Solutions of Equations in One Variable

Secant & Regula Falsi Methods

Numerical Analysis (9th Edition)

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Beamer Presentation Slides
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Outline

1. Secant Method: Derivation & Algorithm

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2. Comparing the Secant & Newton's Methods

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3. The Method of False Position (Regula Falsi)

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Rationale for the Secant Method

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- Newton's method is an extremely powerful technique, but it has a major weakness: the need to know the value of the derivative of f at each approximation.
- Frequently, $f'(x)$ is far more difficult and needs more arithmetic operations to calculate than $f(x)$.

Derivation of the Secant Method

$$f'(p_{n-1}) = \lim_{x \to p_{n-1}} \frac{f(x) - f(p_{n-1})}{x - p_{n-1}}.$$

Circumvent the Derivative Evaluation

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Using this approximation for $f'(p_{n-1})$ in Newton's formula gives

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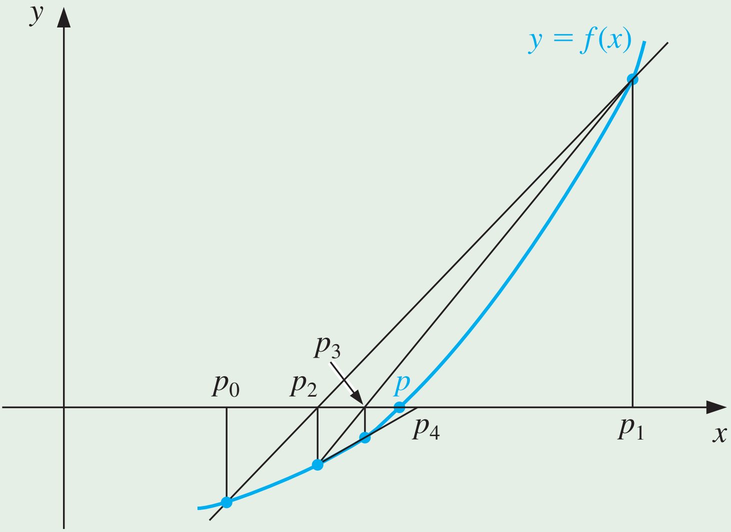
This technique is called the Secant method

Secant Method: Using Successive Secants



The Secant Method

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- Note that only one function evaluation is needed per step for the Secant method after p_2 has been determined.
- In contrast, each step of Newton's method requires an evaluation of both the function and its derivative.

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 $q_0 = q_1$; $p_1 = p$; $q_1 = f(p)$

7 OUTPUT ('The method failed after N_0 iterations, $N_0 =$', N_0);
(*The procedure was unsuccessful*) STOP

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Comparing the Secant & Newton's Methods

Example: $f(x) = \cos x - x$

Use the Secant method to find a solution to $x = \cos x$, and compare the approximations with those given by Newton's method with $p_0 = \pi/4$.

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$$p_n = p_{n-1} - \frac{(p_{n-1} - p_{n-2})(\cos p_{n-1} - p_{n-1})}{(\cos p_{n-1} - p_{n-1}) - (\cos p_{n-2} - p_{n-2})}, \quad \text{for } n \geq 2.$$

Comparing the Secant & Newton's Methods

Newton's Method for $f(x) = \cos(x) - x$, $p_0 = \frac{\pi}{4}$

n	p_{n-1}	$f(p_{n-1})$	$f'(p_{n-1})$	p_n	$\lvert p_n - p_{n-1}\rvert$
1	0.78539816	-0.078291	-1.707107	0.73953613	0.04586203
2	0.73953613	-0.000755	-1.673945	0.73908518	0.00045096
3	0.73908518	-0.000000	-1.673612	0.73908513	0.00000004
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- Because of the agreement of p_3 and p_4 we could reasonably expect this result to be accurate to the places listed.

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Secant Method for $f(x) = \cos(x) - x$, $p_0 = 0.5$, $p_1 = \frac{\pi}{4}$

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2	0.500000000	0.785398163	0.736384139	0.0490140246
3	0.785398163	0.736384139	0.739058139	0.0026740004
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- Here, the convergence of the Secant method is much faster than functional iteration but slightly slower than Newton's method.
- This is generally the case. Order of Convergence

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- Both methods require good first approximations but generally give rapid acceleration.

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- Although it is not a method we generally recommend, it illustrates how bracketing can be incorporated.

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  - If not, choose p_3 as the x-intercept of the line joining $(p_0, f(p_0))$ and $(p_2, f(p_2))$, and then interchange the indices on p_0 and p_1.

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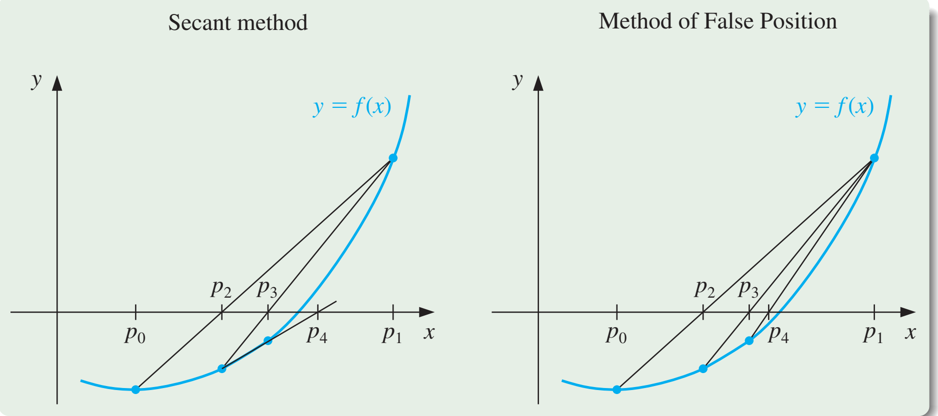
Secant Method & Method of False Position



In this illustration, the first three approximations are the same for both methods, but the fourth approximations differ.

The Method of False Position: Algorithm

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      OUTPUT (p); (*The procedure was successful*): STOP
   5. Set $i = i + 1$; $q = f(p)$
   6. If $q \cdot q_1 < 0$ then set $p_0 = p_1$; $q_0 = q_1$

The Method of False Position: Algorithm

To find a solution to $f(x) = 0$, given the continuous function f on the interval $[p_0, p_1]$ (where $f(p_0)$ and $f(p_1)$ have opposite signs) tolerance TOL and maximum number of iterations N_0.

1 Set $i = 2$; $q_0 = f(p_0)$; $q_1 = f(p_1)$.

2 While $i \leq N_0$ do Steps 3–7:

   3 Set $p = p_1 - q_1(p_1 - p_0)/(q_1 - q_0)$. (*Compute* p_i)
   4 If $|p - p_1| < TOL$ then
   OUTPUT (p); (*The procedure was successful*): STOP
   5 Set $i = i + 1$; $q = f(p)$
   6 If $q \cdot q_1 < 0$ then set $p_0 = p_1$; $q_0 = q_1$
   7 Set $p_1 = p$; $q_1 = q$

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   3. Set $p = p_1 - q_1(p_1 - p_0)/(q_1 - q_0)$. (*Compute* p_i)
   4. If $|p - p_1| < TOL$ then
      OUTPUT (p); (*The procedure was successful*): STOP
   5. Set $i = i + 1$; $q = f(p)$
   6. If $q \cdot q_1 < 0$ then set $p_0 = p_1$; $q_0 = q_1$
   7. Set $p_1 = p$; $q_1 = q$
8. OUTPUT ('Method failed after N_0 iterations, $N_0 =$', N_0);
   (*The procedure was unsuccessful*): STOP

The Method of False Position: Numerical Calculations

Comparison with Newton & Secant Methods

Use the method of False Position to find a solution to $x = \cos x$, and compare the approximations with those given in a previous example which Newton's method and the Secant Method.

The Method of False Position: Numerical Calculations

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To make a reasonable comparison we will use the same initial approximations as in the Secant method, that is, $p_0 = 0.5$ and $p_1 = \pi/4$.

The Method of False Position: Numerical Calculations

Comparison with Newton's Method & Secant Method

	False Position	**Secant**	**Newton**
n	p_n	p_n	p_n
0	0.5	0.5	0.7853981635
1	0.7853981635	0.7853981635	0.7395361337
2	0.7363841388	0.7363841388	0.7390851781
3	0.7390581392	0.7390581392	0.7390851332
4	0.7390848638	0.7390851493	0.7390851332
5	0.7390851305	0.7390851332	
6	0.7390851332		

The Method of False Position: Numerical Calculations

Comparison with Newton's Method & Secant Method

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5	0.7390851305	0.7390851332	
6	0.7390851332		

Note that the False Position and Secant approximations agree through p_3 and that the method of False Position requires an additional iteration to obtain the same accuracy as the Secant method.

The Method of False Position

Final Remarks

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- The added insurance of the method of False Position commonly requires more calculation than the Secant method, . . .
- just as the simplification that the Secant method provides over Newton's method usually comes at the expense of additional iterations.

Questions?

Reference Material

Order of Convergence of the Secant Method

Exercise 14, Section 2.4

It can be shown (see, for example, Dahlquist and Å. Björck (1974), pp. 228–229), that if $\{p_n\}_{n=0}^{\infty}$ are convergent Secant method approximations to p, the solution to $f(x) = 0$, then a constant C exists with

$$|p_{n+1} - p| \approx C\,|p_n - p|\,|p_{n-1} - p|$$

for sufficiently large values of n. Assume $\{p_n\}$ converges to p of order α, and show that

$$\alpha = (1 + \sqrt{5})/2$$

(*Note:* This implies that the order of convergence of the Secant method is approximately 1.62).

Return to the Secant Method

Dahlquist, G. and Å. Björck (Translated by N. Anderson), *Numerical methods*, Prentice-Hall, Englewood Cliffs, NJ, 1974.