Quantum Error Correction - Notes

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1 Week 2: Harlow Toy Model

- References:
  - https://arxiv.org/pdf/1607.03901.pdf (Harlow paper)
  - https://pos.sissa.it/305/002/pdf (Harlow presentation)
  - https://arxiv.org/pdf/quant-ph/9901025.pdf#page=5&zoom=100,0,0 (Quantum secret sharing)
- Harlow presents a toy model of error correction, which we go through
- Suppose we wish to protect the qutrit state

$$|\psi\rangle = \sum_{i=0}^{2} a_i |i\rangle \tag{1.1}$$

  against erasure
- We project to the code subspace spanned by logical qutrits

$$\begin{aligned}|0_L\rangle &= \frac{1}{\sqrt{3}}(|000\rangle + |111\rangle + |222\rangle)\\ |1_L\rangle &= \frac{1}{\sqrt{3}}(|012\rangle + |120\rangle + |201\rangle)\\ |2_L\rangle &= \frac{1}{\sqrt{3}}(|021\rangle + |102\rangle + |210\rangle)\end{aligned} \tag{1.2}$$

- There's two points of note here:
  1. This code subspace is symmetric under cyclic permutations of the physical qubits
  2. Each individual qutrit is maximally mixed - it is equal parts $|0\rangle$, $|1\rangle$, $|2\rangle$ (this means it functions as a *quantum secret-sharing code* too)
- To actually prepare these logical qutrits, we adjoin two ancillary qutrits in the state

$$|\chi\rangle_{23} = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle) \tag{1.3}$$

- We define a unitary operation U_{12} on the first two qutrits via the permutation

$$\begin{array}{lll} |00\rangle \to |00\rangle & |11\rangle \to |01\rangle & |22\rangle \to |02\rangle \\ |01\rangle \to |12\rangle & |12\rangle \to |10\rangle & |20\rangle \to |11\rangle \\ |02\rangle \to |21\rangle & |10\rangle \to |22\rangle & |21\rangle \to |20\rangle \end{array} \tag{1.4}$$

- Then, note that this unitary implements

$$|i_L\rangle = U_{12}(|i\rangle_1 \otimes |\chi\rangle_{23}) \tag{1.5}$$

  for all physical basis elements $|i\rangle$

- So, the state can be encoded just by doing

$$|\psi_L\rangle = U_{12}(|\psi\rangle_1 \otimes |\chi\rangle_{23}) \tag{1.6}$$

- We can similarly encode with a unitary on the (1,3) or (2,3) subsystems by symmetry
- This explicitly provides the erasure protection procedure: given only two subsystems of $|\psi_L\rangle$, we can just apply the corresponding $U_{ij}^\dagger$ and we get back the original $|\psi\rangle$ in one of the subsystems
- Sticking with Harlow's numbering, suppose the third qutrit is erased, so we recover $|\psi\rangle$ perfectly in the first qutrit
- The no-cloning theorem then implies that we should not be able to work out **any** information about $|\psi\rangle$ given knowledge of just the third qutrit alone
- This is quite obvious from (1.6) and from (1.2): the third qutrit in the encoded state is always maximally mixed/proportional to the identity, so contains zero information about the unencoded state on its own as expected
- What can we say about the state $|\chi\rangle$?
- All we can really say is that $|\chi\rangle$ has to be entangled
- Suppose $|\chi\rangle = |\alpha\rangle \otimes |\beta\rangle$, so is in a product state; then, (1.6) becomes

$$|\psi_L\rangle = U_{12}(|\psi\rangle_1 \otimes |\alpha\rangle_2 \otimes |\beta\rangle_3) \tag{1.7}$$

  so the third qutrit is sitting in the state $|\beta\rangle$ **for all** states in the code subspace, so it provides no info about $|\psi\rangle$ which the second qubit didn't already know

- This can also be viewed under the lens of *quantum secret sharing* - Harlow's toy model is a $((2,3))$ threshold scheme, so we should be able to recover full state information from any two qubits, but **no** information from any one share
- $|\chi\rangle$ clearly needs entanglement for the former to hold
- This can equivalently be phrased by saying that a U_{23} and U_{13} can't both simultaneously exist if $|\chi\rangle$ is a product state
- This correctability can also be framed in terms of *logical operators*
- Consider the operator on a physical qutrit as

$$O\,|i\rangle = \sum_j (O)_{ji}\,|j\rangle \tag{1.8}$$

- We can (in some sense) project this operator up into the code subspace by just defining an operator with the same matrix elements $(O)_{ij}$ acting on the basis for the code subspace:

$$O_L\,|i_L\rangle = \sum_j (O)_{ji}\,|j_L\rangle \tag{1.9}$$

- We can then just arbitrarily define the action of this logical operator on the full 3-qutrit space which the code subspace is a subspace of

- Now, consider the case where $O = O_1$ acts only on the first qutrit in the toy model

- We can define a logical operator on the code subspace which behaves equivalently to O_1 but which has non-trivial support only on the first and second qutrits:

$$O_{12} \coloneqq U_{12} O_1 U_{12}^\dagger \tag{1.10}$$

- Thus any operator acting on a single qutrit can be represented as a logical operator with support on any two of the qutrits

- Harlow says that this is an analogue of *subregion duality* in AdS/CFT

- He also says it is an analogue of *radial commutativity*, meaning that any logical operator O_L on $\mathcal{H}_{\text{code}}$ commutes with any operator acting on a single qutrit

- But O_{12} clearly commutes with any operator X_3 acting on qutrit 3, and similar for O_{13} and O_{23}; since all of these act identically to O_L on $\mathcal{H}_{\text{code}}$, it must be that O_L commutes with all single qutrit operators

- More concretely, for any two states $|\psi_L\rangle$ and $|\phi_L\rangle$ in $\mathcal{H}_{\text{code}}$, we have

$$\langle \psi_L | [O_L, X] | \phi_L \rangle = 0 \tag{1.11}$$

  where X is any single qutrit operator

- A version of the *Ryu-Takayanagi formula* also holds in this code

- This formula is

$$S(\rho_A) = \text{Tr}(\rho \mathcal{L}_A) + S_{bulk}(\rho_{\mathcal{E}_A}) \tag{1.12}$$

- To see the link, suppose we have an arbitrary mixed state ρ_L on $\mathcal{H}_{\text{code}}$, which is the encoding of a physical mixed state ρ

- If we define

$$\rho_1 = \sum_{i,j} a_{i,j} |i\rangle \langle j| \tag{1.13}$$

  on the first subsystem, then by (1.5) we have that

$$\rho_L = U_{12}(\rho_1 \otimes |\chi\rangle \langle \chi|_{23}) \tag{1.14}$$

- Define $\rho_L^3 \coloneqq \text{Tr}_{12}(\rho_L)$ and $\rho_L^{12} \coloneqq \text{Tr}_3(\rho_L)$; we then explicitly compute

$$\begin{aligned} \rho_L^3 &= \frac{1}{3}(|0\rangle \langle 0| + |1\rangle \langle 1| + |2\rangle \langle 2|) = \frac{1}{3} I_3 \\ \rho_L^{12} &= U_{12} \left(\rho_1 \otimes \frac{1}{3} I_2 \right) U^\dagger \end{aligned} \tag{1.15}$$

- This gives von Neumann entropies

$$\begin{aligned} S(\rho_L^3) &= -3 \times \frac{1}{3} \log \frac{1}{3} = \log 3 \\ S(\rho_L^{12}) &= S(\rho_1) + S\left(\frac{1}{3} I_2\right) = S(\rho_L) + \log 3 \end{aligned} \tag{1.16}$$

- In this case, (1.12) holds: 'the logical degree of freedom in the bulk is not in the entanglement wedge of the third physical qutrit, but is in the entanglement wedge of the union of the first and second'
- Thus we expect it contributes entropy only to $S(\rho_L^{12})$, which is what we observe
- WHAT DOES THIS MEAN AAAAA

2 Stabiliser Codes

2.1 The Stabiliser Formalism

- Consider the EPR state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{2.1}$$

- This state satisfies $X_1X_2\,|\psi\rangle = |\psi\rangle$ and $Z_1Z_2\,|\psi\rangle = |\psi\rangle$; we say it is *stabilised* by X_1X_2 and Z_1Z_2
- It's easy to check that in fact $|\psi\rangle$ is the unique state stabilised by these operators
- The idea of the formalism is that it's often easier to work with the operators which stabilise a state rather than the state itself
- The key to the stabiliser formalism is lots of group theory, with the group of principal interest being the *Pauli group* G_n on n qubits
- This is defined on a single qubit as consisting of all the Pauli matrices together with multiplicative factors of ± 1 and $\pm i$:

$$G_1 \coloneqq \{\pm I, \pm iI, \pm X, \pm iX, \pm Y, \pm iY, \pm Z, \pm iZ\} \tag{2.2}$$

- The Pauli group on n qubits is then just the n fold tensor product of any elements of G_1
- Suppose $S \leq G_n$ is a subgroup, and define V_S as the set of n qubit states fixed by S: $|\psi\rangle \in V_S \implies S\,|\psi\rangle = |\psi\rangle$
- V_S is called the *vector space stabilised by* S, and S is the *stabiliser* of V_S
- As an example, consider $n = 3$ and $S = \{I, Z_1Z_2, Z_2Z_3, Z_1Z_3\}$
- Z_1Z_2 fixes $\mathrm{Span}(|000\rangle, |001\rangle, |110\rangle, |111\rangle)$, and Z_2Z_3 fixes $\mathrm{Span}(|000\rangle, |100\rangle, |011\rangle, |111\rangle)$
- Note that $|000\rangle$ and $|111\rangle$ are in both these lists, so $V_S = \mathrm{Span}(|000\rangle, |111\rangle)$
- We determined V_S by looking at the subspaces which were stabilised by just two operators here, which is a more general example of using generators to describe a group
- We write $G = \langle g_1, \ldots, g_l\rangle$ for the group generated by the l elements above; in this example, $S = \langle Z_1Z_2, Z_2Z_3\rangle$ since $Z_1Z_3 = (Z_1Z_2)(Z_2Z_3)$ and $I = (Z_1Z_2)^2$
- So to find V_S, we only need to find the states stabilised by the generators
- Not every subgroup of the Pauli group is the stabiliser for a non-trivial vector space
- For example, consider $\{\pm I, \pm X\} \leq G_1$; the only solution to $-I\,|\psi\rangle = |\psi\rangle$ is $|\psi\rangle = 0$

- It turns out that necessary and sufficient conditions for a subgroup S to stabilise a non-trivial space are
  1. All elements of S commute
  2. $-I \notin S$
- We can use the stabiliser formalism to describe error correcting codes
- For example, the 7 qubit Steane code has code space stabilised by the 7 generators

$$\begin{aligned} g_1 &= X_4X_5X_6X_7 \\ g_2 &= X_2X_3X_6X_7 \\ g_3 &= X_1X_3X_5X_7 \\ g_4 &= Z_4Z_5Z_6Z_7 \\ g_5 &= Z_2Z_3Z_6Z_7 \\ g_6 &= Z_1Z_3Z_5Z_7 \end{aligned} \tag{2.3}$$

- A useful way of presenting generators $g_1, \ldots, g_l$ is via the *check matrix*
- This is an $l \times 2n$ matrix whose rows correspond to the generators, and the LHS contains 1s to indicate which generators contain Xs and the RHS for Zs, so a 1 on both sides indicates a Y
- We usually use $r(g)$ to denote the $2n$-dimensional row vector rep of an element g of the Pauli group
- Define a $2n \times 2n$ matrix by

$$\Lambda = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \tag{2.4}$$

  and then elements g and g' of G_n commute iff $r(g)\Lambda r(g')^T = 0$ modulo 2
- We have the following proposition: let $S = \langle g_1, \ldots, g_l \rangle$ be such that $-I$ is not in S. Then, the generators are independent iff the rows of the corresponding check matrix are linearly independent
- An even more useful proposition is that if S is as above where all generators are independent, and if we fix $i \in \{1, \ldots, l\}$, then $\exists g \in G_n$ such that $gg_ig^\dagger = -g_i$ and $gg_jg^\dagger = g_j$ for all $j \neq i$
- This proposition is leveraged to prove that if $S = \langle g_1, \ldots, g_{n-k} \rangle$ is generated by $n-k$ independent and commuting elements from G_n, and $-I \notin S$, then V_S is a 2^k-dimensional vector space

2.2 Unitary Gates

- Suppose we apply a unitary U to V_S stabilised by S
- For any $|\psi\rangle \in V_S$ and $g \in S$, we have

$$U|\psi\rangle = Ug|\psi\rangle = UgU^\dagger U|\psi\rangle \tag{2.5}$$

  meaning $U|\psi\rangle$ is stabilised by $UgU^\dagger$
- This means that UV_S is stabilised by $USU^\dagger$

- Moreover, if S is generated by $g_1, \ldots, g_l$, then $USU^\dagger$ is generated by $Ug_1U^\dagger, \ldots$, so we only need to see how this unitary affects the generators

- For certain Us, the generators transform particularly nicely

- Suppose for example we apply a Hadamard to a single qubit, and note that

$$HXH^\dagger = Z, \quad HYH^\dagger = -Y, \quad HZH^\dagger = X \tag{2.6}$$

- So, imagine we have n qubits in a state stabilised by $\langle Z_1, \ldots, Z_n \rangle$ (this is clearly just $|0\rangle^{\otimes n}$)

- Applying the Hadamard to each qubit in turn, the resultant state is stabilised by $\langle X_1, \ldots, X_n \rangle$, being $|+\rangle^{\otimes n}$

- The cool thing here is that we need 2^n amplitudes to specify the final state, but only n generators

- This isn't that surprising: after applying the Hadamards, the resultant state is still not entangled so it's not too surprising that we can obtain a compact description

- However, let's consider the controlled not gate

- Denote U the controlled not with qubit 1 as control; then, a bit of algebra shows

$$UX_1U^\dagger = X_1X_2 \tag{2.7}$$

- Similar calculations show $UX_2U^\dagger = X_2$, $UZ_1U^\dagger = Z_1$, and $UZ_2U^\dagger = Z_1Z_2$

- So we can describe CNOT and Hadamards through the stabiliser formalism

- We can also describe the phase gate, which is the operation

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \tag{2.8}$$

- We find

$$SXS^\dagger = Y, \quad SZS^\dagger = Z \tag{2.9}$$

- It turns out that **any** operation taking $G_n \to G_n$ under conjugation can be written as some composition of Hadamards, CNOTS, and phase gates

- We call the set of U such that $UG_nU^\dagger = G_n$ the *normaliser* of G_n, denoted $N(G_n)$

- We claim therefore that $N(G_n)$ is generated by Hadamards, CNOTs, and phase gates

- Unfortunately, most gates are not in $N(G_n)$, in particular the $\pi/8$ gate and Toffoli gate

- This makes analysing quantum circuits involving these much harder in the stabiliser formalism

- It turns out though that error correction can be done without these, so analysing stabiliser codes can be made very nice in this formalism

2.3 Measurement

- Measurements can also be described within the stabiliser formalism
- Suppose we measure $g \in G_n$, and assume g is a product of Pauli matrices with no factor of -1 or $\pm i$ out the front
- Assume further that the system is in state $|\psi\rangle$ with stabiliser $\langle g_1, \ldots, g_n\rangle$; how does this stabiliser transform under the measurement?
- There's two cases:
  1. g commutes with all generators of the stabiliser
  2. g anticommutes with one or more of the generators of the stabiliser
- In case 1, either g or $-g$ is in the stabiliser: since $g_j g |\psi\rangle = g g_j |\psi\rangle = g |\psi\rangle$ for each stabiliser generator, $g |\psi\rangle \in V_S$ and so is proportional to $|\psi\rangle$
- Since $g^2 = I$, it follows that $g |\psi\rangle = \pm |\psi\rangle$, so g or $-g$ is in the stabiliser
- Assuming g is in the stabiliser ($-g$ is analogous), $g |\psi\rangle = |\psi\rangle$, so a measurement of g gives $+1$ with probability 1
- Therefore the measurement does not disturb the system, and leaves the stabiliser invariant
- For the second case where g anticommutes with g_1 and commutes with all other generators, we note the following
- g has eigenvalues ± 1, so the projectors onto the ± 1 eigenspaces are given by $(I \pm g)/2$ respectively, and so the measurement probabilities are

$$\mathbb{P}(+1) = \operatorname{Tr}\left(\frac{I+g}{2} |\psi\rangle \langle\psi|\right), \quad \mathbb{P}(-1) = \operatorname{Tr}\left(\frac{I-g}{2} |\psi\rangle \langle\psi|\right) \tag{2.10}$$

- But, since $g_1 |\psi\rangle = |\psi\rangle$ and $g g_1 = -g_1 g$, we find that

$$\mathbb{P}(+1) = \operatorname{Tr}\left(\frac{I+g}{2} g_1 |\psi\rangle \langle\psi|\right) = \operatorname{Tr}\left(g_1 \frac{I-g}{2} |\psi\rangle \langle\psi|\right) = \mathbb{P}(-1) \tag{2.11}$$

  and so we deduce $\mathbb{P}(\pm 1) = 1/2$
- Suppose we measure $+1$, so the system collapses to $|\psi^+\rangle = (I+g)|\psi\rangle/\sqrt{2}$, which has stabiliser $\langle g, g_2, \ldots, g_l\rangle$, and similarly for -1 with $g \mapsto -g$

2.4 Gottesman-Knill Theorem

- We have implicitly proved so far the following
- Suppose a quantum computation is performed only involving states prepared in the computational basis, Hadamard gates, phase gates, CNOT gates, Pauli gates, and measurements of Pauli observables, together with classical operations conditioned on the outcome of measurements. Such a computation can be efficiently simulated on a classical computer
- This shows that some quantum computations involving highly entangled states can be simulated efficiently classically

2.5 Stabiliser Codes

- An $[n, k]$ *stabiliser code* is defined to be the vector space V_S stabilised by $S \leq G_n$ such that $-I \notin S$ and S has $n-k$ independent, commuting generators $S = \langle g_1, \ldots, g_{n-k} \rangle$
- We denote this code $C(S)$ in what follows
- What are the logical basis states for $C(S)$?
- A systematic way of choosing them is as follows; we could however just arbitrarily choose any 2^k orthonormal vectors in $C(S)$, that said
- Choose operators $\bar{Z}_1, \ldots, \bar{Z}_k \in G_n$ such that $g_1, \ldots, g_{n-k}, \bar{Z}_1, \ldots, \bar{Z}_k$ forms an independent and commuting set
- The $\bar{Z}_j$ operator plays the role of a logical Pauli σ_z operator on logical qubit j, so the logical computational basis state $|x_1, \ldots, x_k\rangle_L$ is defined to be the state with stabiliser

$$\langle g_1, \ldots, g_{n-k}, (-1)^{x_1} \bar{Z}_1, \ldots, (-1)^{x_k} \bar{Z}_k \rangle \tag{2.12}$$

- Similarly, we define $\bar{X}_j$ to be the product of Pauli gates which take $\bar{Z}_j \mapsto -\bar{Z}_j$ under conjugation, and leaves all other $\bar{Z}_i$ and g_i alone when acting by conjugation
- $\bar{X}_j$ has the effect of a NOT gate acting on the jth encoded qubit
- Moreover, $\bar{X}_j$ satisfies $\bar{X}_j g_k \bar{X}_j^\dagger = g_k$, and so commutes with the stabiliser
- $\bar{X}_j$ also commutes with all $\bar{Z}_i$ for $i \neq j$, and anticommutes when $i = j$
- How are error correcting properties of $C(S)$ related to the generators of the stabiliser?
- Suppose we encode a state by a $[n, k]$ stabiliser code $C(S)$ with generators as above, and an error $E \in G_n$ occurs
- When E anticommutes with an element of the stabiliser, E takes $C(S)$ to an orthogonal subspace so the error can be detected by an appropriate projective measurement
- If $E \in S$, then we don't have to worry since E doesn't corrupt the space at all
- If E commutes with all elements of S but $E \notin S$, then we have issues
- The set of $E \in G_n$ such that $Eg = gE$ for all $g \in S$ is called the *centraliser* of S in G_n, denoted $Z(S)$
- For stabiliser groups S which we concern ourselves with, the centraliser coincides with the normaliser of S, $N(S)$, which is just all elements $E \in G_n$ such that $EgE^\dagger \in S$ for all $g \in S$
- These observations motivate the error correction conditions for stabiliser codes
- Let S be the stabiliser for a stabiliser code $C(S)$. Suppose $\{E_j\}$ is a set of operators in G_n such that $E_j^\dagger E_k \notin N(S) - S$ for all j, k. Then $\{E_j\}$ is a correctable set of errors for $C(S)$
- WLOG we can restrict to considering $E_j \in G_n$ such that $E_j^\dagger = E_j$
- This doesn't tell us explicitly how to perform the error correction operation
- Suppose $\{g_1, \ldots, g_{n-k}\}$ is a set of generators for the stabiliser of a $[n, k]$ stabiliser code, and $\{E_j\}$ is a set of correctable errors

- Error detection is performed by measuring the generators of the stabiliser g_1 to g_{n-k} in turn, to obtain the error syndrome, which consists of the results of the measurements β_1 to β_{n-k}
- If E_j occurred, then the syndrome is given by β_l such that $E_j g_l E_j^\dagger = \beta_l g_l$
- If E_j is the unique error having this syndrome, recovery is just performed by applying $E_j^\dagger$
- If there are two errors E_j, $E_{j'}^\dagger$ given rise to the same syndrome, it follows that $E_j P E_j^\dagger = E_{j'} P E_{j'}^\dagger$ where P is the projector onto the code space, so $E_j^\dagger E_{j'} P E_{j'}^\dagger E_j = P$, so $E_j^\dagger E_{j'} \in S$, so applying $E_j^\dagger$ after the error $E_{j'}$ occurs results in a successful recovery
- We define the *weight* of an error $E \in G_n$ as the number of terms in the tensor product which aren't equal to the identity
- So e.g. the weight of $X_1 Z_4 Y_8$ is three
- The *distance* of $C(S)$ is defined to be the minimum weight of an element of $N(S) - S$, and if $C(S)$ is an $[n, k]$ code with distance d, we say that $C(S)$ is an $[n, k, d]$ stabiliser code
- By the error correcting conditions, a code with distance at least $2t + 1$ is able to correct arbitrary errors on any t qubits

2.6 Examples

2.6.1 Bit-Flip Code

- Consider the three qubit bit-flip code, with code subspace spanned by $|000\rangle$ and $|111\rangle$
- The stabiliser is therefore $S = \langle Z_1 Z_2, Z_2 Z_3 \rangle$
- By inspection, every possible product of two elements from the set of errors $\{I, X_1, X_2, X_3\}$ anticommutes with at least one of the generators of S, so by the error correcting conditions the set $\{I, X_1, X_2, X_3\}$ forms a correctable set of errors
- Error detection is effected by measuring the stabiliser generators
- If e.g. X_1 occurred, then the stabiliser becomes $\langle -Z_1 Z_2, Z_2 Z_3 \rangle$, so the syndrome measurement gives the results ± 1
- Similarly, X_2 gives -1 and -1, X_3 ± 1, and I gives $+1$ and $+1$
- In each instance, recovery is done by applying the inverse operation to the error indicated
- This is summarised by:

$Z_1 Z_2$	$Z_2 Z_3$	Error Type	Action
+1	+1	No error	Nothing
+1	-1	Bit 3 flipped	Flip bit 3
-1	+1	Bit 1 flipped	Flip bit 1
-1	-1	Bit 2 flipped	Flip bit 2

- We don't really gain any insight here - we only see the power on more complex examples

3 AdS/CFT (Harlow notes)

- Reference: https://pos.sissa.it/305/002/pdf
- The main statement of AdS/CFT is that 'any CFT in d-dimensions is equivalent to a theory of quantum gravity in a family of asymptotic $\mathrm{AdS}_d \times M$ spacetimes, where M is a compact manifold'
- We start with AdS spaces

3.1 Anti de Sitter Space

- *Anti de Sitter space* is defined as the maximally symmetric spacetime of constant negative curvature
- Consider $d+2$-dimensional Minkowski space with $(2, d)$ signature, which has metric

$$ds^2 = -dT_1^2 - dT_2^2 + dX_1^2 + \ldots + dX_d^2 \tag{3.1}$$

and define a submanifold via

$$T_1^2 + T_2^2 - \mathbf{X}^2 = \ell^2 \tag{3.2}$$

so ℓ has units of length

- We can define global coordinates for this submanifold by

$$\begin{aligned} T_1 &= \sqrt{\ell + r^2} \cos\left(\frac{t}{\ell}\right) \\ T_2 &= \sqrt{\ell + r^2} \sin\left(\frac{t}{\ell}\right) \\ \mathbf{X}^2 &= r^2 \end{aligned} \tag{3.3}$$

which we substitute back into (2.1) to obtain the AdS_{d+1}metric in global coordinates as

$$ds^2 = -\left(1 + \frac{r^2}{\ell^2}\right) dt^2 + \frac{dr^2}{1 + \frac{r^2}{\ell^2}} + r^2 d\Omega_{d-1}^2 \tag{3.4}$$

where $r \in [0, \infty)$ is the usual radial spatial coordinate, $t \in (-\infty, \infty)$ is 'time' in some sense, and Ω_{d-1} is the angular part of the metric for the hypersphere S^{d-1}

- AdS_{d+1}space has several important properties
- Firstly, it solves Einstein's equations

$$G_{\mu\nu} \coloneqq R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{3.5}$$

with the energy-momentum tensor given by

$$T_{\mu\nu} = -\rho_0 g_{\mu\nu}, \qquad \rho_0 = -\frac{d(d-1)}{16\pi G \ell^2} \tag{3.6}$$

- We usually work in units of ℓ, so $\ell = 1$ in what follows, and we can reintroduce through dimensional analysis if needed
- A second property is that it has a large symmetry group: $SO(d, 2)$, the Lorentz group of $d+2$ dimensions

- This group acts *transitively* on AdS_{d+1}(any two points are connected by a symmetry), so $AdS_{d+1}$is *homogenous*
- It is also *locally isotropic*: the subgroup which fixes a chosen $p \in AdS_{d+1}$ can take any $X \in T_p(AdS_{d+1})$ to a multiple of any other tangent vector at p
- To work out the causal structure of AdS_{d+1}, we define a new coordinate $\rho \in (0, \pi/2)$ by

$$r = \tan\rho \tag{3.7}$$

which transforms the metric (2.4) to

$$ds^2 = \frac{1}{\cos^2\rho}(-dt^2 + d\rho^2 + \sin^2\rho\, d\Omega_{d-1}^2) \tag{3.8}$$

- Null geodesics by definition satisfy $ds^2 = 0$, so they do not care about the *Weyl factor* (multiplication by a scalar function) of the metric, so the causal structure of $AdS_{d+1}$is equivalent to the cylinder with metric

$$ds^2 = -dt^2 + d\rho^2 + \sin^2\rho\, d\Omega_{d-1}^2 \tag{3.9}$$

- AdS_{d+1}space also has an *asymptotic boundary* at $r = \infty$ (or $\rho = \frac{\pi}{2}$)
- This boundary is just the boundary of a hypercylinder, so has topology $\mathbb{R} \times S^{d-1}$
- This means that a null geodesic sent from the 'center' of the cylinder hits the boundary and returns to its original spatial coordinate in finite proper time
- We therefore think of AdS_{d+1}space as behaving in some sense like a 'finite box', despite having infinite spatial volume
- If matter is present, the global metric (2.4) will be perturbed, so it is natural to define *asymptotically-AdS spacetimes*, which have an asymptotic boundary isomorphic to $\mathbb{R} \times S^{d-1}$ and metric asymptotically approaching (2.4) as we approach said boundary
- More generally, we can study spacetimes which approach $AdS_{d+1} \times M$ at the asymptotic boundary, where M is compact with finite volume as $r \to \infty$
- **Important example:** AdS-Schwarzschild geometry
  - This has metric

$$\begin{aligned} ds^2 &= -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2\, d\Omega_{d-1}^2 \\ f(r) &= 1 + r^2 - \frac{16\pi GM}{(d-1)\Omega_{d-1}}\frac{1}{r^{d-2}} \end{aligned} \tag{3.10}$$

  - For $d > 2$, this gives the **unique** set of spherically symmetric solutions to Einstein's equations
  - As $r \to \infty$, this approaches (2.4), but at small r, weird things happen
  - The full geometry at small r describes two asymptotically AdS boundaries connected by a wormhole
- A natural thing to do on AdS_{d+1}space is to quantise a free scalar field
- This has action

$$S = -\frac{1}{2}\int d^{d+1}x\, \sqrt{-g}(\partial_\mu\phi\partial_\nu\phi g^{\mu\nu} + m^2\phi^2) \tag{3.11}$$

where $g = \det g_{\mu\nu}$ is the determinant of the metric tensor

- This yields equation of motion

$$\nabla^2\phi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right) = m^2\phi \tag{3.12}$$

which has a basis in global coordinates as

$$f_{\omega\ell\mathbf{m}}(r,t,\Omega) = \Psi_{\omega\ell}(r)e^{-i\omega t}Y_{\ell\mathbf{m}}(\Omega) \tag{3.13}$$

where $\omega^2 = m^2$, $Y_{\ell\mathbf{m}}$ are spherical harmonics, defined to obey

$$\nabla^2 Y_{\ell\mathbf{m}} = -\ell(\ell+d-2)Y_{\ell\mathbf{m}} \tag{3.14}$$

for the spherical Laplacian, and $\Psi_{\mathbf{m}\ell}$ obeys

$$(1+r^2)\Psi'' + \left(\frac{d-1}{r}(1+r^2)+2r\right)\Psi' + \left(\frac{\omega^2}{1+r^2} - \frac{\ell(\ell+d-2)}{r^2} - m^2\right)\Psi = 0 \tag{3.15}$$

- At small r, this asymptotically becomes

$$\Psi'' + \frac{d-1}{r}\Psi' - \frac{\ell(\ell+d-2)}{r^2}\Psi = 0 \tag{3.16}$$

which has solutions

$$\Psi \propto r^{-\frac{d-2}{2}\pm\frac{1}{2}\sqrt{(d-2)^2+4\ell(\ell+d-2)}} \tag{3.17}$$

- At large r, we instead have

$$r^2\Psi'' + (d+1)r\Psi' - m^2\Psi = 0 \tag{3.18}$$

with solutions

$$\Psi \propto r^{-\left(\frac{d}{2}\pm\frac{1}{2}\sqrt{d^2+4m^2}\right)} \tag{3.19}$$

- Smoothness at $r=0$ requires we choose the solution with $+$
- At $r\to\infty$, the sign is specified by boundary conditions
- For $m^2 \geq 0$ and $d \geq 2$, the only choice which maintains unitarity and preserves symmetry is the $+$ sign, so

$$\Psi_{\omega\ell}(r) \sim r^{-\Delta}, \quad \Delta \coloneqq \frac{d}{2} + \frac{1}{2}\sqrt{d^2+4m^2} \tag{3.20}$$

- This is known as *standard quantisation*
- The full equation (2.15) can be solved by hypergeometric functions, and imposes quantisation of ω as well - we find

$$\omega = \omega_{n\ell} \coloneqq \Delta + \ell + 2n \tag{3.21}$$

where $n \in \mathbb{N}$

- We are now in position to write down the full solution of the scalar field theory in terms of creation/annihilation operators as

$$\phi(r,t,\Omega) = \sum_{n,\ell,\mathbf{m}}\left(f_{\omega_{n\ell}\ell\mathbf{m}}a_{n\ell\mathbf{m}} + f^*_{\omega_{n\ell}\ell\mathbf{m}}a^\dagger_{n\ell\mathbf{m}}\right) \tag{3.22}$$

with the creation/annihilation operators obeying

$$\left[a_{n\ell\mathbf{m}}, a^\dagger_{n'\ell'\mathbf{m'}}\right] = \delta_{nn'}\delta_{\ell\ell'}\delta_{mm'} \tag{3.23}$$

- This has the usual particle interpretation from QFT
- The quantisation of ω means that the particle spectrum is discrete - this is **very** different to Minkowski space, where we have a continuous spectrum of particle energy and momentum
- Note the analogy to the QM 'particle in a box', which also has a discrete spectrum; this is further evidence that we can think of AdS_{d+1}as being in some sense similar to a finite box

3.2 Conformal Field Theories

- *Conformal field theories (CFTs)* are QFTs invariant under *conformal transformations* (angle-preserving, but not necessarily distance preserving transformations)
- More concretely, the conformal group includes translations, Lorentz transformations, and scaling/dilations
- This includes *special conformal transformations*, defined as

$$x^{\mu\prime} = \frac{x^\mu + a^\mu x^2}{1 + 2a_\nu x^\nu + a^2 x^2} \tag{3.24}$$

- The generators of the conformal group $SO(d,2)$ are:

$$\begin{aligned} D &:= -ix_\mu \partial^\mu && \text{(dilations)} \\ P_\mu &:= -i\partial_\mu && \text{(translations)} \\ K_\mu &:= i(x^2\partial_\mu - 2x_\mu x_\nu \partial^\nu) && \text{(special conformal)} \\ M_{\mu\nu} &:= i(x_\mu \partial_\nu - x_\nu \partial_\mu) && \text{(Poincare)} \end{aligned} \tag{3.25}$$

- In CFTs we are often interested in *primary operators*, which are local operators which transform as

$$\begin{aligned} e^{iD\alpha}\mathcal{O}(x)e^{-iD\alpha} &= e^{\alpha\Delta}\mathcal{O}(e^\alpha x) \\ e^{iK_\mu \alpha^\mu}\mathcal{O}(0)e^{-iK_\mu\alpha^\mu} &= \mathcal{O}(0) \end{aligned} \tag{3.26}$$

  where Δ is a scalar known as the *scaling dimension* of $\mathcal{O}$
- If we act on a primary operator $\mathcal{O}$ repeatedly by derivatives, we obtain *descendant operators*, which have scaling dimension equal to Δ plus the number of derivatives
- Descendant operators are never primary unless they are identically zero
- In a CFT, the set of primary operators and their descendants at any x are in bijection with a complete basis of the Hilbert space of the CFT quantised on S^{d-1}, called the *state-operator correspondence*
- The map from operators to states is defined by performing a path integral over a Euclidean ball centered on the operator; this is invertible, since if we have a state on the boundary of the ball, we can dilate to shrink the ball down to a point, which defines an operator producing the same state once we scale up again
- If the operator has dimension Δ, the state on S^{d-1} has energy $\Delta + E_0$, where E_0 is the ground state energy
- Depending on the dimension, we can't always set $E_0 = 0$

- This relation comes from choosing polar coordinates near to the Euclidean origin, and then setting $\rho = e^\tau$:

$$ds^2 = d\rho^2 + \rho^2 d\Omega_{d-1}^2 = e^{2\tau}(d\tau^2 + d\Omega_{d-1}^2) \tag{3.27}$$

so dilations $\rho' = e^\alpha \rho$ are equivalent to Euclidean cylinder time translations $\tau' = \tau + \alpha$

3.3 The Correspondence

- The AdS/CFT correspondence states: any CFT on $\mathbb{R} \times S^{d-1}$ is equivalent to a theory of quantum gravity in asymptotically $AdS_{d+1} \times M$ spacetime, with M some compact manifold
- Immediately, there's two big questions:
    1. What is the map between observables on both sides?
    2. Which CFTs define 'semiclassical' gravity theories, where the Planck length ℓ_p is much smaller than the AdS scale ℓ and gravity is approximately described by Einstein's equations?
- The solution to Q1 is called the *dictionary*
- The simplest way to view the duality is a Hilbert space isomorphism

$$\phi : \mathcal{H}_{AdS} \to \mathcal{H}_{CFT} \tag{3.28}$$

- We've already seen that the asymptotic symmetry group of $AdS_{d+1}$and the conformal group in d dimensions are both isomorphic to $SO(d,2)$, so our first line in the dictionary is that the unitary operators implementing these symmetries are related by ϕ:

$$\phi \circ U_{AdS} = U_{CFT} \circ \phi \tag{3.29}$$

- More generally, any operator on one side can be mapped to one on the other using ϕ
- We choose a basis where ϕ is just the identity, and assume this from now on
- To formulate more of the dictionary, we need to answer Q2: if the bulk space is not semiclassical, we don't know what other observables to study
- We therefore say that a d dimensional CFT has a *semiclassical dual near the vacuum* if there exists a finite set of CFT primary operators $\{\mathcal{O}_i\}$ and a local bulk effective action $S_{\text{eff}}[\phi_i, \Lambda]$, where Λ is a cutoff which is large compared to $1/\ell$, but is at most $1/\ell_p := G^{-\frac{1}{d-1}}$, and $\{\phi_i\}$ are a finite set of bulk fields including the metric g such that

$$\int D\phi_i \, e^{iS_{\text{eff}}[\phi_i,\Lambda]} \mathcal{O}_{i_1}(t_1,\Omega_1)\ldots\mathcal{O}_{i_n}(t_n,\Omega_n) \approx \langle \mathcal{O}_{i_1}(t_1,\Omega_1)\ldots\mathcal{O}_{i_n}(t_n,\Omega_n)\rangle_{CFT} \tag{3.30}$$

to all orders in $1/\ell\Lambda$, provided n is $O(1)$ in this parameter. The $\mathcal{O}_i$ on the LHS are given by the 'extrapolate dictionary'

$$\lim_{r\to\infty} r^{\Delta_i}\phi_i(r,t,\Omega) = \mathcal{O}_i(t,\Omega) \tag{3.31}$$

where Δ_i is the scaling dimension of $\mathcal{O}_i$. The bulk field configurations we integrate over obey asymptotically AdS boundary conditions, with the $i\epsilon$ prescription chosen to project onto the vacuum at early and late times, and the CFT expectation values are computed in the ground state on S^{d-1}

- This definition ensures that 'bulk particle physics' arises from the CFT appropriately
- (2.31) should be thought of as being analogous to the LSZ reduction formula on flat space
- This definition isn't quite enough to produce everything we know about bulk quantum gravity
- For example, it says nothing about black holes
- We should at least have that the thermal entropy of the CFT on the sphere at high temps. reproduces the black hole entropy we would derive from S_{eff}
- A definition that ensures this is as follows: a d-dimensional CFT has a *semiclassical dual* if in addition to having a semiclassical dual near the vacuum, (2.30) also holds for more general asymptotically AdS boundary conditions which allow the induced metric on the boundary to be arbitrary but fixed, and the expectation values of the $\mathcal{O}_i$s then need to match the CFT correlators on the boundary

3.4 Gapped Large-N Theories

- A natural question to ask is which CFTs have semiclassical duals?
- We have some examples from string theory (most notably, asymptotic $AdS_5 \times S^5$), but we want a more general criterion on CFTs which guarantee a semiclassical dual existing
- We don't have such a condition yet, but we have one that looks to be sufficient
- A *gapped large-N CFT* is a family of CFTs indexed by N such that
  - There is a finite set of 'single-trace' primaries $\mathcal{O}_i$ such that, if they are normalised by $\langle \mathcal{O}_i \mathcal{O}_i \rangle \sim N^0$, then
  $$\langle \mathcal{O}_i \mathcal{O}_j \mathcal{O}_k \rangle \lessapprox \frac{1}{N^{\#/2}} \tag{3.32}$$
  where # is $O(N^0)$
  - There is only one single-trace primary of spin 2 and scaling dimension d, the energy-momentum tensor $T_{\mu\nu}$, and its three point function has a nonzero term which is $O(1/N^{\#/2})$
  - For any collection of single-trace operators $\{\mathcal{O}_{i_1}, \ldots, \mathcal{O}_{i_n}\}$ with $n \sim N^0$, there is an associated 'multi-trace' primary $\mathcal{O}_{i_1 \ldots i_n}$ with dimension $\Delta_{i_1} + \ldots + \Delta_{i_n}$
  - At leading order in $1/N$, correlators of singla dn multi trace operators can all be computed by Wick contraction, and corrections to this statement are at most order $N^{-\#(n-2)/2}$, where n is the number of single trace operators plus the number of multitrace operators counted including multiplicity
  - All operators with $\Delta \sim N^0$ are single/multi trace primaries and their descendants
- We now claim that any gapped large-N theory has a semiclassical dual, and the bulk fields ϕ_i correspond to the single-trace primaries $\mathcal{O}_i$
- This has not been proven, but there's significant evidence in favour of it