How to Use the Bisection Algorithm

Quick Overview

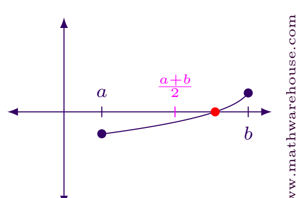
- The bisection algorithm approximates the location of an x-intercept (a root) of a continuous function.
- The bisection algorithm depends on the Intermediate Value Theorem.
- The algorithm is *iterative*. This means that the result from using it once will help us get a better result when we use the algorithm a second time.

Basic Idea

Suppose $f(x)$ is continuous over $[a, b]$, and $f(a)$ and $f(b)$ have opposite signs (see the image below). Then, the Intermediate Value Theorem tells us that the function will achieve every value between $f(a)$ and $f(b)$ at least once somewhere in $[a, b]$.

Since $f(a)$ and $f(b)$ have opposite signs, then we know 0 is somewhere in-between. So the IVT guarantees that somewhere in $[a, b]$ the function will equal 0 (again, see the image below).

We approximate the location of the root by finding the midpoint of the interval at $x = \frac{a+b}{2}$ (see image below).



The function crosses the x-axis between a and b

The Algorithm

Suppose $f(x)$ is continuous over $[a, b]$ and the function values at the endpoints have different signs.

1. Find the midpoint of $[a, b]$. Call it x_1.
   (a) If $f(x_1) = 0$, we're done.
   (b) If not, then x_1 is our first approximation to the root of the function. It has a maximum error of $\frac{1}{2}$ the length of the interval.
2. Find a smaller interval where $f(x)$ has opposite signs at the endpoints. This new interval will either be $[a, x_1]$, or $[x_1, b]$.
3. Using this smaller interval, repeat Steps 1 and 2 until the error is small enough.

Example 1

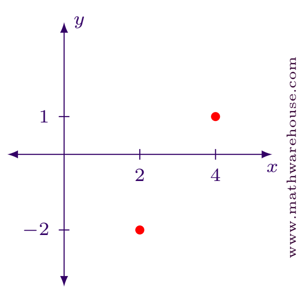
Using the Bisection Algorithm, find three approximations of the root of $f(x) = \frac{1}{4}x^2 - 3$. Determine the maximum error possible in using each approximation.

Solution

Step 1) Verify the Bisection Algorithm can be used.

We first note that the function is continuous everywhere on it's domain.

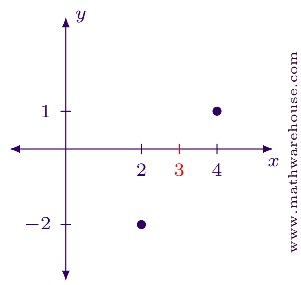
Next, we pick an interval to work with. If we pick $x = 2$, we see that $f(0) = -2 < 0$ and if we pick $x = 4$ we see $f(4) = 1 > 0$. So we can start with the interval $[2, 4]$.



1st interval for Example 1

Step 2) Find the first approximation to the root and its associated error.

The first approximation to the root is the midpoint of our starting interval. In this case, the midpoint of $[2, 4]$ is at $x = 3$.



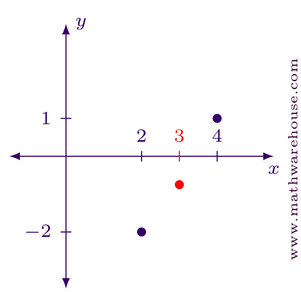
1st approximation of the root

Maximum Error: Since the root has to be between $x = 2$ and $x = 4$, using $x = 3$ as an approximation for the root means the farthest away the root could possibly be is a distance of ± 1 unit (the plus/minus is because our approximation could be too big or too small).

In general, the maximum error in using a particular approximation is half the interval length.

Step 3) Use the midpoint to find a smaller interval so we can improve our approximation.

Notice that $f(3) = \frac{1}{4}(3)^2 - 3 = -\frac{3}{4} < 0$. Updating our graph, we now have three points on it.



Finding a smaller interval

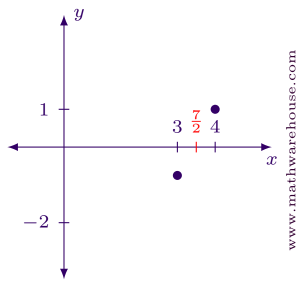
Examining this graph, we see that the root must lie between $x = 3$ and $x = 4$. Consequently, this is our new interval.

Step 4) Find the second approximation and its associated error.

The midpoint of the interval $[3, 4]$ is at $x = \frac{7}{2}$. This is the second approximation.

Max Error: Our interval has a length of 1 unit, so the maximum possible error in using $x = \frac{7}{2}$ as an approximation will be $\pm 1/2$ of a unit.

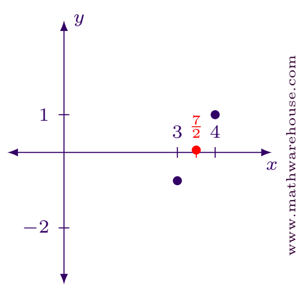
Our graph now looks like this:



The second approximation

Step 5) Find the next interval.

We note that $f\left(\frac{7}{2}\right) = \frac{1}{16} > 0$. Plotting this on our graph we see the following.



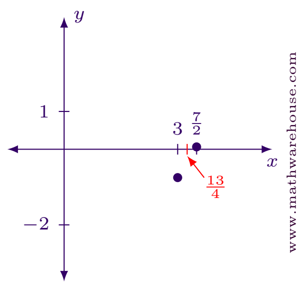
Finding a smaller interval

Examining this graph, we see that the root must lie between $x = 3$ and $x = \frac{7}{2}$. This becomes our next interval.

Step 6) Find the third approximation and its associated error.

The midpoint of the interval $\left[3, \frac{7}{2}\right]$ is at $x = \frac{13}{4}$, as shown on the graph below. This is the next approximation.

Max Error: The interval has a length of $1/2$, so the maximum possible error is $\pm 1/4$ of a unit.



The third approximation

Answer The table below summarizes the approximations we found and their associated errors.

Approximation	x-value	Possible Error
1^{st}	$x = 3$	± 1
2^{nd}	$x = \frac{7}{2}$	$\pm\frac{1}{2}$
3^{rd}	$x = \frac{13}{4}$	$\pm\frac{1}{4}$

Example 2

Use the bisection algorithm to approximate the solution to the equation below to within less than 0.1 of its real value. Assume x is in radians.

$$\sin x = 6 - x$$

Solution

Step 1) Rewrite the equation so it is equal to 0.

$$x - 6 + \sin x = 0$$

The function we'll work with is $f(x) = x - 6 + \sin x$. Notice that the function is continuous everywhere.

Step 2) Find an initial interval to work with.

Setting up a table of values, we see the following.

x	$f(x)$
0	$f(0) = -6$
1	$f(1) \approx -4.2$
2	$f(2) \approx -3.1$
3	$f(3) \approx -2.9$
4	$f(4) \approx -2.8$
5	$f(5) \approx -2$
6	$f(6) \approx -0.3$
7	$f(7) \approx 1.7$

The first time we see a positive function value is at $x = 7$. So, we can use $[6, 7]$ as the initial interval.

Step 3) Find the first approximation and its associated error.

Interval	Midpoint	Max Error
$[6, 7]$	6.5	± 0.5

1st Approximation: $x = 6.5$

Associated error: ± 0.5 units.

Step 4) Find the 2nd interval. Use this new interval to determine the 2nd approximation. The first line of the table is included for completeness. The new work is on the second line.

Finding the New Interval				Next Approximation	
$f(\text{left})$	$f(\text{mid})$	$f(\text{right})$	New Interval	Midpoint	Max Error
		Starting Interval:	$[6, 7]$	6.5	± 0.5
$f(6) \approx -0.28$	$f(6.5) \approx 0.72$	$f(7) \approx 1.66$	$[6, 6.5]$	6.25	± 0.25

2nd Interval: $[6, 6.5]$

2nd Approximation: $x = 6.25$

Associated Error: ± 0.25 units.

Step 5) Repeat Step 4 until the associated error is less than 0.1 units.

Finding the 3rd Approximation

f(left)	f(mid)	f(right)	New Interval	Midpoint	Max Error
			Finding the New Interval	**Next Approximation**	
		Starting Interval:	$[6, 7]$	6.5	± 0.5
$f(6) \approx -0.28$	$f(6.5) \approx 0.72$	$f(7) \approx 1.66$	$[6, 6.5]$	6.25	± 0.25
$f(6) \approx -0.28$	$f(6.25) \approx 0.22$	$f(6.5) \approx 0.72$	$[6, 6.25]$	6.125	± 0.125

3rd Interval: $[6, 6.25]$

3rd Approximation: $[6, 6.125]$

Associated Error: ± 0.125 units.

Finding the 4th Approximation

f(left)	f(mid)	f(right)	New Interval	Midpoint	Max Error
			Finding the New Interval	**Next Approximation**	
		Starting Interval:	$[6, 7]$	6.5	± 0.5
$f(6) \approx -0.28$	$f(6.5) \approx 0.72$	$f(7) \approx 1.66$	$[6, 6.5]$	6.25	± 0.25
$f(6) \approx -0.28$	$f(6.25) \approx 0.22$	$f(6.5) \approx 0.72$	$[6, 6.25]$	6.125	± 0.125
$f(6) \approx -0.28$	$f(6.125) \approx -0.03$	$f(6.25) \approx 0.22$	$[6.125, 6.25]$	6.1875	± 0.0625

4th Interval: $[6, 6.25]$

4th Approximation: $[6, 6.125]$

Associated Error: ± 0.125 units.

Answer The solution to the equation is approximately 6.1875. This approximation is accurate to within ± 0.0625 units.

Example 3

Find the third approximation from the bisection algorithm to approximate the value of $\sqrt[3]{2}$.

Solution

Step 1) Find (make) a non-linear function with a root at $\sqrt[3]{2}$.

We are interested in knowing the approximate value of $x = \sqrt[3]{2}$. We use this equation to build a non-linear function with a root at the appropriate value.

$$\begin{aligned} x &= \sqrt[3]{2} \\ x^3 &= 2 \\ x^3 - 2 &= 0 \end{aligned}$$

We'll use the function $f(x) = x^3 - 2$. Notice that the function is continuous everywhere.

Step 2) Find a starting interval.

We notice that at $x = 0$, the function is negative, and at $x = 2$ the function is positive. We'll use $[0, 2]$ as our starting interval.

Step 3) Setup and work through the table as in the previous example. The approximations are in blue. The new intervals are in red.

Finding the New Interval				Next Approximation	
$f(\text{left})$	$f(\text{mid})$	$f(\text{right})$	New Interval	Midpoint	Max Error
	Starting Interval:		$[0,2]$	1	± 1
$f(0) = -2$	$f(1) = -1$	$f(2) = 6$	$[1,2]$	1.5	± 0.5
$f(1) = -1$	$f(1.5) \approx 1.4$	$f(2) = 6$	$[1, 1.5]$	1.25	± 0.25

Answer $\sqrt[3]{2} \approx 1.25$ with a possible error of ± 0.25. (Even with only 3 approximations, we're pretty close! The calculator tells us $\sqrt[3]{2} \approx 1.25992$)

Accuracy and Iterations

We know the first approximation is within $0.5(b - a)$ of the actual value of the root. Similarly,

the second approximation is within $0.5^2(b - a)$ of the actual value, and

the third approximation is within $0.5^3(b - a)$ of the actual value.

the fourth approximation is within $0.5^4(b - a)$ of the actual value.

In general, the n^{th} approximation will be within $0.5^n(b - a)$ of the actual value. This allows us to determine ahead of time how many iterations are needed to achieve a desired degree of accuracy, as in the following example.

Example 4

The function $f(x) = x^4 - 5$ has a positive root that is less than 3. If we started the bisection algorithm with the interval $[0, 3]$, how many iterations would it take before our approximation is within 10^{-4} of the actual value?

Solution

Step 1) Solve $0.5^n(b - a) = 0.01$ for n when $a = 0$ and $b = 3$.

$$\begin{aligned}
0.5^n(3 - 0) &= 10^{-4} \\
0.5^n(3) &= 10^{-4} \\
0.5^n &= \frac{10^{-4}}{3} \\
n\ln(0.5) &= \ln\left(\frac{1}{30000}\right) \\
-n\ln 2 &= -\ln(30000) \\
n &= \frac{\ln 30000}{\ln 2} \\
&\approx 14.87
\end{aligned}$$

We would need at least 15 iterations to ensure the accuracy desired.

Questions

Question 1 Find the 4th approximation of the positive root of the function $f(x) = x^4 - 7$ using the bisection algorithm.

Question 2 Find the third approximation of the root of the function $f(x) = \frac{1}{2}x - \sqrt[3]{x+1}$ using the bisection algorithm.

Question 3 Approximate the negative root of the function $f(x) = x^2 - 7$ to within 0.1 of its actual value.

Question 4 Approximate the value of the root of $f(x) = -3x^3 + 5x^2 + 14x - 16$ near $x = 3$ to within 0.05 of its actual value.

Question 5 Find the 4th approximation to the solution of the equation below using the bisection algorithm.

$$x^2 - x - 2 = x$$

Question 6 Find the 5th approximation to the solution to the equation below, using the bisection algorithm.

$$x^4 - 2 = x + 1$$

Question 7 The equation below should have a solution that is larger than 5. Use the bisection algorithm to approximate this solution to within 0.1 of its actual value.

$$x^3 + 18x - 6 = 9x^2 - 2x + 7$$

Question 8 The only real solution to the equation below is negative. Approximate the value of this solution to within 0.05 units of its actual value.

$$x^3 + 7x^2 - 2 = 2x^2 - 7x - 7$$

Question 9 Use the bisection algorithm to approximate the value of $\sqrt{71}$. Find the 4th approximation.

Question 10 Use the bisection algorithm to approximate the value of $\frac{1}{\sqrt[5]{3}}$. Find the 3rd approximation.

Question 11 Use the bisection algorithm to approximate the value of $\sqrt{125}$ to within 0.125 units of the actual value.

Question 12 Use the bisection algorithm to approximate the value of $\frac{\sqrt[4]{12500}}{2}$ to within 0.1 units of the actual value.

Question 13 Suppose we used the bisection method on $f(x)$, with an initial interval of $[2, 5]$. How many iterations would it take before the maximum error would be less than 0.01 units?

Question 14 Suppose we used the bisection method on $f(x)$, with an initial interval of $[-1, 1]$. How many iterations would it take before the maximum error would be less than 0.02 units?

Answers

Question 1 Solution

Step 1) Since the function is continuous everywhere, find an appropriate starting interval.

Then, notice that $f(1) = -6 < 0$, but $f(2) = 9 > 0$. Let's use $[1, 2]$ as the starting interval.

Step 2) Set up and use the table of values as in the examples above. The approximations are in blue, the new intervals are in red.

Finding the New Interval				Next Approximation	
$f(\text{left})$	$f(\text{mid})$	$f(\text{right})$	New Interval	Midpoint	Max Error
		Starting Interval:	$[1, 2]$	1.5	± 0.5
$f(1) = -6$	$f(1.5) \approx -2$	$f(2) = 9$	$[1.5, 2]$	1.75	± 0.25
$f(1.5) \approx -2$	$f(1.75) \approx 2.4$	$f(2) = 9$	$[1.5, 1.75]$	1.625	± 0.125
$f(1.5) \approx -2$	$f(1.625) \approx -0.03$	$f(1.75) \approx 2.4$	$[1.625, 1.75]$	1.6875	± 0.0625

Answer The positive root of $f(x) = x^4 - 7$ is at approximately $x = 1.6875$. This approximation is off by at most ± 0.0625 units.

Question 2 Solution

Step 1) Since the function is continuous everywhere, determine an appropriate starting interval.

Set up a table of values to help us find an appropriate interval.

x	$f(x)$
0	$f(0) = -1$
1	$f(1) \approx -0.8$
2	$f(2) \approx -0.4$
3	$f(3) \approx -0.1$
4	$f(4) \approx 0.3$

This table indicates the root is between $x = 3$ and $x = 4$, so a good starting interval is $[3, 4]$

Step 2) Set up and use a table to track the appropriate values.

	Finding the New Interval			Next Approximation	
$f(\text{left})$	$f(\text{mid})$	$f(\text{right})$	New Interval	Midpoint	Max Error
		Starting Interval:	$[3, 4]$	3.5	± 0.5
$f(3) \approx -0.1$	$f(3.5) \approx 0.1$	$f(4) \approx 0.3$	$[3, 3.5]$	3.25	± 0.25
$f(3) \approx -0.1$	$f(3.25) \approx 0.01$	$f(3.5) \approx 0.1$	$[3, 3.25]$	3.125	± 0.125

Answer The function has a root at approximately $x = 3.125$ with a maximum possible error of ± 0.125 units.

Question 3 Solution

Step 1) Since the function is continuous everywhere, determine an appropriate starting interval.

At $x = -2$ the function value is $f(-2) = -3$, and at $x = -3$ the function value is $f(-3) = 2$. We'll use $[-3, -2]$ as the starting interval.

Step 2) Determine the first approximation and the maximum possible error in using it.

1st Approximation: The midpoint of the first interval is $x = -2.5$.

Associated Error: ± 0.5 units.

Step 3) Determine the second interval, second approximation and the associated maximum error.

Finding the 2nd Interval

	x	$f(x)$
Current left-endpoint	-3	$f(-3) = 2$
Midpoint	-2.5	$f(-2.5) \approx -0.8$
Current right-endpoint	-2	$f(-2) = -3$

Second Interval: $[-3, -2.5]$

Second Approximation: The midpoint of the second interval is $x = -2.75$.

Associated Error: ± 0.25 units

Step 4) Repeat Step 3 with the new interval. Continue to repeat until the maximum error is less than 0.1.

Finding the 3rd Interval

	x	$f(x)$
Current left-endpoint	-3	$f(-3) = 2$
Midpoint	-2.75	$f(-2.75) \approx 0.6$
Current right-endpoint	-2.5	$f(-2.5) \approx -0.8$

Third interval: $[-2.75, -2.5]$.

Third approximation: The midpoint is $x = -2.625$

Associated error: ± 0.125 units.

Finding the 4th Interval

	x	$f(x)$
Current left-endpoint	-2.75	$f(-2.75) \approx 0.6$
Midpoint	-2.625	$f(-2.625) \approx -0.1$
Current right-endpoint	-2.5	$f(-2.5) \approx -0.8$

4th interval: $[-2.75, -2.625]$.

4th approximation: Midpoint is at $x = -2.6875$

Associated error: ± 0.0625 units.

Answer The negative root of the function is at approximately $x = -2.6875$ with a maximum error of only ± 0.0625 units.

Question 4 Solution

Step 1) Determine an appropriate starting interval, the first approximation and its associated maximum error value.

First, notice that the function is continuous everywhere. Then, since we're told that the root is near $x = 3$ we can check that $f(3) = -10$.

Checking $x = 4$ we find that $f(4) = -72$, but at $x = 2$ the function value is $f(2) = 8$.

First Interval: $[2, 3]$

First Approximation: The midpoint is at $x = 2.5$

Associated error: ± 0.5 units

Step 2) Determine the second interval, the second approximation and the associated maximum error.

Finding the 2nd Interval

	x	$f(x)$
Current left-endpoint	2	$f(2) = 8$
Midpoint	2.5	$f(2.5) \approx 3$
Current right-endpoint	3	$f(3) = -10$

Second Interval: $[2.5, 3]$

Second Approximation: The midpoint is at $x = 2.75$

Associated Error: ± 0.25 units.

Step 3) Repeat Step 2 until the maximum possible error is less than 0.05 units.

Finding the 3rd Interval

	x	$f(x)$
Current left-endpoint	2.5	$f(2.5) \approx 3$
Midpoint	2.75	$f(2.75) \approx -2$
Current right-endpoint	3	$f(3) = -10$

Third interval: $[2.5, 2.75]$.

Third approximation: The midpoint is at $x = 2.625$

Associated error: ± 0.125 units.

Finding the 4th Interval

	x	$f(x)$
Current left-endpoint	2.5	$f(2.5) \approx 3$
Midpoint	2.625	$f(2.625) \approx 0.9$
Current right-endpoint	2.75	$f(2.75) \approx -2$

4th interval: $[2.625, 2.75]$

4th approximation: The midpoint is $x = 2.6875$.

Associated error of ± 0.0625 units.

Finding the 5th Interval

	x	$f(x)$
Current left-endpoint	2.625	$f(2.625) \approx 0.9$
Midpoint	2.6875	$f(2.6875) \approx -0.5$
Current right-endpoint	2.75	$f(2.75) \approx -2$

5th interval: $[2.625, 2.6875]$.

5th approximation: The midpoint is $x = 2.65625$

Associated error: ± 0.03125 units.

Answer The root of the function is approximately $x = 2.65625$ and has an associated maximum error of only ± 0.03125 units.

Question 5 Solution

Step 1) Identify the function by getting the equation equal to zero.

$$x^2 - 2x - 2 = 0$$

The function we'll use is $f(x) = x^2 - 2x - 2$.

Step 2) Determine an appropriate starting interval, the first approximation and its associated maximum error.

At $x = 0$ the function value is $f(0) = -2$, while at $x = 3$ the function value is $f(3) = 1$.

1st Interval: $[0, 3]$

1st Approximation: The midpoint is at $x = 1.5$.

Associated Error: ± 1.5 units.

Step 3) Determine the second interval, second approximation and its associated maximum error.

Finding the 2nd Interval

	x	$f(x)$
Current left-endpoint	0	$f(0) = -2$
Midpoint	1.5	$f(1.5) = -2.75$
Current right-endpoint	3	$f(3) = 1$

2nd Interval: $[1.5, 3]$

2nd Approximation: $x = 2.25$

Associated Error: ± 0.75 units

Step 4) Repeat Step 3 twice to complete the iterations of the bisection method for this question.

Finding the 3rd Interval

	x	$f(x)$
Current left-endpoint	1.5	$f(1.5) = -2.75$
Midpoint	2.25	$f(2.25) \approx -1.4$
Current right-endpoint	3	$f(3) = 1$

3rd Interval: $[2.25, 3]$

3rd Approximation: $x = 2.625$

Associated Error: ± 0.375 units

Finding the 4th Interval

	x	$f(x)$
Current left-endpoint	2.25	$f(2.25) = -1.4375$
Midpoint	2.625	$f(2.625) = -0.359375$
Current right-endpoint	3	$f(3) = 1$

4th Interval: $[2.625, 3]$

4th Approximation: $x = 2.8125$

Associated Error: ± 0.1875 units

Answer The solution to the equation is approximately $x = 2.8125$ with a maximum error of 0.1875 units.

Question 6 Solution

Step 1) Identify the function we will use by rewriting the equation so it is set equal to zero.

$$x^4 - x - 3 = 0$$

The function we will use is $f(x) = x^4 - x - 3$.

Step 2) Identify the first interval, the first approximation and its associated maximum error.

Notice that at $x = 0$ the function value is $f(0) = -3$.

Also, at $x = 2$ the function value is $f(2) = 11$.

1st Interval: $[0, 2]$

1st Approximation: $x = 1$

Associated Error: ± 1 unit

Step 3) Identify the 2nd interval, approximation and associated error.

Finding the 2nd Interval

	x	$f(x)$
Current left-endpoint	0	$f(0) = -3$
Midpoint	1	$f(1) = -3$
Current right-endpoint	2	$f(2) = 11$

2nd Interval: $[1, 2]$

2nd Approximation: $x = 1.5$

Associated Error ± 0.5 units

Step 4) Repeat Step 3 until you've found the 5th approximation.

Finding the 3rd Interval

	x	$f(x)$
Current left-endpoint	1	$f(1) = -3$
Midpoint	1.5	$f(1.5) \approx 0.6$
Current right-endpoint	2	$f(2) = 11$

3rd Interval: $[1, 1.5]$

3rd Approximation: $x = 1.25$

Associated Error: ± 0.25 units

Finding the 4th Interval

	x	$f(x)$
Current left-endpoint	1	$f(1) = -3$
Midpoint	1.25	$f(1.25) \approx -1.8$
Current right-endpoint	1.5	$f(1.5) \approx 0.6$

4th Interval: $[1.25, 1.5]$

4th Approximation: $x = 1.375$

Associated Error: ± 0.125 units

Finding the 5th Interval

	x	$f(x)$
Current left-endpoint	1.25	$f(1.25) \approx -1.8$
Midpoint	1.375	$f(1.375) \approx -0.8$
Current right-endpoint	1.5	$f(1.5) \approx 0.6$

- 5th Interval: $[1.375, 1.5]$
- 5th Approximation: $x = 1.4375$
- Associated Error: ± 0.0625 units

Answer The solution to the equation is approximately $x = 1.4375$. This approximation has an maximum error of at most 0.0625 units.

Question 7 Solution

Step 1) Identify the function we'll use by rewriting the equation so it is equal to zero.

$$x^3 - 9x^2 + 20x - 13 = 0$$

The function is $f(x) = x^3 - 9x^2 + 20x - 13$.

Step 2) Determine the first interval, 1st approximation, and its associated error.

We know the solution is larger than 5, but we don't know how much larger. We set up a small table of values to help us out.

x	$f(x)$
5	-13
6	-1
7	29

From this table we can select the first interval and determine the first approximation.

1st Interval: $[6, 7]$

1st Approximation: $x = 6.5$

Associated Error: ± 0.5 units

Step 3) Find the second interval, second approximation and the associated error.

Finding the 2nd Interval

	x	$f(x)$
Current left-endpoint	6	$f(6) = -1$
Midpoint	6.5	$f(6.5) \approx 11.4$
Current right-endpoint	7	$f(7) = 29$

2nd Interval: $[6, 6.5]$

2nd Approximation $x = 6.25$

Associated Error: ± 0.25 units

Step 4) Repeat Step 3 until the maximum error is less than the given tolerance of 0.1.

Finding the 3rd Interval

	x	$f(x)$
Current left-endpoint	6	$f(6) = -1$
Midpoint	6.25	$f(6.25) \approx 4.6$
Current right-endpoint	6.5	$f(6.5) \approx 11.4$

3rd Interval: $[6, 6.25]$

3rd Approximation: $x = 6.125$

Associated Error: ± 0.125 units

Finding the 4th Interval

	x	$f(x)$
Current left-endpoint	6	$f(6) = -1$
Midpoint	6.125	$f(6.125) \approx 1.6$
Current right-endpoint	6.25	$f(6.25) \approx 4.6$

4th Interval: $[6, 6.125]$

4th Approximation $x = 6.0625$

Associated Error: ± 0.0625 units

Answer: The solution to the equation is approximately $x = 6.0625$ with a maximum error of 0.0625 units.

Question 8 Solution

Step 1) Rewrite the equation so we can identify the function we are working with.

$$x^3 + 5x^2 + 7x + 5 = 0$$

So, $f(x) = x^3 + 5x^2 + 7x + 5$

Step 2) Identify the first interval, the first approximation, and the associated error.

We know the solution is negative, but that is all. Let's set up a table of values to get an idea of where our first interval should be.

x	$f(x)$
-1	2
-2	3
-3	2
-4	-7

1st Interval: $[-3, -4]$

1st Approximation: $x = -3.5$

Associated Error: ± 0.5 units

Step 3) Identify the 2nd interval, 2nd approximation and the associated maximum error.

Finding the 2nd Interval

	x	$f(x)$
Current left-endpoint	-4	$f(-4) = -7$
Midpoint	-3.5	$f(-3.5) \approx -1.1$
Current right-endpoint	-3	$f(-3) = 2$

2nd Interval: $[-3.5, -3]$

2nd Approximation: $x = -3.25$

Associated Error: ± 0.25 units

Step 4) Repeat Step 3 until the maximum error is less 0.05 units.

Finding the 3rd Interval

	x	$f(x)$
Current left-endpoint	-3.5	$f(-3.5) \approx -1.1$
Midpoint	-3.25	$f(-3.25) \approx 0.7$
Current right-endpoint	-3	$f(-3) = 2$

3rd Interval: $[-3.5, -3.25]$

3rd Approximation: $x = -3.375$

Associated Error: ± 0.125

Finding the 4th Interval

	x	$f(x)$
Current left-endpoint	-3.5	$f(-3.5) \approx -1.1$
Midpoint	-3.375	$f(-3.375) \approx -0.1$
Current right-endpoint	-3.25	$f(-3.25) \approx 0.7$

4th Interval: $[-3.375, -3.25]$

4th Approximation: $x = -3.3125$

Associated Error: ± 0.0625 units

Finding the 5th Interval

	x	$f(x)$
Current left-endpoint	-3.375	$f(-3.375) \approx -0.1$
Midpoint	-3.3125	$f(-3.3125) \approx 0.3$
Current right-endpoint	-3.25	$f(-3.25) \approx 0.7$

5th Interval: $[-3.375, -3.3125]$

5th Approximation: $x = -3.34275$

Associated Error: ± 0.03125 units

Answer The equation has a solution at approximately $x = -3.34275$ with a maximum error in the approximation of at most 0.03125 units.

Question 9 Solution

Step 1) Find a non-linear function whose root is at $\sqrt{7}$.

$$x = \sqrt{71}$$
$$x^2 = 71$$
$$x^2 - 71 = 0$$

We'll use $f(x) = x^2 - 71$.

Step 2) Find the first interval, first approximation and its associated maximum error.

We know $\sqrt{71}$ is larger than 8, but less than 9. We'll use $[8, 9]$ as the first interval.

- 1st Interval: $[8, 9]$
- 1st Approximation: $x = 8.5$
- Associated Error: ± 0.5 units

Step 3) Find the second interval, second approximation and the associated maximum error.

Finding the 2nd Interval

	x	$f(x)$
Current left-endpoint	8	$f(8) = -7$
Midpoint	8.5	$f(8.5) = 1.25$
Current right-endpoint	9	$f(9) = 10$

2nd Interval: $[8, 8.5]$

2nd Approximation: $x = 8.25$

Associated Error: ± 0.25 units

Step 4) Repeat Step 3 until you've found the 4th approximation.

Finding the 3rd Interval

	x	$f(x)$
Current left-endpoint	8	$f(8) = -7$
Midpoint	8.25	$f(8.25) \approx -2.9$
Current right-endpoint	8.5	$f(8.5) = 1.25$

3rd Interval: $[8.25, 8.5]$

3rd Approximation $x = 8.375$

Associated Error: ± 0.125 units

Finding the 4th Interval

	x	$f(x)$
Left endpoint	8.25	$f(8.25) \approx -2.9$
Midpoint	8.375	$f(8.375) \approx -0.9$
Right endpoint	8.5	$f(8.5) = 1.25$

4th Interval: $[8.375, 8.5]$

4th Approximation: $x = 8.4375$

Associated Error: ± 0.0625 units

Answer: $\sqrt{71} \approx 8.4375$ with a maximum error in this approximation of 0.0625.

Question 10 Solution

Step 1) Determine the nonlinear function we will use for the bisection algorithm.

$$\begin{aligned} x &= \frac{1}{\sqrt[5]{3}} \\ x^5 &= \frac{1}{3} \\ 3x^5 &= 1 \\ 3x^5 - 1 &= 0 \end{aligned}$$

We will use $f(x) = 3x^5 - 1$.

Step 2) Find the first interval, first approximation and the associated error.

Since $0 < \frac{1}{\sqrt[5]{3}} < 1$, we should be able to use $[0, 1]$ as the first interval. A quick check of the function values confirms this.

x	$f(x)$
0	-1
1	2

1st Interval: $[0, 1]$

1st Approximation: $x = 0.5$

Associated Error: ± 0.5 units

Step 3) Determine the second interval, second approximation and the associated error.

Finding the 2nd Interval

	x	$f(x)$
Current left-endpoint	0	$f(0) = -1$
Midpoint	0.5	$f(0.5) \approx -0.9$
Current right-endpoint	1	$f(1) = 2$

2nd Interval: $[0.5, 1]$

2nd Approximation: $x = 0.75$

Associated Error: ± 0.25 units

Step 4) Find the third interval, third approximation and its associated error.

Finding the 3rd Interval

	x	$f(x)$
Current left-endpoint	0.5	$f(0.5) \approx -0.9$
Midpoint	0.75	$f(0.75) \approx -0.2$
Current right-endpoint	1	$f(1) = 2$

Third Interval: $[0.75, 1]$
Third Approximation: $x = 0.875$ with an error of 0.125 units.

Answer: $\sqrt[5]{3} \approx 0.875$ with a maximum error of 0.125 units.

Question 11 Solution

Step 1) Find a nonlinear function with a root at $\sqrt{125}$.

$$x = \sqrt{125}$$
$$x^2 = 125$$
$$x^2 - 125 = 0$$

We'll use $f(x) = x^2 - 125$.

Step 2) Determine an appropriate starting interval. It's midpoint will be the first approximation.

Since $11^2 = 121$ and $12^2 = 144$ we know $11 < \sqrt{125} < 12$.

First Interval: $[11, 12]$

First approximation: $x = 11.5$

Associated Error: ± 0.5 units

Step 3) Determine the second interval, the second approximation, and the associated error value.

Finding the 2nd Interval

	x	$f(x)$
Current left-endpoint	11	$f(11) = -4$
Midpoint	11.5	$f(11.5) = 7.25$
Current right-endpoint	12	$f(12) = 19$

2nd Interval: $[11, 11.5]$
2nd Approximation: $x = 11.25$ with a maximum error of 0.25 units.

Step 4) Determine the third interval, the third approximation, and the associated error value.

Finding the 3rd Interval

	x	$f(x)$
Current left-endpoint	11	$f(11) = -4$
Midpoint	11.25	$f(11.25) \approx 1.6$
Current right-endpoint	11.5	$f(11.5) = 7.25$

Third Interval: $[11, 11.25]$
Third Approximation: $x = 11.125$ with a maximum error of 0.125.

Answer: $\sqrt{125} \approx 11.125$ with a maximum error of 0.125 units.

Question 12

Step 1) Find a nonlinear function with a root at $\frac{\sqrt[4]{12500}}{2}$.

$$\begin{aligned} x &= \frac{\sqrt[4]{12500}}{2} \\ x^4 &= \frac{12500}{16} \\ x^4 &= \frac{3125}{4} \\ 4x^4 &= 3125 \\ 4x^4 - 3125 &= 0 \end{aligned}$$

We'll use the function $f(x) = 4x^4 - 3125$.

Step 2) Determine the appropriate starting interval, the first approximation and the associated error.

Since $10^4 = 10{,}000$ is about the right size, we try $f(10) = 36{,}875$

By comparison, $f(5) = -625$, so the best starting interval is somewhere between $x = 5$ and $x = 10$. Let's make a table of values to help us narrow things down.

x	$f(x)$
5	-625
6	2059

Well, that was convenient.

1st Interval: $[5, 6]$

1st Approximation: $x = 5.5$

Associated Error: ± 0.5 units

Step 3) Find the second interval, approximation, and associated error.

Finding the 2nd Interval

	x	$f(x)$
Current left-endpoint	5	$f(5) = -625$
Midpoint	5.5	$f(5.5) = 535.25$
Current right-endpoint	6	$f(6) = 2059$

2nd Interval: $[5, 5.5]$

2nd Approximation: $x = 5.25$

Associated Error: ± 0.25 units

Step 4) Repeat Step 3 until the maximum error is smaller than the allowed tolerance.

Finding the 3rd Interval

	x	$f(x)$
Current left-endpoint	5	$f(5) = -625$
Midpoint	5.25	$f(5.25) \approx -86.2$
Current right-endpoint	5.5	$f(5.5) = 535.25$

Third Interval: $[5.25, 5.5]$

Third Approximation: The midpoint of the 3rd interval is $x = 5.375$

Associated Error: ± 0.125 units

Finding the 4th Interval

	x	$f(x)$
Current left-endpoint	5.25	$f(5.25) \approx -86.2$
Midpoint	5.375	$f(5.375) \approx 213.7$
Current right-endpoint	5.5	$f(5.5) = 535.25$

Fourth Interval: $[5.25, 5.375]$

Fourth Approximation: The midpoint of the 4th interval is $x = 5.3125$

Associated Error: ± 0.0625 units

Answer: $\frac{1}{2} \cdot \sqrt[4]{12500} \approx 5.3125$ with a maximum error of 0.0625.

Question 13 Solution

Step 1) Solve $0.5^n(b-a)$ for n when $a = 2$ and $b = 5$.

$$
\begin{aligned}
0.5^n(5-2) &= 0.01 \\
0.5^n \cdot 3 &= \frac{1}{10} \\
\left(\frac{1}{2}\right)^n &= \frac{1}{30} \\
n \cdot \ln\left(\frac{1}{2}\right) &= \ln\left(\frac{1}{30}\right) \\
n \cdot (\ln 1 - \ln 2) & \ln 1 - \ln 30 \\
-n \ln 2 &= -\ln 30 \\
n &= \frac{\ln 30}{\ln 2} \\
&\approx 4.90732
\end{aligned}
$$

Answer We will need to use at least 5 iterations in order to ensure the accuracy.

Question 14

Step 1) Solve $0.5^n(b-a) = 0.02$ for n when $a = -1$ and $b = 1$.

$$
\begin{aligned}
0.5^n\,(1-(-1)) &= 0.02 \\
\left(\frac{1}{2}\right)^n \cdot 2 &= \frac{1}{50} \\
\left(\frac{1}{2}\right)^n &= \frac{1}{100} \\
n\ln\left(\frac{1}{2}\right) &= \ln\left(\frac{1}{100}\right) \\
n\,(\ln 1 - \ln 2) &= \ln 1 - \ln 100 \\
-n\ln 2 &= -\ln 100 \\
n &= \frac{\ln 100}{\ln 2} \\
&\approx 6.64473
\end{aligned}
$$

Answer: We will need at least 7 iterations before the error tolerance is reached.

Next Lesson: What is a Derivative?

Previous Lesson: The Intermediate Value Theorem

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