Errors

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Overview

PRESENTATION SLIDES

Errors

- Not all Mathematical problems can be solved analytically
  - $2x + 5 = 8$ easy to solve
  - $2^{\sqrt{x}} + 5 = 8$ not as easy
- Sometimes we actually give up solving it
- And we just approximate it
- Hence use numerical rather than analytical approaches
- Issue of how far from the actual answer we are is important
- The study of errors forms an important part of numerical analysis.
- There are several ways in which errors can be introduced in the solution of the problem.

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- Precision
  - How close the values are to each other

Sources of errors

- Rounding
  - Occurs due to computing device's inability to deal with inexact numbers. $\pi = \pi = 3.14159265\cdots$ can be rounded to four significant digits as $\pi = 3.142$
  - Maximum error in rounding is 0.5×10^{-n}. n is the number of rounded places
- Truncating
  - Occurs because some series (finite or infinite) is truncated/choped to a fewer number of terms. e.g. A surd $\pi = 3.14159265\cdots$ can be truncated to four significant figures to $\pi = 3.141$
  - Maximum error in truncation is 1×10^{-n}.

Error Propagation

- Calculating the uncertainty or error of an approximation against the actual value it is trying to approximate. Represented as *absolute value* showing how far the approximation is or *relative* error shown as a proportion
- Absolute Error
  - $\delta x = |x - x'|$
- Relative Error
  - $RE = \frac{\delta x}{x} = |\frac{x-x'}{x}|$
- Percentage Error
  - $PE = RE \times 100 = |\frac{x-x'}{x}| \times 100$

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- Percentage Error
  - $PE = RE \times 100 = |\frac{x-x'}{x}| \times 100$
- Which is better?

Errors in Addition & Subtraction

Theorem (Errors in Addition)

For variables $x_1 \pm \delta x_1$ and $x_2 \pm \delta x_2$ where δx_i is the maximum possible error. The maximum possible error when they are added is $(\delta x_1 + \delta x_2)$.

Proof.

Since δx_1 and δx_2 are $\pm$, the possible combinations on addition are $(\delta x_1 + \delta x_2)$, $(\delta x_1 - \delta x_2)$, $(-\delta x_1 + \delta x_2)$ and $(-\delta x_1 - \delta x_2)$. The extremes are the first and last. Hence the sum $\in \pm(\delta x_1 + \delta x_2)$ □

Theorem (Errors in Addition)

For variables $x_1 \pm \delta x_1$ and $x_2 \pm \delta x_2$ where δx_i is the maximum possible error. The maximum possible error when they are subtracted is $(\delta x_1 + \delta x_2)$.

Proof.

Same as addition □

Errors in Multiplication & Division

Theorem (Errors in Multiplication and Division)

The relative error in division and multiplication are the same.

Proof.

Consider $x_1 \pm \delta x_1$ and $x_2 \pm \delta x_2$. For multiplication, the absolute error is $(x_1 + \delta x_1)(x_2 + \delta x_2) - x_1 x_2 = x_1 \delta x_2 + x_2 \delta x_1$ (Note: $\delta x_1 \delta x_2$ tends to zero). The relative error is $\frac{x_1 \delta x_2 + x_2 \delta x_1}{x_1 x_2} = \frac{\delta x_1}{x_1} + \frac{\delta x_2}{x_2}$. For division, the absolute error is $\frac{x_1 + \delta x_1}{x_2 + \delta x_2} - \frac{x_1}{x_2} \equiv \frac{x_1 + \delta x_1}{x_2 + \delta x_2} \times \frac{x_2 - \delta x_2}{x_2 - \delta x_2} - \frac{x_1}{x_2}$ which simplifies to $\frac{x_2 \delta x_1 - x_1 \delta x_2}{x_2^2}$. The relative error is $\frac{x_2 \delta x_1 - x_1 \delta x_2}{x_2^2} \Big/ \frac{x_1}{x_2}$ which simplifies to
$\frac{\delta x_1}{x_1} - \frac{\delta x_2}{x_2} \equiv \frac{\delta x_1}{x_1} + \frac{\delta x_2}{x_2}$ since errors are $\pm$ □

Generic use of Binomial Theorem

- We have looked at errors in simple cases of addition/subtraction/multiplication/division
- We may need error expressions for fairly complex cases: roots, reciprocals, big powers (eg x^{25}), etc
- Generally we employ the binomial theorem for any index
- Approximate higer powers of δx to be very small

Theorem (Binomial theorem for any index n)

$$(1+p)^n = 1 + np + \frac{n(n-1)}{2!}p^2 + \frac{n(n-1)(n-2)}{3!}p^3 + \cdots\cdots$$

All we do is to write the error expression as $(1+\alpha)^k$ and simplify

Some samples

power 100

If the error in x is δx, the error in x^{100} can be got by $(x+\delta x)^{100} - x^{100}$. This can be rewritten as $x^{100}(1+\frac{\delta x}{x})^{100} - x^{100} = x^{100}((1+(\frac{\delta x}{x}))^{100} - 1)$. Expanding to $x^{100}\left[1 + 100(\frac{\delta x}{x}) + \frac{100\times 99}{2!}(\frac{\delta x}{x})^2 + \frac{100\times 99\times 98}{3!}(\frac{\delta x}{x})^3 \cdots - 1\right]$ and simplifying to $x^{100} \times 100(\frac{\delta x}{x}) \equiv 100x^{99}\delta x$

Try convince your self

That the error for the

- square root of x is $\frac{\delta x}{2\sqrt{x}}$
- reciprocal of x is $\frac{\delta x}{x^2}$
- square of the cuberoot of x is $\frac{2\delta x}{3\sqrt[3]{x}}$

Examples

Given the numbers $a = 2.3$, $b = 4.65$ and $c = 2.10$. a and b were generated by rounding while c by truncating. Find the maximum possile error in

1. $a + b$
2. $a + b - c$
3. $\frac{ac}{b+c}$
4. $(b - c)^{23}$

The maximum possible errors δa, δb and δc are 0.5×10^{-1}, 0.5×10^{-2} and 1×10^{-2} respectvely

1. Maximum error in $a + b$ $= 0.05 + 0.005 = 0.055$
2. Max error in $a + b - c$ $= 0.05+0.005+0.01 = 0.065$
3. Error in
   - multiplication $= ac(\frac{\delta a}{a} + \frac{\delta c}{c}) = 0.128$
   - addition $= \delta b + \delta c = 0.015$
   - division $= \frac{ac}{b+c} \times (\frac{\delta(ac)}{ac} + \frac{\delta(b+c)}{b+c}) =$ 0.026
4. Error in
   - $b - c = \delta b + \delta c = 0.015$
   - $(b - c)^{23} = 23(b - c)^{22}\delta(b - c) = 5.02 \times 10^7$

The End