ECE 329 **Homework 9 — Solutions** **Due: Tuesday, Oct 31, 2017, 6pm**

1. For each of the following plane TEM waves in free space:

a) For plane waves propagating in free space, E and H are related by

$$H = \frac{1}{\eta_0}\hat{v} \times E,$$

where $\hat{v}$ is the unit vector in the wave propagation direction and it is in the same direction as the Poynting vector. Therefore,

$$H_1 = \frac{1}{\eta_0}\hat{x} \times E_1 = \frac{1}{\eta_0}[\hat{z}10cos(\omega t - \beta x)]\ (A/m).$$

The phasors are

$$\tilde{\mathbf{E}}_1 = \hat{y}10e^{-j\beta x}(V/m),$$
$$\tilde{\mathbf{H}}_1 = \hat{z}\frac{10}{\eta_0}e^{-j\beta x}(A/m).$$

The instantaneous power across a 1 m^2 area is simply the Poynting vector multiplied by the unit area

$$|S| \cdot 1 = |E \times H| = \left|E \times (\frac{1}{\eta_0}\hat{v} \times E)\right| = \frac{1}{\eta_0}|E|^2 (W),$$

and we have

$$|S| \cdot 1 = \frac{1}{\eta_0}\left|E^2\right| = \frac{1}{\eta_0}\left[100cos^2(\omega t - \beta x)\right] = \frac{100}{\eta_0}cos^2(\omega t - \beta x)\ (W).$$

The time-average power across a $1m^2$area can be calculated as

$$< P_1 >= \frac{1}{T}\int_0^T |S| \cdot 1dt = \frac{50}{\eta_0}\ (W).$$

b) For plane waves propagating in free space, E and H are also related by

$$E = \eta_0 H \times \hat{v},$$

where $\hat{v}$ is the unit vector in the wave propagation direction. Therefore,

$$E_2 = \eta_0 H \times (-\hat{z}) = \eta_0[\hat{y}5cos(\omega t + \beta x + \frac{\pi}{3}) - \hat{x}sin(\omega t + \beta x - \frac{\pi}{6})]\ (V/m).$$

The phasors are

$$\tilde{\mathbf{E}}_2 = \eta_0(\hat{y}5e^{j\beta z + j\frac{\pi}{3}} - \hat{x}5e^{j\beta z - j\frac{\pi}{2} - j\frac{\pi}{6}}) = \eta_0(\hat{y}5e^{j\beta z + j\frac{\pi}{3}} - \hat{x}5e^{j\beta z - j\frac{2\pi}{3}})\,(V/m),$$
$$\tilde{\mathbf{H}}_2 = \hat{x}5e^{j\beta z + j\frac{\pi}{3}} + \hat{y}5e^{j\beta z - j\frac{\pi}{2} - j\frac{\pi}{6}} = \hat{x}5e^{j\beta z + j\frac{\pi}{3}} + \hat{y}5e^{j\beta z - j\frac{2\pi}{3}}\ (A/m).$$

Again, the instantaneous power across a 1 m^2 area is the Poynting vector multiplied by the unit area

$$|S| \cdot 1 = |E \times H| = |(\eta_0 H \times \hat{v}) \times H| = \eta_0 |H|^2 (W).$$

Therefore,

$$\begin{aligned}|S_2|\cdot 1 &= \eta_0\,|H_2|^2 = \eta_0[25cos^2(\omega t+\beta x+\frac{\pi}{3})+25sin^2(\omega t+\beta x-\frac{\pi}{6})]\\ &= 50\eta_0 cos^2(\omega t+\beta x+\frac{\pi}{3})\ (W).\end{aligned}$$

Then, we obtain the time-average power across a $1m^2$ area

$$< P_2 >= \frac{1}{T}\int_0^T |S_2|\cdot 1dt = 25\eta_0\ (W).$$

c)

$$E_3 = \eta_0 H_3\times\hat{z} = \eta_0[-\hat{y}sin(\omega t-\beta z+\frac{\pi}{4})-\hat{x}2sin(\omega t-\beta z-\frac{3\pi}{4})]\ (V/m).$$

$$\tilde{\mathbf{E}}_3 = \eta_0(-\hat{y}e^{-j\beta z-j\frac{\pi}{2}+j\frac{\pi}{4}}-\hat{x}2e^{-j\beta z-j\frac{\pi}{2}-j\frac{3\pi}{4}}) = \eta_0(-\hat{y}e^{-j\beta z-j\frac{\pi}{4}}-\hat{x}2e^{-j\beta z-j\frac{5\pi}{4}})\ (V/m),$$

$$\tilde{\mathbf{H}}_3 = \hat{x}e^{-j\beta z-j\frac{\pi}{2}+j\frac{\pi}{4}}-\hat{y}2e^{-j\beta z-j\frac{\pi}{2}-j\frac{3\pi}{4}} = \hat{x}e^{-j\beta z-j\frac{\pi}{4}}-\hat{y}2e^{-j\beta z-j\frac{5\pi}{4}}\ (A/m).$$

$$\begin{aligned}|S_3|\cdot 1 &= \eta_0\,|H_3|^2 = \eta_0[sin^2(\omega t-\beta z+\frac{\pi}{4})+4sin^2(\omega t-\beta z-\frac{3\pi}{4})]\\ &= 5\eta_0 sin^2(\omega t-\beta z+\frac{\pi}{4})\ (W).\end{aligned}$$

$$< P_3 >= \frac{5}{2}\eta_0\ (W).$$

d)

$$H_4 = \frac{1}{\eta_0}(-\hat{y})\times E_4 = \frac{1}{\eta_0}[\hat{z}2cos(\omega t+\beta y-\frac{\pi}{2})+\hat{x}2sin(\omega t+\beta y)]\ (A/m).$$

$$\tilde{\mathbf{E}}_4 = \hat{x}2e^{+j\beta y-j\frac{\pi}{2}}-\hat{z}2e^{+j\beta y-j\frac{\pi}{2}}\ (V/m),$$

$$\tilde{\mathbf{H}}_4 = \frac{1}{\eta_0}(\hat{z}2e^{+j\beta y-j\frac{\pi}{2}}+\hat{x}2e^{+j\beta y-j\frac{\pi}{2}})\ (A/m).$$

$$\begin{aligned}|S_4|\cdot 1 &= \frac{1}{\eta_0}\,|E_4|^2 = \frac{1}{\eta_0}[4cos^2(\omega t+\beta y-\frac{\pi}{2})+4sin^2(\omega t+\beta y)]\\ &= \frac{8}{\eta_0}sin^2(\omega t+\beta y)\ (W).\end{aligned}$$

$$< P_4 >= \frac{4}{\eta_0}\ (W).$$

e)

$$E_5 = \eta_0 H_5\times\hat{y} = \eta_0[\hat{z}cos(\omega t-\beta y)-\hat{x}sin(\omega t-\beta y-\frac{\pi}{4})]\ (V/m).$$

$$\tilde{\mathbf{E}}_5 = \eta_0(\hat{z}e^{-j\beta y}-\hat{x}e^{-j\beta y-j\frac{\pi}{2}-j\frac{\pi}{4}}) = \eta_0(\hat{z}e^{-j\beta y}-\hat{x}e^{-j\beta y-j\frac{3\pi}{4}})\ (V/m),$$

$$\tilde{\mathbf{H}}_5 = \hat{x}e^{-j\beta y}+\hat{z}e^{-j\beta y-j\frac{\pi}{2}-j\frac{\pi}{4}} = \hat{x}e^{-j\beta y}+\hat{z}e^{-j\beta y-j\frac{3\pi}{4}}\ (A/m).$$

$$\begin{aligned}|S_5|\cdot 1 &= \eta_0\,|H_5|^2 = \eta_0[cos^2(\omega t-\beta y)+sin^2(\omega t-\beta y-\frac{\pi}{4})]\\ &= \eta_0 cos^2(\omega t-\beta y)+\eta_0 cos^2(\omega t-\beta y-\frac{\pi}{4})\ (W).\end{aligned}$$

$$< P_5 >= \frac{\eta_0}{2}+\frac{\eta_0}{2} = \eta_0\ (W).$$

2. By comparing with the given expression and the general expression

$$\gamma\eta = j\omega\mu \text{ and } \frac{\gamma}{\eta} = \sigma + j\omega\epsilon$$

as well as

$$\mu = \frac{\gamma\eta}{j\omega},\ \sigma = \mathrm{Re}\{\frac{\gamma}{\eta}\},\ \epsilon = \frac{1}{\omega}\mathrm{Im}\{\frac{\gamma}{\eta}\}$$

Using these relations, for a plane wave propagating in a non-magnetic material ($\mu = \mu_0$) with

$$\mathbf{H} = \hat{x}3e^{y}\cos(8\pi 10^6 t + \sqrt{3}y - \frac{\pi}{3}) \text{ A/m}$$

determine:

a) By comparing with the given expression and the general expression

$$\mathbf{H} = \hat{x}H_0 e^{\alpha y}\cos(\omega t + \beta y - \phi) \text{ (A/m)},$$

we find that $\alpha = 1$ (Np/m), $\beta = \sqrt{3}$ (rad/m), $\gamma = \alpha + j\beta = 1 + j\sqrt{3}$. As for the unit of γ, it should be a linear combination of (Np/m) and (rad/m).

b) The angular frequency $\omega = 8\pi\times 10^6$ (rad/s), the frequency $f = 4\times 10^6$ (Hz), the wavelength $\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\sqrt{3}}$ (m), and the phase velocity $\upsilon_p = \frac{\omega}{\beta} = \frac{8\pi}{\sqrt{3}} \times 10^6$ (m/s).

c) From $\gamma\eta = j\omega\mu$, we have

$$\eta = \frac{j\omega\mu}{\gamma} = \frac{j\times 8\pi\times 10^6\times 4\pi\times 10^{-7}}{1+j\sqrt{3}} = \frac{4\pi^2}{5}(\sqrt{3}+j)\ (\Omega).$$

$$\because\ \frac{\gamma}{\eta} = \frac{1+j\sqrt{3}}{\frac{4\pi^2}{5}(\sqrt{3}+j)} = 0.110 + j0.0633\ (S/m).$$

$$\therefore \begin{cases} \epsilon &= \frac{1}{\omega}Im(\frac{\gamma}{\eta}) = \frac{1}{8\pi\times 10^6}\times 0.0633 = 2.52\times 10^{-9}\ (F/m) \\ \sigma &= Re(\frac{\gamma}{\eta}) = 0.110\ (S/m) \end{cases}$$

d)

$$\tilde{\mathbf{H}} = \hat{x}3e^{(1+j\sqrt{3})y - j\frac{\pi}{3}}\ (A/m).$$

$$E = \eta H \times \hat{\upsilon},$$

$$\tilde{\mathbf{E}} = \eta\tilde{\mathbf{H}}\times(-\hat{y}) = (\frac{8\pi^2}{5}\cdot e^{j\frac{\pi}{6}})\cdot(-\hat{z})\cdot 3e^{(1+j\sqrt{3})y-j\frac{\pi}{3}} = -\hat{z}\frac{24\pi^2}{5}e^{(1+j\sqrt{3})y-j\frac{\pi}{6}}\ (V/m).$$

e)

$$\begin{aligned} < E\times H > &= \frac{1}{2}Re(\tilde{\mathbf{E}}\times\tilde{H}*) \\ &= \frac{1}{2}Re(-\hat{y}\frac{72\pi^2}{5}e^{2y+j\frac{\pi}{6}}) \\ &= -\hat{y}\frac{36\pi^2}{5}e^{2y}(\frac{\sqrt{3}}{2}) \\ &= -\hat{y}\frac{18\sqrt{3}\pi^2}{5}e^{2y}\ (W/m^2). \end{aligned}$$

f) For the amplitude of the electricfield at a given distance y to reduce by a factor of 1/e relative to its value at y = 0, we must solve the following expression for y:

$$\frac{|E(x)|}{|E(x=0)|} = \frac{\frac{24\pi^2}{5}e^y}{\frac{24\pi^2}{5}} = e^{-1}$$

so that

$$y = -1\ (m).$$

Note that the absolute magnitude of this distance is always equal to $1/\alpha$ and is known as the penetration depth.

g) The total time-averaged power dissipated within a specied volume V (in W) is the volume integral of the time-averaged dissipated power density (in W/m^3):

$$P_{dis} = \int_V < J \cdot E > dV$$

Since J = σE, the dissipated power density J·E is easily calculated, but the volume integral of its time-average is not.

According to the integral form of the Poynting theorem (the law of electromagnetic energy conservation), the sum of the time-averaged dissipated power P_{dis} and the flux of time- averaged propagated power, <S> = <E×H>, through the surface S bounding the volume V must equal zero (note that the divergence theorem is used to derive the integral form from the dierential form):

$$\oint_s < E \times H > \cdot dS + \int_V < J \cdot E > dV = 0$$

so that

$$P_{dis} = -\oint_s < E \times H > \cdot dS$$

Since <E × H> is along the −y direction, only the surfaces S_1 : y = 0 and S_2 : y = 1 have non-zero contributions to the total surface flux integral. Thus,

$$\begin{aligned} P_{dis} &= -[\int_{s_1} < E \times H > |_{y=0} \cdot (-\hat{y}) dS + \int_{s_2} < E \times H > |_{y=1} \cdot (-\hat{y}) dS] \\ &= -[\int_{s_1} \frac{18\sqrt{3}\pi^2}{5} dS - \int_{s_2} \frac{18\sqrt{3}\pi^2}{5} e^2 dS] \\ &= \frac{18\sqrt{3}\pi^2}{5}(e^2 - 1)\ (W) \end{aligned}$$

Note that the value for P_{dis} is positive, indicating that, indeed, electromagnetic power is dissipated (via its conversion to kinetic energy of the free charges in the conducting material), rather than injected.

h) Stored energy per unit volume is the stored energy density (in $\frac{J}{m^3}$) , which is given by $\frac{1}{2}\epsilon E \cdot E + \frac{1}{2}\mu H \cdot H$, such that the time rate change of this quantity $\frac{d}{dt}[\frac{1}{2}\epsilon E \cdot E + \frac{1}{2}\mu H \cdot H]$ has units of $\frac{W}{m^3}$ and represents stored power density. For this cosinusoidal waveeld, instantaneous stored energy density is proportional to $cos^2(\omega t + \beta x + \phi)$, such that the instantaneous stored power density is proportional to $cos(\omega t + \beta x + \phi)\ sin(\omega t + \beta t + \phi)$. The time average of this function will be 0 at all positions in space, including at (1,1,1).

Alternatively, one can express the time average of the stored power density at a given point as the time rate change of the time-averaged stored energy density, i.e. take the time average of $[\frac{1}{2}\epsilon E{\cdot}E + \frac{1}{2}\mu H{\cdot}H]$ first and then the time derivative, which is thus the time average of a time-independent function (the time average of $cos^2(\omega t + \beta x + \phi)$ is $\frac{1}{2}$), which also evaluates to 0.

3. We observe that

$$\frac{\sigma}{\omega\epsilon} = \frac{4}{2\pi \times 3 \times 10^4 \times 81 \times 8.85 \times 10^{-12}} \gg 1,$$

which means ocean water can be treated as a good conductor at 30 kHz.

a) In a good conductor

$$\alpha \approx \sqrt{\pi f \mu \sigma} = \sqrt{\pi \times 3 \times 10^4 \times 4\pi \times 10^{-7} \times 4} = 0.688\ (Np/m)$$

$$\beta \approx \sqrt{\pi f \mu \sigma} = \sqrt{\pi \times 3 \times 10^4 \times 4\pi \times 10^{-7} \times 4} = 0.688\ (rad/m)$$

$$|\eta| = \sqrt{\frac{\omega\mu}{\sigma}} = 0.243$$

becasue it is a good conductor, $\tau = 45\circ$

$$\eta = 0.243\angle 45\circ\ (\Omega)$$

b) Assume the distance between the submarine and the ship is d, we want

$$e^{-\alpha d} \geqslant 1\% = 0.01$$

$$\therefore \quad -\alpha d \geqslant ln(0.01),$$

$$\therefore \quad d \leqslant \frac{-ln(0.01)}{\alpha} \approx 6.69\ (m)$$

c) Based on β the in Part a), the wavelength is

$$\lambda = \frac{2\pi}{\beta} = 9.13\,(m)$$

d) At f = 300 Hz, since

$$\frac{\sigma}{\omega\epsilon} = \frac{4}{2\pi \times 300 \times 81 \times 8.85 \times 10^{-12}} = 2.96 \times 10^6 \gg 1,$$

we can still treat ocean water as a good conductor at 300 Hz. By using the general formula:

$$\alpha \approx \sqrt{\pi f \mu \sigma} = \sqrt{\pi \times 3 \times 10^2 \times 4\pi \times 10^{-7} \times 4} = 0.0688\ (Np/m)$$

$$\beta \approx \sqrt{\pi f \mu \sigma} = \sqrt{\pi \times 3 \times 10^2 \times 4\pi \times 10^{-7} \times 4} = 0.0688\ (rad/m)$$

$$|\eta| = \sqrt{\frac{\omega\mu}{\sigma}} = 0.0243$$

$$\eta = 0.0243\angle 45\circ \ (\Omega)$$

$$d \leqslant \frac{-ln(0.01)}{\alpha} \approx 66.9 \ (m)$$

$$\lambda = \frac{2\pi}{\beta} = 91.3\,(m).$$

4.

a)

$$f = \frac{V_p\beta}{2\pi} = 3.18 \times 10^6 \ (Hz)$$

b)

$$\lambda = \frac{2\pi}{\beta} = 31.4\,(m)$$

c)

$$\epsilon_r = \frac{1}{V_p^2\epsilon_0\mu} = \frac{9}{4} = 2.25$$

d)

$$\sigma = \frac{2\alpha}{\eta_0}\sqrt{\frac{\epsilon_r}{\mu_r}} = 3.98 \times 10^{-6} \ (S/m)$$

e)

$$y = \frac{1}{\alpha} = 1000 \ (m)$$

f)

$$\tau = \frac{\sigma}{2\omega\epsilon} = \frac{3.98 \times 10^{-6}}{2 \times 2\pi \times 3.18 \times 10^6 \times 2.25 \times 8.85 \times 10^{-12}} = 0.005$$

$$\begin{aligned}
\tilde{\mathbf{H}} &= \frac{20}{|\eta|} e^{-(0.001+j0.2)y} e^{-j\tau}\hat{z} \\
&= \frac{1}{8\pi} e^{-(0.001+j0.2)y-j0.005}\hat{z} \ (A/m).
\end{aligned}$$