ECE 329 Homework 3 Solution Due: Tuesday Sept 19, 2017, 6PM

1. Given $\mathbf{E} = z\hat{x} - y\hat{y} + 2x\hat{z}$ V/m

$$\nabla \times \mathbf{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & -y & 2x \end{vmatrix} = -\hat{y} \neq 0$$

So it is not an electrostatic field.

2. Given$V = \sin(x)(y+2)z^2$ V

$$\mathbf{E} = -\nabla V = -cos(x)(y+2)z^2\,\hat{x} - sin(x)z^2\,\hat{y} - 2sin(x)(y+2)z\,\hat{z}\ V/m$$

$$\rho = \epsilon_0 \nabla \cdot \mathbf{E} = \epsilon_0 sin(x)(y+2)(z^2-2)\ C/m^3$$

$$\nabla \times \mathbf{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -cos(x)(y+2)z^2 & -sin(x)z^2 & -2sin(x)(y+2)z \end{vmatrix} = 0$$

3. Given that $\mathbf{E} = x\hat{x} + z\cos(y)\hat{y} + \sin(y)\hat{z}$ V/m

$$V(1,2,3) - V(0,0,0) = -\int_0^P \mathbf{E}\cdot d\mathbf{l} = -\int_0^1 E_x(x,0,0)dx - \int_0^2 E_y(1,y,0)dy - \int_0^3 E_z(1,2,z)dz = -\frac{1}{2} - 3sin(2)\ V$$

$$V(1,2,3) = 2.5 - 3sin(2)\ V$$

4. Given the two fields $\mathbf{E} = y\hat{x} \pm x\hat{y}$ V/m,

a) $+$ sign:

$$\oint_c \mathbf{E}\cdot d\mathbf{l} = \int_1^{-1} y|_{y=-1}dx + \int_{-1}^{1} (y\hat{x} + x\hat{y})|_{x=y} \cdot (\hat{x}+\hat{y})dx + \int_1^{-1} x|_{x=1}dy = 0$$

- sign:

$$\oint_c \mathbf{E}\cdot d\mathbf{l} = \int_1^{-1} y|_{y=-1}dx + \int_{-1}^{1} (y\hat{x} - x\hat{y})|_{x=y} \cdot (\hat{x}+\hat{y})dx + \int_1^{-1} -x|_{x=1}dy = 4$$

b) $+$ sign:

$$\nabla \times \mathbf{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x & 0 \end{vmatrix} = 0$$

so

$$\iint_s (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = 0 = \oint_c \mathbf{E}\cdot d\mathbf{l}$$

- sign:

$$\nabla \times \mathbf{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = -2\hat{z}$$

and

$$\iint_s (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = (\nabla \times \mathbf{E}) \cdot Area \cdot d\mathbf{S} = 4$$

note that $d\mathbf{S} = -\hat{z} \cdot dS$ according to the right hand rule, so

$$\iint_s (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = 4 = \oint_c \mathbf{E} \cdot d\mathbf{l}$$

5. Using vector line integral equation from question 3, the voltage drop is given by

$$V_p = \int_2^{-2} -4\hat{z} \cdot \hat{z} dz = 16\ V$$

6. Following Exmaple 5 in Lecture 5,

$$V(r) = \int_{z=r}^{\infty} \frac{Q}{4\pi\epsilon_o z^2} \hat{z} \cdot \hat{z} dz = -\frac{Q}{4\pi\epsilon_o z}|_r^{\infty} = \frac{Q}{4\pi\epsilon_0 r}$$

7. Consider a static sheet of charge $\rho(x, y, z) = \rho_s \delta(z)$ C/m^3.

   a) Since

   $$\mathbf{E} = \begin{cases} \frac{\rho_s}{2\epsilon_0} \hat{z} & z > 0 \\ -\frac{\rho_s}{2\epsilon_0} \hat{z} & z < 0 \end{cases}$$

   so

   $$\mathbf{D} = \epsilon_{\mathbf{0}} \mathbf{E} = \begin{cases} \frac{\rho_s}{2} \hat{z} & z > 0 \\ -\frac{\rho_s}{2} \hat{z} & z < 0 \end{cases}$$

   b) With the substitutions,

   $$\mathbf{D_1} = \hat{\mathrm{a_n}} \frac{\rho_\mathbf{s}}{2}$$

   $$\mathbf{D_2} = -\hat{\mathrm{a_n}} \frac{\rho_\mathbf{s}}{2}$$

   from which it is easy to see that $\hat{a_n} \cdot (\mathbf{D_1} - \mathbf{D_2}) = \rho_s$

8. a) $\rho_s = \hat{y} \cdot (\mathbf{D}_{2+} - \mathbf{D}_{2-}) = 1 - 2 = -1\ C/m^2$

   b) Since $E_z^{2+} = E_z^{2-}$, and all the fields are in free space, $D_z^{2+} = -2\ C/m^2$, so $\mathbf{D}_{2+} = \hat{y} - 2\hat{z}\ C/m^2$

   c) Since $D_y^{0+} - D_y^{0-} = 3$, $D_y^{0-} = -1\ C/m^2$,so $\mathbf{D}_{0-} = -\hat{y} - 2\hat{z}\ C/m^2$

9. The proof is shown as below,

$$\nabla f = \hat{x} + z\hat{y} + y\hat{z}$$

$$\nabla \times \nabla f = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1 & z & y \end{vmatrix} = 0$$

$$\nabla \times \mathbf{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+z)y & 0 & -z \end{vmatrix} = y\hat{y} - (x+z)\hat{z}$$

so

$$\nabla \cdot (\nabla \times \mathbf{A}) = 1 - 1 = 0$$

since

$$\nabla \times \nabla \times \mathbf{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & y & -(x+z) \end{vmatrix} = \hat{y}$$

and

$$\nabla \cdot \mathbf{A} = y - 1$$

$$\nabla(\nabla \cdot \mathbf{A}) = \hat{y}$$

$$\nabla^2 \mathbf{A} = 0$$

therefore

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot A) - \nabla^2 \mathbf{A}$$