ECE 329 Homework 1 Solution Due: Sept 5, 2017, 5PM

1. Consider the 3D vectors

$$\begin{aligned}\vec{A} &= 2\hat{x} + \hat{y} - 2\hat{z},\\ \vec{B} &= \hat{x} + \hat{y} - \hat{z},\\ \vec{C} &= \hat{x} - 2\hat{y} + 3\hat{z},\end{aligned}$$

   (a) The vector

$$\vec{A} + \vec{B} - 4\vec{C} = \hat{x} + 2\hat{y} - 3\hat{z} - 4(\hat{x} - 2\hat{y} + 3\hat{z}) = -\hat{x} + 1\hat{0}y - 15\hat{z}$$

   (b) The vector *magnitude*

$$|\vec{A} + \vec{B} - 4\vec{C}| = \sqrt{1^2 + 10^2 + 15^2} = 18.06$$

   (c) The unit vector $\hat{u}$ along vector

$$\vec{A} + \vec{B} - 4\vec{C} = -\hat{x} + 10\hat{y} - 15\hat{z}$$

$$|\vec{A} + \vec{B} - 4\vec{C}| = 18.06$$

$$\hat{u} = \frac{\vec{A} + \vec{B} - 4\vec{C}}{|\vec{A} + \vec{B} - 4\vec{C}|} = \frac{-\hat{x} + 10\hat{y} - 15\hat{z}}{18.06} = -0.055\hat{x} + 0.55\hat{y} - 0.83\hat{z}$$

   (d) The *dot product*

$$\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A} = (2\hat{x}+\hat{y}-2\hat{z})\cdot(\hat{x}+\hat{y}-\hat{z}) = 2\times1+1\times1+(-2)\times(-1) = 5$$

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   (e) The *cross product*

$$\vec{B}\times\vec{C} = \begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ 1 & 1 & -1\\ 1 & -2 & 3\end{vmatrix} = \hat{x}\,(3-2) - \hat{y}\,(3+1) + \hat{z}\,(-2-1) = \hat{x} - 4\hat{y} - 3\hat{z}$$

$$\vec{C}\times\vec{B} = \begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ 1 & -2 & 3\\ 1 & 1 & -1\end{vmatrix} = \hat{x}\,(2-3) - \hat{y}\,(-1-3) + \hat{z}\,(1+2) = -\hat{x} + 4\hat{y} + 3\hat{z}$$

2. In the three cases, we have the same E and B at the origin, but different v. Thus, we have different F. In each case, v and F must satisfy F = q(E + v B). Therefore, we have the following three equations

$$\begin{cases} \quad 3\vec{z} = E \\ \quad \hat{z} = E + \hat{y}\times B \\ 3\hat{z} + 4\hat{y} = E + 2\hat{z}\times B \end{cases}$$

from which we obtain

$$\begin{cases} E &= 3\hat{z} \\ \hat{y} \times B &= -2\hat{z} \\ \hat{z} \times B &= 2\hat{y} \end{cases}$$

If we assume $B = \hat{x}B_x + \hat{y}B_y + \hat{z}B_z$, then

$$\begin{cases} \hat{y} \times (\hat{x}B_x + \hat{y}B_y + \hat{z}B_z) &= -2\hat{z} \\ \hat{z} \times (\hat{x}B_x + \hat{y}B_y + \hat{z}B_z) &= 2\hat{y} \end{cases}$$

Further

$$\begin{cases} -\hat{z}B_x + \hat{x}B_z &= -2\hat{z} \\ \hat{y}B_x - \hat{x}B_y &= 2\hat{y} \end{cases}$$

Therefore, we have

$$\begin{cases} B_x &= 2 \\ B_y &= B_z = 0 \end{cases}$$

In summary, $E = 3\hat{z}$(V/m) and $B = 2\hat{x}(Wb/m^2)$.

3. Let us use S1, S2, S3, S4, S5 and S6 to denote the surfaces x = 0, x = 1, y = 0, y = 1, z = 0 and z = 1, respectively. We consider S1 and S2 first. The unit vector on S1 pointing away from the volume is along $-\hat{x}$ direction, so we have

$$\int_{S_1} J \cdot dS = \int_0^1 \int_0^1 [y^2(\hat{x} + \hat{y} + \hat{z})]\,|_{x=0} \cdot (-\hat{x}) dydz = -\frac{1}{3}\,(A).$$

For S2, the unit vector pointing away from the volume is along $+\hat{x}$ direction, therefore

$$\int_{S_2} J \cdot dS = \int_0^1 \int_0^1 [y^2(\hat{x} + \hat{y} + \hat{z})]\,|_{x=1} \cdot (+\hat{x}) dydz = \frac{1}{3}\,(A).$$

Similarly, for S3, S4, S5 and S6, we can obtain

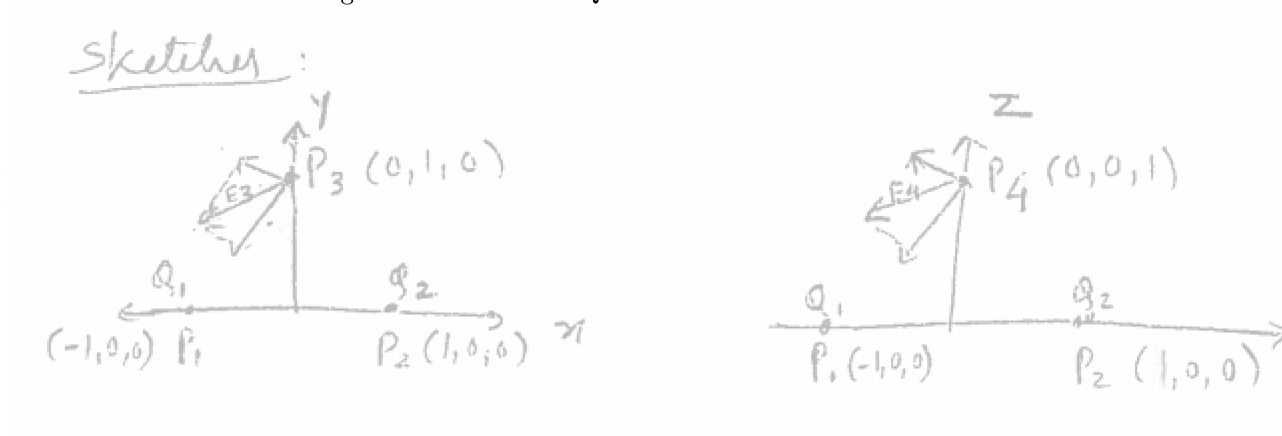
$$\int_{S_3} J \cdot dS = \int_0^1 \int_0^1 [y^2(\hat{x} + \hat{y} + \hat{z})]\,|_{y=0} \cdot (-\hat{y}) dxdz = 0\,(A),$$

$$\int_{S_4} J \cdot dS = \int_0^1 \int_0^1 [y^2(\hat{x} + \hat{y} + \hat{z})]\,|_{y=1} \cdot (+\hat{y}) dxdz = 1\,(A),$$

$$\int_{S_5} J \cdot dS = \int_0^1 \int_0^1 [y^2(\hat{x} + \hat{y} + \hat{z})]\,|_{z=0} \cdot (-\hat{z}) dxdy = -\frac{1}{3}\,(A),$$

$$\int_{S_6} J \cdot dS = \int_0^1 \int_0^1 [y^2(\hat{x} + \hat{y} + \hat{z})]\,|_{z=1} \cdot (+\hat{z}) dxdy = \frac{1}{3}\,(A).$$

Figure 1: Sketches for Q4



Therefore,

$$\oint_S J \cdot dS = \sum_{i=1}^{6} \int_{s_i} J \cdot dS = 1\,(A).$$

4. We know$Q_1 = -4\pi\varepsilon_0$, so $Q_2 = -Q_1/2 = 2\pi\varepsilon_0$. The electric eld at the point P_3 is the superposition of those induced by Q1 and Q2, namely

$$\begin{aligned} E_3 &= \sum_{i=1}^{2} \frac{Q_i}{4\pi\varepsilon_0 |r_3 - r_i|^2} \cdot \frac{r_3 - r_i}{|r_3 - r_i|} \\ &= \frac{-1}{|\hat{x} + \hat{y}|^3}(\hat{x} + \hat{y}) + \frac{0.5}{|-\hat{x} + \hat{y}|^3}(-\hat{x} + \hat{y}) \\ &= \frac{-1.5\hat{x} - 0.5\hat{y}}{2\sqrt{2}} = \frac{-3\hat{x} - \hat{y}}{4\sqrt{2}}\,(V/m). \end{aligned}$$

The electric field at the point P_4 can be obtained in a similar way

$$\begin{aligned} E_4 &= \sum_{i=1}^{2} \frac{Q_i}{4\pi\varepsilon_0 |r_4 - r_i|^2} \cdot \frac{r_4 - r_i}{|r_4 - r_i|} \\ &= \frac{-1}{|\hat{x} + \hat{z}|^3}(\hat{x} + \hat{z}) + \frac{0.5}{|-\hat{x} + \hat{z}|^3}(-\hat{x} + \hat{z}) \\ &= \frac{-1.5\hat{x} - 0.5\hat{z}}{2\sqrt{2}} = \frac{-3\hat{x} - \hat{z}}{4\sqrt{2}}\,(V/m). \end{aligned}$$

5. Gauss's law for electric flux density D states that

$$\oint_S \mathbf{D} \cdot \mathbf{dS} = \int_V \rho dV$$

over any closed surface S enclosing a volume V where electric charge density is specified by $\rho(x,\, y,\, z)\, [\frac{C}{m^3}]$. Here, we will compute the electric flux density $\oint_S \mathbf{D} \cdot \mathbf{dS}$ over the surface of a cube of volume $V = L^3$thta is centered at the origin.

(a) If $\rho(x,\, y,\, z) = -2\frac{C}{m^3}$ (within the cube), the total electric flux can be computed as follows:

$$\begin{aligned}\oint_S \mathbf{D} \cdot \mathbf{dS} &= \int_V \rho dV \\ &= -2L^3\,(C)\end{aligned}$$

(b) If $\rho(x,\, y,\, z) = x^2\frac{C}{m^3}$ (within the cube), the total electric flux can be computed as follows:

$$\begin{aligned}\oint_S \mathbf{D} \cdot \mathbf{dS} &= \int_V x^2 dV \\ &= \int_{-L/2}^{L/2}\int_{-L/2}^{L/2}\int_{-L/2}^{L/2} x^2 dx\, dy\, dz = \frac{1}{3}x^3\,|_{-L/2}^{L/2} \cdot L^2 = \frac{1}{12}L^5\,(C)\end{aligned}$$