Operator-Algebra Quantum Erasure Correction and Holography

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22/06/2023

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Quantum Error Correction

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- Qubits are very sensitive to changes in the environment - suffer **decoherence**
- **Quantum error correction**: finds ways to correct errors at every stage of a computation
- Potential approach to **fault-tolerant** quantum computers
- Rough idea: spread information of a single (**logical**) quantum state across a highly entangled (**physical**) composite state

Holography

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- Famous example: **AdS/CFT correspondence**
- Relationship between gravitational theory on an **Anti-de Sitter space** and a **conformal field theory** on its boundary
- **Holographic dictionary**: provides 1-to-1 correspondence between bulk and boundary

Holographic Error Correction

- 2015: Ahmed Almheiri, Xi Dong, and Daniel Harlow publish paper describing certain aspects of AdS/CFT in the language of error correction[1]

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- Quantum **erasure** correcting codes satisfy algebraic version of the **Ryu-Takayanagi (RT) formula**
- Links **entropy** of a boundary CFT state on subregion A, and the entropy in the gravitational region 'visible' from A
- Unexpected link between entanglement properties of quantum error correcting codes and geometry of spacetime

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- Problem: every third qutrit is erased with **certainty**
- Solution: Alice sends **three** physical qutrits in **code subspace**, spanned by

$$\begin{aligned} |\tilde{0}\rangle &= \frac{1}{\sqrt{3}}(|000\rangle + |111\rangle + |222\rangle) \\ |\tilde{1}\rangle &= \frac{1}{\sqrt{3}}(|012\rangle + |120\rangle + |201\rangle) \\ |\tilde{2}\rangle &= \frac{1}{\sqrt{3}}(|021\rangle + |102\rangle + |210\rangle) \end{aligned} \tag{2}$$

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- With access to only first two physical qutrits, Bob can construct intended logical qutrit in the first physical register!

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- Given by

$$O_{12} = U_{12}^{\dagger} \tilde{O} U_{12} \tag{7}$$

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- In example, $\mathcal{H} = \mathcal{H}_{12} \otimes \mathcal{H}_3$
- Correctability phrased in terms of operators (Heisenberg picture) or states (Schrodinger picture)

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- von Neumann algebra M: subset of linear operators on a Hilbert space $\mathcal{H}$, closed under addition, multiplication, Hermitian conjugation, and contains all scalar multiples of the identity
- Also have **commutant** algebra M': algebra of operators commuting with M
- Can generalise (a lot of) objects 'on M', such as generalised **algebraic entropy** or even an algebraic version of the holographic RT formula

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- Correctability of M on A: for any $\tilde{O} \in M$, there is a O_A on $\mathcal{H}_A$ such that for any $|\tilde{\psi}\rangle \in \mathcal{H}_{\text{code}}$, we have

$$O_A |\tilde{\psi}\rangle = \tilde{O} |\tilde{\psi}\rangle , \quad O_A^\dagger |\tilde{\psi}\rangle = \tilde{O}^\dagger |\tilde{\psi}\rangle \tag{9}$$

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- **Complementary recovery** M' is correctable on $\overline{A}$; for any $\tilde{O}' \in M'$, there is an $O_{\overline{A}}$ on $\mathcal{H}_{\overline{A}}$ such that

$$O_{\overline{A}} |\tilde{\psi}\rangle = \tilde{O}' |\tilde{\psi}\rangle , \quad O_{\overline{A}}^\dagger |\tilde{\psi}\rangle = \tilde{O}'^\dagger |\tilde{\psi}\rangle \tag{10}$$

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Theorem

Suppose $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\overline{A}}$ has a code subspace $\mathcal{H}_{code}$, which has a von Neumann algebra M acting on it. Moreover, suppose that (A, M) has complementary recovery. Then, the pairs (A, M) and $(\overline{A}, M')$ both have the same RT formula. The converse also holds.

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- Sometimes phrased as '*spacetime is an error correcting code*'

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- Many generalisations: **approximate** erasure correction, **non-isometric** codes etc.
- Most prolific area: **tensor networks**, which are more interesting from holography point of view
- AdS space is directly 'tiled' with erasure correcting codes based on contracting tensors - provides concrete link between entanglement of codes and geometry of spacetime