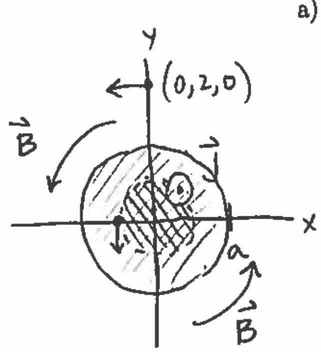
1. (25 pts) According to Ampere's Law, the magnetic **H** field generated by a cylindrically symmetric current distribution along the z-axis points in the azimuthal ($\hat{\phi}$) direction, and its magnitude varies with distance r from the z-axis in terms of the $+\hat{z}$-directed current I_{encl} passing through a circular contour centered on the z-axis:

$$H_\phi(r) = \frac{I_{encl}}{2\pi r} = \frac{B_\phi}{\mu}$$

$\mu = \mu_0$

a) PART ONE: Consider a volumetric current density $\mathbf{J} = 3\hat{z}\ \mathrm{A/m^2}$ embedded in free space which is symmetric around the z-axis and uniformly distributed across a cross sectional area of radius $a = 1$ m.



i. (3 pts) In terms of cartesian coordinate unit directions, what is **B** at $(x, y, z) = (0, 2, 2)$?

$r = 2$ (distance along $\hat{z}$ is irrelevant in cylindrical coords)

$$I_{encl} = \int_0^2 \vec{J}\cdot d\vec{s} = \int_0^a 3\,ds = 3\int^a ds = 3(\pi a^2) = I_{total}.$$

$$\vec{B} = -\hat{x}\mu_0 H_\phi = -\hat{x}\mu_0 \frac{3\pi a^2}{2\pi(2)} = -\hat{x}\frac{3}{4}\mu_0 a^2\ [\mathrm{Wb/m^2}]$$

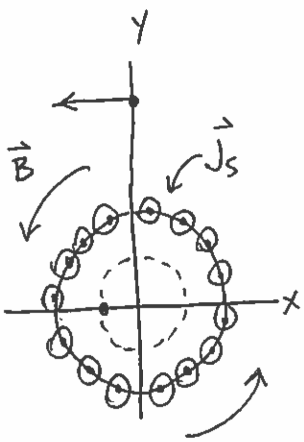
ii. (3 pts) In terms of cartesian coordinate unit directions, what is **B** at $(x, y, z) = (-\frac{1}{2}, 0, 0)$?

$r = \frac{1}{2}$

$$I_{encl} = \int_0^{1/2} \vec{J}\cdot d\vec{s} = 3\left(\pi\left(\tfrac{1}{2}\right)^2\right) = \frac{3}{4}\pi$$

$$\vec{B} = -\hat{y}\mu_0 H_\phi = -\hat{y}\mu_0 \frac{\frac{3}{4}\pi}{2\pi(1/2)} = -\hat{y}\frac{3}{4}\mu_0$$

b) PART TWO: Now consider that the same total current I (in A) is instead carried as a surface current on a hollow cylindrical shell having the same radius $a = 1$ m as the thick wire described in PART ONE above.



i. (5 pts) What is the current density on this cylindrical surface?

total current $I = 3(\pi a^2)\,[A]$ is spread around the circumference of the shell $= 2\pi a$.

$$\text{so } \vec{J_s} = \hat{z}\frac{3\pi a^2}{2\pi a} = \hat{z}\frac{3}{2}a\ [\mathrm{A/m}]$$

ii. (5 pts) How would your answers in PART ONE above change for this new current configuration?

Outside the shell, I_{encl} is the same.

so No CHANGE in $\vec{B}(0,2,2)$

Inside the shell, $I_{encl} = 0$.

so $\vec{B}(-\frac{1}{2}, 0, 0) = 0$ Field components cancel by symmetry

2. (30 pts) Short answer questions, each is independent of the others.

a) (5 pts) A surface current density $\mathbf{J} = f(t)\hat{z}$ on the $y = 0$ plane generates a TEM wave that propagates into free space on both sides. The following expression is NOT the wave equation for the plane wave just described:

$$\nabla^2 \mathbf{E} = \frac{\partial^2 E_x}{\partial y^2} + \cancel{\frac{\partial^2 E_x}{\partial z^2}}^{\,0} = \frac{\partial^2 E_x}{\partial t^2}$$

Identify and explain the errors in this wave equation and write the correct expression below.

Source @ $y=0 \Rightarrow \hat{v}_p = \pm\hat{y}$ so $\vec{E}$ only depends on y. $\frac{\partial}{\partial z} \to \phi$

$\vec{J}$ along $\hat{z} \Rightarrow \vec{E}$ along $\hat{z}$ so $\vec{E} = E_z(y,t)$. not E_x.

$$\boxed{\nabla^2 \vec{E} = \frac{\partial^2 E_z}{\partial y^2} = \frac{\partial^2 E_z}{\partial t^2}}$$

b) (5 pts) Which statement best explains the boundary condition between two perfect dielectric media having different magnetic permeabilities μ_1 and μ_2?

~~i.~~ All components of the **B** and **H** fields are continuous across the boundary.

~~ii.~~ The normal and the tangential components of the **B** field are continuous across the boundary.

~~iii.~~ The normal and the tangential components of the **H** field are continuous across the boundary.

iv. The normal component of **B** and the tangential components of **H** are continuous across the boundary.

~~v.~~ Only the normal component of **B** is continuous across the boundary.

~~vi.~~ None of the above.

no free charges to carry current at the boundary: $\vec{J}_s = 0$

boundary condition equations:

$\hat{n} \cdot (\vec{B}^+ - \vec{B}^-) = 0$ so $B_n^+ = B_n^-$

normal $\vec{B}$ continuous

$\hat{n} \times (\vec{H}^+ - \vec{H}^-) = \vec{J}_s = 0$

tangential H continuous.

since $\vec{H} = \vec{B}/\mu$, normal $\vec{H}$ and tangential $\vec{B}$ will be discontinuous since $\mu^+ \neq \mu^-$

c) (5 pts) A shorted co-axial cable of length l, inner conductor radius a, outer conductor radius b, is filled with a material having permittivity ϵ and permeability μ. A total current I flows along the inner conductor and returns through the outer conductor. The inductance L exhibits which of the following behaviors? Choose all that apply.

i. L increases as b increases

ii. L increases as l increases

iii. L increases as μ increases

~~iv.~~ L increases as I increases

v. none of the above

$$L = \frac{\Phi}{I} = \mu l \frac{\ln(b/a)}{2\pi}$$

$\boxed{i, ii, iii}$

d) (5 pts) If $\mathbf{J} = 2\sin(z)\hat{z}$ A/m^2, then the charge density at the origin is

i. constant

ii. increasing

iii. decreasing

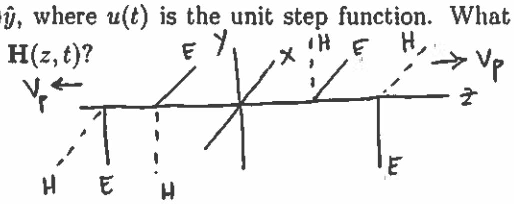
$$-\frac{\partial \rho}{\partial t} = \nabla \cdot \vec{J} = \frac{\partial J_z}{\partial z} = 2\cos(z)$$

at $z=0$ $\frac{\partial \rho}{\partial t} = -2$

$\boxed{iii}$

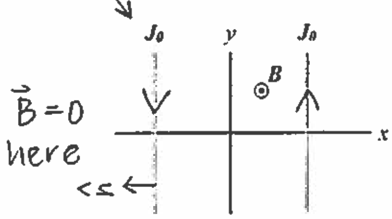
e) (5 pts) A TEM wave propagating in free space is known to have the following electric field waveform: $\mathbf{E}(z,t) = u(t \pm \frac{z}{c})\hat{x} - u(t \pm \frac{z}{c})\hat{y}$, where $u(t)$ is the unit step function. What is the waveform of the associated magnetic field $\mathbf{H}(z,t)$?

$$\frac{|E|}{|H|} = \eta_0 \qquad \hat{e} \times \hat{h} = \hat{v}_p$$



$$\vec{H}(z,t) = \frac{1}{\eta_0} u(t \mp \frac{z}{c})(\pm\hat{y}) + \frac{1}{\eta_0} u(t \mp \frac{z}{c})(\pm\hat{x})$$

f) (5 pts) A pair of infinite current sheets are positioned on the $x = -2$ m plane and $x = +2$ m plane and carry equal and opposite current density of constant magnitude J_0. The region between the sheets is filled with a perfect dielectric material having permittivity $9\epsilon_0$ and permeability $4\mu_0$. The magnetic field in the region between the current sheets is known to be $\mathbf{B} = 8\mu_0\hat{z}$ Wb/m^2. What is the magnitude and direction of the surface current on the $x = -2$ m plane? You are not required to derive the field strength of a single current sheet using Ampere's Law to answer this question.



direction $= -\hat{y}$

$$\underbrace{|B| = \mu|J_s|}_{} = 4\mu_0 J_0 = 8\mu_0$$

so $J_0 = 2$ A/m

$|\vec{H}| = \frac{|J_s|}{2}$ due to one sheet.

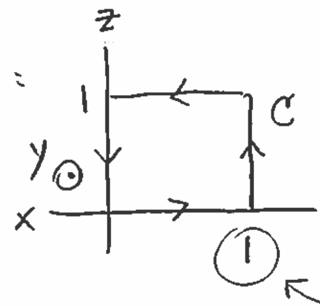
4

$$\text{-or-} \quad \hat{n} \times (\vec{H}^+ - \vec{H}^-) = \vec{J}_s = \underbrace{-\hat{x} \times (0 - |B|/\mu\,\hat{z})}_{-\hat{y}|B|/\mu = -\hat{y}J_0} = -\hat{y}J_0$$

$\Rightarrow J_0 = 2$ A/m

Fall 2017 329 Exam 2

3. Conducting wire shown as:



Typo! was supposed to be -1

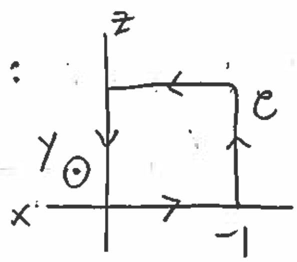
Embedded in

$$\vec{B} = 5y\cos(\omega t)\,\hat{x} + 5x\sin(\omega t)\,\hat{y} + 10e^{-t}\,\hat{z} \quad \text{Wb/m}^2$$

The position of the loop and orientation of $d\vec{s}$ (RH rule w/ C) is critical to answers for parts (a)–(c).

Full credit was given to the following answers:

- CASE 1:



here, $d\vec{s} = +\hat{y}\,dx\,dz$

only B_y component contributes to magnetic flux through the loop

a) At $t=0$, $B_y = 5x\sin(\omega t)$ is negative $(-1 < x < 0)$ and increasing in magnitude $\frac{d}{dt}|\sin(\omega t)| > 0$

Induced current will counteract this change in flux by generating $\vec{B}'$ with a positive B_y component.

This current must flow along C $\Rightarrow$ TRUE.

(Therefore, I expect derived $\mathcal{E} > 0$ below)

b) $\Psi(t) = \int \vec{B} \cdot d\vec{s} = \int_0^1 \int_{-1}^0 B_y\,\hat{y} \cdot \hat{y}\,dx\,dz$ $\quad \hat{y} \cdot \hat{y} = +1$

$$= \int_0^1 dz \int_{-1}^0 5x\sin(\omega t)\,dx$$

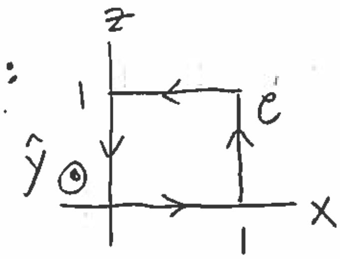
$$= (1)(5\sin(\omega t))\left[\tfrac{1}{2}x^2\Big|_{-1}^0\right]$$

$$= -\frac{5}{2}\sin(\omega t) \quad [\text{Wb}]$$

c) $\mathcal{E} = \frac{-d\Psi}{dt} = -\frac{d}{dt}\left(-\frac{5}{2}\sin(\omega t)\right) = \frac{5}{2}\omega\cos(\omega t)$ [V]

(indeed $\mathcal{E} > 0$ at $t = 0$)

- CASE 2:



NOTE: by moving $\hat{x}$, coordinate system is now left-handed, so this solution is physically wrong.

a) loop $\vec{ds} = +\hat{y}\,dx\,dz$ as before.

but now B_y is positive and increasing at $t=0$.

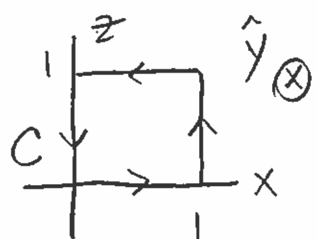
$\Rightarrow$ current will flow opposite to C to induce $B_y < 0$ (Lenz law)

b) $\Psi(t) = \int_0^1\int_0^1 B_y\hat{y}\cdot\hat{y}\,dx\,dz = \int_0^1 dz\int_0^1 5x\sin(\omega t)\,dx$

$= +\frac{5}{2}\sin(\omega t)$ [Wb]

c) $\mathcal{E} = \frac{-d\Psi}{dt} = -\frac{5}{2}\omega\cos(\omega t)$ [V] ($\mathcal{E} < 0$ at $t = 0$ is consistent w/ a))

- CASE 3:



For RH coordinate system, reverse $\hat{x}$ and $\hat{y}$.

a) now $\vec{ds} = -\hat{y}\,dx\,dz$. B_y is still positive & increasing at $t=0$, so current must generate negative B_y. $\Rightarrow$ along C $\Rightarrow$ TRUE

b) $\Psi(t) = \int_0^1\int_0^1 B_y\hat{y}\cdot(-\hat{y})\,dx\,dz$

$= -\frac{5}{2}\sin(\omega t)$ [Wb].

c) $\mathcal{E} = +\frac{5}{2}\omega\cos(\omega t)$ [V]

d) Regardless of coordinate system, consider changing material around loop to $\mu = 1.5\mu_0$.

Assuming $\vec{B}$ is the same as above, $\Phi(t)$ stays the same, and $-\frac{d\Phi}{dt} = \mathcal{E}$ also stays the same.

So current direction stays the <u>same</u>.

Induced current magnitude $I = \mathcal{E}/R$ stays the <u>same</u>.

<u>NOTE</u>! I is driven by an electric field in the loop frame: $\mathcal{E} = \oint \vec{E} \cdot d\vec{l}$, generates $\vec{B}_{induced}$ that <u>will</u> change magnitude in the magnetic material.

e) One test for $\vec{B}$ viability is whether it satisfies Maxwell eqn

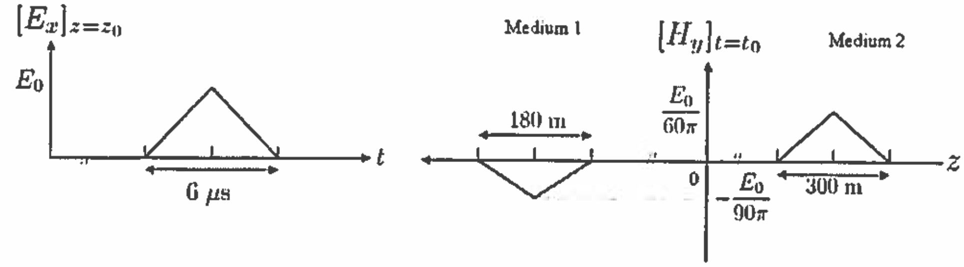
$$\nabla \cdot \vec{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0.$$ This field <u>does</u>.

It is also realistic to have temporal variation that is co-sinusoidal in one direction and decaying in another — fields from different sources superpose.

However, this field strength $\rightarrow \infty$ for increasing distances from origin. This is not physically realistic unless the expression for B is only valid locally.

4. (25 pts) An infinite plane current sheet of uniform density $\mathbf{J_s} = -J_s(t)\hat{x}$ A/m is sandwiched on the $z = 0$ plane between two perfect dielectric media, having material parameters ϵ_1 and μ_1 in Region 1 (on the $-\hat{z}$ side) and ϵ_2 and μ_2 in Region 2 (on the $+\hat{z}$ side). $J_s(t)$ is a triangular pulse which begins at $t = 0$ and lasts a duration of 6 μs.

The figure below depicts the associated wavefield $E_x(z_0, t)$ at some location $z = z_0 > 0$ and the associated wavefield $H_y(z, t_0)$ at some time $t = t_0 > 0$.



Based on the information in the figures, answer the following questions:

a) (4 pts) What are the intrinsic impedances η_1 and η_2 in medium 1 and medium 2, respectively?

$\frac{|E|}{|H|} = \eta$ by inspection: $|H_1| = \frac{|E_0|}{90\pi}$ so $\eta_1 = 90\pi \ \Omega$

$|H_2| = \frac{|E_0|}{60\pi}$ so $\eta_2 = 60\pi \ \Omega$

b) (6 pts) What are the propagation velocities v_{p1} and v_{p2} in medium 1 and medium 2, respectively?

Pulse duration of 6 μs spans Δz where $v_p = \Delta z / 6\mu s$.

so $v_{p_1} = \frac{180 \text{ m}}{6 \mu s} = 30 \text{ m}/\mu s$

$v_{p2} = \frac{300 \text{m}}{6 \mu s} = 50 \text{ m}/\mu s$

c) (3 pts) If the maximum magnitude of $J_s(t)$ is 4 A/m, what is E_0?

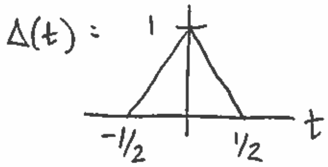
$|E_0| = \frac{\eta}{2}|J_0| \ \left[\frac{V}{m}\right]$

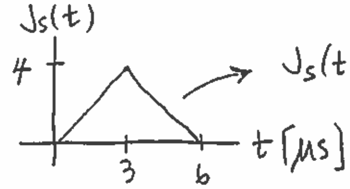
$$= \begin{cases} \frac{\eta_1(4)}{2} = 180\pi \ [V/m] & z<0 \\ \frac{\eta_2(4)}{2} = 120\pi \ [V/m] & z>0 \end{cases}$$

Can also relate $|J_s|$ to $|H|$ via boundary condition eqn:

$\left(\vec{J}_s = \hat{n} \times (H^+ - H^-)\right)_{max} = 4 = \frac{E_0}{60\pi} + \frac{E_0}{90\pi} \Rightarrow E_0 = 144\pi$

but this approach assumes $E_0(z>0) = E_0(z<0)$, which is not true (Full credit given regardless)





$J_s(t) = 4\Delta\left(\frac{t-3}{6}\right)$ [A/m]

d) (7 pts) Write the general expression for the propagating electric field $\mathbf{E}(z,t)$ in terms of an appropriately scaled and shifted triangular pulse function $\Delta(t)$. You should express your answer in terms of E_0, v_{p1}, etc., rather than using the numerical values you found above.

$$\vec{E}(z,t) = \begin{cases} \frac{J_0\eta_1}{2}\Delta\left(\frac{t+z/v_{p1}-3}{6}\right)(+\hat{x}) \quad [\text{V/m}] & z<0 \\ \frac{J_0\eta_2}{2}\Delta\left(\frac{t-z/v_{p2}-3}{6}\right)(+\hat{x}) \quad [\text{V/m}] & z>0 \end{cases}$$

e) (5 pts) Is it possible to determine the material parameters μ_1 and ϵ_1 from the information given? If so, how? If not, what other information is needed?

YES. $\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}}$ $\quad v_{p1} = \frac{1}{\sqrt{\mu_1\varepsilon_1}}$

so $\varepsilon_1 = \frac{1}{\eta_1 v_{p1}}$

$\mu_1 = \frac{\eta_1}{v_{p1}}$