MT 2106: Numerical Analysis - Langrange

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2015/2016 Sem II

Derivatives with Lagrange Polynomial

Consider the Lagrange polynomial:

$$f(x_j) = \Sigma_{k=0}^{n} \; L_k(x) f_k, \text{where } L_k(x) = \Pi_{j=0,\; j\neq k}^{n} \; \frac{(x - x_j)}{(x_k - x_j)}$$

To obtain general derivative formulae for $(n+1)$ distinct points, $\{x_0, x_1, ..., x_n\}$, the equation above is differentiated (ignoring the error term) to give :

$$f'(x_j) = \Sigma_{k=0}^{n} \; L_k'(x_j) f_k, \; \text{ the } (n+1) \text{ -point formula for } f'(x_j)$$

Derivatives with Lagrange Polynomial

The most common formulae are those involving

- three and
- five evaluation points

Note: Using more evaluation points produces greater accuracy, although the number of functional evaluations and growth of round-off error are a disadvantage.

Derivatives with Lagrange Polynomial

To derive the three point formulae; consider Lagrange's quadratic interpolating polynomial, with $n = 2$ for the points $\{x_0, x_1, x_2\}$

$$f(x_j) = \Sigma_{k=0}^{2} \; L_k(x) f_k, \text{where } L_k(x) = \Pi_{j=0,\; j \neq k}^{n} \; \frac{(x - x_j)}{(x_k - x_j)}$$

$$= (\frac{(x - x_1)}{x_0 - x_1})(\frac{(x - x_2)}{x_0 - x_2}) f_0 + (\frac{(x - x_0)}{x_1 - x_0})(\frac{(x - x_2)}{(x_1 - x_2)}) f_1 + (\frac{(x - x_0)}{x_2 - x_0})(\frac{(x - x_1)}{(x_2 - x_1)}) f_2$$

Derivatives with Lagrange Polynomial

Differentiating the respective kth Lagrange coefficient polynomials for each $j = 0, 1, 2$ gives:

$$f'(x_j) = L'_0(x_j)f_0 + L'_1(x_j)f_1 + L'_2(x_j)f_2$$

Giving:

$$f'(x_j) = \frac{(2x_j - x_1 - x2)}{(x_0 - x_1)(x_0 - x_2)}f_0 + \frac{(2x_j - x_0 - x2)}{(x_1 - x_0)(x_1 - x_2)}f_1 + \frac{(2x_j - x_0 - x1)}{(x_2 - x_0)(x_2 - x_1)}f_2$$

Derivatives with Lagrange Polynomial

Assume that the nodes are equally spaced with,

- $x_1 = x_0 + h$
- $x_2 = x_0 + 2h$, for some $h \neq 0$

Substituting in $f'(x_j)$ above,
With $x_j = x_0$, and x_1, x_2 as above:

$$f'(x_0) = \frac{1}{h}[-\frac{3}{2}f(x_0) + 2f(x_1) - \frac{1}{2}f(x_2)]$$

Derivatives with Lagrange Polynomial

For $x_j = x_1$,

$$f'(x_1) = \frac{1}{h}[-\frac{1}{2}f(x_0) + \frac{1}{2}f(x_2)]$$

And $x_j = x_2$ gives,

$$f'(x_2) = \frac{1}{h}[\frac{1}{2}f(x_0) - 2f(x_1) + \frac{3}{2}f(x_2)]$$

Since $x_1 = x_0 + h$ and $x_2 = x_0 + 2h$, the formulae can be expressed as:

Three-point formulae

$$f'(x_0) = \frac{1}{h}[-\frac{3}{2}f(x_0) + 2f(x_0 + h) - \frac{1}{2}f(x_0 + 2h)]$$

$$f'(x_0 + h) = \frac{1}{h}[-\frac{1}{2}f(x_0) + \frac{1}{2}f(x_0 + 2h)]$$

$$f'(x_0 + 2h) = \frac{1}{h}[\frac{1}{2}f(x_0) - 2f(x_0 + h) + \frac{3}{2}f(x_0 + 2h)]$$

Giving three formulae for approximating $f'(x_0)$

Three-point formulae

$$f'(x_0) = \frac{1}{2h}[-3f(x_0) + 4f(x_0 + h) - f(x_0 + 2h)]...\mathbf{I}$$

$$f'(x_0) = \frac{1}{2h}[-f(x_0 - h) + f(x_0 + h)]...\mathbf{II}$$

$$f'(x_0) = \frac{1}{2h}[f(x_0 - 2h) - 4f(x_0 - h) + 3f(x_0)]...\mathbf{III}$$

Three-point formulae

These formulae give the (n + 1) three point derivative formulae for:

- **I** - Forward difference with points x_0, $x_0 + h$, $x_0 + 2h$,
- **II** - Central difference with points x_0, $x_0 - h$, $x_0 + h$, and
- **III** - Backward difference with $x_0 - 2h$, $x_0 - h$, x_0

Considerations:

- The error in **II** is half the error in **I** and **III**, because equation **II** uses data on both sides of x_0, and the other two use data on only one side
- Also, **f** needs to be evaluated only twice in **II** compared to thrice in **I** and **III**

Example

Consider $f(x) = xe^x$ for $x = [1.8, 1.9, ..., 2.2]$

1. Use the formulae derived above to find $f'(2.0)$
2. Compare the solutions obtained with the exact value
   Hint: A table may be handy, useful for solution representation

Five-point Formulae

These are evaluated at two more points than the three-point formulae and derived in the same manner

Assignment: Derive any one of the five-point formulae

Note: The five-point formulae are superior than the three-point formulae, ignoring the error terms