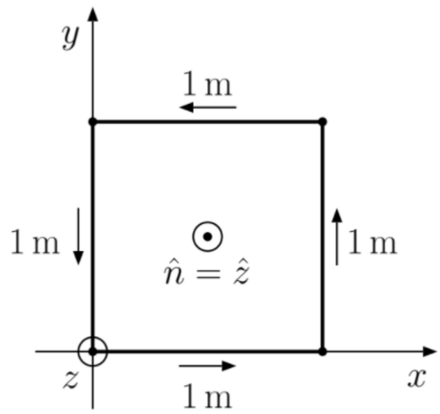
1. Since the closed paths are not varying in time and the magnetic field $\mathbf{B}$ is independent of position, we can rewrite Faraday's law as follows

$$\mathcal{E} = \oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d\mathbf{B}}{dt} \cdot \int_S d\mathbf{S} = \left(-\frac{d\mathbf{B}}{dt} \cdot \hat{n}\right) \cdot \text{Area},$$

where $\hat{n}$ is the unit vector normal to the surface, and

$$\frac{d\mathbf{B}}{dt} = B_0 \left((\sin(\omega t) + \omega t \, \cos(\omega t))\, \hat{y} - \omega \sin(\omega t)\, \hat{z} \right).$$

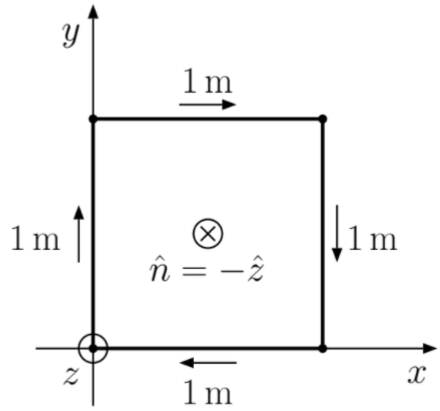
a) For the rectangular path shown in the figure below,



the area of the enclosed surface S is $1\,\text{m}^2$ and the unit vector normal to S is $\hat{n} = \hat{z}$. Therefore, the electromotive force is

$$\mathcal{E} = -\frac{d\mathbf{B}}{dt} \cdot \hat{z} = B_0 \omega \sin(\omega t)\,\text{V}.$$

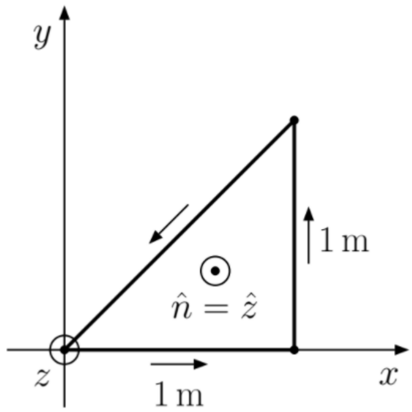
b) If we consider the same rectangle of part (a), but the direction of the path is reversed, we have that $\hat{n} = -\hat{z}$.



As a result, the electromotive force is

$$\mathcal{E} = \frac{d\mathbf{B}}{dt} \cdot \hat{z} = -B_0 \omega \sin(\omega t)\,\mathrm{V}.$$

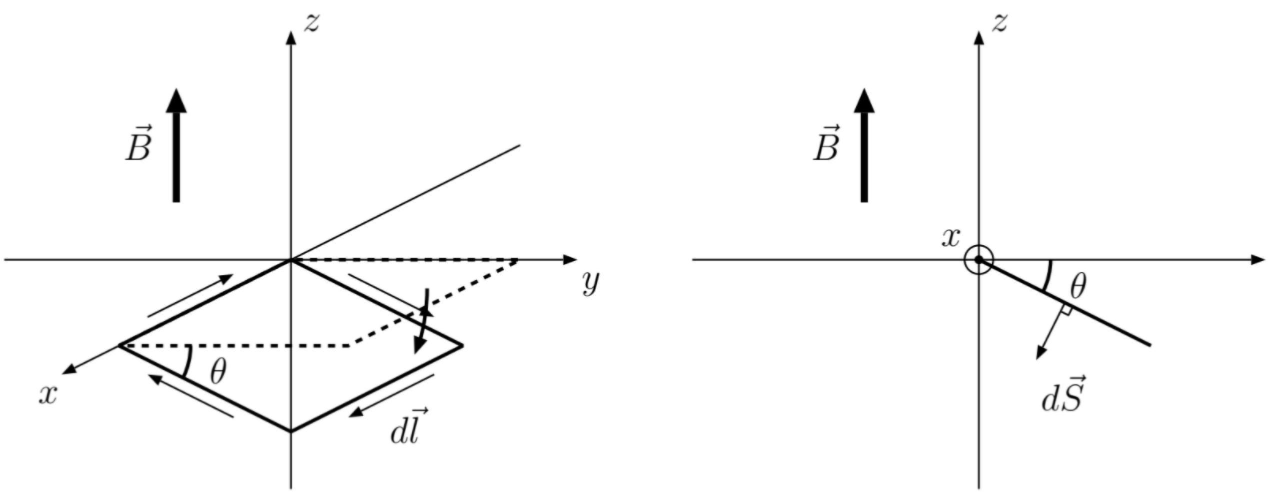
c) For the triangular path shown in the figure below,



the area of the enclosed surface S is $\frac{1}{2}\,\mathrm{m}^2$ and the unit vector normal to S is $\hat{n} = \hat{z}$. Then, the electromotive force is

$$\mathcal{E} = -\frac{d\mathbf{B}}{dt} \cdot \hat{z} \cdot \frac{1}{2} = \frac{1}{2} B_0 \omega \sin(\omega t)\,\mathrm{V}.$$

2. Consider a square loop of wire of some finite resistance R with $2\,\mathrm{cm}^2$ surface area that is located in a region of constant magnetic field $\mathbf{B} = 4\hat{z}\,\mathrm{Wb/m}^2$. The loop can rotate around the x-axis as shown in the following figure.



a) If $\theta = 0°$, the differential area is $d\mathbf{S} = -dS\hat{z}$, and therefore, the magnetic flux is

$$\Psi = \int_S \mathbf{B} \cdot d\mathbf{S} = \mathbf{B} \cdot \int_S d\mathbf{S} = 4\hat{z}\,\frac{\mathrm{Wb}}{\mathrm{m}^2} \cdot (-2 \times 10^{-4}\,\mathrm{m}^2\hat{z}) = -8 \times 10^{-4}\,\mathrm{Wb}.$$

b) For any angle θ, the differential area is $d\mathbf{S} = -dS\left(\cos\theta\hat{z} + \sin\theta\hat{y}\right)$, and therefore, the magnetic flux is

$$\Psi = \int_S \mathbf{B} \cdot d\mathbf{S} = 4\hat{z} \cdot \left(-2 \times 10^{-4}\left(\cos\theta\hat{z} + \sin\theta\hat{y}\right)\right) = -8 \times 10^{-4}\cos\theta\,\mathrm{Wb}.$$

c) The induced emf is given by $\mathcal{E} = \int_C \mathbf{E} \cdot d\mathbf{l}$. Hence, we write

$$\mathcal{E} = -\frac{d\Psi}{dt} = -8 \times 10^{-4}\sin\theta\frac{d\theta}{dt}.$$

Considering $\theta = \frac{\pi}{4}$ rad and $\frac{d\theta}{dt} = \pi$ rad/s, we get

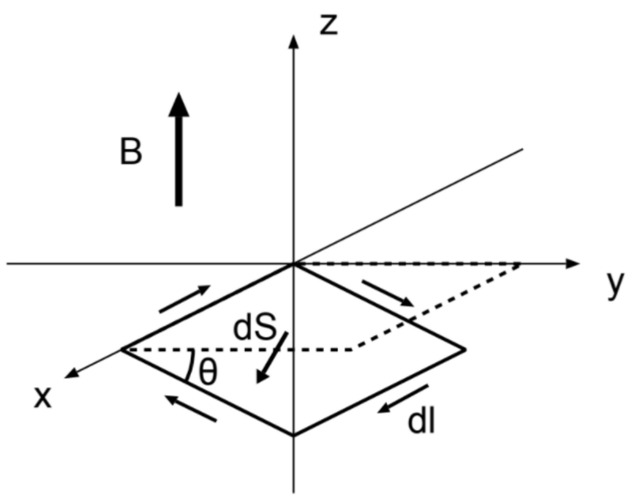
$$\mathcal{E}(\theta = \frac{\pi}{4}) = -8 \times 10^{-4}\left(\sin\frac{\pi}{4}\right)(\pi) = -4\sqrt{2}\pi \times 10^{-4}\,\mathrm{V} \approx -1.777\,\mathrm{mV}.$$

d) There are two approaches to finding the direction of induced current; using Faraday's Law directly or using Lenz Law.

i. Faraday's Law

$$\begin{aligned}\oint_C \mathbf{E} \cdot d\mathbf{l} &= -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S} \\ \mathcal{E} &= -\frac{d\Psi}{dt}\end{aligned}$$

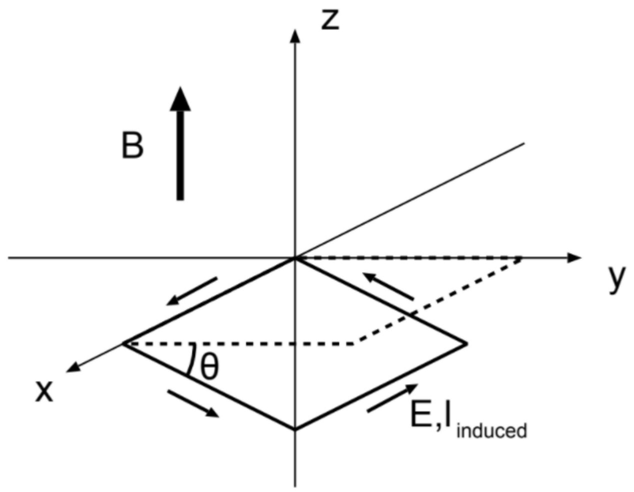
Based on the direction chosen for $d\mathbf{S}$ the direction of $d\mathbf{l}$ is given by the righ hand rule as shown in the figure.



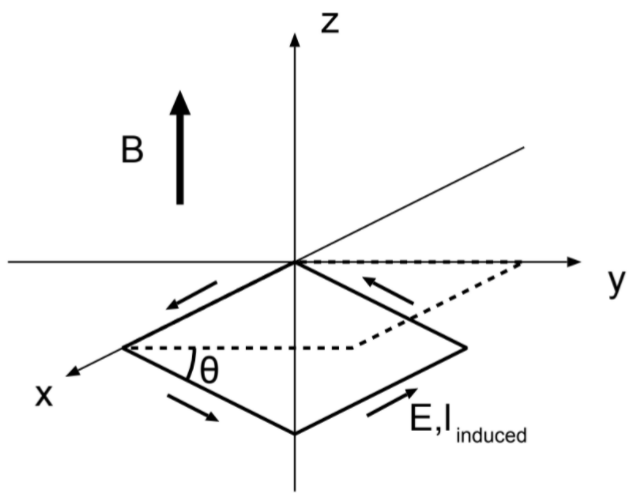
Since the induced emf $\mathcal{E}$ is calculated to be negative the induced electric field $\mathbf{E}$ must be in the opposite direction of $d\mathbf{l}$ due to the dot product.

$$\mathcal{E} = \oint_C \mathbf{E} \cdot d\mathbf{l} < 0$$

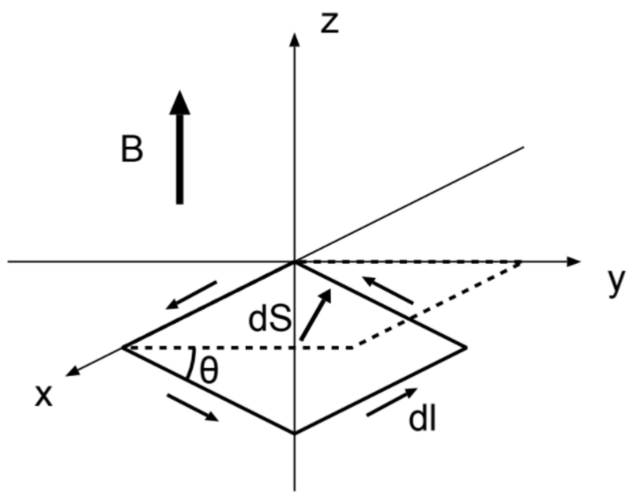
Ohm's Law $\mathbf{J} = \sigma \mathbf{E}$ shows that the induced current will also be directed along the induced electric field. The resulting positive current flow is shown below.



ii. Lenz Law states an induced emf $\mathcal{E}$ is produced that will give rise to an induced current whose magnetic field opposes the original change in magnetic flux. Considering $\theta = \frac{\pi}{4}$ rad and $\frac{d\theta}{dt} = \pi$ rad/s, we can observe that the original magnetic flux is decreasing. Lenz Law tells us that a current will be induced to oppose these changes. Therefore the current must flow as shown below.



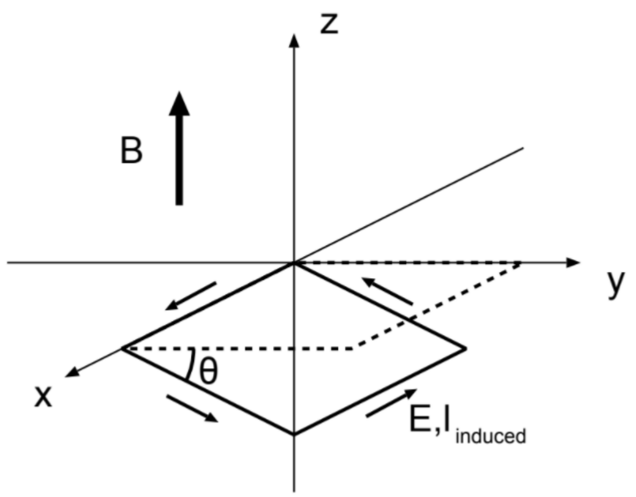
e) If the opposite direction is chosen for $d\mathbf{S}$ the direction of $d\mathbf{l}$ also changes from the right hand rule.



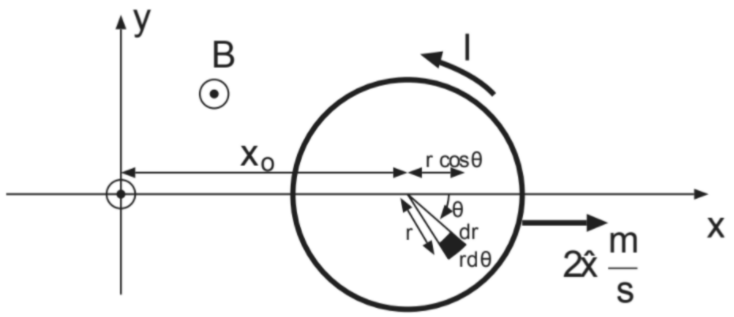
The induced emf $\mathcal{E}$ will be positive so the induced electric field $\mathbf{E}$ must be in the same direction of $d\mathbf{l}$ due to the dot product.

$$\mathcal{E} = \oint_C \mathbf{E} \cdot d\mathbf{l} > 0$$

Ohm's Law $\mathbf{J} = \sigma \mathbf{E}$ shows that the induced current will also be directed along the induced electric field. The resulting positive current flow is shown below.



3. The geometry of the problem is shown in the figure below.



a) Looking at the described geometry, the magnetic flux is given by

$$\begin{aligned}\Psi = \int_S \mathbf{B}\cdot d\mathbf{S} &= \int_0^r\int_0^{2\pi} 25\times 10^{-6}(1+\frac{x_o + r\cos\theta}{L})r d\theta dr,\\ &= 25\times 10^{-6}\left(1+\frac{x_o}{L}\right)\pi r^2\end{aligned}$$

Thus, the emf $\mathcal{E}$ is

$$\begin{aligned}\mathcal{E} &= -\frac{d\Psi}{dt} = -25\pi r^2\times 10^{-6}\times\frac{1}{L}\times\frac{dx_o}{dt} = -25\pi\times 10^{-6}\times\frac{1}{1000}\times 2\\ &\approx -157.1\,\text{nV}.\end{aligned}$$

b) The magnitude of the loop current with resistance $2\,\Omega$ is

$$I = \frac{|\mathcal{E}|}{R} = 78.54\,\text{nA}.$$

4.

a) At time t, the area of the loop= $L(z_0 + v_0 t)$; for clockwise contour (following ABCD), $d\mathbf{S} = dS(-\hat{x})$.

$$\begin{aligned}\Psi &= \int_S \mathbf{B}\cdot d\mathbf{S}\\ &= B_0 L(z_0 + v_0 t)\\ \mathcal{E} &= -\frac{d\Psi}{dt} = -B_0 L v_0\,\text{V}\end{aligned}$$

b) $\mathcal{E} < 0$ so the current flows opposite to the original contour, i.e. along DCBA (counterclockwise),

$$\begin{aligned}I_0 &= \frac{\mathcal{E}}{R} = \frac{-B_0 L v_0}{R}\,\text{A}\\ |I_0| &= \frac{B_0 L v_0}{R}\,\text{A}\end{aligned}$$

c) Lorentz force per unit length

$$\begin{aligned} d\mathbf{F} &= I_0 d\mathbf{l} \times \mathbf{B} \\ &= I_0(dy\hat{y}) \times B_0(-\hat{x}) \\ &= I_0 B_0 dy \hat{z} \end{aligned}$$

Therefore the total $\mathbf{F}$

$$\mathbf{F} = \int d\mathbf{F} = \int_0^L I_0 B_0 dy \hat{z} = I_0 B_0 L \hat{z}\,\mathrm{N}$$

For I_0 as above

$$\mathbf{F} = \frac{B_0^2 L^2 v_0}{R}(-\hat{z})\,\mathrm{N}$$

d) Since the current flows along the contour ABCD, I' is in the $+\hat{y}$ direction in the armature. Similar to part (c),

$$\begin{aligned} \mathbf{F} &= I' L\hat{y} \times B_0(-\hat{x}) \\ &= I' B_0 L \hat{z}\,\mathrm{N} \\ \mathbf{a} &= \frac{I' B_0 L}{M}\hat{z}\,\mathrm{m/s^2} \end{aligned}$$

Alternatively, we could find the total force by superposing the Lorentz forces on each charge in the armature:

$$\mathbf{F} = (q\mathbf{v} \times \mathbf{B}_0) NLA$$

where N is the number density of the charges in the armature, A is its cross-sectional area, and LA is its volume. Since $\mathbf{J} = qN\mathbf{v} = \frac{I'}{A}\hat{z}$, we can arrive at the same result above.

5.

a) The current in the outer loop can be calculated by $I_a = \frac{V}{R}$, where $R \equiv \frac{1}{G} = \frac{d}{A\sigma}$ as you may remember from Lecture 10. Thus, we write

$$I_a = \frac{V}{R} = \frac{V}{\frac{2\pi a}{\sigma \pi r^2}} = \frac{5}{\frac{2\times 0.1}{4\times 10^7 \times 0.001^2}} = 1000\,\mathrm{A}.$$

b) Refering to the Page 8 in Lecture 13, the magnetic flux density generated by the outer loop only has z component along the z axis

$$B_z = \frac{\mu_o I_a a^2}{2(a^2+z^2)^{\frac{3}{2}}}.$$

At the origin, $B_z(z=0) = \frac{\mu_o I_a}{2a}\,\frac{Wb}{\mathrm{m}^2}$. Since $b \ll a$, we assume that the B_z across the inner loop is constant. Thus,

$$\Psi_{a\to b} = \int_S \mathbf{B}\cdot d\mathbf{S} = \int_S B_z(z=0)\hat{z}\cdot\hat{z}dS = \frac{\mu_o I_a}{2a}\pi b^2 = \frac{\mu_o \pi b^2}{2a} I_a\,\mathrm{Wb}.$$

c) The numerical value of $L_{a\to b}$ is

$$L_{a\to b} = \frac{\Psi_{a\to b}}{I_a} = \frac{\mu_o \pi b^2}{2a}\,\mathrm{H} \approx 0.1234\,\mathrm{nH}$$

d) Applying induced emf formula $\mathcal{E} = -\frac{d\Psi}{dt}$, we get

$$\mathcal{E}_{a\to b} = -\frac{d\Psi_{a\to b}}{dt} = -\frac{d}{dt}\int_S B \cdot dS = -\frac{d}{dt}(B_z A_b) = -A_b \frac{d}{dt} B_z.$$

Here, B_z is given by

$$B_z = \frac{\mu_o I_a a^2}{2(a^2 + z^2)^{\frac{3}{2}}},$$

Therefore, we can write

$$\frac{d}{dt}B_z = -\frac{3}{2}\frac{\mu_o I_a a^2 z}{(a^2+z^2)^{\frac{5}{2}}}\frac{d}{dt}z = -\frac{3}{2}\frac{\mu_o I_a a^2 z}{(a^2+z^2)^{\frac{5}{2}}}v_z,$$

from which we obtain

$$\mathcal{E}_{a\to b} = \frac{3}{2}\frac{\mu_o I_a a^2 z \pi b^2}{(a^2+z^2)^{\frac{5}{2}}}v_z \text{ V}.$$

If we consider, $z = v_z t$, we can write $\mathcal{E}_{a\to b}$ as a function of time t as

$$\mathcal{E}_{a\to b}(t) = \frac{3}{2}\frac{\mu_o I_a a^2 v_z^2 t \pi b^2}{(a^2+v_z^2 t^2)^{\frac{5}{2}}} \text{ V}.$$

e) As the induced emf $\mathcal{E}$ is given by $\mathcal{E} = -\frac{d\Psi_t}{dt}$, also the current I by $I = \frac{\mathcal{E}}{R}$. Then, the induced current in the inner loop will be

$$I_b = -\frac{1}{R}\frac{d\Psi_t}{dt},$$

where $\Psi_t = \Psi_{a\to b} + \Psi_s$, $\Psi_{a\to b} = -\int \mathcal{E}_{a\to b}(t)\, dt$ and $\Psi_s = LI_b$ ($L = 0.25\,\mu$H). Thus, we can write

$$I_b = -\frac{1}{R}\frac{d\Psi_{a\to b}}{dt} - \frac{L}{R}\frac{dI_b}{dt}.$$

Rearranging the above equation, it will give us the differential equation as

$$\frac{dI_b}{dt} + \frac{R}{L}I_b = -\frac{1}{L}\frac{d\Psi_{a\to b}}{dt},$$

where right hand side is the emf $\mathcal{E}_{a\to b}(t)$ of part (d) divided by L.