Week 5 - Notes

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1 Harlow Shows that RT $\implies$ Complementary Recovery

- The first thing to consider is a small variation in the entropy:
$$\begin{aligned}
\delta S(\tilde{\rho}, M) &= S(\tilde{\rho} + \delta\tilde{\rho}, M) - S(\tilde{\rho}, M) \\
&= -\hat{\mathrm{Tr}}\left(\left(\hat{\tilde{\rho}}_M + \delta\hat{\tilde{\rho}}_M\right)\log\left(\hat{\tilde{\rho}}_M + \delta\hat{\tilde{\rho}}_M\right)\right) + \hat{\mathrm{Tr}}\left(\hat{\tilde{\rho}}_M \log\hat{\tilde{\rho}}_M\right) \\
&= -\hat{\mathrm{Tr}}\left(\hat{\tilde{\rho}}_M\left(\log\hat{\tilde{\rho}}_M + \hat{\tilde{\rho}}_M^{-1}\delta\hat{\tilde{\rho}}_M\right)\right) - \hat{\mathrm{Tr}}\left(\delta\hat{\tilde{\rho}}_M\log\hat{\tilde{\rho}}_M\right) + \hat{\mathrm{Tr}}\left(\hat{\tilde{\rho}}_M\log\hat{\tilde{\rho}}_M\right) + \mathcal{O}\left(\delta\hat{\tilde{\rho}}_M^2\right) \\
&= -\hat{\mathrm{Tr}}\left(\delta\hat{\tilde{\rho}}_M\right) - \hat{\mathrm{Tr}}\left(\delta\hat{\tilde{\rho}}_M\log\hat{\tilde{\rho}}_M\right) + \mathcal{O}\left(\delta\hat{\tilde{\rho}}_M^2\right) \\
&= -\mathrm{Tr}\left(\delta\tilde{\rho}\log\left(\tilde{\rho}_A \otimes I_{\overline{A}}\right)\right) + \mathcal{O}\left(\delta\tilde{\rho}^2\right) \\
&= -\mathrm{Tr}\left(\delta\tilde{\rho}\log\tilde{\rho}_A\right) + \mathcal{O}\left(\delta\tilde{\rho}^2\right)
\end{aligned} \tag{1.1}$$
using that $\mathrm{Tr}(\delta\tilde{\rho}) = 0$ and the standard log expansion
- We can play the same game more generally if we take the variation $\delta\tilde{\rho}$ about a state σ:
$$S(\tilde{\sigma} + \delta\tilde{\rho}, M) - S(\tilde{\sigma}, M) = -\mathrm{Tr}\left(\delta\tilde{\rho}\log\tilde{\sigma}_A\right) \tag{1.2}$$
- A similar calculation shows
$$\delta S(\tilde{\rho}) = -\mathrm{Tr}(\delta\tilde{\rho}\log\tilde{\rho}) \implies S(\tilde{\sigma} + \delta\tilde{\rho}) - S(\tilde{\sigma}) = -\mathrm{Tr}\left(\delta\tilde{\rho}\log\tilde{\sigma}\right) \tag{1.3}$$
- Recall the RT formula on A:
$$S(\tilde{\rho}_A) = \mathrm{Tr}\tilde{\rho}\mathcal{L}_A + S(\tilde{\rho}, M) \tag{1.4}$$
- So, taking the variation $\delta\tilde{\rho}$ on $\tilde{\sigma}$, we get
$$\begin{aligned}
&S(\tilde{\sigma}_A + \delta\tilde{\rho}_A) - S(\tilde{\sigma}_A) = \mathrm{Tr}\left((\tilde{\sigma} + \delta\tilde{\rho})\mathcal{L}_A\right) - \mathrm{Tr}\left(\tilde{\sigma}\mathcal{L}_A\right) + S(\tilde{\sigma} + \delta\tilde{\rho}, M) - S(\tilde{\sigma}, M) \\
&\implies -\mathrm{Tr}\left(\delta\tilde{\rho}_A\log\tilde{\sigma}_A\right) = \mathrm{Tr}\left(\delta\tilde{\rho}\left(\mathcal{L}_A - \log\tilde{\sigma}\right)\right)
\end{aligned} \tag{1.5}$$
to linear order
- Since both sides are linear in $\delta\tilde{\rho}$, we can integrate over all such variations on both sides, giving us
$$-\mathrm{Tr}\left(\tilde{\rho}_A\log\tilde{\sigma}_A\right) = \mathrm{Tr}\left(\tilde{\rho}\left(\mathcal{L}_A - \log\tilde{\sigma}_A\right)\right) \tag{1.6}$$
- We can now calculate relative entropies as:
$$\begin{aligned}
S(\tilde{\rho}_A|\tilde{\sigma}_A) &= \mathrm{Tr}\left(\tilde{\rho}_A\log\tilde{\rho}_A\right) - \mathrm{Tr}\left(\tilde{\rho}_A\log\tilde{\sigma}_A\right) \\
&= -S(\tilde{\rho}_A) + \mathrm{Tr}\left(\tilde{\rho}(\mathcal{L}_A - \log\tilde{\sigma}_A)\right) \\
&= -S(\tilde{\rho}_A) + S(\tilde{\rho}_A) - S(\tilde{\rho}, M) - \mathrm{Tr}\left(\tilde{\rho}\log\tilde{\sigma}_A\right) \\
&= -S(\tilde{\rho}, M) - \hat{\mathrm{Tr}}\left(\tilde{\rho}_M\log\tilde{\sigma}_M\right) \\
&= S(\tilde{\rho}|\tilde{\sigma}, M)
\end{aligned} \tag{1.7}$$

- A similar argument on $\overline{A}$ shows that

$$S(\tilde{\rho}_{\overline{A}}|\tilde{\sigma}_{\overline{A}}) = S(\tilde{\rho}|\tilde{\sigma}, M') \tag{1.8}$$

- These two between them imply the second condition of Harlow 5.1
- Consider a state $|\tilde{\psi}\rangle \in \mathcal{H}_{\text{code}}$, an operator $X_{\overline{A}}$ on $\mathcal{H}_{\overline{A}}$, and an operator $\tilde{O} \in M$
- Since M is spanned by its Hermitian elements, we take $\tilde{O}$ to be Hermitian
- We now consider

$$\langle\tilde{\psi}|e^{-i\lambda\tilde{O}}X_{\overline{A}}e^{i\lambda\tilde{O}}|\tilde{\psi}\rangle = \langle\tilde{\psi}|e^{-i\lambda\tilde{O}}P_{\text{code}}X_{\overline{A}}P_{\text{code}}e^{i\lambda\tilde{O}}|\tilde{\psi}\rangle \tag{1.9}$$

since $\tilde{O}\,|\tilde{\psi}\rangle = P_{\text{code}}\tilde{O}P_{\text{code}}P_{\text{code}}\,|\tilde{\psi}\rangle = P_{\text{code}}\tilde{O}\,|\tilde{\psi}\rangle$

- Consider the states

$$|\tilde{\psi}(\lambda)\rangle \equiv e^{i\lambda\tilde{O}}\,|\tilde{\psi}\rangle \tag{1.10}$$

and note that for any $\tilde{O}' \in M'$, the expectation $\langle\tilde{\psi}(\lambda)|\tilde{O}'|\tilde{\psi}(\lambda)\rangle$ is independent of λ

- Recalling that for any state ρ, there is a unique state $\rho_{M'} \in M'$ such that $\mathbb{E}_\rho(x') = \mathbb{E}_{\rho_{M'}}(x')$ for any $x' \in M'$, this means that the states $\tilde{\psi}(\lambda)_{M'}$ corresponding to $\tilde{\psi}(\lambda) = |\tilde{\psi}(\lambda)\rangle\,\langle\tilde{\psi}(\lambda)|$ on M' are independent of λ too
- So, for any two λ, λ', $\tilde{\psi}(\lambda)_{M'} = \tilde{\psi}(\lambda')_{M'}$, which means

$$S(\tilde{\psi}(\lambda)|\tilde{\psi}(\lambda'), M') = 0 \tag{1.11}$$

- So from (1.8):

$$0 = S(\tilde{\psi}(\lambda)|\tilde{\psi}(\lambda'), M') = S(\tilde{\psi}(\lambda)_{\overline{A}}|\tilde{\psi}(\lambda')_{\overline{A}}) \tag{1.12}$$

and so $\text{Tr}_A\left(|\tilde{\psi}(\lambda)\rangle\,\langle\tilde{\psi}(\lambda)|\right)$ is independent of λ too

- This can only hold if $|\tilde{\psi}(\lambda)\rangle$ itself is independent of λ
- Now return to (1.9); since this is therefore independent of λ, its first variation with respect to λ must vanish
- Therefore:

$$\delta\,\langle\tilde{\psi}|e^{-i\lambda\tilde{O}}P_{\text{code}}X_{\overline{A}}P_{\text{code}}e^{i\lambda\tilde{O}}|\tilde{\psi}\rangle \propto \langle\tilde{\psi}|[P_{\text{code}}X_{\overline{A}}P_{\text{code}}, \tilde{O}]|\tilde{\psi}\rangle = 0 \tag{1.13}$$

which just implies $P_{\text{code}}X_{\overline{A}}P_{\text{code}} = X'P_{\text{code}}$ for some $X' \in M'$, which is just the second condition as claimed

- Since we can repeat this argument with $M \leftrightarrow M'$ and $A \leftrightarrow \overline{A}$, we get that a two sided RT formula implies complementary recovery

2 Pollack 4.8

- We jump straight in with Pollack's theorem 4.8

Theorem 2.1. *Consider an encoding isometry V, a subregion A, and a von Neumann algebra M so that (V, A, M) have complementary recovery. Then (V, A, M) and $(V, \overline{A}, M')$ both have an RT formula with the same area operator L, and L is in the center Z_M.*

Proof. Suppose M induces decomposition $\mathcal{H}_L = \oplus_\alpha(\mathcal{H}_{L_\alpha} \otimes \mathcal{H}_{\overline{L}_\alpha})$. Then, we can decompose M and M' together as

$$M = \oplus_\alpha(\mathcal{L}(\mathcal{H}_{L_\alpha}) \otimes I_{\overline{L}_\alpha}), \quad M' = \oplus_\alpha(I_{L_\alpha} \otimes \mathcal{L}(\mathcal{H}_{\overline{L}_\alpha})) \tag{2.1}$$

Define $|\alpha, i, j\rangle = |i_\alpha\rangle_{L_\alpha} \otimes |j_\alpha\rangle_{\overline{L}_\alpha}$ as the compatible basis for $\mathcal{H}_L$. By earlier lemmas, we know there exist factorisations of the form

$$\mathcal{H}_A = \oplus_\alpha(\mathcal{H}_{A_1^\alpha} \otimes \mathcal{H}_{A_2^\alpha}) \oplus \mathcal{H}_{A_3}, \quad \mathcal{H}_{\overline{A}} = \oplus_\alpha(\mathcal{H}_{\overline{A}_1^\alpha} \otimes \mathcal{H}_{\overline{A}_2^\alpha}) \oplus \mathcal{H}_{\overline{A}_3} \tag{2.2}$$

and unitaries U_A and $U_{\overline{A}}$ such that

$$\begin{aligned}(U_A \otimes I_{\overline{A}})V |\alpha, i, j\rangle &= |\psi_{\alpha,i}\rangle_{A_1^\alpha} \otimes |\chi_{\alpha,j}\rangle_{A_2^\alpha \overline{A}} \\ (I_A \otimes U_{\overline{A}})V |\alpha, i, j\rangle &= |\overline{\chi}_{\alpha,i}\rangle_{A\overline{A}_2^\alpha} \otimes |\overline{\psi}_{\alpha,j}\rangle_{\overline{A}_1^\alpha}\end{aligned} \tag{2.3}$$

If we apply $(U_A \otimes I_{\overline{A}})$ and $(I_A \otimes U_{\overline{A}})$ sequentially, we see that

$$\begin{aligned}(I_A \otimes U_{\overline{A}})(U_A \otimes I_A)V |\alpha, i, j\rangle &= |\psi_{\alpha,i}\rangle_{A_1^\alpha} \otimes (I_{A_2^\alpha} \otimes U_{\overline{A}}) |\chi_{\alpha,j}\rangle_{A_2^\alpha \overline{A}} \\ (U_A \otimes I_A)(I_A \otimes U_{\overline{A}})V |\alpha, i, j\rangle &= (U_A \otimes I_{\overline{A}_2^\alpha}) |\overline{\chi}_{\alpha,i}\rangle_{A\overline{A}_2^\alpha} \otimes |\overline{\psi}_{\alpha,j}\rangle_{\overline{A}_1^\alpha}\end{aligned} \tag{2.4}$$

In order for both of these to be true simultaneously, there therefore must exist states $|\chi_\alpha\rangle_{A_2^\alpha \overline{A}_2^\alpha}$ such that $(I_{A_2^\alpha} \otimes U_{\overline{A}}) |\chi_{\alpha,j}\rangle_{A_2^\alpha \overline{A}} = |\chi_\alpha\rangle_{A_2^\alpha \overline{A}_2^\alpha} \otimes |\overline{\psi}_{\alpha,j}\rangle_{\overline{A}_1^\alpha}$, which implies

$$(U_A \otimes U_{\overline{A}})V |\alpha, i, j\rangle = |\psi_{\alpha,i}\rangle_{A_1^\alpha} \otimes |\chi_\alpha\rangle_{A_2^\alpha \overline{A}_2^\alpha} \otimes |\overline{\psi}_{\alpha,j}\rangle_{\overline{A}_1^\alpha} \tag{2.5}$$

This result will imply the RT formula when we consider a logical operator in this basis. Suppose ρ is a state with support on $\mathcal{H}_L$; we will compute $S(M, \rho)$ and $S(\mathrm{Tr}_{\overline{A}}(V\rho V^\dagger))$ and take the difference.
Recall that instead of considering $S(\mathrm{Tr}_{\overline{A}}(V\rho V^\dagger))$ we might as well consider $S(\mathrm{Tr}_{\overline{A}}(V\rho_M V^\dagger))$: to see why, set $O_A \in \mathcal{L}(\mathcal{H}_A)$, and compute

$$\mathrm{Tr}(O_A \cdot \mathrm{Tr}_{\overline{A}}(V\rho V^\dagger)) = \mathrm{Tr}((O_A \otimes I_{\overline{A}}) \cdot V\rho V^\dagger) = \mathrm{Tr}(V^\dagger(O_A \otimes I_{\overline{A}})V \cdot \rho) \tag{2.6}$$

by cyclicity of the trace. But $V^\dagger(O_A \otimes I_{\overline{A}})V \in M$, and for any $O \in M$ we have $\mathrm{Tr}(O\rho) = \mathrm{Tr}(O\rho_M)$, we can just consider ρ_M in the above. Since the states $\mathrm{Tr}_{\overline{A}}(V\rho V^\dagger)$ and $\mathrm{Tr}_{\overline{A}}(V\rho_M V^\dagger)$ for all observables in M, they are the same state and so have the same entropy. Acting with a unitary on $\mathcal{H}_A$ and $\mathcal{H}_{\overline{A}}$ separately does not change the unitary, we have:

$$S(\mathrm{Tr}_{\overline{A}}(V\rho_M V^\dagger)) = S(\mathrm{Tr}_{\overline{A}}((U_A \otimes U_{\overline{A}})V\rho_M V^\dagger(U_A \otimes U_{\overline{A}})^\dagger)) \tag{2.7}$$

The next step is to define isometries $\tilde{V}_\alpha : (\mathcal{H}_{L_\alpha} \otimes \mathcal{H}_{\overline{L}_\alpha}) \to (\mathcal{H}_{A_1^\alpha} \otimes \mathcal{H}_{\overline{A}_1^\alpha})$ using the states $|\psi_{\alpha,i}\rangle_{A_1^\alpha}$ and $|\overline{\psi}_{\alpha,j}\rangle_{\overline{A}_1^\alpha}$:

$$\tilde{V}_\alpha |\alpha, i, j\rangle \equiv |\psi_{\alpha,i}\rangle_{A_1^\alpha} \otimes |\psi_{\alpha,j}\rangle_{\overline{A}_1^\alpha} \tag{2.8}$$

which is certainly an isometry since $|\psi_{\alpha,i}\rangle_{A_1^\alpha}$ and the other one are bases for $\mathcal{H}_{A_1^\alpha}$ and $\mathcal{H}_{\overline{A}_1^\alpha}$ respectively.
$\tilde{V}_\alpha$ then lets us rewrite (2.5) as

$$(U_A \otimes U_{\overline{A}})V |\alpha, i, j\rangle = \tilde{V}_\alpha |\alpha, i, j\rangle \otimes |\chi_\alpha\rangle_{A_2^\alpha \overline{A}_2^\alpha} \tag{2.9}$$

which lets us further simplify $(U_A \otimes U_{\overline{A}})V\rho_M V^\dagger (U_A \otimes U_{\overline{A}})^\dagger$:

$$
\begin{aligned}
&(U_A \otimes U_{\overline{A}})V\rho_M V^\dagger (U_A \otimes U_{\overline{A}})^\dagger)\\
&= \sum_\alpha p_\alpha \cdot (U_A \otimes U_{\overline{A}})V\rho_{A_\alpha} V^\dagger (U_A \otimes U_{\overline{A}})^\dagger\\
&= \sum_\alpha p_\alpha \cdot \frac{1}{p_\alpha}\sum_{i,j}\sum_{i',j'} \rho[\alpha]_{i,j,i',j'} (U_A \otimes U_{\overline{A}})V \left|\alpha,i,j\right\rangle \left\langle \alpha,i',j'\right| V^\dagger (U_A \otimes U_{\overline{A}})^\dagger\\
&= \sum_\alpha p_\alpha \cdot \frac{1}{p_\alpha}\sum_{i,j}\sum_{i',j'} \rho[\alpha]_{i,j,i',j'} \tilde{V}_\alpha \left|\alpha,i,j\right\rangle \left\langle \alpha,i',j'\right| \tilde{V}_\alpha^\dagger \otimes \left|\chi_\alpha\right\rangle \left\langle \chi_\alpha\right|\\
&= \sum_{p_\alpha} \cdot \tilde{V}_\alpha \rho_\alpha \tilde{V}_\alpha^\dagger \otimes \left|\chi_\alpha\right\rangle \left\langle \chi_\alpha\right|
\end{aligned}
\tag{2.10}
$$

Each of the states $\tilde{V}_\alpha \rho_\alpha \tilde{V}_\alpha^\dagger \otimes \left|\chi_\alpha\right\rangle \left\langle \chi_\alpha\right|$ are normalised and act on different blocks, we can compute the entropy as

$$
\begin{aligned}
S(\mathrm{Tr}_{\overline{A}}(V\rho V^\dagger)) &= -\sum_\alpha p_\alpha \log p_\alpha + \sum_\alpha p_\alpha S(\mathrm{Tr}_{\overline{A}}(\tilde{V}_\alpha \rho_\alpha \tilde{V}_\alpha^\dagger \otimes \left|\chi_\alpha\right\rangle \left\langle \chi_\alpha\right|))\\
&= \sum_\alpha p_\alpha \log(p_\alpha^{-1}) + \sum_\alpha p_\alpha S(\mathrm{Tr}_{\overline{A}}(\tilde{V}_\alpha \rho_\alpha \tilde{V}_\alpha^\dagger)) + \sum_\alpha p_\alpha S(\mathrm{Tr}_{\overline{A}}(\left|\chi_\alpha\right\rangle \left\langle \chi_\alpha\right|))
\end{aligned}
\tag{2.11}
$$

Now, since $\left|\psi_{\alpha,j}\right\rangle$ is independent of i, we further have that

$$
S(\mathrm{Tr}_{\overline{A}}(\tilde{V}_\alpha \rho_\alpha \tilde{V}_\alpha^\dagger)) = S(\mathrm{Tr}_{\overline{A}_1^\alpha}(\tilde{V}_\alpha \rho_\alpha \tilde{V}_\alpha^\dagger)) = S(\mathrm{Tr}_{\overline{L}_\alpha}(\rho_\alpha))
\tag{2.12}
$$

Now, the first two terms of (2.11) are just $S(\rho, M)$, so we have

$$
S(\mathrm{Tr}_{\overline{A}}(V\rho V^\dagger)) - S(\rho, M) = \sum_\alpha p_\alpha S(\mathrm{Tr}_{\overline{A}}(\left|\chi_\alpha\right\rangle \left\langle \chi_\alpha\right|))
\tag{2.13}
$$

The RHS is linear in p_α, so must be linear in ρ also. This means there exists an area operator $\mathcal{L}$ such that the RHS can be written as $\mathrm{Tr}(\rho\mathcal{L})$. Explicitly, we define

$$
\begin{aligned}
I_\alpha &\equiv \sum_{i,j} \left|\alpha,i,j\right\rangle \left\langle \alpha,i,j\right|\\
\mathcal{L} &\equiv \sum_\alpha S(\mathrm{Tr}_{\overline{A}}(\left|\chi_\alpha\right\rangle \left\langle \chi_\alpha\right|)) \cdot I_\alpha
\end{aligned}
\tag{2.14}
$$

which gives $\mathcal{L} \in M$, and so $\mathrm{Tr}(\rho\mathcal{L}) = \mathrm{Tr}(\rho_M\mathcal{L})$, and we can write

$$
\begin{aligned}
\mathrm{Tr}(\rho_M\mathcal{L}) &= \mathrm{Tr}\left(\sum_\alpha p_\alpha \rho_\alpha \cdot \sum_\alpha S(\mathrm{Tr}_{\overline{A}}(\left|\chi_\alpha\right\rangle \left\langle \chi_\alpha\right|)) \cdot I_\alpha\right)\\
&= \sum_\alpha p_\alpha S(\mathrm{Tr}_{\overline{A}}(\left|\chi_\alpha\right\rangle \left\langle \chi_\alpha\right|)) \cdot \mathrm{Tr}(\rho_\alpha I_\alpha)\\
&= S(\mathrm{Tr}_{\overline{A}}(V\rho V^\dagger)) - S(\rho, M)
\end{aligned}
\tag{2.15}
$$

so (V, A, M) satisfy an RT formula with area operator $\mathcal{L}$, and $\mathcal{L} \in M$. We can do the same derivation for $(V, \overline{A}, M')$ with $i \leftrightarrow j$, and since $\mathcal{L} \in M'$, $\mathcal{L} \in Z_M$ as claimed. $\square$