AdS/CFT Holography Stuff - Notes

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1 Some GR

- This is **not** a full intro to GR, just the machinery we need to understand holographic error correction
- We take a *spacetime* to be a non-flat D-dimensional pseudo-Riemannian manifold M; it has coordinates $\{x^\mu\}_{\mu=0}^{D-1}$, and the *line element* is defined through the metric $g_{\mu\nu}$ via

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{1.1}$$

- At any point $p \in M$, the metric is a matrix/rank 2 tensor, and when $g_{\mu\nu} = \delta_{\mu\nu}$ we just have flat Euclidean space, which has line element

$$ds^2_{\text{Euc}} = (dx^0)^2 + \ldots + (dx^{D-1})^2 \equiv dx_\mu dx^\mu$$

- In GR, usually one metric element $g_{aa} < 0$, which we call *timelike*; when this is true, we say M is *Lorentzian*
- For example, the SR line element is

$$ds^2_{\text{SR}} = -(dx^0)^2 + \ldots + (dx^{D-1})^2 = dx_\mu dx^\mu \tag{1.2}$$

- The line element allows us to calculate the length of a curve $\gamma(\lambda) = x^\mu(\lambda)$ parametrised by lambda by the standard arc length integral formula:

$$|\gamma| \equiv \int_0^1 d\lambda \sqrt{ds^2} = \int_0^1 d\lambda \sqrt{g_{\mu\nu} \frac{dx^\mu(\lambda)}{d\lambda} \frac{dx^\nu(\lambda)}{d\lambda}} \tag{1.3}$$

- We can consider e.g. the set of all curves connecting two points - individual curves are coordinate dependent, but the whole set depends on the global geometry and choice of point only
- This means any quantity we can compute given the data of the set is coordinate independent; a really common quantity is the distance between two points
- For Euclidean/Riemannian manifolds, we can compute the length of the shortest curve connecting two points - for Lorentzian manifolds, we can add zig-zags in the timelike direction to shorten the curve
- So to generalise, we need to find the length of **extremal** rather than minimal curves
- To find these curves (or *geodesics*), we do some standard calculus of variations, and find stationary points of the functional (1.3) viewed as a functional of γ

- The solution is given by the Euler-Lagrange equations with the action $ds = \sqrt{ds^2}$
- These are called the *geodesic equations* in this context
- The resultant curves in GR are those traced out by freely-falling/non-accelerating observers
- In Lorentzian manifolds, the geodesic distance between points gives a *causal structure* on the manifold: we can classify all pairs of points by the labels *timelike*, *spacelike*, and *null*
- These correspond to positive, negative, and zero geodesic distance
- When two points are timelike separated, we call the geodesic distance between them the *proper time* between them
- Causal structure is a global property of the manifold, and is independent of how we parametrise the metric
- All of this can be generalised to higher-dimensions, and considering surfaces/volumes etc.
- Instead of (1.3), we have some higher-dimensional integral which we can just vary again to find extremal surfaces/volumes etc.
- For ease, we work in 2 spacelike and 1 timelike dimension - $D = 2 + 1$
- This explains the notation in the *Ryu-Takayanagi formula* (the main thing we want to understand):

$$S_A(\rho) = \frac{1}{4G_N} \min_{\gamma_A} Area(\gamma_A) \tag{1.4}$$

- A is a subregion of a spatial slice of the boundary ∂M - a dimension $D-2$ (*codimension* 2) object, and the minimisation is over areas of extremal dimension $D-2$ in the bulk spacetime that touch the boundary at the edge of A
- Our universe has $D = 4 = 3 + 1$, so A would be 2D in this case, but in $D = 3 = 2 + 1$, $D - 2 = 1$ so the boundary is diffeomorphic to a circle, A is a subset of the circle, and the relevant extremal objects are curves labelled as γ_A
- Nonetheless, we call the operator computing their length an *area operator* because in a general spacetime the invariant objects have codimension 2

1.1 Einstein Equations

- GR relates the geometry of M to the matter living inside it via the *Einstein field equations*, which read

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu} \tag{1.5}$$

- What are all these objects? Well:
  - $G_{\mu\nu}$ is the *Einstein curvature tensor*, defined via $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ where $R_{\mu\nu}$ is the Ricci tensor and R is the Ricci scalar
  - $T_{\mu\nu}$ is the *stress-energy tensor*. In a particular coordinate basis, we interpret T_{00} as the energy density (or flux of 0-momentum/energy across surfaces of constant time), T_{0i} as energy flux across surfaces $x^i =$ constant, T^{i0} as momentum density in the x^i direction, and T^{ij} as the flux of x^i momentum across surfaces of constant x^j
  - G_N is just Newton's gravitational constant

- Note these are D^2 equations if we range over μ, ν, but we have symmetry under $\mu \leftrightarrow \nu$, so there are only $D(D-1)/2$ truly independent equations
- We often write the Einstein equations as an initial value problem: if we know the metric, its derivatives, and $T_{\mu\nu}$ at a particular point in time, we can use the field equations to calculate the metric at a later time

1.2 Diffeomorphism Invariance

- There is not a 1:1 correspondence between metrics and geometries - several metrics may describe the same geometry
- This is broadly familiar: in Newtonian physics, we can choose any frame of reference/co-ordinate system etc. and we can tell that two seemingly different frames give rise to the same physics
- Similarly in SR, physics is invariant under Lorentz transformations
- So in GR, to answer the question of whether two metrics describe the same geometries, we should compute invariants and see whether they match in the two different situations
- These invariants are the proper distances along geodesics, which are to do with the causal structure of the metric
- Two metrics with the same causal structure are *diffeomorphic*
- A slightly different viewpoint is that the metric doesn't only contain physical info about a spacetime, it also contains redundant info that doesn't matter to an observer
- These are known as *degrees of gauge freedom*
- When we go from a metric to the physics (geodesics), we 'gauge out' the redundant info and leave only the physical ones
- Two metrics are *gauge equivalent* if they're related by a gauge transformation or diffeomorphism
- We refer to *gauge orbits*, which are equivalence classes of gauge equivalent metrics
- We *gauge fix* by specifying information restricting the gauge orbit to a subset of metrics
- If only one metric remains, we have fully fixed the gauge
- Not all gauge transformations preserve all 'physical' information, which is a particular issue when considering metrics on manifolds with boundaries
- For intuition, think about electrodynamics: the behaviour of charged particles is governed by Maxwell's equations, but the E and B fields are not gauge invariant, only certain combinations (e.g. $E^2 + B^2$) are
- Analogously to the Einstein equations, Maxwell's equations are differential equations so their solution in a bounded region depends on the boundary conditions
- The boundary conditions represent an additional set of physical information, so the set of gauge transformations preserving all physical quantities will be smaller than if we didn't worry about boundary conditions
- The same issue still occurs even if the boundary is at infinity

- We ask whether the gauge transformation has any effect at infinity or if it is distinguishable from the identity at infinity
- If it just looks like the identity, we call it a *small gauge transformation*, and keep in mind that *large* gauge transformations are physical but small ones are not
- In GR, a small gauge transformation of the metric takes $g_{\mu\nu} \mapsto g_{\mu\nu} + \delta g_{\mu\nu}$, where $\lim_{x^\mu \to 0} \delta g_{\mu\nu}(x^\mu) = 0$, but a large gauge transformation can (for example) change the total invariant mass contained within some region
- An easy way to gauge fix in GR is to fix a time direction on the manifold, or identify points on the same 'spatial slice' at fixed time
- In 4 spacetime dimensions, this is called a 3+1 *decomposition* or a *foliation* of the spacetime into spatial slices
- When the spacetime has a boundary, some foliations will coincide on the boundary and others will not - only those which do coincide describe the same physics
- The RT formula applies to spacetimes with a spatial boundary
- Invariant quantities are those which are unchanged by small diffeomorphisms, leaving the boundary unchanged
- The invariant quantities of interest for the RT formula are areas of surfaces which end on the boundary - in 2+1 dimensions, these are geodesics which are between points on the boundary

1.3 Quantum Gravity

- GR is classical - given enough data to describe a situation at an initial time, we can use the theory to find a description at a later or earlier time
- Quantum mechanics is similar - given a Hamiltonian and initial wave function, we can evolve in time by the Schrodinger equation
- For sensible Hamiltonians, the expectation values of observables evolve smoothly, but when measuring an observable itself, we project onto one of its eigenstates - only by repeated measurements do we have access to the expectation value
- Finding a full theory of quantum gravity is **really** hard, but we can say some things with confidence
- In the quantum theory of a point particle, the classical observables are operators which measure position, velocity etc.
- The classical observables of a field theory are similarly the operators which measure field values and their derivatives
- For a gravitational theory, the classical observables then are those which when we apply then to a state on a classical geometry measure the metric, stress tensor etc.
- This means that at worst, we expect the Einstein equations to be the classical limit of some non-classical theory

- Loosely, this means we should schematically have an operator equation looking like

$$\hat{G}_{\mu\nu} \text{“} = \text{”} 8\pi G_N \hat{T}_{\mu\nu} \tag{1.6}$$

  where the hats denote operators

- What this really means is that for classical states $|\Psi\rangle$ which are simultaneous eigenstates of both operators, we have

$$\hat{G}_{\mu\nu} |\Psi\rangle = G_{\mu\nu} |\Psi\rangle = 8\pi G_N \hat{T}_{\mu\nu} |\Psi\rangle = 8\pi G_N T_{\mu\nu} |\Psi\rangle \tag{1.7}$$

- This automatically implies

$$\langle \hat{G}_{\mu\nu} \rangle_\Psi = 8\pi G_N \langle \hat{T}_{\mu\nu} \rangle_\Psi \tag{1.8}$$

- To move away from classical/exact eigenstates, we have two options:
  1. Use perturbation theory to examine what happens if we measure operators on states close to classical states
  2. We can use the linearity of QM to discuss *superpositions* of separate geometries
- There's a slight issue with the second: linearity of QM only applies to states in a fixed Hilbert space
- For example, for the quantum mechanical theory of a single particle in a potential, we have a position operator on this Hilbert space and we know how it acts on position eigenstates and general states written in the position basis, but it does **not** tell us how to act on single particle states in a different potential
- To answer this, we could e.g. couple the particle to an external field, so the original potential is recovered for a fixed state of the field
- In this new, larger Hilbert space, we can talk about a superposition of a particle in one potential with a particle in a different potential
- To do this, we need to understand how to embed the original Hilbert space into the larger one
- This is (finally) where *holography* comes in

1.4 Holography

- A holographic theory is one where a gravitational theory can be described using a non-gravitational one 'at the boundary' in some sense
- These theories can be used to describe superpositions of states describing different geometries
- Annoyingly, all our best-known holographic theories have *negative* curvature, whereas our universe seemingly has *positive* curvature
- We therefore can't just interpret our universe as a state in a holographic Hilbert space

1.4.1 The Holographic Dictionary

- The main objects in a quantum theory are states and observables
- When a state describes a spatial slice of a classical spacetime geometry satisfying the Einstein equations, it is an eigenstate of some specific operators, with eigenvalues corresponding to diffeomorphism-invariant quantities
- We can then compute the expectations of these operators on states close to the classical ones, and linearity then allows us to find the expectations on superpositions of near-classical states
- In *anti-de Sitter (AdS)* geometries (negative curvature) in D dimensions, the expectation values of all these operators can alternatively be computed using operators in a non-gravitational $D-1$ dimensional theory
- One major result is that there is an exact dictionary for matching operators in the gravitational theory at points near the AdS boundary to operators in the boundary theory, and a precise way of integrating over points on the boundary to reconstruct operators deeper in the bulk
- There's two important properties of these boundary theories:
  1. This type of correspondence makes sense only if the boundary theory has the same symmetries as the gravitational one at its boundary. These are the small gauge symmetries of diffeomorphisms which leave the boundary at spatial infinity unchanged - for the metric describing hyperbolic space, the group of transformations which do this is the *conformal group*, and so we have conformal field theories (CFTs)
  2. Hyperbolic metrics have a natural length scale - the *anti-de Sitter length* - and so too do Einstein's equations, the Planck scale. The ratio of these two scales is a dimensionless number which also appears on the boundary CFT as the number of fields in the theory. In spacetimes which look classical, the ratio needs to be very large - loosely, it's the ratio between cosmological and quantum gravitational scales
- As a result of this second point, boundary CFTs that can describe classical-like geometries have a huge number of fields, and are often referred to as large-N CFTs
- It can be shown that in conformal theories, the larger the number of fields, the stronger they couple to each other: so when used to describe classical geometries, the AdS/CFT correspondence relates gravity in asymptotically AdS spacetimes to the behaviour of strongly-coupled CFTs
- This may seem like mere coincidence, but there's a few reasons why this is an exciting correspondence:
  - Some quantities are more easily calculated in a gravitational theory than a non-gravitational one and vice versa; for example, the RT formula exploits this. It's easy to compute the areas of extremal curves from the metric, but to compute the entropy of a CFT state is non-trivial
  - We can freely compute the expectations of states which **aren't** close to classical - ones which do not even remotely solve the Einstein equations. These states still live in the same Hilbert space as the near-classical states

1.5 The Ryu-Takayanagi (RT) Formula

- Let's return to the RT formula (1.4): this is specifically the version that applies to a holographic state dual to a classical geometry

$$S_A(\rho) = \frac{1}{4G_N} \min_{\gamma_A} Area(\gamma_A) \tag{1.9}$$

- We now know what the RHS means: we can find geodesics/extremal surfaces from the metric, and consider those which hit the boundary at precisely the edge of the boundary region A
- If there are multiple such geodesics, we simply choose the one with smallest area
- The LHS refers to the boundary CFT
- The state ρ is in the Hilbert space of a large-N CFT, with N related to G_N as discussed above, and S_A is the entanglement entropy of it on A
- More generally, the RT formula is

$$S_A(\rho) = S_{\text{bulk},A}(\rho) + \text{Tr}(L\rho) \tag{1.10}$$

  which applies to states with entanglement in the bulk and superpositions of geometries
- Here, L is an operator in the quantum-gravitational Hilbert space, with eigenstates dual to classical geometries and eigenvalues being the areas of the minimal surfaces which meet the boundary subregion
- There are many states which live on the same curved spacetime, but the bulk entanglement term identifies which equivalence class of these states is described by ρ
- This means neither the LHS or RHS of (1.10) are the expectation value of some observable, but the difference between the boundary and bulk entropies **is**
- It turns out L can be obtained given access either to the reduced state in A or its complement $\overline{A}$

1.6 The Causal and Entanglement Wedges

- We want to know which operators in the bulk can be reconstructed given access to a specific boundary region
- Due to the causal structure of Lorentzian manifolds, only those parts of the boundary which are causally related to a point or region of the bulk can affect an operator there
- This is formalised by the notion of the *causal wedge* (see notes page 23 for diagram)
- On a time-slice of the boundary, choose a spatial sub-region A: the causal wedge $\mathcal{C}[A]$ of A is the bulk region bounded by both the boundary domain of dependence of A and the set of bulk geodesics which start and end on it
- It's often useful to not work with the full causal wedge, but some particular spatial slice within it
- If the RT surface is spacelike, all spatial slices in the causal wedge end on the RT surface itself but they hit the boundary at different times

- It's usually most convenient to choose a spatial slice intersecting the boundary at A itself
- If the spacetime is static (for example), we can pick a spatial slice extending between A and its RT surface, which by causality contains all info needed to reproduce the whole causal wedge (see notes page 24 for diagram)
- We know that given access to the entropy of a CFT subregion we can use the RT formula to work out the area of the relevant extremal surface
- We expect that if we know the full reduced density matrix in addition to just the entropy, we can construct e.g. the entire RT surface, and we can also use this info to construct the RT surfaces of smaller bits of the subregion - we should be able to find the metric everywhere in the part of the bulk between the boundary region and the RT surface
- This is formalised by the concept of the *entanglement wedge* $\mathcal{E}[A]$: this is the domain of dependence of the bulk region bounded by A and by the RT surface of A
- The entanglement wedge is determined by the RT surface, which has area equal to the von Neumann entropy of the boundary theory in A, hence entanglement
- Under the assumption the bulk is a 'sensible' geometry, $\mathcal{C}[A] \subseteq \mathcal{E}[A]$
- For a pure boundary state, the RT surface of A is the same as that of $\overline{A}$: the operator L which gives its area is both in the set of operators on A and on $\overline{A}$
- Causality says that all operators in one of these sets commutes with all operators in the other, so L is in the centre of A
- One example is the identity operator, but in general, when there is some gauge symmetry in the bulk, there will be non-trivial elements in the centre
- The non-triviality of the area operator tells us the bulk is gravitational, and hence has diffeomorphism invariance

1.7 Complementary Recovery and Radial Commutativity

- If we divide the boundary into two regions, an operator acting at a particular bulk point must lie in the causal wedge of at least one of the regions, and only in the causal wedge of both when the point is in the RT surface
- This is *complementary recovery*: given a subregion, we can reconstruct **all** operators in the causal wedge but none outside it
- What happens if we allow the boundary A to vary but the bulk operator remains fixed?
- In general, the operator is in the causal wedge of multiple regions, so if we had access to a boundary region large enough that lots of its subregions can reconstruct the operator, knowledge of the operator is redundantly encoded in the state: we don't need the full state on A to reconstruct it
- We say the state exhibits *subregion duality*, in which many subregions can reconstruct the same operator
- If we erase a piece of the boundary which is much smaller than A, almost every bulk operator can still be exactly reconstructed - see the link to error correction?
- For non-pathological spacetimes, the RT surfaces of small subregions don't extend very far into the bulk: we need large regions to penetrate deeply

- If a bulk operator is outside a subregion's causal wedge, that means it commutes with every operator in the wedge
- In particular, it commutes with operators acting on the boundary itself
- Every boundary operator acting on a point in the boundary lives inside the causal wedge of any boundary subregion containing that point; in particular, it lives inside the causal wedge of an arbitrarily small subregion around the point
- Therefore, any bulk operator living away from the boundary must commute with all boundary operators acting at single points on the boundary: this is called *radial commutativity*
- CFTs admit an *operator product expansion*: the product of operators acting at multiple points can be written as a sum of local operators acting at single points
- Each of these local operators must therefore commute with the bulk operator - therefore, a bulk operator commutes with every operator in the boundary theory
- This paradox only arises if we treat bulk operators and boundary operators as having the same Hilbert space, but in fact the bulk space is encoded into the CFT space
- We therefore must map the bulk into the boundary using an isometry V