MT 2106: Numerical Analysis - Intergration

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Numerical Integration

Analytically,

$$\int_a^b f(x)\mathrm{d}x = [F(x) + A]_a^b = F(b) - F(a), \text{where}$$

- $f(x)$ is the integrand,
- $F(x)$ the antiderivative/ primitive of $f(x)$ (or definite integral), and
- $F(b) - F(a)$ is a finite value of the definite integral

Numerical Integration

In numerical integration, the step $[F(x) + A]_a^b$ is skipped and $F(b) - F(a)$ is approximated directly

- This is applicable when the continuous function representing the antiderivative/ primitive cannot be easily obtained

Numerical Integration

Given the data:

x	x_0	x_1	...	x_n
y	y_0	y_1	...	y_n

$\int_{x_0}^{x_n} y \mathrm{d}x$ can be obtained using numerical techiques

- If $f(x)$ is fairly complicated and it is difficult/impossible to find $F(x)$, numerical integration can be used
- Numerical integration is the only choice if the data is in the form of table above, and the interest is to find $\int_{x_0}^{x_n} y \mathrm{d}x$

Numerical Integration

To approximate $\int_{x_0}^{x_n} y \, \mathrm{d}x$, the basic method of **numerical quadrature** is used

This uses the sum

$$\Sigma_{i=0}^{n} \; a_i f(x_i)$$

- Most of the methods of quadrature are based on the interpolation polynomials already discussed
- A set of distinct nodes $\{x_0, ..., x_n\}$ are selected from the interval [a, b]

Numerical Integration

- The Lagrange interpolating polynomial is integrated, together with its error term over the interval [a, b], implying

$$\int_a^b f(x)\,\mathrm{d}x \approx \Sigma_{i=0}^{n}\ a_i f(x_i)$$

$$\int_a^b f(x)\,\mathrm{d}x = \Sigma_{i=0}^{n}\ a_i f(x_i) + E_f$$

Techniques

Popular numerical techniques for approximating integrals include:

- Trapezoidal rule
- Simpson's rule

These are produced from the first and second Lagrange polynomials, with nodes equally spaced

Techniques

Others (not considered here) are:

- Quadrature rules
- Gauss-Laguerre
- Gauss-Hermite
- Gauss-Tshebychev
- Gauss-Quadrature
- Romberg Integration

Trapezoidal Rule

To find $\int_{x_0}^{x_n} f(x)\mathrm{d}x$, the area covered by $y = f(x)$, and the x-axis, $x = x_0$, $x = x_n$, is approximated, by subdividing it into a number of trapezia each with width h

- On each of the sub-intervals, we approximate $y = f(x)$ linearly by a straight line
- Error associated:

$$\frac{h^3}{12} f''(\xi), \text{ where } \xi \in [x_0, x_n]$$

Trapezoidal Rule

$$\int_a^b f(x)\,\mathrm{d}x = \frac{h}{2}[f(x_0) + f(x_1)] - \frac{h^3}{12} f''(\xi)$$

Error term for Trapezoidal rule involves term in f'', $\therefore$ the rule gives an exact result when applied to a function whose second derivative is zero, that is,

The Trapezoidal rule gives exact results for polynomials of degree one or less

Exercise

1. Evaluate $\int_1^7 f(x)\mathrm{d}x$ using the trapezoidal rule and:

x	1	2	3	4	5	6	7
y	2.105	2.808	3.614	4.604	5.857	7.45	9.817

2. Find the area below the curve $y = f(x)$ in the interval $[7.47, 7.52]$ given that:

x	7.47	7.48	7.49	7.5	7.51	7.52
y	1.93	1.95	1.98	2.01	2.03	2.06