Week 4 - Notes

Ben Karsberg

2021-22

1 Harlow Second Theorem

- Harlow's 2nd theorem generalise erasure correction that allows A to access only *partial* information about the encoded state
- He calls this 'subsystem quantum erasure correction'
- The rough idea is to consider a code subspace which itself factorises as $\mathcal{H}_{\text{code}} = \mathcal{H}_a \otimes \mathcal{H}_{\overline{a}}$ and then only ask for recovery of the state of $\mathcal{H}_a$

  Theorem 1.1 (Subsystem Error Correction). *Suppose $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\overline{A}}$, and $\mathcal{H}_{code} = \mathcal{H}_a \otimes \mathcal{H}_{\overline{a}}$ is a subspace of $\mathcal{H}$. Choose orthonormal bases $|i_a\rangle$ of $\mathcal{H}_a$ and $|\overline{i}_{\overline{a}}\rangle$ of $\mathcal{H}_{\overline{a}}$. Then, the following 4 statements are equivalent:*

  1. *For any operator O_a acting on $\mathcal{H}_a$, there exists an operator O_A with an equivalent action; that is, for any $|\psi_L\rangle \in \mathcal{H}_{code}$, we have*
  $$\begin{aligned} O_A |\psi_L\rangle &= O_a |\psi_L\rangle \\ O_A^\dagger |\psi_L\rangle &= O_a^\dagger |\psi_L\rangle \end{aligned} \tag{1.1}$$
  2. *For any operator $X_{\overline{A}}$ on $\mathcal{H}_{\overline{A}}$, we have*
  $$P_{code} X_{\overline{A}} P_{code} = (I_a \otimes X_{\overline{a}}) P_{code} \tag{1.2}$$
  where $X_{\overline{a}}$ is an operator on $\mathcal{H}_{\overline{a}}$
  3. *Define two auxiliary systems R and $\overline{R}$ with $\mathcal{H}_R = \mathcal{H}_a$ and $\mathcal{H}_{\overline{R}} = \mathcal{H}_{\overline{a}}$. Define the state $|\phi\rangle = \frac{1}{\sqrt{|R||\overline{R}|}} \sum_{i,j} |i\rangle_R |\overline{j}\rangle_{\overline{R}} |i_a \overline{j}_{\overline{a}}\rangle_{A\overline{A}}$. Then, in the state $|\phi\rangle$ we have*
  $$\rho_{R\overline{RA}}[\phi] = \rho_R[\phi] \otimes \rho_{\overline{RA}}[\phi] \tag{1.3}$$
  4. *$|a| \leq |A|$, and if we decompose $\mathcal{H}_A = (\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}) \oplus \mathcal{H}_{A_3}$ with $|A_1| = |a|$ and $|A_3| < |a|$, there exists a unitary transformation U_A on $\mathcal{H}_A$ and a set of orthonormal states $|\chi_j\rangle_{A_2\overline{A}} \in \mathcal{H}_{A_2\overline{A}}$ such that*
  $$|i_a \overline{j}_{\overline{a}}\rangle = U_A(|i\rangle_{A_1} \otimes |\chi_j\rangle_{A_2\overline{A}}) \tag{1.4}$$
  where $|i\rangle_{A_1}$ is an orthonormal basis for $\mathcal{H}_{A_1}$.

- The proof is similar to before, but we still go through it for completeness since Harlow does not

Proof. (1) $\implies$ (2): Contradiction again. Suppose there was an $X_{\overline{A}}$ such that $P_{\text{code}} X_{\overline{A}} P_{\text{code}} \neq (I_a \otimes X_{\overline{a}}) P_{\text{code}}$ for any operator $X_{\overline{a}}$. Schur's lemma then implies that there must be an operator O_a on $\mathcal{H}_a$ which does not commute with $X_{\overline{A}}$ and a state $|\psi_L\rangle \in \mathcal{H}_{\text{code}}$ such that

$$\langle\psi_L|[P_{\text{code}} X_{\overline{A}} P_{\text{code}}, O_a]|\psi_L\rangle = \langle\psi_L|[X_{\overline{A}}, O_a]|\psi_L\rangle \neq 0 \tag{1.5}$$

But such an O_a cannot have a representation O_A on $\mathcal{H}_A$ since it would then commute with $X_{\overline{A}}$, which contradicts (1).
(2) $\implies$ (3): First, note that (2) implies

$$\langle i_a \bar{i}_{\overline{a}}|X_{\overline{A}}|j_a \bar{j}_{\overline{a}}\rangle = \langle i_a \bar{i}_{\overline{a}}|P_{\text{code}} X_{\overline{A}} P_{\text{code}}|j_a \bar{j}_{\overline{a}}\rangle = \langle i_a \bar{i}_{\overline{a}}|I_a \otimes X_{\overline{a}}|j_a \bar{j}_{\overline{a}}\rangle = \delta_{ij} \langle \bar{i}_{\overline{a}}|X_{\overline{a}}|\bar{j}_{\overline{a}}\rangle \tag{1.6}$$

and so, for arbitrary operators O_R and $O_{\overline{R}}$ on $\mathcal{H}_R$ and $\mathcal{H}_{\overline{R}}$ respectively, we have

$$\begin{aligned} \langle\phi|O_R O_{\overline{R}} X_{\overline{A}}|\phi\rangle &= \frac{1}{|R||\overline{R}|} \langle i|O_R|j\rangle_R \langle\bar{i}|O_{\overline{R}}|\bar{j}\rangle_{\overline{R}} \langle i_a \bar{i}_{\overline{a}}|X_{\overline{A}}|j_a \bar{j}_{\overline{a}}\rangle_{A\overline{A}} \\ &= \frac{1}{|R||\overline{R}|} \langle i|O_R|i\rangle_R \langle\bar{i}|O_{\overline{R}}|\bar{j}\rangle \langle \bar{i}_{\overline{a}}|X_{\overline{a}}|\bar{j}_{\overline{a}}\rangle_{A\overline{A}} \end{aligned} \tag{1.7}$$

However, also note that

$$\langle\phi|O_R|\phi\rangle = \frac{1}{|R|} \langle i|O_R|i\rangle_R \tag{1.8}$$

and

$$\langle\phi|O_{\overline{R}} X_{\overline{A}}|\phi\rangle = \frac{1}{|\overline{R}|} \langle\bar{i}|O_{\overline{R}}|\bar{j}\rangle_{\overline{R}} \langle \bar{i}_{\overline{a}}|X_{\overline{a}}|\bar{j}_{\overline{a}}\rangle_{A\overline{A}} \tag{1.9}$$

which together mean that

$$\langle\phi|O_R O_{\overline{R}} X_{\overline{A}}|\phi\rangle = \langle\phi|O_R|\phi\rangle \langle\phi|O_{\overline{R}} X_{\overline{A}}|\phi\rangle \tag{1.10}$$

and so provided $|\phi\rangle$ has no non-vanishing connected correlators for any such O_R, $O_{\overline{R}}$, $X_{\overline{A}}$, then

$$\rho_{R\overline{RA}}[\phi] = \rho_R[\phi] \otimes \rho_{\overline{RA}}[\phi] \tag{1.11}$$

as required.
(3) $\implies$ (4): First, note that $|\phi\rangle$ is a purification of $\rho_{R\overline{RA}}[\phi]$ by definition. Moreover, $|\phi\rangle$ maximally entangles R with $\overline{RA}$, so $\rho_R[\phi] = I_R/|R|$, and (3) becomes

$$\rho_{R\overline{RA}}[\phi] = \frac{I_R}{|R|} \otimes \rho_{\overline{RA}}[\phi] \tag{1.12}$$

(basically same) □