1. Since the field is in $-\hat{z}$ direction, we can know that the plate at z = 0 m is holding a surface charge density of $-2[C/m^2]$. The electric field associated with surface charge density of ρ_s is

$$\mathbf{E} = \hat{z}\frac{\rho_s}{2\epsilon_0}sgn(z)$$

The field between the two plates is $\mathbf{E} = -\frac{2}{\epsilon_0}\hat{z}\,[V/m]$ and $\mathbf{E} = 0\,[V/m]$ elsewhere

a) The displacement in the gap

$$\mathbf{D} = \epsilon_0\frac{-2}{\epsilon_0}\hat{z} = -2\hat{z}\,[C/m^2]$$

By definition,

$$\mathbf{D} = \epsilon\mathbf{E} = \epsilon_0\mathbf{E} + \mathbf{P}$$

The gap is occupied by vacuum, $\epsilon = \epsilon_0$

$$\mathbf{P} = 0\,[C/m^2]$$

From $E = -\nabla V$, we can know $V(z) = \frac{2z}{\epsilon_0} + C$. Given that $V(0) = 0$,

$$V(z) = \frac{2z}{\epsilon_0}\,[V]$$

Voltage drop between the two copper plates is

$$V(2) - V(0) = \frac{4}{\epsilon_0}$$

So capacitance per unit area is

$$\frac{C}{A} = \frac{\epsilon_0}{2}\,[F/m^2]$$

b) The displacement does not change even if vacuum is replaced by pure water because the charge densities on the plates are the same. So V

$$\mathbf{E} = \frac{-2}{\epsilon} = -\frac{2}{80\epsilon_0} = -\frac{1}{40\epsilon_0}\hat{z}\,[V/m]$$

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$$

$$-2\hat{z} = -\frac{1}{40}\hat{z} + \mathbf{P}$$

$$\mathbf{P} = -\frac{79}{40}\hat{z}\,[C/m^2]$$

From $E = -\nabla V$, we can know $V(z) = \frac{z}{40\epsilon_0} + C$. Given $V(0) = 0$,

$$V(z) = \frac{z}{40\epsilon_0} \, [V]$$

Voltage drop between the two copper plates is

$$V(2) - V(0) = \frac{1}{20\epsilon_0}$$

so capacitance per unit area is

$$\frac{C}{A} = 40\epsilon_0 \, [F/m^2]$$

c) If the conductivity of the material between the parallel plates becomes dierent from zero then $\mathbf{E} = \mathbf{D} = 0$ and $\mathbf{P} = 0$ as the steady-state equilibrium is reached. Also, the whole conducting system is at the same potential, so $V = 0$. This happens because in a conducting medium all equilibrium elds vanish after the rearrangements of the net charge on the bounding surface. In this particular case, the salt water shorts out the original eld between the plates. Since there is no potential dierence between the two plates, and the two plates do not carry any charge, the capacitance is undened.

2. We will look at each region at a time.

a) For r $\leq$ a , it is a perfect conductor. For a conductor, there is no electric field within it when equilibrium is reached, and all the charges should be located on the surface at $r = a$

$$\mathbf{D_1} = 0 \, [C/m^2] \; \mathbf{E}_1 = 0 \, [V/m] \; \mathbf{P_1} = 0 \, [C/m^2]$$

$$\rho_s|_{r=a} = \frac{-2}{4\pi a^2} = -\frac{1}{2\pi a^2} \, [C/m^2]$$

b) For a $<$ r $<$ b, applying Guass's Law $\oint_S \mathbf{D} \cdot \mathbf{dS} = Q_{enclosed}$

$$\mathbf{D_2} = \frac{-2}{4\pi r^2} = -\frac{1}{2\pi r^2}\hat{r} \, [C/m^2]$$

$$\mathbf{E_2} = \frac{\mathbf{D_2}}{\epsilon} = -\frac{1}{8\epsilon_0 \pi r^2}\hat{r} \, [V/m]$$

$$\mathbf{P_2} = \mathbf{D}_2 - \epsilon_0 \mathbf{E_2} = -\frac{3}{8\pi r^2}\hat{r} \, [C/m^2]$$

c) For b $\leq$ a $\leq$ c, it is a conductor, so there is no electric field within it when equilibrium is reached. By Guass's Law, there should be a total charge of +2C uniformly distributed on the surface r = b. Since the net charge of the conductor is 1C, there is a total charge of -1C uniformly distributed on the surface r = c, so

$$\mathbf{D_3} = 0\,[C/m^2]\ \mathbf{E}_3 = 0\,[V/m]\ \mathbf{P_3} = 0\,[C/m^2]$$

$$\rho_s|_{r=b} = \frac{2}{4\pi b^2} = \frac{1}{2\pi b^2}\,[C/m^2]$$

$$\rho_s|_{r=c} = \frac{-1}{4\pi c^2} = -\frac{1}{4\pi c^2}\,[C/m^2]$$

d) For r > c, since the net charge of the object is -1C

$$\mathbf{D_4} = \frac{-1}{4\pi r^2}\hat{r}\,[C/m^2]$$

$$\mathbf{E_4} = \frac{\mathbf{D_4}}{\epsilon_0} = \frac{-1}{4\epsilon_0 \pi r^2}\hat{r}\,[V/m]$$

$$\mathbf{P}_4 = 0$$

3. Consider a simplified model of a **vacuum diode** consisting of a **cathode** in the$x = 0$ plane and an **anode** in the$x = d$ plane, where the anode is held to a constant potential $V_a = 4$V relative to the cathode. The region $0 < x < d$ between the cathode and the anode supports a charge density$\rho(x)$ accounting for the electrons in transit from the cathode (where they are emitted) to the anode. If the potential distribution in the region $0 < x < d$ is given by $V(x) = V_a(x/d)^{4/3}$, find the following:

a) Since

$$\mathbf{E} = -\nabla V = -V_a \frac{4}{3d}(\frac{x}{d})^{\frac{1}{3}}\hat{x}\,[V/m]$$

Evaluating at x = d/4,

$$\mathbf{E}(\frac{d}{4}) = -\frac{16}{3d}(\frac{1}{4})^{\frac{1}{3}}\hat{x}\,[V/m]$$

b) Applying Guass's Law,

$$\rho = \nabla \cdot \mathbf{D} = -\frac{16\epsilon_0}{9d^2}(\frac{x}{d})^{-\frac{2}{3}}\,[C/m^2]$$

So

$$\rho(x = \frac{d}{2}) = -\frac{16\epsilon_0}{9d^2}(\frac{1}{2})^{-\frac{2}{3}}\,[C/m^3]$$

c) By boundary condition,

$$\rho_s = \hat{x} \cdot (\mathbf{D_{d+}} - \mathbf{D_{d-}})$$

Since $D_{d+} = 0$, and

$$\mathbf{D_{d-}} = -\frac{16\epsilon_0}{3d}(\frac{x}{d})^{\frac{1}{3}}|_{x=d}\hat{x} = -\frac{16\epsilon_0}{3d}\hat{x}\,[C/m^2]$$

$$\rho_s = \frac{16\epsilon_0}{3d}\,[C/m^2]$$

4. .

a) Considering the medium in each slab to be homogeneous, we can refer to Laplace's equation, $\nabla^2 V = 0$. Hence, the potential in each dielectric slab will be written as

$$V(z) = \begin{cases} A_1 z + B_1 & 0 < z < d \\ A_2 z + B_2 & d < z < z_0 \end{cases}$$

from which we find the elecric field as

$$\mathbf{E}(z) = -\nabla V = \begin{cases} -A_1 \hat{z} & 0 < z < d \\ -A_2 \hat{z} & d < z < z_0 \end{cases}$$

Given that $V = 0$ at $z = 0$, we will have $B_1 = 0$. Similarly, since $V = Vp$ at $z = z_0$, we get

$$V_p = A_2 z_0 + B_2$$

The boundary conditions state that the electric field must be continuous along the interface between the two dielectrics. Thus, we can write

$$A_2 d + B_2 = A_1 d \Rightarrow A_2 d + B_2 - A_1 d = 0$$

for the potential at $z = d$. Applying the other boundary condition stating that there must be no change between the normal components of the displacement vector D within the two dielectrics due to the fact that there are no mobile free carriers along it, we can write

$$\hat{n} \cdot (\mathbf{D_1} - \mathbf{D_2}) = -\hat{z} \cdot (\epsilon_1(-A_1\hat{z}) - \epsilon_2(-A_2\hat{z}))$$

$$0 = \epsilon_1 A_1 - \epsilon_2 A_2$$

Using the last three equations, we find

$$A_1 = \frac{\epsilon_2 V_p}{z_0\epsilon_1 + d(\epsilon_2 - \epsilon_1)}$$

$$A_2 = \frac{\epsilon_1 V_p}{z_0\epsilon_1 + d(\epsilon_2 - \epsilon_1)}$$

$$B_2 = \frac{d(\epsilon_2 - \epsilon_1)V_p}{z_0\epsilon_1 + d(\epsilon_2 - \epsilon_1)}$$

from which the electric potential can be written as

$$V(z) = \begin{cases} \frac{\epsilon_2 V_p}{z_0\epsilon_1 + d(\epsilon_2-\epsilon_1)} z & 0 < z < d \\ \frac{\epsilon_1 V_p}{z_0\epsilon_1 + d(\epsilon_2-\epsilon_1)} z + \frac{d(\epsilon_2-\epsilon_1)V_p}{z_0\epsilon_1 + d(\epsilon_2-\epsilon_1)} & d < z < z_0 \end{cases}$$

b) Referring to $E = -\nabla V$, the electric field inside the dielectrics is given by

$$\mathbf{E}(z) = \begin{cases} -\frac{\epsilon_2 V_p}{z_0\epsilon_1 + d(\epsilon_2-\epsilon_1)} \hat{z} & 0 < z < d \\ -\frac{\epsilon_1 V_p}{z_0\epsilon_1 + d(\epsilon_2-\epsilon_1)} \hat{z} & d < z < z_0 \end{cases}$$

Given that $z_o = 3d = 3\text{m}, V_p = 6\text{V}$, $\epsilon_1 = \epsilon_o$, and $\epsilon_2 = 2\epsilon_o$, we find that

$$\mathbf{E}(z) = \begin{cases} -3\hat{z} & 0 < z < d \\ -1.5\hat{z} & d < z < z_0 \end{cases}$$

The surface charge density at $z = z_0$ is given by

$$\begin{aligned} \rho_s|_{z=z_0} &= \mathbf{D} \cdot \hat{n}|_{z=z_0} \\ &= D_z^+(z_0) - D_z^-(z_0) \end{aligned}$$

where $D_z^+ = 0$. Therefore, we can write

$$\rho_s(z_0) = 3\epsilon_0 \, [C/m^2]$$

5. Using Laplace's equation, $\nabla^2 V = 0$, the potential in each dielectric slab will be written as

$$V(z) = \begin{cases} A_1 z & 0 < z < 2 \\ A_2 z + B_2 & 2 < z < 5 \end{cases}$$

from which we find the elecric field as

$$\mathbf{E}(z) = -\nabla V = \begin{cases} -A_1\hat{z} & 0 < z < 2 \\ -A_2\hat{z} & 2 < z < 5 \end{cases}$$

Since

$$V(5) = 5A_2 + B_2 = 0$$

$$\hat{x} \cdot (\mathbf{D_{2+}} - \mathbf{D_{2-}}) = \rho_s \implies -2\epsilon_0 A_2 + \epsilon_0 A_1 = 7\epsilon_0$$

$$V(2+) = V(2-) \implies 2A_2 + B_2 = 2A_1$$

we can determine that

$$\begin{cases} A_1 = 3 \\ A_2 = -2 \\ B_2 = 10 \end{cases}$$

So

$$\mathbf{D} = \begin{cases} -3\epsilon_0\hat{x}\,[C/m^2] & 0 < z < 2 \\ 4\epsilon_0\hat{x}\,[C/m^2] & 2 < z < 5 \end{cases}$$

Thus,

$$\rho_s = \begin{cases} -3\epsilon_0\,[C/m^2] & x = 0 \\ -4\epsilon_0\,[C/m^2] & x = 5 \end{cases}$$

6. .

a) Using the formula given in the problem and Lecture 10, the resistance R of this copper wire is calculated as

$$R = \frac{d}{A\sigma} \approx 0.567\,[\Omega]$$

b) The voltage drop across the wire is given by $V = E \cdot d = R \cdot I$, where I is the current flowing along the wire. Assuming I = 1 A, the electric field within the wire is

$$E = \frac{R \cdot I}{d} \approx 5.4 \times 10^{-3}\,[V/m]$$

c) Referring to the hint given in the problem, the current density may be expressed as$\mathbf{J} = \sigma\mathbf{E} = N_e q_e \mathbf{v_e}$ where $q_e = -1.6 \times 10^{-19}$C. Then, the mean speed of an electron is given by

$$|\mathbf{v_e}| = \frac{\sigma|\mathbf{E}|}{N_e|q_e|} \approx 2.33 \times 10^{-5}\,[m/s]$$

d) The time it would take an electron to travel from one end of the wire to the other is simply

$$t = \frac{d}{|\mathbf{v_e}|} \approx 5.35 \times 10^{6}\,[s]$$

7. .

a) Applying the integral form of Gauss's law to a sphere of radius r (where a<r<b), we get

$$4\pi r^2 E_r = \frac{Q}{\epsilon}$$

from which we find

$$\mathbf{E} = E_r \hat{r} = \frac{Q}{4\pi\epsilon r^2}\hat{r}$$

Taking the path integral of the electric field from a to b

$$V_{ab} = V_a - V_b = -\int_b^a \mathbf{E}\cdot\hat{r}dr$$

$$= \int_a^b \frac{Q}{4\pi\epsilon r^2}dr$$

we find the potential drop V from the inner to outer shell as

$$V = \frac{Q}{4\pi\epsilon}(\frac{b-a}{ab})$$

Since $Q = CV$, we obtain

$$\frac{1}{C} = \frac{1}{4\pi\epsilon}(\frac{b-a}{ab})$$

Implying

$$C = 4\pi\epsilon(\frac{ab}{b-a})$$

b) Taking the limit of capacitance C as $b \to \infty$, we obtain

$$C = 4\pi\epsilon(\frac{ab}{b}) = 8\pi\epsilon_0 a \approx 222\,[pF]$$

c) Since

$$dR = \frac{dr}{4\pi r^2\sigma}$$

$$R = \int_a^b dR = \int_a^b \frac{dr}{4\pi r^2\sigma} = \frac{b-a}{4\pi\sigma ab}$$

So

$$G = \frac{1}{R} = \frac{4\pi\sigma ab}{b-a} = \frac{\sigma}{\epsilon}C$$