Week 4 - Harlow 3 in Pollack Language

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1 von Neumann Algebras - Relevant Bits

- The von Neumann classification theorem tells us that for any von Neumann algebra M on $\mathcal{H}_{\text{code}}$, we have a decomposition

$$\mathcal{H}_{\text{code}} = \oplus_\alpha (\mathcal{H}_{a_\alpha} \otimes \mathcal{H}_{\bar{a}_\alpha}) \tag{1.1}$$

  where M is given by the set of all operators $\tilde{O}$ which are block diagonal in α, and that within each block they act as $\tilde{O}_{a_\alpha} \otimes I_{\bar{a}_\alpha}$

- In matrix form:

$$\tilde{O} = \begin{pmatrix} \tilde{O}_{a_1} \otimes I_{\bar{a}_1} & 0 & \dots \\ 0 & \tilde{O}_{a_2} \otimes I_{\bar{a}_2} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \tag{1.2}$$

  for any $\tilde{O} \in M$

- The *commutant* of M, denoted M', is the set of all operators which commute with M, which are block-diagonal of the form

$$\tilde{O}' = \begin{pmatrix} I_{a_1} \otimes \tilde{O}'_{\bar{a}_1} & 0 & \dots \\ 0 & I_{a_2} \otimes \tilde{O}'_{\bar{a}_2} \dots & \\ \vdots & \vdots & \ddots \end{pmatrix} \tag{1.3}$$

- The *center* $Z_M \equiv M \cap M'$ consists of operators of the form

$$\tilde{\Lambda} = \begin{pmatrix} \lambda_1 (I_{a_1} \otimes I_{\bar{a}_1}) & 0 & \dots \\ 0 & \lambda_2 (I_{a_2} \otimes I_{\bar{a}_2}) & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \tag{1.4}$$

  where the $\lambda_\alpha \in \mathbb{C}$

- When M is the set of all operators on a tensor factor, Z_M is trivial and M is called a *factor*

- Harlow introduces orthonormal bases $|\widetilde{\alpha, i}\rangle$ and $|\widetilde{\alpha, \bar{i}}\rangle$ for $\mathcal{H}_{a_\alpha}$ and $\mathcal{H}_{\bar{a}_\alpha}$ respectively

- This gives us an orthonormal basis for the full $\mathcal{H}_{\text{code}}$ space:

$$|\widetilde{\alpha, i\bar{i}}\rangle \equiv |\widetilde{\alpha, i}\rangle \otimes |\widetilde{\alpha, \bar{i}}\rangle \tag{1.5}$$

- Given state $\tilde{\rho}$ and von Neumann algebra M on $\mathcal{H}_{\text{code}}$, we can define an algebraic entropy for $\tilde{\rho}$ on M which reduces to standard von Neumann entropy when M is a factor

- We first define
$$p_\alpha \tilde{\rho}_{a_\alpha} \equiv \mathrm{Tr}_{\overline{a}_\alpha} \tilde{\rho}_{\alpha\alpha} \tag{1.6}$$
where $\tilde{\rho}_{\alpha\alpha}$ are the diagonal blocks of $\tilde{\rho}$, and $p_\alpha \in [0,1]$ chosen so $\mathrm{Tr}_{a_\alpha} \tilde{\rho}_{a_\alpha} = 1$

- This further implies $\sum_\alpha p_\alpha = 1$

- We then define algebraic entropy as
$$S(\tilde{\rho}, M) \equiv -\sum_\alpha p_\alpha \log p_\alpha + \sum_\alpha p_\alpha S(\tilde{\rho}_\alpha) \tag{1.7}$$

- We similarly define the entropy of $\tilde{\rho}$ on M' via
$$p_\alpha \tilde{\rho}_{\overline{a}_\alpha} \equiv \mathrm{Tr}_{a_\alpha} \tilde{\rho}_{\alpha\alpha} \tag{1.8}$$
and then
$$S(\tilde{\rho}, M') \equiv -\sum_\alpha p_\alpha \log p_\alpha + \sum_\alpha p_\alpha S(\tilde{\rho}_{\overline{a}_\alpha}) \tag{1.9}$$

1.1 Harlow Theorem 3

- This is Harlow's most general theorem on error correction

Theorem 1.1. *Suppose $\mathcal{H}$ is a finite dimensional Hilbert space, which has a tensor product structure $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\overline{A}}$, and suppose $\mathcal{H}_{code} \subseteq \mathcal{H}$ is a subspace on which we have a von Neumann algebra M. Set $|\widetilde{\alpha, i\bar{i}}\rangle$ to be an orthonormal basis of $\mathcal{H}_{code}$ which is compatible with the block decomposition induced by M. Also define $|\phi\rangle \equiv \frac{1}{\sqrt{|R|}} \sum_{\alpha,i,\bar{i}} |\alpha, i\bar{i}\rangle_R |\widetilde{\alpha, i\bar{i}}\rangle_{A\overline{A}}$, where R is an auxiliary system with $\mathcal{H}_R = \mathcal{H}_{code}$. Then, the following 4 statements are equivalent:*

1. *For any operator $\tilde{O} \in M$, there exists an operator O_A on $\mathcal{H}_A$ with an equivalent action on $\mathcal{H}_{code}$; that is, for any $|\tilde{\psi}\rangle \in \mathcal{H}_{code}$, we have*
$$\begin{aligned} O_A |\tilde{\psi}\rangle &= \tilde{O} |\tilde{\psi}\rangle \\ O_A^\dagger |\tilde{\psi}\rangle &= \tilde{O}^\dagger |\tilde{\psi}\rangle \end{aligned} \tag{1.10}$$

2. *For any operator $X_{\overline{A}}$ on $\mathcal{H}_{\overline{A}}$, we have*
$$P_{code} X_{\overline{A}} P_{code} = X' P_{code} \tag{1.11}$$
where $X' \in M'$, and P_{code} is the projector onto $\mathcal{H}_{code}$.

3. *For any $\tilde{O} \in M$, we have*
$$[O_R, \rho_{R\overline{A}}(\phi)] = 0 \tag{1.12}$$
where O_R is the unique operator on $\mathcal{H}_R$ such that
$$\begin{aligned} O_R |\phi\rangle &= \tilde{O} |\phi\rangle \\ O_R^\dagger |\phi\rangle &= \tilde{O}^\dagger |\phi\rangle \end{aligned} \tag{1.13}$$
which acts with the same matrix elements on R as $\tilde{O}^T$ does on $\mathcal{H}_{code}$.

4. $\sum_\alpha |\mathcal{H}_{a_\alpha}| \leq |\mathcal{H}_A|$, *and we can decompose*

$$\mathcal{H}_A = \oplus_\alpha (\mathcal{H}_{A_1^\alpha} \otimes \mathcal{H}_{A_2^\alpha}) \oplus \mathcal{H}_{A_3} \tag{1.14}$$

with $|A_1^\alpha| = |a_\alpha|$, *and there exists a unitary* U_A *on* $\mathcal{H}_A$ *and sets of orthonormal states* $|\chi_{\alpha,\bar{i}}\rangle_{A_2^\alpha \overline{A}} \in \mathcal{H}_{A_2^\alpha \overline{A}}$ *such that*

$$\widetilde{|\alpha, i\bar{i}\rangle} = U_A\bigg(|\alpha, i\rangle_{A_1^\alpha} \otimes |\chi_{\alpha,\bar{i}}\rangle_{A_2^\alpha \overline{A}} \bigg) \tag{1.15}$$

where $|\alpha, i\rangle_{A_1^\alpha}$ *is an orthonormal basis for* $\mathcal{H}_{A_1^\alpha}$.

- This theorem characterises how well a code subspace can correct a subalgebra M for the erasure of $\overline{A}$
- It reduces to subsystem correction if M is a factor, and to standard correction of $M = \mathcal{L}(\mathcal{H}_{\text{code}})$

Proof. (1) $\implies$ (2): Suppose for sake of contradiction that $P_{\text{code}} X_{\overline{A}} P_{\text{code}} = x' P_{\text{code}}$ where x' is an operator on $\mathcal{H}_{\text{code}}$ but is not in M'. Then there must exist an $\tilde{O} \in M$ and a state $|\tilde{\psi}\rangle \in \mathcal{H}_{\text{code}}$ such that

$$\langle\tilde{\psi}|[x', \tilde{O}]|\tilde{\psi}\rangle = \langle\tilde{\psi}|[X_{\overline{A}}, \tilde{O}]|\tilde{\psi}\rangle \neq 0 \tag{1.16}$$

but such an $\tilde{O}$ cannot have a corresponding O_A as such an O_A would automatically commute with $X_{\overline{A}}$.
(2) $\implies$ (3): Say $\tilde{O} \in M$, and $X_{\overline{A}}$ and Y_R are arbitrary operators on $\mathcal{H}_{\overline{A}}$ and $\mathcal{H}_R$ respectively. We then have:

$$\begin{aligned}
\mathrm{Tr}_{R\overline{A}}\left(O_R \rho_{R\overline{A}}(\phi) X_{\overline{A}} Y_R\right) &= \langle\phi|X_{\overline{A}} Y_R O_R|\phi\rangle \\
&= \langle\phi|X_{\overline{A}} Y_R \tilde{O}|\phi\rangle \\
&= \langle\phi|\tilde{O} X_{\overline{A}} Y_R|\phi\rangle \\
&= \langle\phi|O_R X_{\overline{A}} Y_R|\phi\rangle \\
&= \mathrm{Tr}_{R\overline{A}}(\rho_{R\overline{A}}(\phi) O_R X_{\overline{A}} Y_R)
\end{aligned} \tag{1.17}$$

where the first equality is due to a small calculation, the second is by definition of O_R, the third is by the fact that $X_{\overline{A}}$ commutes with $\tilde{O}$ by (3), and the last is just a calculation. This can only hold for arbitrary $X_{\overline{A}}$ and Y_R if $[O_R, \rho_{R\overline{A}}(\phi)] = 0$.
(3) $\implies$ (4): Our basis $|\alpha, i\bar{i}\rangle_R$ for $\mathcal{H}_R$ gives a decomposition for $\mathcal{H}_{R\overline{A}} \equiv \mathcal{H}_R \otimes \mathcal{H}_{\overline{A}}$ as

$$\mathcal{H}_{R\overline{A}} = \oplus_\alpha \left(\mathcal{H}_{R_\alpha} \otimes \mathcal{H}_{\overline{R}_\alpha} \otimes \mathcal{H}_{\overline{A}}\right) \tag{1.18}$$

From (3), we know that $[O_R, \rho_{R\overline{A}}(\phi)] = 0$ for all such O_R as defined. This means that the reduced state of $\rho_{R\overline{A}}(\phi)$ on R alone must be $I_R/|R|$. We therefore have

$$\rho_{R\overline{A}}(\phi) = \oplus_\alpha \left[\frac{|R_\alpha||\overline{R}_\alpha|}{|R|} \left(\frac{I_{R_\alpha}}{|R_\alpha|} \otimes \rho_{\overline{R}_\alpha \overline{A}}\right)\right] \tag{1.19}$$

for some state $\rho_{\overline{R}_\alpha \overline{A}}$, with the coefficient out the front to ensure this is a valid state satisfying $\mathrm{Tr}_{R\overline{A}}(\rho_{R\overline{A}}(\phi)) = 1$. Moreover, since $\rho_R = I_R/|R|$, we must have $\mathrm{Tr}_{\overline{A}}(\rho_{\overline{R}_\alpha \overline{A}}) = I_{\overline{R}_\alpha}/|\overline{R}_\alpha|$.
Now, since $|\phi\rangle$ purifies $\rho_{R\overline{A}}$ on A, and in a purification the dimension of the purifying system is necessarily as big as the rank of the state being purified (cf Schmidt decomposition), so denoting the rank of $\rho_{\overline{R}_\alpha \overline{A}}$ as $|\rho_{\overline{R}_\alpha \overline{A}}|$, we have

$$\sum_\alpha |R_\alpha||\rho_{\overline{R}_\alpha \overline{A}}| \leq |A| \tag{1.20}$$

This means we can indeed decompose

$$\mathcal{H}_A = \oplus_\alpha \left(\mathcal{H}_{A_1^\alpha} \otimes \mathcal{H}_{A_2^\alpha}\right) \oplus \mathcal{H}_{A_3} \tag{1.21}$$

where $|A_1^\alpha| = |R_\alpha| = |a_\alpha|$ and $|A_2^\alpha| \geq |\rho_{\overline{R}_\alpha \overline{A}}|$. For each α, we can therefore purify $\rho_{\overline{R}_\alpha \overline{A}}$ on A_2^α, and since $\mathrm{Tr}_{\overline{A}}(\rho_{\overline{R}_\alpha \overline{A}}) = I_{\overline{R}_\alpha}/|\overline{R}_\alpha|$, this purification looks like

$$|\psi_\alpha\rangle_{\overline{R}_\alpha A_2^\alpha \overline{A}} = \frac{1}{\sqrt{|\overline{R}_\alpha|}} \sum_{\bar{i}} |\alpha, \bar{i}\rangle_{\overline{R}_\alpha} |\chi_{\alpha,\bar{i}}\rangle_{A_2^\alpha \overline{A}} \tag{1.22}$$

where the $|\chi_{\alpha,\bar{i}}\rangle$s are mutually orthonormal on $A_2^\alpha \overline{A}$. This means a purification for $\rho_{R\overline{A}}$ on the full A system is

$$|\phi'\rangle = \sum_{\alpha, i\bar{i}} \frac{1}{\sqrt{|R_\alpha|}} |\alpha, i\rangle_{R_\alpha} |\alpha, i\rangle_{A_1^\alpha} |\psi_\alpha\rangle_{\overline{R}_\alpha A_2^\alpha \overline{A}} = \frac{1}{\sqrt{|R|}} \sum_{\alpha, i\bar{i}} |\alpha, i\bar{i}\rangle_R |\alpha, i\rangle_{A_1^\alpha} |\chi_{\alpha,\bar{i}}\rangle_{A_2^\alpha \overline{A}} \tag{1.23}$$

and since $|\phi\rangle$ and $|\phi'\rangle$ are two purifications of $\rho_{R\overline{A}}$ on A, they must differ by some unitary U_A.

(4) $\implies$ (1): Just define

$$O_A \equiv U_A(\oplus_\alpha(O_{A_1^\alpha} \otimes I_{A_2^\alpha}))U_A^\dagger \tag{1.24}$$

where $O_{A_1^\alpha}$ acts on $\mathcal{H}_{A_1^\alpha}$ in the same way as $\tilde{O}_{a_\alpha}$ from (1.2) does on $\mathcal{H}_{a_\alpha}$. □

- Harlow defines a *subalgebra code with complementary recovery* as one where not only can we represent any element of M on A as in (1), but we can also represent any element of M' on $\overline{A}$

- Equivalence of (1) and (4) then tells us that

$$|\widetilde{\alpha, i\bar{i}}\rangle = U_A U_{\overline{A}} \bigg(|\alpha, i\rangle_{A_1^\alpha} |\alpha, \bar{i}\rangle_{\overline{A}_1^\alpha} |\chi_\alpha\rangle_{A_2^\alpha \overline{A}_2^\alpha} \bigg) \tag{1.25}$$

where we've decomposed

$$\mathcal{H}_{\overline{A}} = \oplus_\alpha \bigg(\mathcal{H}_{\overline{A}_1^\alpha} \otimes \mathcal{H}_{\overline{A}_2^\alpha} \bigg) \oplus \mathcal{H}_{\overline{A}_3} \tag{1.26}$$

where $|\overline{A}_1^\alpha| = |\bar{a}_\alpha|$

1.2 RT Formula Stuff

- Consider arbitrary encoded state $\tilde{\rho}$ in a subalgebra code with complementary recovery on A and $\overline{A}$

- Recall that in order to define an algebraic entropy of $\tilde{\rho}$ on M, we considered the diagonal blocks $\tilde{\rho}_{\alpha\alpha}$ of $\tilde{\rho}$ and defined

$$p_\alpha \tilde{\rho}_{a_\alpha} \equiv \mathrm{Tr}_{\bar{a}_\alpha} \tilde{\rho}_{\alpha\alpha} \tag{1.27}$$

so that

$$S(\tilde{\rho}, M) \equiv -\sum_\alpha p_\alpha \log p_\alpha + \sum_\alpha p_\alpha S(\tilde{\rho}_{a_\alpha}) \tag{1.28}$$

- Now, consider (1.25): we can rewrite it as

$$|\widetilde{\alpha, i\bar{i}}\rangle = |\widetilde{\alpha, i}\rangle_{a_\alpha} \otimes |\widetilde{\alpha, \bar{i}}\rangle_{\bar{a}_\alpha} = U_A U_{\overline{A}} \bigg(|\alpha, i\rangle_{A_1^\alpha} |\alpha, \bar{i}\rangle_{\overline{A}_1^\alpha} |\chi_\alpha\rangle_{A_2^\alpha \overline{A}_2^\alpha} \bigg) \tag{1.29}$$

- So, if we define $\rho_{A_1^\alpha}$ and $\rho_{\overline{A}_1^\alpha}$ to have the same matrix elements on $\mathcal{H}_{A_1^\alpha}$ and $\mathcal{H}_{\overline{A}_1^\alpha}$ as $\tilde{\rho}_{a_\alpha}$ and $\tilde{\rho}_{\overline{a}_\alpha}$ do on $\mathcal{H}_{a_\alpha}$ and $\mathcal{H}_{\overline{a}_\alpha}$, we see that for arbitrary $\tilde{\rho}$ with diagonal blocks $\tilde{\rho}_{\alpha\alpha}$, we have

$$\tilde{\rho}_{\alpha\alpha} = p_\alpha \tilde{\rho}_{a_\alpha} \otimes \tilde{\rho}_{\overline{a}_\alpha} = U_A U_{\overline{A}} \Big(p_\alpha \rho_{A_1^\alpha} \otimes \rho_{A_2^\alpha} \otimes |\chi_\alpha\rangle \langle \chi_\alpha| \Big) U_{\overline{A}}^\dagger U_A^\dagger \tag{1.30}$$

- We therefore obtain reduced density matrices on A and $\overline{A}$ as

$$\begin{aligned} \tilde{\rho}_A &\equiv \mathrm{Tr}_{\overline{A}} \tilde{\rho} = U_A \Big(\oplus_\alpha \left(p_\alpha \rho_{A_1^\alpha} \otimes \chi_{A_2^\alpha} \right) \Big) U_A^\dagger \\ \tilde{\rho}_{\overline{A}} &\equiv \mathrm{Tr}_A \tilde{\rho} = U_{\overline{A}} \Big(\oplus_\alpha \left(p_\alpha \rho_{\overline{A}_1^\alpha} \otimes \chi_{\overline{A}_2^\alpha} \right) \Big) U_{\overline{A}}^\dagger \end{aligned} \tag{1.31}$$

  where $\chi_{A_2^\alpha} = \mathrm{Tr}_{\overline{A}_2^\alpha} |\chi_\alpha\rangle \langle \chi_\alpha|$ etc

- So, defining area operator

$$\mathcal{L}_A \equiv \oplus_\alpha S(\chi_{A_2^\alpha}) I_{a_\alpha \overline{a}_\alpha} \tag{1.32}$$

  we find Ryu-Takayanagi formulae

$$\begin{aligned} S(\tilde{\rho}_A) &= \mathrm{Tr} \tilde{\rho} \mathcal{L}_A + S(\tilde{\rho}, M) \\ S(\tilde{\rho}_{\overline{A}}) &= \mathrm{Tr} \tilde{\rho} \mathcal{L}_A + S(\tilde{\rho}, M') \end{aligned} \tag{1.33}$$

- Note that the area operator is now non-trivial, since $S(\chi_{A_2^\alpha})$ can take different values for different α

- Moreover, $\mathcal{L}_A$ is clearly in Z_M

- Harlow also shows that obeying RT formulae in this way also implies complementary recovery, so in some sense complementary recovery and this 'two-sided RT formula' for A and $\overline{A}$ are equivalent

- Pollack shows that the existence of complementary recovery determines the von Neumann algebra M uniquely - so why don't we do away with M?

- He suggests the possibility of a 'one-sided RT formula' obeyed by A but not $\overline{A}$ - is this mathematically possible?

- Might give this a go lol

2 The Pollack Way

2.1 Complementary Recovery

- Harlow thinks of quantum erasure correction in terms of a subspace $\mathcal{H}_{\text{code}} \subseteq \mathcal{H}$; Pollack et al. work instead in terms of a *logical space* $\mathcal{H}_L$ which is thought of as being separate from $\mathcal{H}$ with the same dimensionality as $\mathcal{H}_{\text{code}}$

- An *encoding isometry* $V : \mathcal{H}_L \to \mathcal{H}$ then takes logical states and encodes them in the physical Hilbert space

- The image of V is $\mathcal{H}_{\text{code}}$

- We intuitively think of this as fixing a basis for $\mathcal{H}_{\text{code}}$ as $\mathcal{H}_{\text{code}}$ is invariant under the action $V \to VU_L$ for some unitary U_L on $\mathcal{H}_L$

- This view makes the notion of the center of a von Neumann algebra on $\mathcal{H}_L$ simpler to understand, and in constructing explicit examples it is easier to write down V rather than $\mathcal{H}_{\text{code}}$
- In holography, there are two properties that are relevant for an RT formula: the subregion A which determines entropy S_A and the visible bulk degrees of freedom which determine the entropy $S_{\text{bulk},A}$
- The visible degrees of freedom are denoted by a von Neumann algebra M, and clearly (V, A, M) are related, so we establish the following nomenclature

  Definition 2.1 (Correctable and Private Algebra)**.** Suppose $V : \mathcal{H}_L \to \mathcal{H}$ is an encoding isometry V for some quantum error correcting code, and A is a subregion of $\mathcal{H}$ inducing the factorisation $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\overline{A}}$. A von Neumann algebra $M \subseteq \mathcal{L}(\mathcal{H}_L)$ is said to be:

  – **Correctable** from A with respect to V if $M \subseteq V^\dagger(\mathcal{L}(\mathcal{H}_A) \otimes I_A)V$; for every $O_L \in M$ there exists an $O_A \in \mathcal{L}(\mathcal{H}_A)$ such that $O_L = V^\dagger(O_A \otimes I_{\overline{A}})V$.

  – **Private** from A with respect to V if $V^\dagger(\mathcal{L}(\mathcal{H}_A) \otimes I_{\overline{A}})V \subseteq M'$; for every $O_A \in \mathcal{L}(\mathcal{H}_A)$, $V^\dagger(O_A \otimes I_{\overline{A}})V$ commutes with every operator in M.

- Intuitively: think of M being correctable from A if every operator in M maps to a non-trivial operator on the tensor factor A only, and private from A if the preimage of any operator in A commutes with all of M
- If M is correctable, then it is a set of logical operators which can be performed on the encoded state given access to A only
- A hermitian element of M then corresponds to an observable on $\mathcal{H}_L$ which can be measured from A - M tells about which parts of a logical state can be recovered from knowledge of A only
- Conversely, if M is private then the observables in M tell us which parts of a logical state are invisible from A
- We then can talk about *complementary recovery* in this language

  Definition 2.2. A code with encoding isometry $V : \mathcal{H}_L \to \mathcal{H}$, a subregion of the physical Hilbert space A, and a von Neumann algebra $M \subseteq \mathcal{L}(\mathcal{H}_L)$, together taken as a triplet (V, A, M) exhibit **complementary recovery** if:

  – M is correctable from A with respect to V: $M \subseteq V^\dagger(\mathcal{L}(\mathcal{H}_A) \otimes I_{\overline{A}})V$

  – M' is correctable from $\overline{A}$ with respect to V: $M' \subseteq V^\dagger(I_A \otimes \mathcal{L}(\mathcal{H}_{\overline{A}}))V$.

- So far, this doesn't restrict the form M can take very much: note that if $N \subset M$ is a subalgebra and M is correctable, then so too is N
- It would therefore seem logical that if (V, A, M) has complementary recovery, then so too does (V, A, N)
- We actually find that complementary recovery is so restrictive on M that it gets determined uniquely, and such subalgebras N do not exhibit complementary recovery
- This is important for holography because the von Neumann algebra M is important in the RT formula: it tells us how to define the entropy of the bulk degrees of freedom visible from A $S_{\text{bulk},A}$ via an algebraic entropy $S(M, A)$

- For this to make sense, M needs to be uniquely determined by V and the subregion A
- To show this, Pollack quotes the following result:

  Theorem 2.1 (Correctability $\leftrightarrow$ Privacy)**.** *A von Neumann algebra M is correctable from A with respect to V if and only if M is private from $\overline{A}$ with respect to V.*

- The proof of this is quite straightforward

  Proof. ($\Longrightarrow$): Suppose $\exists O_{\overline{A}} \in \mathcal{L}(\mathcal{H}_{\overline{A}})$ such that $V^\dagger(I_A \otimes I_{\overline{A}})V$ doesn't commute with some $O_L \in M$. But since M is correctable from A, we know $\exists O_A \in \mathcal{L}(\mathcal{H}_A)$ such that $O_L = V^\dagger(O_A \otimes I_{\overline{A}})V$, which clearly does commute with $V^\dagger(I_A \otimes I_{\overline{A}})V$, so such an $O_{\overline{A}}$ cannot exist and thus M is private from $\overline{A}$ with respect to V.
  ($\Longleftarrow$): We know that for all $O_{\overline{A}} \in \mathcal{L}(\mathcal{H}_{\overline{A}})$, $V^\dagger(I_A \otimes O_{\overline{A}})V$ commutes with all $O_L \in M$. But this is only true for arbitrary O_L and $O_{\overline{A}}$ if O_L can be written as $O_L = V^\dagger(O_A \otimes I_{\overline{A}})V$ by Schur's lemma, which is just the definition of M being correctable on A with respect to V. □

- This lemma essentially shows that for erasures, correctability is in some sense complementary to privacy: correctability on A implies that $\overline{A}$ is erased
- Pollack says that informally, a subsystem B of a Hilbert space $\mathcal{H} = A \otimes B$ is private if it completely decoheres after the action of a quantum channel
- Pollack now proves that the von Neumann algebra M defined by complementary recovery is in fact unique

  Theorem 2.2 (Uniqueness of complementary von Neumann algebra)**.** *Suppose V is an encoding isometry and A is a subregion inducing $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\overline{A}}$. Let $M = V^\dagger(\mathcal{L}(\mathcal{H}_A) \otimes I_{\overline{A}})V$ be the image of operators on $\mathcal{H}_A$ projected onto $\mathcal{H}_L$. If M is a von Neumann algebra, then is is the unique von Neumann algebra satisfying complementary recovery on V and A. If M is not a von Neumann algebra, then no von Neumann algebra satisfying complementary recovery exists.*

  Proof. The theorem essentially has two conditions:

  – **Existence**: If $M = V^\dagger(\mathcal{L}(\mathcal{H}_A) \otimes I_{\overline{A}})V$ is a von Neumann algebra, then (V, A, M) have complementary recovery.
  – **Uniqueness**: If $N \subseteq V^\dagger(\mathcal{L}(\mathcal{H}_A) \otimes I_{\overline{A}})V$ is a von Neumann algebra, then (V, A, N) do not have complementary recovery.

  which we prove in turn.
  Existence: Assume $M = V^\dagger(\mathcal{L}(\mathcal{H}_A) \otimes I_{\overline{A}})V$ is a von Neumann algebra, so the commutant M' is well defined. The first condition of complementary recovery (M is correctable on A with respect to V) is trivial by definition of M. Also note that by the bicommutant theorem:

  $$V^\dagger(\mathcal{L}(\mathcal{H}_A) \otimes I_{\overline{A}})V \subseteq M = M'' \tag{2.1}$$

  so by definition of privacy, M' is private from A with respect to V. So by theorem (2.1), M' is correctable from $\overline{A}$ with respect to V, which is the second condition for complementary recovery.
  Uniqueness: Suppose $N \subsetneq V^\dagger(\mathcal{L}(\mathcal{H}_A) \otimes I_{\overline{A}})V$ is a von Neumann algebra which is correctable from A but not equal to the full set of correctable operators. For sake of contradiction, assume (V, A, N) have complementary recovery. By the second condition

of complementary recovery, N' is correctable from $\overline{A}$ with respect to V, so by theorem (2.1), N' is private from A with respect to V. This means:

$$V^\dagger(\mathcal{L}(\mathcal{H}_A)\otimes I_{\overline{A}})V \subseteq N'' = N \tag{2.2}$$

by the bicommutant theorem, and so we have

$$N \subsetneq V^\dagger(\mathcal{L}(\mathcal{H}_A)\otimes I_{\overline{A}})V \subseteq N \tag{2.3}$$

which is a contradiction since $N \subseteq N$ is a tautology, so such an N cannot exist. □

- Complementarity seems like a natural property for quantum error correcting codes to have
- It turns out that complementary recovery implies an RT formula, which is surprising
- The fact that von Neumann algebras with complementary recovery can fail to exist is quite non-trivial - to this end, we give an example of a code without complementary recovery

Example 2.1. *We let the physical space be that of two qubits: $\mathcal{H} = \mathcal{H}_2 \otimes \mathcal{H}_2$, and let the logical space be a qutrit $\mathcal{H}_L = span(|0\rangle, |1\rangle, |2\rangle)$. We denote the first qubit of $\mathcal{H}$ by A, and then give encoding isometry*

$$V = |00\rangle\langle 0| + |01\rangle\langle 1| + |10\rangle\langle 2| \tag{2.4}$$

The set of correctable operators is then

$$V^\dagger(\mathcal{L}(\mathcal{H}_A)\otimes I_{\overline{A}})V = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & a \end{pmatrix} \tag{2.5}$$

where $a, b, c, d \in \mathbb{C}$. This set itself is not closed under multiplication, so cannot be a von Neumann algebra, so we take M to be the largest von Neumann algebra contained within $V^\dagger(\mathcal{L}(\mathcal{H}_A)\otimes I_{\overline{A}})V$, which is given by

$$M = \begin{pmatrix} a & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & a \end{pmatrix} \tag{2.6}$$

(V, A, M) satisfy the first condition of complementary recovery (M is correctable from A with respect to V), but not the second (M' is correctable from $\overline{A}$ with respect to V), since

$$\begin{pmatrix} a & 0 & b \\ 0 & d & 0 \\ c & 0 & e \end{pmatrix} = M' \nsubseteq V^\dagger(I_A \otimes \mathcal{L}(\mathcal{H}_{\overline{A}}))V = \begin{pmatrix} a & 0 & b \\ 0 & a & 0 \\ c & 0 & e \end{pmatrix} \tag{2.7}$$

If we had chosen M to be a smaller subalgebra, then M' would be larger and in particular would contain the above. But since M' is already not contained in the relevant object, there cannot exist a von Neumann algebra with complementary recovery.

2.2 The RT Formula and Properties

- In the language we have so far, we can *define* what an RT formula is, with no reference to holography

Definition 2.3 (RT Formula)**.** Say V is an encoding isometry, A is a subregion, and M is a von Neumann algebra on $\mathcal{H}_L$. We say (V, A, M) has an **RT formula** if there exists an **area operator** $L \in \mathcal{L}(\mathcal{H}_L)$ such that for any state ρ with support on $\mathcal{H}_L$, we have:

$$S(\mathrm{Tr}_{\overline{A}}(V\rho V^\dagger)) = S(M, \rho) + \mathrm{Tr}(\rho L) \tag{2.8}$$

If $L \propto I$, we say (V, A, M) has a trivial RT formula.

- The existence of an RT formula in a code is closely connected to complementary recovery
- Recalling that a von Neumann algebra implies a Wedderburn decomposition on the Hilbert space it acts on ($\mathcal{H} = \oplus_\alpha(\mathcal{H}_{A_\alpha} \otimes \mathcal{H}_{\overline{A}_\alpha})$), Pollack finds that when a von Neumann algebra is correctable from A with respect to V, then the Hilbert space $\mathcal{H}_A$ also decomposes, where the decomposition of this and of $\mathcal{H}_L$ are related
- This is formalised by the following lemma - note the similarity to Harlow 3

Theorem 2.3. *Suppose $V : \mathcal{H}_L \to \mathcal{H}$ is an encoding isometry, A is a subregion inducing $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\overline{A}}$, and M is a von Neumann algebra on $\mathcal{H}_L$ which is correctable from A with respect to V.*
Suppose M induces the decomposition $\mathcal{H}_L = \oplus_\alpha(\mathcal{H}_{L_\alpha} \otimes \mathcal{H}_{\overline{L}_\alpha})$, so that

$$M = \oplus_\alpha(\mathcal{L}(\mathcal{H}_{L_\alpha}) \otimes I_{\overline{L}_\alpha}) \tag{2.9}$$

Set $\{|\alpha, i, j\rangle\}$ to be an orthonormal basis of $\mathcal{H}_L$ which is compatible with M, so α enumerates the diagonal blocks and within each block we have $|\alpha, i, j\rangle = |i_\alpha\rangle_{L_\alpha} \otimes |j_\alpha\rangle_{\overline{L}_\alpha}$ where $\{|i_\alpha\rangle_{L_\alpha}\}$ and $\{|j_\alpha\rangle_{\overline{L}_\alpha}\}$ are orthonormal bases for $\mathcal{H}_{L_\alpha}$ and $\mathcal{H}_{\overline{L}_\alpha}$ respectively.
Then, there exists a factorisation $\mathcal{H}_A = \oplus_\alpha(\mathcal{H}_{A_1^\alpha} \otimes \mathcal{H}_{A_2^\alpha}) \oplus \mathcal{H}_{A_3}$ and a unitary U_A on $\mathcal{H}_A$ such that the state $(U_A \otimes I_{\overline{A}})V\,|\alpha, i, j\rangle$ factorises as

$$(U_A \otimes I_{\overline{A}})V\,|\alpha, i, j\rangle = |\psi_{\alpha,i}\rangle_{A_1^\alpha} \otimes |\chi_{\alpha,j}\rangle_{A_2^\alpha \overline{A}} \tag{2.10}$$

where $|\psi_{\alpha,i}\rangle$ is independent of j and $|\chi_{\alpha,j}\rangle$ is independent of i.

- Note that this is essentially Harlow 3 in the language of Pollack, and this theorem specifically says (1) $\implies$ (4)
- To see why this is true, recall the first condition of Harlow's: for any operator $\tilde{O}$ on M, there exists an operator O_A on $\mathcal{H}_A$ with the same action on $\mathcal{H}_{\text{code}}$, so for any $|\tilde{\psi}\rangle \in \mathcal{H}_{\text{code}}$, we have $O_A\,|\tilde{\psi}\rangle = \tilde{O}\,|\tilde{\psi}\rangle$
- This is just the statement that M is correctable from A with respect to V!
- Indeed, if M is correctable from A with respect to V, then for every $O_L \in M$ there is a corresponding $O_A \in \mathcal{L}(\mathcal{H}_A)$ such that $O_L = V^\dagger(O_A \otimes I_{\overline{A}})V$
- Therefore, for any $|\psi\rangle \in \mathcal{H}_L$ and corresponding $V\,|\psi\rangle \in V(\mathcal{H}_L)$, we clearly have:

$$(O_A \otimes I_{\overline{A}})V\,|\psi\rangle = (VO_LV^\dagger)(V\,|\psi\rangle) = O_L\,|\psi\rangle \tag{2.11}$$

  which is just Harlow's second condition as claimed

- The proof therefore straightforwardly adapts
- The next thing Pollack does is show that complementary recovery implies a 'two-sided RT formula'; that is, (V, A, M) and $(V, \overline{A}, M')$ have an RT formula with the same area operator