APPENDIX

A. *The expression in Line **4** of **Algorithm 2***

Line 4 of **Algorithm 2** is: p_i^{h+1}=arg min $L_\rho(p_i)$. L_ρ is denoted in (27). According to the first-order optimality condition,

$$\frac{\partial L_\rho}{\partial p_i}=0 \tag{1-1}$$

To achieve consensus within the distributed framework, each node is required to exchange and aggregate information with its local neighbors. We define the auxiliary variables $\tilde{\lambda}_{i,l}^{G,h}$ and $\tilde{\xi}_{i,l}^{G,h}$ to represent the weighted sum of neighboring states:

$$\tilde{\lambda}_{i,l}^{G,h}=\sum_{j\in\mathcal{N}} w_{(i,j)}\lambda_{j,l}^{G,h},\forall i\in\mathcal{N},l\in\mathcal{L} \tag{1-2}$$

$$\tilde{\xi}_{i,l}^{G,h}=\sum_{j\in\mathcal{N}} w_{(i,j)}\xi_{j,l}^{G,h},\forall i\in\mathcal{N},l\in\mathcal{L} \tag{1-3}$$

In order to maintain equivalence with the centralized trading model, we collect the terms related to p_i and ignore the constant terms, $L_\rho(p_i)$ can be rewritten as the following standard form:

$$L_\rho(p_i)=e_i\cdot|a_i\cdot p_i+b_i^h|+p_i^T\cdot c_i\cdot p_i+d_{i,h}^T\cdot p_i \tag{1-4}$$

where all parameters in the formula are expressed as follows

$$\begin{aligned}
&a_i=\delta_{i,i}^T=\delta_{i,l}^T\big|_{l=i}=[\delta_{(i,1),i},\delta_{(i,2),i},...,\delta_{(i,N),i}];\\
&b_{i,h}=\sum_{m\in\mathcal{N}}\delta_{m,m}^T\cdot p_m^h-\delta_{i,i}^T\cdot p_i^h;\\
&c_i=\frac{1}{2}\cdot(\rho\cdot B_i^T\cdot B_i+\rho+\frac{1}{2}\cdot\rho\cdot\sum_{l\in\mathcal{L}}\delta_{i,l}\cdot\delta_{i,l}^T);\\
&d_{i,h}=-\pi_i\cdot B_i^T+B_i^T\cdot\upsilon_i^h-\rho\cdot B_i^T\cdot\hat{p}_i^h\\
&\qquad+\sum_{j\in N}\left(\mu_{(i,j)}^h-\rho\cdot p_i^h+\frac{1}{2}\cdot\rho\cdot\xi_{(i,j)}^h\right)\\
&\qquad+\frac{1}{2}\cdot\sum_{l\in\mathcal{L}}\begin{pmatrix}\delta_{i,l}\cdot(\lambda_{1\,i,l}^{\mathrm{G},h}-\widetilde{\lambda_{2\,i,l}^{\mathrm{G},h}})\\ +\rho\cdot\delta_{i,l}\cdot(\widetilde{\xi_{1\,i,l}^{\mathrm{G},h}}-\widetilde{\xi_{2\,i,l}^{\mathrm{G},h}})\\ -\rho\cdot\delta_{i,l}\cdot\delta_{i,l}^T\cdot p_i^h\end{pmatrix};\\
&e_i=C_{i,t}=C_{l,t}\big|_{l=i};
\end{aligned} \tag{1-5}$$

Since the objective function contains an absolute value term, we solve the problem by analyzing the following three cases:

1) if $a_i\cdot p_i+b_{i,h}=0$, the problem in line 4 can be rewritten as

$$\begin{aligned}&\min_{p_i^{\mathrm{eq}}}\ p_i^{\mathrm{eq\,T}}\cdot c_i\cdot p_i^{\mathrm{eq}}+d_{i,h}^{\mathrm{T}}\cdot p_i^{\mathrm{eq}}\\ &\text{s.t.}\quad a_i\cdot p_i^{\mathrm{eq}}+b_{i,h}=0\end{aligned} \tag{1-6}$$

Then p_i can be written as

$$p_i^{\mathrm{eq}}=-\frac{1}{2}\cdot c_i^{-1}\cdot d_{i,h}+c_i^{-1}\cdot\frac{a_i\cdot c_i^{-1}\cdot d_{i,h}-2\cdot b_{i,h}}{2\cdot a_i\cdot c_i^{-1}\cdot a_i^T}\cdot a_i^{\mathrm{T}} \tag{1-7}$$

2) if $a_i\cdot p_i+b_{i,h}>0$, the problem in line 4 can be rewritten as

$$\min_{p_i^+}\ e_i\cdot(a_i\cdot p_i^++b_{i,h})+p_i^{+\mathrm{T}}\cdot c_i\cdot p_i^++d_{i,h}^{\mathrm{T}}\cdot p_i^+ \tag{1-8}$$

Then p_i can be written as

$$p_i^+=-\frac{1}{2}c_i^{-1}(d_{i,h}+e_ia_i^{\mathrm{T}}) \tag{1-9}$$

3) if $a_i\cdot p_i+b_{i,h}<0$, the problem in line 4 can be rewritten as

$$\min_{p_i^-}\ -e_i\cdot(a_i\cdot p_i^-+b_{i,h})+p_i^{-\mathrm{T}}\cdot c_i\cdot p_i^-+d_{i,h}^{\mathrm{T}}\cdot p_i^- \tag{1-10}$$

Then p_i can be written as

$$p_i^-=-\frac{1}{2}c_i^{-1}(d_{i,h}-e_ia_i^{\mathrm{T}}) \tag{1-11}$$

After verifying the validity of the solution, the final solution can be expressed as

$$\begin{aligned}p_i^{h+1}&=\arg\min_{p\in\{p_i^{h+1,\mathrm{eq}},p_i^{h+1,+},p_i^{h+1,-}\}_{valid}}L_\rho(p_i)\\
&=\begin{cases}-\frac{1}{2}\cdot c_i^{-1}\cdot d_{i,h}+c_i^{-1}\cdot\frac{a_i\cdot c_i^{-1}\cdot d_{i,h}-2\cdot b_{i,h}}{2\cdot a_i\cdot c_i^{-1}\cdot a_i^T}\cdot a_i^{\mathrm{T}}, \text{ if } a_i\cdot p_i+b_{i,h}=0\\ -\frac{1}{2}c_i^{-1}(d_{i,h}+e_ia_i^{\mathrm{T}}), \text{ if } a_i\cdot p_i+b_{i,h}>0\\ -\frac{1}{2}c_i^{-1}(d_{i,h}-e_ia_i^{\mathrm{T}}), \text{ if } a_i\cdot p_i+b_{i,h}<0\end{cases}\end{aligned} \tag{1-12}$$

B. *Proof of Lemma 1*

By non-expansiveness of projection onto a closed convex set, for any (u,v)

$$\left\|\Pi_{\mathbb{R}_+}(u)-\Pi_{\mathbb{R}_+}(v)\right\|\le\|u-v\| \tag{1-13}$$

Let u=$(W\otimes I)\lambda_G$+$\rho\xi_G$, v=$(W\otimes I)\,\tilde{\lambda}_G+\rho\,\tilde{\xi}_G$. Then

$$\begin{aligned}\left\|\Pi_{\mathbb{R}_+}(u)-\Pi_{\mathbb{R}_+}(v)\right\|&\le\|u-v\|\\
&=\left\|(W\otimes I)(\lambda_G-\tilde{\lambda}_G)+\rho(\xi_G-\tilde{\xi}_G)\right\|\\
&\le\left\|(W\otimes I)(\lambda_G-\tilde{\lambda}_G)\right\|+\rho\left\|\xi_G-\tilde{\xi}_G\right\|\\
&\le\left\|\lambda_G-\tilde{\lambda}_G\right\|+\rho\left\|\xi_G-\tilde{\xi}_G\right\|\end{aligned} \tag{1-14}$$

which is the desired bound. The proof is completed.

C. *Proof of Lemma 2*

We first write the updates for each type of variable in deviation-from-average form and then stack them.

i) GIC multipliers λ_G^h and tracker ξ_G^h.

It is easy to derived that

$$\begin{aligned}e_{\xi_G}^{h+1}&=\xi_G^{h+1}-\bar{\xi}_G^{h+1}\\
&=W\xi_G^h+G(p^{h+1})-G(p^h)-\bar{\xi}_G^{h+1}\\
&=\left(W-\frac{1}{N}11^\top\right)\otimes e_{\xi_G}^h+\left(I-\frac{1}{N}11^\top\right)\otimes\Delta G(p^{h+1})\end{aligned} \tag{1-15}$$

where $\Delta G(p^{h+1})$:= $\mathrm{col}(G_i(p_i^{h+1})-G_i(p_i^h))_{i\in\mathcal{N}}$. The second term depends only on the changes in the primal variables between two consecutive iterations and hence can be expressed as a linear function of the stacked increments Δp^h. Therefore there exists a constant matrix H_G such that

$$e_{\xi_G}^{h+1}=F_{\xi_G}e_{\xi_G}^h+H_G\Delta p^h \tag{1-16}$$

where $F_{\xi_G}:=\left(W-\frac{1}{N}11^\top\right)\otimes I$. By *Assumption 2*, the eigenvalues of $W-\frac{1}{N}11^\top$ lie strictly inside the unit circle, so $\rho(F_{\xi_G})$ <1 on the consensus–orthogonal subspace.

For the GIC multipliers, eq.(38) is same as

$$\lambda_{i,l}^{G,h+1}=\Pi_{\mathbb{R}_+}\left(\sum_{j\in\mathcal{N}}w_{(i,j)}\lambda_{j,l}^{G,h}+\rho\xi_{i,l}^{G,h+1}\right) \tag{1-17}$$

Stack these across nodes:

$$\lambda_G^{h+1}=\Pi_{\mathbb{R}_+}\left((W\otimes I)\lambda_G^h+\rho\xi_G^{h+1}\right) \tag{1-18}$$

Let $\bar{\lambda}_G^h$ be the consensus embedding of the average of λ_G^h, and define $e_{\lambda_G}^h := \lambda_G^h - \bar{\lambda}_G^h$. Applying *Lemma 1* with

$$u = (W \otimes I)\lambda_G^h + \rho\xi_G^{h+1}, \quad v = (W \otimes I)\bar{\lambda}_G^h + \rho\bar{\xi}_G^{h+1} \tag{1-19}$$

we obtain the non-expansive bound

$$\| e_{\lambda_G}^{h+1} \| \le \| (W \otimes I)e_{\lambda_G}^h + \rho e_{\xi_G}^{h+1} \| \tag{1-20}$$

Since the consensus component is annihilated by $W - \frac{1}{N}11^\top$, we have

$$(W \otimes I)e_{\lambda_G}^h = \left(W - \frac{1}{N}11^\top\right) \otimes I e_{\lambda_G}^h \tag{1-21}$$

Thus (1-20) shows that the GIC multiplier error is driven by the contractive matrix

$$F_{\lambda_G} := \left(W - \frac{1}{N}11^\top\right) \otimes I \tag{1-22}$$

and by the GIC tracker error $e_{\xi_G}^{h+1}$. In particular, we can write this dependence in the compact form

$$e_{\lambda_G}^{h+1} = F_{\lambda_G} e_{\lambda_G}^h + G_{\lambda_G} e_{\xi_G}^{h+1} \tag{1-23}$$

for a suitable constant matrix G_{λ_G}. Again, *Assumption 2* implies $\rho(F_{\lambda_G})$ <1.

ii) GEC multipliers μ^h and tracker ξ_e^h.

For each edge (i,j), the GEC tracker ξ_e^h is denoted as in (32), and the associated multiplier $\mu_{(i,j)}^h$ is updated in (34) by an ADMM-like rule involving this residual. The pair (ξ_e^h, $\mu_{(i,j)}^h$) thus forms a two-node consensus and correction mechanism. Stacking all such edge-wise updates gives a linear operator analogous to W-$\mathbf{11}^\mathrm{T}/N$ on a lifted edge space (the incidence matrix of the graph). As in [1, Lemma 2], when we project these dynamics onto the orthogonal complement of the consensus subspace for edge variables, we get a contraction matrix F_{μ,ξ_e} with spectral radius strictly less than 1.

iii) LIC multipliers v^h.

The LIC multipliers v^h are updated according to (31) by purely local residuals, without any averaging through W. Therefore they do not contribute to consensus errors in the sense induced by W. When we stack all errors into e^h, we can treat the LIC part as having a zero block in the homogeneous matrix, i.e.

$$e_v^{h+1} = 0 \cdot e_v^h + H_v \Delta p^h \tag{1-24}$$

so the corresponding block of F is zero and the LIC residual effects are absorbed into the input term $H_v\Delta p^h$. Collecting the three parts, we write

$$e^{h+1} = Fe^h + G\Delta\xi^h + H\Delta p^h \tag{1-25}$$

where F is block-diagonal with blocks F_{λ_G}, F_{μ,ξ_e} and zero blocks for LIC. G and H collect the contributions from the tracker updates and primal increments. By *Assumption 2*, all nontrivial blocks of F associated with inter-node communication are strict contractions. Therefore, the overall F is strictly contractive on the consensus-orthogonal subspace, and we have $\rho(F) < 1$. The proof is completed.

D. *Proof of Lemma 3*

We proceed in steps.

i) Primal variables.

At each iteration, the primal variables p_i^h and $\hat{p}_i^h$ are obtained as minimizers of local convex subproblems (28)-(30). By *Assumption 1*, the objective functions are convex and the feasible sets are nonempty, closed, and convex. The quadratic penalty terms in the augmented Lagrangian ensure that each subproblem has a nonempty compact set of minimizers, so the primal update mapping is well-defined and maps bounded sets into bounded sets.

Similar arguments apply to the edge variables p_e^h that appear in the GEC constraints: their updates are also solutions of convex subproblems with quadratic penalties on the residuals $p_{(i,j)}+p_{(j,i)}$. Hence $\{p_e^h\}$ is bounded.

ii) LIC and GEC multipliers v^h and μ^h.

The LIC multipliers v_i^h are updated according to (31) by adding a multiple of the LIC residual. The GEC multipliers $\mu_{(i,j)}^h$ are updated via (34) with residual $\xi_{(i,j)}^h$=$p_{(i,j)}^h$+ $p_{(j,i)}^h$. Because the primal variables are bounded by step (i), the residuals are bounded as well, and the dual ascent steps for v_i^h and $\mu_{(i,j)}^h$ have bounded increments. Moreover, the underlying augmented Lagrangian has the same structure as in standard ADMM for linearly constrained convex problems. From [1, Lemma 3 and Theorem 2], which analyze exactly such update schemes, the dual variables of LIC and GEC remain bounded whenever the primal variables are bounded and the problem satisfies *Assumption 1*. We can apply those results directly to each LIC and edgewise GEC subsystem, thus obtaining boundedness of $\{v^h\}$, $\{\mu^h\}$.

iii) GIC trackers and multipliers ξ^h and λ_G^h.

Stack (35) over all nodes:

$$\xi_G^{h+1} = (W \otimes I)\xi_G^h + \Delta G(p^{h+1}) \tag{1-26}$$

Because p^h is bounded and each $G_i(\cdot)$ is continuous on a bounded set, the sequence $\{ G_i(p_i^h) \}$ is bounded and $\{ \Delta G(p^{h+1}) \}$ is bounded as well. The matrix $W\otimes I$ has spectral norm at most 1. Thus, the linear system defining ξ_G^h has bounded input and non-expansive homogeneous part, which implies boundedness of $\{\xi_G^h\}$. This is a standard property of linear time-invariant systems with stable or marginally stable homogeneous part and bounded input.

For the GIC multipliers, applying *Lemma 1* with the reference pair (λ_G, ξ_G^h)=(λ_G^h, ξ_G^{h+1}) and ($\tilde{\lambda}_G$, $\tilde{\xi}_G$)=(0,0) yields

$$\left\|\lambda_G^{h+1}\right\| \le \left\|\lambda_G^h\right\| + \rho\left\|\xi_G^{h+1}\right\| \tag{1-27}$$

Thus $\| \lambda_G^h \|$ cannot grow faster than linearly with h. To rule out linear growth and obtain uniform boundedness, we appeal to the Lyapunov analysis in [1].

Specifically, consider first the algorithm obtained by treating the GIC constraints as additional linear coupling constraints with associated multipliers λ_G and without the projection onto $\mathbb{R}_+^m$. This intermediate algorithm fits exactly into the framework of [1]: its dual updates are affine, its augmented Lagrangian has the same structure, and the same Lyapunov function V^h as in [1, Lemma 3] can be constructed. The proof of [1, Lemma 3] then shows that this Lyapunov function is nonincreasing and that all dual variables, including the GIC-related multipliers, are bounded and have square-summable increments. Our actual algorithm differs from the intermediate one only in that, after the affine dual update, we apply the projection $\Pi_{\mathbb{R}_+^m}$ to λ_G. In other words, the GIC dual update can be written as

$$\begin{aligned}\lambda_{G,temp}^{h+1} &= (W \otimes I)\lambda_G^h + \rho\xi_G^{h+1} \\ \lambda_G^{h+1} &= \Pi_{\mathbb{R}_+^m}\left(\lambda_{G,temp}^{h+1}\right)\end{aligned} \tag{1-28}$$

Projection onto a closed convex set is firmly nonexpansive and, in particular, nonexpansive with respect to any reference point $\lambda_G^\star$:

$$\| \lambda_G^{h+1} - \lambda_G^\star \| \le \| \lambda_{G,temp}^{h+1} - \lambda_G^\star \| \tag{1-29}$$

The Lyapunov function V^h in [1] is built from the augmented Lagrangian plus squared distances of the primal–dual variables to a KKT point. Hence replacing $\lambda_{G,temp}^{h+1}$ by its projection λ_G^{h+1} cannot increase V^h. Consequently, the descent inequality for V^h derived in [1, Lemma 3] continues to hold for our projected GIC update, possibly with the GIC residual added to the residual vector.

Therefore, exactly as in [1, Lemma 3], the Lyapunov function is bounded from below and nonincreasing, and the GIC multiplier sequence λ_G^h remains bounded and has square-summable increments. In particular,

$$\sup_h |\lambda_G^h| < \infty \tag{1-30}$$

iv) GEC trackers $\xi_{(i,j)}^h$.

Since we already showed that $\{p_e^h\}$ is bounded, combined with (32) the sequence $\{\xi_e^h\}$ is bounded as well.

Combining steps *i)–iv)* above, all sequences listed are bounded. The proof is completed. ♦

E. *Proof of Theorem 1*

We prove the three parts in order.

i) Vanishing primal residuals.

For LIC and GEC, the updates of (p^h, $\hat{p}^h$, p_e^h, v^h, μ^h) form standard ADMM steps on linear equality constraints with convex objectives and indicator functions. Under *Assumption 1*, the convergence result of [1, Theorem 2] implies that the LIC residuals $\{ B_i p_i^h - \hat{p}_i^h \}$ and the GEC residuals $\{ p_{(i,j)}^h + p_{(j,i)}^h \}$ converge to zero.

For GIC, the residual is $\sum_i G_i(p_i^h)$, i.e., equivalently the violation of the inequality constraint. From *Lemma 3*, p^h and λ_G^h are bounded. Consider again the intermediate "unprojected" algorithm from the proof of Lemma 3, in which the GIC constraints are treated as additional linear constraints and the multipliers are updated affinely. For that algorithm, the Lyapunov function constructed in [1, Lemma 3] can be extended by simply including the GIC residual in the residual vector. The proof in [1] then shows that the sum of squared residuals including GIC is finite.

Our actual algorithm applies the projection $\Pi_{\mathbb{R}_+^m}$ to the intermediate GIC multipliers after the affine update. As argued in *Lemma 3*, this projection is firmly nonexpansive and cannot increase the Lyapunov function of [1]. Therefore the same Lyapunov inequality as in [1, Lemma 3] holds for the projected iterates, and the sum of squared residuals remains finite. Since each squared residual is nonnegative, finiteness of their sum implies that every primal residual (LIC, GEC, and GIC) must converge to zero.

ii) Vanishing consensus/duplication errors.

By *Lemma 2*, the stacked error vector e^h satisfies (42) with $\rho(F)$<1. The Lyapunov function used in [1, Theorem 2], extended by including GIC residuals and the corresponding dual variables, yields that the increments of the primal and tracker variables are square-summable, i.e.,

$$\sum_{h=0}^{\infty}\left(\| \Delta p^h \|^2 + \| \Delta \xi^h \|^2 \right) < \infty \tag{1-31}$$

As in the proof of [1, Theorem 2], this follows from the same descent inequality for the Lyapunov function, which is not affected and in fact only strengthened by the additional GIC projection step, due to its firm nonexpansiveness. Square summability implies $\Delta p^h \to 0$ and $\Delta\xi^h \to 0$, such that the input is

$$u^h := G\Delta\xi^h + H\Delta p^h \tag{1-32}$$

in the recursion for e^h converges to zero. For a linear recursion with a strictly contractive homogeneous part and a vanishing input, standard linear systems arguments imply $e^h \to 0$. In particular, all consensus/duplication errors vanish asymptotically.

iii) Primal–dual limit points are KKT.

From *Lemma 3*, the sequence (p^h, $\hat{p}^h$, p_e^h, r^h) is bounded, so it has at least one cluster point (p^*, $\hat{p}^*$, p_e^*, r^*). By *i*), all primal residuals converge to zero, so the limit satisfies all LIC, GEC, and GIC constraints. By part *ii*), all duplication errors vanish, so we can identify the limiting multipliers with global dual variables (v^*,μ^*,λ_G^*). The subgradient optimality conditions for each local subproblem imply that, at the limit point,

$$0 \in \partial_{p_i} \mathcal{L}_\rho (p^*, \hat{p}^*, p_e^*, v^*, \mu^*, \lambda_G^*) \tag{1-33}$$

where $\mathcal{L}_\rho$ denotes the augmented Lagrangian in (27). Together with feasibility of LIC, GEC, and GIC, and the fact that duplication errors vanish, this yields stationarity and primal feasibility.

For GIC, we also need complementary slackness. At a limit point (p^*,λ_G^*), the GIC multiplier update becomes the fixed-point equation

$$\lambda_G^* = \Pi_{\mathbb{R}_+^m}\left(\lambda_G^* + \rho, G(p)\right) \tag{1-34}$$

projection onto the nonnegative orthant $\mathbb{R}_+^m$ is characterized by

$$y = \Pi_{\mathbb{R}_+^m}(y - \gamma z) \quad \Leftrightarrow \quad 0 \le y \perp z \le 0 \tag{1-35}$$

Applying this relation with y=λ_G^* and z=-$G(p^*)$, we obtain

$$0 \le \lambda_G^* \perp G(p^*) \le 0 \tag{1-36}$$

i.e., componentwise complementarity $\lambda_{G,j}^* G_j(p^*)$ =0 for all *j*.

$$p^* := \mathrm{col}(p_i^*)_{i \in \mathcal{N}}, G(p^*) := \sum_i G_i(p_i^*) \tag{1-37}$$

Therefore, the GIC constraint is $G(p^*)$≤0. Combining primal feasibility, dual feasibility, stationarity, and complementary slackness shows that

$$(p^*, \hat{p}^*, p_e^*, v^*, \mu^*, \lambda_G^*) \tag{1-38}$$

is a KKT point of (26). If the KKT point is unique, then the whole sequence must converge to this unique point rather than oscillating among multiple cluster points, because any cluster point is a KKT point and the sequence is bounded. This proves all statements of the theorem. The proof is completed. ♦

REFERENCES

[1] S. Zhu, T. Ding, C. Chen, M.-Y. Chow, and X. Guan, “DEED-ADMM: A Scalable Distributed Algorithm for Economic Dispatch in Multi-energy Systems with Energy Storage,” *IEEE Trans. Autom. Sci. Eng.*, vol. 22, 2025.