1. Two parts of this problem are independent:

a) (10 pts) Using Maxwell's equations and the vector identity div(curl**A**) = 0 for any vector field **A**, derive the continuity equation: $\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$. Hint: start with Ampere's Law.

$\nabla \cdot (\nabla \times \vec{A}) = 0$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\nabla \cdot (\nabla \times \vec{H}) = \nabla \cdot \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right)$$

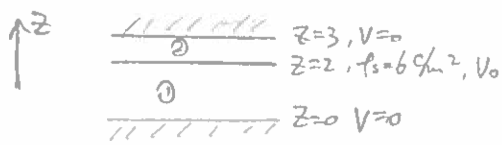
$$0 = \nabla \cdot \vec{J} + \nabla \cdot \frac{\partial \vec{D}}{\partial t}$$

$$0 = \nabla \cdot \vec{J} + \frac{\partial}{\partial t}(\nabla \cdot \vec{D})$$

$$0 = \nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t}$$

ie. $\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$ ✓

b) (15 pts) Conducting plates placed on $z = 0$ and $z = 3$ m surfaces (of infinite extent) are grounded and assigned zero electrostatic potential. A third surface on $z = 2$ m plane supports a surface charge density of $\rho_s = 6$ C/m^2 and is at at an electrostatic potential V_o. Determine V_o if the region $0 < z < 3$ m is occupied by free space.



In Region ①. ②, $\rho = 0 \Rightarrow \nabla^2 V = 0 \Rightarrow V$ is linear ~~sep~~ in ①. ②

Also at $z=2$, V should be continuous.

for region ① : Assume $V = az + b \Rightarrow V(0) = b \Rightarrow b = 0$

$\therefore \quad V = az \Rightarrow V(2) = 2a$

For region ② . Assume $V = cz + d$

$V(2) = 2c + d = 2a$ ①

$V(3) = 3c + d = 0$ ②

Also $-\nabla V = \vec{E} \Rightarrow$ Region ① $\vec{E} = -a\,\hat{z}$

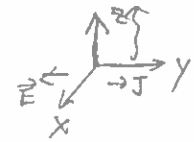
Region ② $\vec{E} = -c\,\hat{z}$

$((D^+ - D^-) = \rho_s)$

Maxwell's B.C $\Rightarrow \quad -a + \frac{6}{\varepsilon_0} = -c$ ③

Solving ①②③ $\Rightarrow a = +\frac{2}{\varepsilon_0}$ $b = 0$, $c = -\frac{4}{\varepsilon_0}$ $d = \frac{12}{\varepsilon_0}$

$\Rightarrow V_o = V(2) = 2a = \frac{4}{\varepsilon_0}$ (V) ✓



2. Consider a time-varying surface current density $\mathbf{J}_s(t) = u(t)\sin(2\pi f t)\hat{y}$ A/m, with $f = 10^6$ Hz, residing on the $z = 0$ surface embedded in free space. In this expression $u(t)$ is the unit-step function modulating the sine signal with a frequency $\omega = 2\pi f$ rad/s.

a) (4 pts) What is the vector wave field $\mathbf{E}(z,t)$ caused by $\mathbf{J}_s(t)$ in the region $z > 0$?

$$\vec{E}(z,t) = -\frac{\eta_0}{2} u\left(t-\frac{z}{c}\right) \sin\left(2\pi f\left(t-\frac{z}{c}\right)\right)\hat{y} \ (V/m)$$

$$= -60\pi u\left(t-\frac{z}{c}\right) \sin\left(2\pi f\left(t-\frac{z}{c}\right)\right)\hat{y} \ (V/m)$$

where $c = 3\times10^8$ m/s, $f = 10^6$ Hz

b) (4 pts) What is the accompanying wavefield $\mathbf{H}(z,t)$ in the same region?

$$\vec{H}(z,t) = \frac{1}{2}\hat{x}\, u\left(t-\frac{z}{c}\right)\sin\left(2\pi\times10^6\left(t-\frac{z}{c}\right)\right) \ (A/m)$$

where $c = 3\times10^8$ m/s.

c) (3 pts) What is the period $T = \frac{2\pi}{\omega}$ of the modulated $\mathbf{E}(z,t)$ and $\mathbf{H}(z,t)$ waveforms?

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{2\pi f} = \frac{1}{f} = 10^{-6}\,(s) = 1\,\mu s.$$

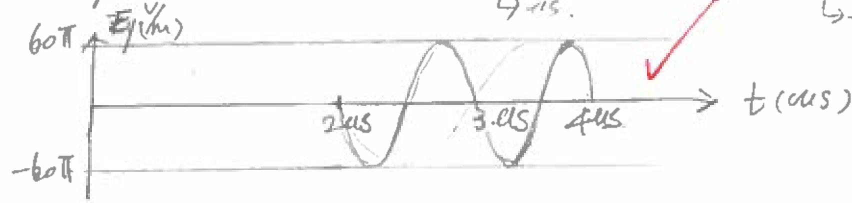
d) (3 pts) What is the wavelength λ of the modulated $\mathbf{E}(z,t)$ and $\mathbf{H}(z,t)$ waveforms?

$$\lambda = \frac{c}{f} = \frac{3\times10^8}{10^6} = 300\ m.$$

e) (5 pts) Plot $E_y(z,t)$ vs t at $z = 600$ m over the time interval $0 < t < 4\ \mu s$. Label both axes carefully. **Hint:** E_y is initially zero until the wavefront arrives at 600 m.

$$E_y(600,t) = -60\pi\, u\left(t-\frac{600}{300}\right)\sin\left(2\pi f\left(t-\frac{600}{300}\right)\right)$$

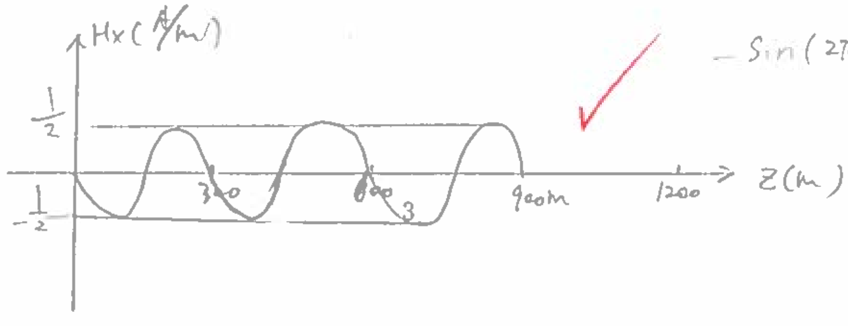
μs μs



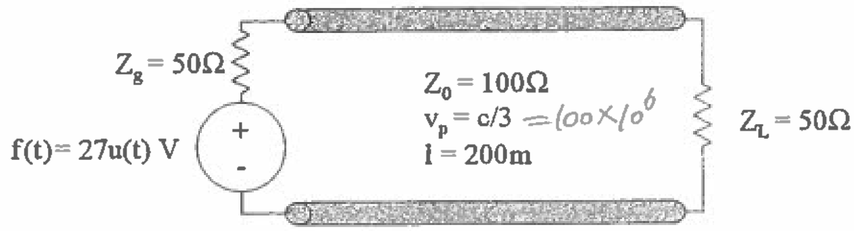
f) (6 pts) Plot $H_x(z,t)$ vs z at $t = 3\,\mu s$ over the region $0 < z < 1200$ m. Label both axes carefully. **Hint:** H_x is zero at distances where the wavefront has not yet reached.

$$H_x(z, 3\mu s) = \frac{1}{2}\hat{x}\, u\left(3-\frac{z}{300}\right)\sin\left(2\pi f\left(3-\frac{z}{300}\right)\right)$$

$$-\sin\left(2\pi f\,\frac{1}{300}(z-900)\right)$$



3. Consider the TL circuit shown below:



The source in the circuit is specified as $f(t) = 27u(t)$ V, where $u(t)$ is the unit-step function. The transmission line has a length $l = 200$ m, propagation velocity of $v_p = \frac{c}{3} = 10^8$ m/s, and characteristic impedance $Z_o = 100\,\Omega$.

a) (2 pts) Find the injection coefficient τ_g.

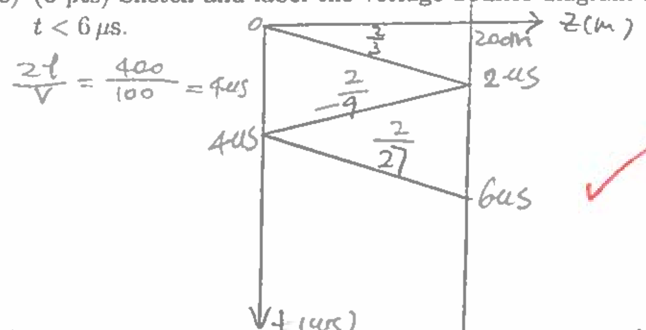
$$Z_g = \frac{Z_0}{Z_g + Z_0} = \frac{100}{150} = \frac{2}{3}$$

b) (4 pts) Find the reflection coefficients, Γ_L and Γ_g, for the load and source ends.

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{50-100}{50+100} = -\frac{50}{150} = -\frac{1}{3}$$

$$\Gamma_g = \frac{Z_g - Z_0}{Z_g + Z_0} = \frac{50-100}{50+100} = -\frac{1}{3}$$

c) (6 pts) Sketch and label the voltage bounce diagram in terms of products of τ_g, Γ_L, and Γ_g for $t < 6\,\mu$s.



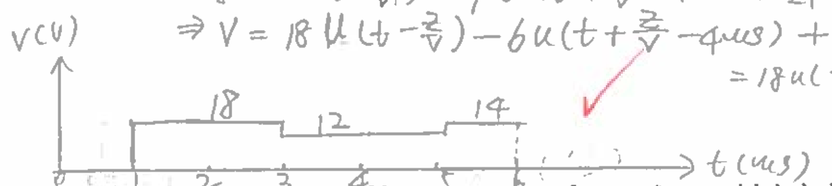
d) (7 pts) Sketch the voltage at $z = 100$ m versus time for $t < 6\,\mu$s and label the axes.

$$V_s = \frac{2}{3}\delta\left(t - \frac{z}{v}\right) - \frac{2}{9}\delta\left(t + \frac{z}{v} - 4\mu s\right) + \frac{2}{27}\delta\left(t - \frac{z}{v} - 4\mu s\right)$$

$$\Rightarrow V = 18u\left(t - \frac{z}{v}\right) - 6u\left(t + \frac{z}{v} - 4\mu s\right) + 2u\left(t - \frac{z}{v} - 4\mu s\right)$$

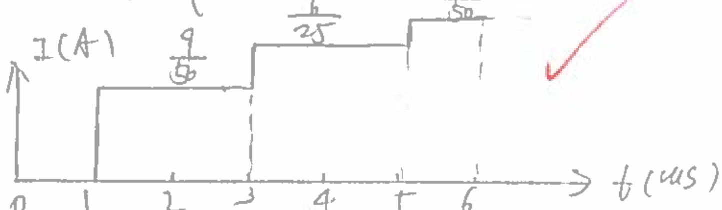
$$= 18u(t-1) - 6u(t-3) + 2u(t-5)$$

(V)

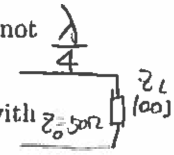


e) (6 pts) Sketch the current at $z = 100$ m versus time for $t < 6\,\mu$s and label the axes.

$$I = \frac{1}{100}\left(18u(t-1) + 6u(t-3) + 2u(t-5)\right) \text{ (A)}$$



4. In this problem do (a) or (b) but not both — please put a large "X" over the part that you do not want to be graded:



a) A TL with length $l = \frac{\lambda}{4}$ and characteristic impedance $Z_o = 50\,\Omega$ is terminated by a load with impedance $Z_L = 100\,\Omega$.

$I_L = -j\frac{V_{in}}{Z_o}$

i. (10 pts) What is the load current I_L if the input voltage at the generator terminals of the transmission line is $V_{in} = 100$ V?

$$I_L = -j\frac{V_{in}}{Z_o} = -j\frac{100}{50} = -j2\ (A)$$

ii. (10 pts) What is the current I_{in} at the generator terminals under the same conditions?

$$Z_{in} = \frac{Z_o^2}{Z_L} = \frac{50^2}{100} = 25\ (\Omega)$$

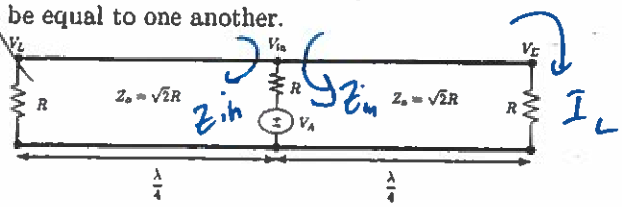
$$I_{in} = V_{in}/Z_{in} = \frac{100}{25} = 4\ (A)$$

iii. (5 pts) What is the average power absorbed by $Z_L = 100\,\Omega$?

$$\langle P\rangle = \frac{1}{2}Re\{V_L \cdot I_L^*\} = \frac{1}{2}Re\{-j2\cdot 100\cdot j2$$

$$= \frac{1}{2}\cdot 400 = 200\ (W)$$

b) In the distributed circuit shown below, whetre a pair of quarter-wave transformers with $Z_o = \sqrt{2}R$ are utilized, the symmetry of the circuit dictates that load voltages V_L at both ends must be equal to one another.



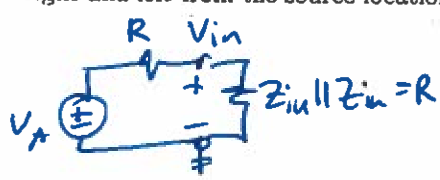
$$Z_{in} = \frac{(\sqrt{2}R)^2}{R} = \frac{2R^2}{R} = 2R$$

Quarter wave transfor impedance formula

$$Z_{in} \| Z_{in} = R$$

Determine:

i. (15 pts) The input voltage phasor V_{in}, after determining the total impedance seen by the source circuit (with an open circuit voltage V_A and internal resistance R) — express V_{in} in terms of V_A. **Hint:** use voltage division after parallel combining the impedances seen to the right and left from the source location towards the two loads (on the right and on the left)



$\Rightarrow V_{in} = \frac{1}{2}V_A$ by voltage division

ii. (10 pts) Load voltage phasor V_L in terms of V_{in}. **Hint:** make uses of current forcing formula.

$V_{in} = \frac{1}{2}V_A$

$$\Rightarrow I_L = -j\frac{V_{in}}{Z_o} = -j\frac{\frac{1}{2}V_A}{\sqrt{2}R} \Rightarrow V_L = I_L R = -j\frac{V_A}{2\sqrt{2}}$$

5. Answer the following questions using the Smith Chart (SC) seen on the right. Assume $Z_o = 50\,\Omega$. The SC should be marked appropriately to support the answers.

a) (3 pts) Mark and label the point on the SC that corresponds to the normalized *impedance* of an open.

$z = \infty$

b) (3 pts) Mark and label the point on the SC that corresponds to the normalized *admittance* of an open.

$Y = 0$

c) (3 pts) Mark and label the point on the SC that corresponds to an *impedance match*.

$\Rightarrow \Gamma = 0$

d) (3 pts) Draw the constant $|\Gamma|$ circle if VSWR=3.

1 e) (3 pts) Determine the maximum $|y|$ if VSWR=3.

$y = \frac{1}{z} \Rightarrow |y|_{max}$ corresponds to $|z|_{min}$

$|z| = \left|\frac{1+\Gamma}{1-\Gamma}\right| = \frac{|1+\Gamma|}{|1-\Gamma|}$, min occurs at $\Gamma = -|\Gamma| \Rightarrow |y|_{max}$ is at $\Gamma = |\Gamma_L|$

$\Rightarrow |y|_{max} = 3$

1 f) (3 pts) Determine the minimum $|z|$ if VSWR=~~3~~ $\frac{|1+\Gamma|}{|1-\Gamma|}$

$|z| = \left|\frac{1+\Gamma}{1-\Gamma}\right|$, min occurs at at $\Gamma = -|\Gamma_L|$

$|z|_{min} = 0.335$

g) (3 pts) Determine Z_L if VSWR=3 and the load is located at a site of minimum $|V(d)|$ on the line.

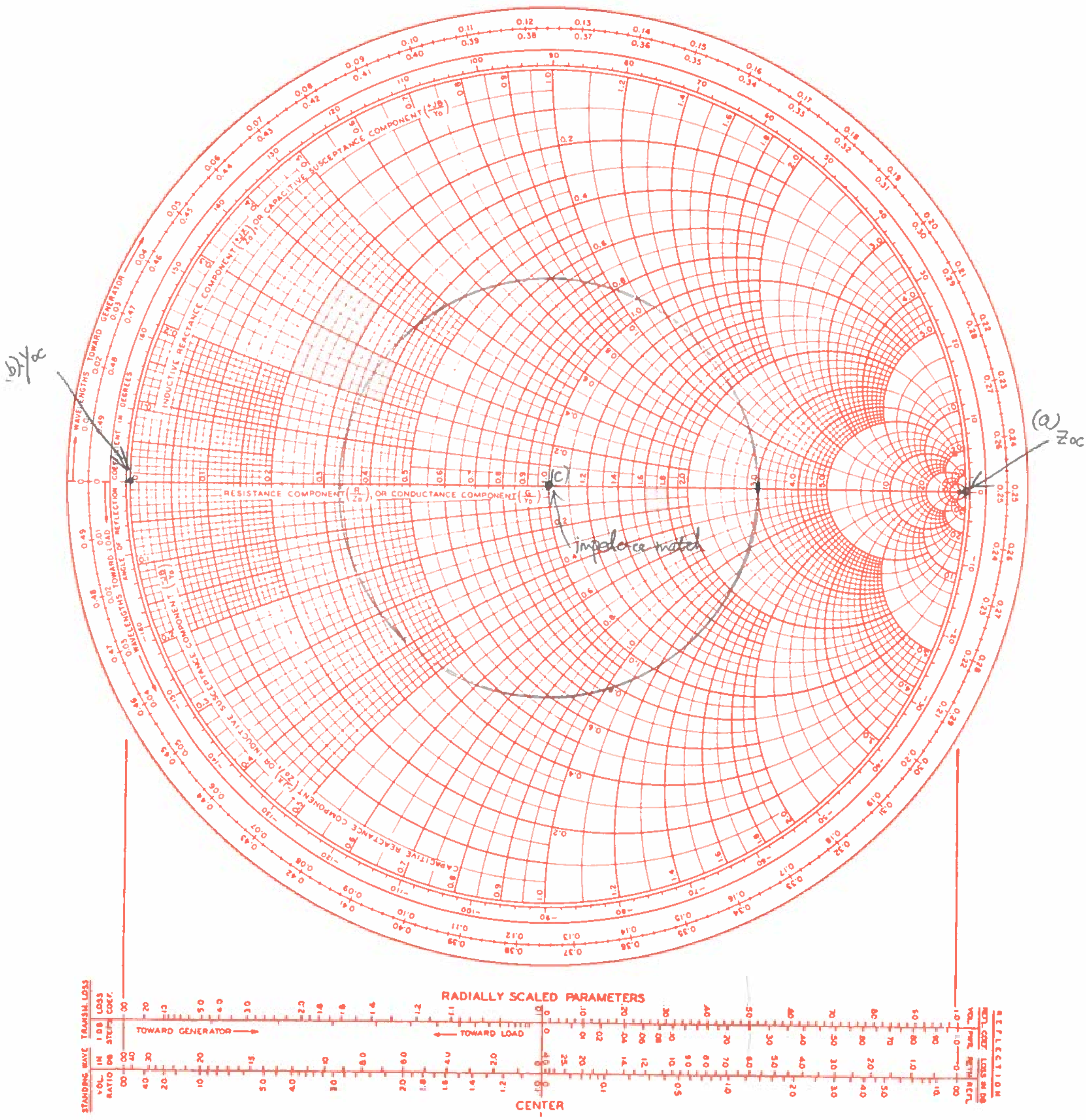
$Z_L = 0.335 \cdot 50 = 16.75\,\Omega.$

3

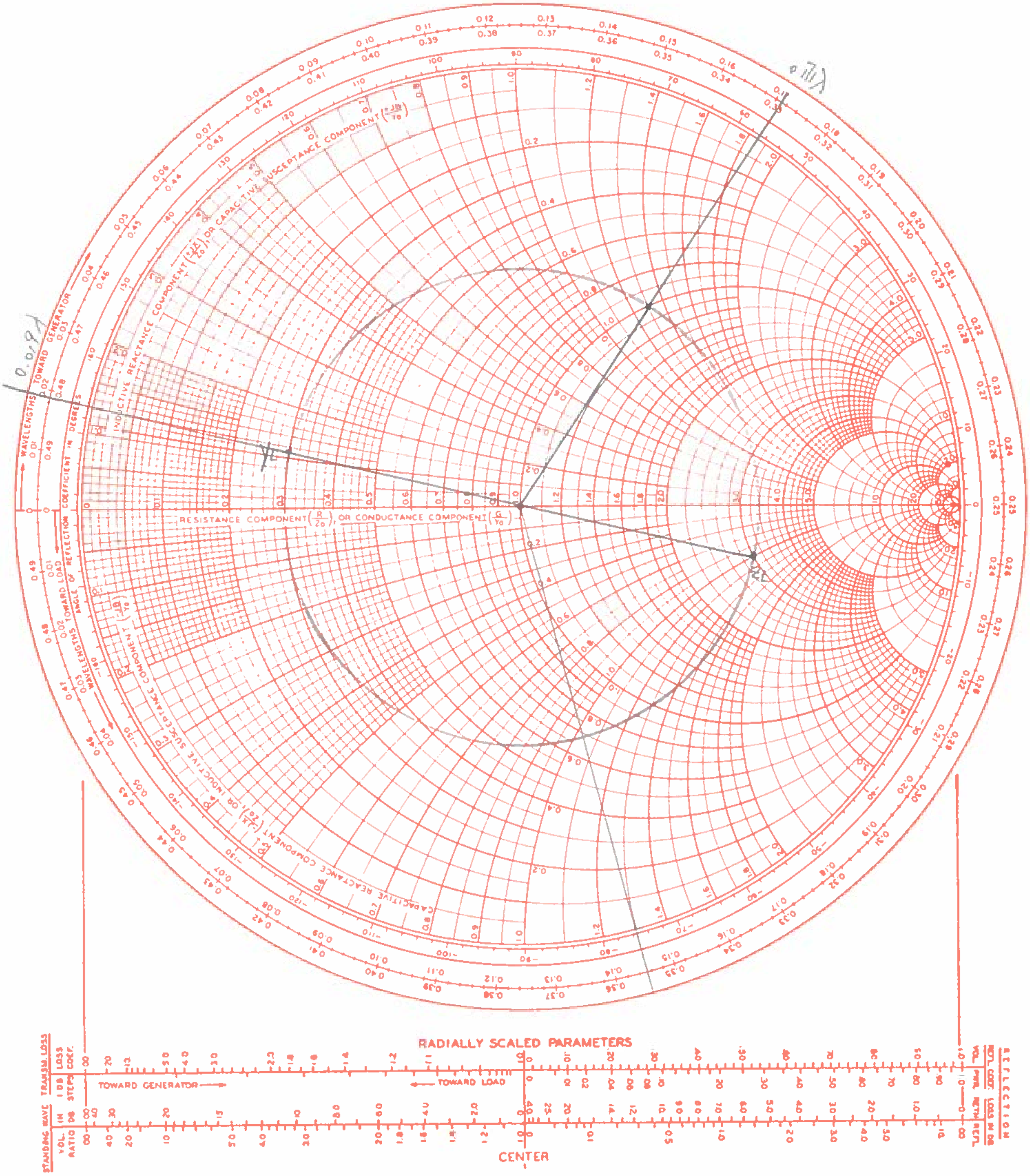
h) (4 pts) Determine Γ_L if VSWR=3 and the load is located at a site of minimum $|V(d)|$ on the line.

$\Gamma_L = -|\Gamma_L| = -\left(\frac{3-1}{3+1}\right) = -0.5$

IMPEDANCE OR ADMITTANCE COORDINATES



IMPEDANCE OR ADMITTANCE COORDINATES



6. $Z_L = 150 - j50\,\Omega$ is to be matched to a $Z_o = 50\,\Omega$ line using the single-stub matching technique.

a) (4 pts) Enter and mark z_L on the SC.

$z_L = \frac{Z_L}{Z_0} = 3 - j$

b) (4 pts) Mark y_L on the SC.

c) (4 pts) Determine the shortest distance d away from the load to attach a shorted stub for matching purposes.

clockwise $\rightarrow 0.171\lambda - 0.019\lambda = 0.152\lambda = d$

i.e. $d = 0.152\lambda$

d) (4 pts) What is the corresponding $y(d)$ before shunt connection of the shorted stub is made?

$y(d) = 1 + j1.3$

e) (4 pts) What is the normalized input admittance y_{stub} for the shorted stub to achieve a match?

$y_{stub} = -j1.3$

f) ~~(5 pts)~~ What is the length ℓ_{stub} of the stub that produces the required y_{stub} in (e) assuming $Z_o = 50\,\Omega$ for the stub?

$\parallel$ Z_0 of the TL.

start rotating from $\Gamma = 1\angle 0°$, clockwise, to $-j1.3$

$\ell = (0.355 - 0.25)\lambda = 0.105\lambda$.