1.

a) Seeing that this is an RC circuit, the time constant, τ, is equal to RC. We know that $\tau_{slab} = 1.5\tau_{air}$ and and have some slab of width l and area A. Then, we can find the time constant.

$$\begin{aligned} \tau &= RC \\ &= (\rho\frac{l}{A})(\epsilon\frac{A}{l}) \\ &= \rho\epsilon \end{aligned}$$

Now we can compare τ with τ_0 to solve for ϵ. It is assumed that the slab has the same resistivity, ρ, as air.

$$\begin{aligned} \tau_{slab} &= 1.5\tau_{air} \\ \rho_{slab}\epsilon_{slab} &= 1.5\rho_{air}\epsilon_0 \\ \epsilon_{slab} &= 1.5\epsilon_0 \end{aligned}$$

b) If instead of a slab with width l, the slab width is now half of l, we can still use an RC circuit but now have a capacitance of air in series with the capacitance of the slab. We know that $C_{air} = \epsilon_0\frac{A}{l}$, and therefore $C_{1/2\,air} = 2\epsilon_0\frac{A}{l}$ and $C_{1/2\,slab} = 2\epsilon_r\epsilon_o\frac{A}{l}$. Let us use the formula for capacitance in series to solve this:

$$\begin{aligned} \frac{1}{C_{eq}} &= \frac{1}{C_{1/2\,air}} + \frac{1}{C_{1/2\,slab}} \\ &= \frac{1}{2C_{air}} + \frac{1}{2\epsilon_r C_{air}} \\ C_{eq} &= \frac{2\epsilon_r C_{air}}{1+\epsilon_r} \end{aligned}$$

Once again we can compare τ_{eq} with τ_{air} to solve for ϵ_r. It is assumed that the slab has the same resistance, R, as air.

$$\begin{aligned} \tau_{eq} &= 1.5\tau_{air} \\ R_{eq}C_{eq} &= 1.5R_{air}C_{air} \\ \frac{2\epsilon_r C_{air}}{1+\epsilon_r} &= 1.5C_{air} \\ \epsilon_r &= 3 \end{aligned}$$

Thus $\epsilon_{slab} = \epsilon_r\epsilon_0 = 3\epsilon_0$.

c) Seeing that this is an RL circuit, $\tau = L/R$. We know that $\tau_{rod} = 0.998\tau_{air}$ and have some slab of length l and area A which is inside a solenoid with parameters K and N. Knowing the inductance of a cylindrical solenoid:

$$L = \frac{\mu K N^2 A}{l} \quad \rightarrow \quad \tau = \frac{\mu K N^2 A}{Rl}$$

Once again we can compare τ_{eq} with τ_{air} to solve for μ. It is assumed that the slab has the same resistance, R, as air.

$$\begin{aligned}
\tau_{rod} &= 0.998\tau_{air} \\
\frac{\mu_{rod}KN^2A}{R_{rod}l} &= 0.998\frac{\mu_0 KN^2A}{R_{air}l} \\
\mu &= 0.998\mu_0
\end{aligned}$$

Since $\mu_{rod} < \mu_0$, this slab is dimagnetic.

2. Starting with the left-hand side of the vector identity given in the problem, we can write

$$\begin{aligned}
\mathbf{H}\cdot\nabla\times\mathbf{E} - \mathbf{E}\cdot\nabla\times\mathbf{H} &= (4e^{-\alpha z}\hat{x})\cdot\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 2e^{-\alpha z} & 0 \end{vmatrix} - (2e^{-\alpha z}\hat{y})\cdot\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4e^{-\alpha z} & 0 & 0 \end{vmatrix} \\
&= (4e^{-\alpha z}\hat{x})\cdot(2\alpha e^{-\alpha z}\hat{x}) - (2e^{-\alpha z}\hat{y})\cdot(-4\alpha e^{-\alpha z}\hat{y}) \\
&= 16\alpha e^{-2\alpha z}.
\end{aligned}$$

The right-hand side of the vector identity is solved by

$$\begin{aligned}
\nabla\cdot(\mathbf{E}\times\mathbf{H}) &= \nabla\cdot\left(2e^{-\alpha z}\hat{y}\times 4e^{-\alpha z}\hat{x}\right) \\
&= \nabla\cdot\left(-8e^{-2\alpha z}\hat{z}\right) \\
&= \frac{\partial}{\partial z}\left(-8e^{-2\alpha z}\right) = 16\alpha e^{-2\alpha z}.
\end{aligned}$$

Consequently, the vector identity is verified.

3.

a) Referring to the hint, we should first find the divergence of the current density. Thus, we write

$$\nabla\cdot\mathbf{J} = \frac{\partial}{\partial x}\left(5z^2\right) + \frac{\partial}{\partial y}\left(4x^3y\right) + \frac{\partial}{\partial z}\left(3z(y-y_o)^2\right) = 4x^3 + 3(y-y_o)^2.$$

Then, taking the integral of both sides of the continuity equation $\frac{\partial\rho}{\partial t} = -\nabla\cdot\mathbf{J}$, we get

$$\rho(\mathbf{r},t) = -\left(4x^3 + 3(y-y_o)^2\right)t + \rho_o\,\frac{\mathrm{C}}{\mathrm{m}^3}.$$

Evaluating $\rho(\mathbf{r},t)$ at $\mathbf{r}=\mathbf{0}$ and given that $\rho_o = 0$ and $y_o = 2$, we find

$$\rho(\mathbf{0},t) = -12t\,\frac{\mathrm{C}}{\mathrm{m}^3}.$$

b) Since the units of J_x, J_y, and J_z are $\mathrm{A/m^2}$, we get

$$\begin{aligned}
[J_x] = \left[5z^2\right] &= \frac{\mathrm{A}}{\mathrm{m}^2} &\rightarrow\quad [5] &= \frac{\mathrm{A}}{\mathrm{m}^4}, \\
[J_y] = \left[4x^3y\right] &= \frac{\mathrm{A}}{\mathrm{m}^2} &\rightarrow\quad [4] &= \frac{\mathrm{A}}{\mathrm{m}^6}, \\
[J_z] = \left[3z(y-y_o)^2\right] &= \frac{\mathrm{A}}{\mathrm{m}^2} &\rightarrow\quad [3] &= \frac{\mathrm{A}}{\mathrm{m}^5}.
\end{aligned}$$

4.

a) In a homogeneous conductor where $\mathbf{J} = \sigma \mathbf{E}$, Gauss' law $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_o}$ implies

$$\nabla \cdot \mathbf{J} = \sigma \frac{\rho}{\epsilon_o}.$$

Then, using the continuity equation $\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$, we can show that

$$\frac{\partial \rho}{\partial t} + \frac{\sigma}{\epsilon_o}\rho = 0.$$

b) The resulting differential equation in part (a) can be rewritten as

$$\frac{1}{\rho}\frac{\partial \rho}{\partial t} = -\frac{\sigma}{\epsilon_o} \quad \rightarrow \quad \frac{\partial \ln \rho}{\partial t} = -\frac{\sigma}{\epsilon_o}.$$

Integrating over time from 0 to t, we get

$$\int_0^t \frac{\partial \ln \rho}{\partial t} dt = -\int_0^t \frac{\sigma}{\epsilon_o} dt \quad \rightarrow \quad \ln \rho - \ln \rho_o = -\frac{\sigma}{\epsilon_o} t,$$

from which we obtain

$$\rho = \rho_o e^{-\frac{\sigma}{\epsilon_o} t},$$

where ρ_o is the charge density distribution at time $t = 0$. Given that $\rho_o = \sin(100x)\,\frac{\mathrm{C}}{\mathrm{m}^3}$, we find

$$\rho = \sin(100x)\, e^{-\frac{\sigma}{\epsilon_o} t}\, \frac{\mathrm{C}}{\mathrm{m}^3} \qquad \text{for } t \geq 0.$$

c) The time it takes for ρ to reduce to $0.01 \sin(100x)\,\mathrm{C/m}^3$ is calculated as,

$$e^{-\frac{\sigma}{\epsilon_o} t} = 0.01 \quad \rightarrow \quad t = -\frac{\epsilon_o}{\sigma} \ln 0.01 = 8.15 \times 10^{-19}\,\mathrm{s}.$$

d)

i. At $t = 0$ and from Gauss's law, $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_o}$, we know that there is a non-zero electric field $\mathbf{E}$ associated to the non-zero charge density distribution $\rho(\mathbf{r}, t)$. From page 3 of Lecture 10, we know that the stored electrostatic energy per unit volume has a non-zero value, i.e. $w = \frac{1}{2}\epsilon_o \mathbf{E} \cdot \mathbf{E}$.

ii. As $t \rightarrow \infty$ without any external source, the charge density $\rho \rightarrow 0$, and so does the electric field $\mathbf{E} \rightarrow 0$. It means that the electrostatic energy per unit volume is 0.

The stored energy at $t = 0$ can be seen as the stored energy in a capacitor C. The conductor has a finite conductivity σ and will have a resistance $R \propto 1/\sigma$. From ECE210 we know that the energy stored in a capacitor in an RC circuit will completely dissipate through the resistor R in the absence of any other energy source.

5.

a) An electric field given by

$$\mathbf{E} = \cos(\omega t - \beta x)\hat{y}\,\frac{\mathrm{V}}{\mathrm{m}}$$

is propagating at a velocity $v = \frac{\omega}{\beta} = c$. The medium has $\mu = \mu_r\mu_o = \mu_o$, which implies $\mu_r = 1$ and thus using $\frac{1}{\sqrt{\epsilon_r\mu_r}} = 1$, we get $\epsilon_r = 1$. Using Faraday's law $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$, first we get

$$\nabla \times \mathbf{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & \cos(\omega t - \beta x) & 0 \end{vmatrix} = \frac{\partial}{\partial x}\left(\cos(\omega t - \beta x)\right)\hat{z} = \beta \sin(\omega t - \beta x)\hat{z}$$

Now equating the above result to $-\frac{\partial \mathbf{B}}{\partial t}$, we get

$$\frac{\partial \mathbf{B}}{\partial t} = -\beta \sin(\omega t - \beta x)\hat{z}$$

Integrating both sides of the above equation and dividing by $\mu = \mu_0 = 1.2566 \times 10^{-6}$ will give us

$$\begin{aligned} \mathbf{H} &= \frac{\mathbf{B}}{\mu_o} = -\frac{\beta}{\mu_o}\int (\sin(\omega t - \beta x)\hat{z})dt \\ &= \frac{\beta}{\mu_o \omega}\cos(\omega t - \beta x)\hat{z} \\ &= \frac{1}{c\mu_o}\cos(\omega t - \beta x)\hat{z} \\ &= 2.65 \times 10^{-3}\cos(\omega t - \beta x)\hat{z}\,\frac{\mathrm{A}}{\mathrm{m}} \end{aligned}$$

b) A magnetic field given by

$$\mathbf{H} = \cos(\omega t + \beta y)\hat{x}\,\frac{\mathrm{A}}{\mathrm{m}}$$

is propagating at a velocity $v = \frac{\omega}{\beta} = \frac{2}{3}c$. The medium is homogeneous with $\epsilon = \epsilon_r\epsilon_o$ and $\mu = \mu_r\mu_o$, and thus using $\frac{1}{\sqrt{\epsilon_r\mu_r}} = \frac{2}{3}$, we get $\mu_r = \frac{9}{4} \times \frac{1}{\epsilon_r} = \frac{9}{4} \times \frac{1}{2.25} = 1$ ($\therefore \mu = \mu_0$). Using Ampere's law $\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$ (where $\mathbf{J} = \sigma\mathbf{E} = \mathbf{0}$ as $\sigma = 0$), first we get

$$\nabla \times \mathbf{H} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos(\omega t + \beta y) & 0 & 0 \end{vmatrix} = -\frac{\partial}{\partial y}\left(\cos(\omega t + \beta y)\right)\hat{z} = \beta \sin(\omega t + \beta y)\hat{z}$$

Now equating the above result to $\frac{\partial \mathbf{D}}{\partial t}$, we get

$$\frac{\partial \mathbf{D}}{\partial t} = \beta \sin(\omega t + \beta y)\hat{z}$$

Integrating both sides of the above equation and dividing by ϵ will give us

$$\begin{aligned} \mathbf{E} &= \frac{\beta}{2.25\epsilon_o}\int (\sin(\omega t + \beta y)\hat{z})dt \\ &= -\frac{1}{2.25\epsilon_o}\frac{\beta}{\omega}\cos(\omega t + \beta y)\hat{z} \\ &= -\frac{2}{3\epsilon_o c}\cos(\omega t + \beta y)\hat{z} \\ &= -251.15\cos(\omega t + \beta y)\hat{z}\,\frac{\mathrm{V}}{\mathrm{m}} \end{aligned}$$

6.

a) Referring to page 4 of Lecture 18, a valid solution, $\mathbf{E} = \hat{y}E_y(x,t)\,\frac{\mathrm{V}}{\mathrm{m}}$, to the wave equation should satisfy

$$\frac{\partial^2 E_y}{\partial x^2} = \mu\epsilon \frac{\partial^2 E_y}{\partial t^2}$$

Now we test if $\mathbf{E} = \cos^2(\omega t - \beta x)\hat{y} = \frac{1}{2}(1 + \cos(2\omega t - 2\beta x))\hat{y}\,\frac{\mathrm{V}}{\mathrm{m}}$ follows this relation.

$$\begin{aligned}
\frac{\partial^2 E_y}{\partial x^2} &= \frac{1}{2}\frac{\partial^2}{\partial x^2}(1 + \cos(2\omega t - 2\beta x)) \\
&= \frac{\partial}{\partial x}(\beta \sin(2\omega t - 2\beta x)) \\
&= -2\beta^2 \cos(2\omega t - 2\beta x)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 E_y}{\partial t^2} &= \frac{1}{2}\frac{\partial^2}{\partial t^2}(1 + \cos(2\omega t - 2\beta x)) \\
&= \frac{\partial}{\partial t}(-\omega \sin(2\omega t - 2\beta x)) \\
&= -2\omega^2 \cos(2\omega t - 2\beta x)
\end{aligned}$$

Note that $\frac{\beta}{\omega} = \frac{1}{v}$ and $v = \frac{1}{\sqrt{\mu\epsilon}}$, so $\frac{\partial^2 E_y}{\partial x^2}$ is given by

$$\begin{aligned}
\frac{\partial^2 E_y}{\partial x^2} &= -2(\frac{\omega}{v})^2 \cos(2\omega t - 2\beta x) \\
&= (\mu\epsilon)(-2\omega^2 \cos(2\omega t - 2\beta x)) \\
&= \mu\epsilon \frac{\partial^2 E_y}{\partial t^2}
\end{aligned}$$

In addition, the given $\mathbf{E}$ is in the direction of $\hat{y}$ and propagates along the x-axis, which is orthogonal to $\hat{y}$. Therefore, it is a valid solution to the wave equation.
We can also treat $\mathbf{E}$ as a superposition of $\mathbf{E}_1$ and $\mathbf{E}_2$, where $\mathbf{E}_1 = \frac{1}{2}\hat{y}\,\frac{\mathrm{V}}{\mathrm{m}}$ and $\mathbf{E}_2 = \frac{1}{2}\cos(2\omega t - 2\beta x)\hat{y}\,\frac{\mathrm{V}}{\mathrm{m}}$. $\mathbf{E}_1$ is a constant field which doesn't depend on position or time, while $\mathbf{E}_2$ is apparently a valid solution with $\omega_2 = 2\omega$ and $\beta_2 = 2\beta$. Consequently, $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$ should also be a valid solution to the wave equation, based on the law of superposition.

b) Applying the same principle as in (a) to $\mathbf{E} = \cos(\omega t)\cos(\beta x)\hat{y} = \hat{y}E_y(x,t)\,\frac{\mathrm{V}}{\mathrm{m}}$,

$$\begin{aligned}
\frac{\partial^2 E_y}{\partial x^2} &= \frac{\partial^2}{\partial x^2}(\cos(\omega t)\cos(\beta x)) \\
&= \frac{\partial}{\partial x}(-\beta \cos(\omega t)\sin(\beta x)) \\
&= -\beta^2 \cos(\omega t)\cos(\beta x)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 E_y}{\partial t^2} &= \frac{\partial^2}{\partial t^2}(\cos(\omega t)\cos(\beta x)) \\
&= \frac{\partial}{\partial t}(-\omega \sin(\omega t)\cos(\beta x)) \\
&= -\omega^2 \cos(\omega t)\cos(\beta x)
\end{aligned}$$

Similarly, given $\frac{\beta}{\omega} = \frac{1}{v}$ and $v = \frac{1}{\sqrt{\mu\epsilon}}$, we have

$$\frac{\partial^2 E_y}{\partial x^2} = \mu\epsilon\frac{\partial^2 E_y}{\partial t^2}$$

Recall that $\mathbf{E} = \cos(\omega t)\cos(\beta x)\hat{y} = \frac{1}{2}(\cos(\omega t + \beta x) + \cos(\omega t - \beta x))\hat{y}$. Given by the law of superposition, this is a valid solution.

In future lectures, you will learn that this field is actually a standing wave, which can be viewed as the sum of an incident wave and its reflection on a perfectly conducting mirror.