$$

Relative Permittivity = "Dielectric Constant" $\varepsilon_r = (1 + \chi_e)$

Dielectric Permittivity of the Material $\varepsilon = \varepsilon_0 \varepsilon_r$

$$\boxed{\vec{D} = \varepsilon \vec{E}_{total}}$$

Units for ε_r: None

Ampere's Law in a Dielectric

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

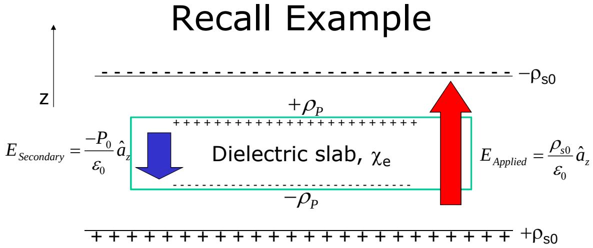
with

$$\vec{D} = \varepsilon \vec{E}_{total}$$

No need to worry about polarization current - it is all incorporated into the "new" definition for **D**

Generally we can use Maxwell's equations to solve for **D** and then calculate $\mathbf{E}_{total}$

Recall Example



$$\vec{E}_{Total} = \frac{P_0}{\varepsilon_0 \chi_e} = \frac{\rho_{S0}/\varepsilon_0}{1+\chi_e}$$

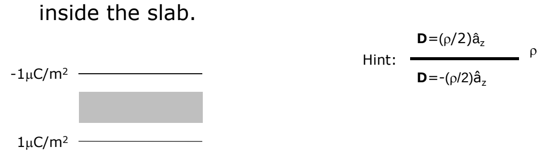
E-field strength reduced by $(1+\chi_e)$

$$\vec{D} = \varepsilon \vec{E}_{Total} = \varepsilon_0 (1+\chi_e) \vec{E}_{Total} = \rho_{S0}$$

D-field strength is same as free space

Infinite plane dielectric slab

- An infinite plane dielectric slab lies between two infinite plane sheets of uniform charge density of $\rho = \pm 1\mu C/m^2$. Find **D**, **E**, and **P** inside the slab.



From D4.3 (p 226) of old book

Challenge Question: Spherical shell dielectric

- If we have: $\begin{cases} r < a & perfect\ conductor \\ a < r < b & perfect\ dielectric\ \varepsilon = 3\varepsilon_0 \\ r > b & perfect\ conductor \end{cases}$

with equally and oppositely charged PCs and **D** points radially inward, which is true:

(a) V(b) > V(a)

(b) V(b) = V(a)

(c) V(b) < V(a)

LG's question

Lecture 9 Summary

- Electric dipole moment $\mathbf{p}=q\mathbf{d}$
- Polarization or electric dipole moment per unit volume

  $\mathbf{P}=N\mathbf{p}=\varepsilon_0\chi_e\mathbf{E}_{total}=\varepsilon_0\chi_e(\mathbf{E}_a+\mathbf{E}_s)$
  - Simple linear isotropic dielectric
    - Reduces **E**-field strength by $(1+\chi_e)$
    - **D** has same value as free space
  - Polarization current $\mathbf{J}_p=d\mathbf{P}/dt$
  - New definition $\mathbf{D}=\mathbf{P}+\varepsilon_0\mathbf{E}_{total}=\varepsilon_0\mathbf{E}_{total}$
- Next class
  - Capacitance and Conductance

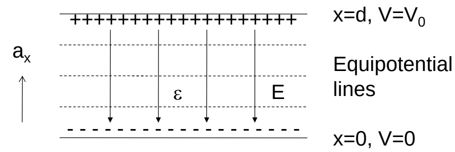
ECE 329
Lectures 10-11
Sections 6.3, 5.1

Capacitance and Conductance

Conductivity and Susceptibility

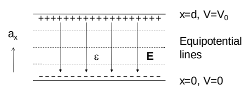
Adapted from Prof. Cunningham's Notes

Parallel-plate Capacitor



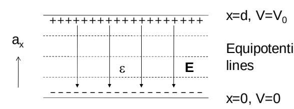
Capacitance is: $C = Q/V_0$

Steps to Find Capacitance



- Laplace Equation $\vec{\nabla}^2 V = 0$ $\boxed{V(x) = V_0 \frac{x}{d}}$
- Find V using boundary conditions
- Find **E** using $\vec{E} = -\vec{\nabla} V$
- Find **D** using $\vec{D} = \varepsilon \vec{E}$
- Get surface charge density on <u>one</u> conductor using BC $\rho_s = \vec{a}_n \bullet (D_{n1} - D_{n2})$ $\boxed{\rho = \varepsilon V_0 / d}$
- Charge $Q = (Area)(\rho_s)$
- Capacitance $C = Q/V_0$ $\boxed{C = \frac{\varepsilon A}{d}}$

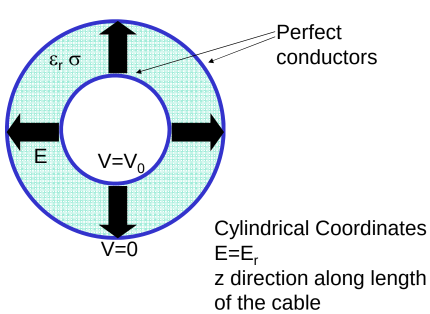
Non-uniform permittivity



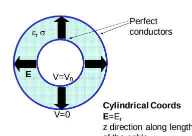
$$\varepsilon(x) = \frac{\varepsilon_0}{1 + x/2d}$$

From P5.20 (p352) of old book

Coaxial Cable



Coaxial Cable



$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = 0$$

$$\rho = \begin{cases} \varepsilon V_0 / (a \ln(b/a)), r = a \\ -\varepsilon V_0 / (b \ln(b/a)), r = b \end{cases}$$

$$\boxed{C = \frac{2\pi\varepsilon L}{\ln(b/a)}}$$

Challenge Question: Coaxial cable capacitance

- Consider a coax with ($\varepsilon_{old}=3\varepsilon_0$) for a<r<b. If we remove the dielectric and replace it with free space ($\varepsilon_{new}=\varepsilon_0$) but **keep the same amount of charge** on each PC, which is **false**:

(a) the voltage across the capacitor is reduced by 3x

(b) the capacitance is reduced by 3x

(c) the E-field for a<r<b is increased by 3x

(d) the D-field for a<r<b is unchanged

LG's question

Challenge Question: Coaxial cable capacitance

- Consider a coax with ($\varepsilon_{old}=3\varepsilon_0$) for a<r<b. If we remove the dielectric and replace it with free space ($\varepsilon_{new}=\varepsilon_0$) but **keep the same voltage** across the coax, which is **false**:

(a) the charge across the capacitor is reduced by 3x

(b) the capacitance is reduced by 3x

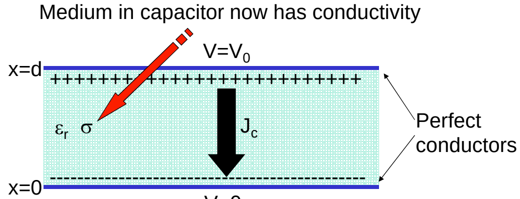
(c) the E-field for a<r<b is reduced by 3x

(d) the D-field for a<r<b is reduced by 3x

LG's question

Conductance

Medium in capacitor now has conductivity



Now a current can flow x=d to x=0

$$\vec{J}_c = \sigma\vec{E} = \sigma\left(\frac{V_0}{d}\right)(-\vec{a}_x)$$

Current density (A/m^2)

Conductance

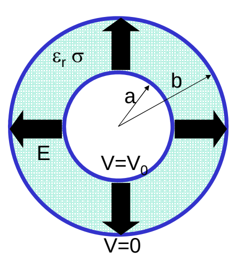
$$G = \frac{|I_c|}{V_0}$$

Conductance
Units: Siemens

For the parallel plate capacitor

$$G = \frac{\sigma A}{d}$$

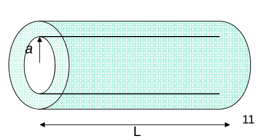
Conductance/Length for Coaxial Cable



$$\vec{E} = \frac{V_0}{\ln(b/a)}\left(\frac{1}{r}\right)\vec{a}_r$$

so

$$\vec{J}_c = \sigma\vec{E} = \frac{\sigma V_0}{\ln(b/a)}\left(\frac{1}{r}\right)\vec{a}_r \qquad \left[\frac{A}{m^2}\right]$$



What is radial current (I_c) for a fixed length of the cable?

Conductance/Length of Coaxial Cable

$$I_c = \int \vec{J} \bullet d\vec{S} \qquad [A]$$

$$I_c = \int_{\phi=0}^{2\pi}\int_{z=0}^{L} J_c(r d\phi dz)$$

$$I_c = \frac{2\pi\sigma V_0 L}{\ln(b/a)}$$

$$G = \frac{I_c}{V_0} = \frac{2\pi\sigma L}{\ln(b/a)} \qquad [Siemens]$$

$$\mathcal{G} = \frac{G}{L} = \frac{2\pi\sigma}{\ln(b/a)} \qquad [S/m]$$

Steps for Finding Conductance

- Find Electric Field
- Find Conduction Current Density (A/m^2) $\vec{J}_c = \sigma \vec{E}$
- Conduction Current (A) $I_c = \int \vec{J} \bullet d\vec{S}$
- Conductance $G = \dfrac{I_c}{V_0}$
- Conductance/Length

(Optional) The Laplace Transform

- Can be used for Transmission Line analysis
  - Method to solve Initial Value Differential Equations

$$\mathfrak{L}\{f(t)\} = F(s) = \int_{0^-}^{\infty} e^{-st} f(t)dt$$

- Most useful property - it converts differential equations in time to algebraic ones in s-space:

$$\mathfrak{L}\{f'(t)\} = sF(s) - f(0)$$

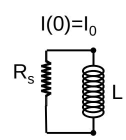
$$\mathfrak{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

(Optional) The Laplace Transform

- Key Transformations are the following:

$f(t) = \mathfrak{L}^{-1}\{F(s)\}$	$F(s) = \mathfrak{L}\{f(t)\}$
1	1/s
e^{at}	1/(s-a)
t^n	$n!/s^{n+1}$
u(t-c)	e^{-cs}/s
u(t-c)f(t-c)	$e^{-cs}F(s)$
$e^{ct}f(t)$	F(s-c)
f(ct)	1/c F(s/c)
$\delta(t-c)$	e^{-cs}
$\int_0^t f(t-\tau)g(\tau)d\tau$	F(s)G(s)

(Optional) Application of the Laplace Transform



$$IR + L\frac{dI}{dt} = 0$$

$$\mathfrak{L}\left\{IR + L\frac{dI}{dt}\right\} = 0$$

$$F(s)R + (sF(s) - I_0)\cdot L = 0$$

$$F(s) = \frac{I_0 L}{R + sL} = \frac{I_0}{s + R/L}$$

$$\mathfrak{L}\{e^{at}\} = \frac{1}{s-a} \Rightarrow \boxed{I(t) = I_0 e^{-\frac{R}{L}t}}$$

Lecture 10 Summary

- Capacitance $C = Q/V_0$
- Conductance $G = |I_c|/V_0$
- Next Up
  - Conductivity and Susceptibility

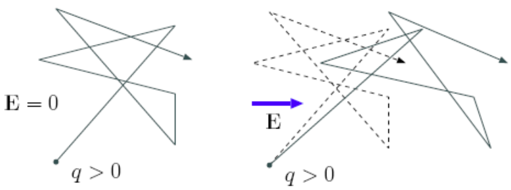
Lecture 11

Bound charge

Modeling χ and ε

The Lorentz-Drude model

- With applied **E**, the mean position of free charge q>0 drifts in direction E with mean velocity **v**: balance of acceleration due to E and friction from collisions with lattice at random intervals with mean time τ



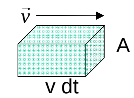
$$m\frac{d\vec{v}}{dt} = q\vec{E} - m\frac{\vec{v}}{\tau}$$

The Lorentz-Drude model: DC conductivity

- If **E**=0, charge eventually slows to **v**=0: $\vec{v}(t) = v_0 e^{-t/\tau}$
- But, with a constant **E**, the steady state solution is:

$$\vec{v}(t=\infty) = \frac{q\tau}{m}\vec{E} = \mu\vec{E} \qquad \mu = \frac{q\tau}{m}$$

charge mobility. For N charges per unit volume, the total current becomes:



$$\vec{I} = \frac{dQ}{dt} = \frac{q\,N\,\Delta V}{\Delta t} = \frac{qNA\vec{v}\Delta t}{\Delta t} = qNA\vec{v}$$

$$\boxed{\vec{J} = \frac{\vec{I}}{A} = qN\vec{v} = \frac{Nq^2\tau}{m}\vec{E}}$$

What does this look like?

Ohm's Law!

$$\boxed{\vec{J} = \frac{\vec{I}}{A} = qN\vec{v} = \frac{Nq^2\tau}{m}\vec{E}}$$

$$\boxed{\vec{J} = \sigma\vec{E} \text{ where } \sigma = \frac{Nq^2\tau}{m} = \frac{Nq^2}{m\varpi}}$$

$$\textit{where } \varpi \equiv \frac{1}{\tau} \textit{ is the collision frequency}$$

Review of PHASORS

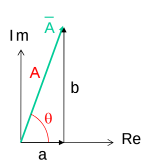
Spock, set your TI-89 to "STUN", not "KILL"!!!

Sorry Captain, I promise not to harm another ECE329 student. Next time, I will set the Phasor to "AWAKEN" instead!

Phasor Review - Complex #'s

$$a + jb = \overline{A} = Ae^{j\theta}$$

$$Ae^{j\theta} = A\cos\theta + jA\sin\theta$$



$$A = \sqrt{a^2 + b^2}$$

$$\theta = \tan^{-1}\frac{b}{a}$$

Phasor Review - Complex #'s

$$Ae^{jx} = A\cos(x) + jA\sin(x)$$

$$A\cos(x) = \mathrm{Re}[Ae^{jx}]$$

$$A\sin(x) = \mathrm{Im}[Ae^{jx}] = \mathrm{Re}[-jAe^{jx}] = \mathrm{Re}[Ae^{j(x-\pi/2)}]$$

Write $5\cos(\omega t) + 10\sin(\omega t - 30^\circ)$ as a phasor

Phasor Review with vectors

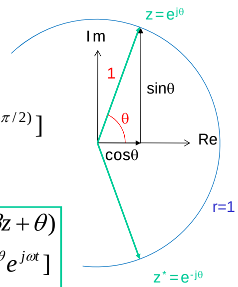
$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\cos(\theta) = \mathrm{Re}[e^{j\theta}]$$

$$\sin(\theta) = \mathrm{Re}[-je^{j\theta}] = \mathrm{Re}[e^{j(\theta-\pi/2)}]$$

$$\mathrm{Re}[z] = (z+z^*)/2$$

$$\boxed{\begin{aligned} E_x(z,t) &= E_x(z)\cos(\omega t \mp \beta z + \theta) \\ &= \mathrm{Re}[E_x(z)e^{\mp j\beta z}e^{j\theta}e^{j\omega t}] \\ &= \mathrm{Re}[\overline{E}_x(z)e^{j\omega t}] \end{aligned}}$$



The Lorentz-Drude model: AC conductivity

- If **E** is time varying, we use phasors to analyze the response:

$$\vec{E}(t) = \mathrm{Re}\{\tilde{E}e^{j\omega t}\}$$

$$\vec{v}(t) = \mathrm{Re}\{\tilde{v}e^{j\omega t}\}$$

$$\vec{J}(t) = \mathrm{Re}\{\tilde{J}e^{j\omega t}\}$$

- Note that : $\tilde{E}$ and $\tilde{J}$ are vectors and complex valued

(e.g. $\tilde{E} = 3.2e^{j\pi/4}\hat{x} - 1.3e^{j\pi/6}\hat{y}$)

The Lorentz-Drude model: AC conductivity

- We can now transform the force equation from a differential one to an algebraic one (d/dt = jω):

$$m\frac{d\vec{v}}{dt} = q\vec{E} - m\frac{\vec{v}}{\tau} \Rightarrow mj\omega\tilde{v} = q\tilde{E} - m\varpi\tilde{v}$$

and thus, we get:

$$\tilde{v} = \frac{q\tilde{E}}{mj\omega + m\varpi} \Rightarrow \boxed{\tilde{J} = qN\tilde{v} = \frac{Nq^2}{m(j\omega+\varpi)}\tilde{E}}$$

$$\boxed{\sigma = \frac{Nq^2}{m(j\omega+\varpi)}}$$

Challenge Question: AC conductivity

- We found: $\boxed{\sigma = \dfrac{Nq^2}{m(j\omega+\varpi)}}$

I. At all finite frequencies, energy is being lost to Joule heating

II. At low frequency (DC), J and E are in phase

III. At very high frequency ($\omega >> \varpi$), J and E are 90° out of phase

Which of the following is true:

(a) I only, (b) II only, (c) III only,

(d) I, II, and III, (e) None are true

LG's question

(Preview) Time Averaged Poynting Vector

$$\vec{E} = \mathrm{Re}\left[\overline{E}e^{jwt}\right] = \frac{1}{2}(\overline{E}e^{jwt} + \overline{E}^*e^{-jwt}) \qquad \vec{H} = \mathrm{Re}\left[\overline{H}e^{jwt}\right]$$

$$\left\langle\vec{P}\right\rangle = \left\langle\vec{E}\times\vec{H}\right\rangle = \left\langle\frac{\overline{E}e^{jwt} + \overline{E}^*e^{-jwt}}{2} \times \frac{\overline{H}e^{jwt} + \overline{H}^*e^{-jwt}}{2}\right\rangle$$

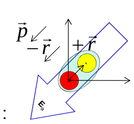
$$= \frac{1}{4}\left\langle\overline{E}\times\overline{H}e^{2jwt} + \overline{E}\times\overline{H}^* + \overline{E}^*\times\overline{H} + \overline{E}^*\times\overline{H}^*e^{-2jwt}\right\rangle$$

$$\boxed{\left\langle\vec{P}\right\rangle = \frac{1}{2}\mathrm{Re}\left[\overline{E}\times\overline{H}^*\right]}$$

Susceptibility

- A perfect dielectric is defined by σ=0 so since $\boxed{\sigma = \dfrac{Nq^2}{m(j\omega+\varpi)}}$, there can't be any free charge inside (N=0)
- The dielectric can have bound charge and we know it can be polarized:

$$p = q\vec{d} = -q\vec{r}$$



if the nucleus is located at the origin

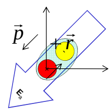
- The electron's motion is described by:

$$\vec{F} = m\vec{a} = m\frac{d^2\vec{r}}{dt^2} = -q\vec{E} - 2m\alpha\frac{d\vec{r}}{dt} - m\omega_0^2\vec{r}$$

where $-m\omega_0^2$**r** describes the spring like restoring force to the nucleus and $-2m\alpha$**v** is a friction like damping force

DC Susceptibility

- For DC, d/dt = 0 so we get



$$\vec{r} = -\frac{q\vec{E}}{m\omega_0^2}$$

and so: $$p = -q\vec{r} = \frac{q^2}{m\omega_0^2}\vec{E}$$

and thus: $$P = N_d\, p = \frac{N_d q^2}{m\omega_0^2}\vec{E} = \varepsilon_0 \chi_e \vec{E}$$

where: $$\boxed{\chi_e = \frac{N_d q^2}{m\omega_0^2 \varepsilon_0}}$$

Note: N_d (# of dipoles per vol) ≠ N (# of free charge per vol)

Can use phasors to derive AC susceptibility: $\chi_e(\omega)$

Lecture 11 Summary

- AC Conductivity: $\boxed{\sigma = \frac{Nq^2}{m(j\omega + \varpi)}}$
- DC Susceptibility: $\boxed{\chi_e = \frac{N_d q^2}{m\omega_0^2 \varepsilon_0}}$
- Next Up
  - Magnetic Force, Ampere's law, current sheets, Faraday's law (1.6, 2.4, 2.3)

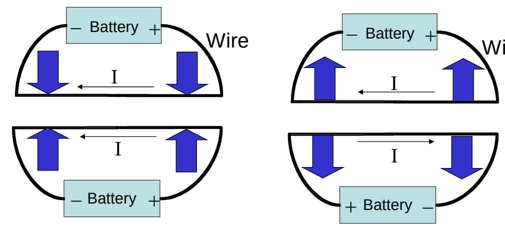
Lectures 12-14
Sections 1.6, 2.4, 2.1, 2.3

Magnetic Fields
Biot-Savart Law
Ampere's Law
Displacement Current
Faraday's Law

Adapted from Prof. Cunningham's Notes

Magnetism and Electricity

Andre Marie Ampere - 1820



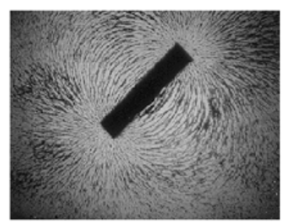
Parallel Currents in Same Direction ATTRACT

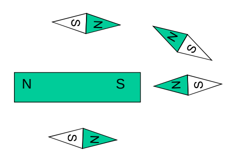
Parallel Currents in Opposite Direction REPEL

Ampere's observations

- The magnitude of F is
  - proportional to the product of the currents AND to the product of their lengths
  - inversely proportional to the square of the distance
  - depends on the medium
- The direction of F on current 1 is
  - perpendicular to d$\mathbf{l}_1$
  - perpendicular to d$\mathbf{l}_2$ x a_{21}
- The forces d$\mathbf{F}_1$ and d$\mathbf{F}_2$ are not always equal and opposite

Magnetic Field of Bar Magnet



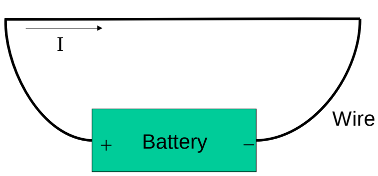


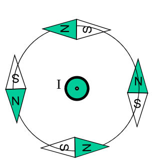
UNLIKE poles ATTRACT
LIKE poles REPEL

(same as electric charges)

Magnetism and Electricity

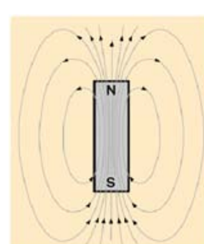
Hans Oersted - 1821





Magnets line up in the presence of a nearby current

Magnetic Flux Lines



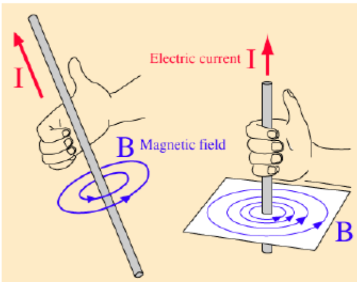
Magnetic flux lines form a VECTOR FIELD

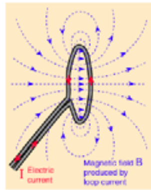
Density of lines indicates MAGNITUDE

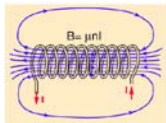
Direction = the way our compass would point

Unlike electric field lines which begin on positive charges and end on negative charges, magnetic flux lines **NEVER** begin or end

Magnetic Flux Lines



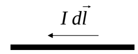




Units for **B** are Tesla = Newt/(Amp-meter) or Webers/m^2 (Magnetic flux density)

Current = Moving Charge

What is CURRENT? CHARGES IN MOTION!!



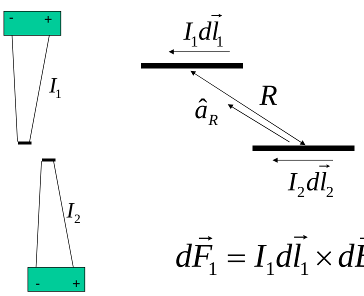
$$I\,d\vec{l} = q\vec{v}$$

$$\frac{coul}{\text{sec}} \cdot m = coul \cdot \frac{m}{\text{sec}}$$

So the current in a wire, ***I***, flowing across a magnetic field will feel a force…

$$F_M = (I\,d\vec{l}\,) \times \vec{B} = q\vec{v} \times \vec{B}$$

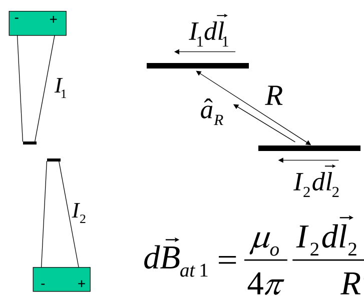
Ampere's Force Law



$$d\vec{F}_1 = I_1 d\vec{l}_1 \times d\vec{B}_1$$

Force on current 1 due to current 2 depends on the Magnetic field at 1 due to current 2.

Biot-Savart Law for finding **B**



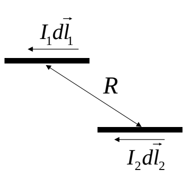
$$d\vec{B}_{at\,1} = \frac{\mu_o}{4\pi} \frac{I_2 d\vec{l}_2 \times \hat{a}_R}{R^2}$$

Magnetic field at point 1 due to current 2

Force between two wires

Combining these results,

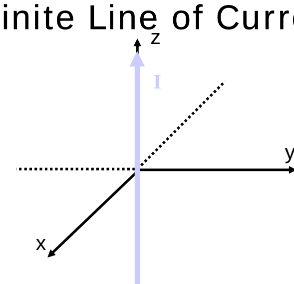
$$dF = (I_1 d\vec{l}_1) \times \underbrace{\left(\frac{\mu_0}{4\pi} \cdot \frac{I_2 d\vec{l}_2 \times \hat{a}_R}{R^2} \right)}_{\text{Magnetic flux density caused by \#2 at \#1}}$$



$$dF_{at\,1} = (I_1 d\vec{l}_1) \times \vec{B}$$

To find the total force on wire 1, we add up (integrate) the contributions from segments d**l**$_2$ for every d**l**$_1$ (double integral)

Example: Find **B** for an Infinite Line of Current

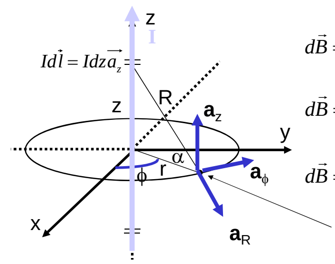


Wire carrying current in +z direction = I (A)
Find **B** for any arbitrary point P(r, ϕ, z) in cylindrical coords

Patented 5-Step Program for Problem Solving

1. MAKE A **LARGE CLEAR** DRAWING
   a. Also draw cross-sections if the problem is in 3D
   b. Pick a coordinate system that is appropriate for the symmetry of the problem
2. Divide current distributions into tiny pieces
3. Find d**B** of one tiny piece
4. Use SYMMETRY to eliminate any components that cancel (i.e. add to ZERO)
5. INTEGRATE to add contribution of ALL the tiny pieces

Step 1 - draw a picture
Step 2 - divide the line into segments d**l**
Step 3 - find d**B** for one small segment



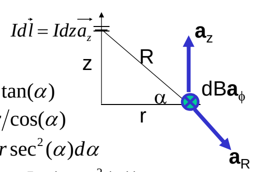
$$d\vec{B} = \frac{\mu_o}{4\pi}\frac{Id\vec{l}\times\hat{a}_R}{R^2}$$

$$d\vec{B} = \frac{\mu_o}{4\pi}\frac{Idz\sin(90+\alpha)}{R^2}\hat{a}_\phi$$

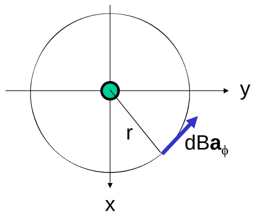
$$d\vec{B} = \frac{\mu_o}{4\pi}Idz\frac{r}{R^3}\hat{a}_\phi$$

Step 4: Use symmetry to eliminate components
Step 5: Integrate over the whole object

TANGENTIAL View



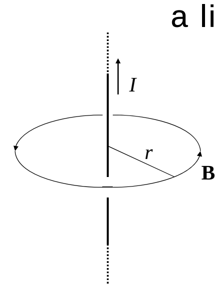
TOP View



$$z = r\tan(\alpha)$$
$$R = r/\cos(\alpha)$$
$$dz = r\sec^2(\alpha)d\alpha$$
$$d\vec{B} = \frac{\mu_o I}{4\pi}\frac{r(r\sec^2(\alpha))}{(r/\cos(\alpha))^3}d\alpha\hat{a}_\phi$$

$$\vec{B} = \frac{\mu_o I}{4\pi r}\hat{a}_\phi\int_{-\pi/2}^{\pi/2}\cos(\alpha)d\alpha = \frac{\mu_o I}{2\pi r}\hat{a}_\phi$$

Magnetic flux density (**B**) around a line of current



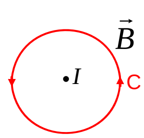
$$\vec{B} = \frac{\mu_o I}{2\pi r}\vec{a_\phi}$$

Right hand rule
Curl fingers around current

Practice sketching field lines:

"Static" Ampere's Law

Let's integrate B over any circular path centered on the wire:



$$\oint_C \vec{B}\bullet d\vec{l} = \int_{\phi=0}^{2\pi}\left(\frac{\mu_0 I}{2\pi r}\hat{a}_\phi\right)\bullet\left(rd\phi\hat{a}_\phi\right)$$
$$= \int_{\phi=0}^{2\pi}\left(\frac{\mu_0 I}{2\pi}\right)$$
$$= \mu_0 I$$

So… $\boxed{\oint_C \frac{\vec{B}}{\mu_0}\bullet d\vec{l} = I}$ for any radius circle

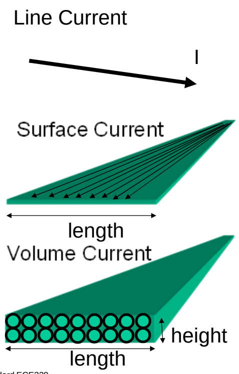
New Definition: Magnetic Field Intensity Vector

$$\oint_C \vec{H}\bullet d\vec{l} = I_{enclosed}$$

$$\boxed{\vec{H} = \frac{\vec{B}}{\mu_0}}$$

Units: (A/m)

Current Distributions



$$\vec{I} = Amps(A)$$

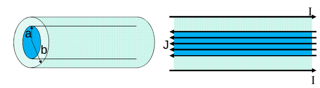
$$\vec{J}_s = \frac{A}{m}$$

$$\vec{J}_v = \frac{A}{m^2}$$

Challenge Question

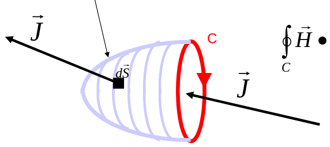
- A coaxial cable has a solid center wire and an outer cylindrical shell wire. A uniform current density J flows in the direction $-\mathbf{a}_x$ on the inner wire and the total current returns on the outer shell. In which region(s) will the B-field be constant?

(a) r<a
(b) a<r<b
(c) r>b
(d) both (b) and (c)



Static Ampere's Law

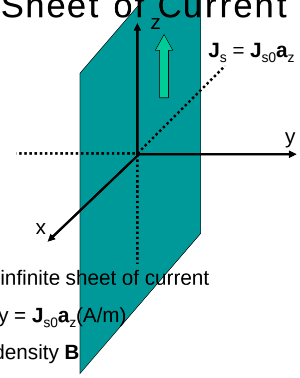
Open Surface, where the Path forms a "Mouth"



$$\oint_C \vec{H} \bullet d\vec{l} = I_{enclosed} = \iint_S \vec{J} \bullet d\vec{S}$$

In general, all the current entering the "mouth" will end up passing out of any surface that is formed around the mouth

Example: Infinite Plane Sheet of Current

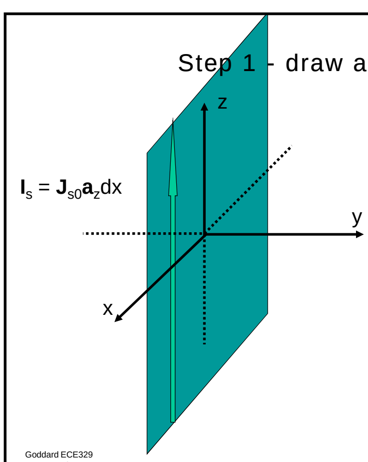


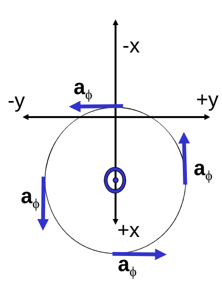
xz plane is an infinite sheet of current

Current density = $\mathbf{J}_{s0}\mathbf{a}_z$ (A/m)

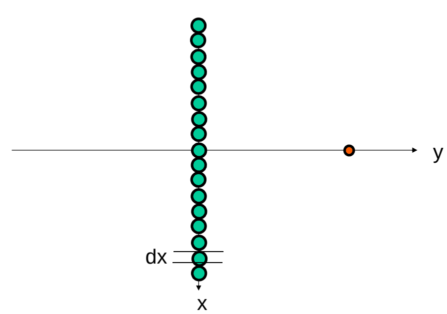
Solve for flux density **B**

Step 1 - draw a picture

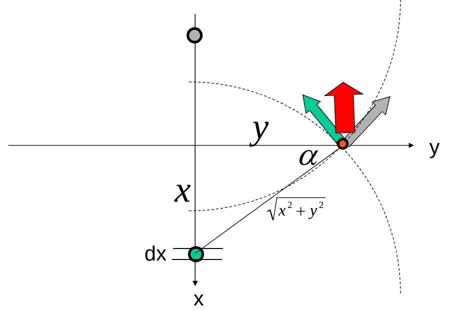




Step 2 - divide into segments

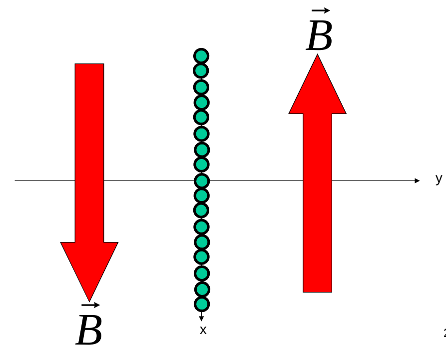


Step 3 - find d**B** for one small segment

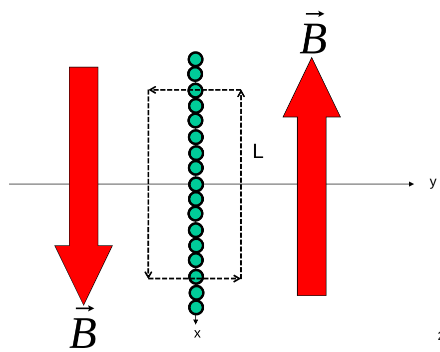


At every point (x,y), only the x-component remains, why?
For y>0, it is in -$\mathbf{a}_x$ direction

Step 4 – use symmetry to eliminate components that are zero



Aha! Now that we know B is along a_x, we can apply Ampere's Law to a simple path



And now for the math …

$\mathbf{J}_s = \mathbf{J}_{s0}\mathbf{a}_z$

$$\oint_C \frac{\vec{B}}{\mu_0} \bullet d\vec{l} = I \qquad \Rightarrow \quad \frac{B}{\mu_0}L + \frac{B}{\mu_0}L = J_{s0}L$$

$$B = \frac{1}{2}\mu_0 J_{s0}$$

$$\boxed{\vec{B} = \frac{1}{2}\mu_0 J_{s0}(\pm \hat{a}_x)}$$

General Formula for **B** due to infinite current sheet

The direction of current flow can be in ANY direction
The current sheet might not be on a coordinate plane

$$B = \frac{\mu_0}{2}\vec{J}_s \times \hat{a}_n$$

Unit vector normal
to the surface

Sample problem

- Infinite plane sheets of current lie in the x=0, y=0, and z=0 planes with uniform surface current densities $J_{s0}\mathbf{a}_z$, $2J_{s0}\mathbf{a}_x$, and $-J_{s0}\mathbf{a}_x$, respectively. Find **B** at the points: (a) (1,2,2), (b) (2,-2,-1) and (c) (-2,1,-2).

Problem taken from D1.20 p58 in old book

Lecture 12 Summary

- Ampere's Force Law $d\vec{F}_1 = I_1 d\vec{l}_1 \times d\vec{B}_1$
- Biot-Savart Law $d\vec{B}_{at\,1} = \dfrac{\mu_o}{4\pi}\dfrac{I_2 d\vec{l}_2 \times \hat{a}_R}{R^2}$
- Infinite line of current $\vec{B} = \dfrac{\mu_o I}{2\pi r}\vec{a}_\phi$
- Lorentz Force Equation $\vec{F}_{TOTAL} = q\vec{E} + q\vec{v}\times\vec{B}$
- Next class
  - Ampere's Law (Section 2.4)

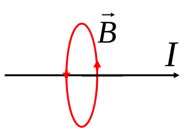
Lecture 13
Section 2.4

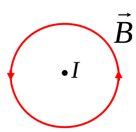
Ampere's Law
Displacement Current

Adapted from Prof. Cunningham's Notes

"Static" Ampere's Law

Using the Biot-Savart Law, we solved for the B field around a straight wire:



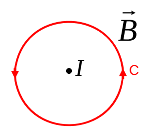


$$\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{a}_\phi$$

Magnetic flux density
(Wb/m²)

"Static" Ampere's Law

We integrated B over any circular path centered on the wire:



$$\oint_C \vec{B}\bullet d\vec{l} = \int_{\phi=0}^{2\pi}\left(\frac{\mu_0 I}{2\pi r}\hat{a}_\phi\right)\bullet\left(rd\phi\hat{a}_\phi\right)$$

$$= \int_{\phi=0}^{2\pi}\left(\frac{\mu_0 I}{2\pi}\right)$$

$$= \mu_0 I$$

So… $\boxed{\oint_C \frac{\vec{B}}{\mu_0}\bullet d\vec{l} = I}$ for any radius circle

Ampere's Law Physical Meaning

$\boxed{\oint_C \frac{\vec{B}}{\mu_0}\bullet d\vec{l} = I_{enclosed}}$ Taking the integral of the quantity (B/μ_o) around a closed path equals the enclosed current.

$$\oint_C \frac{\vec{B}}{\mu_0}\bullet d\vec{l} = Magneto\,Motive\,Force\,(MMF)$$

similar to:

$$\oint_C \vec{E}\bullet d\vec{l} = Electro\,Motive\,Force\,(EMF)$$

Caution: MMF does no work!

New Definition:
Magnetic Field Intensity Vector

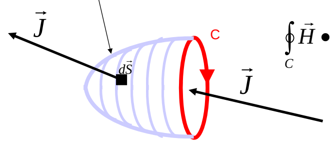
$$\oint_C \frac{\vec{B}}{\mu_0}\bullet d\vec{l} = I_{enclosed}$$

$$\boxed{\vec{H} = \frac{\vec{B}}{\mu_0}}$$

Units: (A/m)

Static Ampere's Law

Open Surface, where the Path forms a "Mouth"



$$\oint_C \vec{H} \bullet d\vec{l} = I_{enclosed} = \iint_S \vec{J} \bullet d\vec{S}$$

According to this static law, all the current entering the "mouth" will end up passing out of any surface that is formed around the mouth. Is this true?

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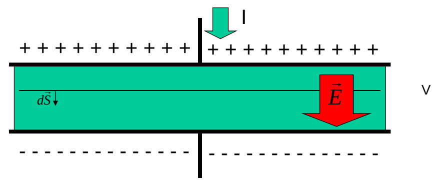
Lectures 12-14

Problem with Static Ampere's Law

Consider a simple capacitor
- No DC current goes through
- But, AC voltage results in current flow



How does AC current get through a capacitor if it is not conducted through by a wire?

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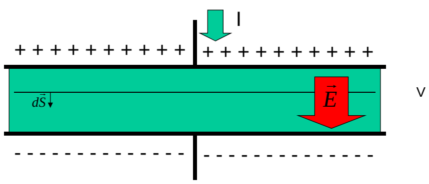
Problem with Static Ampere's Law



There is a second "way" to get current to flow

Somehow, a TIME-VARYING E field results in current flow

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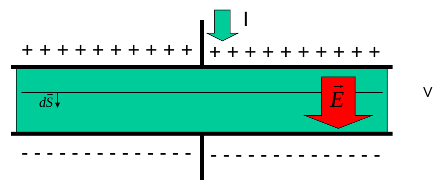
Problem with Static Ampere's Law



Flux of E field lines crossing a surface, S

$$\psi_E = \oint_S \varepsilon_0 \vec{E} \bullet d\vec{S}$$

Recall the Definition: Electric Flux
Units: (C)

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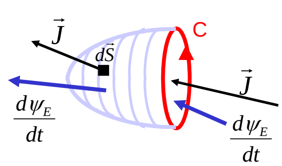
James Clerk Maxwell's Genius Breakthrough

There are TWO sources of MMF:
1. Flow of charges due to current
2. Time-varying electric field

Called (by Maxwell)
"Displacement Current"



$$\oint_C \vec{H} \bullet d\vec{l} = \iint_S \vec{J} \bullet d\vec{S} + \frac{d}{dt}\iint_S \vec{D} \bullet d\vec{S}$$

MMF (Amps) = "Regular" Current (Amps) + Displacement Current (Amps)

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"New" Definition: Displacement Flux Density Vector

$\vec{E}$ "Electric Field Intensity Vector" V/m

$\vec{D} = \varepsilon_0 \vec{E}$ "Displacement Flux Density Vector"

Units: $\vec{D} = \frac{Charge}{Area} = \frac{C}{m^2}$

$\psi_E = \iint_S \varepsilon_0 \vec{E} \bullet d\vec{S} = \iint_S \vec{D} \bullet d\vec{S}$ Coul

$\frac{d\psi_E}{dt} = \frac{d}{dt}\iint_S \vec{D} \bullet d\vec{S}$ Coul/sec (Amps)

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Displacement current from a time varying E-field

- Find the displacement current crossing an area $A=0.1m^2$ in the xy plane from the –z to +z side for:

$$\mathbf{E}=E_0\, t\, e^{-t^2}\mathbf{a}_z$$

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From Discussion 2.8 (p 106) of old book

Ampere's Law (Non static)

The Third Maxwell Equation

$$\oint_C \vec{H}\bullet d\vec{l} = \iint_S \vec{J}\bullet d\vec{S} + \frac{d}{dt}\iint_S \vec{D}\bullet d\vec{S}$$

MMF (Amps) = "Regular" Current (Amps) + Displacement Current (Amps)

$$\nabla\times\vec{H} = \vec{J} + \frac{d\vec{D}}{dt}$$

After using Stokes' theorem to convert to differential form

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Ampere's Law Rules

1. Right Hand Rule:
   Choose direction of C so dS points OUT of the surface
2. Must use same surface when evaluating surface integrals for conduction current and displacement current

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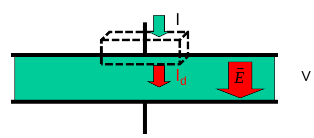
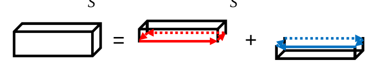
Displacement Current



- The displacement current bridges the gap in the capacitor plates
  - Regular current flowing into the CLOSED surface = displacement current flowing out

$$-\oint\!\!\oint_S \vec{J}\bullet d\vec{S} = \frac{d}{dt}\oint\!\!\oint_S \vec{D}\bullet d\vec{S}$$



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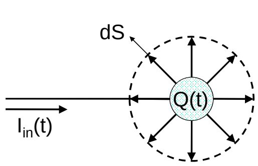
Displacement Current

- Current flow changes the amount of charge
  - Since the charge changes, the electric flux out of the surface changes, i.e. a displacement current



$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 R^2}\hat{a}_R$$

$$\psi_E = \iint_S \varepsilon_0 \vec{E}\bullet d\vec{S} = \varepsilon_0 E(Surf\ Area)$$

$$= \varepsilon_0 \frac{Q}{4\pi\varepsilon_0 R^2}(4\pi R^2) = Q$$

$$I_d = \frac{d\psi_E}{dt} = \frac{dQ}{dt}$$

$$I_d = \frac{dQ}{dt} = I_{in} \text{ because current out = current in}$$

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Displacement Current from time varying charge

- 3 point charges, $Q_1(t)$, $Q_2(t)$, and $Q_3(t)$ are at the corners of an equilateral triangle and connected by wires. Currents of I and 3I flow from Q_1 to Q_2 and Q_3 respectively. The displacement current emanating from a small surface surrounding Q_2 is -2I. Find: (a) the current flowing from Q_2 to Q_3 and (b) the displacement current from a small surface surrounding Q_1 and surrounding Q_3.

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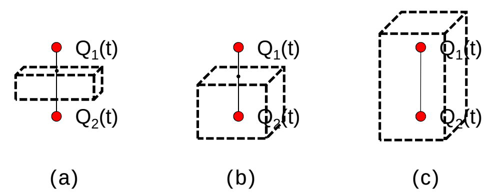
From Discussion 2.9 (p 106) of old book

Challenge Question 1

- Current moves along a wire connecting two point charges. For which closed surface can the displacement current be non-zero?



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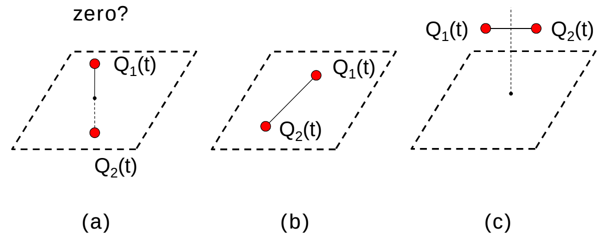
Challenge Question 2

- Current moves along a wire connecting two point charges. For which infinite plane (open surface) can the displacement current be non-zero?



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Lecture 13 Summary

- Ampere's Circuit Law
- Next class
  - Faraday's Law (Sections 2.1 and 2.3)

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Lecture 14
Sections 2.1 and 2.3

Review of Line Integrals

Faraday's Law

Adapted from Prof. Cunningham's Notes

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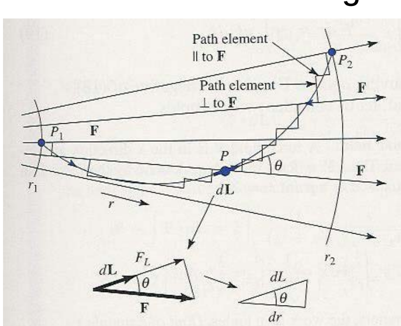
Work is a Line Integral



$$W_{AB} = \sum_{j=1}^{n} \vec{\mathbf{F}}_j \cdot \Delta\vec{\mathbf{l}}_j = \int_{\mathbf{A}}^{\mathbf{B}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{l}}$$

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Conservative Forces

- Conservative means the work done by the force is independent of path
  - E.g. Gravity, Static Electric/Magnetic
- No work done along any closed loop
- Described by a potential energy
  - Energy conservation
    - Work done increases KE & decreases PE
    - Friction, Drag & Time Dependent Electric or Magnetic Forces are non-conservative
- Curl-free (non-rotational)
  - Field strength does not vary perpendicular to the field direction

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The Force From a **Static** EM Field is Conservative

- Definition of Voltage

$$V_B - V_A = \frac{W_{AB}}{q} = -\sum_{j=1}^{n} \mathbf{E}_j \bullet \Delta \mathbf{l}_j$$

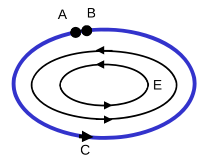
Voltage drop from *B* to *A* is equal to the work you need to do to move a unit charge from *A* to *B* against the electric field **E**.

- In the limit $n \to \infty$,

$$V_B - V_A = -\int_A^B \mathbf{E} \bullet d\mathbf{l}$$

= Line integral of E from *A* to *B*. Well defined since integral is independent of path.

Non-conservative Fields: Electromotive Force (emf)



$$emf = \oint_C \mathbf{E} \bullet d\mathbf{l}$$

= Line integral of E around the closed path *C* (counter-clockwise).

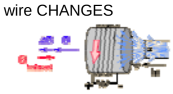
EMF can be non-zero if EM field varies in time.
EMF is a difference in potential that can give rise to an electric current. Think of it as a battery between *A* and *B*.

Summary of Conservative Fields

- Conservative fields
  - Line integral around a closed path is ZERO
  - Gravity, static EM Field
- Non-conservative fields
  - Line integral around a closed path is NONZERO
  - Friction, time-varying EM Field

Faraday's Law

When the magnetic flux enclosed by a loop of wire CHANGES WITH TIME, a current is produced in the loop



The EMF in the loop is the NEGATIVE of the rate of change of the magnetic flux enclosed in the loop

$$\oint_C \vec{E} \bullet d\vec{l} = -\frac{d}{dt} \iint_S \vec{B} \bullet d\vec{S}$$

Our third Maxwell Equation!!!

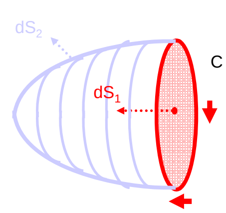
$$emf = -\frac{d\psi_B}{dt}$$

Lenz's Law

- The direction of the induced EMF always OPPOSES the CHANGE in magnetic flux that produces it.
- It opposes the **CHANGE** in flux, not the flux itself!

  - Explains why it is "-d/dt"

The curve C and surface S

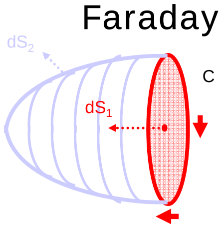


Closed Path, C enclosing open surface S_1

It also encloses open surface S_2

- Outward magnetic flux thru the closed surface: $S_1 \cup S_2$ is zero
  - The flux out any open surface S_2 = minus the flux out S_1 = plus the flux into S_1 so all surfaces S bounded by C have the same flux

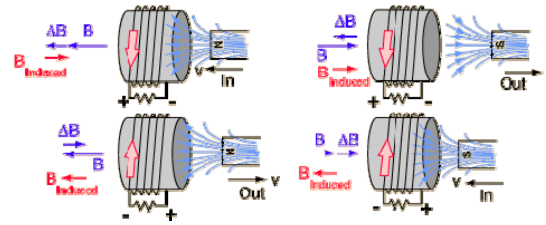
Faraday's Law Rules



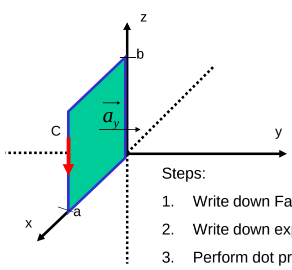
$$\oint_C \vec{E} \bullet d\vec{l} = -\frac{d}{dt} \iint_S \vec{B} \bullet d\vec{S}$$

- Right Hand Rule
  - Right hand curls around C so thumb points in direction of d**S**

Experiments for Faraday's Law



Induced emf around rectangular loop in a time-varying **B** field



Rectangular wire loop
In the xz-plane

$$\vec{B} = B_0 \cos \omega t \hat{a}_y$$

Steps:

1. Write down Faraday's law
2. Write down expression for dS. DIRECTION!!
3. Perform dot product B*dS
4. Solve double integral over limits of the loop
5. Take time derivative of result. Put in "-" sign!

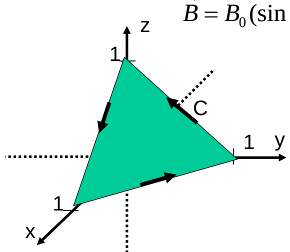
Faraday's Law Rules

- EMF increases in proportion to the number of turns of wire
  - If the loop, C, contains N turns of wire, the EMF is multiplied by N

$$EMF = -N \frac{d\psi_B}{dt}$$

Example Problem: Induced emf around closed path

$\vec{B} = B_0(\sin \omega t \hat{a}_x - \cos \omega t \hat{a}_y)$ Wb/m²

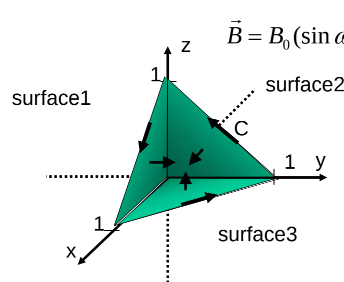


Setting up the limits of integration here would be painful!

We can make the problem easier by solving for the flux going into the three surfaces on the coordinate planes

From Discussion 2.5b p100 in old book

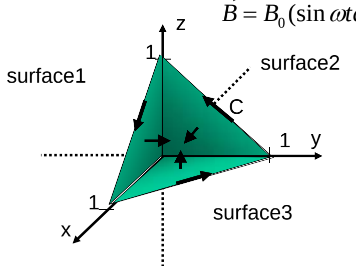
$$\vec{B} = B_0(\sin \omega t \hat{a}_x - \cos \omega t \hat{a}_y)$$



Steps

1. Identify surfaces on the coordinate axes that are bounded by the same closed path C
2. Solve for $\iint_S \vec{B} \bullet d\vec{S}$ separately for each surface
3. Add up contribution of each surface for the final result

$$\vec{B} = B_0(\sin\omega t\hat{a}_x - \cos\omega t\hat{a}_y)$$



Solving for $\iint_S \vec{B}\bullet d\vec{S}$ of each component surface

1. Write down expression for dS. PAY ATTENTION TO DIRECTION!
2. Perform dot product with B. Is the dot product zero for a particular surface?
3. Shortcut if B is uniform over the surface. Then $\iint_S \vec{B}\bullet d\vec{S}$ =(Surface Area)(B)
4. Otherwise, perform double integral over the dimension limits of the surface.

Example Problem: Motional EMF

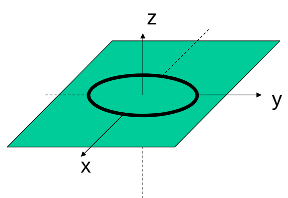
- A non-uniform static magnetic field given by $B=B_0/x$ $\mathbf{a}_z$ exists in the region x>0. A square loop with side length, s, and situated in the xy plane (x>0) moves in the $+\mathbf{a}_x$ direction with speed v. Find the induced emf in the loop.

Adapted from Problem 2.11 (p 123) of old book

Challenge Question

- A loop, radius R, centered at the origin, sits in the xy plane in the presence of a uniform field $\mathbf{B}=B_0\mathbf{a}_z$ (B_0>0). If the loop radius begins to decrease, which direction is the induced emf?

(a) $\mathbf{a}_\phi$
(b) $-\mathbf{a}_\phi$
(c) $\mathbf{a}_z$
(d) cannot be determined



Comparing Faraday and Ampere's Law

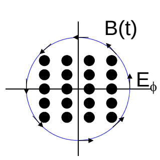
- Faraday's Law
  - Time varying magnetic fields generate emf (voltage)
- Ampere's Law
  - Time varying electric fields generate magnetic fields
  - Electric currents generate magnetic fields (Ampere's static law)

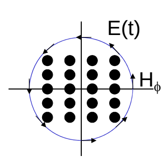
EMF and MMF

$$\oint_C \vec{E}\bullet d\vec{l} = EMF \qquad \oint_C \vec{H}\bullet d\vec{l} = MMF$$

E=Volts/m | H=Amps/m

EMF=Volts | MMF=Amps





Lecture 14 Summary

- Faraday's Law: ____________
  - The generated emf opposes ____________________ and for a loop with N turns is ___ times larger
  - The direction for C and dS determined by right hand rule
- Next class
  - Magnetic Vector Potential (Section 10.6 – just pp. 406-407)
  - Inductance (Section 6.3)

ECE 329
Lectures 15-17
Sections 10.6, 6.3, 2.5, 5.5, 5.2

Magnetic Vector Potential

Inductance

Conservation of Charge (Continuity)

Boundary Conditions

Magnetic Materials

Adapted from Prof. Cunningham's Notes

Magnetic Potentials

$\vec{\nabla} \bullet \vec{B} = 0$ Gauss' Law

If the divergence is zero, then **B** can be written as the curl of a vector (not obvious **A** should exist, but it does)

New Definition: Magnetic Potential Vector

$$\boxed{\vec{B} = \vec{\nabla} \times \vec{A}}$$

Oddly familiar:

$$\vec{\nabla} \bullet \left(\vec{\nabla} \times \vec{A}\right) = 0$$

Faraday's Law

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{A})$$

$$\vec{\nabla} \times \vec{E} = -\vec{\nabla} \times \frac{\partial \vec{A}}{\partial t}$$

$$\vec{\nabla} \times (\vec{E} + \frac{\partial \vec{A}}{\partial t}) = 0$$

If curl is zero, then can be written as the gradient of a scalar

$$\boxed{\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla \Phi}$$

Oddly familiar:

$$\vec{\nabla} \times \nabla \Phi = 0$$

Electric and Magnetic Potentials

$$\boxed{\vec{E} = -\nabla \Phi - \frac{\partial \vec{A}}{\partial t}}$$

Electric Potential Magnetic Potential

$$\boxed{\vec{B} = \vec{\nabla} \times \vec{A}}$$

With these definitions, we automatically satisfy Faraday's Law & Gauss' Magnetic Law

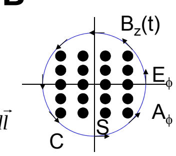
Instead of 6 unknowns: (E_x, E_y, E_z) & (B_x, B_y, B_z) we have 4: (A_x, A_y, A_z) and V

The magnetic potential **A** and its relations to **E** and **B**

$$EMF = -\frac{d\psi_B}{dt} = -\frac{d}{dt}\iint_S \vec{B} \cdot d\vec{S}$$

$$= -\frac{d}{dt}\iint_S (\nabla \times \vec{A}) \cdot d\vec{S} = -\frac{d}{dt}\oint_C \vec{A} \cdot d\vec{l} = -\oint_C \frac{\partial \vec{A}}{\partial t} \cdot d\vec{l}$$

assuming the loop geometry is constant



$$EMF = \oint_C \vec{E} \cdot d\vec{l} = \oint_C (-\nabla \Phi - \frac{\partial \vec{A}}{\partial t}) \cdot d\vec{l} = -\oint_C \frac{\partial \vec{A}}{\partial t} \cdot d\vec{l}$$

why can we drop gradient of Φ?

Gauss' Electric Law for Potentials

$$\vec{\nabla} \bullet \vec{D} = \rho$$

$$\vec{\nabla} \bullet \varepsilon \vec{E} = \rho$$

$$\vec{\nabla} \bullet \vec{E} = \frac{\rho}{\varepsilon}$$ Assuming ε is constant

$$\boxed{\vec{\nabla} \bullet \left(-\nabla \Phi - \frac{\partial \vec{A}}{\partial t}\right) = \frac{\rho}{\varepsilon}}$$ 1 equation

Ampere's Law for Potentials

$$\vec{\nabla}\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t}$$

$$\vec{\nabla}\times\frac{\vec{B}}{\mu}=\vec{J}+\varepsilon\frac{\partial\vec{E}}{\partial t}$$

$$\vec{\nabla}\times\vec{B}=\mu\vec{J}+\mu\varepsilon\frac{\partial\vec{E}}{\partial t}$$

$$\vec{\nabla}\times\vec{B}-\mu\varepsilon\frac{\partial\vec{E}}{\partial t}=\mu\vec{J}$$

$$\boxed{\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right)-\mu\varepsilon\frac{\partial}{\partial t}\left(-\nabla\Phi-\frac{\partial\vec{A}}{\partial t}\right)=\mu\vec{J}}$$

3 equations

Special Case: Static Fields

Gauss

$$\boxed{\vec{\nabla}\bullet\left(-\nabla\Phi-\frac{\partial\vec{A}}{\partial t}\right)=\frac{\rho}{\varepsilon}}$$

$$\boxed{\vec{\nabla}\bullet(\nabla\Phi)=-\frac{\rho}{\varepsilon}}$$

Ampere

$$\boxed{\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right)-\mu\varepsilon\frac{\partial}{\partial t}\left(-\nabla\Phi-\frac{\partial\vec{A}}{\partial t}\right)=\mu\vec{J}}$$

$$\boxed{\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right)=\mu\vec{J}}$$

We studied this one already (Poisson):
Relationship between a charge
distribution and the potential field

Challenge Question: Gauge Transformation

- Suppose **E** and **B** can be represented by the scalar and vector potentials: Φ and **A**.

$$\boxed{\vec{E}=-\nabla\Phi-\frac{\partial\vec{A}}{\partial t}} \qquad \boxed{\vec{B}=\vec{\nabla}\times\vec{A}}$$

Which of the following is true:

(a) Φ and **A** are uniquely defined

(b) $\vec{A}'=\vec{A}+\nabla\lambda,\ \Phi'=\Phi-\frac{\partial\lambda}{\partial t}$ also represents **E** and **B**

(c) The divergence of **A** must be 0

(d) The laplacian of Φ must be $-\rho/\varepsilon$

LG's question

Lecture 15a Summary

- Since div curl **A** = 0 and curl grad f =0,

$$\boxed{\vec{E}=-\nabla\Phi-\frac{\partial\vec{A}}{\partial t}} \qquad \boxed{\vec{B}=\vec{\nabla}\times\vec{A}}$$

satisfy Faraday's and Gauss' Mag. Laws

- For static fields, Poisson's equation is

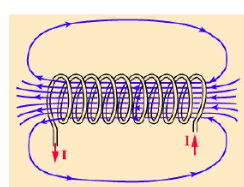
$$\boxed{\nabla^2\Phi=-\frac{\rho}{\varepsilon}}$$

ECE 329
Lecture 15b
Section 6.3

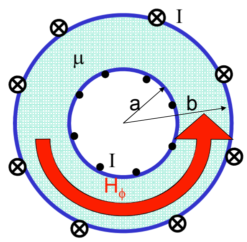
Inductance

Adapted from Prof. Cunningham's Notes

Example: Find B for an infinitely long solenoid with n turns per unit length



Inductance of a Coax Cable



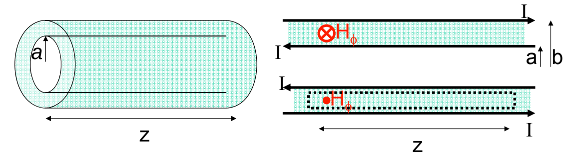
Now, instead of applying a voltage across the inner and outer conductor, a current, I, flows down the length of the outer conductor and returns in the opposite direction through the inner conductor

Results in magnetic field

$$H_\phi = \frac{I}{2\pi r}$$

in between the coax

Inductance of Coaxial Cable



$$\vec{B} = \mu\vec{H} = \frac{\mu I}{2\pi r}\vec{a}_\phi$$ Magnetic Flux Density $\left[\frac{Wb}{m^2}\right]$

$$\psi = \int B \bullet dS = \int_{r=a}^{b}\int_{z=0}^{z}\left(\frac{\mu I}{2\pi r}\right)(drdz)$$ Magnetic Flux $[Wb]$

$$\psi = \frac{\mu I z}{2\pi}\ln(b/a)$$

Inductance

$$L = \frac{\psi}{I}$$ Units: Henry (H)

$$L = \frac{\mu z}{2\pi}\ln(b/a)$$

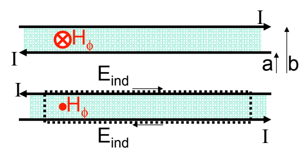
$$\mathcal{L} = \frac{L}{z} = \frac{\mu}{2\pi}\ln(b/a)$$ Inductance/Length (H/m)

Induced emf

Faraday's Law

$$emf = \oint \vec{E} \bullet d\vec{l} = -\frac{d\psi_B}{dt}$$

$$emf = -\frac{d(LI)}{dt} = -L\frac{dI}{dt}$$



Assuming dI/dt>0

The inductance of the wire creates an emf that opposes rapid changes in the current

Steps for Finding Inductance

- Find H(r)
- Find B $\vec{B} = \mu\vec{H}$
- Find Magnetic flux by integrating $\psi = \int B \bullet dS$
- Inductance $L = \frac{\psi}{I}$
- Inductance/Length

Relationships between Capacitance, Conductance & Inductance

Notice in the above examples,

$$\mathcal{C} = \varepsilon \cdot GeometricalFactor$$
$$\mathcal{L} = \mu / GeometricalFactor$$
$$\mathcal{G} = \sigma \cdot GeometricalFactor$$

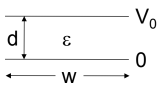
This is true in general for any pair of infinitely long, parallel perfect conductors and so we have the following:

$$\mathcal{L}\mathcal{C} = \mu\varepsilon \qquad \mathcal{G}/\mathcal{C} = \sigma/\varepsilon$$

If you know one (L, C, or G), you can find the other two from the material parameters

Challenge Question: Parallel plate capacitor

- For the parallel plate capacitor, we found



$$C = \frac{\varepsilon A}{d} \Rightarrow \mathcal{C} = \frac{\varepsilon w}{d}$$

If the plate separation d increases, which is true:

(a) $\mathcal{G}$ and $\mathcal{L}$ will both increase

(b) $\mathcal{C}$ and $\mathcal{G}$ will both increase

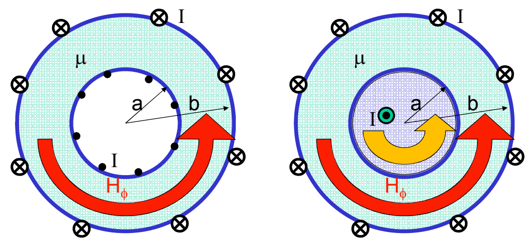
(c) $\mathcal{L}$ will increase, but $\mathcal{C}$ will decrease

(d) $\mathcal{G}$ will increase, but $\mathcal{L}$ will decrease

LG's question

(Optional) Self Inductance

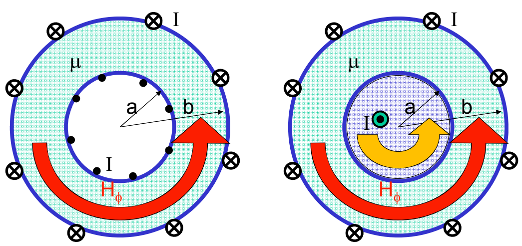
Last Example New Example



There now is also H field inside the inner wire that will also contribute to inductance

(Optional) Self Inductance

Last Example New Example

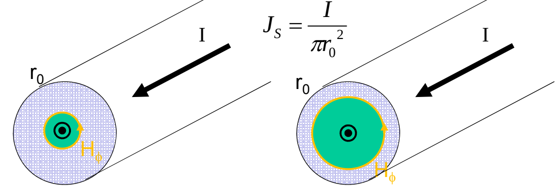


$$L = \frac{\psi}{I} \qquad L = \frac{\Lambda}{I}$$

"Flux Linkage"

(Optional) Flux Linkage

$$J_S = \frac{I}{\pi r_0^2}$$



This flux line is "linked" to a small amount of current

This flux line is "linked" to a large amount of current

AND, since flux ψ_B is dependent on r

Flux here is LOW

Flux here is HIGH

(Optional) Definition of Flux Linkage

Λ=(Flux Magnitude)·(Fraction of Current Linked to the Flux Line)

Accounts for two factors:

1. The magnitude of the flux line
2. The amount of current linked to a flux line

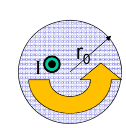
The self inductance is then defined as:

$$L_{\text{int}} = \frac{\Lambda}{I_{total}} = \int_S \frac{d\Lambda}{I_{total}} = \int_S \frac{N \cdot d\psi}{I_{total}} = \frac{1}{I_{total}^2} \int_S I_{linked}\, d\psi$$

Fraction of linked current: $N = \dfrac{I_{linked}}{I_{total}}$

(Optional) Example: Self-Inductance of a Wire

<u>Step 1:</u> Ampere's Law inside the Wire



$$\oint \vec{H} \bullet dl = I_{enclosed}$$

$$\oint \frac{\vec{B}}{\mu} \bullet dl = I_{enclosed}$$

$$\frac{1}{\mu} \int_{\phi=0}^{2\pi} B_\phi (r d\phi) = I_{enclosed}$$

$$\frac{1}{\mu} B_\phi (2\pi r) = I_{enclosed}$$

$$\frac{1}{\mu} B_\phi (2\pi r) = J_S (Area\_Enclosed)$$

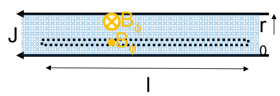
$$\frac{1}{\mu} B_\phi (2\pi r) = \left(\frac{I}{\pi r_0^2} \right) (\pi r^2) \qquad (r < r_0)$$

(Optional) Example, continued

$$\frac{1}{\mu}B_\phi(2\pi r) = \left(\frac{I}{\pi r_0^2}\right)(\pi r^2) \qquad (r < r_0)$$

$$B_\phi = \frac{\mu I}{2\pi}\frac{r}{r_0^2} \qquad \text{(Wb/m}^2\text{)}$$

Step 2: Determine One Differential Piece of Flux



$$d\psi_B = B(r)\cdot dr\cdot l \qquad \text{(Wb)}$$

$$d\psi_B = \frac{\mu I}{2\pi}\frac{r}{r_0^2}\cdot dr\cdot l$$

(Optional) Example, continued

Step 3: Differential piece of flux linkage

$$d\Lambda = d\psi_B \cdot N$$

$$d\Lambda = d\psi_B\left(\frac{\pi r^2}{\pi r_0^2}\right)$$

N=Fraction of current inside radius r

$$d\Lambda = \frac{\mu I}{2\pi}\frac{r^3}{r_0^4}\cdot dr\cdot l$$

Step 4: Total flux linkage

$$\Lambda = \frac{\mu I\cdot l}{2\pi r_0^4}\int_0^{r_0} r^3 dr = \frac{\mu I\cdot l}{8\pi}$$

(Optional) Example, continued

Step 5: Internal inductance

$$L = \frac{\Lambda}{I} = \frac{\mu l}{8\pi} \qquad \text{(units: Henry (H))}$$

Internal inductance per unit length of the wire:

$$\frac{L}{l} = \frac{\mu}{8\pi} \qquad \text{(units: H/m)}$$

Lecture 15b Summary

- Capacitance $C = Q/V_0$
- Conductance $G = |I_c|/V_0$
- Inductance $L = \psi/I$
- Relationships $\mathcal{L}\mathcal{C} = \mu\varepsilon \qquad \mathcal{G}/\mathcal{C} = \sigma/\varepsilon$
- Self-inductance $L_{\text{int}} = \int_S \frac{N d\psi}{I_{total}} = \int_S \frac{I_{linked}}{I_{total}^2} d\psi$
- Next Up
  - Conservation of Charge
  - Boundary Conditions

Lecture 16
Sections 2.5 and 5.5

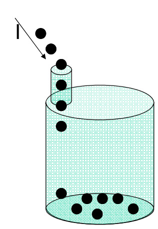
Conservation of Charge

Review of Maxwell's Equations in Integral Form

Boundary Conditions

Adapted from Prof. Cunningham's Notes

Conservation of Charge



Say we have a container that can accumulate charge

Start pouring charges into the container

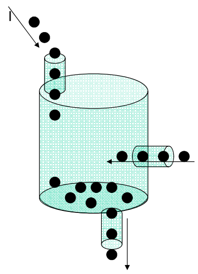
Flow of charges is a current, I, in Amps

If the charges don't leave the container the charge inside the container increases

Current flow IN = Charge INCREASE

Current flow OUT = Charge DECREASE

Conservation of Charge



In general, we can pour charges in from more than one direction, or take some out from other parts of the container

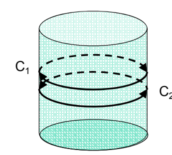
Net Rate of Current flow OUT = Net Rate of Charge DECREASE

$$\oint\!\!\oint_S \vec{J} \bullet d\vec{S} = -\frac{dQ_{enc}}{dt} = -\frac{d}{dt}\iiint_V \rho dV$$

or, $\nabla \cdot \vec{J} = -\dfrac{d\rho}{dt}$ using the divergence theorem

Conservation of Charge

Can be derived by combining two of Maxwell's equations



$$0 = \oint_{C_1} \vec{H} \bullet d\vec{l} + \oint_{C_2} \vec{H} \bullet d\vec{l}$$

$$= \iint_{S_1} \vec{J} \bullet d\vec{S} + \frac{d}{dt}\iint_{S_1} \vec{D} \bullet d\vec{S} + \iint_{S_2} \vec{J} \bullet d\vec{S} + \frac{d}{dt}\iint_{S_2} \vec{D} \bullet d\vec{S}$$

$$= \oint\!\!\oint_S \vec{J} \bullet d\vec{S} + \frac{d}{dt}\oint\!\!\oint_S \vec{D} \bullet d\vec{S}$$

$$= \oint\!\!\oint_S \vec{J} \bullet d\vec{S} + \frac{d}{dt}\iiint_V \rho dV$$

$$\boxed{\therefore \oint\!\!\oint_S \vec{J} \bullet d\vec{S} = -\frac{d}{dt}\iiint_V \rho dV = -\frac{dQ_{enc}}{dt}}$$

Conservation of Charge

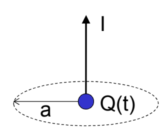
Differential equation derivation is much faster!

$$0 = \nabla \cdot \left(\nabla \times \vec{H}\right) = \nabla \cdot \left(J + \frac{\partial \vec{D}}{\partial t}\right) = \nabla \cdot J + \frac{\partial \left(\nabla \cdot \vec{D}\right)}{\partial t} = \nabla \cdot J + \frac{\partial \rho}{\partial t}$$

$$\boxed{\therefore \nabla \cdot J = -\frac{\partial \rho}{\partial t}}$$

Application of multiple Maxwell's Equations

- Current I flows from a point charge Q(t) at the origin along the z-axis off to infinity. Find the counterclockwise MMF for a circular path of radius a in the xy plane centered at the origin. Hint: consider a sphere and a hemisphere



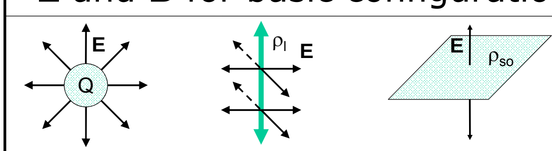
From Example 2.5 (p 111) of old book

Application of charge conservation

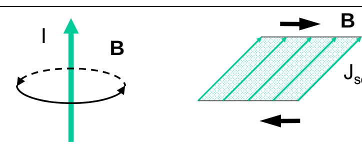
- For **J**=<x,y,z>, find the rate of decrease of charge contained in the unit cube: corner vertices (0,0,0) and (1,1,1).

From Problem 2.24a (p 126) of old book

E and **B** for basic configurations



$\vec{E} = \dfrac{Q}{4\pi\varepsilon_0 R^2}\hat{a}_R$ $\quad \vec{E} = \dfrac{\rho_l}{2\pi\varepsilon_0 r}\hat{a}_r$ $\quad \vec{E} = \dfrac{\rho_{so}}{2\varepsilon_0}(\pm\hat{a}_z)$



$\vec{B}_\phi = \dfrac{\mu_0 I}{2\pi R}$ $\quad B = \dfrac{\mu_0}{2}\vec{J}_s \times \hat{a}_n$

Lecture 16a Summary

- Maxwell's Equations
  - Gauss Magnetic: ____________
  - Gauss Electric: ____________
  - Faraday: ____________
  - Ampere + Maxwell:

____

Lecture 16b
Sections 5.5

Boundary Conditions

Adapted from Prof. Cunningham's Notes

Maxwell's Eqns - Integral form

$$\oint_C \vec{E} \bullet d\vec{l} = -\frac{d}{dt}\iint_S \vec{B} \bullet d\vec{S}$$

$$\oint_C \vec{H} \bullet d\vec{l} = \iint_S \vec{J} \bullet d\vec{S} + \frac{d}{dt}\iint_S \vec{D} \bullet d\vec{S}$$

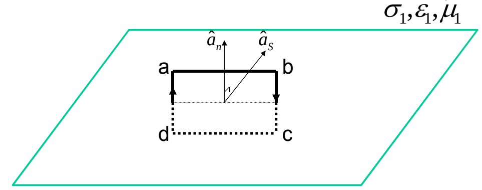
$$\oiint_S \vec{B} \bullet d\vec{S} = 0$$

$$\oiint_S \vec{D} \bullet d\vec{S} = \iiint_V \rho dV$$

They are valid for ALL closed paths and closed surfaces, EVEN WHEN THEY SPAN A BOUNDARY BETWEEN TWO MATERIALS

Closed Path Through a Boundary

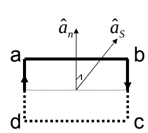
Medium 1 (above)
$\sigma_1, \varepsilon_1, \mu_1$



Medium 2 (below)
$\sigma_2, \varepsilon_2, \mu_2$

Closed path: abcd
Apply Faraday's Law and Ampere's Law to the closed path

Normal Vectors



Medium 1 (above)
$\sigma_1, \varepsilon_1, \mu_1$

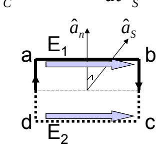
Medium 2 (below)
$\sigma_2, \varepsilon_2, \mu_2$

$\hat{a}_n$ Vector NORMAL to the boundary. Points INTO medium 1

$\hat{a}_S$ Vector normal to the path, TANGENT to the interface. Use right hand rule for path to define direction

Faraday's Law at Boundary

$$\oint_C \vec{E} \bullet d\vec{l} = -\frac{d}{dt}\iint_S \vec{B} \bullet d\vec{S} = 0$$



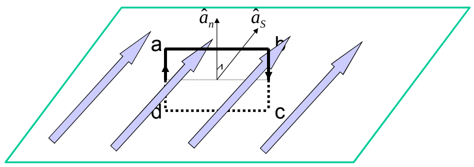
Medium 1 (above)
$\sigma_1, \varepsilon_1, \mu_1$

Medium 2 (below)
$\sigma_2, \varepsilon_2, \mu_2$

Take limit as *ad* and *bc* go to zero
Consider remaining E_1 and E_2 TANGENT TO SURFACE

$$E_1 = E_2 \;\; i.e. E_t \text{ is continuous}$$

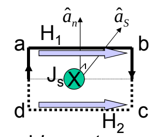
The closed path can enclose surface current



Example:
1. Current on surface of a conductor

Ampere's Law at Boundary

$$\oint_C \vec{H}\bullet d\vec{l} = \iint_S \vec{J}\bullet d\vec{S} + \frac{d}{dt}\iint_S \vec{D}\bullet d\vec{S} = I_{enclosed}$$



Medium 1 (above) $\sigma_1, \varepsilon_1, \mu_1$

Medium 2 (below) $\sigma_2, \varepsilon_2, \mu_2$

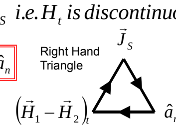
Take limit as *ad* and *bc* go to zero

Consider remaining H_1 and H_2 TANGENT TO SURFACE

$$H_1 - H_2 = J_S \;\; i.e. H_t \text{ is discontinuous because of } J_S$$

$$\boxed{(\vec{H}_1 - \vec{H}_2)_t = \vec{J}_S \times \hat{a}_n}$$

Right Hand Triangle

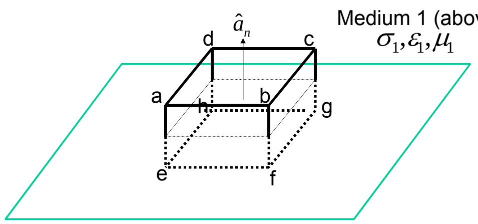


$$\boxed{\hat{a}_n \times (\vec{H}_1 - \vec{H}_2)_t = \vec{J}_S}$$

Note: we know nothing about H_n

Closed surface through the volume

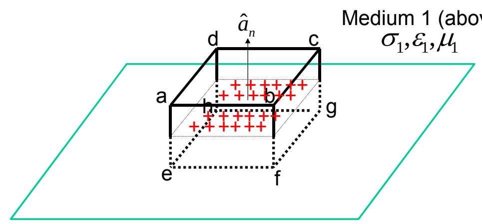
Medium 1 (above) $\sigma_1, \varepsilon_1, \mu_1$



Medium 2 (below) $\sigma_2, \varepsilon_2, \mu_2$

The closed volume can enclose surface charges

Medium 1 (above) $\sigma_1, \varepsilon_1, \mu_1$

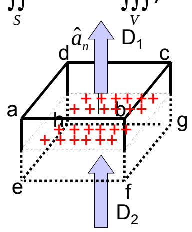


Medium 2 (below) $\sigma_2, \varepsilon_2, \mu_2$

Example
1. Free charges on the surface of a conductor

Gauss' Law for D at Boundary

$$\oiint_S \vec{D}\bullet d\vec{S} = \iiint_V \rho dV$$



Take limit as *ae*, *bf*, *cg*, and *dh* go to zero

Consider D_1 and D_2 NORMAL TO SURFACE

$$D_1 - D_2 = \rho \;\; i.e. D_n \text{ is discontinuous because of } \rho_S$$

Finding surface charge

- Given D on the surface of a perfect conductor, find the surface charge, ρ, at that point (Hint: what is the vector $\mathbf{a}_n$ for each surface):

(a) $\mathbf{D}=D_0$<1,-2,2> pointing away from surface

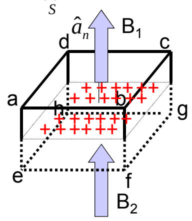
(b) $\mathbf{D}=D_0$<1,0,√3> pointing towards surface

(c) $\mathbf{D}=D_0$<0.8,0,0.6> pointing away

From D4.13 (p262) of old book

Gauss' Law for B at Boundary

$$\oiint_S \vec{B} \bullet d\vec{S} = 0$$



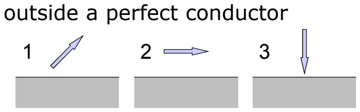
Take limit as *ae*, *bf*, *cg*, and *dh* go to zero

Consider B_1 and B_2 NORMAL TO SURFACE

$$B_1 = B_2 \text{ i.e. } B_n \text{ is continuous}$$

Challenge Question: Boundary conditions

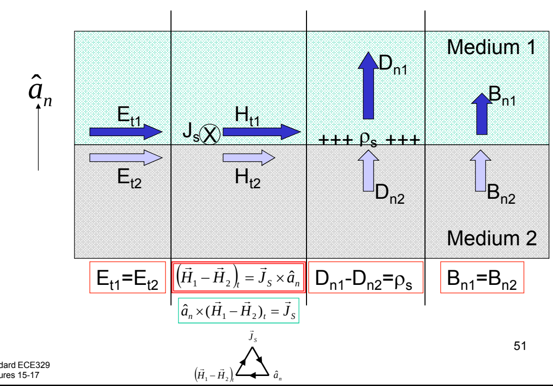
- Which of the following are realizable as the field outside a perfect conductor



(a) 1, 2, and 3 can be either **E** or **H**
(b) 1 can be either **E** or **H**, but 2 is **E**, 3 is **H**
(c) 1 can be either **E** or **H**, but 2 is **H**, 3 is **E**
(d) 1 can be neither **E** nor **H**, but 2 is **E**, 3 is **H**
(e) 1 can be neither **E** nor **H**, but 2 is **H**, 3 is **E**

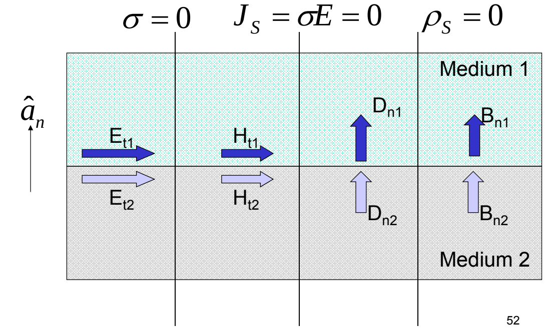
LG's question

Remember this drawing!!



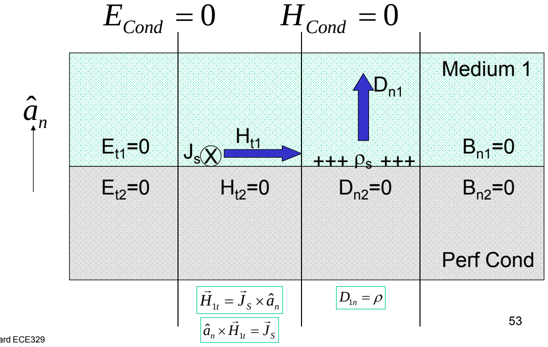
Boundary between Two Perfect Dielectrics

$$\sigma = 0 \quad J_S = \sigma E = 0 \quad \rho_S = 0$$



Surface of Perfect Conductor

$$E_{Cond} = 0 \quad H_{Cond} = 0$$



(Time permitting) BCs for a rectangular cavity resonator

- The region 0<x<a, 0<y<b, 0<z<d is a perfect dielectric $\varepsilon=4\varepsilon_0$ and the boundary is a perfect conductor on all 6 sides. Inside the resonator, the fields are:

$$\vec{E} = E_0 \sin\frac{\pi x}{a} \sin\frac{\pi z}{d} \cos\omega t \hat{a}_y$$

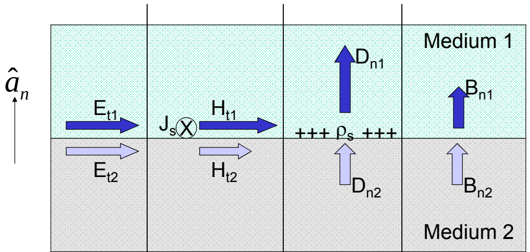
$$\vec{H} = H_{01} \sin\frac{\pi x}{a} \cos\frac{\pi z}{d} \sin\omega t \hat{a}_x - H_{02} \cos\frac{\pi x}{a} \sin\frac{\pi z}{d} \sin\omega t \hat{a}_z$$

Find ρ_s and $\mathbf{J}_s$ on all 6 walls.

From P4.29 (p277) of old book

Lecture 16b Summary

- **Never** use the differential form of Maxwell's equations at a boundary – only use integral form



$E_{t1}=E_{t2}$ | $(\vec{H}_1-\vec{H}_2)_t = \vec{J}_S \times \hat{a}_n$ | $D_{n1}-D_{n2}=\rho_s$ | $B_{n1}=B_{n2}$

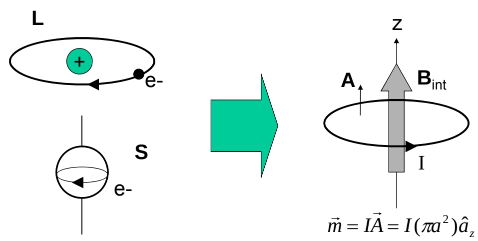
$\hat{a}_n \times (\vec{H}_1-\vec{H}_2)_t = \vec{J}_S$

ECE 329
Lecture 17
Section 5.2

Magnetic Materials

Adapted from Prof. Cunningham's Notes

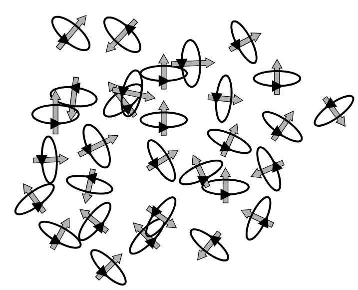
Magnetic Moments at the atomic scale



$\vec{m} = I\vec{A} = I(\pi a^2)\hat{a}_z$

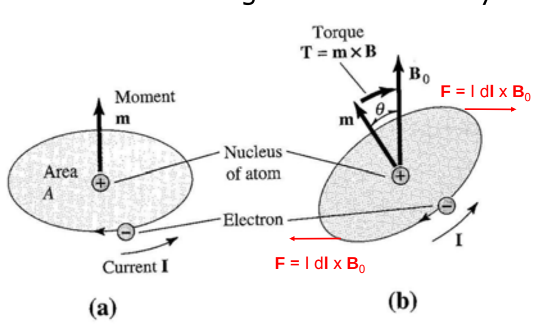
Internal magnetic fields are produced by electrons orbiting the nucleus, **L**, or by the internal spin of electrons, **S**
The atom has a magnetic dipole moment, **m**

Net magnetic moment

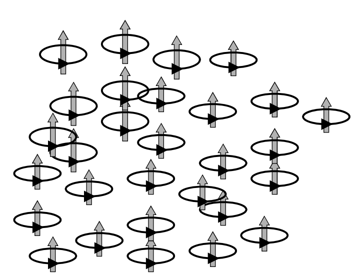


A volume of material contains many magnetic moments. They might be randomly oriented.

External **B**-field can rotate Magnetic Moments or change Orbital Velocity



Net magnetic moment



So after momentarily applying an external field, they can get magnetized and are mostly aligned in one direction. The internal field can keep them aligned to each other

Magnetization Vector

$$\vec{M} = N\vec{m}$$

Magnetic dipole moment per unit volume

N = # atoms per unit volume

m = average dipole moment per molecule

Units for **M**: $(1/m^3)\ (A{*}m^2) = A/m$

$$\vec{B}_{\text{int}} = \mu_0 \vec{M}$$

is the magnetic flux per unit area from the dipoles

Magnetic Susceptibility

B INSIDE the material is a function of how strong and how well-aligned all the magnetic moments are

The INTERNAL magnetic flux is INDUCED by the application of an EXTERNAL magnetic field

Total flux = Applied + Secondary

$$\vec{B}_{total} = \mu_0(\vec{H}_{ext} + \vec{M})$$

Some materials are more easily "magnetized" than others

$$\vec{M} = \chi_m \vec{H}_{ext}$$

Definition of magnetic susceptibility (has no units)

Relative Magnetic Permeability

$$\vec{B}_{total} = \mu_0(\vec{H}_{ext} + \vec{M})$$
$$\vec{M} = \chi_m \vec{H}_{ext}$$

$$\vec{D} = \varepsilon_0 \vec{E}_{tot} + \vec{P}$$
$$\vec{P} = \varepsilon_0 \chi_e \vec{E}_{tot}$$

$$\vec{B}_{total} = \mu_0(1+\chi_m)\vec{H}_{ext}$$
$$\vec{B} = \mu\vec{H}$$
$$\mu = \mu_0(1+\chi_m) = \mu_0\mu_r$$
$$\mu_r = 1+\chi_m$$

$$\vec{D} = \varepsilon_0(1+\chi_e)\vec{E}_{tot}$$
$$\vec{D} = \varepsilon\vec{E}_{tot}$$
$$\varepsilon = \varepsilon_0(1+\chi_e) = \varepsilon_0\varepsilon_r$$
$$\varepsilon_r = 1+\chi_e$$

Diamagnetic Materials (χ_m<0)

- With $\mathbf{H}_{ext}$, the electron orbital speed changes depending on the relative orientation of **v** and $\mathbf{H}_{ext}$
  - Equivalent to a weak magnetic dipole that OPPOSES $\mathbf{H}_{ext}$
- Magnetic susceptibility is a NEGATIVE number

Examples:

Copper	$\chi_m = -0.94\text{x}10^{-5}$
Lead	$\chi_m = -1.70\text{x}10^{-5}$
Water	$\chi_m = -0.88\text{x}10^{-5}$

Paramagnetic Materials (χ_m>0) Positive susceptibility

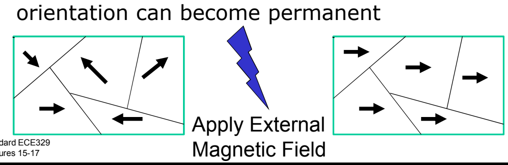
- With no $\mathbf{H}_{ext}$, domains of orbital electrons and spinning electrons exist
- However, the domains are physically oriented in random directions (as a function of time), so overall $\mathbf{B}_{int}$=0
- With $\mathbf{H}_{ext}$, domains reorient themselves to generate $\mathbf{B}_{int}$ that ALIGNS WITH H_{ext}
- Magnetic susceptibility is a POSITIVE number

Examples:

Platinum	$\chi_m = +2.90\text{x}10^{-5}$
Aluminum	$\chi_m = +2.10\text{x}10^{-5}$
Liquid Oxygen	$\chi_m = +3.50\text{x}10^{-5}$

Ferromagnetic Materials (χ_m>>1)

- Very high susceptibility
- Microscopic "domains" that have strongly oriented magnetic dipoles
- Direction of magnetic dipole differs from one domain to another
- Under an externally applied H field, the domains can orient coherently (i.e. in the same direction)
- Using a strong enough applied field, the domain orientation can become permanent

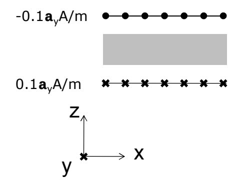


Material		$\mu_r=1+\chi_m$
Bismuth	Diamagnetic	0.99983
Silver	Diamagnetic	0.99993
Lead	Diamagnetic	0.99993
Copper	Diamagnetic	0.999991
Water	Diamagnetic	0.999991
Vacuum	Nonmagnetic	1†
Air	Paramagnetic	1.00000(
Aluminum	Paramagnetic	1.00002
Palladium	Paramagnetic	1.0008
2-81 Permalloy powder (2 Mo, 81 Ni)‡	Ferromagnetic	130
Cobalt	Ferromagnetic	250
Nickel	Ferromagnetic	600
Ferroxcube 3 (Mn-An-ferrite powder)	Ferromagnetic	1,500
Mild steel (0.2 C)	Ferromagnetic	2,000
Iron (0.2 impurity)	Ferromagnetic	5,000
Silicon Iron (4 Si)	Ferromagnetic	7,000
78 Permalloy (78.5 Ni)	Ferromagnetic	100,000
Mumetal (75 Ni, 5 Cu, 2 Cr)	Ferromagnetic	100,000
Purified iron (0.05 impurity)	Ferromagnetic	200,000
Superalloy (5 Mo, 79 Ni)§	Ferromagnetic	1,000,000

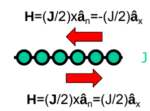
†By definition.
‡Percentage composition. Remainder is iron and impurities.
§Used in transformer applications with continuous tape-wound (gapless) cores.

Magnetization

- An infinite plane ferromagnetic slab (μ_r=100) lies between two infinite plane sheets of uniform current density of **J**=±0.1**a**$_y$A/m. Find **H**, **B**, and **M** inside the slab and compare to if the slab were non-magnetic (μ_r=1).



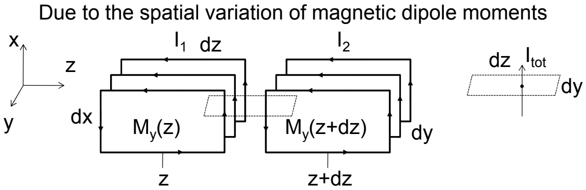
Hint:



From D4.6 (p238) of old book

Magnetization Current

Due to the spatial variation of magnetic dipole moments



$$M_y(z)=\frac{m_y}{Volume}=\frac{IA}{Volume}=\frac{I_{1x}(dxdz)}{dxdydz}=\frac{I_{1x}}{dy} \qquad M_y(z+dz)=\frac{-I_{2x}}{dy}$$

$$I_{tot}=I_{1x}+I_{2x}=(M_y(z)-M_y(z+dz))dy$$

$$J_M=\frac{I_{tot}}{dydz}=-\frac{\partial M_y}{\partial z}\hat{a}_x \qquad \boxed{\vec{J}_M=\vec{\nabla}\times\vec{M}}$$

Ampere's Law in Mag. Material

$$\vec{\nabla}\times\left(\frac{\vec{B}}{\mu_0}\right)=\vec{J}+\vec{J}_M+\frac{\partial\vec{D}}{\partial t}$$

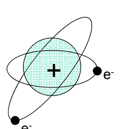
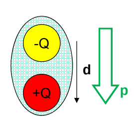
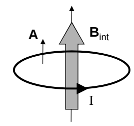
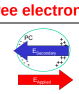
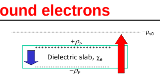
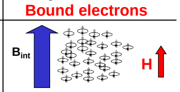
In magnetic medium, we need to include J_M with the total current

$$\vec{\nabla}\times\left(\frac{\vec{B}}{\mu_0}\right)=\vec{J}+\vec{\nabla}\times\vec{M}+\frac{\partial\vec{D}}{\partial t}$$

$$\vec{\nabla}\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t} \qquad \boxed{H\equiv\frac{\vec{B}}{\mu_0}-\vec{M}}$$

$$\vec{B}=\mu_0(\vec{H}+\vec{M})$$

3 types of materials

Conductors	Dielectrics	Magnetic
 Free electrons	 Polarized atoms/molecules Bound electrons	 Magnetic moments Bound electrons
		
E=0 inside ρ=0 inside ρ=ρ$_s$ only surface charge V is same throughout E$_{outside}$ is ⊥ to surface	E≠0 inside but it is reduced **E**$_{tot}$=**E**$_a$+**E**$_s$ **D**=ε**E**$_{tot}$=**P**+ε$_0$**E**$_{tot}$	**B**$_{tot}$=**B**$_a$+**B**$_s$ **B**$_{tot}$=μ**H**=μ$_0$(**H**+**M**)

Lecture 17 Summary

- Magnetic dipole moment **m**=I**A**
- Magnetization or magnetic dipole moment per unit volume

  M=N**m**=χ_m**H**$_{external}$

  - Simple linear isotropic magnetic material
    - Strengthens **B**-field strength by (1+χ_m)
    - **H** has same value as free space
  - Magnetization current **J**$_M$=∇x**M**
  - New definition **H**=**B**/μ_0-**M** and **B**=μ**H**
- Upcoming schedule: Plane Waves
  - Sections 4.1, 4.2, 4.4, 4.5, 5.3, and 5.4

Lectures 18-20
Sections 4.1, 4.2, 4.4, 4.5
Section 4.6

Uniform Plane Waves in Free Space
Poynting's Theorem

Adapted from Prof. Cunningham's Notes

Summary of Maxwell's Equations

Faraday's Law	$\oint_C \vec{E} \bullet d\vec{l} = -\frac{d}{dt}\iint_S \vec{B} \bullet d\vec{S}$	$\nabla \times \vec{E} = -\frac{d\vec{B}}{dt}$
Ampere's Law	$\oint_C \vec{H} \bullet d\vec{l} = \iint_S \vec{J} \bullet d\vec{S} + \frac{d}{dt}\iint_S \vec{D} \bullet d\vec{S}$	$\nabla \times \vec{H} = \vec{J} + \frac{d\vec{D}}{dt}$
Gauss' Law	$\oiint_S \vec{B} \bullet d\vec{S} = 0$	$\nabla \bullet \vec{B} = 0$
Gauss' Law	$\oiint_S \vec{D} \bullet d\vec{S} = \iiint_V \rho dV$	$\nabla \bullet \vec{D} = \rho$
Continuity Eq.	$\oiint_S \vec{J} \bullet d\vec{S} = -\frac{d}{dt}\iiint_V \rho dV$	$\nabla \bullet \vec{J} = -\frac{d\rho}{dt}$

Motivation for Waves

- Maxwell's equations say…
  - Time variation in **J**(t) leads to spatial variation of **H**(t)

$$\nabla \times \vec{H} = \vec{J} + \frac{d\vec{D}}{dt}$$

  - Time variation in **H**(t) leads to spatial variation in **E**(t)

$$\nabla \times \vec{E} = -\frac{d\vec{B}}{dt}$$

  - Time variation in **E**(t) leads to spatial variation in **H**(t)

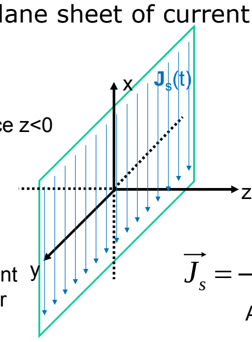
$$\nabla \times \vec{H} = \vec{J} + \frac{d\vec{D}}{dt}$$

- This recursion relationship between **E**(t) and **H**(t) leads to electromagnetic wave propagation

Field Source

- Infinite plane sheet of current at z=0

Free space z<0



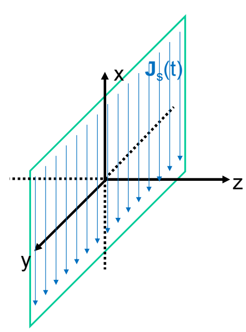
Free space z>0

J_s = surface current in Amps per meter

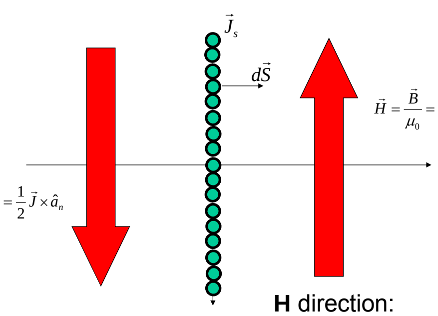
$$\vec{J_s} = -J_s(t)\vec{a_x}$$

At z=0

Which direction do we expect the generated fields to point?



Hmm, we did the static case



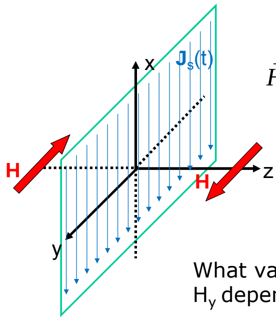
$$\vec{H} = \frac{\vec{B}}{\mu_0} = \frac{1}{2}\vec{J} \times \hat{a}_n$$

$$\vec{H} = \frac{\vec{B}}{\mu_0} = \frac{1}{2}\vec{J} \times \hat{a}_n$$

H direction:

$$\perp \vec{J}_s \text{ and } \perp d\vec{S}$$

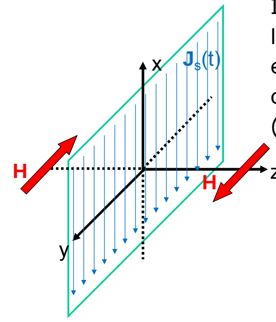
So we expect **H** to be along the ±y-direction



$$\vec{H} = H_y \hat{a}_y$$

What variables can H_y depend on??

H_y only depends on z and t



Infinite plane looks the same everywhere so H_y can't depend on (x,y) coordinates

$$\vec{H} = H_y(z,t)\vec{a_y}$$

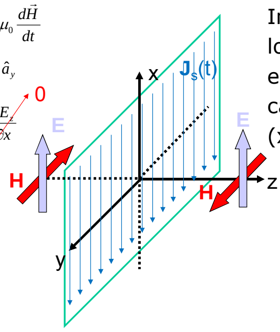
E depends only on z and t, but in what direction is **E**?

$$\nabla \times \vec{E} = -\frac{d\vec{B}}{dt} = -\mu_0 \frac{d\vec{H}}{dt}$$

$$= -\mu_0 \frac{dH_y}{dt}\hat{a}_y$$

$$\left(\nabla \times \vec{E}\right)_y = \frac{\partial E_x}{\partial z} - \cancelto{0}{\frac{\partial E_z}{\partial x}}$$

$$\boxed{\frac{\partial E_x}{\partial z} = -\mu_0 \frac{dH_y}{dt}}$$



Infinite plane looks the same everywhere so **E** can't depend on (x,y) coordinates

$E_x \neq 0$, what about E_y and E_z?

$$0 = -\frac{dB_x}{dt} = \left(\nabla \times \vec{E}\right)_x = \cancelto{0}{\frac{\partial E_z}{\partial y}} - \frac{\partial E_y}{\partial z}$$

So $\frac{\partial E_y}{\partial z} = 0 \Rightarrow E_y = const,$
but $const = 0$ since we can turn off our source, i.e. set J(t) = 0.

$$0 = -\frac{dB_z}{dt} = \left(\nabla \times \vec{E}\right)_z = \cancelto{0}{\frac{\partial E_y}{\partial x}} - \cancelto{0}{\frac{\partial E_x}{\partial y}}$$

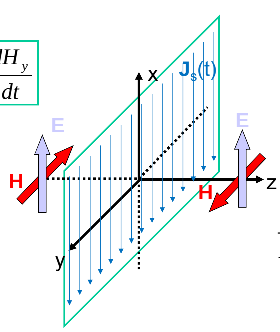
No new information.

$$0 = \frac{\rho}{\varepsilon_0} = \nabla \bullet \vec{E} = \cancelto{0}{\frac{\partial E_x}{\partial x}} + \cancelto{0}{\frac{\partial E_y}{\partial y}} + \frac{\partial E_z}{\partial z}$$

So $\frac{\partial E_z}{\partial z} = 0 \Rightarrow E_z = const = 0$

E then must be along the x-direction

$$\boxed{\frac{\partial E_x}{\partial z} = -\mu_0 \frac{dH_y}{dt}}$$

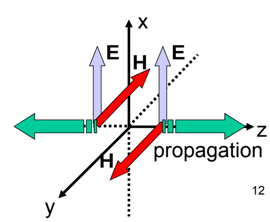


$$\vec{E} = E_x(z,t)\vec{a_x}$$

Summary so far …

- We have **E** and **H** that are
  - Perpendicular to each other
  - Perpendicular to the direction of propagation
  - Magnitude is constant ("uniform") in any plane perpendicular to the propagation direction

$$\vec{J_s} = -J_x(t)\vec{a_x}$$
$$\vec{E} = E_x(z,t)\vec{a_x}$$
$$\vec{H} = H_y(z,t)\vec{a_y}$$



Following Maxwell's Footsteps

- We will SIMULTANEOUSLY solve Maxwell's equations to find **E** and **H** caused **J**

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \vec{E} = E_x(z,t)\overrightarrow{a_x}$$

$$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t} \qquad \vec{H} = H_y(z,t)\overrightarrow{a_y}$$

$$\vec{J_s} = -J_x(t)\overrightarrow{a_x}$$

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Apply the two Maxwell's Equations

- Performing the cross product, only two equations contain E_x or H_y

$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t}$$

$$\frac{\partial H_y}{\partial z} = -J_x - \frac{\partial D_x}{\partial t}$$

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Simplify to E and H

- Use constitutive relationships $\mathbf{D}=\varepsilon_0\mathbf{E}$ and $\mathbf{B}=\mu_0\mathbf{H}$ for free space to express **D** and **B** in terms of **E** and **H**
- We write the equations for everywhere EXCEPT at z=0 (where the current source is) so J_x goes away - for now

$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t} = -\mu_0\frac{\partial H_y}{\partial t}$$

$$\frac{\partial H_y}{\partial z} = -\frac{\partial D_x}{\partial t} = -\varepsilon_0\frac{\partial E_x}{\partial t}$$

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Eliminate H in favor of E only

$$\frac{\partial E_x}{\partial z} = -\mu_0\frac{\partial H_y}{\partial t}$$

$$\frac{\partial H_y}{\partial z} = -\varepsilon_0\frac{\partial E_x}{\partial t}$$

$$\frac{\partial^2 E_x}{\partial z^2} = -\mu_0\frac{\partial}{\partial z}\left(\frac{\partial H_y}{\partial t}\right) = -\mu_0\frac{\partial}{\partial t}\left(\frac{\partial H_y}{\partial z}\right) = -\mu_0\frac{\partial}{\partial t}\left(-\varepsilon_0\frac{\partial E_x}{\partial t}\right)$$

- Coupled 1st order partial differential equation (PDE) → single 2nd order PDE

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This is the "wave equation"

$$\frac{\partial^2 E_x}{\partial z^2} = \mu_0\varepsilon_0\frac{\partial^2 E_x}{\partial t^2}$$

- Why is it called a wave equation?
  - The solution to this differential equation will be a function whose shape moves like a wave in the z-direction
  - How do we solve it?
    - Two techniques: Separation of variables (see book) or factorable operators (my notes)
  - We will do the solution for **E**, and then we can come back and plug the solution into Maxwell's equation to get **H**

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Change of variables

- Makes the math a little cleaner

$$\tau = z\sqrt{\mu_0\varepsilon_0}$$ Has units of time

$$\frac{\partial^2 E_x}{\partial \tau^2} = \frac{\partial^2 E_x}{\partial t^2}$$ Still the wave eqn

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Two possible solutions

$$\frac{\partial^2 E_x}{\partial \tau^2} = \frac{\partial^2 E_x}{\partial t^2}$$

$$\frac{\partial^2 E_x}{\partial \tau^2} - \frac{\partial^2 E_x}{\partial t^2} = 0$$

$$"x^2 - y^2 = (x+y)(x-y)"$$

$$\left(\frac{\partial}{\partial \tau} + \frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial \tau} - \frac{\partial}{\partial t}\right) E_x = 0$$

$$\frac{\partial E_x}{\partial \tau} = -\frac{\partial E_x}{\partial t} \qquad \frac{\partial E_x}{\partial \tau} = +\frac{\partial E_x}{\partial t}$$

Now we have two first order diff eqns that each can be a solution to the wave equation

Two possible solutions

$$\frac{\partial E_x}{\partial \tau} = -\frac{\partial E_x}{\partial t} \qquad \frac{\partial E_x}{\partial \tau} = +\frac{\partial E_x}{\partial t}$$

If E_x is a function of $(t-\tau)$ the equation is satisfied

If E_x is a function of $(t+\tau)$ the equation is satisfied

$$E_x(\tau,t) = Af(t-\tau) \qquad E_x(\tau,t) = Bg(t+\tau)$$

A = a constant

B = a constant

Combining the two possible solutions…

$$E_x(\tau,t) = Af(t-\tau) + Bg(t+\tau)$$

Changing variables *back* to *z*

$$E_x(\tau,t) = Af(t-\tau) + Bg(t+\tau)$$

$$E_x(z,t) = \underbrace{Af(t - z\sqrt{\mu_0\varepsilon_0})}_{\text{Traveling wave propagating in the +z direction}} + \underbrace{Bg(t + z\sqrt{\mu_0\varepsilon_0})}_{\text{Traveling wave propagating in the -z direction}}$$

Solving for **H**

$$E_x(z,t) = Af(t - z\sqrt{\mu_0\varepsilon_0}) + Bg(t + z\sqrt{\mu_0\varepsilon_0})$$

Recall… $\frac{\partial H_y}{\partial z} = -\varepsilon_0 \frac{\partial E_x}{\partial t}$

$$H_y(z,t) = \frac{1}{\sqrt{\mu_0/\varepsilon_0}}\left[Af(t - z\sqrt{\mu_0\varepsilon_0}) - Bg(t + z\sqrt{\mu_0\varepsilon_0})\right]$$

Two definitions

$$v_p = c = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = \text{speed of light} = 3\times10^8 \ (m/s)$$

$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx 377 \ (ohms)$ Intrinsic impedance of free space

Rewriting the solution

$$E_x(z,t) = Af(t - \frac{z}{v_p}) + Bg(t + \frac{z}{v_p})$$

$$H_y(z,t) = \frac{1}{\eta_0}\left[Af(t - \frac{z}{v_p}) - Bg(t + \frac{z}{v_p})\right]$$

Solution is a superposition of traveling waves

- one going in +z direction
- one going in -z direction

Challenge Problem: Velocity of propagation

- The velocities of propagation for the following waves: f = $(0.05y-t)^2$, g = u(t+0.02x), h = $\cos(2\pi 10^8 t - 2\pi z)$ are:

(a) 0.05$\mathbf{a}_y$, -0.02$\mathbf{a}_x$, 10^8 $\mathbf{a}_z$
(b) 0.05$\mathbf{a}_y$, -0.02$\mathbf{a}_x$, -10^8 $\mathbf{a}_z$
(c) 20$\mathbf{a}_y$, -50$\mathbf{a}_x$, 10^8 $\mathbf{a}_z$
(d) 20$\mathbf{a}_y$, 50$\mathbf{a}_x$, 10^8 $\mathbf{a}_z$
(e) 20$\mathbf{a}_y$, -50$\mathbf{a}_x$, -10^8 $\mathbf{a}_z$

From D3.11 (p 171) of old book

Two useful identities

Forward wave $\mathbf{a_z}$

$$E_x(z,t) \equiv f(t - \frac{z}{v_p}) \Rightarrow$$

$$\boxed{E_x(z,t) = E_x(0, t - \frac{z}{v_p}) = E_x(z - v_p t, 0)}$$

Backwards wave -$\mathbf{a_z}$

$$E_x(z,t) \equiv g(t + \frac{z}{v_p}) \Rightarrow$$

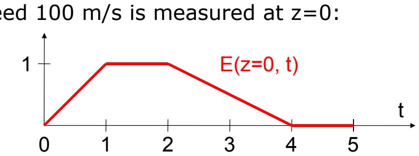
$$\boxed{E_x(z,t) = E_x(0, t + \frac{z}{v_p}) = E_x(z + v_p t, 0)}$$

These identities simply say that the forward wave moves like $z=v_p t$ and the backwards wave like $z=-v_p t$

They allow you to express the wave solutions in terms of the wave at a fixed position or at a fixed time

Moving waveform

- A wave traveling in the -$\mathbf{a}_z$ direction with speed 100 m/s is measured at z=0:



Find the wave amplitude at:
(a) z=200m, t=0.2s
(b) z=-300m, t=3.4s
(c) z=100m, t=0.6s

From D3.13 (p 171) of old book

Lecture 18 Summary

- Differentiate Maxwell's Equations

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

to derive wave equation: $\frac{\partial^2 E_x}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_x}{\partial t^2}$

which has solution:

$$E_x(z,t) = \underbrace{Af(t - \frac{z}{v_p})}_{\text{Traveling wave +z direction}} + \underbrace{Bg(t + \frac{z}{v_p})}_{\text{Traveling wave -z direction}}$$

Lecture 19
Sections 4.4, 4.5

Uniform Plane Waves in Free Space

Adapted from Prof. Cunningham's Notes

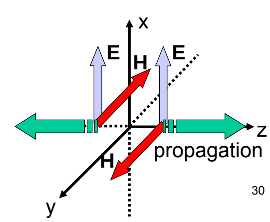
Summary so far …

- We have **E** and **H** that are
  - Perpendicular to each other
  - Perpendicular to the direction of propagation
  - Magnitude is constant ("uniform") in any plane perpendicular to the propagation direction

$$\vec{J}_s = -J_x(t)\vec{a}_x$$

$$\vec{E} = E_x(z,t)\vec{a}_x$$

$$\vec{H} = H_y(z,t)\vec{a}_y$$



Our solution so far …

$$E_x(z,t) = Af(t - \frac{z}{v_p}) + Bg(t + \frac{z}{v_p})$$

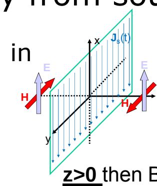
$$H_y(z,t) = \frac{1}{\eta_0}\left[Af(t - \frac{z}{v_p}) - Bg(t + \frac{z}{v_p}) \right]$$

Solution is a superposition of traveling waves

- one going in +z direction
- one going in -z direction

For a valid solution, the waves move away from source

- Generated waves travel in the +z direction for z>0 and −z direction for z<0



z<0 then A=0

$$E_x(z,t) = Bg(t + \frac{z}{v_p})$$

$$H_y(z,t) = -\frac{B}{\eta_0} g(t + \frac{z}{v_p})$$

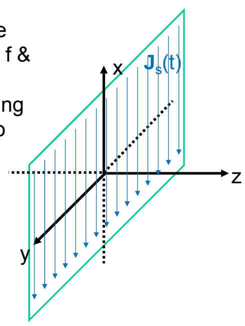
z>0 then B=0

$$E_x(z,t) = Af(t - \frac{z}{v_p})$$

$$H_y(z,t) = \frac{A}{\eta_0} f(t - \frac{z}{v_p})$$

Final Step: Boundary conditions from the current source

We have to replace unknown functions f & g and constants A & B with something that relates them to the current source



$$\vec{J}_s = -J_s(t)\vec{a}_x$$

At z=0

Near the boundary z → 0 from the z<0 and z>0 sides

$z=0^-$

$$E_x(z,t) = Bg(t)$$

$$H_y(z,t) = -\frac{B}{\eta_0} g(t)$$

$z=0^+$

$$E_x(z,t) = Af(t)$$

$$H_y(z,t) = \frac{A}{\eta_0} f(t)$$

Apply Faraday's Law to closed path cutting through sheet that is parallel to current flow

$$\oint_C \vec{E} \bullet \vec{dl} = -\frac{d}{dt}\int_S \vec{B} \bullet \vec{dS}$$

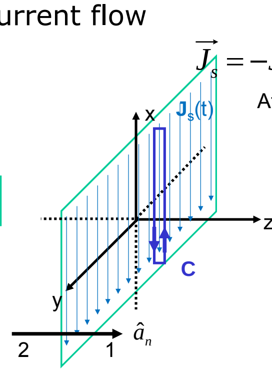
$$\oint_C \vec{E} \bullet \vec{dl} = 0$$

$$\boxed{\vec{E}_t \text{ is continuous}}$$

$$\boxed{E_{1t} = E_{2t}}$$

$$\vec{J}_s = -J_s(t)\vec{a}_x$$

At z=0



Use E_t continuous to match the solutions for z<0 and z>0

$$\boxed{\vec{E}_t \text{ is continuous}}$$

$$E_x(z=0^+)\Delta x - E_x(z=0^-)\Delta x = 0$$

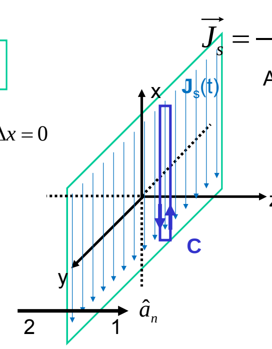
$$Af(t) - Bg(t) = 0$$

$$Af(t) = Bg(t)$$

$$\boxed{E_{1t} = E_{2t}}$$

$$\vec{J}_s = -J_s(t)\vec{a}_x$$

At z=0



Eliminate Bg in favor of Af

z=0⁻

$$E_x(z,t) = Af(t)$$

$$H_y(z,t) = -\frac{A}{\eta_0} f(t)$$

z=0⁺

$$E_x(z,t) = Af(t)$$

$$H_y(z,t) = \frac{A}{\eta_0} f(t)$$

Apply Ampere's Law to closed path cutting through sheet that is perpendicular to current

$$\oint_C \vec{H} \bullet d\vec{l} = I_{enclosed} + \frac{d}{dt}\iint_S \varepsilon_0 \vec{E} \bullet d\vec{S}$$

$$\oint_C \vec{H} \bullet d\vec{l} = I_{enclosed}$$

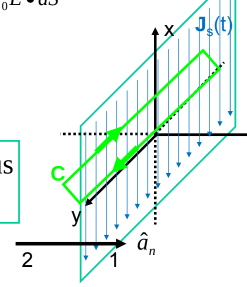
$\vec{H}_t$ is discontinuous at current sheet

$$\left(\vec{H}_1 - \vec{H}_2\right)_t = \vec{J}_S \times \hat{a}_n$$

$$\vec{J}_s = -J_s(t)\vec{a}_x$$

At z=0

J_s is a surface current in Amps per meter



Use $H_{||}$ discontinuity to relate the solutions to the current sheet

$\vec{H}_t$ is discontinuous at sheet

$$H_y(z=0^+)\Delta y - H_y(z=0^-)\Delta y = J_s \Delta y$$

$$\frac{A}{\eta_0} f(t) + \frac{A}{\eta_0} f(t) = J_s(t)$$

$$\left(\vec{H}_1 - \vec{H}_2\right)_y = \left(\vec{J}_S \times \hat{a}_n\right)_y$$

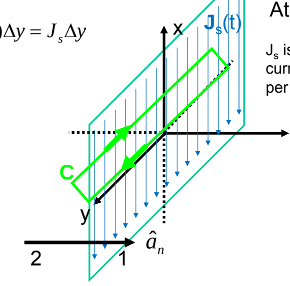
$$\frac{2A}{\eta_0} f(t) = J_s(t)$$

$$Af(t) = \frac{\eta_0}{2} J_s(t)$$

$$\vec{J}_s = -J_s(t)\vec{a}_x$$

At z=0

J_s is a surface current in Amps per meter



The final answer at last

z<0

$$E_x(z,t) = \frac{\eta_0}{2} J_S\left(t + \frac{z}{v_p}\right)$$

$$H_y(z,t) = -\frac{1}{2} J_S\left(t + \frac{z}{v_p}\right)$$

z>0

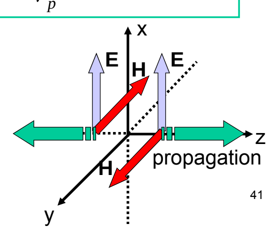
$$E_x(z,t) = \frac{\eta_0}{2} J_S\left(t - \frac{z}{v_p}\right)$$

$$H_y(z,t) = \frac{1}{2} J_S\left(t - \frac{z}{v_p}\right)$$

Combining the expressions…

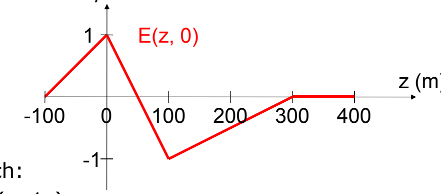
$$\vec{E}(z,t) = \frac{\eta_0}{2} J_S\left(t \mp \frac{z}{v_p}\right)\vec{a}_x$$

$$\vec{H}(z,t) = \pm\frac{1}{2} J_S\left(t \mp \frac{z}{v_p}\right)\vec{a}_y \qquad z \gtrless 0$$



Transforming time and space

- A wave travelling in the $\mathbf{a}_z$ direction with speed 100m/s is measured at t=0:



Sketch:
(a) E(z, 1s)
(b) E(0, t)
(c) E(200m, t)

From Problem 3.22 (p 200) of old book

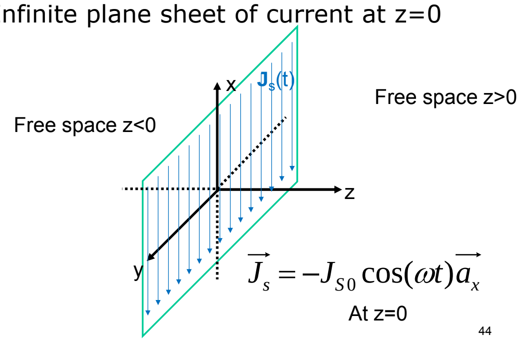
Lecture 19a Summary

- Wave emanates from z=0 so must travel $+a_z$ for z>0 and $-a_z$ for z<0
- Use integral form of Maxwell's Eqns to get BOUNDARY CONDITIONS

$$\boxed{\begin{array}{l}\vec{E}_t \text{ is continuous}\\ E_x(0^+) = E_x(0^-)\end{array}} \qquad \boxed{\begin{array}{l}\vec{H}_t \text{ is discontinuous}\\ \mathrm{H_y}(0^+) - \mathrm{H_y}(0^-) = J_S\end{array}}$$

Sinusoidal Plane Waves

- Infinite plane sheet of current at z=0



$$\vec{J}_s = -J_{S0}\cos(\omega t)\vec{a}_x$$

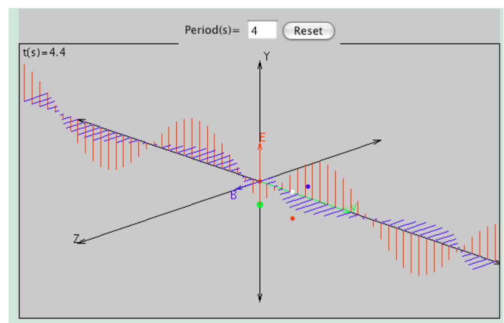
At z=0

Solution …

$$\vec{J}_s = -J_{S0}\cos(\omega t)\vec{a}_x \qquad \vec{E}(z,t) = \frac{\eta_0}{2}J_S(t \mp \frac{z}{v_p})\vec{a}_x \qquad \vec{H}(z,t) = \pm\frac{1}{2}J_S(t \mp \frac{z}{v_p})\vec{a}_y$$

$$\boxed{\begin{array}{ll}\vec{E}(z,t) = \dfrac{\eta_0 J_{S0}}{2}\cos(\omega t \mp \beta z)\vec{a}_x & \\ & z \gtrless 0\\ \vec{H}(z,t) = \pm\dfrac{J_{S0}}{2}\cos(\omega t \mp \beta z)\vec{a}_y & \\ & \beta = \dfrac{\omega}{v_p}\end{array}}$$

Web Demo



http://www.phy.ntnu.edu.tw/java/emWave/emWave.html

Wave Parameters

$$\vec{E}(z,t) = \frac{\eta_0 J_{S0}}{2}\cos(\omega t \mp \beta z)\vec{a}_x$$

Phase	$\phi = \omega t \mp \beta z$	(radians)
Angular Frequency	$\omega = \frac{\partial\phi}{\partial t}$	(radians/sec)
Linear Frequency	$f = \frac{\omega}{2\pi}$	(1/sec)
Phase Constant	$\beta = \mp\frac{\partial\phi}{\partial z}$	(radians/m)
Wavelength	$\lambda = \frac{2\pi}{\beta}$	(m)
Phase Velocity	$v_p = \frac{\omega}{\beta}$	(m/sec)

Sinusoidal wave parameters

For a wave in free space, find the following:

a. f if the phase of the field at a point changes 3π in 0.1μs

b. λ if the phase changes 0.04π in 1m

c. f if λ=25m

d. λ if f=5MHz

From D3.14 (p 178) of old book

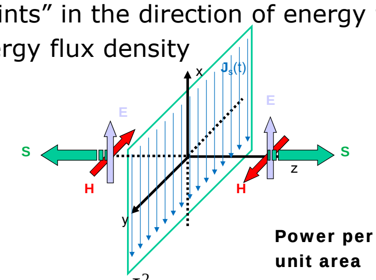
A few additional important properties

$v_p = \lambda f$ Speed of light in free space

$\frac{|\vec{E}|}{|\vec{H}|} = \eta_0$ Intrinsic impedance of free space

Poynting Vector

- "Points" in the direction of energy flow
- Energy flux density

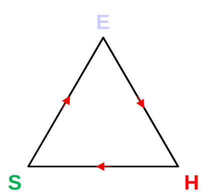


Power per unit area

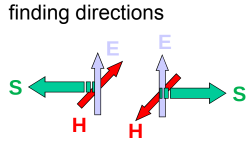
$$\vec{S} = \vec{E} \times \vec{H} = \pm \frac{\eta_0 J_{S0}^2}{4} \cos^2(\omega t \mp \beta z)\vec{a_z} \qquad z \gtrless 0$$

Notice the Triad (permutations)

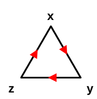
$$\hat{S} = \hat{E} \times \hat{H}$$
$$\hat{E} = \hat{H} \times \hat{S}$$
$$\hat{H} = \hat{S} \times \hat{E}$$



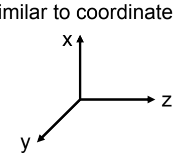
Very useful for finding directions



$$\hat{a}_z = \hat{a}_x \times \hat{a}_y$$
$$\hat{a}_x = \hat{a}_y \times \hat{a}_z$$
$$\hat{a}_y = \hat{a}_z \times \hat{a}_x$$



Similar to coordinate axes



Challenge problem: Finding field directions

- If **H**=$H_0 \cos(6\pi x 10^8 t + 2\pi y)$ $\mathbf{a}_x$ A/m, then the directions of: (1) **H** at t=0, y=0, (2) propagation, and (3) **E** at t=0, y=0 are:

(a) $\mathbf{a}_x$, $-\mathbf{a}_y$, $-\mathbf{a}_x$
(b) $\mathbf{a}_x$, $\mathbf{a}_y$, $\mathbf{a}_z$
(c) $\mathbf{a}_x$, $\mathbf{a}_y$, $-\mathbf{a}_x$
(d) $\mathbf{a}_x$, $-\mathbf{a}_y$, $-\mathbf{a}_z$
(e) $\mathbf{a}_x$, $\mathbf{a}_y$, $-\mathbf{a}_z$

From D3.15 (p 178) of old book

Example: Antenna Array

- An antenna array consists of two or more antenna elements spaced appropriately and excited with currents of appropriate amplitude and phase. Find **E** everywhere if:

  $\mathbf{J}_{s1} = -J_{so}\cos\omega t\ \mathbf{a}_x$ at z=0
  $\mathbf{J}_{s2} = -J_{so}\sin\omega t\ \mathbf{a}_x$ at z=$\lambda/4$

From Example 3.12 (p 177) of old book

Lecture 19 Summary

- Sinusoidal waves

$$\phi = \omega t \mp \beta z \qquad \omega = \frac{\partial \phi}{\partial t} \qquad f = \frac{\omega}{2\pi}$$

$$\beta = \mp \frac{\partial \phi}{\partial z} \qquad \lambda = \frac{2\pi}{\beta} \qquad v_p = \frac{\omega}{\beta}$$

$$v_p = \lambda f \qquad \left|\frac{E}{H}\right| = \eta_0$$

- Poynting vector (power per unit area) is in direction of energy flow

$$\vec{S} = \vec{E} \times \vec{H}$$

- **E**, **H**, **S** form a triad

Lecture 20
Section 4.6

Power Flow/Energy Storage
Poynting's Theorem

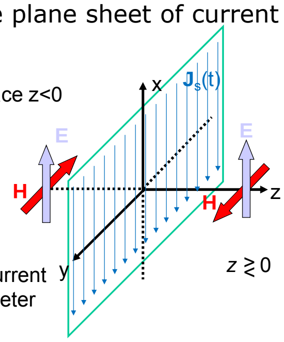
Adapted from Prof. Cunningham's Notes

Field Source

- Infinite plane sheet of current at z=0

Free space z<0

Free space z>0



$\vec{J}_s = -J_x(t)\vec{a}_x$

At z=0

$\vec{E} = E_x(z,t)\vec{a}_x$

$\vec{H} = H_y(z,t)\vec{a}_y$

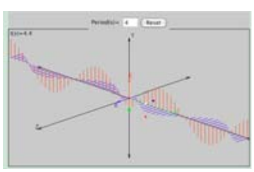
J_s = surface current in Amps per meter

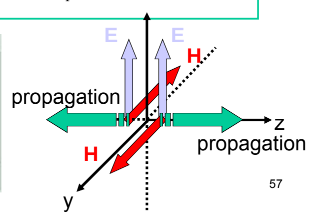
Combining the expressions…

$$\vec{E}(z,t) = \frac{\eta_0}{2} J_S(t \mp \frac{z}{v_p})\vec{a}_x$$

$$\vec{H}(z,t) = \pm\frac{1}{2} J_S(t \mp \frac{z}{v_p})\vec{a}_y$$

$z \gtrless 0$





Definition: Poynting Vector

The POWER provided by the current source is used to generate the propagating **E** and **H** fields.

The **E** and **H** fields are carrying power with them as they propagate

$$\vec{S} = \vec{E} \times \vec{H}$$

Definition for the Power Flow Density of an EM Field

Units for **S**: Watts/m^2

Integral & Differential Forms

$$\vec{S} = \vec{E} \times \vec{H}$$

"Instantaneous" Poynting vector

We can calculate power density magnitude and direction for any single place and time if we know **E** and **H** at that place and time.

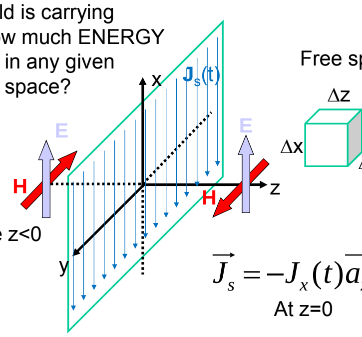
$$\oint\!\!\oint_S \vec{S} \bullet d\vec{S} = \oint\!\!\oint_S \left(\vec{E} \times \vec{H}\right) \bullet d\vec{S}$$

Power flow **out** of a CLOSED surface (units = Watts)

There's ENERGY in the field!

If the EM field is carrying POWER, how much ENERGY is contained in any given VOLUME of space?

Free space z>0



Free space z<0

$\vec{J}_s = -J_x(t)\vec{a}_x$

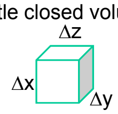
At z=0

How much ENERGY?

$$\vec{S}=\vec{E}\times\vec{H}=|E_x||H_y|\hat{a}_z$$

Use integral form to get a special case of Poynting's Theorem and calculate the power flow OUT of our little closed volume

$$\oint\!\!\oint_S \vec{S}\bullet d\vec{S}=$$



How much energy is "stored" in **E** and **H** for this volume dV?

$$\oint\!\!\oint_S \vec{S}\bullet d\vec{S}=\frac{\partial[E_xH_y]}{\partial z}\Delta V=$$

Hint: Use slide 14 result:

$$\frac{\partial E_x}{\partial z}=-\frac{\partial B_y}{\partial t}=-\mu_0\frac{\partial H_y}{\partial t}$$

$$\frac{\partial H_y}{\partial z}=-J_x-\frac{\partial D_x}{\partial t}=-J_x-\varepsilon_0\frac{\partial E_x}{\partial t}$$

A question we may ask on an exam

The power is not a constant value as a function of time. Remember - it has that $\cos^2(t)$ dependence.

What is the TIME-AVERAGED power being carried by the EM field?

Do derivation - it contains a trig identity

Power calculations

- For **H**=$H_0\cos(6\pi\text{x}10^7t - 0.2\pi z)$ $\mathbf{a}_y$ A/m, find:

(a) the instantaneous power flow across a 1 m^2 area, A, in the z=0 plane at t=0

(b) the instantaneous power across A at t=1/8μs

(c) the time averaged power flow across A

From D3.19 (p 191) of old book

Challenge Question: Power flow

- For **H**=$H_0\cos(6\pi\text{x}10^7t - 0.2\pi z)$ $\mathbf{a}_y$ A/m, and $A_{@z0}$ is a $1m^2$ area at the plane $z=z_0$ which statement is true:

(a) the instantaneous power flow crossing $A_{@1m}$ in the $\mathbf{a}_z$ direction can be negative

(b) the time averaged power flow across $A_{@1m}$ in the $\mathbf{a}_z$ direction can be negative

(c) the instantaneous power flow across $A_{@z0}$ at t=0 depends on the position z_0

(d) the time averaged power flow across $A_{@z0}$ at t=0 depends on the position z_0

LG's question

Poynting's Theorem

Begin with: $\nabla\bullet\vec{S}=\nabla\bullet(\vec{E}\times\vec{H})=\vec{H}\bullet(\nabla\times\vec{E})-\vec{E}\bullet(\nabla\times\vec{H})$

Poynting's Theorem

Thus, $\nabla\bullet\vec{S}=-\frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon_0E^2\right)-\frac{\partial}{\partial t}\left(\frac{1}{2}\mu_0H^2\right)-\vec{E}\bullet\vec{J}$

$$\nabla\bullet\vec{S}+\vec{E}\bullet\vec{J}=-\frac{\partial u_e}{\partial t}-\frac{\partial u_m}{\partial t}$$

$$\boxed{-\frac{\partial}{\partial t}(u_e+u_m)=\nabla\bullet\vec{S}+\vec{E}\bullet\vec{J}}$$

$$u_e=\frac{1}{2}\varepsilon_0E^2,\quad u_m=\frac{1}{2}\mu_0H^2$$

u_e=electric field energy density
u_m=magnetic field energy density

Poynting's Theorem

$$-\frac{\partial}{\partial t}(u_m + u_e) = \nabla \bullet \vec{S} + \vec{E} \bullet \vec{J}$$

Integrate over the volume and Apply Divergence Theorem: $\oint\!\!\oint_S \vec{S} \bullet d\vec{S} = \iiint_V \nabla \bullet \vec{S}\ dV$

$$\boxed{-\frac{\partial}{\partial t}\iiint_V (u_m + u_e) dV = \oint\!\!\oint_S \vec{S} \bullet d\vec{S} + \iiint_V \vec{E} \bullet \vec{J}\ dV}$$

Rate the fields **LOSE** energy = Power flow **OUT** of surface + Rate of work done **BY** the fields

↑ Can be + or -

↑ Can be + or -

↑ **E·J** is non-negative: J=σE so **E·J** = σ|E|² ≥ 0 Zero only when σ=0, i.e. in a perfect dielectric

Rate of work done by the fields

$\vec{E}$ → ⊕ $dq = \rho dV$

$\vec{E}$ → $d\vec{l}$ ⊕ $d\vec{l} = \vec{v}dt$

Work done moving dq a distance dl is dW=**dFdl**

$$d\vec{F} = dq\vec{E}$$

Rate of work done **BY** the fields moving a small charge dq:

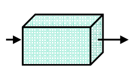
$$\frac{dW}{dt} = d\vec{F} \cdot \frac{d\vec{l}}{dt} = d\vec{F} \cdot \vec{v} = dq\vec{E} \cdot \vec{v} = \vec{E} \cdot (\rho\vec{v})dV = (\vec{E} \cdot \vec{J})dV$$

$$\therefore \iiint_V \vec{E} \bullet \vec{J}\ dV = \text{Rate of work done by the fields (Joule heating)}$$

Poynting's Theorem for Perfect Dielectric

$$\vec{J} = \sigma\vec{E} = 0 \quad \Rightarrow \quad \oint\!\!\oint_S \vec{S} \bullet d\vec{S} = -\frac{\partial}{\partial t}\iiint_V (u_m + u_e) dV$$

If net power flows out, the energy stored inside must decrease



$P_{out}^{net} > 0$

$$\text{Energy stored} = \iiint_V (u_m + u_e) dV \text{ is decreasing}$$

Challenge Question: Poynting's Theorem

- If **H**=H_0cos(ct + x) $\mathbf{a}_y$ A/m and **E**=$\eta_0 H_0$cos(ct + x) $\mathbf{a}_z$ V/m in the ***free space*** region x>0, which statement is **true** for V, the volume bounded by the x=0 and x=1 planes

(a) the net outward power flow is zero at all times

(b) the fields do work and thus lose energy

(c) the total electric field energy inside is constant in time

(d) the time averaged electric field energy density at each position x is constant in time

(e) the total electric and magnetic field energy density at fixed position x is constant in time

LG's question

Time Averaged Poynting Vector

$$\vec{E} = \text{Re}\left[\tilde{E}e^{jwt}\right] = \frac{\tilde{E}e^{jwt} + \tilde{E}^* e^{-jwt}}{2}$$

$$\langle \vec{S} \rangle = \langle \vec{E} \times \vec{H} \rangle = \left\langle \frac{\tilde{E}e^{jwt} + \tilde{E}^* e^{-jwt}}{2} \times \frac{\tilde{H}e^{jwt} + \tilde{H}^* e^{-jwt}}{2} \right\rangle$$

$$= \frac{1}{4}\left\langle \tilde{E} \times \tilde{H}e^{2jwt} + \tilde{E} \times \tilde{H}^* + \tilde{E}^* \times \tilde{H} + \tilde{E}^* \times \tilde{H}^* e^{-2jwt} \right\rangle$$

$$\boxed{\langle \vec{S} \rangle = \frac{1}{2}\text{Re}\left[\tilde{E} \times \tilde{H}^*\right]} = \frac{1}{2}\text{Re}\left[(\eta\tilde{H})\tilde{H}^*\right]\hat{S} = \frac{1}{2}|H|^2\ \text{Re}[\eta]\hat{S}$$

$$\boxed{\text{"}\tilde{E} = \eta\tilde{H}\text{" but } \tilde{E}, \tilde{H} \text{ point in different directions}}$$

Lecture 20 Summary

- Poynting's Theorem:
  - Rate that stored field energy is lost = rate that energy flows out boundary surface + rate that the fields do work (Joule heating)

$$-\frac{\partial}{\partial t}\iiint_V (w_e + w_m) dV = \oint\!\!\oint_S \vec{S} \bullet d\vec{S} + \iiint_V \vec{E} \bullet \vec{J} dV$$

  - Time averaged Poynting vector

$$\langle \vec{S} \rangle = \frac{1}{2}\text{Re}\left[\tilde{E} \times \tilde{H}^*\right]$$

- Next up: Waves in materials
  - Sections 5.3-5.4

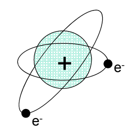
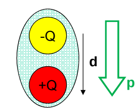
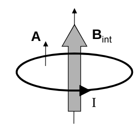
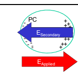
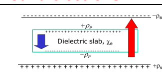
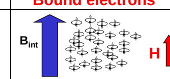
Lectures 21-24
Sections 5.3-5.4

Plane Waves in Materials

Also Section 1.4: Polarization

Adapted from Prof. Cunningham's Notes

3 types of materials

Conductors	Dielectrics	Magnetic
 Free electrons	 Polarized atoms/molecules Bound electrons	 Magnetic moments Bound electrons
		
E=0 inside ρ=0 inside $\rho=\rho_s$ only surface charge V is same throughout $\mathbf{E}_{outside}$ is $\perp$ to surface	**E**≠0 inside but it is reduced $\mathbf{E}_{tot}=\mathbf{E}_a+\mathbf{E}_s$ $\mathbf{D}=\varepsilon\mathbf{E}_{tot}=\mathbf{P}+\varepsilon_0\mathbf{E}_{tot}$	$\mathbf{B}_{tot}=\mathbf{B}_a+\mathbf{B}_s$ $\mathbf{B}_{tot}=\mu\mathbf{H}=\mu_0(\mathbf{H}+\mathbf{M})$

New Relations

$$\vec{J}=\sigma\vec{E}$$

$$\vec{D}=\varepsilon\vec{E} \qquad \varepsilon=\varepsilon_0\varepsilon_r$$

$$\vec{B}=\mu\vec{H} \qquad \mu=\mu_0\mu_r$$

Inside a material, Maxwell's Equations become:

Free Space	Inside Material
$\vec{\nabla}\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t}$	$\vec{\nabla}\times\vec{H}=\sigma\vec{E}+\varepsilon\frac{\partial\vec{E}}{\partial t}$
$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$	$\vec{\nabla}\times\vec{E}=-\mu\frac{\partial\vec{H}}{\partial t}$
$\vec{\nabla}\bullet\vec{D}=\rho$	$\vec{\nabla}\bullet(\varepsilon\vec{E})=\rho$
$\vec{\nabla}\bullet\vec{B}=0$	$\vec{\nabla}\bullet\vec{H}=0$

A single material can have conductive, dielectric, and magnetic properties AT THE SAME TIME

$$\sigma\neq 0$$
$$\varepsilon_r\neq 1$$
$$\mu_r\neq 1$$

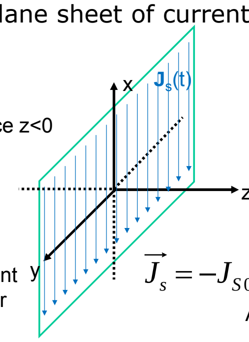
How does an EM wave propagate through this?

Field Source

- Infinite plane sheet of current at z=0

Free space z<0

Free space z>0



J_s = surface current in Amps per meter

$$\vec{J}_s=-J_{S0}\cos(\omega t)\vec{a}_x$$

At z=0

Solution

$$\overrightarrow{E}(z,t) = \frac{\eta_0 J_{S0}}{2}\cos(\omega t \mp \beta z)\overrightarrow{a_x}$$

$$\overrightarrow{H}(z,t) = \pm\frac{J_{S0}}{2}\cos(\omega t \mp \beta z)\overrightarrow{a_y}$$

$z \gtrless 0$

$$\beta = \frac{\omega}{v_p}$$

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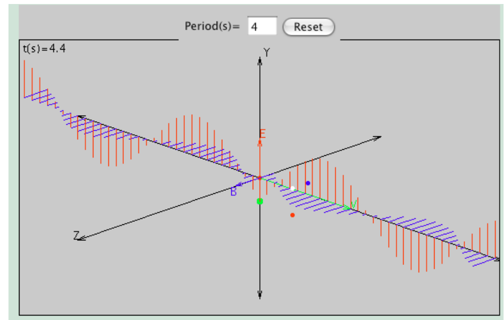
Web Demo



http://www.phy.ntnu.edu.tw/java/emWave/emWave.html 8

Key characteristics in FREE SPACE

- No attenuation
- E_x and H_y are IN PHASE for linear polarization
- Impedance: $|\vec{E}| = \eta_0 |\vec{H}|$
- Travel at Speed of Light: $v_p = c = \omega/\beta$
- Perpendicular: **E** ⊥ **H** ⊥ **S**

$$\vec{S} = \vec{E} \times \vec{H}$$

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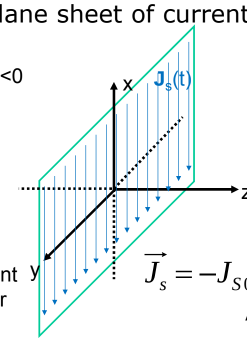
Field Source in a MATERIAL

- Infinite plane sheet of current at z=0

Material z<0
σ, μ, ε

Material z>0
σ, μ, ε



J_s = surface current in Amps per meter

$$\overrightarrow{J_s} = -J_{S0}\cos(\omega t)\overrightarrow{a_x}$$

At z=0

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Similar Procedure for Maxwell's Equations

- We will <u>SIMULTANEOUSLY</u> solve Maxwell's equations to find **E** and **H** caused by **J**
  - Note: **J**=σ**E**≠0 inside the material

$$\nabla \times \vec{E} = -\mu\frac{\partial \vec{H}}{\partial t}$$

$$\nabla \times \vec{H} = \sigma\vec{E} + \varepsilon\frac{\partial \vec{E}}{\partial t}$$

$$\overrightarrow{E} = E_x(z,t)\overrightarrow{a_x}$$

$$\overrightarrow{H} = H_y(z,t)\overrightarrow{a_y}$$

$$\overrightarrow{J_s} = -J_x(t)\overrightarrow{a_x}$$

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Apply the two Maxwell's Equations

- Performing the cross product, only two equations contain E_x or H_y

$$\frac{\partial E_x}{\partial z} = -\mu\frac{\partial H_y}{\partial t}$$

$$\frac{\partial H_y}{\partial z} = \boxed{-\sigma E_x} - \varepsilon\frac{\partial E_x}{\partial t}$$

The key difference

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Reminder from Lecture 11

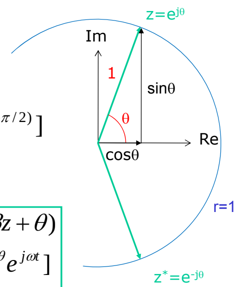
Phasor Review

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\cos(\theta) = \mathrm{Re}[e^{j\theta}]$$

$$\sin(\theta) = \mathrm{Re}[-je^{j\theta}] = \mathrm{Re}[e^{j(\theta-\pi/2)}]$$

$$\mathrm{Re}[z] = (z+z^*)/2$$



$$\boxed{\begin{aligned} E_x(z,t) &= E_x(z)\cos(\omega t \mp \beta z + \theta) \\ &= \mathrm{Re}[E_x(z)e^{\mp j\beta z}e^{j\theta}e^{j\omega t}] \\ &= \mathrm{Re}[\overline{E}_x(z)e^{j\omega t}] \end{aligned}}$$

Solve PDEs with Phasors

- Technique simplifies the algebra

$$E_x(z,t) = \mathrm{Re}[\widetilde{E}_x(z)e^{j\omega t}] \qquad H_y(z,t) = \mathrm{Re}[\widetilde{H}_y(z)e^{j\omega t}]$$

$$\frac{\partial E_x}{\partial t} = \mathrm{Re}[j\omega\widetilde{E}_x(z)e^{j\omega t}] \qquad \boxed{\frac{\partial}{\partial t} \equiv j\omega}$$

$$\left.\begin{aligned} \frac{\partial E_x}{\partial z} &= -\mu\frac{\partial H_y}{\partial t} \\ \frac{\partial H_y}{\partial z} &= \boxed{-\sigma E_x} - \varepsilon\frac{\partial E_x}{\partial t} \end{aligned}\right\} \quad \begin{aligned} \frac{\partial \widetilde{E}_x}{\partial z} &= -\mu(j\omega)\widetilde{H}_y \\ \frac{\partial \widetilde{H}_y}{\partial z} &= -\sigma\widetilde{E}_x - \varepsilon(j\omega)\widetilde{E}_x \end{aligned}$$

Differentiate again in z

$$\left.\begin{aligned} \frac{\partial \widetilde{E}_x}{\partial z} &= -\mu(j\omega)\widetilde{H}_y \\ \frac{\partial \widetilde{H}_y}{\partial z} &= -\sigma\widetilde{E}_x - \varepsilon(j\omega)\widetilde{E}_x \end{aligned}\right\} \quad \begin{aligned} \frac{\partial^2 \widetilde{E}_x}{\partial z^2} &= -\mu(j\omega)\frac{\partial \widetilde{H}_y}{\partial z} \\ &= -\mu(j\omega)[-\sigma\widetilde{E}_x - \varepsilon(j\omega)\widetilde{E}_x] \\ &= j\omega\mu(\sigma + j\omega\varepsilon)\widetilde{E}_x \end{aligned}$$

$$\boxed{\frac{\partial^2 \widetilde{E}_x}{\partial z^2} = \overline{\gamma}^2\widetilde{E}_x}$$

$$\overline{\gamma} \equiv \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} = \alpha + j\beta$$

Solve simple PDE and re-write Phasors as Cosines

$$\frac{\partial^2 \widetilde{E}_x}{\partial z^2} = \overline{\gamma}^2\widetilde{E}_x \qquad \boxed{\widetilde{E}_x(z) = \overline{A}e^{-\overline{\gamma}z} + \overline{B}e^{+\overline{\gamma}z}} \qquad \overline{\gamma} = \alpha + j\beta$$

$$\begin{aligned} \widetilde{E}_x(z) &= \overline{A}e^{-\alpha z}e^{-j\beta z} + \overline{B}e^{\alpha z}e^{j\beta z} \\ &= Ae^{j\theta}e^{-\alpha z}e^{-j\beta z} + Be^{j\phi}e^{\alpha z}e^{j\beta z} \end{aligned}$$

$$\vec{E}_x(z,t) = \mathrm{Re}[\widetilde{E}_x(z)e^{j\omega t}]$$

$$\boxed{\vec{E}_x(z,t) = \underbrace{Ae^{-\alpha z}\cos(\omega t - \beta z + \theta)}_{z>0} + \underbrace{Be^{+\alpha z}\cos(\omega t + \beta z + \phi)}_{z<0}}$$

Also Solve for H_y

$$\boxed{\vec{E}_x(z,t) = \underbrace{Ae^{-\alpha z}\cos(\omega t - \beta z + \theta)}_{z>0} + \underbrace{Be^{+\alpha z}\cos(\omega t + \beta z + \phi)}_{z<0}}$$

$$\frac{\partial \widetilde{E}_x}{\partial z} = -\mu(j\omega)\widetilde{H}_y \qquad \overline{\eta} \equiv \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = |\overline{\eta}|e^{j\tau}$$

$$\boxed{\widetilde{H}_y(z,t) = \underbrace{\frac{A}{|\overline{\eta}|}e^{-\alpha z}\cos(\omega t - \beta z + \theta - \tau)}_{z>0} - \underbrace{\frac{B}{|\overline{\eta}|}e^{+\alpha z}\cos(\omega t + \beta z + \phi - \tau)}_{z<0}}$$

Apply Faraday's Law to closed path cutting through sheet that is parallel to current flow

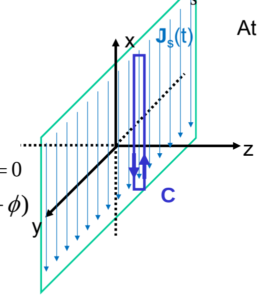
$$\oint_C \vec{E} \bullet \vec{dl} = -\frac{d}{dt}\int_S \vec{B} \bullet \vec{dS}$$

$$\oint_C \vec{E} \bullet \vec{dl} = 0$$

$$\boxed{\vec{E}_{\parallel} \text{ is continuous}}$$

$$\vec{J}_s = -J_s(t)\vec{a}_x$$

At z=0



$$E_x(z=0^+)\Delta x - E_x(z=0^-)\Delta x = 0$$

$$A\cos(\omega t + \theta) = B\cos(\omega t + \phi)$$

$$\therefore A = B \text{ and } \theta = \phi$$

Apply Ampere's Law to closed path cutting through sheet that is perpendicular to current

$$\oint_C \vec{H} \bullet d\vec{l} = I_{enclosed} + \frac{d}{dt}\iint_S \varepsilon_0 \vec{E} \bullet d\vec{S}$$

$$\oint_C \vec{H} \bullet d\vec{l} = I_{enclosed}$$

$$\vec{J}_s = -J_s(t)\vec{a}_x$$

At z=0

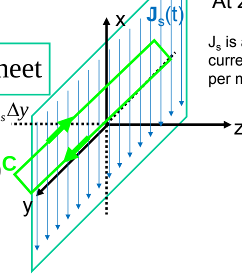
J_s is a surface current in Amps per meter

$\boxed{\vec{H}_{\parallel} \text{ is discontinuous at sheet}}$

$$H_y(z=0^+)\Delta y - H_y(z=0^-)\Delta y = J_s \Delta y$$

$$\frac{2A}{|\eta|}\cos(\omega t + \theta - \tau) = J_{S0}\cos(\omega t)$$

$$\therefore A = \frac{|\eta| J_{S0}}{2} \text{ and } \theta = \tau$$



Final Solution for E_x and H_y

$$\vec{E}(z,t) = \frac{|\eta| J_{S0}}{2} e^{\mp \alpha z}\cos(\omega t \mp \beta z + \tau)\hat{a}_x$$

$$\vec{H}(z,t) = \frac{\pm J_{S0}}{2} e^{\mp \alpha z}\cos(\omega t \mp \beta z)\hat{a}_y$$

$z \gtrless 0$

Strength of fields drops exponentially according to the attenuation constant

Magnitude of E and H related through magnitude of the complex impedance, $\bar{\eta}$

E and H are out of phase by the phase of the complex impedance, $\tau = \arg(\bar{\eta})$

Phasor Solution for E_x and H_y

$$J_x(z=0,t) = \mathrm{Re}[-\tilde{J}_{S0}e^{j\omega t}]$$

$$E_x(z,t) = \mathrm{Re}[\tilde{E}_x(z)e^{j\omega t}]$$

$$H_y(z,t) = \mathrm{Re}[\tilde{H}_y(z)e^{j\omega t}]$$

$$\tilde{E}_x(z) = \frac{\bar{\eta}\tilde{J}_{S0}}{2} e^{\mp \bar{\gamma} z}\hat{a}_x$$

$$\tilde{H}_y(z) = \frac{\pm \tilde{J}_{S0}}{2} e^{\mp \bar{\gamma} z}\hat{a}_y$$

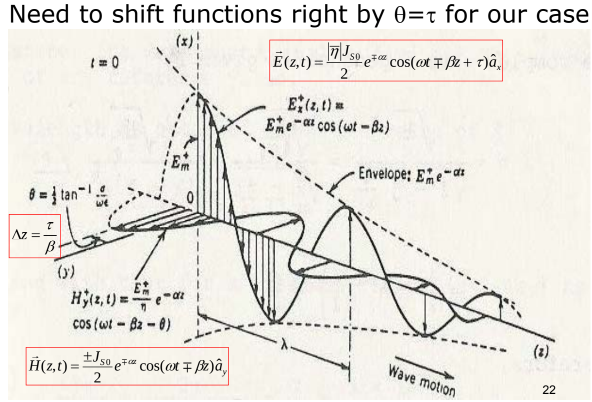
$z \gtrless 0$

$\tilde{H}$ is in phase with the current source

$\boxed{\text{"} \tilde{E}_x = \bar{\eta}\tilde{H}_y \text{"}}$

holds for amplitude and for phase
but they point in different directions

In this drawing, the current sheet is not at z=0
Need to shift functions right by $\theta = \tau$ for our case



What is the same?

- Frequency of the wave = ω
- Perpendicular: **E** $\perp$ **H** $\perp$ **S**

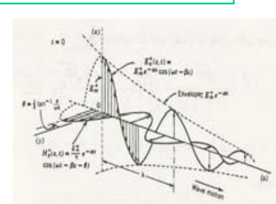
$$\vec{S} = \vec{E} \times \vec{H}$$

What is different?

1. The magnitude of the wave is ATTENUATED

Gets weaker by $e^{-\alpha z}$ for a wave travelling in +z

α = "Attenuation Constant"
Units = 1/m or Np/m or dB/m



What is different?

2. Magnitude relationship between E and H

$$|\vec{E}| = |\overline{\eta}||\vec{H}|$$

$$|\overline{\eta}| \neq \eta_0$$

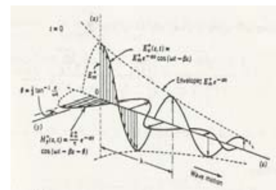
$\overline{\eta}$ = "Complex Impedance" of the material (because it is a complex number)

$|\overline{\eta}|$ = Magnitude of the complex impedance (Units = Ohms)

What is different?

3. **E** and **H** are OUT OF PHASE by τ

τ = "Phase Offset"
Units = radians



What is different?

4. Speed of propagation is no longer c in free space

$$v_p = \omega/\beta \qquad \text{Is still good}$$

But β will be related to properties of the material and actually, β can depend on ω

If relation $\beta(\omega)$ is nonlinear, then the material has dispersion (different frequencies travel at different speeds, e.g. colors separate in a prism due to this dispersion) and thus a pulse (Fourier series - sum of waves) can change its shape as the pulse propagates

Challenge Question: Propagation in σ=0 medium

- Given the phasors: $\tilde{E}_x(z) = \frac{\overline{\eta}\tilde{J}_{S0}}{2} e^{\mp\overline{\gamma}z}\hat{a}_x \quad \tilde{H}_y(z) = \frac{\pm\tilde{J}_{S0}}{2} e^{\mp\overline{\gamma}z}\hat{a}_y$

and definitions: $\overline{\eta} \equiv \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = |\overline{\eta}|e^{j\tau} \quad \overline{\gamma} \equiv \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} = \alpha + j\beta$

If σ=0 and $\varepsilon=2\varepsilon_0$,
consider the veracity of the statements:

I. The fields will attenuate as they propagate
II. **E** and **H** will be out of phase
III. The velocity of propagation will be c

(a) I only, (b) II only, (c) III only, (d) I, II, and III, (e) none are true

LG's question

Lecture 21-22a Summary

- Differentiate Maxwell's Equations

$$\vec{\nabla} \times \vec{E} = -\mu\frac{\partial \vec{H}}{\partial t} \qquad \vec{\nabla} \times \vec{H} = \sigma\vec{E} + \varepsilon\frac{\partial \vec{E}}{\partial t}$$

to get complex wave equation: $\frac{\partial^2 \tilde{E}_x}{\partial z^2} = \overline{\gamma}^2 \tilde{E}_x$

which has decaying solutions:

$$\vec{E}(z,t) = \frac{|\overline{\eta}|J_{S0}}{2} e^{\mp\alpha z}\cos(\omega t \mp \beta z + \tau)\hat{a}_x \qquad \tilde{E}_x(z) = \frac{\overline{\eta}\tilde{J}_{S0}}{2} e^{\mp\overline{\gamma}z}\hat{a}_x$$

$$\vec{H}(z,t) = \frac{\pm J_{S0}}{2} e^{\mp\alpha z}\cos(\omega t \mp \beta z)\hat{a}_y \qquad \tilde{H}_y(z) = \frac{\pm\tilde{J}_{S0}}{2} e^{\mp\overline{\gamma}z}\hat{a}_y$$

ECE 329
Lectures 22b-23

Plane Waves in Materials

Adapted from Prof. Cunningham's Notes

Hmm, Almost the entire Greek Alphabet!!!

σ
ε
μ

These constants are material properties and affect wave propagation: α, β, γ, τ, η

α
β
γ
τ
$\overline{\eta}$

Wave Equation for E_x

$$\frac{\partial^2 \tilde{E}_x}{\partial z^2} = \underbrace{j\omega\mu(\sigma + j\omega\varepsilon)}_{\text{(Propagation Constant)}^2}\tilde{E}_x$$

(Propagation Constant)^2
Most important definition from last class

$$\frac{\partial^2 \tilde{E}_x}{\partial z^2} = \overline{\gamma}^2 \tilde{E}_x$$

For wave in free space, wave eqn was: $\frac{\partial^2 E_x}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_x}{\partial t^2}$

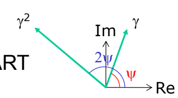
Propagation Constant

$$\frac{\partial^2 \tilde{E}_x}{\partial z^2} = \overline{\gamma}^2 \tilde{E}_x$$

$$\overline{\gamma} = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} = \alpha + j\beta = |\gamma| e^{j\psi}$$

It's a COMPLEX NUMBER

REAL PART + IMAGINARY PART



$\text{Re}[\overline{\gamma}^2] < 0, \ \text{Im}[\overline{\gamma}^2] > 0$

$\therefore \ \overline{\gamma}^2$ is in Quadrant II $\Rightarrow \ 45^\circ \le \psi \le 90^\circ$

$\therefore \ \beta \ge \alpha \ge 0$

Propagation Constant

$$\overline{\gamma} = \alpha + j\beta = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$$

Calculating $\overline{\gamma}^2$ gives 2 equations and 2 unknowns α, β

$$\overline{\gamma}^2 = \alpha^2 - \beta^2 + 2j\alpha\beta = -\omega^2\mu\varepsilon + j\omega\mu\sigma$$

$$\therefore \ \alpha^2 - \beta^2 = -\omega^2\mu\varepsilon \ \text{ and } \ 2\alpha\beta = \omega\mu\sigma$$

Solving for α and β using Mathematica:

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}}\left(\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right)^{1/2}$$ Attenuation Constant

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}}\left(\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right)^{1/2}$$ Phase Constant

The key relationships

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}}\left(\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right)^{1/2}$$

attenuation
$e^{-\alpha z}$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}}\left(\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right)^{1/2}$$

velocity
$v_p = \omega/\beta$

ATTENUATION AND VELOCITY ARE A FUNCTION OF ω

Complex Material Impedance

$$\overline{\eta} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}$$

$$\overline{\eta} = |\overline{\eta}| e^{j\tau}$$

Magnitude of Complex Impedance

Phase difference between E_x and H_y

Solving for $|\eta|$ and τ

$$\frac{\partial^2 \tilde{E}_x}{\partial z^2} = \bar{\gamma}^2 \tilde{E}_x$$

$$\boxed{\tilde{E}_x(z) = \begin{cases} \bar{A}e^{-\bar{\gamma}z} & z > 0 \\ \bar{B}e^{+\bar{\gamma}z} & z < 0 \end{cases}}$$

$$\frac{\partial \tilde{E}_x}{\partial z} = -\mu(j\omega)\tilde{H}_y$$

$$\tilde{H}_y(z) = \begin{cases} \dfrac{-\bar{\gamma}\bar{A}e^{-\bar{\gamma}z}}{-j\omega\mu} = \dfrac{\bar{\gamma}\tilde{E}}{j\omega\mu} = \dfrac{\tilde{E}}{\bar{\eta}} & z > 0 \\[2ex] \dfrac{+\bar{\gamma}\bar{B}e^{+\bar{\gamma}z}}{-j\omega\mu} = -\dfrac{\tilde{E}}{\bar{\eta}} & z < 0 \end{cases}$$

$$\boxed{\therefore\ \tilde{E} = \pm\bar{\eta}\tilde{H} \text{ where } \bar{\eta} = \frac{j\omega\mu}{\bar{\gamma}}}$$

$$\bar{\gamma} \equiv \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$$

$$\boxed{\therefore \bar{\eta} \equiv \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}}$$

$$\boxed{|\eta| = \sqrt{\frac{\omega\mu}{\sqrt{\sigma^2 + (\omega\varepsilon)^2}}}}$$

$$\bar{\eta}^2 = |\bar{\eta}|^2 e^{j2\tau} = \frac{j\omega\mu(\sigma - j\omega\varepsilon)}{\sigma^2 + (\omega\varepsilon)^2} = \frac{\omega\mu(\omega\varepsilon + j\sigma)}{\sigma^2 + (\omega\varepsilon)^2}$$

$$\boxed{\therefore \tau = \frac{1}{2}\tan^{-1}\left(\frac{\sigma}{\omega\varepsilon}\right)}$$

Propagation in Dielectric

- For a uniform plane wave with f=10^6 Hz in a nonmagnetic medium ($\mu=\mu_0$), the propagation constant is: $\gamma=0.05+0.1j$ m^{-1}. Find:

(a) Distance for field to attenuate by e^{-1}

(b) Distance for field to change phase by 1 rad

(c) Distance that constant phase moves in 1μs

(d) The ratio of |**E**| to |**H**|

(e) The phase difference between **E** and **H**

From D4.8 (p249) of old book

Power flow in Dielectric

- Given $\mu=\mu_0$ and $\vec{H} = H_0 e^{-z}\cos(6\pi 10^7 t - \sqrt{3}z)\hat{a}_y$, find:

(a) The instantaneous power flow across A=1m^2 in the z=0 plane at t=0.

(b) The time averaged power flow across $A_{@z=0}$

(c) The time averaged power flow across $A_{@z=1}$

From D4.9 (p249) of old book

Perfect Dielectric Material

Definition of a PERFECT dielectric: $\boxed{\sigma = 0}$

Propagation Constant:

$$\bar{\gamma} = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$$

$$\bar{\gamma} = j\omega\sqrt{\mu\varepsilon}$$

Perfect Dielectric Material

$$\bar{\gamma} = j\omega\sqrt{\mu\varepsilon}$$

$$\bar{\gamma} = \alpha + j\beta$$

$\alpha = 0$

NO ATTENUATION

$$\beta = \omega\sqrt{\mu\varepsilon}$$

$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\varepsilon}}$$

Speed of wave is less than in free space:

$$\boxed{v_p = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu_0\mu_r\varepsilon_0\varepsilon_r}} = \frac{c}{\sqrt{\mu_r\varepsilon_r}}}$$

Perfect Dielectric Material

$$\bar{\eta} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}}$$

Is a REAL number

$$\bar{\eta} = |\bar{\eta}|e^{j\tau}$$

$\tau = 0$

E and **H** are IN PHASE

$$\boxed{|\bar{\eta}| = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_0\mu_r}{\varepsilon_0\varepsilon_r}} = \eta_0\sqrt{\frac{\mu_r}{\varepsilon_r}}}$$

Impedance is different than free space - could be greater or less, depending on material

Finding parameters

- Given $\mu=\mu_0$ and $\vec{E}=10\cos(3\pi 10^7 t-0.2\pi x)\hat{a}_z$, find:

(a) The frequency

(b) The wavelength

(c) The phase velocity

(d) The relative permittivity

(e) The associated **H** field

From P4.18 (p275) of old book

Imperfect Dielectric

Definition: $\sigma \neq 0$

But material does not conduct enough to really be considered a conductor

So, how much conductivity is "enough"?

Look at the **loss tangent:** $\dfrac{\sigma}{\omega\varepsilon}$

σ → conductivity

$\omega\varepsilon$ → dielectric

Imperfect Dielectric

$$\boxed{\frac{\sigma}{\omega\varepsilon} << 1}$$

Official math definition for what constitutes an imperfect dielectric (small loss tangent)

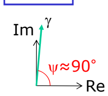
Very important definition

Also valid definitions:

$$\boxed{\begin{array}{l}\psi \approx 90^\circ \\ \alpha << \beta\end{array}}$$

$$\frac{\partial^2 \tilde{E}_x}{\partial z^2} = j\omega\mu(\sigma + j\omega\varepsilon)\tilde{E}_x$$

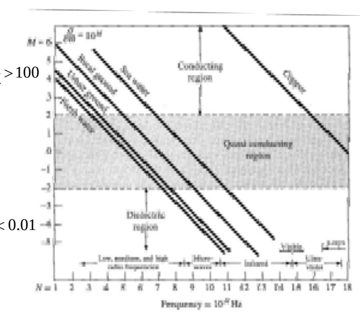
Tells which term is dominant



Behaves like a Dielectric or Conductor depending on ω

$\dfrac{\sigma}{\omega\varepsilon} > 100$

$\dfrac{\sigma}{\omega\varepsilon} < 0.01$



Imperfect Dielectric: How is it different from perfect dielectric?

Only important difference: There IS attenuation in an imperfect dielectric

$$\alpha \neq 0$$

$$\alpha \approx \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$$

Everything else is the same as in a perfect dielectric:

$$\underbrace{\beta \approx \omega\sqrt{\mu\varepsilon} \qquad v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\varepsilon}} \qquad \lambda = 2\pi/\beta}$$

Pretty good approximation as long as $\dfrac{\sigma}{\omega\varepsilon} < 0.1$

Good Conductors

Definition: $\sigma \neq \infty$

But conductivity is still large

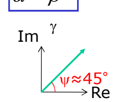
$$\boxed{\frac{\sigma}{\omega\varepsilon} >> 1}$$

Also valid definitions:

$$\boxed{\begin{array}{l}\psi \approx 45^\circ \\ \alpha \approx \beta\end{array}}$$

Official math definition of a good conductor

Large loss tangent



Note that any material with $\sigma>0$ behaves like a good conductor at low enough frequency

Good Conductor

Propagation Constant

$$\bar{\gamma} = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} \approx \sqrt{j\omega\mu\sigma}$$

$$= \sqrt{j}\sqrt{\omega\mu\sigma}$$

Warning: math trick $\sqrt{j} = \frac{1+j}{\sqrt{2}}$

$$\bar{\gamma} = \sqrt{\frac{\omega\mu\sigma}{2}}(1+j) = \alpha + j\beta$$

$$\boxed{\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}}}$$

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Good Conductor

Complex Impedance

$$\bar{\eta} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} \approx \sqrt{\frac{j\omega\mu}{\sigma}}$$

$$= \sqrt{j}\sqrt{\frac{\omega\mu}{\sigma}}$$

$$\bar{\eta} = \frac{1+j}{\sqrt{2}}\sqrt{\frac{\omega\mu}{\sigma}} = (1+j)\sqrt{\frac{\omega\mu}{2\sigma}}$$

$$\boxed{|\bar{\eta}| = \sqrt{\frac{\omega\mu}{\sigma}}} \qquad \boxed{\tau = \pi/4}$$

E and **H** are 45 deg out of phase

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Good Conductor: Skin Depth

α tells us how far the E and H fields can propagate into a good conductor

In a conductor, α=LARGE, so fields drop off in a short distance

SKIN DEPTH = Distance that the field strength is attenuated by e^{-1}

$$\delta = \frac{1}{\alpha} = \sqrt{\frac{2}{\omega\mu\sigma}}$$

Or $$\delta = \frac{1}{\alpha} = \sqrt{\frac{1}{\pi f\mu\sigma}}$$ Using $\omega = 2\pi f$

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Skin Depth

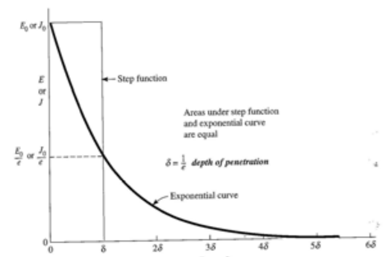

FIGURE 4-10
Relative magnitude of electric field **E** or current density **J** (= σ**E**) as a function of depth of penetration δ for a plane wave traveling in x direction into conducting medium. The abscissa gives the penetration distance x and is expressed in $1/e$ depths (δ). The wavelength in the conductor equals $2\pi\delta$.

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Perfect Conductor

$$\sigma = \infty$$

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}}\left(\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right)^{1/2}$$

$$\boxed{\alpha = \infty} \qquad \boxed{\delta = 0}$$

Infinite attenuation

Means that neither **E** or **H** can propagate into a perfect conductor

NO TIME-VARYING FIELDS (**E** or **H**) CAN EXIST IN A PERFECT CONDUCTOR!!!

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Challenge Question: Material characteristics

- In what type of material would you expect dispersion (velocity depends on frequency)?

(a) free space
(b) perfect dielectric
(c) imperfect dielectric
(d) good conductor
(e) perfect conductor

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LG's question

Finding parameters

- For a uniform plane wave with f=10^5 Hz in a good conductor, the field is attenuated by $e^{-\pi}$ in 2.5m. Find the following:

(a) Distance for a 2π phase change if f=10^5 Hz

(b) Distance a constant phase plane travels in 1μs for f=10^5 Hz

(c) Distance a constant phase plane travels in 1μs for f=10^4 Hz assuming the material properties are the same as at f=10^5 Hz

From D4.11 (p254) of old book

Lecture 21-23 Summary

Perfect Dielectric

Definition: $\boxed{\sigma = 0}$
Attenuation: $\alpha = 0$
Speed: $v_p = c/\sqrt{\mu_r \varepsilon_r} \le c$
E,**H** In Phase: $\tau = 0$
Impedance: $|\bar{\eta}| = \eta_0 \sqrt{\mu_r/\varepsilon_r}$

Imperfect Dielectric

Definition: $\boxed{\sigma/\omega\varepsilon << 1}$
Attenuation: $\boxed{\alpha \approx \sigma/2\sqrt{\mu/\varepsilon}}$
Speed: $v_p \approx c/\sqrt{\mu_r \varepsilon_r} \le c$
E,**H** In Phase: $\tau \approx 0$
Impedance: $|\bar{\eta}| \approx \eta_0 \sqrt{\mu_r/\varepsilon_r}$

Good Conductor

Definition: $\boxed{\sigma/\omega\varepsilon >> 1}$
Attenuation: $\boxed{\alpha \approx \sqrt{\omega\mu\sigma/2}}$
Speed: $\boxed{v_p \approx \sqrt{2\omega/\sigma\mu}}$
E,**H** 45° Phase: $\boxed{\tau \approx \pi/4}$
Impedance: $\boxed{|\bar{\eta}| \approx \sqrt{\omega\mu/\sigma}}$

Perfect Conductor

Definition: $\boxed{\sigma \to \infty}$
Attenuation: $\alpha \to \infty$ $\boxed{\vec{E} \to 0}$
Speed: $v_p \to 0$ $\boxed{\vec{H} \to 0}$
Impedance: $|\bar{\eta}| \to 0$

Lecture 24
Section 1.4

Polarization

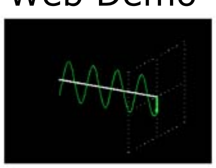
Adapted from Prof. Cunningham's Notes

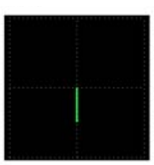
Definition of Polarization

- Describes the "tip" of the time-varying **E** field vector for a particular point in space as it varies in time
- **J** = -$J_{S0}a_x$ example
  - Field with only one component is always linearly polarized

$$E_x(z,t) = \frac{\eta_0 J_{S0}}{2} \cos(\omega t \mp \beta z)$$

Web Demo





http://www.enzim.hu/~szia/cddemo/edemo0.htm

How do we treat this mathematically?

Say we have two superimposed fields

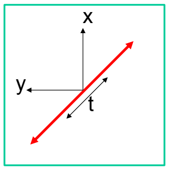
- Propagating in same direction
- Possibly out of phase
- Possibly oriented out of plane with each other

How can we tell if the combined wave will be:

- Linear
- Circular
- Elliptical

Linear Polarization

Plane in space perpendicular to propagation direction



The tip of the E-field vector traces out a straight line
DIRECTION: Constant
MAGNITUDE: Changes w/ time

How do we get this mathematically?

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Linear Polarization

Given two linearly polarized vector fields

$$\vec{E}_1 = E_1\cos(\omega t + \phi)\hat{a}_x$$

E_1 is horizontal linear polarization (green)

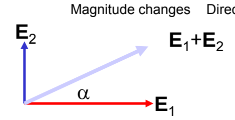
$$\vec{E}_2 = \pm E_2\cos(\omega t + \phi)\hat{a}_y$$

E_2 is vertical linear polarization (red)

If the vectors are IN PHASE or 180° apart, $\mathbf{E}_1+\mathbf{E}_2$ will also be linear polarization

$$\vec{E}_1 + \vec{E}_2 = \cos(\omega t + \phi)(E_1\hat{a}_x \pm E_2\hat{a}_y)$$

Magnitude changes   Direction is constant



$$\alpha = \pm\tan^{-1}\left(\frac{E_2}{E_1}\right)$$

Polarization ANGLE depends on relative magnitudes of E_1 and E_2

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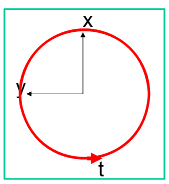
Circular Polarization

Plane in space perpendicular to propagation direction



The tip of the E-field vector traces out a circle
DIRECTION: Changes with time
MAGNITUDE: Constant

How do we get this mathematically?

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Circular Polarization

Given two linearly polarized vector fields

$$\vec{E}_1 = E_0\cos(\omega t + \phi)\hat{a}_x$$

E_1 is horizontal linear polarization (green)

$$\vec{E}_2 = E_0\sin(\omega t + \phi)\hat{a}_y$$

E_2 is vertical linear polarization (red)

The vectors have EQUAL MAGNITUDES
The vectors are OUT OF PHASE by **$\pi/2$**
The vectors are PERPENDICULAR

ALL THREE CONDITIONS MUST BE MET
for $\mathbf{E}_1+\mathbf{E}_2$ to be circular polarization

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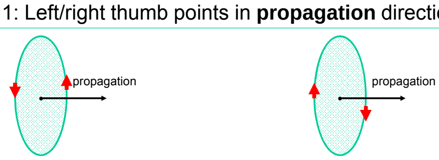
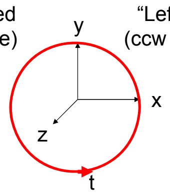
How to tell: Left or Right Handed Circular Polarization

Method 1: Left/right thumb points in **propagation** direction



"Right-Hand" Polarized (cw as seen by source)

"Left-Hand" Polarized (ccw as seen by source)



LHCP if moving in $-a_z$

RHCP if moving in $+a_z$

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Lectures 21-24

How to tell: Left or Right Handed Circular Polarization

Method 2: It's right handed if
Ahead $\hat{E}$ x Behind $\hat{E}$ is in propagation direction

$$\vec{E}_1 = E_0\cos(\omega t \pm \beta z)\hat{x} \quad \textit{Ahead}$$

$$\vec{E}_2 = E_0\sin(\omega t \pm \beta z)\hat{y}$$

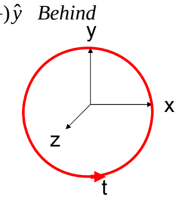
$$= E_0\cos(\omega t \pm \beta z - \frac{\pi}{2})\hat{y} \quad \textit{Behind}$$

$$\hat{E}_1 \times \hat{E}_2 = \hat{z}$$

so RHCP if moving in $+a_z$

else LHCP if moving in $-a_z$

It is behind by $\pi/2$ because ωt needs to be $\pi/2$ larger to be at the same part of the wave



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How to tell: Left or Right Handed Circular Polarization

Method 3: It's right handed if $\mathrm{Re}[\hat{\tilde{E}}] \times \mathrm{Im}[-\hat{\tilde{E}}]$ is in propagation direction

$\vec{E}_1 = E_0 \cos(\omega t \pm \beta z)\hat{x}$ *Ahead*

$\vec{E}_2 = E_0 \cos(\omega t \pm \beta z - \frac{\pi}{2})\hat{y}$ *Behind*

$$\tilde{\vec{E}} = \tilde{\vec{E}}_1 + \tilde{\vec{E}}_2 = E_0 e^{\pm j\beta z}\hat{x} + E_0 e^{\pm j\beta z} e^{-j\frac{\pi}{2}}\hat{y} = E_0 e^{\pm j\beta z}(\hat{x} - j\hat{y})$$

$\hat{\tilde{E}} \equiv \hat{x} - j\hat{y}$

I made up this notation. note it is not a unit vector

$\mathrm{Re}[\hat{\tilde{E}}] = \hat{x}$ gives the Ahead field

$\mathrm{Im}[-\hat{\tilde{E}}] = \hat{y}$ gives the Behind field

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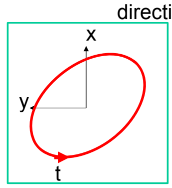
Elliptical Polarization

Most general: two combined linearly polarized waves will result in an elliptically polarized wave - if the conditions for linear or circular are not satisfied.

Plane in space perpendicular to propagation direction



The tip of the E-field vector traces out an ellipse

DIRECTION: Changes with time

MAGNITUDE: Changes

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Sum of two fields

- For $\mathbf{E}_1$=E_0 cos(2πx10^{8}t - 2πz)$\mathbf{a}_x$, and $\mathbf{E}_2$=E_0 cos(2πx10^{8}t - 3πz)$\mathbf{a}_y$, find the polarization of $\mathbf{E}_1$+$\mathbf{E}_2$ at the following points:

(a) (3, 4, 0)
(b) (3, -2, 0.5)
(c) (-2, 1, 1)
(d) (-1, -3, 0.2)

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From D3.17 (p 184) of old book

Challenge Question: Forming Circular Polarization

- For what value of φ, will the following fields add to produce **right** handed circular polarization:

  $\mathbf{E}_1$=E_0 cos(2πx10^{8}t + 2πz + π/3)$\mathbf{a}_x$,
  $\mathbf{E}_2$=E_0 cos(2πx10^{8}t + 2πz + φ)$\mathbf{a}_y$

(a) π/3
(b) -π/3
(c) π/6
(d) -π/6
(e) 5π/6

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LG's question

Writing linearly polarized as a sum of circular polarization

- Rewrite $\mathbf{E}$=E_0cos(ωt+βz)$\mathbf{a}_x$ as a sum of circular polarized fields.

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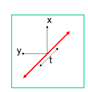
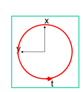
From Problem 3.35a (p 203) of old book

Lecture 24 Summary

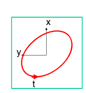
- Polarization describes the "tip" of **E**(t)
- Linear
  - Direction is ________
  - Magnitude __________



- Circular
  - Direction __________
  - Magnitude is ________



- Elliptical
  - Direction __________
  - Magnitude __________



- Next up: Section 5.6: Wave reflection

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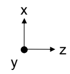
Lectures 21-24

Lectures 25-26
Section 5.6

Reflection and Transmission

Standing Waves

Normal incidence plane wave



Medium 1 $\sigma_1, \varepsilon_1, \mu_1$ | Medium 2 $\sigma_2, \varepsilon_2, \mu_2$

Incident (+) $\longrightarrow$

$$\tilde{E}_1(z) = \bar{E}_1^+ e^{-\bar{\gamma}_1 z}\hat{x}$$

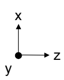
$$\tilde{H}_1(z) = \bar{H}_1^+ e^{-\bar{\gamma}_1 z}\hat{y} = \frac{1}{\bar{\eta}_1}\bar{E}_1^+ e^{-\bar{\gamma}_1 z}\hat{y}$$

What should happen next?

z<0 | z>0

We will analyze the case J_s=0 at z=0 (no applied surface current). For the case $J_s \neq 0$, use superposition of the J_s=0 case with the answer for waves generated by a current sheet.

Reflected and Transmitted Waves are Generated



Medium 1 $\sigma_1, \varepsilon_1, \mu_1$ | Medium 2 $\sigma_2, \varepsilon_2, \mu_2$

Incident (+) $\longrightarrow$ | $\longrightarrow$ Transmitted (+)

$$\tilde{E}_1(z) = \bar{E}_1^+ e^{-\bar{\gamma}_1 z}\hat{x},\ \tilde{H}_1(z) = \frac{1}{\bar{\eta}_1}\bar{E}_1^+ e^{-\bar{\gamma}_1 z}\hat{y} \qquad \tilde{E}_2(z) = \bar{E}_2^+ e^{-\bar{\gamma}_2 z}\hat{x},\ \tilde{H}_2(z) = \frac{1}{\bar{\eta}_2}\bar{E}_2^+ e^{-\bar{\gamma}_2 z}\hat{y}$$

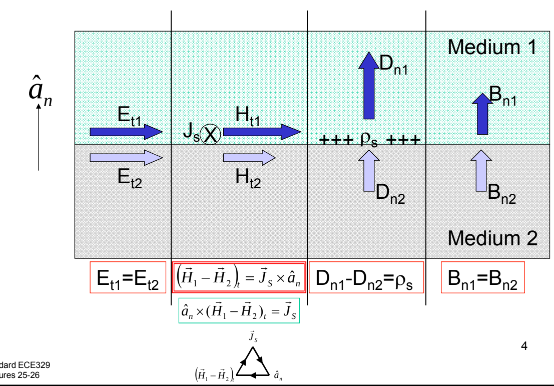
Reflected (-) $\longleftarrow$

$$\tilde{E}_1(z) = \bar{E}_1^- e^{+\bar{\gamma}_1 z}\hat{x},\ \tilde{H}_1(z) = \frac{-1}{\bar{\eta}_1}\bar{E}_1^- e^{+\bar{\gamma}_1 z}\hat{y}$$

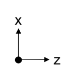
z<0 | z>0

At z=0, $\mathbf{J}_S$=0

Apply Boundary Conditions



Apply Boundary Conditions



Medium 1 $\sigma_1, \varepsilon_1, \mu_1$ | Medium 2 $\sigma_2, \varepsilon_2, \mu_2$

$$\tilde{E}_{1x\,total}(z) = \bar{E}_1^+ e^{-\bar{\gamma}_1 z} + \bar{E}_1^- e^{+\bar{\gamma}_1 z} \qquad \tilde{E}_{2x\,total}(z) = \bar{E}_2^+ e^{-\bar{\gamma}_2 z}$$

$$\tilde{H}_{1y\,total}(z) = \frac{1}{\bar{\eta}_1}\left(\bar{E}_1^+ e^{-\bar{\gamma}_1 z} - \bar{E}_1^- e^{+\bar{\gamma}_1 z}\right) \qquad \tilde{H}_{2y\,total}(z) = \frac{1}{\bar{\eta}_2}\bar{E}_2^+ e^{-\bar{\gamma}_2 z}$$

$\boxed{E_{t1}=E_{t2}}$ $\boxed{H_{t1}-H_{t2}=J_s=0}$

$$\tilde{E}_{1x\,total}(0^-) = \bar{E}_1^+ + \bar{E}_1^- = \bar{E}_2^+ = \tilde{E}_{2x\,total}(0^+)$$

$$\tilde{H}_{1y\,total}(0^-) = \frac{1}{\bar{\eta}_1}\left(\bar{E}_1^+ - \bar{E}_1^-\right) = \frac{1}{\bar{\eta}_2}\bar{E}_2^+ = \tilde{H}_{2y\,total}(0^+)$$

Solve the Equations

$$\bar{E}_1^+ + \bar{E}_1^- = \bar{E}_2^+$$

$$\frac{\bar{\eta}_2}{\bar{\eta}_1}\left(\bar{E}_1^+ - \bar{E}_1^-\right) = \bar{E}_2^+$$

$$\Rightarrow \quad \bar{E}_1^+ + \bar{E}_1^- = \frac{\bar{\eta}_2}{\bar{\eta}_1}\left(\bar{E}_1^+ - \bar{E}_1^-\right)$$

Reflection Coefficient: $\boxed{\therefore \Gamma = \frac{\bar{E}_1^-}{\bar{E}_1^+} = \frac{\bar{\eta}_2 - \bar{\eta}_1}{\bar{\eta}_1 + \bar{\eta}_2}}$

$$\tau = \frac{\bar{E}_2^+}{\bar{E}_1^+} = \frac{\bar{E}_1^+ + \bar{E}_1^-}{\bar{E}_1^+} = 1 + \Gamma$$

Transmission Coefficient: $\boxed{\therefore \tau = \frac{\bar{E}_2^+}{\bar{E}_1^+} = 1 + \Gamma = \frac{2\bar{\eta}_2}{\bar{\eta}_1 + \bar{\eta}_2}}$

Challenge Question: Special cases

- If $\eta_2=\eta_1$, we expect:

Medium 1 $\sigma_1, \varepsilon_1, \mu_1$ | Medium 2 $\sigma_1, \varepsilon_1, \mu_1$

Incident (+) $\longrightarrow$

(a) $\Gamma=0$, $\tau=1$ (0% reflected, 100% transmitted)
(b) $\Gamma=0$, $\tau=-1$ (0% reflected, 100% transmitted)
(c) $\Gamma=1$, $\tau=0$ (100% reflected, 0% transmitted)
(d) $\Gamma=-1$, $\tau=0$ (100% reflected, 0% transmitted)
(e) cannot be determined

LG's question

Challenge Question: Special cases

- If medium 2 is a perfect conductor, we expect:

Medium 1 $\sigma_1, \varepsilon_1, \mu_1$ | Medium 2 $\sigma_2 \rightarrow \infty$

Incident (+) $\longrightarrow$

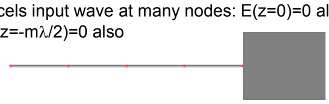
(a) $\Gamma=0$, $\tau=1$ (0% reflected, 100% transmitted)
(b) $\Gamma=0$, $\tau=-1$ (0% reflected, 100% transmitted)
(c) $\Gamma=1$, $\tau=0$ (100% reflected, 0% transmitted)
(d) $\Gamma=-1$, $\tau=0$ (100% reflected, 0% transmitted)
(e) cannot be determined

LG's question

Special Cases of Reflection/Transmission

$$\boxed{\Gamma \equiv \frac{\overline{E}_1^-}{\overline{E}_1^+} = \frac{\eta_2-\eta_1}{\eta_2+\eta_1}, \tau \equiv \frac{\overline{E}_2^+}{\overline{E}_1^+} = 1+\Gamma}$$

- Impedance matched: $\eta_2=\eta_1$, then $\Gamma=0$, $\tau=1$
  - No reflection, entire wave is transmitted
- Perfect dielectrics: $\sigma_1=\sigma_2=0$, then Γ and τ are real since η_1 and η_2 are real
- Perfect conductor for medium 2: $\eta_2 \rightarrow 0$, then $\Gamma=-1$, $\tau=0$
  - No transmission, entire wave is reflected. If medium 1 is a perfect dielectric, a standing wave is set up there since reflected wave perfectly cancels input wave at many nodes: E(z=0)=0 always, but if $\sigma_1=0$, then $E(z=-m\lambda/2)=0$ also



Standing Waves

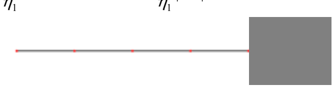
- The reflection from a PC gives $\Gamma=-1$
- If region 1 is a perfect dielectric, we get standing waves whereby the input and reflected wave add destructively at specific points in space (vs. travelling waves f(t ± z/v))

$$\tilde{E}_{1x\,total}(z) = \overline{E}_1^+ e^{-\overline{\gamma}_1 z} + \overline{E}_1^- e^{+\overline{\gamma}_1 z} = \overline{E}_1^+(e^{-j\beta_1 z} - e^{+j\beta_1 z}) = \overline{E}_1^+(-2j\sin(\beta_1 z))$$

$$\tilde{H}_{1y\,total}(z) = \frac{1}{\overline{\eta}_1}\left(\overline{E}_1^+ e^{-\overline{\gamma}_1 z} - \overline{E}_1^- e^{+\overline{\gamma}_1 z}\right) = \frac{\overline{E}_1^+}{\overline{\eta}_1}(e^{-j\beta_1 z} + e^{+j\beta_1 z}) = \frac{\overline{E}_1^+}{\overline{\eta}_1} 2\cos(\beta_1 z)$$

$$\vec{E}(z,t) = \hat{x}\,\mathrm{Re}[\overline{E}_1^+(-2j\sin(\beta_1 z))e^{j\omega t}] = \hat{x}2\left|\overline{E}_1^+\right|\sin(\beta_1 z)\sin(\omega t + \angle\overline{E}_1^+)$$

$$\vec{H}(z,t) = \hat{y}\,\mathrm{Re}[\frac{\overline{E}_1^+}{\overline{\eta}_1}2\cos(\beta_1 z)e^{j\omega t}] = \hat{y}\frac{2}{\eta_1}\left|\overline{E}_1^+\right|\cos(\beta_1 z)\cos(\omega t + \angle\overline{E}_1^+)$$



Standing Waves: on average, the net energy transport is zero

Show < **S** > = < **E** x **H** > = 0. Online notes use phasors. Here we'll use **E** and **H** explicitly (still assume $\sigma_1=0$).

$$\vec{E}(z,t) = 2\left|\overline{E}_1^+\right|\sin(\beta_1 z)\sin(\omega t + \angle\overline{E}_1^+)\hat{x}$$

$$\vec{H}(z,t) = \frac{2}{\eta_1}\left|\overline{E}_1^+\right|\cos(\beta_1 z)\cos(\omega t + \angle\overline{E}_1^+)\hat{y}$$

$$\vec{S}(z,t) = \frac{4}{\eta_1}\left|\overline{E}_1^+\right|^2 \sin(\beta_1 z)\cos(\beta_1 z)\sin(\omega t + \angle\overline{E}_1^+)\cos(\omega t + \angle\overline{E}_1^+)\hat{z}$$

$$= \frac{1}{\eta_1}\left|\overline{E}_1^+\right|^2 \sin(2\beta_1 z)\sin(2(\omega t + \angle\overline{E}_1^+))\hat{z} \qquad \boxed{\therefore \left\langle \vec{S}(z,t)\right\rangle = 0}$$

since $\sin(2\omega t)$ is periodic

At any moment in time, there is energy transport, but it averages to zero.

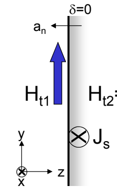
Incident wave induces surface current on PC that radiates the reflected wave

H is discontinuous at z=0, $\mathbf{H}_{t2}$=0 but $\mathbf{H}_{t1}$ oscillates:

$$\vec{H}(z,t) = \hat{y}\frac{2}{\eta_1}\left|\overline{E}_1^+\right|\cos(\omega t + \angle\overline{E}_1^+)$$

Thus, the BC's imply a surface current is induced at z=0:

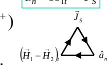
$$\vec{J}_S(t) = \hat{x}\frac{2}{\eta_1}\left|\overline{E}_1^+\right|\cos(\omega t + \angle\overline{E}_1^+)$$



This surface current is now a source that can radiate a wave away from the plane in $-a_z$ (i.e. the reflected wave):

$$\boxed{\hat{a}_n \times \vec{H}_{1t} = \vec{J}_S}$$

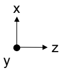
$$\vec{E}_{1reflected}(z,t) = (-\hat{x})\frac{\eta_1}{2}J_s(t - \frac{z}{v_p}) = -\hat{x}\left|\overline{E}_1^+\right|\cos(\omega t + \beta z + \angle\overline{E}_1^+)$$



which is precisely the wave we found when we used $\Gamma=-1$. The part that radiates into $+a_z$ perfectly cancels out the incident field for z>0 which gives us E=0 in the PC as expected.

Slight complication: J was created by $J=\sigma E$, but E=0 at the PC. Note $\sigma=\infty$.

Lectures 25-26 Summary

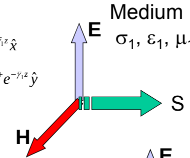


Medium 1 — $\sigma_1,\ \varepsilon_1,\ \mu_1$

Medium 2 — $\sigma_2,\ \varepsilon_2,\ \mu_2$

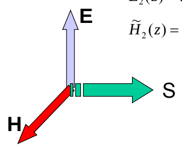
$$\tilde{E}_1(z) = \bar{E}_1^+ e^{-\bar{\gamma}_1 z}\hat{x}$$

$$\tilde{H}_1(z) = \frac{1}{\bar{\eta}_1}\bar{E}_1^+ e^{-\bar{\gamma}_1 z}\hat{y}$$



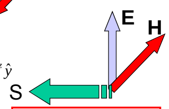
$$\tilde{E}_2(z) = \tau\bar{E}_1^+ e^{-\bar{\gamma}_2 z}\hat{x}$$

$$\tilde{H}_2(z) = \frac{1}{\bar{\eta}_2}\tau\bar{E}_1^+ e^{-\bar{\gamma}_2 z}\hat{y}$$



$$\tilde{E}_1(z) = \Gamma\bar{E}_1^+ e^{+\bar{\gamma}_1 z}\hat{x}$$

$$\tilde{H}_1(z) = \frac{-1}{\bar{\eta}_1}\Gamma\bar{E}_1^+ e^{+\bar{\gamma}_1 z}\hat{y}$$



$$\boxed{\Gamma = \frac{\bar{E}_1^-}{\bar{E}_1^+} = \frac{\bar{\eta}_2 - \bar{\eta}_1}{\bar{\eta}_1 + \bar{\eta}_2}}$$

$$\boxed{\tau = \frac{\bar{E}_2^+}{\bar{E}_1^+} = 1 + \Gamma = \frac{2\bar{\eta}_2}{\bar{\eta}_1 + \bar{\eta}_2}}$$

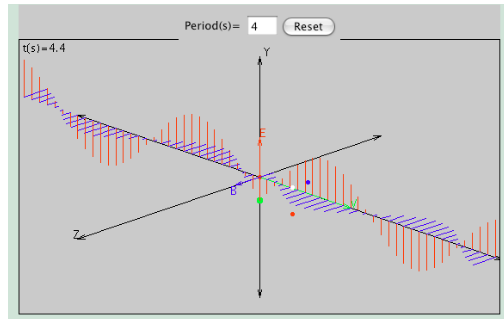
Next: Transmission lines: Section 6.5 & Chapter 7

ECE 329
Lectures 27-30
Section 6.5 and Chapter 7

Transmission Lines
Time Domain Analysis

Adapted from Prof. Cunningham's Notes

Starting Point: Uniform Plane Wave



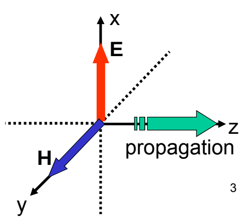
http://www.phy.ntnu.edu.tw/java/emWave/emWave.html

Starting Point: Uniform Plane Wave

- Consider **E** and **H** that are
  - Perpendicular to each other
  - Perpendicular to the direction of propagation
  - Magnitude is constant ("uniform") in the plane perpendicular to the propagation direction
  - And for **perfect dielectric** media:
    - E and H are in phase
    - No attenuation in z-direction

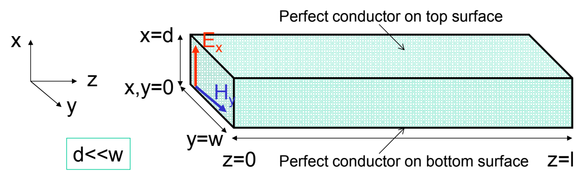
$$\vec{E} = E_x(z,t)\vec{a_x}$$

$$\vec{H} = H_y(z,t)\vec{a_y}$$



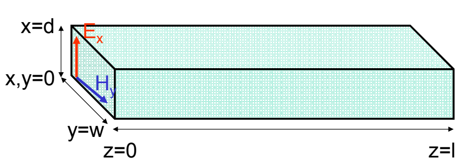
Parallel Plate Transmission Line

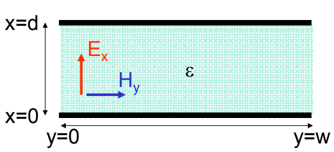
Imagine a rectangular box made of perfect conductors
on the upper and lower surfaces, filled by perfect dielectric medium



- If we place conducting sheets in the path of the uniform plane wave, some of the wave enters the box and is guided by it

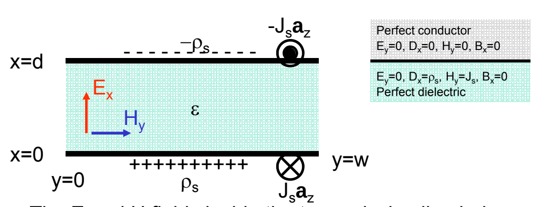
Parallel Plate Transmission Line





Cross section of the transmission line so
wave is propagating into the page

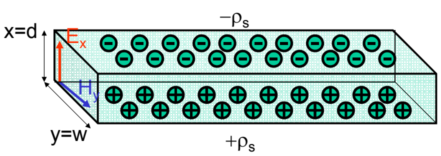
Consider boundary conditions



The E and H fields inside the transmission line <u>induce</u>
charge and current on the upper and lower surfaces
Apply BC's on D_n and H_t to find ρ_s and J_s respectively

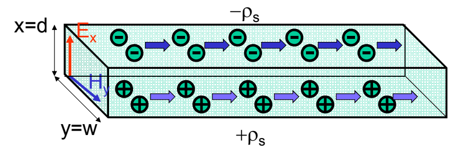
$$|\rho_S| = |\varepsilon E_x| \qquad |J_S| = |H_y|$$

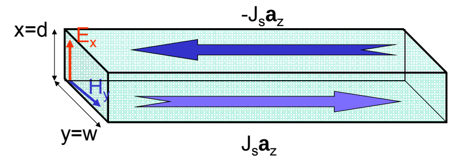
TL Voltage



$$V(z,t) = (d)E_x(z,t)$$

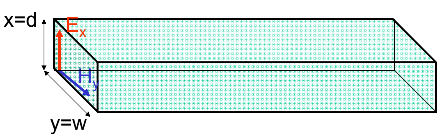
TL Current – <u>Charges on both plates move to the right</u>





$$I(z,t) = wH_y(z,t)$$

TL Power

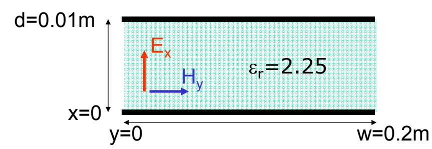


$$P(z,t) = \int_S (\vec{E} \times \vec{H}) \bullet d\vec{S} = \int_{x=0}^{x=d} \int_{y=0}^{y=w} \frac{V}{d} \cdot \frac{I}{w} dxdy$$

$$\boxed{P(z,t) = V(z,t)I(z,t)}$$

same as in circuit theory

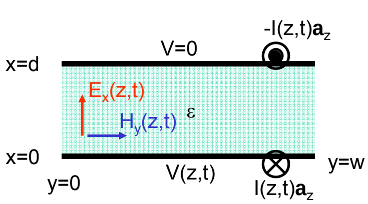
Converting E ↔ V and H ↔ I



Find the power crossing this transverse cross section if
(a) **E**=300π V/m, (b) **H**=7.5 A/m, (c) V=4π V, (d) I=0.5A

From D6.1 (p371) of old book

Another look at transverse plane of TL



V(z,t) and I(z,t) can be used to describe the state of the transmission line instead of E_x(z,t) and H_y(z,t)

Transmission Line Equations

For plane waves in perfect dielectric:

$$\begin{cases} \dfrac{\partial E_x}{\partial z} = -\dfrac{\partial B_y}{\partial t} = -\mu \dfrac{\partial H_y}{\partial t} \\ \dfrac{\partial H_y}{\partial z} = -\dfrac{\partial D_x}{\partial t} = -\varepsilon \dfrac{\partial E_x}{\partial t} \end{cases}$$

But in the transmission line:

$$\begin{cases} E_x(z,t) = \dfrac{V(z,t)}{d} \\ H_y(z,t) = \dfrac{I(z,t)}{w} \end{cases}$$

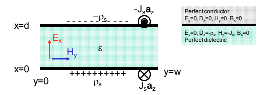
Transmission Line Equations

Substituting…

$$\begin{cases} \dfrac{\partial V}{\partial z} = -\left(\dfrac{\mu d}{w}\right)\dfrac{\partial I}{\partial t} \\[2ex] \dfrac{\partial I}{\partial z} = -\left(\dfrac{\varepsilon w}{d}\right)\dfrac{\partial V}{\partial t} \end{cases}$$

These are the transmission line equations!!
- They describe wave propagation along the TL in terms of currents and voltages
- It is just another way of stating Maxwell's Eqns

Circuit Parameters



Rewrite the TL Equation using circuit parameters

$$\mathcal{L} = \frac{L}{z} = \frac{\text{Inductance}}{\text{Length}} = \frac{Flux/Current}{Length} = \frac{\psi/I}{z} = \frac{B_y zd/H_y w}{z} = \frac{B_y d}{H_y w} = \frac{\mu H_y d}{H_y w} = \frac{\mu d}{w}$$

$$\mathcal{C} = \frac{C}{z} = \frac{\text{Capacitance}}{\text{Length}} = \frac{\text{Charge/Voltage}}{\text{Length}} = \frac{Q/V}{z} = \frac{\rho z w/E_x d}{z} = \frac{\rho w}{E_x d} = \frac{\varepsilon E_x w}{E_x d} = \frac{\varepsilon w}{d}$$

Transmission Line Equation

$$\frac{\partial V}{\partial z} = -\mathcal{L}\frac{\partial I}{\partial t}$$

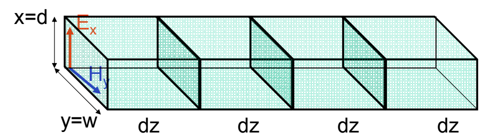
$$\frac{\partial I}{\partial z} = -\mathcal{C}\frac{\partial V}{\partial t}$$

For lossless transmission lines
Perfect dielectric filling
Perfect conductor outer shell

Don't forget that L & C are related:

$$\mathcal{L}\mathcal{C} = \mu\varepsilon$$

Distributed Circuit

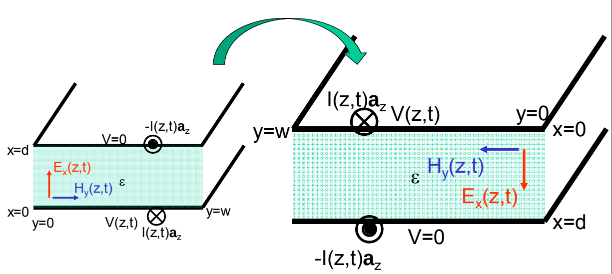


One small slice of the transmission line would have a finite inductance and capacitance.

If we put our small slices together, end to end, to make a whole transmission line, the inductance and capacitance would be distributed over the whole length of the TL.

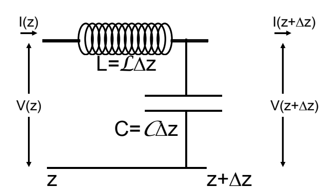
How do we represent a distributed circuit element?

Flip the parallel plate for TL (ground the bottom plate)



Distributed Circuit

One slice of the TL:



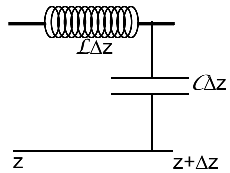
$$V(z+\Delta z) - V(z) = -L\frac{\partial I}{\partial t} \qquad I(z) - I(z+\Delta z) = C\frac{\partial V}{\partial t}$$

$$\therefore \frac{\partial V}{\partial z} = -\mathcal{L}\frac{\partial I}{\partial t} \qquad \therefore \frac{\partial I}{\partial z} = -\mathcal{C}\frac{\partial V}{\partial t}$$

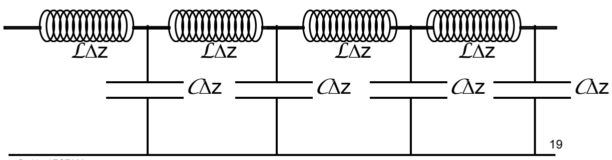
These are the same as the TL equations so this is the equivalent circuit for the TL

Distributed Circuit

One slice of the TL:

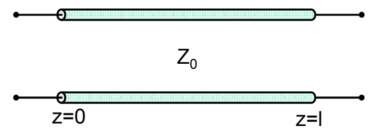


The entire TL:



Characteristic Impedance

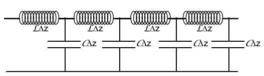
A more convenient way of representing
the distributed circuit



$$Z_0 = \sqrt{\frac{\mathcal{L}}{\mathcal{C}}} = \left|\frac{V(z,t)}{I(z,t)}\right|$$

$$v_p = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mathcal{L}\mathcal{C}}}$$

Challenge Question: Transmission Lines



- For waves propagating down transmission lines, which of the following is **false**:

(a) the impedances Z_0 and η_0 are equal
(b) the propagation speed is ≤ c
(c) the propagation speed in two lines can be different even if the impedance Z_0 is the same
(d) in steady state, the TL acts like plain wires
(e) the fields store energy distributed among the inductive and capacitive segments of the TL

LG's question

Designing coaxial cables

Design a 50Ω coaxial cable if ε_r=2.56 and a=1cm, i.e. find b.

From D6.2a (p372) of old book

ECE 329
Lecture 28

TL Terminated by Resistive Load

Bounce Diagram

Adapted from Prof. Cunningham's Notes

Solution to TL Equation

We can solve the TL Equations the same way that we solved the wave equation for uniform plane waves in free space:

$$V(z,t) = Af(t - \frac{z}{v_p}) + Bg(t + \frac{z}{v_p})$$

$$I(z,t) = \frac{1}{Z_0}\left[Af(t - \frac{z}{v_p}) - Bg(t + \frac{z}{v_p})\right]$$

Solution is a superposition of traveling waves
- one going in +z direction
- one going in -z direction

Simplified shorthand notation

$$V(z,t) = V^+(t - \frac{z}{v_p}) + V^-(t + \frac{z}{v_p})$$

$$I(z,t) = \frac{1}{Z_0}\left[V^+(t - \frac{z}{v_p}) - V^-(t + \frac{z}{v_p})\right]$$

Simplifying even more…

$$V = V^+ + V^-$$

$$I = \frac{1}{Z_0}(V^+ - V^-) = I^+ + I^-$$

Solution to TL equation

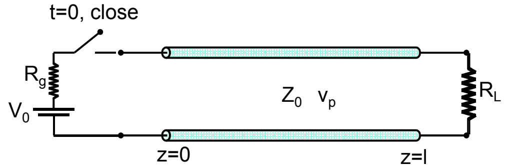
$$V = V^+ + V^-$$

$$I = \frac{1}{Z_0}(V^+ - V^-) = I^+ + I^-$$

$$I^+ = \frac{V^+}{Z_0} \qquad I^- = -\frac{V^-}{Z_0}$$

Solution to TL equation is summation of traveling waves propagating in the +z or -z directions

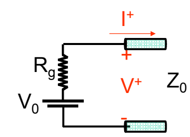
Example: TL + Resistive Load



Example, t=0

$$V_0 - I^+R_g - V^+ = 0$$

$$I^+ = \frac{V^+}{Z_0}$$



$$\boxed{V^+ = \tau_g V_0}$$

τ_g is the injection coefficient

$$\boxed{V^+ = \frac{Z_0}{R_g + Z_0}V_0}$$

$$\boxed{I^+ = \frac{V_0}{R_g + Z_0}}$$

At t=0, a wave originates at z=0 and starts to travel in the +z direction

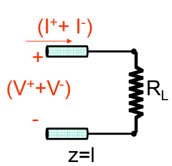
Until the wave propagates to the end and reflects back, there is no V^- wave, and the load resistance R_L has no effect.

Example t=T

$$T = \frac{l}{v_p}$$

Time required for V^+ wave to reach load end of the TL

$$V^+ + V^- = R_L(I^+ + I^-)$$



$$\boxed{V^- = V^+\frac{R_L - Z_0}{R_L + Z_0}}$$

Wave reflects to set up a "-" wave IN ADDITION TO the "+" wave

Equation 7.50a,b should say $V^+ + V^-$ instead of $V^+ - V^-$

Definition: Reflection Coefficient

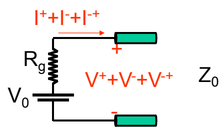
Voltage Reflection Coefficient

$$\Gamma = \frac{V^-}{V^+} = \frac{R_L - Z_0}{R_L + Z_0}$$

Current Reflection Coefficient

$$\frac{I^-}{I^+} = \frac{-(V^-/Z_0)}{(V^+/Z_0)} = -\frac{V^-}{V^+} = -\Gamma$$

Example, t=2T



Reflected wave travels back towards the source, and gets there at t=2T

The wave gets RE-REFLECTED at the source end and travels back toward the load as a “+” wave

The re-reflection process continues forever until steady state conditions are reached

Special Case TL Terminations

Short-circuited line: $R_L=0$

$$\Gamma = \frac{V^-}{V^+} = \frac{R_L - Z_0}{R_L + Z_0} = \frac{0 - Z_0}{0 + Z_0} = -1$$

No voltage across R_L

Open-circuited line: R_L=infinity

$$\Gamma = \frac{V^-}{V^+} = \frac{\infty - Z_0}{\infty + Z_0} = 1$$

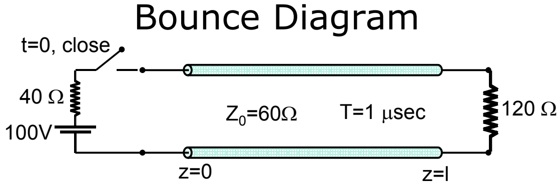
No current across R_L

Impedance-matched line: $R_L= Z_0$

$$\Gamma = \frac{V^-}{V^+} = \frac{Z_0 - Z_0}{Z_0 + Z_0} = 0$$

No reflection at R_L

Bounce Diagram

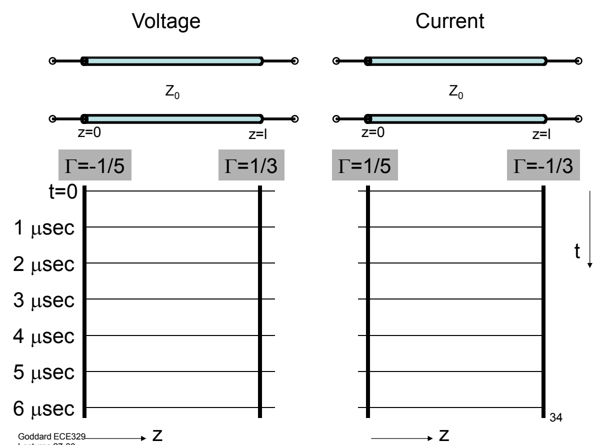


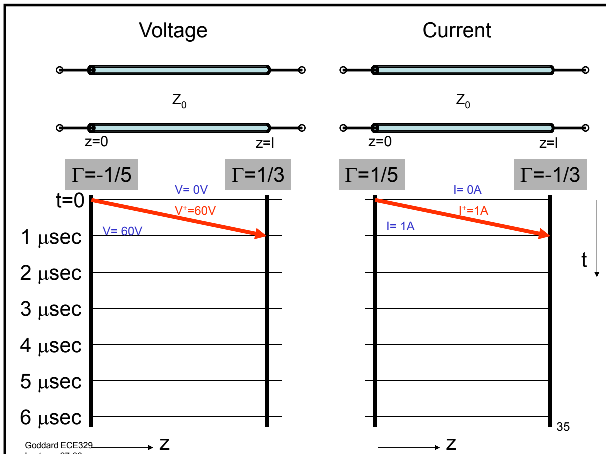
First step: Calculate V^+, I^+, Γ_{load}, Γ_{source}

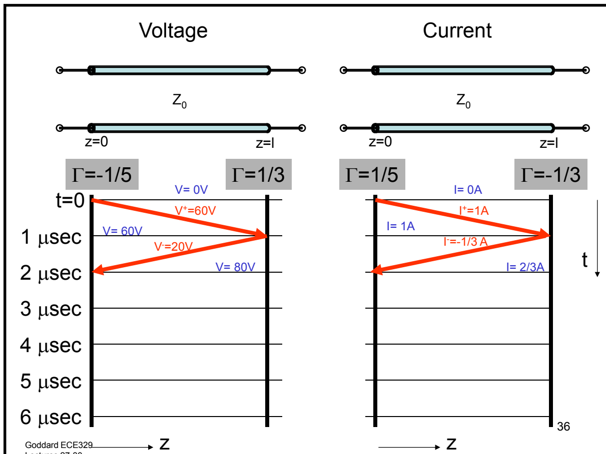
$$V^+ = V_0 \frac{Z_0}{R_g + Z_0} = 100 \frac{60}{40+60} = 60V \qquad \Gamma_L = \frac{R_L - Z_0}{R_L + Z_0} = \frac{120-60}{120+60} = \frac{1}{3}$$

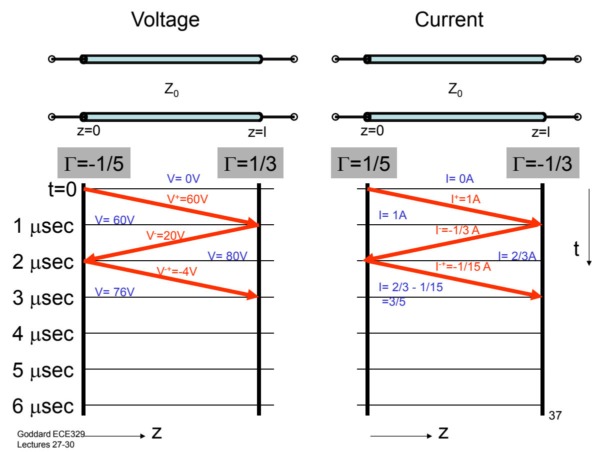
$$I^+ = \frac{V^+}{Z_0} = \frac{60}{60} = 1A \qquad \Gamma_S = \frac{R_S - Z_0}{R_S + Z_0} = \frac{40-60}{40+60} = -\frac{1}{5}$$

Second step: Construct 2 bounce diagrams (Voltage and Current)

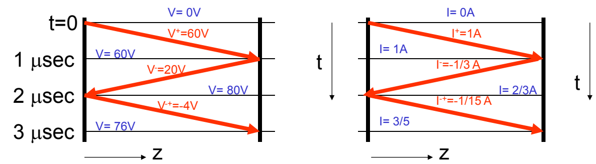






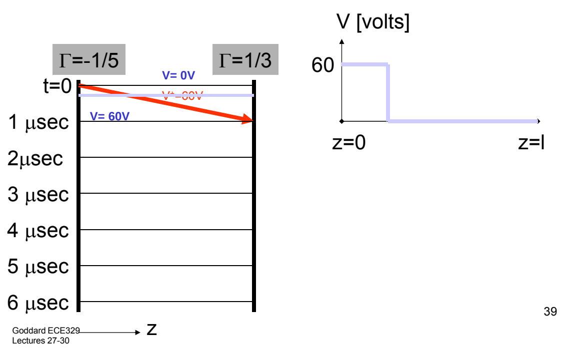


Plotting the Line Voltage/Current

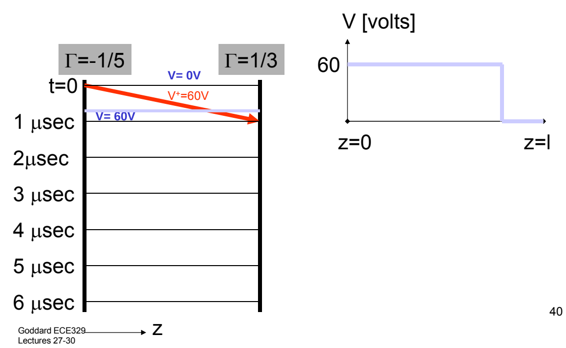


The bounce diagram is a 2D plot! Just have to sketch a cross sectional graph of the data in blue.

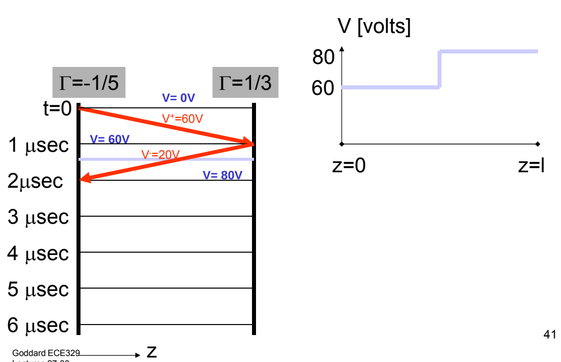
Question: What is V(z) at t=0.25 μsec?



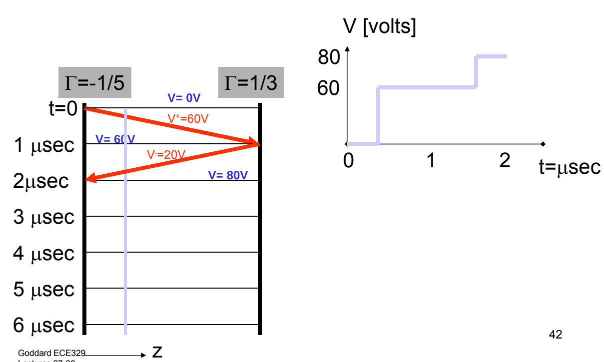
Question: What is V(z) at t=0.75 μsec?



Question: What is V(z) at t=1.5 μsec?



Question: What is V(t) at z=0.25*l m?



ECE 329
Lecture 29

Algebra of the Bounce Diagram

Adapted from Prof. Cunningham's Notes

Steady State

If $\Gamma_s\Gamma_l<1$, then eventually, the magnitude of the reflections will die down, and the voltage and current reach constant, steady state values

$V_{ss}^{+}=60(1-\frac{1}{15}+\frac{1}{15^2}+\ldots)$ Sum of all + waves = 56.25V

$V_{ss}^{-}=20(1-\frac{1}{15}+\frac{1}{15^2}+\ldots)$ Sum of all - waves = 18.75V

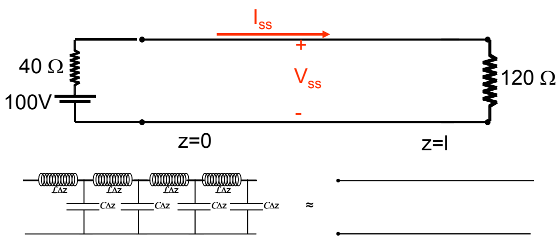
$I_{ss}^{+}=1(1-\frac{1}{15}+\frac{1}{15^2}+\ldots)$ Sum of all + waves = 0.9375A

$I_{ss}^{-}=-\frac{1}{3}(1-\frac{1}{15}+\frac{1}{15^2}+\ldots)$ Sum of all – waves = -0.3125A

$$\sum_{n=0}^{\infty}\left(-\frac{1}{15}\right)^n=\frac{1}{1+1/15}=\frac{15}{16}$$

In SS, TL Looks Like a Wire

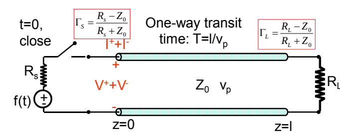
$$V_{SS}=V_{SS}^{+}+V_{SS}^{-}=75V \qquad I_{SS}=I_{SS}^{+}+I_{SS}^{-}=0.625A$$



$V_{ss}=100V\frac{120\Omega}{(40+120)\Omega}=75V$ $\qquad I_{ss}=\frac{100V}{(40+120)\Omega}=0.625A$

Algebra of the Bounce Diagram

Voltage divider: $\tau_S=\frac{Z_0}{R_S+Z_0}$



$V^{-}(0,t)=\Gamma_L V^{+}(0,t-2T)$ ← Reflected wave is the forward wave at time 2T ago and multiplied by reflection coefficient Γ_L

$$V(0,t)=V^{+}(0,t)+V^{-}(0,t)=f(t)-R_s I(0,t)$$

$$I(0,t)=I^{+}(0,t)+I^{-}(0,t)=\frac{V^{+}(0,t)-V^{-}(0,t)}{Z_0}$$

New wave is source + old wave 2T ago multiplied by RT reflection coeff. $\Gamma_L\,\Gamma_s$

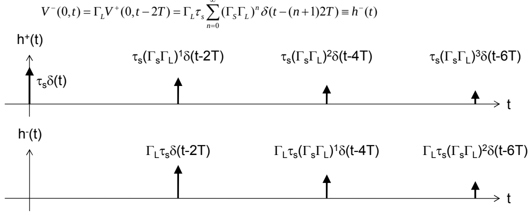
$$\boxed{V^{+}(0,t)=\tau_S f(t)+\Gamma_S\Gamma_L V^{+}(0,t-2T)}$$

Algebra of the Bounce Diagram

For f(t)=δ(t), the solution of: $\boxed{V^{+}(0,t)=\tau_S\delta(t)+\Gamma_S\Gamma_L V^{+}(0,t-2T)}$
is an impulse response:

$$V^{+}(0,t)=\tau_s\sum_{n=0}^{\infty}(\Gamma_S\Gamma_L)^n\delta(t-n2T)\equiv h^{+}(t)$$

$$V^{-}(0,t)=\Gamma_L V^{+}(0,t-2T)=\Gamma_L\tau_s\sum_{n=0}^{\infty}(\Gamma_S\Gamma_L)^n\delta(t-(n+1)2T)\equiv h^{-}(t)$$



Shown for: $\Gamma_S=0.8,\ \Gamma_L=0.9$

Algebra of the Bounce Diagram

$$V^{+}(0,t)=\tau_s\sum_{n=0}^{\infty}(\Gamma_S\Gamma_L)^n\delta(t-n2T)\equiv h^{+}(t)$$

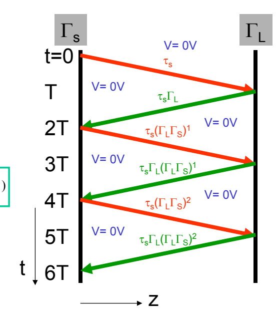
$$V^{-}(0,t)=\Gamma_L\tau_s\sum_{n=0}^{\infty}(\Gamma_S\Gamma_L)^n\delta(t-(n+1)2T)\equiv h^{-}(t)$$

For arbitrary position z, replace t with t ± z/v_p

$$\boxed{V^{+}(z,t)=\tau_s\sum_{n=0}^{\infty}(\Gamma_S\Gamma_L)^n\delta(t-\frac{z}{v_p}-n2T)} \qquad \boxed{I=\frac{1}{Z_0}(V^{+}-V^{-})}$$

$$\boxed{V^{-}(z,t)=\Gamma_L\tau_s\sum_{n=0}^{\infty}(\Gamma_S\Gamma_L)^n\delta(t+\frac{z}{v_p}-(n+1)2T)}$$

Note: the voltage is nonzero only on the bounce lines



Algebra of the Bounce Diagram

For f(t)=δ(t), the solution was:

$$V^+(z,t)=\tau_s\sum_{n=0}^{\infty}(\Gamma_S\Gamma_L)^n\delta(t-\frac{z}{v_p}-n2T)$$

$$V^-(z,t)=\Gamma_L\tau_s\sum_{n=0}^{\infty}(\Gamma_S\Gamma_L)^n\delta(t+\frac{z}{v_p}-(n+1)2T)$$

$$I=\frac{1}{Z_0}(V^+-V^-)$$

For arbitrary f(t), convolve the solution with f:

$$V^+(z,t)=\tau_s\sum_{n=0}^{\infty}(\Gamma_S\Gamma_L)^n f(t-\frac{z}{v_p}-n2T)$$

$$V^-(z,t)=\Gamma_L\tau_s\sum_{n=0}^{\infty}(\Gamma_S\Gamma_L)^n f(t+\frac{z}{v_p}-(n+1)2T)$$

$$I=\frac{1}{Z_0}(V^+-V^-)$$

Note: the voltage can be nonzero in between the bounce lines depending on the function f

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Lectures 27-29 Summary

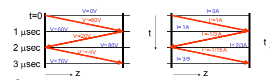
- Transmission Line Equations

$$\frac{\partial \mathcal{V}}{\partial z}=-\mathcal{L}\frac{\partial \mathcal{I}}{\partial t}\qquad \frac{\partial \mathcal{I}}{\partial z}=-C\frac{\partial \mathcal{V}}{\partial t}$$

- Characteristic impedance & Speed

$$Z_0=\sqrt{\frac{\mathcal{L}}{C}}=\frac{|V(z,t)|}{|I(z,t)|}\qquad v_p=\frac{1}{\sqrt{\mu\varepsilon}}=\frac{1}{\sqrt{\mathcal{L}C}}$$

- Bounce Diagram



- Round Trip Equation

$$V^+(0,t)=\tau_S f(t)+\Gamma_S\Gamma_L V^+(0,t-2T)$$

$$V^+(z,t)=\tau_s\sum_{n=0}^{\infty}(\Gamma_S\Gamma_L)^n f(t-\frac{z}{v_p}-n2T)$$

$$I=\frac{1}{Z_0}(V^+-V^-)$$

$$V^-(z,t)=\Gamma_L\tau_s\sum_{n=0}^{\infty}(\Gamma_S\Gamma_L)^n f(t+\frac{z}{v_p}-(n+1)2T)$$

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ECE 329
Lecture 30

TL Discontinuities

(Optional) TL Circuits with Reactive Elements

Adapted from Prof. Cunningham's Notes

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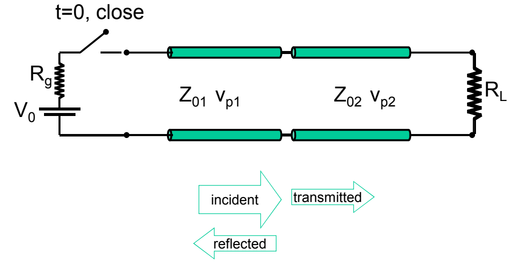
TL Discontinuity

t=0, close



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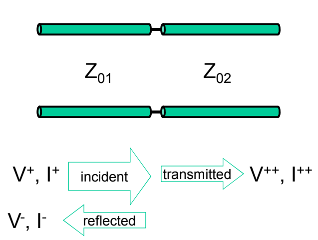
TL Discontinuity

What happens at the boundary between two TL's?



Assume that (+) wave hits the junction from the left

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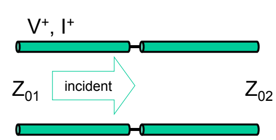
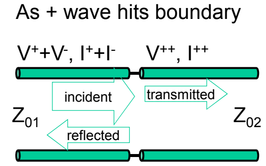
Before + wave hits boundary



As + wave hits boundary



What equations should we write for the boundary?

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Calculation Space

Voltage Reflection Coeff

$$\Gamma = \frac{V^-}{V^+} = \frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}}$$

Fraction of incident voltage that gets reflected back

Voltage Transmission Coeff

$$\tau_v = \frac{V^{++}}{V^+} = \frac{V^+ + V^-}{V^+} = 1 + \frac{V^-}{V^+}$$

$$\tau_v = 1 + \Gamma$$

Fraction of incident voltage that gets transmitted through

Current Transmission Coeff

$$\tau_c = \frac{I^{++}}{I^+} = \frac{I^+ + I^-}{I^+} = 1 + \frac{I^-}{I^+}$$

$$\tau_c = 1 - \Gamma$$

Fraction of incident current that gets transmitted through

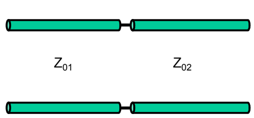
Power

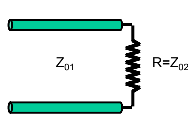
$$P_{incident} = V^+ I^+$$
$$P_{reflected} = V^- I^-$$
$$P_{transmitted} = V^{++} I^{++}$$

Some of the incident power is reflected back and the rest is transmitted through to the second line

TL Discontinuity Looks Like Load Resistor with R=Z_{02}

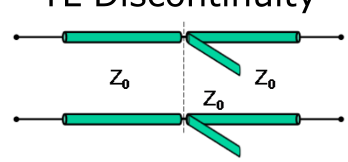
The reflection coefficient for the two configurations are the same, but power is not dissipated in the first case, rather it is transmitted down the line.





$$\Gamma = \frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}} \qquad \Gamma = \frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}}$$

Challenge Question: TL Discontinuity



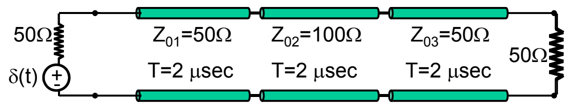
- For a right moving wave (into the splitter), the fraction of power that is reflected is:

(a)1/9, (b) 1/4, (c) 1/3, (d) 1/2, (e) 2/3

Given your answer, what is the power transmitted down each line?

LG's question

Example

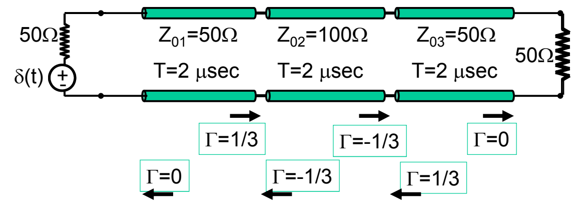


δ(t) is a unit impulse function.

First step: Calculate V^+, I^+, and Γ at each location

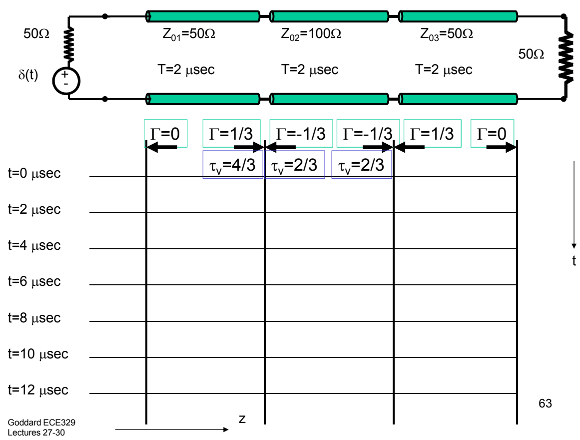
Second step: Draw a bounce diagram to determine magnitude of pulses that come out the end of the third TL

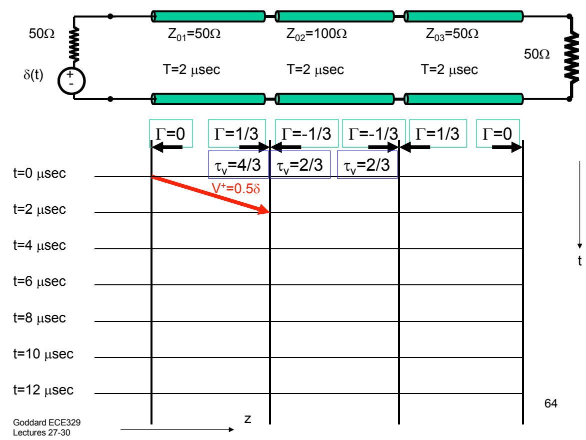
Step 1: V^+, I^+, Γ

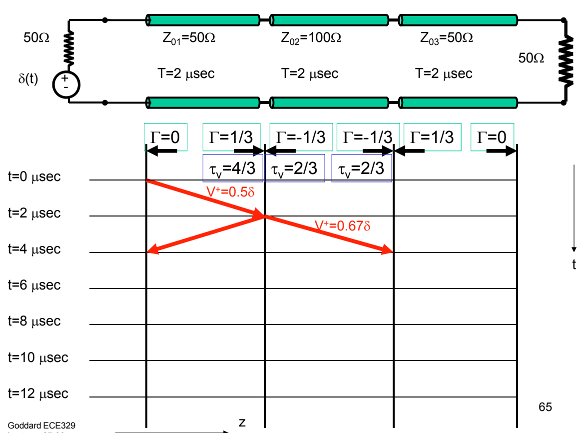


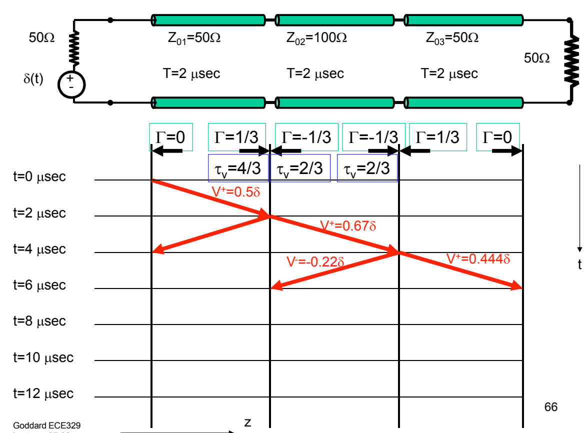
$$V^+ = V_0 \frac{Z_0}{R_g + Z_0} = 1\delta \frac{50}{50+50} = 0.5\delta \ V$$

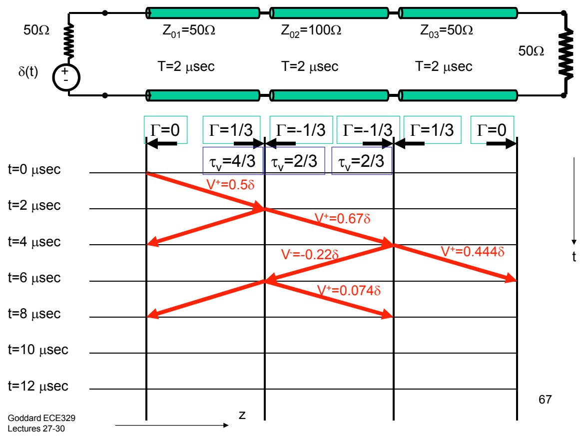
$$I^+ = \frac{V^+}{Z_0} = \frac{0.5\delta}{50} = 0.01\delta \ A$$











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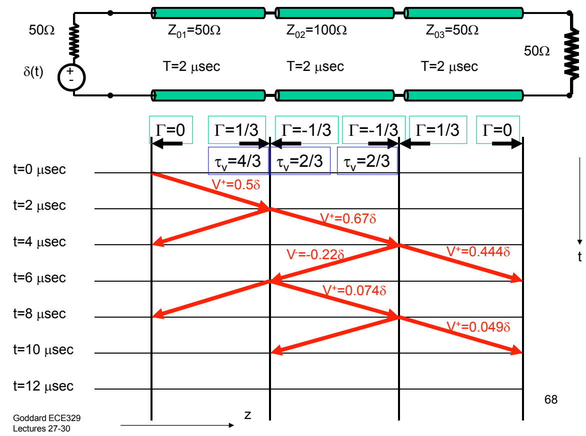

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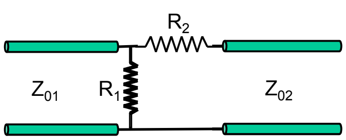
Example 2



V^+ wave incident from the left
What is reflected voltage and current into line 1?
What is transmitted voltage and current into line 2?

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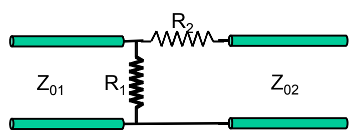
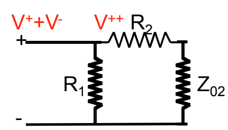


Equivalent circuit "seen" by V+ when it gets to the end of line 1:



$$\Gamma = \frac{V^-}{V^+} = \frac{R_L - Z_{01}}{R_L + Z_{01}}$$

What is R_L for this equivalent circuit?

$$R_L = (R_1)\|(R_2 + Z_{02})$$

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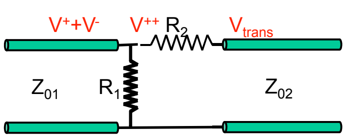
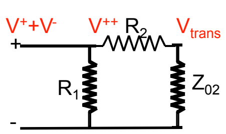


How much voltage gets transmitted through to line 2?



$$\tau_v = \frac{V^{++}}{V^+} = 1 + \Gamma$$

$$V^{++} = V^+(1+\Gamma)$$

$$V_{trans} = \frac{Z_{02}}{Z_{02} + R_2} V^{++}$$

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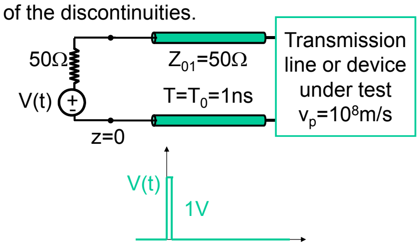
Time Domain Reflectometry

A powerful way to analyze discontinuities and resistive or reactive elements in transmission lines.
Method: Send a very narrow impulse (pulse width << T_0) down the line and measure the reflected voltage waveform. Reverse the bounce diagram calculation to determine the nature of the discontinuities.



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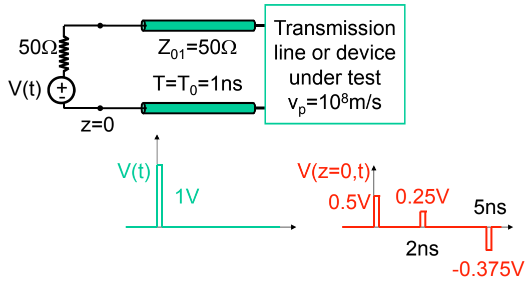
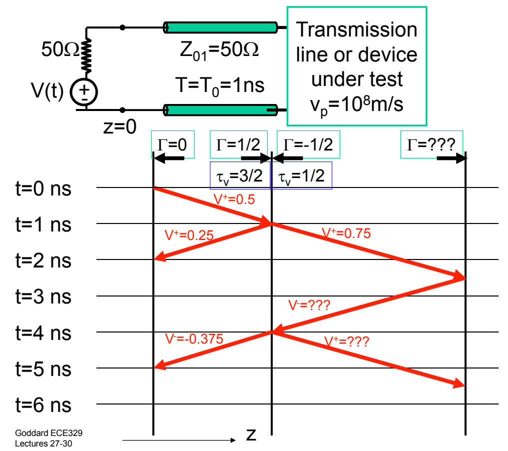
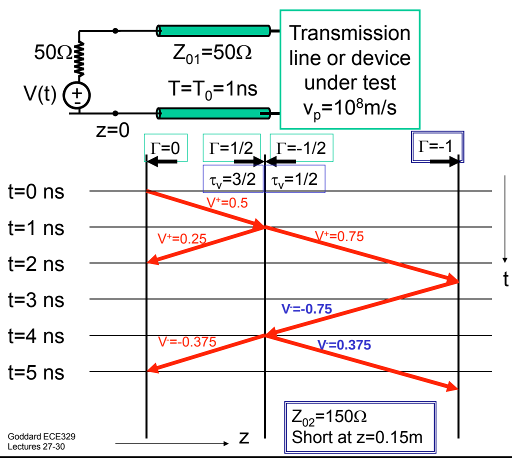
Time Domain Reflectometry

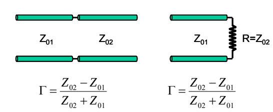






Lecture 30 Summary

- TL discontinuity looks like a load resistor with R=Z_{02} but no power is dissipated



- TDR reverses bounce diagram calculation to infer TL properties
- Next class
  - Chapter 7 (TL in frequency domain)

(Optional)

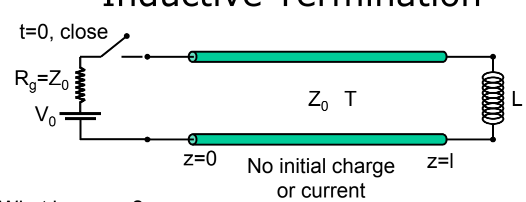
ECE 329
Lecture 30b

TL Circuits with Reactive Elements

Adapted from Prof. Cunningham's Notes

(Optional)

Inductive Termination

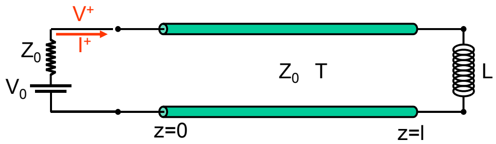


What happens?

- A "+" wave originates at t=0
- The + wave travels down the TL until it reaches the end at time T
- The wave meets the inductor and a reflected "-" wave begins
- The V^- and I^- waves will be a function of time, because of the inductor

(Optional) Inductive Termination

t=0+

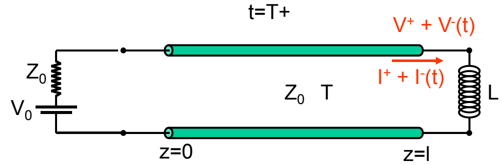


$$V^+ = \frac{V_0}{2}$$

$$I^+ = \frac{V^+}{Z_0} = \frac{V_0}{2Z_0}$$

(Optional) Inductive Termination

t=T+



$$V^+ = \frac{V_0}{2}$$

$$I^+ = \frac{V^+}{Z_0} = \frac{V_0}{2Z_0}$$

$$I^- = -\frac{V^-(t)}{Z_0}$$

For the inductor:

$$V = L\frac{dI}{dt}$$

(Optional) Two ways to solve

- Rigorous mathematical method
  - Laplace Transform
    - We will do this first
- Shortcut method

(Optional) Diff Eqn for the Inductor

$$V = L\frac{dI}{dt} \Rightarrow (V^+ + V^-(t)) = L\frac{d(I^+ + I^-(t))}{dt}$$

$$\frac{V_0}{2} + V^-(t) = L\frac{d}{dt}\left(\frac{V_0}{2Z_0} - \frac{V^-(t)}{Z_0}\right)$$

$$\frac{V_0}{2} = -\frac{L}{Z_0}\frac{dV^-(t)}{dt} - V^-(t)$$

$$\frac{dV^-(t)}{dt} + \frac{Z_0}{L}V^-(t) = -\frac{Z_0}{L}\frac{V_0}{2}$$

Laplace Transform V-(t') to F-(s) $\boxed{t'=t-T}$

$$s\hat{V}^-(s) - V^-(0) + \frac{Z_0}{L}\hat{V}^-(s) = -\frac{Z_0}{L}\frac{V_0}{2}\frac{1}{s}$$

(Optional) Laplace Transform for Inductor

Initial Condition *At t=T, i.e. t'=0, inductor current = 0 since inductor "looks" like an OPEN CIRCUIT*

$$I = I^+ + I^-(T) = 0 \quad \Rightarrow \quad \frac{V_0}{2Z_0} - \frac{V^-(0)}{Z_0} = 0 \quad \Rightarrow \quad V^-(0) = V_0/2$$

$$s\hat{V}^-(s) - \frac{V_0}{2} + \frac{Z_0}{L}\hat{V}^-(s) = -\frac{Z_0}{L}\frac{V_0}{2}\frac{1}{s} \Rightarrow \boxed{\hat{V}^-(s) = \frac{sL - Z_0}{sL + Z_0}\frac{V_0}{2s}}$$

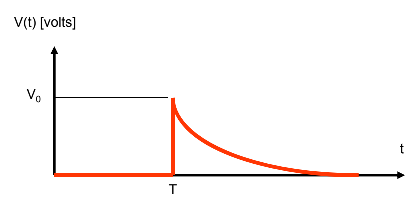
In s-space, we have V-(s) = Γ(s) V+(s) with:

$$\boxed{\Gamma(s) = \frac{Z(s) - Z_0}{Z(s) + Z_0}} \quad \boxed{\hat{V}^+(s) = \frac{V_0}{2s}} \quad \boxed{Z(s) = sL \text{ for an inductor}}$$

(Optional) Invert Laplace Transform

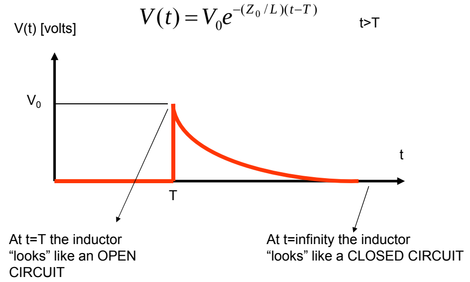
$$\hat{V}(s) = \hat{V}^+ + \hat{V}^- = \frac{V_0}{2s} + \frac{V_0}{2s}\frac{sL - Z_0}{sL + Z_0} = \frac{V_0}{2s}\frac{2sL}{sL + Z_0} = \frac{V_0}{s + Z_0/L}$$

$$\boxed{V(t) = V_0 e^{-(Z_0/L)t'} = V_0 e^{-(Z_0/L)(t-T)}} \quad t>T$$

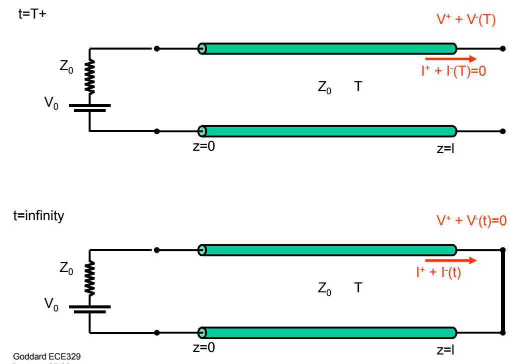


(Optional) Shortcut Method

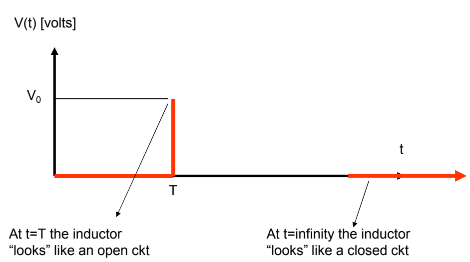
$$V(t) = V_0 e^{-(Z_0/L)(t-T)} \quad t>T$$



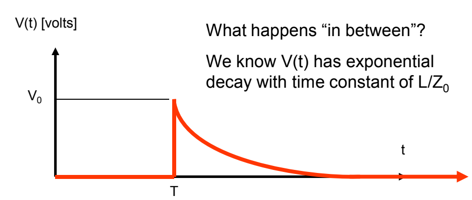
(Optional) Shortcut Method



(Optional) Shortcut Method



(Optional) Shortcut Method



What happens "in between"?

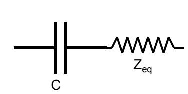
We know V(t) has exponential decay with time constant of L/Z_0

(Optional) Summary of Shortcut Method

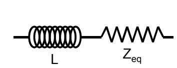
- Solve for voltage and current with inductive or reactive elements in their initial uncharged state
  - Inductor: Open
  - Capacitor: Short
- Solve for voltage and current with elements in their final charged state
  - Inductor: Short
  - Capacitor: Open
- Solve for circuit time constant for exponential function that occurs between initial and final states

(Optional) Shortcut Method

$$\tau = Z_{eq} C$$



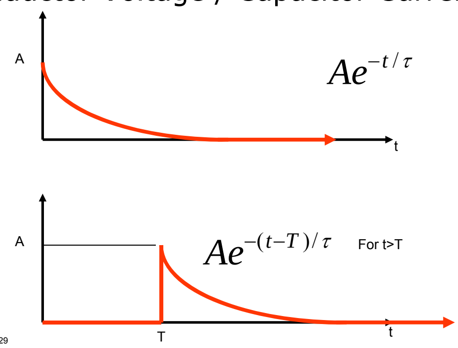
$$\tau = \frac{L}{Z_{eq}}$$



(Optional)

Shortcut Method

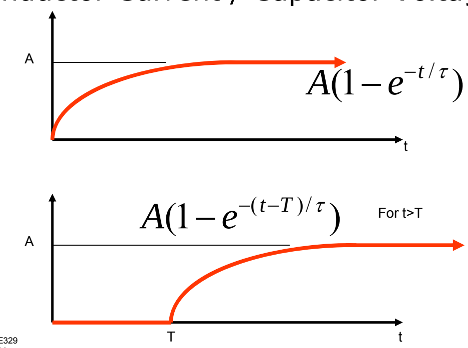
Inductor Voltage / Capacitor Current



(Optional)

Shortcut Method

Inductor Current / Capacitor Voltage



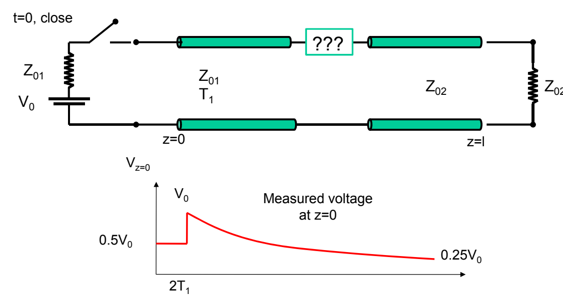
(Optional)

Shortcut Method Warning

- Only for reactive elements charged by DC source with no back reflections

(Optional)

P6.21 (p430) – TDR example



No initial charge or current. Determine the unknown single element (R, L, or C) and the ratio Z_{02}/Z_{01}

(Optional)

Calculation Space

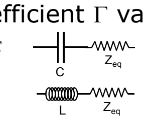
(Optional)

Lecture 30 Summary

- Reactive terminations and discontinuities require solving differential equations or studying start/end behavior & time constants
  - Reflection coefficient Γ varies in time

$\tau = Z_{eq}C$

$\tau = \frac{L}{Z_{eq}}$



- Next class
  - Chapter 7 (TL in frequency domain)

ECE 329
Lectures 31-34

Sections 7.A, 7.3, 7.1
(Section 31-34 in Online Notes)

Line Terminated by an Arbitrary Load

Smith Charts

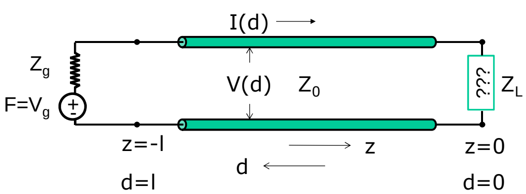
Short and Open Circuited TLs

Half and Quarter Wave Transformers

Example

- Find Γ_L for a TL with $Z_0=60\Omega$ that is terminated with an RLC series combo: $R=30\Omega$, $L=1\mu H$, $C=100pF$ at the following radian frequencies (a) $\omega=10^8$ and (b) $\omega=2\times10^8$
  - Hint: $Z_c=1/(j\omega C)$ and $Z_L=j\omega L$

TL's in Co-sinusoidal steady state



Distributed circuits are best handled with phasor and Fourier techniques and with Smith charts

$$f(t)=\mathrm{Re}[\tilde{F}e^{j\omega t}]$$
$$V(z,t)=\mathrm{Re}[\tilde{V}(z)e^{j\omega t}]$$
$$I(z,t)=\mathrm{Re}[\tilde{I}(z)e^{j\omega t}]$$

Phasors satisfy usual TL equations

$$\left.\begin{aligned}\frac{\partial V}{\partial z}&=-\mathcal{L}\frac{\partial I}{\partial t}\\ \frac{\partial I}{\partial z}&=-\mathcal{C}\frac{\partial V}{\partial t}\end{aligned}\right\} \quad \left.\begin{aligned}\frac{d\tilde{V}}{dz}&=-\mathcal{L}(j\omega\tilde{I})\\ \frac{d\tilde{I}}{dz}&=-\mathcal{C}(j\omega\tilde{V})\end{aligned}\right\} \quad \begin{aligned}\frac{d^2\tilde{V}}{dz^2}&=-\mathcal{L}\mathcal{C}\omega^2\tilde{V}\\ \frac{d^2\tilde{I}}{dz^2}&=-\mathcal{L}\mathcal{C}\omega^2\tilde{I}\end{aligned}$$

$$\boxed{\tilde{V}=V^{\pm}e^{\mp j\beta z}} \quad \boxed{\beta=\omega\sqrt{\mathcal{L}\mathcal{C}}} \quad \boxed{\tilde{V}=V^{+}e^{j\beta d}+V^{-}e^{-j\beta d}}$$

$$\boxed{\tilde{I}=\pm\frac{V^{\pm}}{Z_0}e^{\mp j\beta z}} \quad \boxed{Z_0=\sqrt{\mathcal{L}/\mathcal{C}}} \quad \boxed{\tilde{I}=\frac{1}{Z_0}(V^{+}e^{j\beta d}-V^{-}e^{-j\beta d})}$$

Boundary Condition at Load

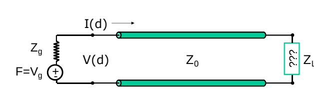
$$\underbrace{\boxed{\Gamma_L=\frac{Z_L-Z_0}{Z_L+Z_0}} \quad \boxed{V^{-}=\Gamma_L V^{+}}}$$

$$\tilde{V}(d)=V^{+}e^{j\beta d}+V^{-}e^{-j\beta d}=V^{+}e^{j\beta d}(1+\Gamma_L e^{-2j\beta d})$$

$$\tilde{I}(d)=\frac{1}{Z_0}(V^{+}e^{j\beta d}-V^{-}e^{-j\beta d})=\frac{V^{+}}{Z_0}e^{j\beta d}(1-\Gamma_L e^{-2j\beta d})$$

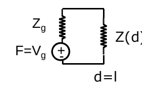
Key Definition: Line Impedance

$$\boxed{Z(d)\equiv\frac{\tilde{V}(d)}{\tilde{I}(d)}}$$



$$\boxed{Z(d)=Z_0\frac{1+\Gamma_L e^{-2j\beta d}}{1-\Gamma_L e^{-2j\beta d}}}$$

Equivalent Circuit

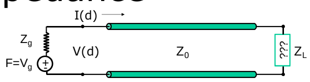


$$\tilde{V}(l)=\tilde{F}\frac{Z(l)}{Z(l)+Z_g}=V^{+}(e^{j\beta l}+\Gamma_L e^{-j\beta l})$$

Allows you to solve for V^+ and thus get V(d,t) and I(d,t)

Challenge Question: Line Impedance

$$Z(d) \equiv \frac{\tilde{V}(d)}{\tilde{I}(d)}$$



$$Z(d) = Z_0 \frac{1+\Gamma_L e^{-2j\beta d}}{1-\Gamma_L e^{-2j\beta d}}$$

- What is the smallest distance d from the load for which input impedance = load impedance?

(a) $d_{min}=\lambda/8$
(b) $d_{min}=\lambda/4$
(c) $d_{min}=\lambda/3$
(d) $d_{min}=\lambda/2$
(e) $d_{min}=\lambda$

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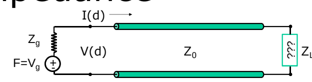
Lectures 31-34

LG's question

Challenge Question: Line Impedance

$$Z(d) \equiv \frac{\tilde{V}(d)}{\tilde{I}(d)}$$



$$Z(d) = Z_0 \frac{1+\Gamma_L e^{-2j\beta d}}{1-\Gamma_L e^{-2j\beta d}}$$

- What is the smallest distance d from the load for which input voltage = load voltage?

(a) $d_{min}=\lambda/8$
(b) $d_{min}=\lambda/4$
(c) $d_{min}=\lambda/3$
(d) $d_{min}=\lambda/2$
(e) $d_{min}=\lambda$

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LG's question

Key Definition: Generalized Reflection Coefficient

$$\Gamma(d) \equiv \frac{\tilde{V}^-(d)}{\tilde{V}^+(d)}$$

$$\Gamma(d) = \frac{V^- e^{-j\beta d}}{V^+ e^{j\beta d}} = \Gamma_L e^{-2j\beta d}$$

Allows you to find the backwards wave if forward wave is known

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Key Definitions: Admittance and Normalized Impedance

Characteristic Admittance $\quad Y_0 \equiv \frac{1}{Z_0}$

Normalized Impedance $\quad z(d) \equiv \frac{Z(d)}{Z_0} \quad z(d) = \frac{1+\Gamma(d)}{1-\Gamma(d)}$

Normalized Admittance $\quad y(d) \equiv \frac{1}{z(d)} \quad y(d) = \frac{1-\Gamma(d)}{1+\Gamma(d)}$

$$\Gamma(d) = \Gamma_L e^{-2j\beta d}$$

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Converting between Impedance and Reflection

$$\Gamma(d) = \Gamma_L e^{-2j\beta d}$$

$$z(d) = \frac{1+\Gamma(d)}{1-\Gamma(d)}$$

Invert to solve for Γ(d)

$$\Gamma(d) = \frac{z(d)-1}{z(d)+1}$$

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Smith Chart

$$\Gamma = \frac{z-1}{z+1}$$

$z = r + jx$

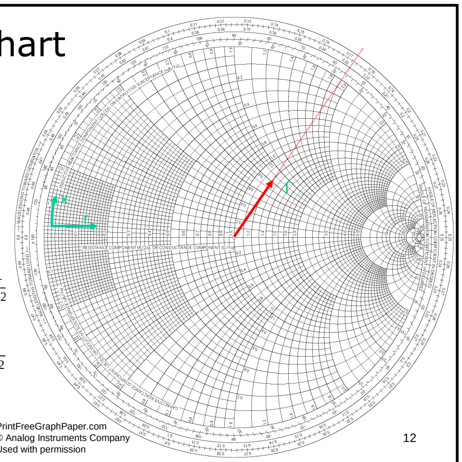
$r \geq 0$

$\Gamma = \Gamma_r + j\Gamma_i$

$$\Gamma_r = \frac{r^2+x^2-1}{(r+1)^2+x^2}$$

$$\Gamma_i = \frac{2x}{(r+1)^2+x^2}$$

$|\Gamma| \leq 1$



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Lecture 32

Short and Open Circuited Lines

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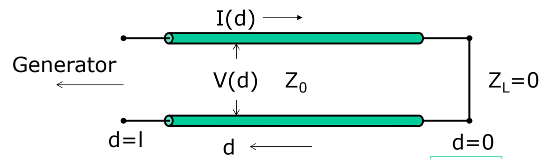
Standing Waves for SC Line



$\boxed{\Gamma_L = -1}$

$$\tilde{V}(d) = V^+(e^{j\beta d} - e^{-j\beta d}) = 2jV^+\sin(\beta d)$$

$$\tilde{I}(d) = \frac{V^+}{Z_0}(e^{j\beta d} + e^{-j\beta d}) = 2Y_0V^+\cos(\beta d)$$

$$Z(d) \equiv \frac{\tilde{V}(d)}{\tilde{I}(d)} = jZ_0\tan\beta d$$

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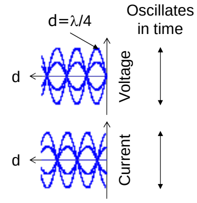
Standing Waves for SC Line

$$V(d,t) = \mathrm{Re}[\tilde{V}(d)e^{j\omega t}] = \mathrm{Re}[2jV^+\sin(\beta d)e^{j\omega t}]$$
$$= \mathrm{Re}[2e^{j\pi/2}|V^+|e^{j\theta}\sin(\beta d)e^{j\omega t}]$$
$$= 2|V^+|\sin(\beta d)\mathrm{Re}[e^{j(\omega t+\theta+\pi/2)}]$$
$$= 2|V^+|\sin(\beta d)\cos(\omega t+\theta+\pi/2)$$
$$= -2|V^+|\sin(\beta d)\sin(\omega t+\theta)$$

$$I(d,t) = 2Y_0|V^+|\cos(\beta d)\cos(\omega t+\theta)$$

$\beta=2\pi/\lambda$



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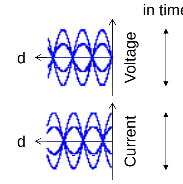
Standing Waves for SC Line

$$V(d,t) = -2|V^+|\sin(\beta d)\sin(\omega t+\theta)$$

$$I(d,t) = 2Y_0|V^+|\cos(\beta d)\cos(\omega t+\theta)$$



V(0,t)=0 always (voltage null)
I(0,t) varies (current maxima)

Dependence is different than traveling wave: $\omega t \pm \beta z$

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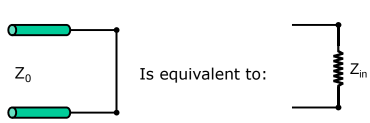
Input Impedance for SC Line

$$Z_{in} = Z(l) = jZ_0\tan\beta l = jZ_0\tan\frac{2\pi f l}{v_p}$$



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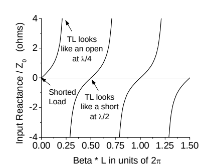
SC Line can act as an inductor or a capacitor depending on βl

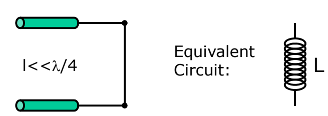
$$Z_{in} = jZ_0\tan\beta l$$

If $\tan(\beta l)>0$, shorted TL is inductive

If $\tan(\beta l)<0$, shorted TL is capacitive



e.g. $\beta l<\pi/2$ or $l<\lambda/4$, TL is inductive



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Lectures 31-34

Show that the equivalent inductance is L=ℒl if λ>>4l

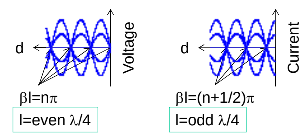
$Z_{in} = jZ_0 \tan \beta l$

$Z_L = j\omega L$

For l=even λ/4, TL is a short For l=odd λ/4, TL is an open

$$Z_{in} = jZ_0 \tan \beta l = \begin{cases} 0 = a\, short\ for\ \beta l = n\pi,\ \ n = 0,1,2,... \\ \infty = an\, open\ for\ \beta l = (n+1/2)\pi \end{cases}$$

If Z_{in}=0, voltage drop is zero, just like a short
If Z_{in}=∞, current is zero, just like an open

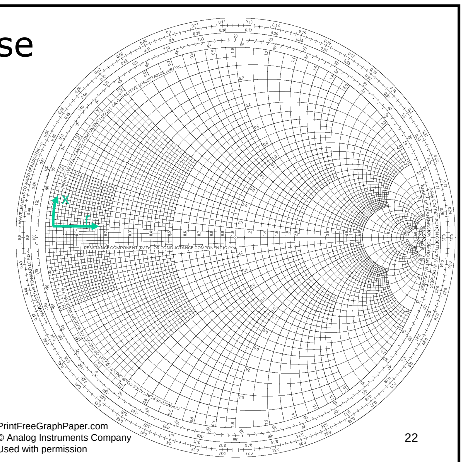


Example

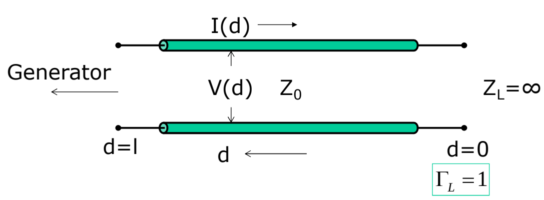
- A TL is shorted and your job is to find the location to fix it. You are equipped with a tunable frequency generator and an ammeter. Compare this to TDR.

Exercise

Locate z(0) and Γ(0) of the shorted line on the S.C., and observe how z(d) and Γ(d) vary as d increases, noting in particular to what happens at d = λ/4 (open conditions are reached) and at d = λ/2 (back to short conditions)



Standing Waves for OC Line



$$\tilde{V}(d) = V^+(e^{j\beta d} + e^{-j\beta d}) = 2V^+\cos(\beta d)$$

$$\tilde{I}(d) = \frac{V^+}{Z_0}(e^{j\beta d} - e^{-j\beta d}) = 2jY_0V^+\sin(\beta d)$$

$$Y(d) \equiv \frac{1}{Z(d)} = \frac{\tilde{I}(d)}{\tilde{V}(d)} = jY_0 \tan \beta d$$

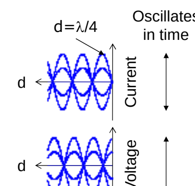
Standing Waves for OC Line

Same phasor algebra as before with current & voltage reversed!

β=2π/λ

$$I(d,t) = -2Y_0|V^+|\sin(\beta d)\sin(\omega t + \theta)$$

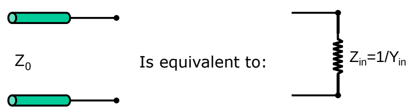
$$V(d,t) = 2|V^+|\cos(\beta d)\cos(\omega t + \theta)$$



I(0,t)=0 always (current null)
V(0,t) varies (voltage maxima)

Input Admittance for OC Line

$$Y_{in} = Y(l) = jY_0 \tan \beta l = jY_0 \tan \frac{2\pi f l}{v_p}$$

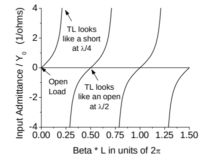


OC Line can act as an inductor or a capacitor depending on βl

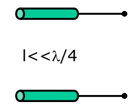
$$Y_{in} = jY_0 \tan \beta l$$

If tan(βl)<0, shorted TL is inductive

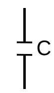
If tan(βl)>0, shorted TL is capacitive



e.g. βl<π/2 or l<λ/4, TL is capacitive



Equivalent Circuit:



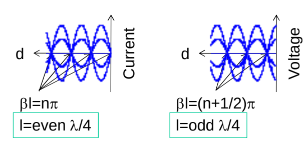
Show that the equivalent capacitance is C=𝒞l if λ>>4l

$$Y_{in} = jY_0 \tan \beta l$$
$$Y_C = j\omega C$$

For l=even λ/4, TL is an open For l=odd λ/4, TL is a short

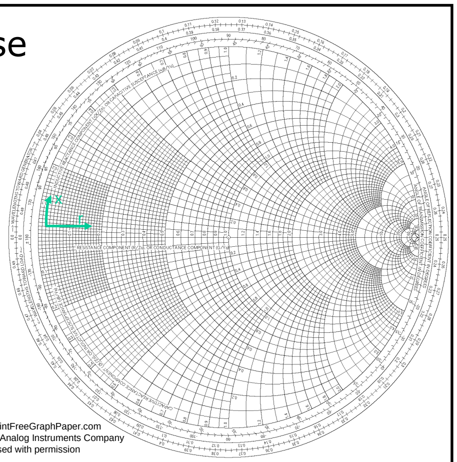
$$Y_{in} = jY_0 \tan \beta l = \begin{cases} 0 = an\, open\ for\ \beta l = n\pi,\ \ n = 0,1,2,... \\ \infty = a\, short\ for\ \beta l = (n+1/2)\pi \end{cases}$$

If $Y_{in}=0$, current is zero, just like an open
If $Y_{in}=\infty$, voltage drop is zero, just like a short



Exercise

Locate z(0) and Γ(0) of the shorted line on the O.C., and observe how z(d) and Γ(d) vary as d increases, noting in particular to what happens at d = λ/4 (short conditions are reached) and at d = λ/2 (back to open conditions)



y(d) is z(d) shifted by λ/4

- Proof:

$$z(d \pm \lambda/4) = \frac{1+\Gamma(d\pm\lambda/4)}{1-\Gamma(d\pm\lambda/4)} = \frac{1+\Gamma(d)e^{\pm 2j\beta\lambda/4}}{1-\Gamma(d)e^{\pm 2j\beta\lambda/4}}$$

$$2\beta\frac{\lambda}{4} = 2\frac{2\pi}{\lambda}\frac{\lambda}{4} = \pi$$

$$= \frac{1+\Gamma(d)e^{\pm j\pi}}{1-\Gamma(d)e^{\pm j\pi}} = \frac{1-\Gamma(d)}{1+\Gamma(d)} = y(d)$$

$$\boxed{y(d) = z(d \pm \lambda/4)}$$

Thus, the Smith Chart can be used for both z and y

Opens and shorts exchange with each other every λ/4

Lectures 31-32 Summary

- As d increases by $\lambda/4$, SC and OC TL switch from being a short (V=0) to an open (I=0)
- Smith Chart is a bilinear transformation of the half plane z(d) ($r\geq 0$) onto the unit circle $|\Gamma|\leq 1$

$$\boxed{\Gamma(d)=\frac{z(d)-1}{z(d)+1}} \qquad \boxed{z(d)=\frac{1+\Gamma(d)}{1-\Gamma(d)}}$$

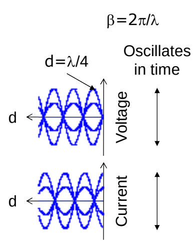
- How to Use Smith Chart
  1. Calculate z(0) and find it on chart using r and x
  2. Find $\Gamma(0)$ as the distance and angle from origin
  3. Move CW along circle of radius $|\Gamma(0)|$ to obtain $\Gamma(d)$
  4. Read off z(d) by looking at grid location (r,x)
  5. If needed, find y(d) on the same circle, 180° away

ECE 329
Lecture 33

Microwave Resonators

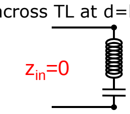
Natural Resonances

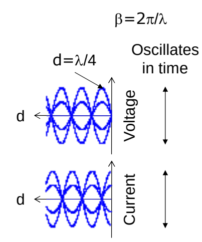
- A lossless T.L. of any length l with open and/or short terminations on either ends can be considered a "microwave resonator"
  - It can sustain unforced voltage and current standing-wave oscillations at a set of discrete resonance frequencies ω_n
  - Example, for SC TL, $z_{in}=\infty$ (open) or 0 (short) has standing waves for the set of λ_n satisfying $l=n\lambda_n/4$



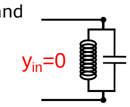
Parallel & Series Resonances

- We can find λ_n and ω_n by applying the appropriate BCs at both ends
- Series resonance if $z_{in}(l)=0$
  - Analogous to an LC circuit in series and requires a short placed across TL at d=l
  - Like a short input

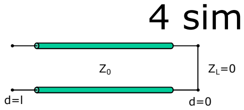




- Parallel resonance: $y_{in}(l)=0$
  - Analogous to an LC circuit in parallel and requires an open across the TL at d=l
  - Like an open input

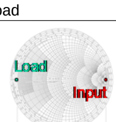


Parallel & Series Resonances 4 simple cases

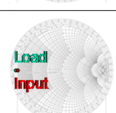


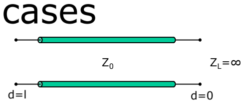
Shorted Load

L=λ/4, 3λ/4, 5λ/4, …
Input is like an open
➔ **Parallel** resonance



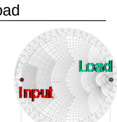
L=λ/2, λ, 3λ/2, …
Input is like a short
➔ **Series** resonance



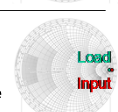


Open Load

L=λ/4, 3λ/4, 5λ/4, …
Input is like a short
➔ **Series** resonance



L=λ/2, λ, 3λ/2, …
Input is like an open
➔ **Parallel** resonance



Examples

a) Find the 3 lowest frequencies for parallel resonances if the load is shorted and v=c, l=3m.

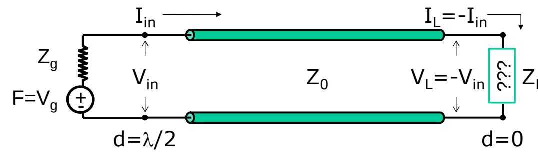
b) Find the resonance frequencies for a 10m long TL that is open circuited at both ends if v = 2/3 c.

ECE 329
Lecture 34

Half-wave and quarter-wave transformers

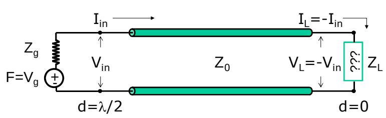
Half-wave transformers

- Given a fixed drive frequency ω, there is a length of line $l=\lambda/2$ such that:



- Note: Current and voltage both invert their algebraic signs

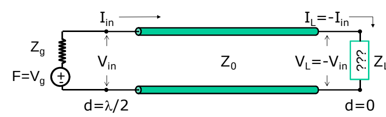
Half-wave transformers



- Proof:

$$e^{\pm j\beta\lambda/2}=e^{\pm j\pi}=-1$$
$$\tilde{V}(d)=V^+e^{j\beta d}+V^-e^{-j\beta d}$$
$$\therefore \tilde{V}_{in}=\tilde{V}(\lambda/2)=-V^+-V^-=-\tilde{V}(0)=-\tilde{V}_L$$
$$\tilde{I}_{in}=\frac{V^+e^{j\beta d}-V^-e^{-j\beta d}}{Z_0}=\frac{-V^++V^-}{Z_0}=-\tilde{I}_L$$

Challenge Question: Half-wave transformers



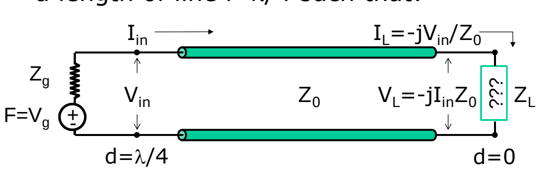
- What is the input impedance of a half-wave transformer?

(a) $Z_{in} = Z_0$
(b) $Z_{in} = Z_0+Z_L$
(c) $Z_{in} = 1/(1/Z_0+1/Z_L)$
(d) $Z_{in} = Z_L$
(e) $Z_{in} = Z_0^2/Z_L$

LG's question

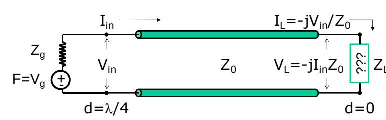
Quarter-wave transformers

- Given a fixed drive frequency ω, there is a length of line $l=\lambda/4$ such that:



- Note: The current through the load does not depend on Z_L (current-forcing)

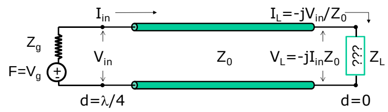
Quarter-wave transformers



- Proof:

$$e^{\pm j\beta\lambda/4}=e^{\pm j\pi/2}=\pm j$$
$$\tilde{V}(d)=V^+e^{j\beta d}+V^-e^{-j\beta d}$$
$$\tilde{V}_{in}=jV^+-jV^-=j\tilde{I}_LZ_0\Rightarrow\tilde{I}_L=-j\tilde{V}_{in}/Z_0$$
$$\tilde{I}_{in}=\frac{jV^+-(-jV^-)}{Z_0}=j\frac{\tilde{V}_L}{Z_0}\Rightarrow\tilde{V}_L=-j\tilde{I}_{in}Z_0$$

Challenge Question: Quarter-wave transformers

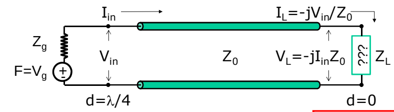


- What is the input impedance of a quarter-wave transformer?

(a) $Z_{in} = Z_0$
(b) $Z_{in} = Z_0+Z_L$
(c) $Z_{in} = 1/(1/Z_0+1/Z_L)$
(d) $Z_{in} = Z_L$
(e) $Z_{in} = Z_0^2/Z_L$

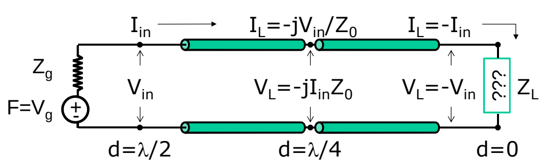
LG's question

Verifying an Identity



- Verify the Lecture 32 result: $\boxed{y(d) = z(d \pm \lambda/4)}$

Half-wave transformer as 2 quarter-wave transformer



Examples

- For Z_L=50+50jΩ, find Z_{in} for a λ/4 transformer with Z_0=50Ω.
- Check your answer with a Smith Chart.
- Find V_{in} if a source with open circuit voltage V_g=100V and Thevenin impedance Z_g=j25Ω is connected
- Find I_L and <P_L>.

Examples

- For Z_L=100Ω, find Z_{in} for a 3λ/4 TL if Z_0=50Ω.
- Check your answer with a Smith Chart.
- Find V_L and I_L if a source with open circuit voltage V_g=j10V and Thevenin impedance Z_g=25Ω is connected.

Lecture 34 Summary

- TL transformers can be used to change the load impedance Z_L to a new value as seen at the input port Z_{in} and thus adjust V_L and I_L
- If the TL is λ/4, then we have current forcing:

$$\tilde{I}_L = -j\tilde{V}_{in}/Z_0 \qquad Z_L Z_{in} = Z_0^2$$
$$\tilde{V}_L = -j\tilde{I}_{in}Z_0$$

- If the TL is λ/2, then I and V both change their own signs:

$$\tilde{I}_L = -\tilde{I}_{in} \qquad Z_L = Z_{in}$$
$$\tilde{V}_L = -\tilde{V}_{in}$$

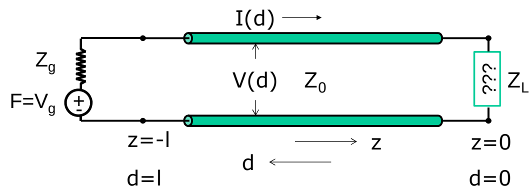
ECE 329
Lectures 35-36
Rao - Sections 7.2, 7.3
Online Notes – 35-36

Line Terminated by an Arbitrary Load

Standing Wave Parameters

Smith Charts

TL's in Co-sinusoidal steady state



Imposed condition on load end will result in some form of standing wave oscillation

$$f(t)=\mathrm{Re}[\tilde{F}e^{j\omega t}]$$
$$V(z,t)=\mathrm{Re}[\tilde{V}(z)e^{j\omega t}]$$
$$I(z,t)=\mathrm{Re}[\tilde{I}(z)e^{j\omega t}]$$

Summary of Equations

$$\Gamma_L=\frac{Z_L-Z_0}{Z_L+Z_0} \qquad V^-=\Gamma_L V^+ \qquad \Gamma(d)=\Gamma_L e^{-2j\beta d}$$

$$\tilde{V}(d)=V^+e^{j\beta d}(1+\Gamma_L e^{-2j\beta d})=V^+e^{j\beta d}(1+\Gamma(d))$$

$$\tilde{I}(d)=\frac{V^+e^{j\beta d}}{Z_0}(1-\Gamma_L e^{-2j\beta d})=\frac{V^+e^{j\beta d}}{Z_0}(1-\Gamma(d))$$

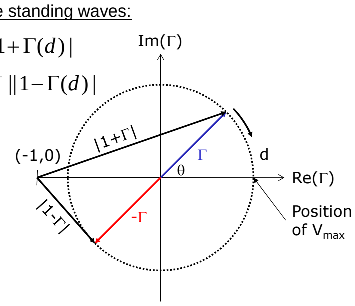
$$Z(d)\equiv\frac{\tilde{V}(d)}{\tilde{I}(d)}=Z_0\frac{1+\Gamma(d)}{1-\Gamma(d)} \qquad z(d)=\frac{1+\Gamma(d)}{1-\Gamma(d)}$$

$$\Gamma(d)=\frac{z(d)-1}{z(d)+1}$$

Standing Wave Parameters

Amplitude of the standing waves:

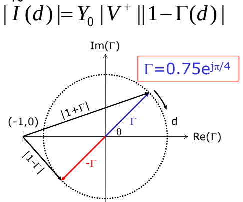
$$|\tilde{V}(d)|=|V^+||1+\Gamma(d)|$$
$$|\tilde{I}(d)|=Y_0|V^+||1-\Gamma(d)|$$

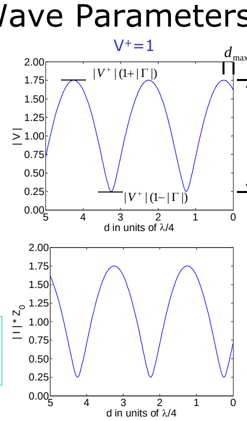


Where would V_{min} be?
What about I_{min} and I_{max}?

Standing Wave Parameters

$$|\tilde{V}(d)|=|V^+||1+\Gamma(d)|$$
$$|\tilde{I}(d)|=Y_0|V^+||1-\Gamma(d)|$$





$$d_{max}=\frac{\lambda}{4\pi}\theta$$

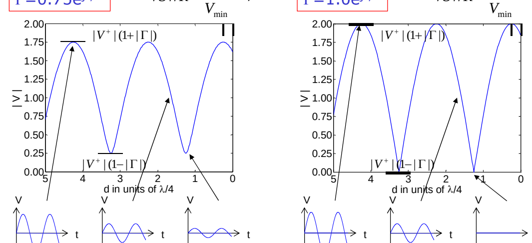
$$VSWR=\frac{V_{max}}{V_{min}}=\frac{1+|\Gamma|}{1-|\Gamma|}$$

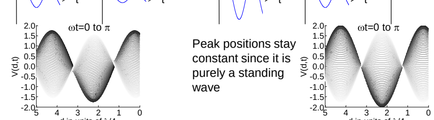
Unlike SC or OC where $|\Gamma|=1$, now we have imperfect nulls for voltage and current b/c $|\Gamma|<1$

Standing Wave Parameters

$\Gamma=0.75e^{j\pi/4}$ $\quad VSWR=\dfrac{V_{max}}{V_{min}}=7$

$\Gamma=1.0e^{j\pi/4}$ $\quad VSWR=\dfrac{V_{max}}{V_{min}}=\infty$

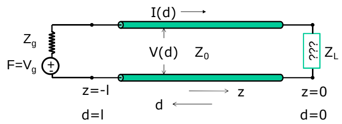




Peak positions move in time since it is a standing wave plus a travelling wave

Peak positions stay constant since it is purely a standing wave

Challenge Question: VSWR



- If $Z_L=Z_0=50\Omega$ and $Z_g=100\Omega$, what will be the VSWR?

(a) ∞

(b) 2

(c) 1

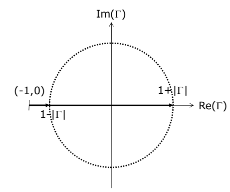
(d) 1/2

(e) 0

LG's question

Standing Wave Useful Facts

$$\boxed{VSWR = \frac{V_{\max}}{V_{\min}} = \frac{1+|\Gamma|}{1-|\Gamma|} = z(d_{\max})}$$



Thus,

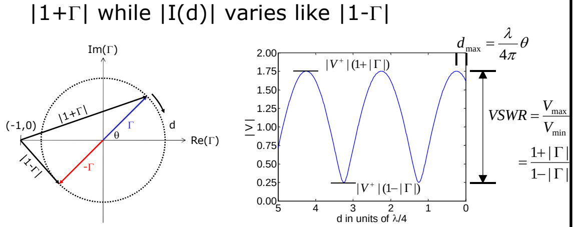
$$\boxed{VSWR = z(d_{\max}) = y(d_{\min})}$$

because

$$\boxed{y(d) = z(d \pm \lambda/4)}$$

Lecture 35 Summary

- As d increases, amplitude |V(d)| varies like |1+Γ| while |I(d)| varies like |1-Γ|



- VSWR = V_{max}/V_{min} = $z(d_{max})$

ECE 329
Lecture 36
Sections 7.2

More Standing Wave Parameters and More Practice with Smith Charts

Example: VSWR measurements

- Find Z_L if VSWR measurements are made on a line with $Z_0=60\Omega$:

(a) SWR = 1.5 and d_{min} = 0

(b) SWR = 3.0 and d_{min} = 3 and 9 cm

(c) SWR = 2.0 and d_{min} = 3 and 7 cm

From D7.5 (p 463) of old book

Calculation Space

Example: Input Impedance

- Find Z_{in} if Z_L=45+60jΩ and Z_0=75Ω and v_p=c for the following cases:

(a) f = 15MHz, l = 5m

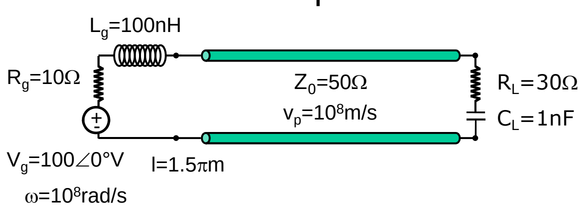
(b) f = 50MHz, l = 3m

(c) f = 37.5MHz, l = 5m

From D7.6 (p 463) of old book

Calculation Space

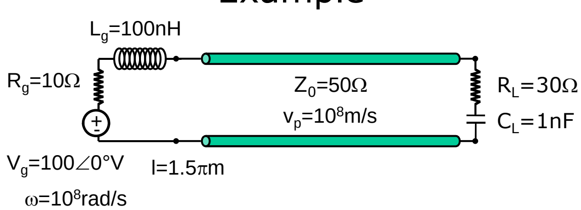
Example



- Using SC, determine: (a) z(0), (b) Γ(0), (c) VSWR, (d) locations of V_{max}

Calculation Space

Example



- Continue the problem and using SC, determine: (e) Γ(l), (f) Z_{in}, (g) V(l), (h) I(l), (i) <P>

Calculation Space

ECE 329
Lectures 37-39
Sections 7.3
Online Notes: 37-39

Average Power

Quarter Wave Transformer Matching

Single Stub Matching

(Optional) Double Stub Matching

Distribution Networks

Lossy Line

Average Power

- In a lossless TL circuit,

$$\langle P_{in}\rangle = \langle P(d)\rangle = \langle P_L\rangle$$

Note that $\langle P(d)\rangle = \frac{1}{2}\mathrm{Re}\left[\tilde{V}(d)\tilde{I}^*(d)\right]$

Hint: Re[z-z*]=0

$$= \ldots = \frac{1}{2}\left(\frac{|V^+|^2}{Z_0} - \frac{|V^-|^2}{Z_0}\right) = \frac{1}{2}\frac{|V^+|^2}{Z_0}(1-|\Gamma_L|^2)$$

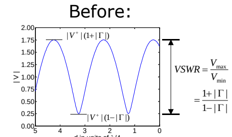
and so $|\Gamma_L|^2$ represents the power reflection coefficient

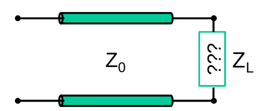
Impedance matching

- When $Z_L \neq Z_0$, power is reflected back to the generator and VSWR>1 (bad)
- Impedance matching achieves VSWR=1 by adjusting the input impedance, Z_{in}, to be equal to the TL characteristic impedance, Z_0

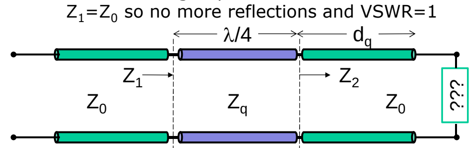
Quarter Wave Matching Transformer

Before:



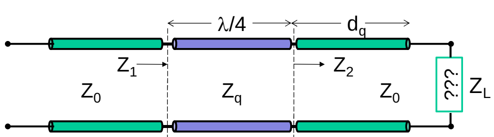


After inserting a quarter-wave transformer, $Z_1=Z_0$ so no more reflections and VSWR=1



Adjust Z_q and d_q for match

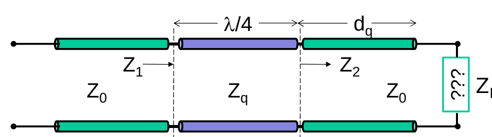
Quarter Wave Matching Transformer



Key Observations:

1. Z_0 and Z_q are real since TL's are lossless
2. $Z_1=Z_0$ for a match so Z_1 must be real
3. $Z_1Z_2=Z_q^2$ since we have a QW transformer
4. Z_2 must be real from equation #3
5. d_q must be at a voltage max or min from #4

Quarter Wave Matching Transformer

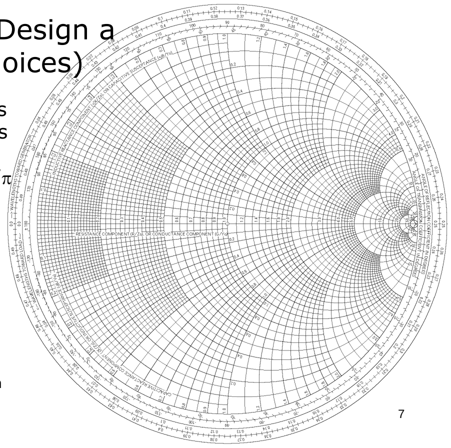


Solution: $Z_q = \sqrt{Z_0 Z_2} = Z_0\sqrt{z_2}$

$$d_q = \begin{cases} d_{\max} = \dfrac{\lambda\theta}{4\pi} \\ d_{\min} = \dfrac{\lambda\theta}{4\pi} + \dfrac{\lambda}{4} \end{cases} \mod \lambda/2$$

$\theta = angle(\Gamma_L)$

Example: Design a QWT (2 choices)

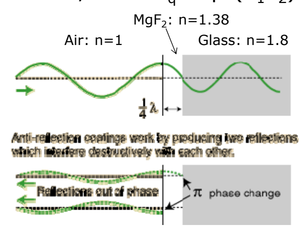


Z_0=50Ω; load is R=30Ω in series with L=1nH for λ=10mm, v_p=c/π

Hints:
Find z(0)
Get $|\Gamma_L|$ & θ
Find d_{max}, d_{min}
Get both Z_2's

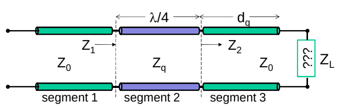
Similar to Optics

- Anti-reflection coatings:
  - thickness=λ/4 and n_q=sqrt(n_1n_2)



http://hyperphysics.phy-astr.gsu.edu/Hbase/phyopt/antiref.html

Challenge Question: QWT matching



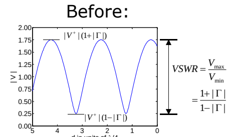
- For QWT matching, which is **false**:

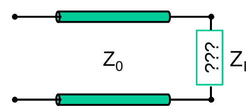
(a) VSWR=1 in segment 1
(b) segment 1 can have any length
(c) there is a voltage min or max at both the left and right edges of segment 2
(d) Γ(d)=1 in segment 3
(e) d_q can be increased by integer multiples of λ/2 without affecting the matching

LG's question

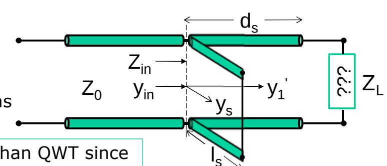
Single Stub Matching

Before:





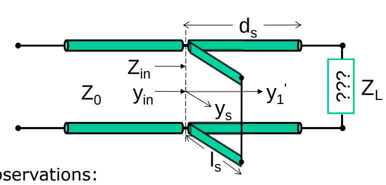
After inserting a shorted stub in parallel, y_{in}=1 so no more reflections and VSWR=1



More convenient than QWT since stub has same impedance

Adjust d_s and l_s for match

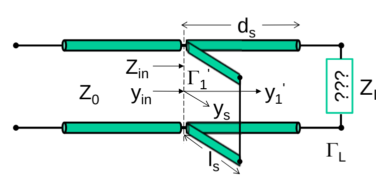
Single Stub Matching



Key Observations:

1. y_{in}=1 for a match because Z_{in}=Z_0 → y_{in}=Z_0/Z_{in}=1
2. y_{in}=y_1'+y_s since admittance adds for parallel elements
3. y_s=1/(j tan βl_s)=jb is purely imaginary for SC line
   a) See Lect 32, slide 18: Z_{in}=jZ_0tan(βl_s)
   b) Needed amount of susceptance, b, depends on $|\Gamma_L|$

(Optional)
Single Stub Matching



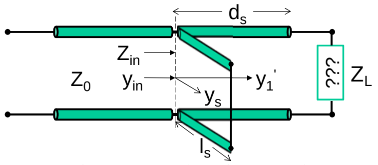
$$y_1' = y_{in} - y_s = 1 - jb$$

$$\Gamma_1' = \frac{1-y_1'}{1+y_1'} = \frac{jb}{2-jb} = \Gamma_L e^{-2j\beta d_s} \Rightarrow \boxed{|\Gamma_L| = \frac{|b|}{\sqrt{4+b^2}}}$$

$$\boxed{\therefore b = \pm\frac{2|\Gamma_L|}{\sqrt{1-|\Gamma_L|^2}}}$$

(Optional)

Single Stub Matching



$\beta = \frac{2\pi}{\lambda}$

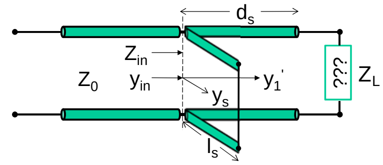
$$\theta = \angle\Gamma_L = \angle\left(\frac{jb}{2-jb}e^{2j\beta d_s}\right) = \left(\angle\frac{jb}{2-jb}\right) + 2\beta d_s$$

$$\angle\frac{jb}{2-jb} = \angle\frac{jb(2+jb)}{4+b^2} = \angle jb + \angle(2+jb) = \pm\frac{\pi}{2} + \tan^{-1}\left(\frac{b}{2}\right) \quad \begin{matrix} if\ b>0 \\ if\ b<0 \end{matrix}$$

$$\boxed{\therefore d_s = \frac{\lambda}{4\pi}\left[\theta \mp \frac{\pi}{2} - \tan^{-1}\left(\frac{b}{2}\right)\right] \mod \lambda/2}$$

(Optional)

Single Stub Matching

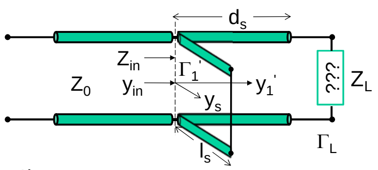


$\beta = \frac{2\pi}{\lambda}$

$$y_s = jb = \frac{1}{j\tan(\beta l_s)} \Rightarrow \tan(\beta l_s) = -\frac{1}{b}$$

$$\boxed{\therefore l_s = \frac{\lambda}{2\pi}\tan^{-1}\left(-\frac{1}{b}\right) \mod \lambda/2}$$

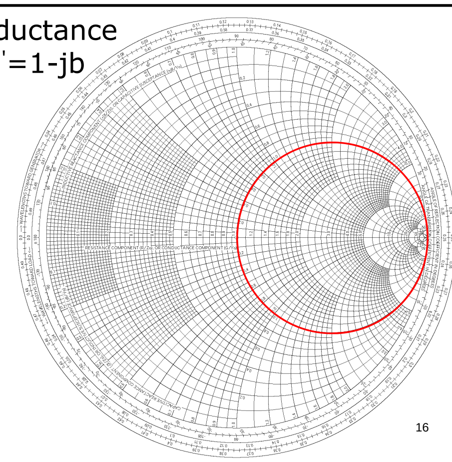
Smith Chart for Single Stub



Key Observations:

1. y_{in}=1 for a match, y_s=jb is purely imaginary for the SC stub
2. Thus $y_1'=y_{in}-y_s$=1-jb must be on the unit conductance circle

Unit Conductance Circle: y_1'=1-jb

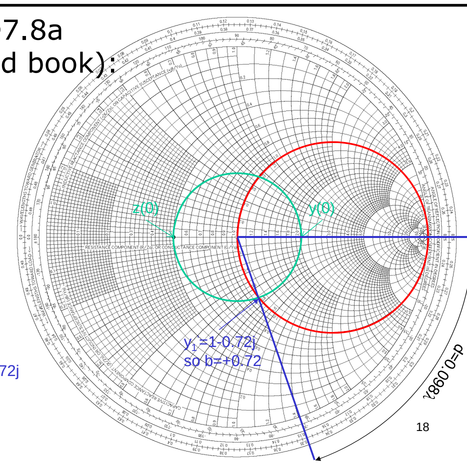


Smith Chart for Single Stub

1. Find z(0) on S.C.
2. Go ½ revolution around regular $|\Gamma(0)|$ circle to find y(0)
3. From y(0), move clockwise towards generator until intersection with unit conductance circle (distance is d_s)
4. Read off $y_1'=y(d_s)$; b=-Im(y_1')
5. Need a stub with susceptance +b so on a clean S.C., find l_s such that $y(l_s)$=jb starting from short: y(0)=∞

Repeat D7.8a (p 472 old book):

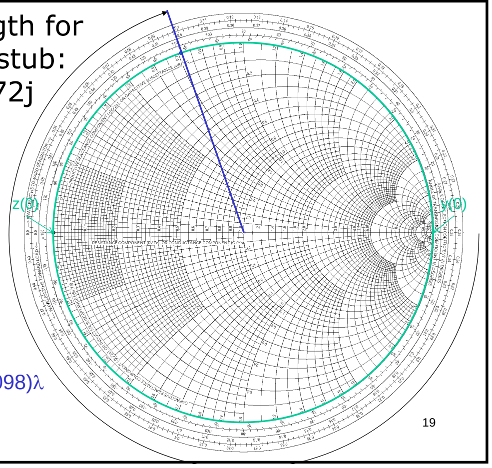
z(0)=0.5



Thus, we need a stub with $y_s(l_s)$=0.72j at a distance of 0.098λ from load

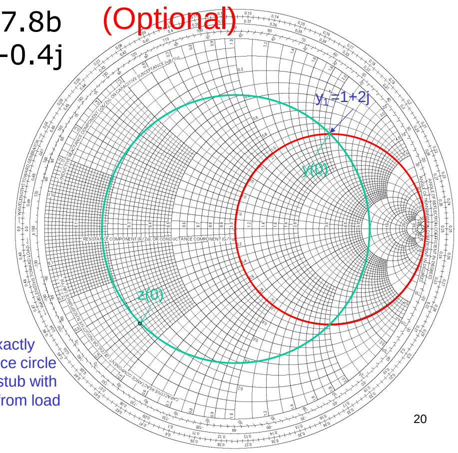
Find length for shorted stub:
$y(l_s)=0.72j$

$z(0)=0$
$y(0)=\infty$



$l_s=(0.25+0.098)\lambda$
$l_s=0.348\lambda$

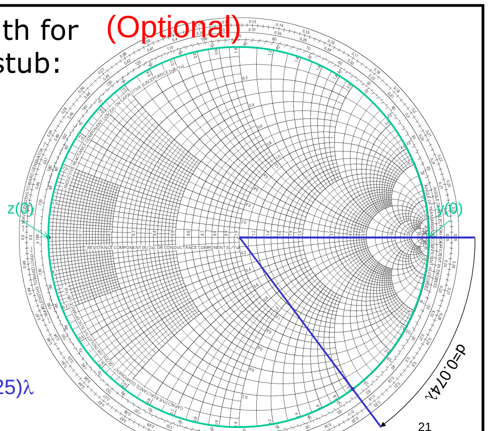
Repeat D7.8b (Optional)
$z(0)=0.2-0.4j$



We are already exactly on unit conductance circle
Thus, we need a stub with $y(l_s)=-2j$ at $d_s=0\lambda$ from load

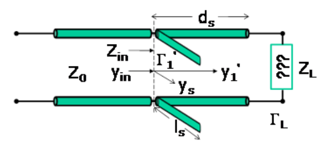
Find length for shorted stub: (Optional)
$y(l_s)=-2j$

$z(0)=0$
$y(0)=\infty$



$l_s=(0.324-0.25)\lambda$
$l_s=0.074\lambda$

Challenge question: Single stub tuning



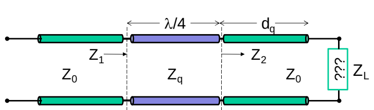
- If the stub is left open instead of being shorted, which is **false**:

(a) the same splice position d_s will work
(b) the needed admittance y_s is unchanged
(c) the admittance y_1' is unchanged
(d) the length l_s must be increased by $\lambda/4$ (mod $\lambda/2$)
(e) the positions of any voltage max or min along the stub are unchanged

LG's question

Lecture 38 Summary

- Quarter wave transformer matching inserts a length of $l=\lambda/4$ of a specific impedance Z_q=sqrt(Z_0Z_2) at a distance d_q from the load, where voltage is min or max



$$Z_q=\sqrt{Z_0Z_2}=Z_0\sqrt{z_2}$$

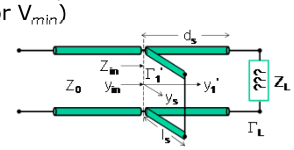
$$d_q=\begin{cases} d_{max}=\dfrac{\lambda\theta}{4\pi} \\ d_{min}=\dfrac{\lambda\theta}{4\pi}+\dfrac{\lambda}{4} \end{cases} \mod \lambda/2$$

Pros: Easy design
Cons: Have to redo QWT for each f

Lecture 38 Summary

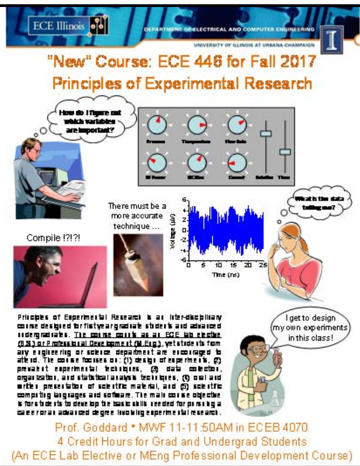
- Single stub matching inserts a shorted stub of the same impedance, Z_0, but with a specific length, l_s, and distance, d_s, from the load (not necessarily at V_{max} or V_{min})

Pros: $Z=Z_0$ on all lines
Cons: Adjusting d_s may be inconvenient



- Find d_s by moving CW from $y(0)$ to the unit conductance circle $y_1'=1-jb$; then find l_s by going CW from $y_s(0)=\infty$ to $y_s(l_s)=+jb$

$$\therefore l_s=\frac{\lambda}{2\pi}\tan^{-1}\left(-\frac{1}{b}\right) \mod \lambda/2$$



(Optional)

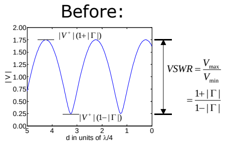
ECE 329
Lecture 38(b)

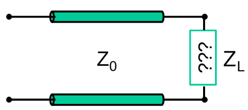
Double Stub Matching

(Optional)

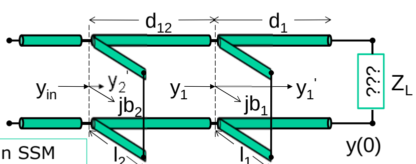
Double Stub Matching

Before:





After inserting two shorted stubs in parallel, $y_{in}=1$ so no more reflections and VSWR=1



More convenient than SSM since stub locations are fixed

Adjust l_1 and l_2 for match

(Optional)

Double Stub Matching

- Derivation proceeds similar to SSM, but algebra gets complicated quickly (p468)

$$y_2' = y_{in} - y_{s2} = 1 - jb_2$$

$$\Gamma_2' = \frac{1-y_2'}{1+y_2'} = \frac{jb_2}{2-jb_2} = \Gamma_1 e^{-2j\beta d_{12}} \Rightarrow \Gamma_1 = \frac{jb_2}{2-jb_2} e^{2j\beta d_{12}}$$

$$y_1 = \frac{1-\Gamma_1}{1+\Gamma_1}$$ Plug in Γ_1 from above

$$y_1' = y_1 - y_{s1} = y_1 - jb_1$$ Plug in y_1 from above

y_1' is known: start with y(0) and move CW by d_1 (predetermined); so just have to solve for b_1 and b_2 satisfying the real/imag parts of $y_1'=y_1-jb_1$

(Optional)

Double Stub Matching

- y_1 depends on b_2 and not b_1 so can solve for b_2 from real part of $y_1'=y_1-jb_1$

$$g' \equiv \mathrm{Re}[y_1'] = \mathrm{Re}[y_1] = \textit{function of } b_2$$

$$\boxed{b_2 = \frac{\cos\beta d_{12} \pm \sqrt{1/g' - \sin^2\beta d_{12}}}{\sin\beta d_{12}}}$$

Note that there will be no solution if $g' > 1/\sin^2\beta d_{12}$

(Optional)

Double Stub Matching

- With b_2 solved, can find b_1 from the imaginary part of $y_1'=y_1-jb_1$

$$b' \equiv \mathrm{Im}[y_1'] = \mathrm{Im}[y_1] - b_1 \Rightarrow$$

$$b_1 = \mathrm{Im}[y_1] - b' = \textit{function of } b_2$$

$$\boxed{b_1 = \frac{b_2^2 \sin 2\beta d_{12} - 2b_2 \cos 2\beta d_{12}}{2 - 2b_2 \sin 2\beta d_{12} + 2b_2^2 \sin^2\beta d_{12}} - b'}$$

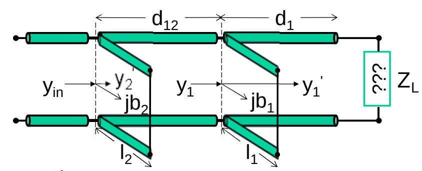
Finally, given b_1 and b_2, we can find l_1 and l_2 using:

$$\boxed{l_s = \frac{\lambda}{2\pi}\tan^{-1}(-\frac{1}{b}) \ \bmod \lambda/2}$$

(Optional)
Example: Design DSM

- $Z_0=50\Omega$, Termination is $Z=30-40j\Omega$ and we choose to fix $d_1=0$ and $d_{12}=0.375\lambda$

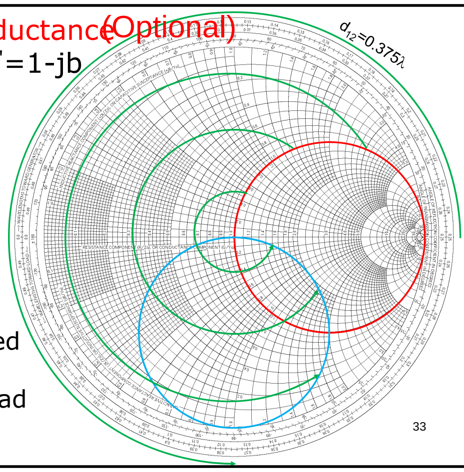
(Optional)
Double Stub Matching w/ S.C.



Key Observations:

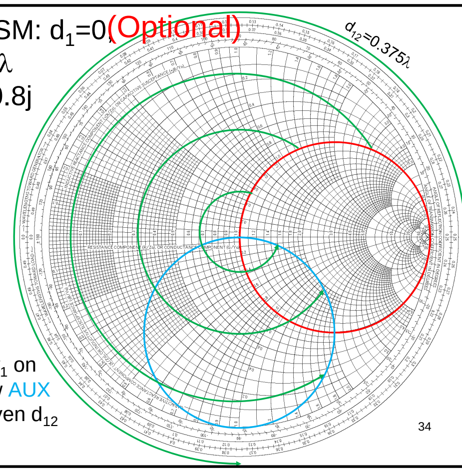
1. For a match, y_2' must be on unit conductance circle
2. Thus, y_1 is on the auxiliary circle
   a) Auxiliary circle is UCC pivoted <u>CCW</u> towards the <u>load</u> by d_{12}

Unit Conductance Circle: $y_2'=1-jb$ (Optional)



Auxiliary Circle y_1 is UCC pivoted d_{12} CCW towards load

Repeat DSM: $d_1=0$ (Optional)
$d_{12}=0.375\lambda$
$z(0)=0.6-0.8j$

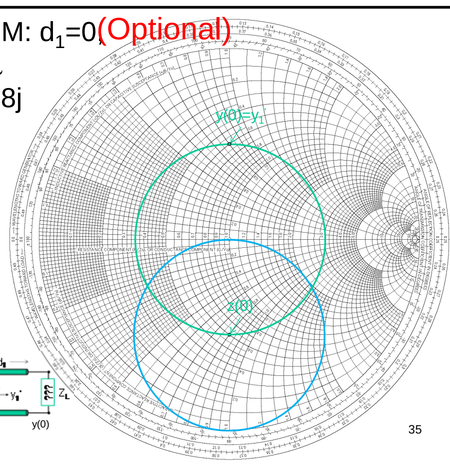


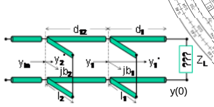
To get y_2' on UCC, need y_1 on AUX so draw AUX first using given d_{12}

Repeat DSM: $d_1=0$ (Optional)
$d_{12}=0.375\lambda$
$z(0)=0.6-0.8j$

$y(0)=0.6+0.8j$

Since $d_1=0$, $y_1'=y(0)$



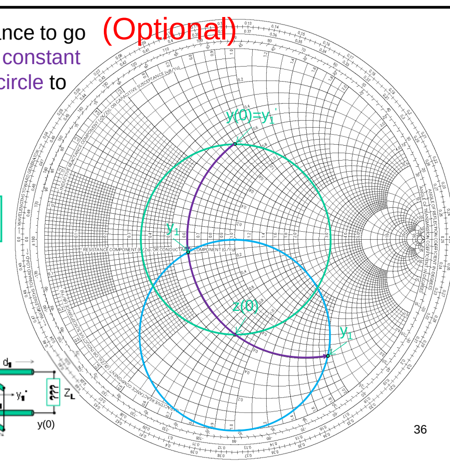


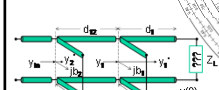
Find susceptance to go from y_1' along constant conductance circle to y_1 on AUX (Optional)

$b_1=-0.89$ or
$b_1=-2.72$

$$\boxed{l_s = \frac{\lambda}{2\pi}\tan^{-1}\left(-\frac{1}{b}\right)}$$

$l_1=0.134\lambda$ or
$l_1=0.056\lambda$





(Optional)

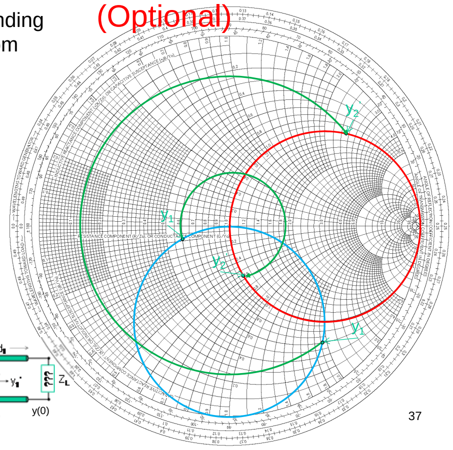
Find corresponding y_2' by going from AUX to UCC

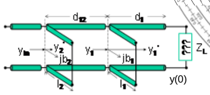
$y_2'=1-0.53j$ or
$y_2'=1+2.5j$

$b_2=0.53$ or
$b_2=-2.5$

$\boxed{l_s = \frac{\lambda}{2\pi}\tan^{-1}(-\frac{1}{b})}$

$l_2=0.327\lambda$ or
$l_2=0.061\lambda$



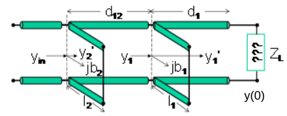


(Optional)

Lecture 38(b) Summary

- Double stub matching inserts two shorted stub of impedance Z_0 with specific lengths, l_1 and l_2, at fixed spots, d_1 and d_1+d_{12}, from the load

Pros: $Z=Z_0$ on all lines, fixed d1, d12
Cons: Many calculations



- Go from y(0) to y_1' a distance d_1 along $|\Gamma(0)|$ circle. Find b_1 by going from y_1' along constant conductance circle to y_1 which is on the AUX circle. Find y_2' by pivoting AUX by d_{12} to UCC and read off b_2=-Im(y_2').
- Calculate l_1 and l_2 using: $\boxed{l_s = \frac{\lambda}{2\pi}\tan^{-1}(-\frac{1}{b})}$

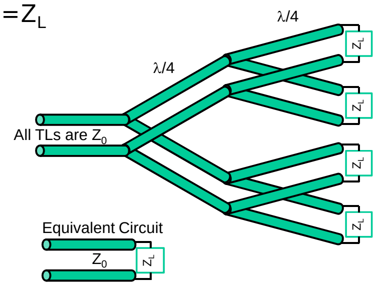
ECE 329
Lecture 39
Online Notes – 38, 39

Distribution Networks

Lossy Line

Corporate Ladder

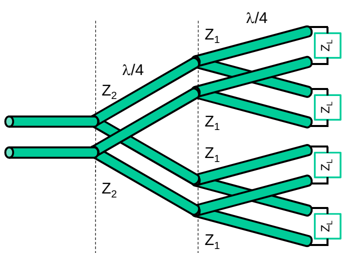
- A network for combining 4 identical loads Z_L into an equivalent single load $Z_{in}=Z_L$



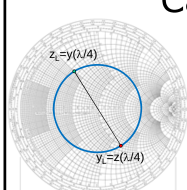
By symmetry, the average power delivered to each load is identical.

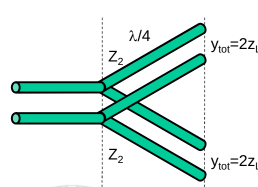
Corporate Ladder

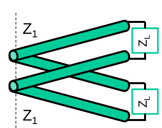
- Verify $Z_{in}=Z_L$ by calculating Z_1 and Z_2



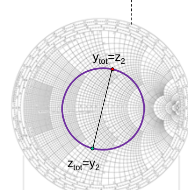
Calculation Space







$y_{tot}=y(\lambda/4)+y(\lambda/4)=2z_L$

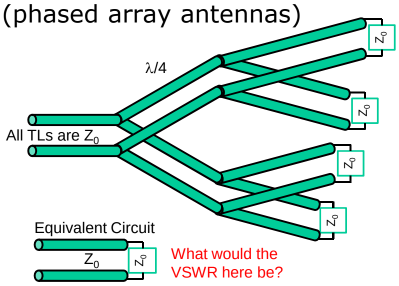


so $z_2=y_{tot}=2z_L$ and thus $y_2=y_L/2$

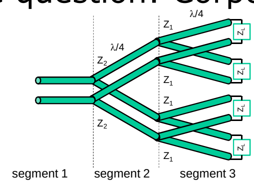
Hence, $y_{in}=y_L$ and $z_{in}=z_L$

Corporate Ladder

- If $Z_L=Z_0$, then the TL segment lengths can be freely varied without affecting Z_{in} (phased array antennas)



Challenge question: Corporate ladder



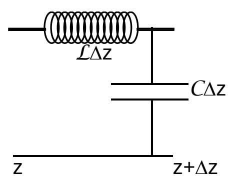
- For the corporate ladder, which is **true**:

(a) there are no reflections anywhere in the ladder
(b) VSWR=1 in segment 1
(c) all the generator power is dissipated at the loads
(d) the power is dissipated evenly among the loads
(e) all segments can be made λ/2 instead of λ/4

LG's question

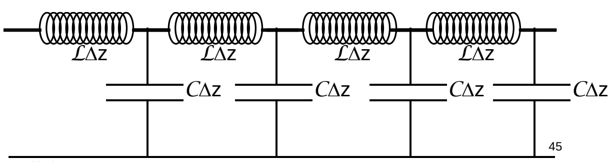
Distributed Circuit of Lossless TL

One slice of the TL:



$$\boxed{\frac{\partial V}{\partial z}=-\mathcal{L}\frac{\partial I}{\partial t}} \quad \boxed{\frac{\partial I}{\partial z}=-C\frac{\partial V}{\partial t}}$$

The entire TL:

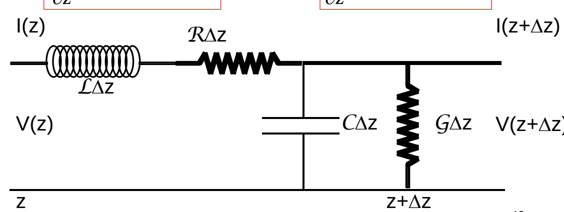


Distributed Circuit of Lossy TL

Model the Ohmic losses in conducting wires and leakage losses in the imperfect dielectric in between

One slice of the lossy TL:

$$\boxed{\frac{\partial V}{\partial z}=-(j\omega\mathcal{L}+\mathcal{R})I} \qquad \boxed{\frac{\partial I}{\partial z}=-(j\omega C+\mathcal{G})V}$$



Solution of Lossy TL

$$\boxed{\tilde{V}(d)=V^+e^{\gamma d}+V^-e^{-\gamma d}} \qquad \gamma=\alpha+j\beta=\sqrt{(j\omega\mathcal{L}+\mathcal{R})(j\omega C+\mathcal{G})}$$

$$\boxed{\tilde{I}(d)=\frac{V^+e^{\gamma d}}{\tilde{Z}_0}-\frac{V^-e^{-\gamma d}}{\tilde{Z}_0}} \qquad \tilde{Z}_0=\sqrt{\frac{j\omega\mathcal{L}+\mathcal{R}}{j\omega C+\mathcal{G}}}$$

These reduce to the lossless results as $\mathcal{R}$ and $\mathcal{G}\rightarrow 0$

$$\gamma=\sqrt{(j\omega\mathcal{L})(j\omega C)}=j\omega\sqrt{\mathcal{L}C}=j\frac{\omega}{v_p}=j\beta$$

$$\tilde{Z}_0=\sqrt{\frac{j\omega\mathcal{L}}{j\omega C}}=\sqrt{\frac{\mathcal{L}}{C}}$$

Lossy TL (high-frequency limit)

$$\gamma=\sqrt{(j\omega\mathcal{L}+\mathcal{R})(j\omega C+\mathcal{G})}$$

$$=j\omega\sqrt{\mathcal{L}C}\sqrt{\left(1+\frac{\mathcal{R}}{j\omega\mathcal{L}}\right)\left(1+\frac{\mathcal{G}}{j\omega C}\right)}$$

$$\approx j\omega\sqrt{\mathcal{L}C}+\frac{1}{2}\left(\frac{\mathcal{R}}{Z_0}+\mathcal{G}Z_0\right)$$

$$\tilde{Z}_0=\sqrt{\frac{j\omega\mathcal{L}+\mathcal{R}}{j\omega C+\mathcal{G}}} \approx \sqrt{\frac{j\omega\mathcal{L}}{j\omega C}}=\sqrt{\frac{\mathcal{L}}{C}}$$

$$\boxed{\beta\approx\omega\sqrt{\mathcal{L}C}} \qquad \boxed{\alpha\approx\frac{1}{2}\left(\frac{\mathcal{R}}{Z_0}+\mathcal{G}Z_0\right)}$$

Similar to lossless case — Signals are attenuated

50Ω and 75Ω coax are so popular because …

- The shunt conductance of the imperfect dielectric is small compared to the series resistance of the conductor, so:

$$\boxed{\alpha \approx \frac{1}{2}\left(\frac{\mathcal{R}}{Z_0} + \mathcal{G} Z_0\right) \approx \frac{1}{2}\frac{\mathcal{R}}{Z_0}}$$

Plugging in the formula for $\mathcal{R}$ and Z_0 of a coax with inner and outer radii a and b, you can show α is minimized when b/a = 3.59 (for fixed b), which works out to 50Ω and 75Ω for a dielectric and air filled coax, respectively. Loss decreases with b, so use thicker coax to reduce loss.

Suggested courses that build on ECE329

		Other Pre-Reqs
ECE 350	Fields and Waves II	
ECE 446	Principles of Experimental Research	313
ECE 451	Advanced Microwave Measurements	350
ECE 452	Electromagnetic Fields	350
ECE 454	Antennas	350
ECE 455	Optical Electronics	350
ECE 453	Wireless Communication Systems	342
ECE 460	Optical Imaging	313

Good luck on the final!

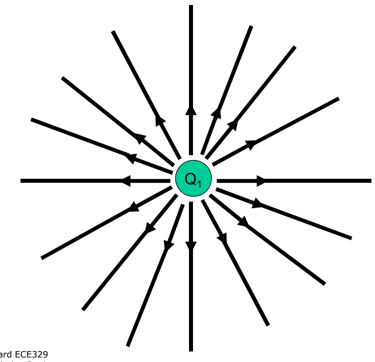
- It was a pleasure teaching ECE329 this semester
  - Thank you for studying so hard ☺

ECE 329

Review for Final Exam

(page and chapter #'s are for the old book)

Coulomb's Law
Electric Field Around a Point Charge

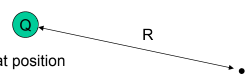


$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 R^2}\hat{a}_R$$

Field strength is proportional to the **density** of field lines

* *Important*

Calculating the Electric Field



Point Charge at position (x_1,y_1,z_1)

Position where we want to calculate electric field at Position (x_2,y_2,z_2)

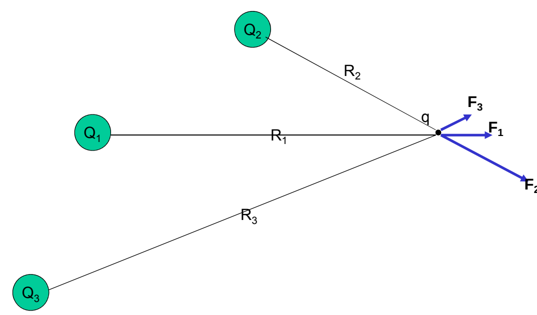
$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 R^2}\hat{a}_R$$

$$R = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

$$\hat{a}_R = \frac{(x_2-x_1)\hat{a}_x + (y_2-y_1)\hat{a}_y + (z_2-z_1)\hat{a}_z}{R}$$

Unit vector pointing along direction from Q to Point

Superposition of Fields

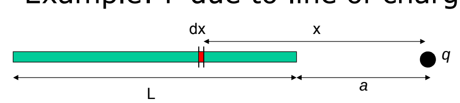


* *Important*

Patented 5-Step Program for Problem Solving

1. MAKE A **LARGE CLEAR** DRAWING
   a. Also draw cross-sections if the problem is in 3D
   b. Pick a coordinate system that is appropriate for the symmetry of the problem
2. Divide charge distributions into tiny pieces
3. Find d**E** of one tiny piece
4. Use SYMMETRY to eliminate any components that cancel (i.e. add to ZERO)
5. INTEGRATE to add contribution of ALL the tiny pieces

Example: F due to line of charge (1)



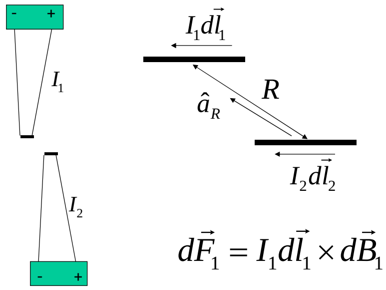
What is the small amount of force, dF, applied by a small sliver of the rod?

Differential charge in one small sliver (coul)

Differential force applied to q

$$dF = \frac{\left[\frac{Q}{L}dx\right]}{4\pi\varepsilon_0 x^2}q\hat{a}_x$$

Ampere’s Force Law



$$d\vec{F}_1 = I_1 d\vec{l}_1 \times d\vec{B}_1$$

Force on current 1 due to current 2

Lorentz Force Equation

If a region of space contains BOTH an **E** field and a **B** field, a moving charge will experience force from both at the same time…

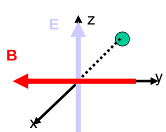
$$\vec{F}_{TOTAL} = \vec{F}_E + \vec{F}_M$$

$$\vec{F}_{TOTAL} = q\vec{E} + q\vec{v} \times \vec{B}$$

Application: Mass Spectrometers

- Part I: Velocity Selector
  - Particles with a specific velocity in crossed EM fields are undeflected

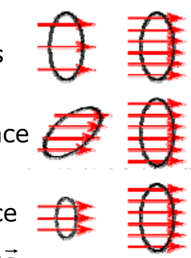
$$\vec{E} = E_0 \hat{a}_z$$
$$\vec{B} = -B_0 \hat{a}_y$$
$$\vec{v} = v_0 \hat{a}_x$$



$$\vec{F}_{TOTAL} = q(E_0 - v_0 B_0)\hat{a}_z = 0 \text{ iff } v_0 = E_0 / B_0$$

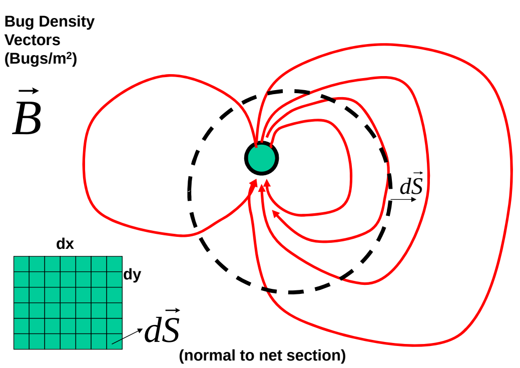
Surface Integrals

- Flux = # of arrows that pass thru a surface; it depends on:
  - The density of vectors
  - The angle of the surface
  - The area of the surface



$$Flux = \vec{B} \bullet d\vec{S}$$

“Closed” Bug Catching Net



Gauss’ Law for B Fields

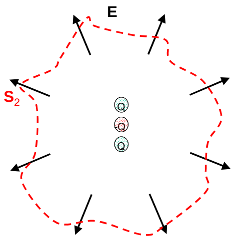
Net flux of magnetic field lines through any closed surface MUST be zero.

$$\oint\!\!\oint_S \vec{B} \bullet d\vec{S} = 0$$

* *Important*

Gauss' Law for E-fields

- Field lines begin on + charges, end on – ones
  - Electric flux out = Net charge enclosed, regardless of shape or location of charges



$$\psi_E = \oint\!\!\oint_S \vec{D} \bullet d\vec{S} = Q_{enclosed}$$

FLUX OUT = CHARGE ENCLOSED

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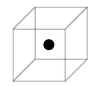
Cumulative Review

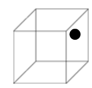
Simple Example 1

6-sided cube with Q at the center:



Flux out of entire box = Q
Flux out of one side = Q/6

6-sided cube with Q NOT at the center:



Flux out of entire box = Q
Flux out of one side = ???

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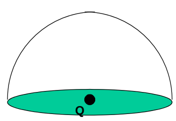
Cumulative Review

Tricky Examples

Flux out of hemisphere with Q at the center = Q/2



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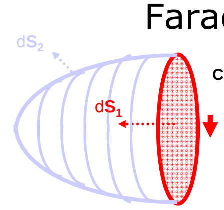
Cumulative Review

* *Important*

Faraday's Law



$$\oint_C \vec{E} \bullet d\vec{l} = -\frac{d}{dt}\iint_S \vec{B} \bullet d\vec{S}$$

- The EMF generated in the loop is the NEGATIVE of the rate of change of the magnetic flux enclosed in the loop
- Right Hand curls around C so thumb points in direction of d**S**

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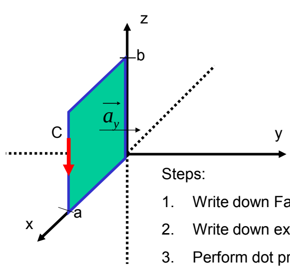
Cumulative Review

Induced emf around rectangular loop in a time-varying **B** field



Rectangular wire loop
In the xz-plane

$$\vec{B} = B_0 \cos \omega t\, \vec{a_y}$$

Steps:

1. Write down Faraday's law $\oint_C \vec{E} \bullet d\vec{l} = -\frac{d}{dt}\iint_S \vec{B} \bullet d\vec{S}$
2. Write down expression for d**S**. DIRECTION!!
3. Perform dot product **B*dS**
4. Solve double integral over limits of the loop
5. Take time derivative of result. Put in "-" sign!

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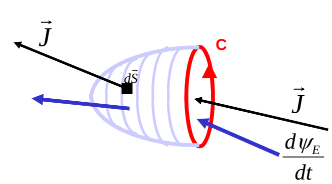
Cumulative Review

* *Important*

Ampere's Law



$$\oint_C \vec{H} \bullet d\vec{l} = \iint_S \vec{J} \bullet d\vec{S} + \frac{d}{dt}\iint_S \vec{D} \bullet d\vec{S}$$

MMF (Amps) = **"Regular" Current (Amps)** + **Displacement Current (Amps)**

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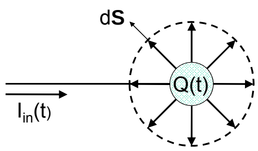
Cumulative Review

Displacement Current

- Current flow changes the amount of charge
  - Since the charge changes, the electric flux out of the surface changes, i.e. a displacement current



$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 R^2}\hat{a}_R$$

$$\psi_E = \iint_S \varepsilon_0 \vec{E} \bullet d\vec{S} = \varepsilon_0 E(Surf\ Area)$$

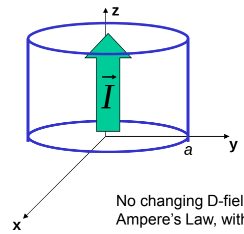
$$= \varepsilon_0 \frac{Q}{4\pi\varepsilon_0 R^2}(4\pi R^2) = Q_{enclosed}$$

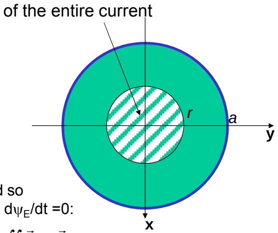
$$I_d = \frac{d\psi_E}{dt} = \frac{dQ_{enclosed}}{dt}$$

$$I_d = \frac{dQ_{enclosed}}{dt} = I_{in} \text{ because current in} = \text{current out}$$

B for an infinitely long solid cylindrical conductor

The path with radius=r<a "encloses" only a portion of the entire current





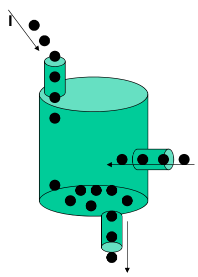
No changing D-field so Ampere's Law, with $d\psi_E/dt$ =0:

$$\oint_C \vec{H} \bullet d\vec{l} = I_{enclosed} = \iint_S \vec{J} \bullet d\vec{S}$$

$$2\pi r H_\phi = \frac{I}{\pi a^2}\pi r^2 \ inside \ or \ = I \ outside$$

** Important*

Conservation of Charge

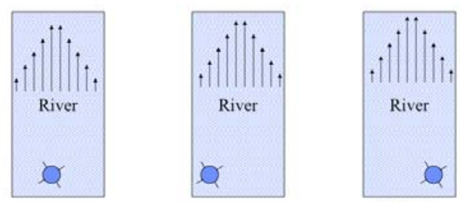


In general, we can pour charges in from more than one direction, or take some out from other parts of the container

Net Rate of Current flow OUT = **Net Rate of Charge DECREASE**

$$\oiint_S \vec{J} \bullet d\vec{S} = -\frac{dQ_{enc}}{dt} = -\frac{d}{dt}\iiint_V \rho dV$$

Curl measures circulation

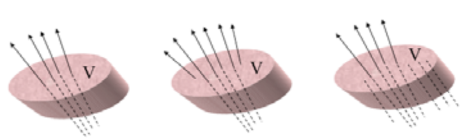


No rotation!

Anti-clockwise rotation.

Clockwise rotation.

Divergence measures # of field lines created

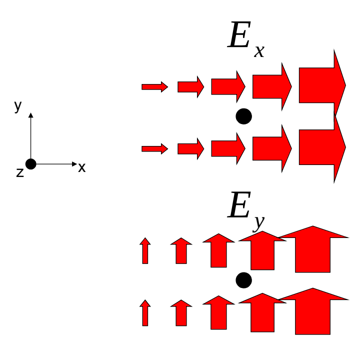


Flux in = flux out so no sources or sinks inside V.

Flux out > flux in Positive divergence. Must be a source inside V.

Flux out < flux in Negative divergence. Must be a sink or drain inside V.

Curl and Divergence



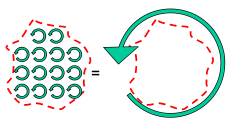
$\frac{dE_x}{dx} \neq 0$

YES DIVERGENCE
NO CURL
Varies ALONG

$\frac{dE_y}{dx} \neq 0$

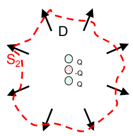
NO DIVERGENCE
YES CURL
Varies ACROSS

Curl and Divergence



$$\iint_S (\nabla\times\vec v)\bullet d\vec S = \oint_C \vec v\bullet d\vec l$$

Stokes' Theorem



$$\iiint_V \nabla\bullet\vec v\ dV = \oiint_{\partial V} \vec v\bullet d\vec S$$

Divergence Theorem

*** *Extremely Important*

Maxwell's Equations

Faraday's Law	$\oint_C \vec E\bullet d\vec l = -\frac{d}{dt}\iint_S \vec B\bullet d\vec S$	$\nabla\times\vec E = -\frac{d\vec B}{dt}$
Ampere's Law	$\oint_C \vec H\bullet d\vec l = \iint_S \vec J\bullet d\vec S + \frac{d}{dt}\iint_S \vec D\bullet d\vec S$	$\nabla\times\vec H = \vec J + \frac{d\vec D}{dt}$
Gauss' Law	$\oiint_S \vec B\bullet d\vec S = 0$	$\nabla\bullet\vec B = 0$
Gauss' Law	$\oiint_S \vec D\bullet d\vec S = \iiint_V \rho dV$	$\nabla\bullet\vec D = \rho$
Continuity Eq.	$\oiint_S \vec J\bullet d\vec S = -\frac{d}{dt}\iiint_V \rho dV$	$\nabla\bullet\vec J = -\frac{d\rho}{dt}$

* *Important*

Realizable Fields

- Is it time dependent (d/dt)?
- Is there any free charge ρ or current density **J**?
- Apply Maxwell's equations

$\nabla\bullet\vec D = \rho$ $\quad\nabla\bullet\vec B = 0$ $\quad\nabla\bullet\vec J = -\frac{d\rho}{dt}$

$\nabla\times\vec E = -\frac{d\vec B}{dt}$ $\quad\nabla\times\vec H = \vec J + \frac{d\vec D}{dt}$

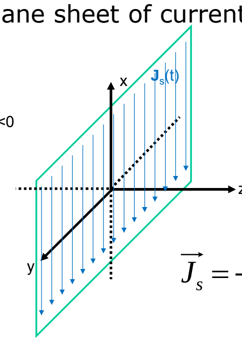
$\vec D = \varepsilon\vec E$ $\quad\vec B = \mu\vec H$

Field Source

- Infinite plane sheet of current at z=0

Free space z<0

Free space z>0



J_s = surface current in Amps per meter

$$\vec J_s = -J_s(t)\vec a_x$$

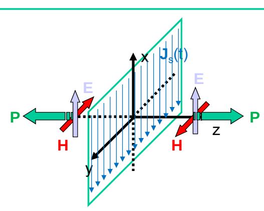
At z=0

Solution

$$\vec E(z,t) = \frac{\eta_0}{2}J_S(t\mp\frac{z}{v_p})\vec a_x$$

$$\vec H(z,t) = \pm\frac{1}{2}J_S(t\mp\frac{z}{v_p})\vec a_y$$

$z \gtrless 0$



* *Important*

Two definitions

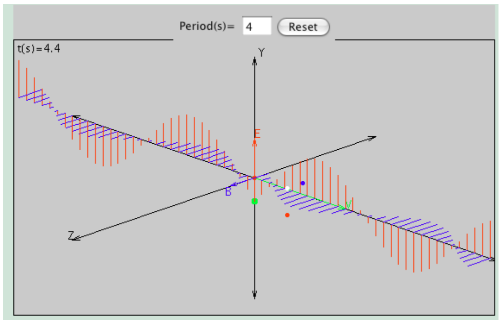
$$v_p = c = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = \text{speed of light} = 3x10^8\ (m/s)$$

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx 377\ (ohms)$$

Intrinsic impedance of free space
$|\mathbf{E}| = \eta_0|\mathbf{H}|$

* Important

Web Demo



http://www.phy.ntnu.edu.tw/java/emWave/emWave.html

* Important

Sinusoidal Plane Waves

$\boxed{\vec{J}_s = -J_{S0}\cos(\omega t)\vec{a}_x}$ $\qquad \vec{E}(z,t) = \frac{\eta_0}{2} J_S(t \mp \frac{z}{v_p})\vec{a}_x \qquad \vec{H}(z,t) = \pm\frac{1}{2} J_S(t \mp \frac{z}{v_p})\vec{a}_y$

$$\vec{E}(z,t) = \frac{\eta_0 J_{S0}}{2}\cos(\omega t \mp \beta z)\vec{a}_x$$

$$\vec{H}(z,t) = \pm\frac{J_{S0}}{2}\cos(\omega t \mp \beta z)\vec{a}_y$$

$z \gtrless 0$

$$\beta = \frac{\omega}{v_p}$$

** Very Important

Wave Parameters

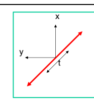
$$\boxed{\vec{E}(z,t) = \frac{\eta_0 J_{S0}}{2}\cos(\omega t \mp \beta z)\vec{a}_x}$$

Phase	$\phi = \omega t \mp \beta z$	(radians)
Angular Frequency	$\omega = \frac{\partial \phi}{\partial t}$	(radians/sec)
Linear Frequency	$f = \frac{\omega}{2\pi}$	(1/sec)
Phase Constant	$\beta = \mp\frac{\partial \phi}{\partial z}$	(radians/m)
Wavelength	$\lambda = \frac{2\pi}{\beta}$	(m)
Phase Velocity	$v_p \equiv \frac{\omega}{\beta} = \lambda f = c$	(m/sec)
Impedance	$\eta_0 = \lvert\vec{E}\rvert / \lvert\vec{H}\rvert$	(Ω)

* Important

Polarization

Linear

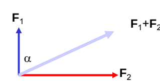


DIRECTION: Constant
MAGNITUDE: Varies

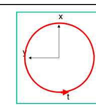
$\vec{F}_1 = F_1\cos(\omega t + \phi)\hat{a}_x$

$\vec{F}_2 = \pm F_2\cos(\omega t + \phi)\hat{a}_y$

The vectors are IN PHASE



Circular

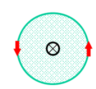


DIRECTION: Varies
MAGNITUDE: Constant

$\vec{F}_1 = F_0\cos(\omega t + \phi)\hat{a}_x$

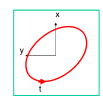
$\vec{F}_2 = F_0\sin(\omega t + \phi)\hat{a}_y$

The vectors must be:
EQUAL MAGNITUDE
OUT OF PHASE by ***π/2***
PERPENDICULAR



"Left-Hand" Polarized CCW as seen by source

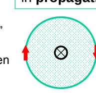
Elliptical



DIRECTION: Varies
MAGNITUDE: Varies

Most general:
If it is not linear
or circular

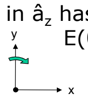
Left/right thumb points in **propagation** direction



"Right-Hand" Polarized CW as seen by source

Writing Fields in Free Space

- Sinusoidal field propagating in $\hat{a}_z$ has left circular polarization, λ=3m, $E(0,0)=E_0\hat{a}_y$



- Answer: $\beta=2\pi/\lambda=2\pi/3$ rad/m;
- $\lambda f=c \rightarrow f=1\text{x}10^8$ Hz $\rightarrow \omega=2\pi f=2\pi\text{x}10^8$ rad/s
- E_y is max first then ¼ period later E_x is max
- $\mathbf{E}=E_0\cos(\omega t-\beta z)\,\hat{a}_y + E_0\sin(\omega t-\beta z)\,\hat{a}_x$
- $\mathbf{H}=E_0/\eta_0\cos(\omega t-\beta z)\,(-\hat{a}_x) + E_0/\eta_0\sin(\omega t-\beta z)\,\hat{a}_y$

35

* Important

Definition: Poynting Vector

The **E** and **H** fields are carrying power with them as they propagate

$$\vec{S} = \vec{E} \times \vec{H}$$

Definition for the Power Flow Density of an EM Field

Units for **S**: Watts/m²

$$\oint_S \vec{S} \bullet \vec{dS} = \oint_S (\vec{E} \times \vec{H}) \bullet \vec{dS}$$

Power flow **out** of a CLOSED surface (units = Watts)

* Important

Poynting's Theorem

$$\oiint_S \vec{S} \bullet d\vec{S} = -\frac{\partial}{\partial t}\iiint_V (w_m + w_e)dV - \iiint_V \vec{E} \bullet J \; dV$$

Power flow **OUT** of surface = Rate of field energy **LOSS** and Rate of work done **BY** the fields

$w_m = \frac{1}{2}\mu_0 H^2$ stored magnetic energy density

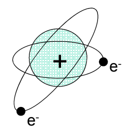
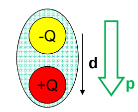
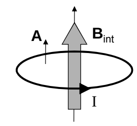
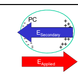
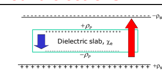
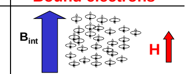
$w_e = \frac{1}{2}\varepsilon_0 E^2$ stored electric energy density

$$\vec{E} = \mathrm{Re}[\overline{E}(z)e^{j\omega t}], \vec{H} = \mathrm{Re}[\overline{H}(z)e^{j\omega t}]$$

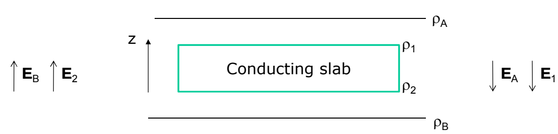
$$\langle \vec{S} \rangle = \mathrm{Re}[\frac{1}{2}\overline{E} \times \overline{H}^*]$$

* Important

3 types of materials

Conductors	Dielectrics	Magnetic
 Free electrons	 Polarized atoms/molecules Bound electrons	 Magnetic moments Bound electrons
		
E=0 inside ρ=0 inside $\rho=\rho_s$ only surface charge V is same throughout $E_{outside}$ is $\perp$ to surface	E≠0 inside but it is reduced $\mathbf{E}_{tot}=\mathbf{E}_a+\mathbf{E}_s$ $\mathbf{D}=\varepsilon\mathbf{E}_{tot}=\mathbf{P}+\varepsilon_0\mathbf{E}_{tot}$	$\mathbf{B}_{tot}=\mathbf{B}_a+\mathbf{B}_s$ $\mathbf{B}_{tot}=\mu\mathbf{H}=\mu_0(\mathbf{H}+\mathbf{M})$

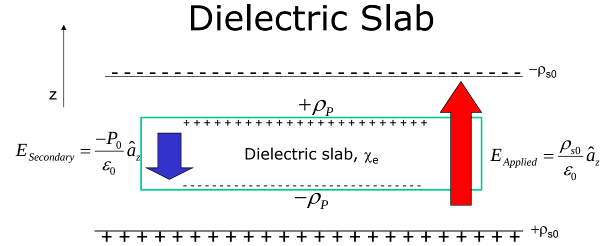
Conducting Slab D4.2 (p217)



- Neutral slab → $\rho_1=-\rho_2$
- $\mathbf{E}_{inside}=0$ → $E_z=(\rho_B+\rho_2-\rho_A-\rho_1)/2\varepsilon_0=0$

$\rho_1=(\rho_B-\rho_A)/2, \; \rho_2=(\rho_A-\rho_B)/2$

Dielectric Slab



$$\vec{E}_{Total} = \frac{P_0}{\varepsilon_0\chi_e} = \frac{\rho_{S0}/\varepsilon_0}{1+\chi_e}$$

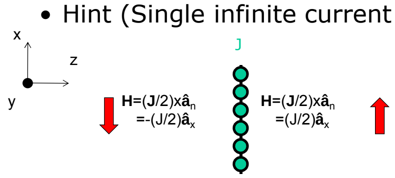
E-field strength reduced by $(1+\chi_e)$

$$\vec{D} = \varepsilon\vec{E}_{Total} = \varepsilon_0(1+\chi_e)\vec{E}_{Total} = \rho_{S0}$$

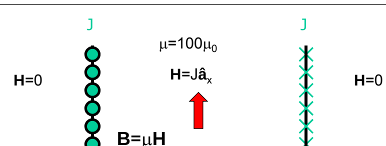
D-field strength is same as free space

Magnetic Sheets D4.6 (p238)

- Hint (Single infinite current sheet)



Answer:



$\mathbf{B}=\mu\mathbf{H}$

$\mathbf{B}=\mu_0(\mathbf{H}+\mathbf{M})$ to solve for $\mathbf{M}=99\mathbf{H}$

*** Extremely Important

Inside a material, Maxwell's Equations become:

Free Space	Inside Material
$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$	$\vec{\nabla} \times \vec{H} = \sigma\vec{E} + \varepsilon\frac{\partial \vec{E}}{\partial t}$
$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$	$\vec{\nabla} \times \vec{E} = -\mu\frac{\partial \vec{H}}{\partial t}$
$\vec{\nabla} \bullet \vec{D} = \rho$	$\vec{\nabla} \bullet (\varepsilon\vec{E}) = \rho$
$\vec{\nabla} \bullet \vec{B} = 0$	$\vec{\nabla} \bullet \vec{H} = 0$

* *Important*

Solve PDEs with Phasors

- Technique simplifies the algebra

$$E_x(z,t) = \mathrm{Re}[\overline{E}_x(z)e^{j\omega t}]$$

$$\frac{\partial E_x}{\partial t} = \mathrm{Re}[\,j\omega\overline{E}_x(z)e^{j\omega t}]$$

$$\left.\begin{aligned}\frac{\partial E_x}{\partial z} &= -\mu\frac{\partial H_y}{\partial t}\\ \frac{\partial H_y}{\partial z} &= \boxed{-\sigma E_x} - \varepsilon\frac{\partial E_x}{\partial t}\end{aligned}\right\} \qquad \begin{aligned}\frac{\partial \overline{E}_x}{\partial z} &= -\mu(j\omega)\overline{H}_y\\ \frac{\partial \overline{H}_y}{\partial z} &= -\sigma\overline{E}_x - \varepsilon(j\omega)\overline{E}_x\end{aligned}$$

** *Very Important*

Final Solution for E_x and H_y

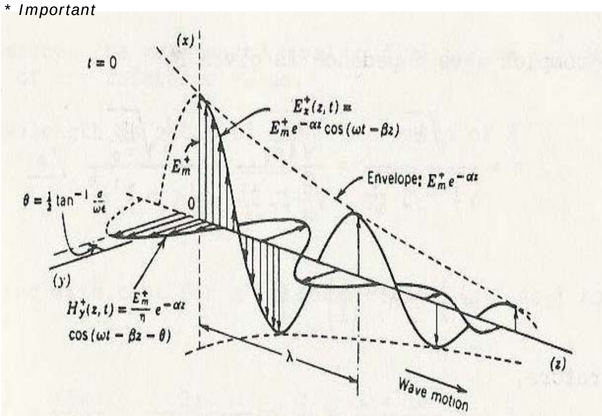
$$\boxed{\begin{aligned}\vec{E}(z,t) &= \frac{|\eta|J_{S0}}{2}e^{\mp\alpha z}\cos(\omega t \mp \beta z + \tau)\hat{a}_x\\ \vec{H}(z,t) &= \frac{\pm J_{S0}}{2}e^{\mp\alpha z}\cos(\omega t \mp \beta z)\hat{a}_y \qquad z \gtrless 0\end{aligned}}$$

Magnitude of E and H related through magnitude of the the complex impedance, $\overline{\eta}$

Strength of fields drops exponentially according to the attenuation constant

E and H are out of phase by the phase of the complex impedance, τ

* *Important*

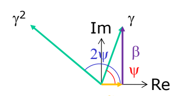


* *Important*

Complex Propagation Constant and Impedance

$$\frac{\partial \overline{E}_x^2}{\partial z^2} = \overline{\gamma}^2\overline{E}_x \qquad \overline{\gamma} = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} = \alpha + j\beta = |\overline{\gamma}|e^{j\psi}$$

Attenuation — Phase constant



$\mathrm{Re}[\overline{\gamma}^2] < 0,\ \mathrm{Im}[\overline{\gamma}^2] > 0$

$\therefore\ \overline{\gamma}^2$ is in Quadrant II

$\Rightarrow\ 45^\circ \le \psi \le 90^\circ$

$\therefore\ \beta \ge \alpha > 0$

$$\overline{\eta} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = |\overline{\eta}|e^{j\tau}$$

Ratio of |E| to |H| — Phase diff. between E and H

$$\boxed{\begin{aligned}&\overline{\gamma}\overline{\eta} = j\omega\mu\\ &\Rightarrow \tau + \psi = \pi/2\\ &\overline{\gamma}/\overline{\eta} = \sigma + j\omega\varepsilon\end{aligned}}$$

Very useful!

** *Very Important*

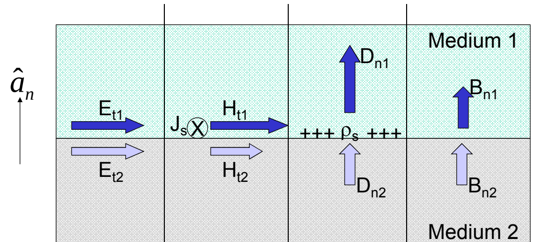
Dielectrics vs Conductors

Perfect Dielectric	Imperfect Dielectric
Definition: $\boxed{\sigma = 0}$	Definition: $\boxed{\sigma/\omega\varepsilon << 1}$
Attenuation: $\alpha = 0$	Attenuation: $\boxed{\alpha \approx \sigma/2\sqrt{\mu/\varepsilon}}$
Speed: $v_p = c/\sqrt{\mu_r\varepsilon_r} \le c$	Speed: $v_p \approx c/\sqrt{\mu_r\varepsilon_r} \le c$
E,**H** In Phase: $\tau = 0$	**E**,**H** In Phase: $\tau \approx 0$
Impedance: $\lvert\overline{\eta}\rvert = \eta_0\sqrt{\mu_r/\varepsilon_r}$	Impedance: $\lvert\overline{\eta}\rvert \approx \eta_0\sqrt{\mu_r/\varepsilon_r}$
Good Conductor	**Perfect Conductor**
Definition: $\boxed{\sigma/\omega\varepsilon >> 1}$	Definition: $\boxed{\sigma \to \infty}$
Attenuation: $\boxed{\alpha \approx \sqrt{\omega\mu\sigma/2}}$	Attenuation: $\alpha \to \infty$ $\boxed{\vec{E} \to 0}$
Speed: $\boxed{v_p \approx \sqrt{2\omega/\sigma\mu}}$	Speed: $v_p \to 0$ $\boxed{\vec{H} \to 0}$
E,**H** 45° Phase: $\boxed{\tau \approx \pi/4}$	**E**,**H** 45° Phase: $\tau \to \pi/4$
Impedance: $\boxed{\lvert\overline{\eta}\rvert \approx \sqrt{\omega\mu/\sigma}}$	Impedance: $\lvert\overline{\eta}\rvert \to 0$

** *Very Important*

Boundary Conditions

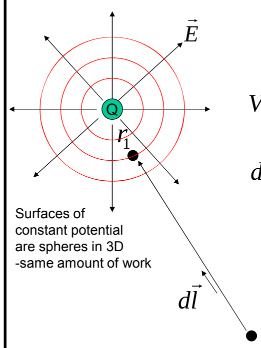
- **Never** use the differential form of Maxwell's equations at a boundary – only use integral form



$\boxed{E_{t1}=E_{t2}}$ $\boxed{H_{t1}-H_{t2}=J_s}$ $\boxed{D_{n1}-D_{n2}=\rho_s}$ $\boxed{B_{n1}=B_{n2}}$

$\boxed{\hat{a}_n \times (\vec{H}_1 - \vec{H}_2) = \vec{J}_S}$

Example: Potentials for a Point Charge



$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{a}_r \qquad \vec{E} = -\nabla V$$

$$V(r) = -\int_{\infty}^{r_1} \vec{E} \bullet d\vec{l}$$

"Absolute" potential at r_1 using zero potential at r = ∞

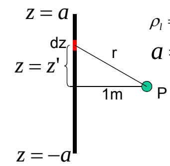
$$d\vec{l} = -|dr|\hat{a}_r = dr\hat{a}_r$$

Since dr<0 going from r=∞

$$V(r) = -\int_{\infty}^{r_1} \frac{Q}{4\pi\varepsilon_0 r^2} dr$$

$$= \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_1} - \frac{1}{\infty}\right) = \boxed{\frac{Q}{4\pi\varepsilon_0 r_1}}$$

Superposition Example: Potential of a Line Charge



$\rho_l = 10^{-10}(C/m)$

$a = 1m$

Find Potential at point "P", 1 m away from the line

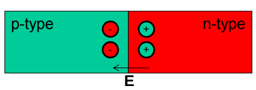
$$V_P = \frac{1}{4\pi\varepsilon_0}\int_{-a}^{a}\frac{dQ}{r}$$

$$V_P = \frac{1}{4\pi\varepsilon_0}\int_{-a}^{a}\frac{\rho_l dz'}{r}$$

$$V_P = \frac{1}{4\pi\varepsilon_0}\int_{-a}^{a}\frac{\rho_l dz'}{\sqrt{z'^2+1^2}}$$

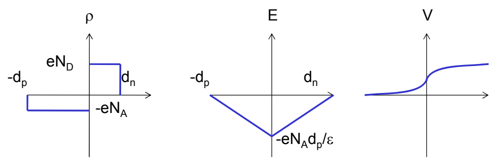
$$V_P = \frac{\rho_l}{4\pi\varepsilon_0}\ln\left(z' + \sqrt{z'^2+1}\right)\Big|_{-a}^{a}$$

Example 5.5 (p-n junction)



$$\nabla^2 V = -\rho/\varepsilon$$

$$\vec{E} = -\nabla V$$

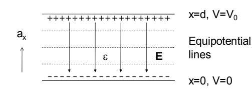


$$\frac{dE}{dx} = \frac{\rho}{\varepsilon} \qquad \frac{dV}{dx} = -E$$

$$E = \int_{-\infty}^{x}\frac{\rho}{\varepsilon}dx \qquad V = -\int_{-\infty}^{x} Edx$$

*** Very Important*

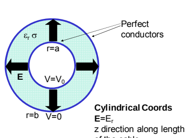
Steps to Find Capacitance



- Laplace Equation $\vec{\nabla}^2 V = 0$
- Find V using boundary conditions
- Find **E** using $\vec{E} = -\vec{\nabla}V$
- Find **D** using $\vec{D} = \varepsilon\vec{E}$
- Get surface charge density on <u>one</u> conductor using BC $\rho_s = \vec{a}_n \bullet (D_{n1} - D_{n2})$
- Charge $Q = (Area)(\rho_s)$
- Capacitance $C = Q/V_0$

$\boxed{V(x) = V_0\frac{x}{d}}$ $\boxed{\rho = \varepsilon V_0/d}$ $\boxed{C = \frac{\varepsilon A}{d}}$

Coaxial Cable



$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_r}{\partial r}\right) = 0$$

$$\Rightarrow r\frac{\partial V_r}{\partial r} = c_1 \Rightarrow V_r = c_1\ln r + c_2$$

$$V(b) = 0 \Rightarrow c_2 = -c_1\ln b$$

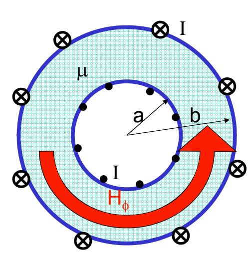
$$V(a) = V_0 \Rightarrow c_1 = V_0/\ln(a/b)$$

$$\boxed{V(r) = -V_0\frac{\ln(r/b)}{\ln(b/a)}}$$

$$E = \frac{-dV}{dr} = \frac{V_0}{r\ln(b/a)}$$

$$\rho = \begin{cases} \varepsilon V_0/(a\ln(b/a)), r = a \\ -\varepsilon V_0/(b\ln(b/a)), r = b \end{cases} \qquad \boxed{C = \frac{2\pi\varepsilon L}{\ln(b/a)}}$$

Inductance of a Coax Cable



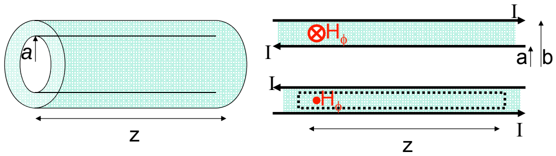
Now, instead of applying a voltage across the inner and outer conductor, a current, I, flows down the length of the outer conductor and returns in the opposite direction through the inner conductor

Results in magnetic field

$$H_\phi = \frac{I}{2\pi r}$$

in between the coax

Inductance of Coaxial Cable



$$\vec{B} = \mu\vec{H} = \frac{\mu I}{2\pi r}\vec{a}_\phi$$ Magnetic Flux Density $\left[\frac{Wb}{m^2}\right]$

$$\psi = \int B \bullet dS = \int_{r=a}^{b}\int_{z=0}^{z}\left(\frac{\mu I}{2\pi r}\right)(drdz)$$ Magnetic Flux $[Wb]$

$$\psi = \frac{\mu I z}{2\pi}\ln(b/a)$$

Inductance

$$L = \frac{\psi}{I}$$ Units: Henry (H)

$$L = \frac{\mu z}{2\pi}\ln(b/a)$$

$$\mathcal{L} = \frac{L}{z} = \frac{\mu}{2\pi}\ln(b/a)$$ Inductance/Length (H/m)

* *Important*

Relationships between Capacitance, Conductance & Inductance

Notice in the above examples,

$$\mathcal{C} = \varepsilon \cdot GeometricalFactor$$
$$\mathcal{L} = \mu / GeometricalFactor$$
$$\mathcal{G} = \sigma \cdot GeometricalFactor$$

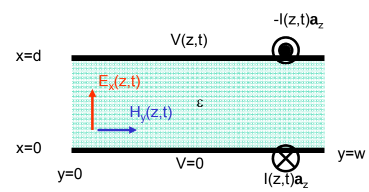
This is true in general and so we have the following:

$$\mathcal{LC} = \mu\varepsilon \qquad \mathcal{G}/\mathcal{C} = \sigma/\varepsilon$$

If you know one (L, C, or G), you can find the other two from the material parameters

Chapter 6

Transmission Line



V(z,t) and I(z,t) can be used to describe the state of the transmission line instead of E_x(z,t) and H_y(z,t)

Transmission Line Equations

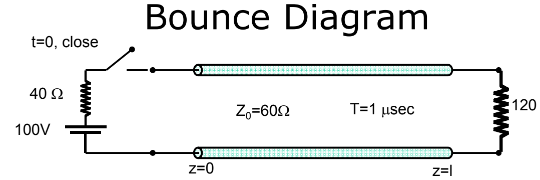
$$\frac{\partial V}{\partial z} = -\left(\frac{\mu d}{w}\right)\frac{\partial I}{\partial t} \qquad \boxed{\frac{\partial V}{\partial z} = -\mathcal{L}\frac{\partial I}{\partial t}}$$

$$\frac{\partial I}{\partial z} = -\left(\frac{\varepsilon w}{d}\right)\frac{\partial V}{\partial t} \qquad \boxed{\frac{\partial I}{\partial z} = -C\frac{\partial V}{\partial t}}$$

These are the transmission line equations!!
- They describe wave propagation along the TL in terms of currents and voltages
- It is just another way of stating Maxwell's Eqns

** *Very Important*

Bounce Diagram



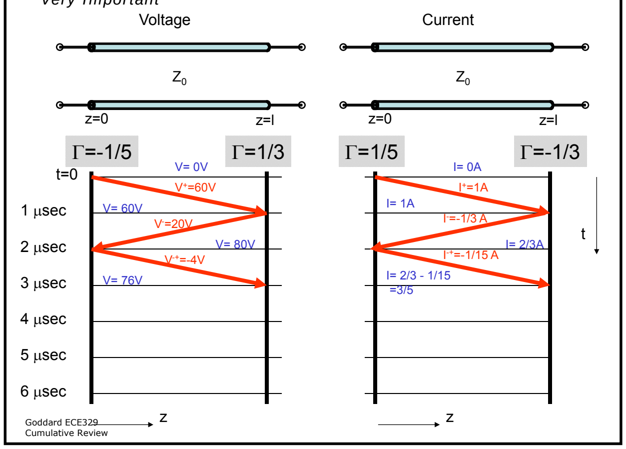
First step: Calculate V^+, I^+, Γ_{load}, Γ_{source}

$$V^+ = V_0\frac{Z_0}{R_g + Z_0} = 100\frac{60}{40+60} = 60V \qquad \Gamma_L = \frac{R_L - Z_0}{R_L + Z_0} = \frac{120-60}{120+60} = \frac{1}{3}$$

$$I^+ = \frac{V^+}{Z_0} = \frac{60}{60} = 1A \qquad \Gamma_S = \frac{R_S - Z_0}{R_S + Z_0} = \frac{40-60}{40+60} = -\frac{1}{5}$$

Second step: Construct 2 bounce diagrams (Voltage and Current)

** *Very Important*



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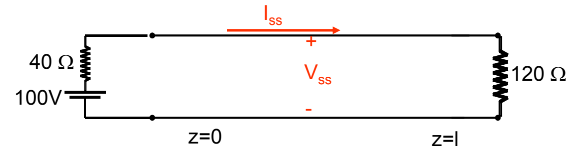
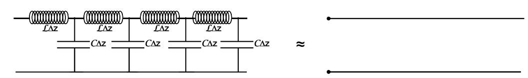
** *Very Important*

In SS, TL Looks Like a Wire

$$V_{SS}=V_{SS}^{+}+V_{SS}^{-}=75V \qquad I_{SS}=I_{SS}^{+}+I_{SS}^{-}=0.625A$$





$$V_{SS}=100V\frac{120\Omega}{(40+120)\Omega}=75V \qquad I_{SS}=\frac{100V}{(40+120)\Omega}=0.625A$$

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** *Very Important*

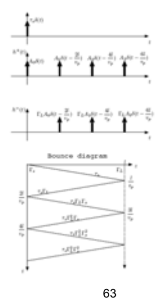
Algebra of the Bounce Diagram

For f(t)=δ(t), the solution of: $V^{+}(0,t)=\tau_S\delta(t)+\Gamma_S\Gamma_L V^{+}(0,t-\frac{2l}{v_p})$
is an impulse response:

$$V^{+}(0,t)=\tau_s\sum_{n=0}^{\infty}(\Gamma_S\Gamma_L)^n\delta(t-n\frac{2l}{v_p})\equiv h^{+}(t)$$

$$V^{-}(0,t)=\Gamma_L V^{+}(0,t-\frac{2l}{v_p})$$

$$=\Gamma_L\tau_s\sum_{n=0}^{\infty}(\Gamma_S\Gamma_L)^n\delta(t-(n+1)\frac{2l}{v_p})\equiv h^{-}(t)$$



For arbitrary position z, replace t with t ± z/v_p

$$V^{+}(z,t)=\tau_s\sum_{n=0}^{\infty}(\Gamma_S\Gamma_L)^n\delta(t-\frac{z}{v_p}-n\frac{2l}{v_p})$$

$$V^{-}(z,t)=\Gamma_L\tau_s\sum_{n=0}^{\infty}(\Gamma_S\Gamma_L)^n\delta(t+\frac{z}{v_p}-(n+1)\frac{2l}{v_p})$$

$$I=\frac{1}{Z_0}(V^{+}-V^{-})$$

Note: the voltage is nonzero only on the bounce lines

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** *Very Important*

Algebra of the Bounce Diagram

For f(t)=δ(t), the solution was:

$$V^{+}(z,t)=\tau_s\sum_{n=0}^{\infty}(\Gamma_S\Gamma_L)^n\delta(t-\frac{z}{v_p}-n\frac{2l}{v_p})$$

$$V^{-}(z,t)=\Gamma_L\tau_s\sum_{n=0}^{\infty}(\Gamma_S\Gamma_L)^n\delta(t+\frac{z}{v_p}-(n+1)\frac{2l}{v_p})$$

$$I=\frac{1}{Z_0}(V^{+}-V^{-})$$

For arbitrary f(t), convolve the solution with f:

$$\boxed{V^{+}(z,t)=\tau_s\sum_{n=0}^{\infty}(\Gamma_S\Gamma_L)^n f(t-\frac{z}{v_p}-n\frac{2l}{v_p})}$$

$$\boxed{V^{-}(z,t)=\Gamma_L\tau_s\sum_{n=0}^{\infty}(\Gamma_S\Gamma_L)^n f(t+\frac{z}{v_p}-(n+1)\frac{2l}{v_p})}$$

$$I=\frac{1}{Z_0}(V^{+}-V^{-})$$

Note: the voltage can be nonzero in between the bounce lines depending on the function f

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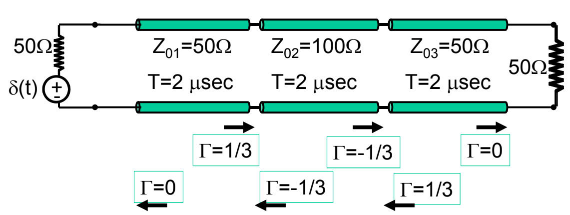
Example 2, Step 1: V⁺, I⁺, Γ



$$V^{+}=V_0\frac{Z_0}{R_g+Z_0}=1\frac{50}{50+50}=0.5\delta$$

$$I^{+}=\frac{V^{+}}{Z_0}=\frac{0.5}{50}=0.01\delta$$

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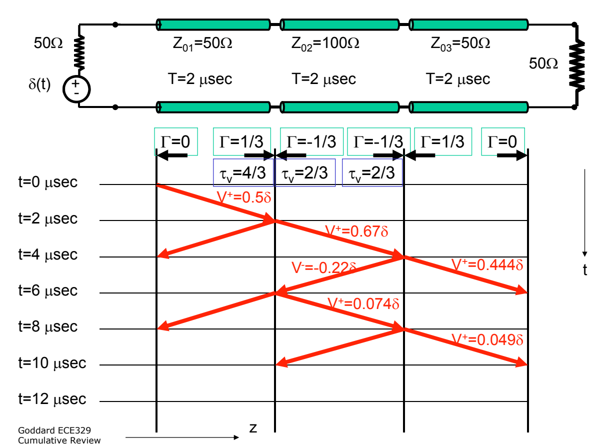

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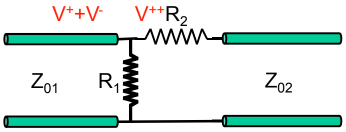
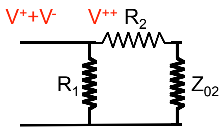
Transmission Line Junction



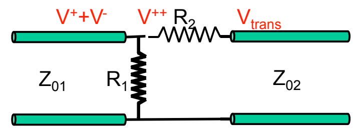
Equivalent circuit "seen" by V+ when it gets to the end of line 1:



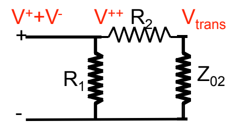
$$\Gamma = \frac{V^-}{V^+} = \frac{R_L - Z_{01}}{R_L + Z_{01}}$$

What is R_L for this equivalent circuit?

$$R_L = (R_1)\|(R_2 + Z_{02})$$



How much voltage gets transmitted through to line 2?



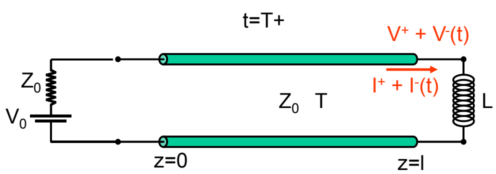
$$\tau_v = \frac{V^{++}}{V^+} = 1 + \Gamma$$

$$V^{++} = V^+(1+\Gamma)$$

$$V_{trans} = \frac{Z_{02}}{Z_{02} + R_2} V^{++}$$

Optional

Inductive Termination



$$V^+ = \frac{V_0}{2}$$

$$I^+ = \frac{V^+}{Z_0} = \frac{V_0}{2Z_0}$$

$$I^- = -\frac{V^-(t)}{Z_0}$$

For the inductor:

$$V = L\frac{dI}{dt}$$

Optional

Diff Eqn for the Inductor

$$V = L\frac{dI}{dt} \Rightarrow (V^+ + V^-(t)) = L\frac{d(I^+ + I^-(t))}{dt}$$

$$\frac{V_0}{2} + V^-(t) = L\frac{d}{dt}\left(\frac{V_0}{2Z_0} - \frac{V^-(t)}{Z_0}\right)$$

$$\frac{V_0}{2} = -\frac{L}{Z_0}\frac{dV^-(t)}{dt} - V^-(t)$$

$$\frac{dV^-(t)}{dt} + \frac{Z_0}{L}V^-(t) = -\frac{Z_0}{L}\frac{V_0}{2}$$

Laplace Transform V-(t') to F-(s) $\boxed{t'=t-T}$

$$s\hat{V}^-(s) - V^-(0) + \frac{Z_0}{L}\hat{V}^-(s) = -\frac{Z_0}{L}\frac{V_0}{2}\frac{1}{s}$$

Optional

Laplace Transform for Inductor

Initial Condition: *At t=T, i.e. t'=0, inductor current = 0 since inductor "looks" like an OPEN CIRCUIT*

$$I = I^+ + I^-(T) = 0 \Rightarrow \frac{V_0}{2Z_0} - \frac{V^-(0)}{Z_0} = 0 \Rightarrow V^-(0) = V_0/2$$

$$s\hat{V}^-(s) - \frac{V_0}{2} + \frac{Z_0}{L}\hat{V}^-(s) = -\frac{Z_0}{L}\frac{V_0}{2}\frac{1}{s} \Rightarrow \boxed{\hat{V}^-(s) = \frac{sL - Z_0}{sL + Z_0}\frac{V_0}{2s}}$$

In s-space, we have V-(s) = Γ(s) V+(s) with:

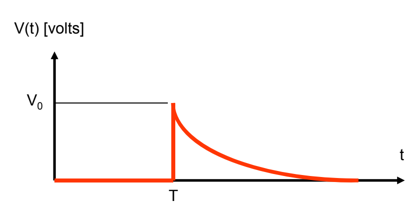
$$\boxed{\Gamma(s) = \frac{Z(s) - Z_0}{Z(s) + Z_0}} \quad \boxed{\hat{V}^+(s) = \frac{V_0}{2s}} \quad \boxed{Z(s) = sL \text{ for an inductor}}$$

Optional

Invert Laplace Transform

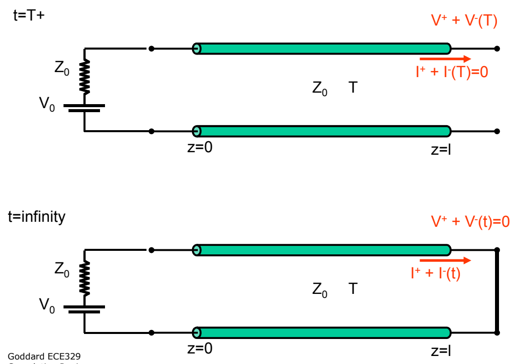
$$\hat{V}(s) = \hat{V}^+ + \hat{V}^- = \frac{V_0}{2s} + \frac{V_0}{2s}\frac{sL - Z_0}{sL + Z_0} = \frac{V_0}{2s}\frac{2sL}{sL + Z_0} = \frac{V_0}{s + Z_0/L}$$

$$\boxed{V(t) = V_0 e^{-(Z_0/L)t'} = V_0 e^{-(Z_0/L)(t-T)}} \quad t>T$$



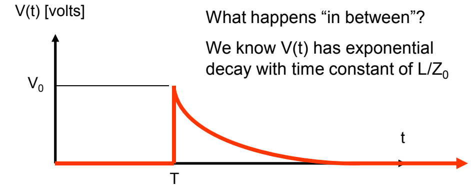
Optional

Shortcut Method



Optional

Shortcut Method

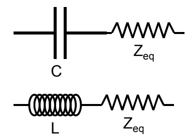


What happens “in between”?

We know V(t) has exponential decay with time constant of L/Z_0

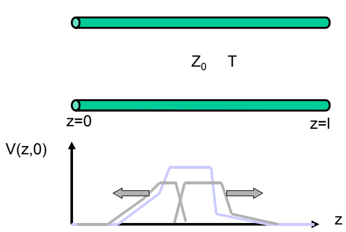
$$\tau = Z_{eq} C$$

$$\tau = \frac{L}{Z_{eq}}$$



Optional

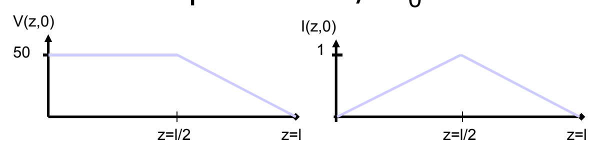
Initial distribution = Forward + Backward waves



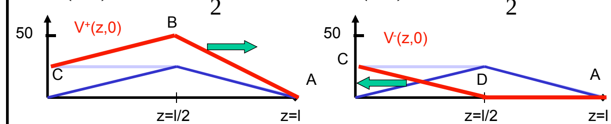
The voltage (made up of charges on the TL) will begin to spread by forming + and – waves.

Optional

Example. TL w/ Z_0=50Ω



$$V^+(z,0) = \frac{V(z,0) + 50I(z,0)}{2} \qquad V^-(z,0) = \frac{V(z,0) - 50I(z,0)}{2}$$



* *Important* *Chapter 7*

Phasors satisfy usual TL equations

$$\left.\begin{aligned} \frac{\partial V}{\partial z} &= -\mathscr{L}\frac{\partial I}{\partial t} \\ \frac{\partial I}{\partial z} &= -\mathscr{C}\frac{\partial V}{\partial t} \end{aligned}\right\} \quad \left.\begin{aligned} \frac{d\bar{V}}{dz} &= -\mathscr{L}(j\omega\bar{I}) \\ \frac{d\bar{I}}{dz} &= -\mathscr{C}(j\omega\bar{V}) \end{aligned}\right\} \quad \begin{aligned} \frac{d\bar{V}}{dz} &= -\mathscr{L}\mathscr{C}\omega^2\bar{V} \\ \frac{d\bar{I}}{dz} &= -\mathscr{L}\mathscr{C}\omega^2\bar{I} \end{aligned}$$

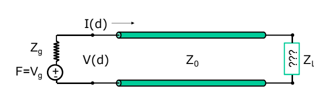
$$\boxed{\bar{V} = V^{\pm} e^{\mp j\beta z}} \qquad \boxed{\beta = \omega\sqrt{\mathscr{L}\mathscr{C}}} \qquad \boxed{\bar{V} = V^+ e^{j\beta d} + V^- e^{-j\beta d}}$$

$$\boxed{\bar{I} = \pm\frac{V^{\pm}}{Z_0} e^{\mp j\beta z}} \qquad \boxed{Z_0 = \sqrt{\mathscr{L}/\mathscr{C}}} \qquad \boxed{\bar{I} = \frac{1}{Z_0}(V^+ e^{j\beta d} - V^- e^{-j\beta d})}$$

** *Very Important*

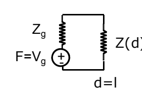
Key Definition: Line Impedance

$$\boxed{Z(d) \equiv \frac{\bar{V}(d)}{\bar{I}(d)}}$$



$$\boxed{Z(d) = Z_0 \frac{1 + \Gamma_L e^{-2j\beta d}}{1 - \Gamma_L e^{-2j\beta d}}}$$

Equivalent Circuit



$$V(l) = F\frac{Z(l)}{Z(l) + Z_g} = V^+(e^{j\beta l} + \Gamma_L e^{-j\beta l})$$

Allows you to solve for V^+ and thus get V(d,t) and I(d,t)

** *Very Important*

Key Definition: Generalized Reflection Coefficient

$$\boxed{\Gamma(d) \equiv \frac{\bar{V}^-(d)}{\bar{V}^+(d)}}$$

$$\boxed{\Gamma(d) = \frac{V^- e^{-j\beta d}}{V^+ e^{j\beta d}} = \Gamma_L e^{-2j\beta d}}$$

Allows you to find the backwards wave if forward wave is known

** *Very Important*

Key Definitions: Admittance and Normalized Impedance

Characteristic Admittance $\boxed{Y_0 \equiv \frac{1}{Z_0}}$

Normalized Impedance $\boxed{z(d) \equiv \frac{Z(d)}{Z_0}}$ $\boxed{z(d) = \frac{1+\Gamma(d)}{1-\Gamma(d)}}$

Normalized Admittance $\boxed{y(d) \equiv \frac{1}{z(d)}}$ $\boxed{y(d) = \frac{1-\Gamma(d)}{1+\Gamma(d)}}$

$$\boxed{\Gamma(d) = \Gamma_L e^{-2j\beta d}}$$

*** *Extremely Important*

Summary of TL Equations

$\boxed{\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}}$ $\boxed{V^- = \Gamma_L V^+}$ $\boxed{\Gamma(d) = \Gamma_L e^{-2j\beta d}}$

$$\bar{V}(d) = V^+ e^{j\beta d}(1+\Gamma_L e^{-2j\beta d}) = V^+ e^{j\beta d}(1+\Gamma(d))$$

$$\bar{I}(d) = \frac{V^+ e^{j\beta d}}{Z_0}(1-\Gamma_L e^{-2j\beta d}) = \frac{V^+ e^{j\beta d}}{Z_0}(1-\Gamma(d))$$

$\boxed{Z(d) \equiv \frac{\bar{V}(d)}{\bar{I}(d)} = Z_0 \frac{1+\Gamma(d)}{1-\Gamma(d)}}$ $\boxed{z(d) = \frac{1+\Gamma(d)}{1-\Gamma(d)}}$

$\boxed{y(d) \equiv \frac{1}{z(d)} = z(d \pm \frac{\lambda}{4})}$ $\boxed{\Gamma(d) = \frac{z(d)-1}{z(d)+1}}$

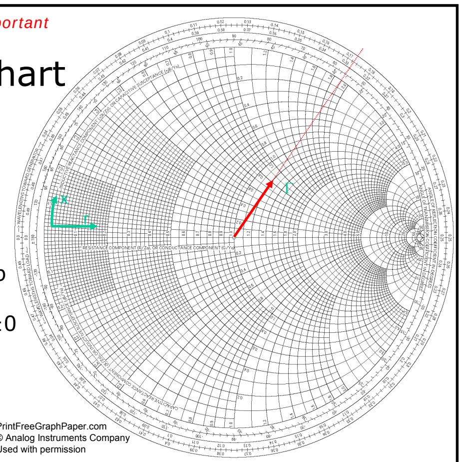
*** *Extremely Important*

Smith Chart

$$\boxed{\Gamma = \frac{z-1}{z+1}}$$

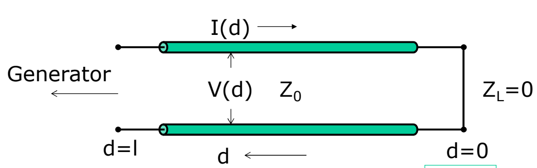
$z = r + jx$

S.C. is a map between the half-plane r≥0 and the disc |Γ|≤1



* *Important*

Standing Waves for SC Line



$$\boxed{\Gamma_L = -1}$$

$$\bar{V}(d) = V^+(e^{j\beta d} - e^{-j\beta d}) = 2jV^+ \sin(\beta d)$$

$$\bar{I}(d) = \frac{V^+}{Z_0}(e^{j\beta d} + e^{-j\beta d}) = 2Y_0 V^+ \cos(\beta d)$$

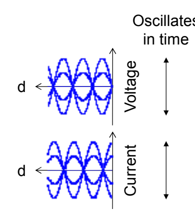
$$Z(d) \equiv \frac{\bar{V}(d)}{\bar{I}(d)} = jZ_0 \tan \beta d$$

* *Important*

Standing Waves for SC Line

$$V(d,t) = -2|V^+|\sin(\beta d)\sin(\omega t + \theta)$$

$$I(d,t) = 2Y_0|V^+|\cos(\beta d)\cos(\omega t + \theta)$$

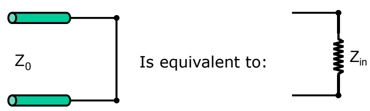


V(0,t)=0 always (voltage null)
I(0,t) varies (current maxima)

Dependence is different than traveling wave: ωt ± βz

Input Impedance for SC Line

$$Z_{in} = Z(l) = jZ_0 \tan \beta l$$



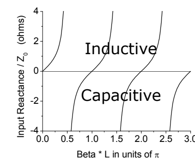
* *Important*

SC Line can act as an inductor or a capacitor depending on βl

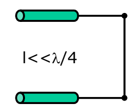
$$Z_{in} = jZ_0 \tan \beta l$$

If tan(βl)>0, shorted TL is inductive

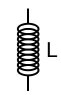
If tan(βl)<0, shorted TL is capacitive



e.g. βl<π/2 or l<λ/4, TL is inductive



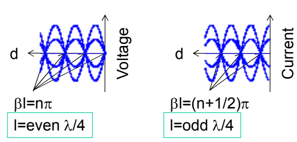
Equivalent Circuit:



For βl=0,π,2π,… TL is a short
For βl=π/2,3π/2,… TL is an open

$$Z_{in} = jZ_0 \tan \beta l = \begin{cases} 0 = a\ short\ for\ \beta l = n\pi,\ n = 0,1,2,... \\ \infty = an\ open\ for\ \beta l = (n+1/2)\pi \end{cases}$$

If Z_{in}=0, voltage drop is zero, just like a short
If Z_{in}=∞, current is zero, just like an open

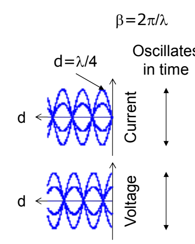


Standing Waves for OC Line

Same phasor algebra as before with current & voltage reversed!

$$I(d,t) = -2Y_0 |V^+| \sin(\beta d) \sin(\omega t + \theta)$$

$$V(d,t) = 2|V^+| \cos(\beta d) \cos(\omega t + \theta)$$

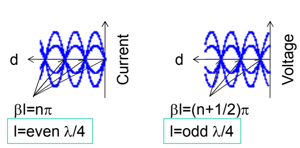


I(0,t)=0 always (current null)
V(0,t) varies (voltage maxima)

For βl=0,π,2π,… TL is an open
For βl=π/2,3π/2,… TL is a short

$$Y_{in} = jY_0 \tan \beta l = \begin{cases} 0 = an\ open\ for\ \beta l = n\pi,\ n = 0,1,2,... \\ \infty = a\ short\ for\ \beta l = (n+1/2)\pi \end{cases}$$

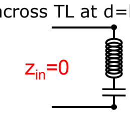
If Y_{in}=0, current is zero, just like an open
If Y_{in}=∞, voltage drop is zero, just like a short

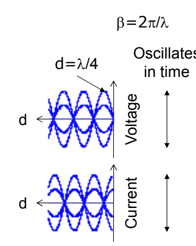


* *Important*

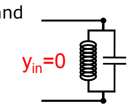
Parallel & Series Resonances

- We can find λ_n and ω_n by applying the appropriate BCs at both ends
- Series resonance if $z_{in}(l)=0$
  - Analogous to an LC circuit in series and requires a short placed across TL at d=l
  - Like a short input



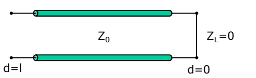


- Parallel resonance: $y_{in}(l)=0$
  - Analogous to an LC circuit in parallel and requires an open across the TL at d=l
  - Like an open input

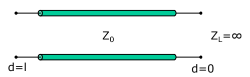


* Important

Parallel & Series Resonances

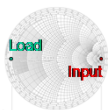


Shorted Load

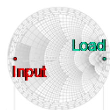


Open Load

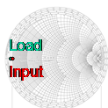
L=λ/4, 3λ/4, 5λ/4, …
Input is like an open
➔ **Parallel resonance**



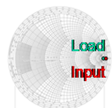
L=λ/4, 3λ/4, 5λ/4, …
Input is like a short
➔ **Series resonance**



L=λ/2, λ, 3λ/2, …
Input is like a short
➔ **Series resonance**



L=λ/2, λ, 3λ/2, …
Input is like an open
➔ **Parallel resonance**



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Cumulative Review

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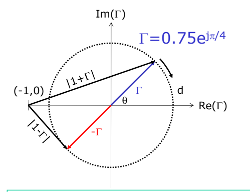
** Very Important

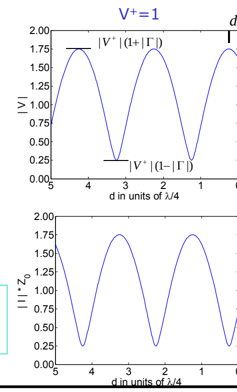
Standing Wave for Arbitrary Load

$$|\bar{V}(d)| = |V^+| \, |1+\Gamma(d)|$$

$$|\bar{I}(d)| = Y_0 \, |V^+| \, |1-\Gamma(d)|$$



Unlike SC or OC where |Γ|=1, now we have imperfect nulls for voltage and current b/c |Γ|<1



$$d_{max} = \frac{\lambda}{4\pi}\theta$$

$$VSWR = \frac{V_{max}}{V_{min}} = \frac{1+|\Gamma|}{1-|\Gamma|}$$

$$VSWR = z(d_{max}) = y(d_{min})$$

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Cumulative Review

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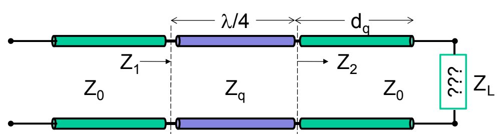
** Very Important

Quarter Wave Transformers



Key Observations:

1. Z_0 and Z_q are real since TL's are lossless
2. Z_1=Z_0 for a match so Z_1 must be real
3. Z_1Z_2=Z_q^2 since we know $\boxed{y(d) = z(d \pm \lambda/4)}$
4. Z_2 must be real from equation #3
5. d_q must be at a voltage max or min from #4

Adjust Z_q and d_q for match

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Cumulative Review

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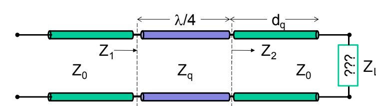
** Very Important

Quarter Wave Transformers

- Quarter wave transformer matching inserts a length of l=λ/4 of a specific impedance Z_q=sqrt(Z_0Z_2) at a distance d_q from the load, where voltage is min or max



$$\boxed{Z_q = \sqrt{Z_0 Z_2}} \qquad \boxed{d_q = \begin{cases} d_{max} = \dfrac{\lambda\theta}{4\pi} \\ d_{min} = \dfrac{\lambda\theta}{4\pi} + \dfrac{\lambda}{4} \end{cases} \mod \lambda/2}$$

Pros: Easy design
Cons: Have to redo QWT for each f

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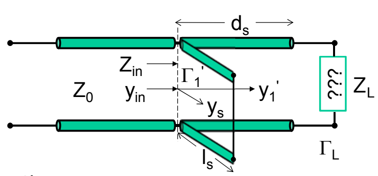
Cumulative Review

** Very Important

Smith Chart for Single Stub



Key Observations:

1. y_{in}=1 for a match, y_s=jb is purely imaginary for the SC stub
2. Thus y_1'=y_{in}-y_s=1-jb must be on the unit conductance circle
   a. Needed amount of susceptance, b, depends on $|\Gamma_L|$

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Cumulative Review

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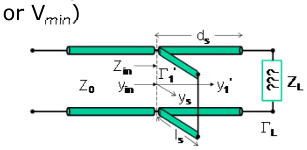
** Very Important

Single Stub Matching

- Single stub matching inserts a shorted stub of the same impedance, Z_0, but with a specific length, l_s, and distance, d_s, from the load (not necessarily at V_{max} or V_{min})

Pros: Z=Z_0 on all lines
Cons: Adjusting d_s may be inconvenient



- Find d_s by moving CW from y(0) to the unit conductance circle y_1'=1-jb; then find l_s by going CW from $y_s(0)$=∞ to $y_s(l_s)$=+jb or using formula:

$$\boxed{\therefore l_s = \frac{\lambda}{2\pi}\tan^{-1}\left(-\frac{1}{b}\right) \mod \lambda/2}$$

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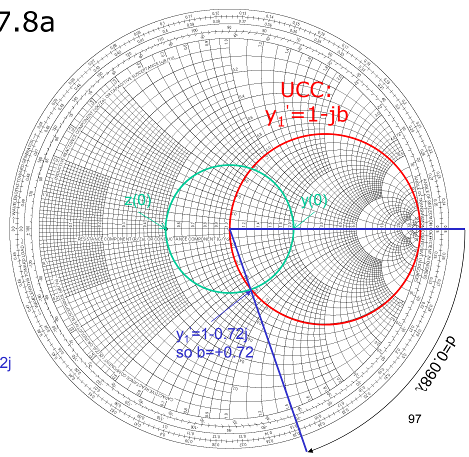
Repeat D7.8a (p 472):

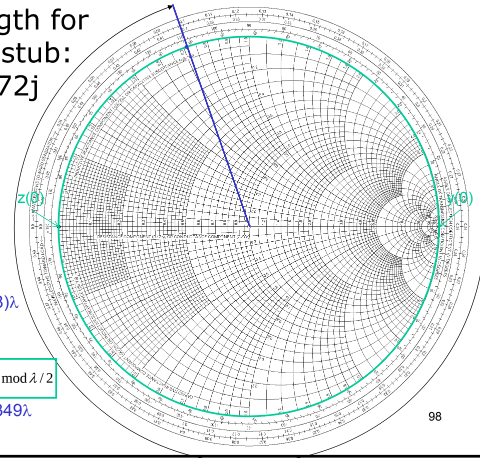
$z(0)=0.5$



Thus, we need a stub with $y_s(l_s)=0.72j$ at a distance of 0.098λ from load

Find length for shorted stub: $y(l_s)=0.72j$

$z(0)=0$
$y(0)=\infty$



$l_s=(0.25+0.098)\lambda$
$l_s=0.348\lambda$

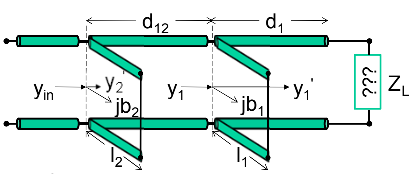
or use:

$$\boxed{l_s=\frac{\lambda}{2\pi}\tan^{-1}(-\frac{1}{b}) \mod \lambda/2}$$

$l_s=-0.151\lambda=0.349\lambda$

Optional

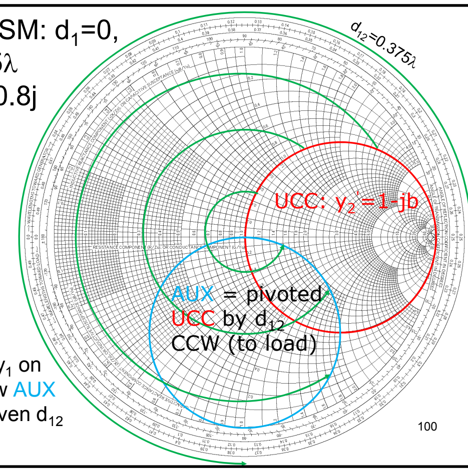
Double Stub Matching w/ S.C.



Key Observations:

1. For a match, y_2' must be on unit conductance circle
2. Thus, y_1 is on the auxiliary circle
   a) AUXILIARY CIRCLE is UCC pivoted <u>CCW</u> towards the <u>load</u> by d_{12}

Repeat DSM: $d_1=0$, $d_{12}=0.375\lambda$ $z(0)=0.6\text{-}0.8j$

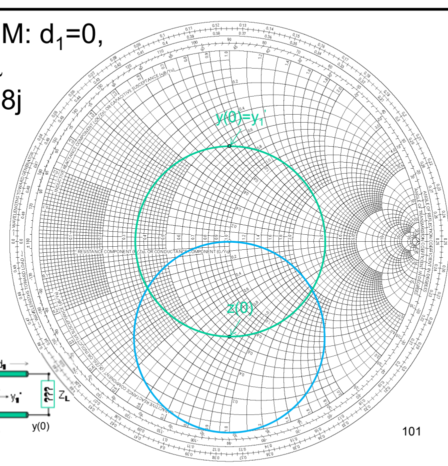


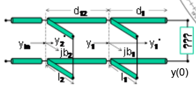
To get y_2' on UCC, need y_1 on AUX so draw AUX first using given d_{12}

Repeat DSM: $d_1=0$, $d_{12}=0.375\lambda$ $z(0)=0.6\text{-}0.8j$

$y(0)=0.6+0.8j$

Since $d_1=0$, $y_1'=y(0)$



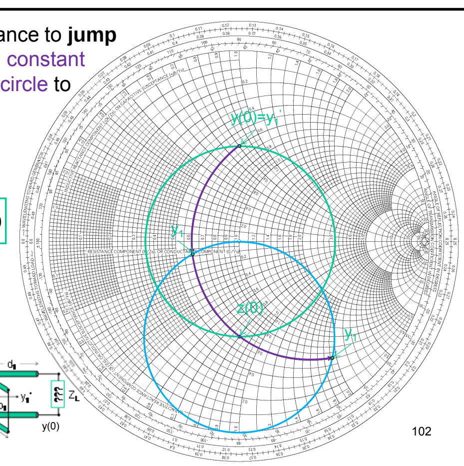


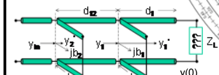
Find susceptance to **jump** from y_1' along constant conductance circle to y_1 on AUX

$b_1=-0.89$ or
$b_1=-2.72$

$$\boxed{l_s=\frac{\lambda}{2\pi}\tan^{-1}(-\frac{1}{b})}$$

$l_1=0.134\lambda$ or
$l_1=0.056\lambda$





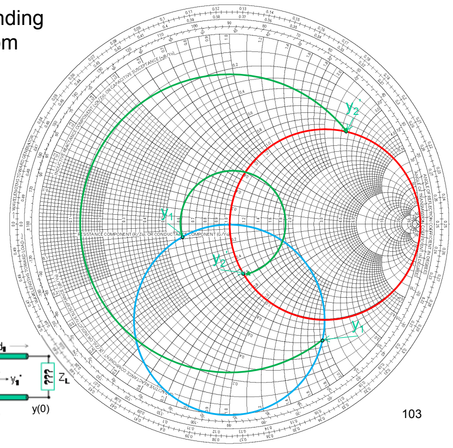
Find corresponding y_2' by going from AUX to UCC

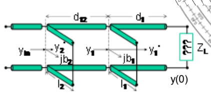
y_2'=1-0.53j or
y_2'=1+2.5j

b_2=0.53 or
b_2=-2.5

$$l_s = \frac{\lambda}{2\pi}\tan^{-1}(-\frac{1}{b})$$

l_2=0.327λ or
l_2=0.061λ



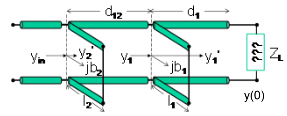


Optional

Double Stub Matching

- Double stub matching inserts two shorted stub of impedance Z_0 with specific lengths, l_1 and l_2, at fixed spots, d_1 and d_1+d_{12}, from the load

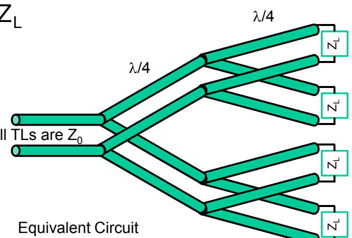
Pros: $Z=Z_0$ on all lines, fixed d1, d12
Cons: Many calculations



- Go from y(0) to y_1' a distance d_1 along $|\Gamma(0)|$ circle. Find b_1 by going from y_1' along constant conductance circle to y_1 which is on the AUX circle. Find y_2' by pivoting AUX by d_{12} to UCC and read off b_2=-Im(y_2').
- Calculate l_1 and l_2 using: $l_s = \frac{\lambda}{2\pi}\tan^{-1}(-\frac{1}{b}) \mod \lambda/2$

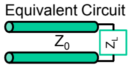
Corporate Ladder

- A network for combining 4 identical loads Z_L into an equivalent single load $Z_{in}=Z_L$



By symmetry, the average power delivered to each load is identical.

Equivalent Circuit

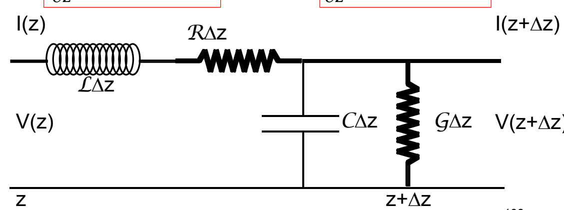


Distributed Circuit of Lossy TL

Model the Ohmic losses in conducting wires and leakage losses in the imperfect dielectric in between

One slice of the lossy TL:

$$\boxed{\frac{\partial V}{\partial z} = -(j\omega\mathcal{L}+\mathcal{R})I} \qquad \boxed{\frac{\partial I}{\partial z} = -(j\omega C+\mathcal{G})V}$$



Solution of Lossy TL

$$\boxed{V(d) = V^+e^{\gamma d} + V^-e^{-\gamma d}} \qquad \gamma = \alpha + j\beta = \sqrt{(j\omega\mathcal{L}+\mathcal{R})(j\omega C+\mathcal{G})}$$

$$\boxed{I(d) = \frac{V^+e^{\gamma d}}{\tilde{Z}_0} - \frac{V^-e^{-\gamma d}}{\tilde{Z}_0}} \qquad \tilde{Z}_0 = \sqrt{\frac{j\omega\mathcal{L}+\mathcal{R}}{j\omega C+\mathcal{G}}}$$

These reduce to the lossless results as $\mathcal{R}$ and $\mathcal{G} \to 0$

$$\gamma = \sqrt{(j\omega\mathcal{L})(j\omega C)} = j\omega\sqrt{\mathcal{L}C} = j\frac{\omega}{v_p} = j\beta$$

$$\tilde{Z}_0 = \sqrt{\frac{j\omega\mathcal{L}}{j\omega C}} = \sqrt{\frac{\mathcal{L}}{C}}$$

Good luck on the final!

- It was a pleasure teaching ECE329 this semester
  - Thank you for studying so hard ☺