# Exponential & Logarithmic Functions

# Exponential Functions

- ***Definition:*** An exponential function is a function in the form,

  $$f(x)=ab^x \text{ where } a \neq 0,\ b > 0\ ,\ b \neq 1.$$

  Letting $a=1$, we have $f(x)=b^x$ .
- **Properties of** $f(x) = b^x$

1. $f(0) = 1$. The function will always take the value of 1 at $x = 0$ .
2. $f(x) \neq 0$ . An exponential function will never be zero.
3. $f(x) > 0$ . An exponential function is always positive.

# Exponential Functions

4. The previous two properties can be summarized by saying that the range of an exponential function is $(0,\infty)$ .
5. The domain of an exponential function is $(-\infty,\infty)$ .
6. If $0 < b < 1$ then,
   a. $f(x) \to 0$ as $x \to \infty$
   b. $f(x) \to \infty$ as $x \to -\infty$
7. If $b > 1$ then,
   a. $f(x) \to \infty$ as $x \to \infty$
   b. $f(x) \to 0$ as $x \to -\infty$

# Exponential Functions

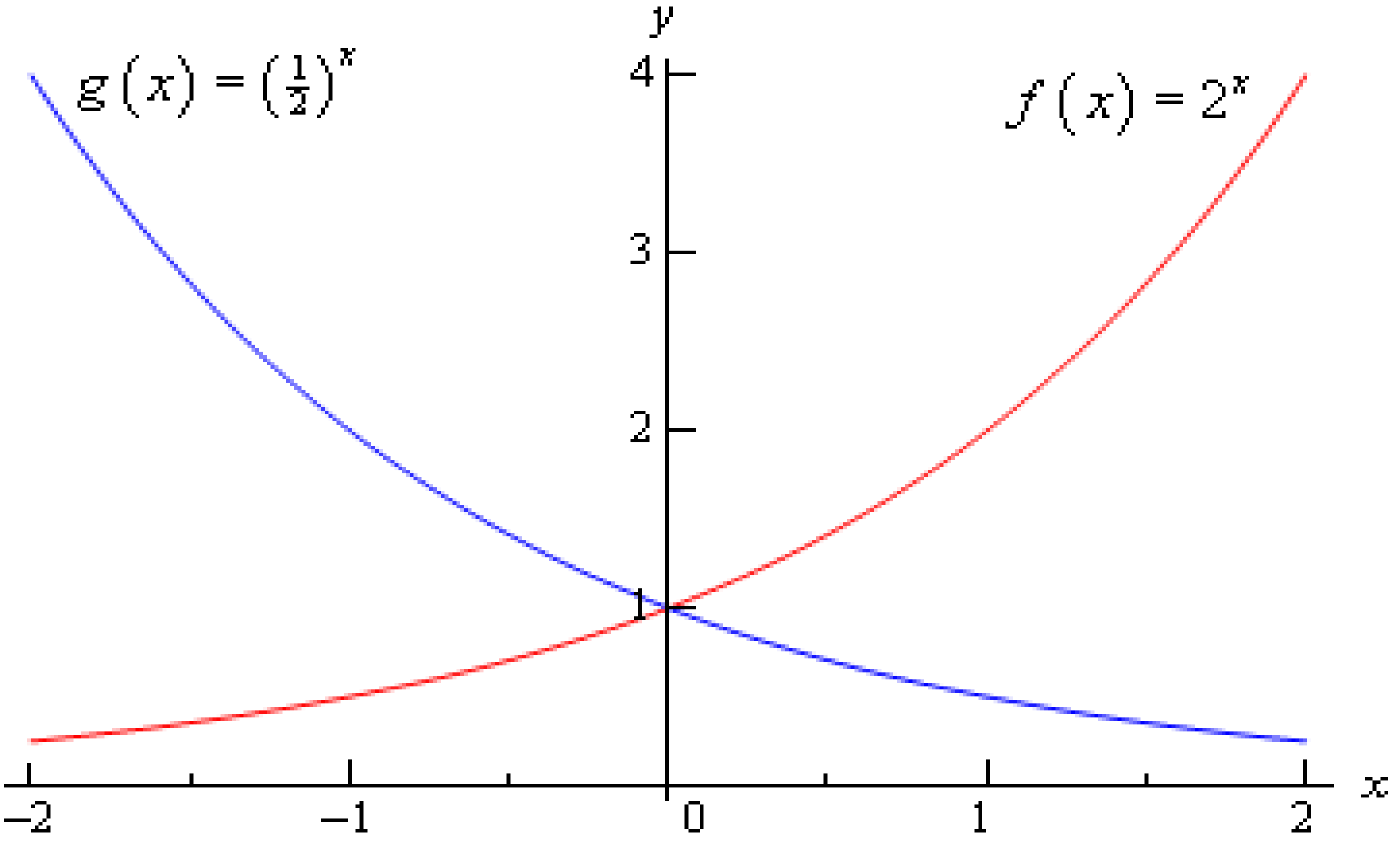
*Example 1* Sketch the graph of $f(x) = 2^x$ and $g(x) = \left(\frac{1}{2}\right)^x$

a table of values for these two functions.

| $x$ | $f(x)$ | $g(x)$ |
|---|---|---|
| -2 | $f(-2) = 2^{-2} = \frac{1}{4}$ | $g(-2) = \left(\frac{1}{2}\right)^{-2} = 4$ |
| -1 | $f(-1) = 2^{-1} = \frac{1}{2}$ | $g(-1) = \left(\frac{1}{2}\right)^{-1} = 2$ |
| 0 | $f(0) = 2^{0} = 1$ | $g(0) = \left(\frac{1}{2}\right)^{0} = 1$ |
| 1 | $f(1) = 2$ | $g(1) = \frac{1}{2}$ |
| 2 | $f(2) = 4$ | $g(2) = \frac{1}{4}$ |

# Exponential Functions

sketch of both of these functions



# Exponential Functions

- **Definition** : The **natural exponential function** is $f(x) = \mathbf{e}^x$ where, $\mathbf{e}$ = 2.71828182845905….
- Since $\mathbf{e} > 1$ then $\mathbf{e}^x \to \infty$ as $x \to \infty$ and $\mathbf{e}^x \to 0$ as $x \to -\infty$.

# Logarithmic Functions

- If $b > 0$ and $b \neq 1$, the exponential function $f(x) = b^x$ is either increasing or decreasing, so it is one-to-one.
- Thus, it has an inverse function $f^{-1}$, which is called the logarithmic function with base $b$ and is denoted by $\log_b$.

# Logarithmic Functions

- If we use the formulation of an inverse function given by

$$f^{-1}(x) = y \quad \Leftrightarrow \quad f(y) = x$$

- Then we have

$y = \log_b x$ is equivalent to $x = b^y$

The first is called logarithmic form and the second is called the exponential form.

The number, $b$, is called the base.

# Logarithmic Functions

- Thus, if $x > 0$, then $\log_b x$ is the exponent to which the base $b$ must be raised to give $x$.

# Logarithmic Functions

- ***Example 1*** Without a calculator give the exact value of each of the following logarithms.

(a) $\log_2 16$

- convert the logarithm to exponential form.

$\log_2 16 = x$ is equivalent to $2^x = 16$

$\log_2 16 = 4$ because $2^4 = 16$

# Logarithmic Functions

**(b)** $\log_9 1/531441$

**(c)** $\log_{3/2} 27/8$

(d) $\log_{10} 0.001$

# Special Logarithms

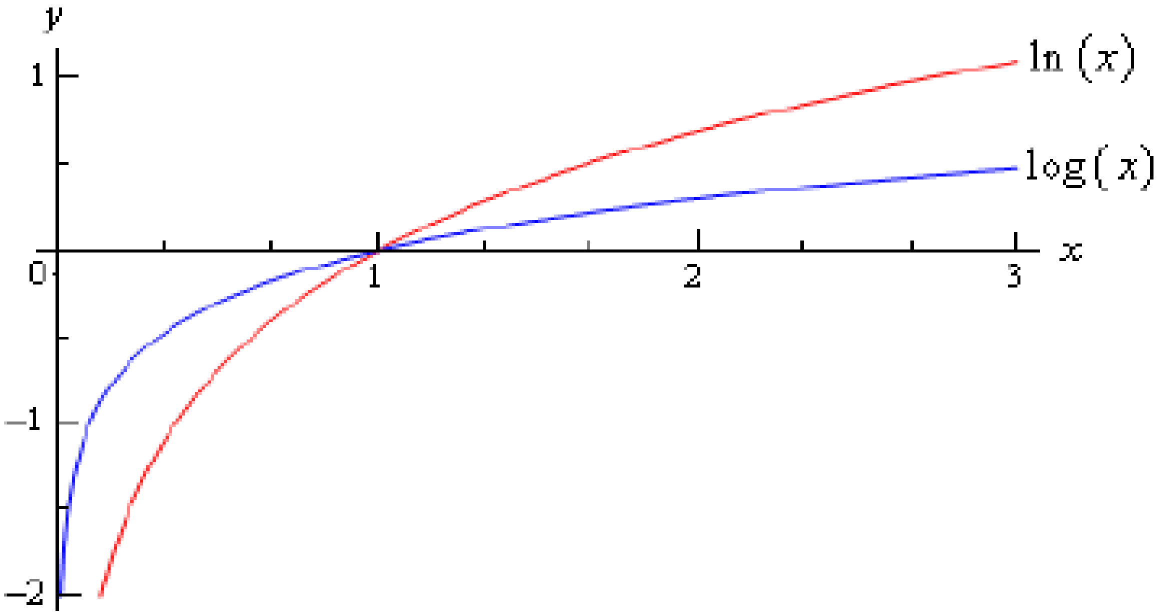
- These are,

| | |
|---|---|
| $\ln x = \log_e x$ | This log is called the natural logarithm |
| $\log x = \log_{10} x$ | This log is called the common logarithm |

- In the natural logarithm the base **e** is the same number as in the natural exponential logarithm

# Special Logarithms

sketch of both of these logarithms




From this graph

$$\ln x \to \infty \quad \text{as} \quad x \to \infty$$

$$\ln x \to -\infty \quad \text{as} \quad x \to 0,\ x > 0$$

# Special Logarithms

*Example 2* Without a calculator give the exact value of each of the following logarithms.

(a) $\ln \sqrt[3]{e}$

(b) $\log 1000$

(c) $\log_{16} 16$

*Solution*

(a) $\ln \sqrt[3]{e} = \frac{1}{3}$ because $e^{\frac{1}{3}} = \sqrt[3]{e}$

(b) $\log 1000 = 3$ because $10^3 = 1000$

(c) $\log_{16} 16 = 1$ because $16^1 = 16$

# Logarithmic Functions

- Some of the basic properties of logarithms.

1. The domain of the logarithm function is $(0, \infty)$. In other words, we can only plug positive numbers into a logarithm! We can't plug in zero or a negative number.
2. $\log_b b = 1$
3. $\log_b 1 = 0$
4. $\log_b b^x = x$
5. $b^{\log_b x} = x$

- Notice that the last two properties tell us that

$f(x) = b^x$ and $g(x) = \log_b x$

are ***inverses*** of each other.

# Logarithmic Functions

- More Properties

6. $\log_b xy = \log_b x + \log_b y$
7. $\log_b \left( \dfrac{x}{y} \right) = \log_b x - \log_b y$
8. $\log_b \left( x^r \right) = r \log_b x$

Note that there is no equivalent property to the first two for sums and differences. In other words,

$$\log_b (x + y) \neq \log_b x + \log_b y$$

$$\log_b (x - y) \neq \log_b x - \log_b y$$

# Logarithmic Functions

- Example 2: simplify the following logarithms

(a) $\ln x^3 y^4 z^5$

(b) $\log_3\left(\dfrac{9x^4}{\sqrt{y}}\right)$

(c) $\log\left(\dfrac{x^2+y^2}{(x-y)^3}\right)$

# Logarithmic Functions

(a) $\ln x^3 y^4 z^5$

Property 6 above can be extended to products of more than two functions. Once we've used Property 6 we can then use Property 8.

$$\begin{aligned} \ln x^3 y^4 z^5 &= \ln x^3 + \ln y^4 + \ln z^5 \\ &= 3\ln x + 4\ln y + 5\ln z \end{aligned}$$

# Logarithmic Functions

- The change of base formula is,

$$\log_b x = \frac{\log_a x}{\log_a b}$$

- It will convert from base *b* to base *a.*
- The two most common change of base formulas are

$$\log_b x = \frac{\ln x}{\ln b} \qquad \text{and} \qquad \log_b x = \frac{\log x}{\log b}$$

- The usual reason for the change of base formula is to compute the value of the logarithm that is in a base that you can't easily deal with.

# Logarithmic Functions

- Example 3: compute $\log_7 50$

$$\log_7 50 = \frac{\ln 50}{\ln 7} = \frac{3.91202300543}{1.94591014906} = 2.0103821378$$

OR

$$\log_7 50 = \frac{\log 50}{\log 7} = \frac{1.69897000434}{0.845098040014} = 2.0103821378$$

- This shows us that regardless of which change of base formula we use, we end up with the same answer.

# Exponential & Logarithm Equations

- We will be solving equations that involve both exponentials and logarithms in them.
- We shall therefore make great use of all the properties of both that we have covered.

# Exponential & Logarithm Equations

*Example 1* Solve $7+15e^{1-3z}=10$.

- The first step is to get the exponential on one side of the equation with a coefficient of one.

$$\begin{aligned} 7+15e^{1-3z} &= 10 \\ 15e^{1-3z} &= 3 \\ e^{1-3z} &= \frac{1}{5} \end{aligned}$$

- Now we need to get z out of the exponent so that we can solve for it. We therefore introduce the natural logarithm on both sides and use the property $\log_b b^x = x$

# Exponential & Logarithm Equations

$$\ln\left(e^{1-3z}\right) = \ln\left(\frac{1}{5}\right)$$

$$1-3z = \ln\left(\frac{1}{5}\right)$$

$$-3z = -1+\ln\left(\frac{1}{5}\right)$$

$$z = -\frac{1}{3}\left(-1+\ln\left(\frac{1}{5}\right)\right) = 0.8698126372$$

# Exponential & Logarithm Equations

*Example 2* Solve $10^{x^2 - x} = 100$.

# Exponential & Logarithm Equations

*Example 3* Solve $x - xe^{5x+2} = 0$.

- The first step is to factor out x from both terms but do not divide x on both sides cause this will lead to loss of a solution

$$x - xe^{5x+2} = 0$$

$$x\left(1 - e^{5x+2}\right) = 0$$

- At this point we have two possibilities either $x = 0$ OR $1 - e^{5x+2} = 0$

# Exponential & Logarithm Equations

- We will now solve the second possibility

$$e^{5x+2} = 1$$

$$5x+2 = \ln 1$$

$$5x+2 = 0$$

$$x = -\frac{2}{5}$$

- We therefore have two solutions

$$x = 0 \text{ and } x = -\frac{2}{5}$$

# Exponential & Logarithm Equations

*Example 4* Solve $5\left(x^2-4\right)=\left(x^2-4\right)e^{7-x}$.

*Example 5* Solve $4e^{1+3x}-9e^{5-2x}=0$.

# Exponential & Logarithm Equations

*Example 6* Solve $3+2\ln\left(\frac{x}{7}+3\right)=-4$.

- The first step is to get the logarithm on one side of the equation with a coefficient of 1.

$$2\ln\left(\frac{x}{7}+3\right)=-7$$

$$\ln\left(\frac{x}{7}+3\right)=-\frac{7}{2}$$

- In order to factor out the x at this point we introduce the exponent ***e*** on both sides.

# Exponential & Logarithm Equations

$$e^{\ln\left(\frac{x}{7}+3\right)} = e^{-\frac{7}{2}}$$

- We shall then use the property $b^{\log_b x} = x$

$$\frac{x}{7}+3 = e^{-\frac{7}{2}}$$

$$\frac{x}{7} = -3+e^{-\frac{7}{2}}$$

$$x = 7\left(-3+e^{-\frac{7}{2}}\right) = -20.78861832$$

- We need to verify that $\frac{x}{7}+3$ is not negative because we can't plug a negative number into a logarithm. ($\frac{x}{7}+3$ > 0)
- Therefore, $x = -20.78861832$

# Exponential & Logarithm Equations

*Example 7* Solve $2\ln\left(\sqrt{x}\right)-\ln\left(1-x\right)=2$.

*Example 8* Solve $\log x+\log\left(x-3\right)=1$.

# Exponential & Logarithm Equations

- When solving equations with logarithms, it is important to check your potential solutions to make sure that they don’t generate logarithms of negative numbers or zero.
- It is also important to make sure that you do the checks in the original equation.