Course Description

Description

- The course gives the students a strong mathematical base to be able to tackle other computer problems.
- The course provides techniques that are commonly used in the general area of computer science. It builds a foundation for other courses that need special mathematical backgrounds.

Aims

The aims of the course are:

- To provide students with a mathematical base that is to be used to solve computer science problems.
- To improve the problem solving skills of students.

Course content

- **Functions**
  - Polynomials (linear, quadratic) rational, composite functions
  - Transcendental functions: logarithmic and exponential functions;
  - The trigonometrical functions and their inverses
  - the hyperbolic functions and their inverses

- **Limits**
  - Informal definition of limits of functions and continuity;
  - one sided limits
  - removable discontinuity
  - Techniques and theorems of evaluating limits
  - Formal definition of limits
  - application to definition and properties of continuous functions
  - Use of the definition in proofs and problem of limits and continuity

Course content

- **Differentiation**

- Definition of a derivative, continuity and differentiability
- Rules and theorems of determining derivatives
- Inverse functions: their derivatives and graphs
- Differentials: applications to approximation. Rolle's theorem, Mean Value Theorem, L'Ohopital's Rule
- Anti-derivatives: Techniques and theorems for determining anti derivatives
- Integration: Define Integral, Riemann sums, the definite integral and area,
- The fundamental theorem of calculus: application to evaluation of definite integrals (by substitution)
- Functions defined by integration: f(t) dt as an anti derivative of f(x), mean value theorem for integrals

Reading material

Main Reference book:

1. Calculus 1 by Paul Dawkins.

Other resources:

2. Calculus and Analytic Geometry(9^{th} Edition) by George B. Thomas, Ross L. Finney Addison Wesley.

Assessment Method

- The assessment will be divided into two parts:
  - **3 Tests which will account for 40% of the final mark.**
  - Final Examination will be out of **60% which adds upto the 100%.**
- **NOTE:**
  - Overall Pass mark is **50%**
  - Several exercises/assignments will be given at the end of each chapter for practice purposes.

Calculus

- Branch in mathematics which focuses on limits, functions, derivatives, integrals and infinite series.
- Constitutes of two major branches:
    - Differential calculus
    - Integral calculus
- These two are related by the fundamental theorem of calculus.
- It specifies the relationship between the two central operations of calculus that is differentiation and integration.

Functions

- What is a function?

An equation will be a function if for any x in the domain of the equation will yield exactly one value of y.

- ***Example 1*** Determine if each of the following are functions.

(i) $y = x^2 + 1$

(ii) $y^2 = x + 1$

Solution

- **(i)** This is a function. Given an *x*, there is only one way to square it and then add 1 to the result. So, no matter what value of *x* you put into the equation, there is only one possible value of *y*.

Solution

(ii) This isn't a function since the exponent is onto the y.

- Let $x=3$ and plug this into the equation $y^2=3+1=4$
- $\Rightarrow$ two possible values of y *i.e* $y = 2$ or $y = -2$.
- Therefore this isn't a function since there are two possible values of y that we get from a single x.

Solution

- Note that this only needs to be the case for a single value of x to make an equation not be a function.
- For instance we could have used $x=-1$ and in this case we would get a single y ($y=0$).
- However, because of what happens at $x=3$ this equation will not be a function.

Function Notation

- Given the following function

$$y=2x^2-5x+3$$

- Using function notation we can write this as any of the following.

$$y(x)=2x^2-5x+3$$

$$g(x)=2x^2-5x+3$$

$$f(x)=2x^2-5x+3$$

Function Notation

- Function notation gives us a nice compact way of representing function values.
- $f(-3)$ *will* represent the value of the function at $x=-3$.

Polynomial Functions

- ***Definitions***
- Polynomial function in one variable of degree n
  - A function with one variable raised to whole number powers (the largest being n) and with real coefficients.
  - The standard form is $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$, $a_n \neq 0$
- Constant function
  - A polynomial function in one variable of degree 0.
  - Polynomial form: $f(x)=a_0$
  - Standard form: $f(x) = c$

Polynomial Functions

- Linear function
  - A polynomial function in one variable of degree 1.
  - Polynomial form: $f(x)= a_1x + a_0$
  - Standard form: $f(x) = ax + b$
- Quadratic function
  - A polynomial function in one variable of degree 2.
  - Polynomial form: $f(x)= a_2x^2 + a_1x + a_0$
  - Standard form : $f(x) = ax^2 + bx + c$
  - Vertex form : $f(x) = a(x-h)^2 + k$

Quadratic Functions

- Parabola
  - The graph of a quadratic function
- Axis of symmetry (for a parabola)
  - The line of symmetry through the center of the parabola
- Vertex
  - The intersection of the axis of symmetry and the parabola. It will be the minimum point on the graph if $a>0$ and the maximum point on the graph if $a<0$.

Quadratic Functions

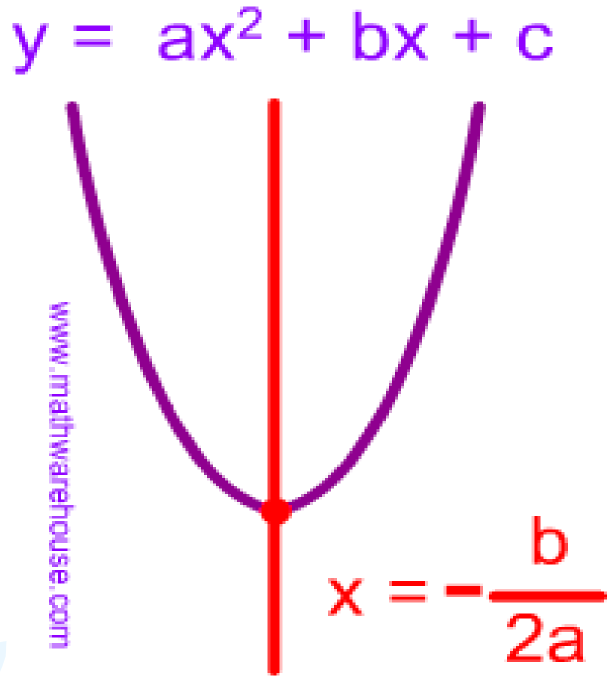
- The standard form of a parabola's equation is generally expressed:

  $f(x) = ax^2 + bx + c$

  - The role of 'a'
    - If a> 0, the parabola opens upwards
    - if a< 0, it opens downwards.
  - The axis of symmetry is the line $x = -b/2a$

Quadratic Functions

Axis of Symmetry from Standard Form



Domain of a function

- ***Domain:*** set of all values that can be plugged into a function and have the function exist and have a real number for a value.
- So, for the domain we need to avoid division by zero, square roots of negative numbers, logarithms of zero and negative numbers.

Range of a function

- Range: set of all possible values that a function can take.

Domain & Range of a function

Example 2 Find the domain and range of each of the following functions.

(a) $f(x)=5x-3$

(b) $g(t)=\sqrt{4-7t}$

(c) $h(x)=-2x^2+12x+5$

(d) $f(z)=|z-6|-3$

(e) $g(x)=8$

Domain & Range of a function

- ***Solution***

(a) $f(x)=5x-3$

- We know that this is a line and that it's not a horizontal line (because the slope is 5 and not zero…).
- Which implies that this function can take on any value and so the range is all real numbers.

Range : $(-\infty,\infty)$

Domain & Range of a function

- This is more generally a polynomial and we know that we can plug any value into a polynomial.
- Therefore the domain is all real numbers.

Domain : $-\infty < x < \infty$ or $(-\infty,\infty)$

Domain & Range of a function

- **(b)** $g\,(t\,) = \sqrt{4 - 7t}$
- This is a square root and we know that square roots are always positive or zero and because we can have the square root of zero in this case,
- $g(4/7) = \sqrt{4 - 7(4/7)} = \sqrt{0} = 0$
- Therefore the range will be,

Range : $[0,\infty)$

Domain & Range of a function

- For the domain require that,

$$4 - 7t \geq 0$$

$$4 \geq 7t$$

$$4/7 \geq t \quad \Rightarrow \quad t \leq 4/7$$

Therefore the domain is then,

Domain : $t \leq 4/7$ or $(-\infty, 4/7]$

Domain & Range of a function

(c) $h(x) = -2x^2+12x+5$

- Here we have a quadratic which is a polynomial and so we again know that the domain is all real numbers.

  Domain : $-\infty < x < \infty$ or $(-\infty,\infty)$

Domain & Range of a function

- Since the coefficient of x^2 is negative therefore the parabola will open downwards implying that the vertex will be the highest point on the graph.

$$x = -b/2a = -\frac{12}{2(-2)} = 3 \qquad y = -2(3)^2 + 12(3) + 5 = 23$$

$\Rightarrow$ The vertex coordinates are (3,23) and this is the highest point on the graph therefore the range will be

$$\text{Range}: (-\infty, 23]$$

Review questions

- Find the domain of each of the following functions.

(a) $f(x) = \dfrac{x-4}{x^2-2x-15}$

(b) $g(t) = \sqrt{6+t-t^2}$

(c) $h(x) = \dfrac{x}{\sqrt{x^2-9}}$

Composite Functions

- Given two functions $f(x)$ and $g(x)$, *then*

  $(f \circ g)(x) = f(g(x))$ forms a composite function of f of g.
- Note that the order is very important.

Composite Functions

- ***Example 3*** Given $f(x)=3x^2-x+10$ and $g(x)=1-20x$ find each of the following.

(a) $(f \circ g)(5)$

(b) $(f \circ g)(x)$

(c) $(g \circ f)(x)$

(d) $(g \circ g)(x)$

Composite Functions

- ***Solution***

(a) $(f \circ g)(5)$

- In this case we've got a number 5 in the place of x which implies that we insert 5 wherever there is x in the function g(x).
- $(f \circ g)(5) = f(g(5))$

$$= f(-99) = 29512$$

Composite Functions

- **(d)** $(g \circ g)(x)$

$$
\begin{aligned}
(g \circ g)(x) &= g(g(x)) \\
&= g(1-20x) \\
&= 1-20(1-20x) \\
&= 400x-19
\end{aligned}
$$

Composite Functions

- ***Example 4*** Given $f(x) = 3x - 2$ and $g(x) = 1/3x + 2/3$ find each of the following.

(a) $(f \circ g)(x)$

(b) $(g \circ f)(x)$

Composite Functions

- ***Solution***

(a) $(f \circ g)(x) = f(g(x))$

$= f(1/3x + 2/3)$

$= 3(1/3x + 2/3) - 2$

$= x + 2 - 2 = x$

(b) $(g \circ f)(x) = g(f(x))$

$= g(3x - 2)$

$= 1/3(3x - 2) + 2/3$

$= x - 2/3 + 2/3 = x$

Composite Functions

- In this case the two compositions where the same i.e

  $(f \circ g)(x)=(g \circ f)(x)=x$

  Which implies that there is a relationship between the two functions.

Inverse Functions

- ***Definition:*** Given two one-to-one functions $f(x)$ and $g(x)$,

  if $(f \circ g)(x) = x$ AND $(g \circ f)(x) = x$

  then we say that $f(x)$ and $g(x)$ are **inverses** of each other.
- A function is called **one-to-one** if no two values of x produce the same y. i.e $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$

Inverse Functions

- More specifically we will say that $g(x)$ is the **inverse** of $f(x)$ and denote it by

$$g(x) = f^{-1}(x)$$

- Likewise $f(x)$ is the **inverse** of $g(x)$ and denote it by

$$f(x) = g^{-1}(x)$$

- Note that $f^{-1}(x) \neq \dfrac{1}{f(x)}$

Inverse Functions

- From the previous example of two functions $f(x)=3x-2$ and

$g(x)=1/3x+2/3$,we saw that

$$(f \circ g)(x)=(g \circ f)(x)=x$$

Considering the following evaluations.

a) $f(-1)=3(-1)-2=-5$, $g(-5)=-5/3+2/3=-1$

b) $g(-1)=-1/3+2/3=1/3$, $f(1/3)=3(1/3)-2=-1$

Inverse Functions

- From the two evaluations it implies that the two functions are inverses of each other.
- From **a)** $\boldsymbol{f^{-1}(x)=g(x)=1/3x+2/3}$
- From **b)** $\boldsymbol{g^{-1}(x)= f(x)=3x-2}$

Finding the Inverse of a function

- Given the function $f(x)$ we want to find the inverse function, $f^{-1}(x)$.

1. First, replace $f(x)$ with y.
2. Replace every x with a y and replace every y with an x.
3. Solve the equation from Step 2 for y.
4. Replace y with $f^{-1}(x)$.
5. Verify by checking that $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$ are both true.

Inverse Functions

- ***Example 1*** Given $f(x) = 3x-2$ find $f^{-1}(x)$.
- ***Solution***

We'll first replace $f(x)$ with y.

$$y = 3x - 2$$

Replace all x's with y and all y's with x.

$$x = 3y - 2$$

Finding the Inverse of a function

- solve for *y*.

  $x+2=3y$

  $y=x/3+2/3$

- Finally replace *y* with $f^{-1}(x)$.

  $f^{-1}(x)=x/3+2/3$

- Lastly we verify that

  $(f \circ f^{-1})(x)=x$ *and* $(f^{-1} \circ f)(x)=x$

Inverse Functions

$(f \circ f^{-1})(x) = f(x/3+2/3)$

$=3[x/3+2/3]-2$

$=x$

$(f^{-1} \circ f)(x) = f^{-1}(3x-2)$

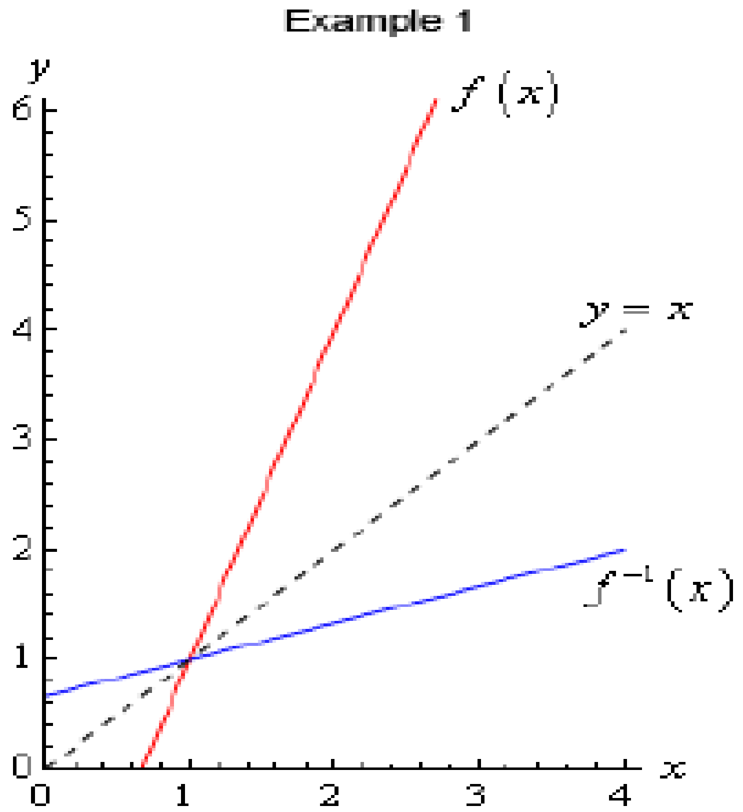
$=[3x-2]/3+2/3$

$= x$

- Therefore

$f^{-1}(x)=x/3+2/3$

Inverse Functions



Inverse Functions

- We can see that the graph of the inverse is a reflection of the actual function about the line $y = x$.
- This will always be the case with the graphs of a function and its inverse.

Inverse Functions

- ***Example 2*** Given $g(x)=\sqrt{x-3}$,find $g^{-1}(x)$.
- ***Example 3*** Given $h(x)=\dfrac{x+4}{2x-5}$,find $h^{-1}(x)$.