Topics in Mathematical Physics

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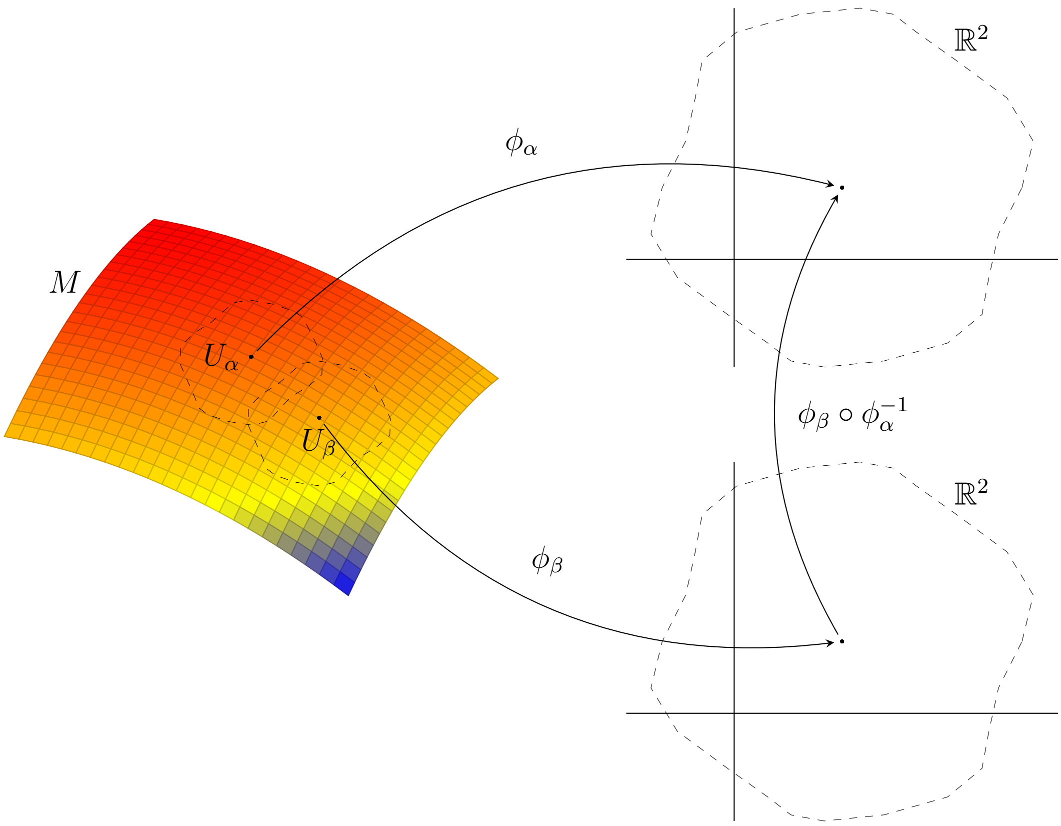
1 Preface

- These notes were written up for the postgraduate masters course in mathematical physics at the University of Edinburgh
- The topic this year was 'Applications of Geometry in Physics'
- It was lectured by James Lucietti, who's notes are unfortunately only available on the Edinburgh intranet
- These notes were largely produced with those official notes as reference
- These notes were written primarily with myself in mind; it has some colloquialisms in it, and some topics are presented in the way I best understand them, and not necessarily the 'best' way
- The notes are also exclusively bullet pointed because I find prose difficult to digest
- Any errors are most certainly mine and not James's - let me know at benkarsberg@gmail.com if you note any bad ones
- Generally, text in *italic* is the definition of something the first time it shows up, and text in **bold** is something I think is important/want to remember/found tricky when writing these up
- Summation convention is implicit throughout this entire set of notes, unless explicitly stated otherwise
- Euclidean spatial vectors are denoted by e.g. $\mathbf{v}$, and 4-vectors are denoted by just $v = (v^0, \mathbf{v})$
- The metric signature is $\eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$ unless stated otherwise
- $\hbar = c = 1$
- I hope these notes are helpful to someone who isn't me - enjoy!

2 Differential Geometry Preliminaries

- This course is all about geometry, almost entirely differential
- Differentiable geometry is a listed prerequisite for this course, but I haven't done any properly, with the exception of pseudo-Riemannian geometry in coordinate bases in a GR course last year
- Therefore, we need to set the scene
- The principal arena for differential geometry is a **differential manifold**:
  - Informally, this is a set M that locally looks like $\mathbb{R}^n$
  - Formally, this is a topological space M equipped with a collection of charts $\{(U_\alpha, \phi_\alpha)\}$ called an **atlas**, where $\{U_\alpha\}$ is an open cover of M and $\phi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^n$ are homeomorphisms
- The image we want in mind is something like this, generalised to $\mathbb{R}^n$ rather than just $\mathbb{R}^2$:



- Essentially, a chart (U, ϕ) assigns coordinates to a point $p \in U$ given by $\phi(p) = (x^1(p), \ldots, x^n(p))$, usually written $\phi = (x^i)$
- Importantly, the **transition functions** $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$ should be $C^\infty(\mathbb{R}^n)$ bijections between open subsets of $\mathbb{R}^n$ (see diagram)
- Essentially, all changes of coordinates are given by a smooth map
- The atlas allows us to define calculus on a manifold, and we can classify a hierarchy of objects on M:

- $C^\infty(M)$ is the set of all smooth functions on M, with the function of a standard commutative algebra
- The **tangent space** T_pM at $p \in M$ is the n-dimensional vector space of directional derivatives at p; that is, the space of all $\mathbb{R}$-linear maps $X_p : C^\infty(M) \to \mathbb{R}$ obeying the Leibnitz/product rule

$$X_p(fg) = X_p(f)g(p) + f(p)X_p(g) \tag{2.1}$$

  for all $f, g \in C^\infty(M)$
- The elements $X_p \in T_pM$ are called **tangent vectors**; if we consider a smooth curve $\gamma : (a, a) \to M$ where $\gamma(0) = p$, we can define a tangent vector to the curve called the velocity $\dot{\gamma} \in T_pM$ by $\dot{\gamma}(f) = \frac{d}{dt} f(\gamma(t))\big|_{t=0}$, and all tangent vectors arise like this
- A **vector field** $X \in \mathfrak{X}(M)$ is a smooth assignment to each $p \in M$ of a tangent vector $X_p \in T_pM$ (or more formally, a section of the **tangent bundle** TM)
- A **differential k-form** α_p at $p \in M$ is a smooth alternating multilinear map $\alpha_p : T_pM \times \ldots \times T_pM \to \mathbb{R}$
- The set of all k-forms is denoted $\Omega^k(M)$, where $\alpha(X_1, \ldots, X_k) \in C^\infty(M)$ and $X_i \in \mathfrak{X}(M)$
- In particular, 1-forms at p are just dual vectors $\alpha_p \in T_p^*M$
- The **wedge produce** of $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^l(M)$ is $\alpha \wedge \beta \in \Omega^{k+l}(M)$ where $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$; in particular, for 1-forms we have

$$(\alpha \wedge \beta)(X, Y) = \alpha(X)\beta(Y) - \beta(X)\alpha(Y) \tag{2.2}$$

- The **exterior derivative** is a $\mathbb{R}$-linear map $d : \Omega^k(M) \to \Omega^{k+1}(M)$ satisfying $d^2 = 0$ and $d(\alpha \wedge \beta = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge (d\beta)$
- The **interior derivative** $\iota_X : \Omega^k(M) \to \Omega^{k-1}(M)$ is defined by $\iota_X \alpha = \alpha(X, \ldots)$ for $X \in \mathfrak{X}(M)$
- In particular, if we regard a smooth function f on M as a 0-form, $(df)(X) = X(f)$
- **Tensors** and **tensor fields** of type (r, s) are very similar to vectors and vector fields, acting as multilinear maps $T : T_p^*M \times \ldots T_p^*M \times T_pM \times \ldots T_pM \to \mathbb{R}$
- The derivative of a smooth map $\phi : M \to N$ is called the **push-forward map** $\phi_* : T_pM \to T_{\phi(p)}N$ given by

$$(\phi_* X_p)(f) = X_p(f \circ \phi) \tag{2.3}$$

  where $f \in C^\infty(N)$
- Similarly, the **pull-back** is $\phi^* : T^*_{\phi(p)}N \to T_p^*M$ given by

$$(\phi^* \alpha_{\phi(p)})(X_p) = \alpha_{\phi(p)}(\phi_* X_p) \tag{2.4}$$

- A smooth bijection $\phi : M \to N$ is called a **diffeomorphism**, and for a diffeomorphism we have $(\phi^{-1})_* = \phi^*$
- The **flow** of a vector field X is a 1-parameter group of diffeomorphisms; i.e. a family $\phi_t : M \to M$ such that $\phi_{t+s} = \phi_t \circ \phi_s$ for all real t, s
- In particular, the integral curve of X through $p \in M$ is given by $\phi_t(p)$ so $X_p(f) = \frac{d}{dt} f(\phi_t(p))|_{t=0}$

- The **Lie derivative** of a tensor field T along a vector field X is

$$\mathcal{L}_X T = \frac{d}{dt}(\phi_t^* T)\bigg|_{t=0} \tag{2.5}$$

  where ϕ_t is the flow of $X \in \mathfrak{X}(M)$

- In particular, we have $\mathcal{L}_X f = X(f)$ for $f \in C^\infty(M)$; $\mathcal{L}_X Y = [X, Y]$ for $Y \in \mathfrak{X}(M)$; $\mathcal{L}_X \alpha = (d\iota_X + \iota_X d)\alpha$ for $\alpha \in \Omega^k(M)$

- Suppose we have a local basis $\{e_a \mid a = 1, \ldots, n\}$ of vectors; we can then expand a vector field as $X = X^a e_a$ where X^a are the components

- The dual basis is $\{f^a \mid a = 1, \ldots, n\}$ defined by $f^a(e_b) = \delta^a_b$; differential 1-forms can then be expanded as $\alpha = \alpha_a f^a$ where $\alpha_a = \alpha(e_a)$ are the components

- More generally, a k-form can be expanded as $\alpha = \frac{1}{k!}\alpha_{a_1 \ldots a_k} f^{a_1} \wedge \ldots \wedge f^{a_k}$ where $\alpha_{a_1 \ldots a_k} = \alpha(e_{a_1}, \ldots, e_{a_k})$

- This generalises to tensors in the expected way

- In particular, a chart (U, ϕ) where $\phi = x^i$ defines a basis of $T_p M$ called the **coordinate basis**, $\left(\frac{\partial}{\partial x^i}\right)_p$

- In this basis, any vector field can be expanded as $X = X^i \frac{\partial}{\partial x^i}$ where $X^i = X(x^i)$ are smooth functions on U

- Similarly, the dual basis of $T_p^* M$ is $(dx^i)_p$, where 1-forms can be expanded as $\alpha = \alpha_i dx^i$, and this can be expanded to a basis for k-forms by $dx^{i_1} \wedge \ldots \wedge dx^{i_k}$ where $i_1 < \ldots < i_k$

3 Symplectic Geometry and Classical Mechanics

- To motivate what follows, consider a classical Hamiltonian system with phase space $\mathbb{R}^{2n}$ and canonical coordinates $(\mathbf{p}, \mathbf{q})$
- Hamilton's equations for the time-evolution of the system are

$$\dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}}, \quad \dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}} \tag{3.1}$$

  where H is the Hamiltonian

- This can be rewritten in matrix form:

$$\begin{pmatrix} \dot{\mathbf{p}} \\ \dot{\mathbf{q}} \end{pmatrix} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \begin{pmatrix} \dfrac{\partial H}{\partial \mathbf{q}} \\ \dfrac{\partial H}{\partial \mathbf{p}} \end{pmatrix} \iff \dot{\mathbf{x}} = \Omega^{-1} \nabla_{\mathbf{x}} H \tag{3.2}$$

  where

$$\Omega = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix} \tag{3.3}$$

  is implicit

- As we shall see, this is the canonical example of a symplectic vector space

3.1 Symplectic Vector Spaces

- A **symplectic vector space** is a pair (V, ω), where V is a $2n$-dimensional real vector space, and $\omega : V \times V \to \mathbb{R}$ is a bilinear form satisfying:
  1. *Skew-symmetry*: $\omega(v, w) = -\omega(w, v)$, $\forall v, w \in V$
  2. *Non-degeneracy*: $\omega(v, w) = 0 \, \forall v \in V \implies w = 0$
- ω is called a **symplectic structure** on V
- Contrasting ω with the standard Euclidean inner product, the only real difference is skew vs non-skew symmetry
- In physics, we usually need a basis to work in - defining the arbitrary basis $\{e_a \,|\, a = 1, \ldots, 2n\}$, the components of ω are given by the usual $\omega_{ab} = \omega(e_a, e_b)$
- From this, it follows that ω_{ab} is an antisymmetric matrix
- Moreover, non-degeneracy of ω is equivalent to

$$\omega_{ab} w^b \implies w^a = 0 \tag{3.4}$$

  which just means ω_{ab} is invertible

- **Example:** the canonical example of a symplectic vector space is $V = \mathbb{R}^{2n}$ and $\omega(\mathbf{v}, \mathbf{w}) = \mathbf{v}^T \Omega \mathbf{w}$, with Ω as defined in the introduction:

$$\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \tag{3.5}$$

- We now show that, in fact, all symplectic vector spaces are isomorphic to this example

- **Theorem:** any symplectic vector space (V, ω) has a **symplectic basis** $\{e_i, f_i \,|\, i = 1, \ldots, n\}$ defined by

$$\omega(e_i, e_j) = \omega(f_i, f_j) = 0 \quad \text{and} \quad \omega(e_i, f_j) = \delta_{ij} \tag{3.6}$$

- **Proof:** by induction
  - For the $n = 1$ case, pick an arbitrary vector $0 \neq e_1 \in V$
  - By non-degeneracy, $\exists f_1 \in V$ such that $\omega(e_1, f_1) \neq 0$, and we can rescale so that $\omega(e_1, f_1) = 1$
  - Then, by skew-symmetry, we see $\omega(e_1, e_1) = \omega(f_1, f_1) = 0$ as required
  - Assume the hypothesis holds for $n = k \geq 1$, and consider the $n = k + 1$ case
  - By the same procedure that we used for the base case, we can construct e_1 and f_1
  - Then, define the *skew-orthogonal complement*:

$$W = \{w \in V \,|\, \omega(w, e_1) = \omega(w, f_1) = 0\} \tag{3.7}$$

  - W has dimension $2k$, and we now wish to show that it is a symplectic vector space
  - First note that any $v \in V$ can be uniquely decomposed as $v = u + w$, where $u \in \text{span}(e_1, f_1)$ and $w \in W$
  - Then consider $\omega|_W$, i.e. the symplectic structure restricted to W; pick $\tilde{w} \in W$, then:

$$\begin{aligned} \omega(\tilde{w}, w) = 0, \ \forall w \in W &\implies \omega(\tilde{w}, v) = 0, \ \forall v \in V \qquad &\text{(by decomposition above)} \\ &\implies \tilde{w} = 0 &\text{(by non-degeneracy)} \end{aligned} \tag{3.8}$$

  - $\omega|_W$ clearly inherits skew-symmetry from ω, so it is a symplectic form, and so W is indeed a symplectic vector space
  - By induction the conclusion follows ■

- Explicitly, this means in the symplectic basis $\omega_{ab} = \Omega_{ab}$, establishing the isomorphism

- The next result shows a radical departure from Euclidean geometry in these spaces

- **Theorem:** let (V, ω) be a $2n$-dimensional symplectic vector space, and $W \subset V$ a *null subspace* so $\omega|_W = 0$. Then $\dim W \leq n$

- **Proof:**
  - By the earlier theorem, we can assume $V = \mathbb{R}^{2n}$ and $\omega = \Omega$
  - Then, W is a null space iff $W \perp \Omega W$, where this is with respect to the Euclidean norm $\mathbf{w} \cdot \mathbf{v} = \mathbf{w}^T \mathbf{v}$
  - Therefore, we have

$$\dim W + \dim \Omega W \leq 2n \tag{3.9}$$

  - However, Ω is invertible so $\dim W = \dim \Omega W$, so we are done ■

- We now want to rewrite the symplectic structure in a symplectic basis, but in a more useful way

- Define the dual symplectic basis $\{e_*^i, f_*^i \,|\, i = 1, \ldots, n\}$ of V^*, so specifically:

$$e_*^i(e_j) = f_*^i(f_j) = \delta_j^i, \quad e_*^i(f_j) = f_*^i(e_j) = 0 \tag{3.10}$$

- Then, the basis of 2-forms is given by the set

$$\{e_*^i \wedge e_*^j,\, e_*^i \wedge f_*^j,\, f_*^i \wedge f_*^j\} \tag{3.11}$$

as usual

- We therefore claim that

$$\omega = e_*^i \wedge f_*^i \tag{3.12}$$

with summation convention implied

- We can easily verify this by its action on basis elements:

$$\omega(e_a, e_b) = (e_*^i \wedge f_*^i)(e_a, e_b) = e_*^i(e_a) f_*^i(f_b) - e_*^i(f_b) f_*^i(e_a) = \delta_a^i \delta_b^i = \delta_{ab} \tag{3.13}$$

- More generally, consider the k-fold wedge product ω^k
- We have $\omega^k = 0$ for $k > n$ as ω^n is the highest ranked form on V, which has dimension $2n$
- **Lemma:** let (V, ω) be a $2n$-dimensional symplectic vector space. Then, in a symplectic basis we have

$$\omega^n = n! e_*^1 \wedge f_*^1 \wedge \ldots \wedge e_*^n \wedge f_*^n \tag{3.14}$$

- **Proof:** by induction (not given, but it's not hard - future Ben, just try and do it yourself for revision xx)

3.2 Symplectic Manifolds and the Cotangent Bundle

- A **symplectic manifold** is a pair (M, ω) where M is a 2-dimensional smooth manifold, and $\omega \in \Omega^2(M)$ is a 2-form satisfying:
  1. $d\omega = 0$ (closed)
  2. $\omega(X, Y) = 0\ \forall X \in \mathfrak{X}(M) \implies Y = 0$ (non-degenerate)
- ω is called a **symplectic form**
- Note that if (M, ω) is a symplectic manifold, then $(T_x M, \omega_x)$ is a symplectic vector space at all points $x \in M$
- **Example:** let $M = \mathbb{R}^{2n}$ with coordinates (p_i, q_i), $i = 1, \ldots, n$, and $\omega = dp_i \wedge dq_i$ (summation implied)
  - $\omega = d(p_i dq_i)$, so it is exact and hence closed
  - Consider the components of ω in the coordinate basis $(\partial/\partial p_i, \partial/\partial q_i)$; we find

$$\begin{aligned} \omega\left(\frac{\partial}{\partial p_j}, \frac{\partial}{\partial p_k}\right) &= \omega\left(\frac{\partial}{\partial q_j}, \frac{\partial}{\partial q_k}\right) = 0 \\ \omega\left(\frac{\partial}{\partial p_j}, \frac{\partial}{\partial q_k}\right) &= dp_i\left(\frac{\partial}{\partial p_j}\right) dq_i\left(\frac{\partial}{\partial q_k}\right) = \delta_{ij}\delta_{ik} = \delta_{jk} \end{aligned} \tag{3.15}$$

    so in particular, $\omega_{ij} = \Omega_{ij}$ in this basis and is non-degenerate
  - In particular, this means ω is a symplectic form and that $(\partial/\partial p_i, \partial/\partial q_i)$ is a symplectic basis at every point
- **Theorem:** any symplectic manifold (M, ω) of dimension $2n$ is orientable with volume form ω^n

- **Proof:**
  - A manifold is **orientable** if there is a consistent definition of clockwise and anti-clockwise (e.g. a Mobius strip would be non-orientable), and a **volume form** is a highest-rank differentiable form on the manifold (of rank equal to the dimension of the manifold)
  - One characterisation of orientability is that a manifold is orientable if and only if there exists a nowhere vanishing top-ranked differential form/volume form
  - Therefore, if we can show $\omega^n = \omega \wedge \ldots \wedge \omega$ is nowhere vanishing, we are done
  - At each $x \in M$, $(T_x M, \omega_x)$ is a symplectic vector space
  - But we know that in symplectic vector spaces

    $$\omega_x^n = n! e_*^1 \wedge f_*^1 \wedge \ldots \wedge e_*^n \wedge f_*^n \tag{3.16}$$

    which is nowhere vanishing ■

- **Example:** consider the manifold given by the unit 2-sphere $M = S^2 = \{\mathbf{x} \in \mathbb{R}^3 \,|\, |\mathbf{x}| = 1\}$
  - The tangent space at $\mathbf{x}$ is $T_{\mathbf{x}} S^2 = span(\mathbf{x})^\perp$
  - The volume form at this point is given by

    $$\omega_{\mathbf{x}}(\mathbf{V}, \mathbf{w}) = \mathbf{x} \cdot (\mathbf{v} \times \mathbf{w}) \tag{3.17}$$

    for $\mathbf{v}, \mathbf{w} \in T_{\mathbf{x}} S^2$

3.2.1 The Cotangent Bundle

- The following discussion is why symplectic manifolds arise in classical mechanics
- Suppose Q is a $2n$-dimensional smooth manifold; its **cotangent bundle** is defined by

  $$T^*Q = \{(p_x, x) \,|\, p_x \in T_x^* Q,\ x \in Q\} \tag{3.18}$$

  which we think of as the set of all points x paired with all their covectors (one-forms)
  - The cotangent bundle is a $2n$-dimensional manifold
  - To show this, consider the chart on Q given by $\phi(x) = (q_i(x))$, containing $x \in Q$
  - We can see that $\Phi(p_x, x) = (p_i(x), q_i(x))$ is then a chart on T^*Q, where p_i are the components of $p_x = p_i(x)(dq_i)_x$
  - To check the transition functions are smooth, consider the projection $\pi : T^*Q \to Q,\ (p_x, x) \to x$; this is smooth and surjective, and the inverse projection has image $\pi^{-1}(x) = T_x^* Q$
  - Therefore, $\phi \circ \pi \circ \Phi^{-1}(p_i(x), q_i(x)) = (q_i(x))$ is smooth
- **Theorem:** T^*Q has a symplectic form ω, given in the above chart by

  $$\omega = dp_i \wedge dq_i \tag{3.19}$$

- **Proof:**
  - Recall that the derivative of a smooth map $\phi : M \to N$ is the push-forward map $\phi_* : T_p M \to T_{\phi(p)} N$, given by

    $$(\phi_* X_p)(f) = X_p(f\phi) \tag{3.20}$$

- Moreover, the pull-back of ϕ is $\phi^* : T^*_{\phi(p)}N \to T^*_pM$, given by

$$(\phi^*\alpha_{\phi(p)})(X_p) = \alpha_{\phi(p)}(\phi_* X_p) \tag{3.21}$$

- The projector above is a smooth map of this form, so its derivative is the push-forward $\pi_* : T_{(p_x,x)}(T^*Q) \to T_xQ$
- We now define $\theta_{(p_x,x)} \in T^*_{(p_x,x)}(T^*Q)$, given by

$$\theta_{(p_x,x)}(\xi) = p_x(\pi_*\xi), \quad \forall \xi \in T_{(p_x,x)}(T^*Q) \tag{3.22}$$

so $\theta_{(p_x,x)} = \pi^* p_x$ is the pull-back of p_x under π, meaning $\theta \in \Omega^1(T^*Q)$ is a one-form on T^*Q
- Define $\omega = d\theta \in \Omega^2(T^*Q)$, which is clearly closed since it is exact
- To show non-degeneracy, we check the components of θ in the chart $\Phi = (p_i, q_i)$ above of T^*Q, defined by chart $\phi = (q_i)$ of Q
- We find first that

$$\theta\left(\frac{\partial}{\partial q_i}\right) = p\left(\pi_*\frac{\partial}{\partial q_i}\right) \tag{3.23}$$

and note in particular that

$$\begin{aligned}
\left(\pi_*\frac{\partial}{\partial q_i}\right)(f) &= \frac{\partial}{\partial q_i}(f\circ\pi) \\
&= \frac{d}{dt} f\circ\pi\circ\Phi^{-1}(p_1,\ldots,p_n,q_1,\ldots,q_i+t,\ldots,q_n)|_{t=0} \\
&= \frac{d}{dt} f\circ\phi^{-1}(q_1,\ldots,q_i+t,\ldots,q_n)|_{t=0} \\
&= \frac{\partial}{\partial q_i}(f)
\end{aligned} \tag{3.24}$$

for all $f \in C^\infty(Q)$
- Therefore, $\theta(\partial/\partial q_i) = p(\partial/\partial q_i) = p_i$
- A similar calculation finds $\pi_*\partial/\partial p_i = 0$, so $\theta(\partial/\partial p_i) = 0$
- Therefore, $\theta = p_i dq_i$ in this chart $\blacksquare$

- This has a very clear link to classical mechanics: Q is the configuration space, and T^*Q is the phase space
- A symplectic manifold also induces a rather natural isomorphism between the tangent and cotangent space
- Specifically, this isomorphism is between the tangent space at each point and its dual
- **Theorem:** let (V, ω) be a symplectic vector space; then there is a 'natural' isomorphism called the musical isomorphism $V \equiv V^*$
- **Proof:**
  - Let $\xi \in V$; we define the linear map $\flat : V \to V^*$ by

$$\xi^\flat(\eta) = \omega(\eta, \xi), \quad \forall \eta \in V \tag{3.25}$$

  - $\flat$ is injective since ω is non-degenerate

– Instead, let $\lambda \in V^*$; we define the linear map $\sharp : V^* \to V$ by

$$\omega(\eta, \lambda^\sharp) = \lambda(\eta), \quad \forall \eta \in V \tag{3.26}$$

– This defines $\lambda^\sharp$ uniquely by non-degeneracy of ω, and it is again injective

– These two maps are mutual inverses:

$$\begin{aligned} \omega(\eta, (\xi^\flat)^\sharp) = \xi^\flat(\eta) = \omega(\eta, \xi), \quad \forall \eta \in V \implies (\xi^\flat)^\sharp = \xi, \ \forall \xi \in V \\ (\lambda^\sharp)^\flat(\eta) = \omega(\eta, \lambda^\sharp) = \lambda(\eta), \quad \forall \eta \in V \implies (\lambda^\sharp)^\flat = \lambda, \ \forall \lambda \in V^* \ \blacksquare \end{aligned} \tag{3.27}$$

- Moreover, the musical isomorphism defines a (2,0) tensor over V:

$$\omega^\sharp(\eta, \lambda) = -\omega(\eta^\sharp, \lambda^\sharp) \tag{3.28}$$

for all $\eta, \lambda \in V^*$

- This is called the **inverse symplectic form**

- Choosing a basis $\{e_a\}$ of V, we find:

$$(\xi^\flat)_a = \omega_{ab}\xi^b, \quad \text{and} \quad \omega_{ab}(\lambda^\sharp)^b = \lambda_a \implies (\lambda^\sharp)^a = \omega^{ab}\lambda_b \tag{3.29}$$

- This is, in a sense, raising and lowering indices using the symplectic form

- This makes sense as to why $\omega^\sharp$ is the inverse symplectic form:

$$(\omega^\sharp)^{ab}\eta_a\lambda_b = -\omega_{ab}(\eta^\sharp)^a(\lambda^\sharp)^b = -\omega_{ab}\omega^{ac}\omega^{bd}\eta_c\lambda_d = \delta^c_b\omega^{bd}\eta_c\lambda_d = \omega^{cd}\eta_c\lambda_d \tag{3.30}$$

- The analogous construction in pseudo-Riemannian geometry (c.f. GR) is defined by the metric tensor: an inner product on V

- This means that for any symplectic manifold (M, ω), there is a musical isomorphism $T_xM \equiv T^*_xM$ for every $x \in M$, which induces a musical isomorphism between vector fields and 1-forms on M

3.3 Hamiltonian Vector Fields and Flows

- Given arbitrary **Hamiltonian function** $H \in C^\infty(M)$, the vector field $X_H = (dH)^\sharp \in \mathfrak{X}(M)$ is the **Hamiltonian vector field**

- The set of all such vector fields is denoted $Ham(M)$

- Equivalently, by the musical isomorphism, we have

$$X_H^\flat = dH \implies X_H^\flat(\eta) = \omega(\eta, X_H) = -\omega(X_H, \eta) = -(\iota_{X_H}\omega)(\eta) \tag{3.31}$$

- Therefore, X_H is Hamiltonian if and only if

$$\iota_{X_H}\omega = -dH \tag{3.32}$$

- Just as a reminder, ι_X denotes the interior derivative of a k-form with respect to X, defined as contraction with X in the first argument

- In a coordinate basis $\{\partial_a : a = 1, \dots, 2n\}$, we find components

$$(\iota_{X_H}\omega)_a = \omega_{ab}X^a_H = -\omega_{ab}X^b_H = (-dH)_a = -\partial_a H \implies \omega_{ab}X^b_H = \partial_a H \tag{3.33}$$

- Equivalently:

$$X_H^a = \omega^{ab}\partial_b H \tag{3.34}$$

- **Example:** consider symplectic manifold $(\mathbb{R}^{2n}, \omega)$ with coordinates (p_i, q_i) and symplectic form $\omega = dp_i \wedge dq_i$
  - Pick some $H \in C^\infty(\mathbb{R}^{2n})$, and we compute the integral curves $\gamma(t)$ of X_H
  - Recall integral curves are defined by

$$\dot{\gamma}(t) = X_H(\gamma(t)) = X_H|_{\gamma(t)} \tag{3.35}$$

  - In our defining chart (p_i, q_i), we have

$$\dot{\gamma}(t) = \dot{p}_i \frac{\partial}{\partial p_i} + \dot{q}_i \frac{\partial}{\partial q_i} \tag{3.36}$$

    along the curve, where we slightly abuse notation to write $\dot{p}_i = \frac{d}{dt} p_i(\gamma(t))$ etc.
  - Therefore, along the curve we must have

$$\iota_{X_H}\omega|_{\gamma(t)} = (dp_i \wedge dq_i)(X_H(\gamma(t))) = dp_i(\dot{\gamma})dq_i - dq_i(\dot{\gamma})dp_i = \dot{p}_i dq_i - \dot{q}_i dp_i \tag{3.37}$$

  - We also have by definition:

$$dH = \frac{dH}{dp_i} dp_i + \frac{dH}{dq_i} dq_i \tag{3.38}$$

  - So by comparison, we deduce Hamilton's equations from purely geometric principles:

$$\dot{p}_i = -\frac{dH}{dq_i}, \quad \dot{q}_i = \frac{dH}{dp_i} \tag{3.39}$$

- This is a new way of looking at Hamilton's equations: solving them is equivalent to finding the integral curves of a certain vector field in phase space
- Now recall: integral curves of a complete vector field $X \in \mathfrak{X}(M)$ are in bijection with a one-parameter family of diffeomorphisms $\phi_t : M \to M$, $t \in \mathbb{R}$ satisfying composition law $\phi_{t+s} = \phi_t \circ \phi_s$ called **flows**
- Given flow ϕ_t, the corresponding vector field X can be reconstructed:

$$X_x(f) = \frac{d}{dt} f(\phi_t(x))\bigg|_{t=0} \quad \forall x \in M,\ f \in C^\infty(M) \tag{3.40}$$

- A **Hamiltonian flow** is the flow of a complete Hamiltonian vector field $X_H \in Ham(M)$
- A **Hamiltonian system** $(M, \omega; H)$ is a symplectic manifold (M, ω) with a chosen function $H \in C^\infty(M)$
- Time-evolution of a Hamiltonian system is defined by the Hamiltonian flow of X_H
- **Lemma:** for $H \in C^\infty(M)$ and $X_H \in Ham(M)$, $\mathcal{L}_{X_H}\omega = 0$ so Hamiltonian vector fields preserve the symplectic form
- **Proof:**
  - Recall Cartan's magic identity for the Lie derivative of k-forms:

$$\mathcal{L}_{X_H}\omega = d\iota_{X_H}\omega + \iota_{X_H} d\omega = -d^2 H = 0 \quad \blacksquare \tag{3.41}$$

- **Theorem:** let ϕ_t be a Hamiltonian flow on symplectic manifold (M, ω); then the symplectic form is preserved by the flow, $\phi_t^* \omega = \omega$
  - At arbitrary $x \in M$, we have

$$\begin{aligned}
(\phi_t^* \omega - \omega)_x &= \int_0^t \frac{d}{du} (\phi_u^* \omega)_x \, du \\
&= \int_0^t \frac{d}{ds} \left(\phi_{u+s}^* \omega\right)_x \Big|_{s=0} du \quad \text{(chain rule)} \\
&= \int_0^t \left(\phi_u^* \frac{d}{ds} \phi_s^* \omega \Big|_{s=0} \right)_x du \\
&= \int_0^t \phi_u^* (\mathcal{L}_{X_H} \omega)_{\phi_u(x)} \, du \\
&= 0
\end{aligned} \tag{3.42}$$

  - The second last line comes from definition of the Lie derivative of a k-form:

$$\mathcal{L}_{X_H} \omega = \frac{d}{dt} \phi_t^* \omega \Big|_{t=0} \quad \blacksquare \tag{3.43}$$

- **Corollary:** a Hamiltonian flow ϕ_t preserves any exterior power of the symplectic form; that is, $\phi_t^* \omega^k = \omega^k$
- **Corollary:** (Liouville's theorem) a Hamiltonian flow ϕ_t preserves the volume form $\phi_t^* \omega^n = \omega^n$
- This means that Hamiltonian flows preserve volumes in phase space
- **Theorem:** the Hamiltonian function $H \in C^\infty(M)$ is preserved by Hamiltonian flow associated to $X_H \in Ham(M)$
- **Proof:**
  - First, note

$$\mathcal{L}_{X_H} H = X_H(H) = dH(X_H) = -(\iota_{X_H} \omega)(X_H)) = -\omega(X_H, X_H) = 0 \tag{3.44}$$

  - Then, by a similar argument to the last theorem, we are done $\blacksquare$
- This result is essentially conservation of energy

3.4 The Poisson Bracket

- Given symplectic manifold (M, ω), we can define the **Poisson bracket** - a binary operation on $C^\infty(M)$
- Specifically, given $f, g \in C^\infty(M)$, the Poisson bracket is

$$\{f, g\} = X_g(f) \tag{3.45}$$

  where X_g is the Hamiltonian vector field associated with g

- Alternatively, we can think of this as the rate of change of f along the Hamiltonian flow ϕ_t of g:

$$\{f, g\}_x = \frac{d}{dt} f(\phi_t(x)) \Big|_{t=0}, \ \forall x \in M \tag{3.46}$$

- This definition is appealing geometrically, but in practice it is more useful to rewrite by noticing the following:

$$X_g(f) = df(X_g) = \iota_{X_g} df = -\iota_{X_g}\iota_{X_f}\omega = -\omega(X_f, X_g) \tag{3.47}$$

  and so

$$\{f, g\} = -\omega(X_f, X_g) \tag{3.48}$$

- This immediately implies that the Poisson bracket acts as $\{\cdot,\cdot\} : C^\infty(M) \to C^\infty(M) \to C^\infty(M)$, is $\mathbb{R}$-*bilinear*, and is *skew-symmetric*
- Moreover, using the musical isomorphism, we notice

$$\omega^\sharp(df, dg) = -\omega\left((df)^\sharp, (dg)^\sharp\right) = -\omega(X_f, X_g) = \{f, g\} \tag{3.49}$$

  which implies the following lemma
- **Lemma:** the Poisson bracket satisfies the *Leibnitz rule*

$$\{f_1 f_2, g\} = f_1\{f_2, g\} + \{f_1, g\}f_2 \tag{3.50}$$

  for all $f_1, f_2, g \in C^\infty(M)$
- As always, to do more useful things with the Poisson bracket, we need to express it in a coordinate basis
- Consider basis $e_a = \partial_a$ defined by chart (x^a), and the dual basis $f^a = dx^a$; then we find

$$\{f, g\} = \omega^\sharp\left(\partial_a f\, dx^a, \partial_b g\, dx^b\right) = \omega^{ab}\partial_a f \partial_b g \tag{3.51}$$

- **Example:** we look again at $(M, \omega) = (\mathbb{R}^{2n}, \omega)$ with coordinates (p_i, q_i) so $\omega = dp_i \wedge dq_i$
  - We know that in the $(\partial/\partial p_i, \partial/\partial q_i)$ basis

$$\omega_{ab} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \implies \omega^{ab} = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \tag{3.52}$$

  - Then, we find

$$\{f, g\} = \omega^{ab}\partial_a f \partial_b g = \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \tag{3.53}$$

    which is the usual definition of it from classical dynamics
- We also find that we can define the Poisson bracket for locally defined functions by just restricting the domain
- **Example:** consider the example from a while back, where Q is an n-dimensional smooth manifold with chart $(q_i(x))$ and the cotangent bundle $T^*Q = \{(p_x, x) : p_x \in T^*_x Q,\ x \in Q\}$ is a $2n$-dimensional symplectic manifold with chart $(p_i(x), q_i(x))$ and $\omega = dp_i \wedge dq_i$
  - The coordinates are functions defined only on the chart $U \subset M$
  - Then by the earlier example, we find the *fundamental Poisson brackets*:

$$\{q_i, q_j\} = \{p_i, p_j\} = 0, \quad \{q_i, p_j\} = \delta_{ij} \tag{3.54}$$

- Now recall that the time-evolution of a Hamiltonian system $(M, \omega; H)$ is given by the flow ϕ_t of X_H

- Therefore for a smooth function $f \in C^\infty(M)$, along the flow we find

$$\frac{d}{dt}\phi_t^* f = X_H(f) = \{f, H\} \tag{3.55}$$

- We usually write this in the shorthand

$$\frac{d}{dt} f = \{f, H\} \tag{3.56}$$

- We can define a **first integral/constant of motion** of $(M, \omega; H)$ is any function $f \in C^\infty(M)$ which is invariant under the flow of X_H; that is, f is a constant of motion iff it Poisson commutes with H: $\{f, H\} = 0$

- **Theorem:** the Hamiltonian H is preserved by the flow of $X_f \in Ham(M)$ if and only if $f \in C^\infty(M)$ is a constant of motion

- **Proof:**

  - Note that
$$X_f(H) = \{H, f\} = -\{f, H\} \tag{3.57}$$
and so $X_f(H) = 0$ iff $\{f, H\} = 0$ as required ■

- This is just Noether's theorem in phase space

- It turns out the Poisson bracket has a *Lie algebra* structure; before we get to this, we need to first relate the commutator of two Hamiltonian vector fields to their Poisson bracket

- Recall the definition of the commutator of two vector fields:

$$[X, Y](f) = X(Y(f)) - Y(X(f)), \quad \forall f \in C^\infty(M) \tag{3.58}$$

- **Theorem:** let X_f and X_g be two Hamiltonian vector fields; then

$$[X_f, X_g] = -X_{\{f,g\}} \tag{3.59}$$

  for $f, g \in C^\infty(M)$

- **Proof:**

  - Recall that X_H is Hamiltonian if and only if there exists $H \in C^\infty(M)$ such that $\iota_{X_H}\omega = -dH$
  - Therefore, we compute

$$\begin{aligned}
\iota_{[X_f, X_g]}\omega &= \left(\mathcal{L}_{X_f}\iota_{X_g} - \iota_{X_g}\mathcal{L}_{X_f}\right)\omega && \text{(Cartan identity)} \\
&= -\mathcal{L}_{X_f} dg && (\iota_{X_g}\omega = -dg \text{ and } \mathcal{L}_{X_f}\omega = 0) \\
&= -d\iota_{X_f} dg && \text{(Cartan's magic identity)} \\
&= -d\left(X_f(g)\right) && \text{(by definition)} \\
&= d\{f, g\} && \text{(by definition)}
\end{aligned} \tag{3.60}$$

  - The proof then follows from the recalled definition of Hamiltonian vector fields and non-degeneracy of ω ■

3.5 Lie Algebra of Hamiltonian and Symplectic Fields

- A **Lie algebra** $\mathfrak{g}$ is a vector space equipped with a bilinear map $[\cdot,\cdot]\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ called the **Lie bracket**, which satisfies:

  1. *Skew-symmetry:* $[x, y] = -[y, x]$
  2. *Jacobi identity:* $[x, [y, z]] + [y, [x, z]] + [z, [x, y]] = 0$

  for all $x, y, z \in \mathfrak{g}$

- The set of vector fields $\mathfrak{X}(M)$ equipped with the commutator (serving as the Lie bracket) is a Lie algebra

- A natural question is whether $Ham(M)$ is a Lie algebra - recall that $[X_f, X_g] = -X_{\{f,g\}}$, so the Lie bracket of Hamiltonian vector fields is Hamiltonian, and hence $Ham(M)$ is closed under $[\cdot,\cdot]$

- So all we need to do is check whether it is a vector space, which it is

- **Theorem:** the set of Hamiltonian vector fields $Ham(M)$ is a real Lie algebra under the Lie bracket of vector fields

- **Proof:**

  – If $X_f, X_g \in Ham(M)$, then $\alpha X_f + \beta X_g = X_{\alpha f+\beta g} \in Ham(M)$ for $\alpha, \beta \in \mathbb{R}$, so $Ham(M)$ is a real vector space
  – Moreover, the above discussion shows it is closed under the commutator of vector fields, which acts as a Lie bracket ■

- Note that we can define X_f for any $f \in C^\infty(M)$, so $Ham(M)$ is an infinite dimensional Lie algebra

- **Theorem:** the Poisson bracket satisfies the Jacobi identity

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0 \tag{3.61}$$

  for all $f, g, h \in C^\infty(M)$

- **Proof:**

  – By definition of the bracket, looking at the three terms in turn:

$$\begin{aligned}\{\{f, g\}, h\} &= -\{h, \{f, g\}\} = -X_{\{f,g\}}(h)\\ \{\{g, h\}, f\} &= -\{\{h, g\}, f\} = -\{X_g(h), f\} = -X_f(X_g(h))\\ \{\{h, f\}, g\} &= \{X_f(h), g\} = X_g(X_f(h))\end{aligned} \tag{3.62}$$

  – Summing over the right hand sides, we find

$$-[X_f, X_g](h) - X_{\{f,g\}}(h) = 0 \tag{3.63}$$

  and so the left hand sides sum to zero too ■

- **Corollary:** $C^\infty(M)$ equipped with the Poisson bracket is a Lie algebra

- Note that $\phi\colon C^\infty(M) \to Ham(M)$, $f \mapsto -X_f$ is a Lie algebra homomorphism, given by $[\phi(f), \phi(g)] = \phi(\{f, g\})$

- The kernel of ϕ is given by constant functions on M, and so by virtue of $\ker\phi$ being non-empty, ϕ is not an isomorphism
- In the absence of additional structure, there is no canonical notion of a Lie bracket for $C^\infty(M)$ - a symplectic structure provides such a bracket
- **Theorem:** if f, g are constants of motion for a Hamiltonian system $(M, \omega; H)$, then $\{f, g\}$ is also a constant of motion
- **Proof:**
  - By definition, $\{f, H\} = \{g, H\} = 0$
  - Then, by Jacobi:

$$\{\{f, g\}, H\} = -\{\{g, H\}, f\} - \{\{H, f\}, g\} = 0 \quad \blacksquare \tag{3.64}$$

- This shows that constants of motion of a Hamiltonian system form a Lie algebra under the Poisson bracket
- From the theorem stating that H is invariant under the flow of X_f iff f is a constant of motion, this therefore corresponds to the Lie subalgebra of $Ham(M)$ preserving H
- Given (M, ω), a **symplectomorphism** is a diffeomorphism $\phi : M \to M$ which preserves the symplectic form $\phi^*\omega = \omega$
- A vector field $X \in \mathfrak{X}(M)$ is **symplectic** if $\mathcal{L}_X\omega = 0$; we denote this as $X \in Sym(M)$
- This means that if the flow $\phi_t : M \to M$ of a vector field X is a symplectomorphism, then X is symplectic
- This follows from $(\mathcal{L}_X\omega)_p = \frac{d}{dt}(\phi_t^*\omega)_p|_{t=0} = 0$ since $\phi_t^*\omega = 0$
- The converse is in fact true, following the same proof as earlier with all the integrals
- We know that Hamiltonian vector fields preserve the symplectic form: $\mathcal{L}_{X_H}\omega = 0$; Hamiltonian vector fields are therefore symplectic
- The converse is sometimes true: if X is symplectic, then

$$0 = \mathcal{L}_X\omega = d\iota_X\omega \quad \Longleftrightarrow \quad \iota_X\omega \text{ is closed} \quad \Longleftrightarrow \quad \iota_X\omega \text{ is locally exact} \tag{3.65}$$

  where the first implication follows from the definition of closure being $d\alpha = 0$, and the second from the *Poincare lemma* (every closed form is exact - the exterior derivative of another form)
- Therefore the converse is true if the de Rham cohomology $H^1(M, \mathbb{R}) = 0$ - every closed 1-form is exact - then there exists $f \in C^\infty(M)$ such that $\iota_X\omega = -df$, so X is Hamiltonian
- **Example:** consider the 2-torus $M = T^2$ with 2π-periodic coordinates (p, q), and symplectic form $\omega = dp \wedge dq$
  - The flow corresponding to $X = \partial/\partial p$ is $\phi_t(p, q) = (p + t, q)$; then $\mathcal{L}_X\omega = 0$ so X is symplectic
  - However, $\iota_X\omega = dq$ is not exact on $\mathrm{T}T^2$ since q is only defined locally (i.e. we have a discontinuous jump in q globally), and so X is not Hamiltonian
  - This is because $H^1(T^2, \mathbb{R}) \neq 0$

- **Theorem:** the set $Sym(M)$ forms a Lie algebra under the Lie bracket of vector fields
- **Proof:**
  - $Sym(M)$ is a real vector space since $\mathcal{L}_X$ is $\mathbb{R}$-linear in X
  - Let $X, Y \in Sym(M)$, and compute

$$\begin{aligned} \iota_{[X,Y]}\omega &= \mathcal{L}_X \iota_Y \omega - \iota_Y \mathcal{L}_X \omega && \text{(Cartan identity)} \\ &= \mathcal{L}_X \iota_Y \omega && (X \in Sym(M) \implies \mathcal{L}_X \omega = 0) \\ &= (d\iota_X + \iota_X d)\iota_Y \omega && \text{(Cartan's magic identity)} \\ &= d(\iota_X \iota_Y \omega) && (Y \in Sym(M) \iff \iota_Y \omega \text{ closed}) \end{aligned} \tag{3.66}$$

  - This means $[X, Y] \in Ham(M)$ with Hamiltonian $\omega(X, Y)$; and therefore $[X, Y] \in Sym(M)$ meaning $Sym(M)$ is a Lie algebra ■
- We can write this conclusion as $[Sym(M), Sym(M)] \subseteq Ham(M)$
- But $Ham(M) \subseteq Sym(M)$ is then an ideal in $Sym(M)$:

$$[Sym(M), Ham(M)] \subseteq Ham(M) \tag{3.67}$$

3.6 Darboux's Theorem

- For the entirety of this section, (M, ω) is a $2n$-dimensional symplectic manifold
- A *symplectic chart* on (M, ω) is a coordinate chart (p_i, q_i), $i = 1, \ldots, n$ on M such that $\omega = dp_i \wedge dq_i$
- In physics, symplectic charts are called canonical coordinates
- **Lemma:** a chart (p_i, q_i) is symplectic iff $\{q_i, p_j\} = \delta_{ij}$, $\{q_i, q_j\} = \{p_i, p_j\} = 0$
- **Proof:**
  - The above Poisson brackets are equivalent to ω^{ab} taking the following form in the dp_i, dq_i basis:

$$\omega^{ab} = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \tag{3.68}$$

  - This is in turn equivalent to ω_{ab} taking the usual canonical form in the $\partial/\partial p_i, \partial/\partial q_i$ basis ■
- **Theorem:** (Darboux) let (M, ω) be a symplectic manifold. Then any point $x \in M$ is contained in a symplectic chart
- **Proof:**
  - This is a long one, so strap in
  - Let $x \in M$, and pick some $p_1 \in C^\infty(M)$ such that $p_1(x) = 0$ but $dp_1|_x \neq 0$
  - This is always possible (e.g. just choose p_1 to be linear)
  - The Hamiltonian vector field X_{p_1} of p_1 (defined by $\iota_{X_{p_1}}\omega = -dp_1$) is unique and non-vanishing at x: $X_{p_1}|_x \neq 0$
  - Define N to be a $(2n-1)$-dimensional submanifold of M containing x, where $(X_{p_1})_x \notin T_x N$; N is *transverse* to X_{p_1} (i.e. we pick a chart adapted to X_{p_1})

- Now let ϕ_1^t be the Hamiltonian flow of X_{p_1}, and consider the map $F: N \times \mathbb{R} \to M$, defined by $F(y,t) = \phi_1^t(y)$ for $y \in N$, $t \in \mathbb{R}$
- Recall that the derivative of a smooth map $\phi: A \to B$ is the pull-back $\phi_*: T_pA \to T_{\phi(p)}B$, given by $(\phi_* X_p)(f) = X_p(f \circ \phi)$
- So the derivative of F at $(x,0)$ is the pull-back $F_*: T_xN \times T_0\mathbb{R} \to T_xM$, and $F_*(Y_x, (d/dt)_0) = (Y_x, (X_{p_1})_x)$
- This has full rank since $(X_{p_1})_x$ is linearly independent of all $Y_x \in T_xN$
- Therefore, F_* is invertible, and by the *inverse function theorem*, F is a local diffeomorphism
- What this means is that there is a neighbourhood (open set) of $(x,0) \in N \times \mathbb{R}$ that is diffeomorphic to a neighbourhood U of x in M
- For any $z \in U$, we can therefore assign a unique time t along the flow ϕ_1^t from $y \in N$, so $z = \phi_1^t(y)$ defines a smooth function $q_1(z) = t$ for $z \in U$
- Note that $q_1|_N = 0$, and we can calculate the derivative at z in the direction of X_{p_1}:
$$(X_{p_1})_z(q_1) = \frac{d}{ds} q_1(\phi_1^s(z))|_{s=0} = \frac{d}{ds} q_1(\phi_1^{s+t}(y))|_{s=0} = 1 \tag{3.69}$$
- Since this holds for any $z \in U$, we have that $X_{p_1}(q_1) = \{q_1, p_1\} = 1$
- For $n = 1$, this completes the construction of a symplectic chart
- For $n > 1$, we use induction
- Assume the theorem holds for $(2n-2)$-dimensional symplectic manifolds
- Note that the dp_1, dq_1 are linearly independent on U since $1 = \{q_1, p_1\} = \omega^\sharp(dq_1, dp_1)$
- So by the *implicit function theorem*, the level set $\bar{M} = \{w \in U \,|\, q_1(w) = p_1(w) = 0\}$ is a $(2n-2)$-dimensional submanifold of M containing x
- We now claim that $(\bar{M}, \omega|_{\bar{M}})$ is a $(2n-2)$-dimensional symplectic manifold
- $\omega|_{\bar{M}}$ inherits skew-symmetry from ω, so we need to check non-degeneracy
- To do this, take $\xi \in T_w\bar{M}$ to be a tangent vector on $\bar{M}$, so $\xi(p_1) = \xi(q_1) = 0$ since p_1 and q_1 are constant on $\bar{M}$
- Then:
$$\omega(X_{p_1}, \xi) = -dp_1(\xi) = 0 \quad \text{and} \quad \omega(X_{q_1}, \xi) = -dq_1(\xi) = 0 \tag{3.70}$$
- This means $T_w\bar{M}$ is skew-orthogonal (with respect to ω) to $span(X_{p_1}, X_{q_1})|_w$ for all $w \in \bar{M}$
- Recall the proof that every symplectic vector space has a symplectic basis: this same method of proof then shows that $\omega|_{\bar{M}}$ is non-degenerate
- Therefore by the induction hypothesis, there are symplectic coordinates (p_i, q_i), $i = 2, \ldots, n$ on a neighbourhood of x in $(\bar{M}, \omega|_{\bar{M}})$
- We can extend these to a neighbourhood of x in M (i.e. to p_1, $q_1 \neq 0$) as follows
- Any point z in a neighbourhood $\tilde{U} \subseteq U$ of x in M can be written uniquely as $z = \phi_1^t \psi_1^s w$ for some $w \in \bar{M}$, where ψ_1^s is the flow of X_{q_1}
- We therefore extend (p_i, q_i), $i = 2, \ldots, n$ to $\tilde{U}$ by keeping their values constant along flows of X_{p_1} and X_{q_1}:
$$X_{q_1}(q_i) = X_{q_1}(p_i) = X_{p_1}(q_i) = X_{p_1}(p_i) = 0 \tag{3.71}$$
so the coordinates of z are the same as w; that is, $p_i(z) = p_i(w)$, $q_i(z) = q_i(w)$

- Then $(p_1, \dots, p_n, q_1, \dots, q_n)$ are coordinates for $\tilde{U}$ of x in M; we now need to show these are symplectic
- We know already that $\{q_1, p_1\} = 1$, and note that $\{q_i, q_1\} = X_{q_1}(q_i) = 0$ and $\{p_i, q_1\} = \{q_i, p_1\} = \{p_i, p_1\} = 0$ similarly for $i = 2, \dots, n$ on $\tilde{U}$
- By the induction hypothesis, $\{q_i, p_j\} = \delta_{ij}$ on $\bar{M} \cap \tilde{U}$
- For $i, j = 2, \dots, n$ on $\tilde{U}$, we have:

$$X_{q_1}(\{q_i, p_j\}) = \{\{q_i, p_j\}, q_1\} = \{q_i, \{p_j, q_1\}\} + \{p_j, \{q_1, q_i\}\} = 0 \tag{3.72}$$

  and similarly, $X_{p_1}(\{q_i, p_j\}) = 0$
- Therefore, since the functions $\{q_i, p_j\}$ are unchanged on the flows of X_{p_1} and X_{q_1}, they still satisfy $\{q_i, p_j\} = \delta_{ij}$ on $\tilde{U}$
- Similar arguments hold for the remaining Poisson brackets, and so we are done ■

- Note that this shows that symplectic manifolds are locally isomorphic to symplectic vector spaces

3.7 Integrable Systems

- Loosely speaking, an integrable system is one which can be solved exactly; but what does this precisely mean?

- Liouville gave a definition: a $2n$-dimensional Hamiltonian system $(M, \omega; H)$ is *Liouville integrable* if:

  1. There exists $f_i \in C^\infty(M)$, $i = 1, \dots, n$ in *involution*; that is,

  $$\{f_i, f_j\} = 0, \quad \forall i, j = 1, \dots, n \tag{3.73}$$

  2. The functions f_i are functionally independent on the level sets

  $$M_{\mathbf{c}} = \{x \in M \mid f_i(x) = c_i,\ i = 1, \dots, n\} \tag{3.74}$$

  (i.e. the one forms df_i are linearly independent on this set)

- Note that a $2n$-dimensional symplectic manifold can have at most n independent functions in involution

- If f_i, $i = 1, \dots, k$ are such a set of functions, we find

$$\omega(X_{f_i}, X_{f_j}) = -\{f_i, f_j\} = 0 \tag{3.75}$$

  so *span*$(X_{f_1}, \dots, X_{f_k})$ is a k-dim. null subspace of TM (i.e. all functions get mapped to zero), hence $k \leq n$

- **Theorem:** (Liouville-Arnold) let $(M, \omega; H)$ be a $2n$-dimensional Hamiltonian system which is Liouville integrable. Then:

  1. $M_{\mathbf{c}}$ is a smooth, n-dimensional submanifold of M, which is invariant under the flow of X_H; that is, if ϕ_t is the flow of X_H and $x \in M_{\mathbf{C}}$, then $\phi_t(x) \in M_{\mathbf{c}}$
  2. If $M_{\mathbf{c}}$ is compact and connected, it is diffeomorphic to an n-torus $T^n = \{(\theta_1, \dots, \theta_n) \bmod 2\pi\}$
  3. The flow of X_H on a compact $M_{\mathbf{c}} \cong T^n$ takes the form $\dot{\theta}_i = \omega_i(\mathbf{c})$
  4. In a neighbourhood of $M_{\mathbf{c}} \cong T^n$, there exists a symplectic chart (I_i, θ_i) (*action-angle coordinates*) where $I_i = I_i(\mathbf{c})$ are constants of motion

- **Proof (1):**
  - By definition of Liouville integrable, the df_i are linearly independent on $M_{\mathbf{c}}$
  - So by the **implicit function theorem**, the level sets $M_{\mathbf{c}}$ are smooth, n-dimensional submanifolds of M
  - We know H is a constant of motion, so we can set $H = f_1$
  - Then, using the fact that the f_i are in involution, we have

$$X_H(f_i)) = -\{H, f_i\} = -\{f_1, f_i\} = 0 \tag{3.76}$$

    which shows that the $M_{\mathbf{c}}$ are tangent to X_H, and hence invariant submanifolds of its flow ■

- Before proving (2), we need a lemma
- **Lemma:** The Hamiltonian vector fields X_{f_i}, $i = 1, \ldots, n$ are tangent to $M_{\mathbf{c}}$, pairwise Lie commute, and are linearly independent at every point in $M_{\mathbf{c}}$
- **Proof:**
  - To prove pairwise commutation, consider the Hamiltonian vector fields X_{f_i} for $i = 1, \ldots, n$
  - We have:

$$[X_{f_i}, X_{f_j}] = -X_{\{f_i, f_j\}} = 0 \tag{3.77}$$

    by involution
  - To show linear independence, recall that ω is non-degenerate ($\omega(X, Y) = 0 \forall X \in \mathfrak{X}(M) \implies Y = 0$), and that $\iota_{X_{f_i}}\omega = -df_i$
  - Then since the df_i are linearly independent on $M_{\mathbf{c}}$ by definition of Liouville integrability, so too are the X_{f_i}
  - For tangent, note that $X_{f_i}(f_j) = -\{f_i, f_j\} = 0$, which means the X_{f_i} are tangent to $M_{\mathbf{c}}$
  - Therefore, X_{f_i} give a nowhere vanishing, pairwise commuting *frame* of vector fields on $M_{\mathbf{c}}$ (i.e. a basis for the tangent space at every point) ■
- We should note that there's actually a little more hidden under the surface of this lemma
- Since $X_{f_i}(f_j) = 0$, it follows that $M_{\mathbf{c}}$ is invariant under the flow of all the X_{f_i}
- Moreover, $\omega(X_{f_i}, X_{f_j}) = -\{f_i, f_j\} = 0$, which implies that $M_{\mathbf{c}}$ is an n-dimensional **null** submanifold; that is, $T_x M_{\mathbf{x}}$ is a null subspace of $T_x M$ for all $x \in M_{\mathbf{c}}$ (also known as a *Lagrangian submanifold*)
- We now use the following lemma for compact and connected manifolds
- **Lemma:** a compact, connected, n-dimensional smooth manifold N admitting a pairwise commuting frame of vector fields is diffeomorphic to T^n
- **Proof:**
  - Denote the commuting frame of vector fields by X_i, $i = 1, \ldots, n$, and its flows by ϕ_i^t
  - The frame is commuting by assumption, so the flows ϕ_i^t and ϕ_j^s commute too

- We can therefore define diffeomorphisms $\phi^{\mathbf{t}} : M \to M$ by $\phi^{\mathbf{t}} = \phi_1^{t_1} \circ \phi_2^{t_2} \circ \ldots \circ \phi_n^{t_n}$, where $\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{R}^n$, and $\phi^{\mathbf{t}+\mathbf{s}} = \phi^{\mathbf{t}}\phi^{\mathbf{s}}$
- Now we fix a point $p \in N$, and define a smooth map $\phi : \mathbb{R}^n \to N$ by $\phi(\mathbf{t}) = \phi^{\mathbf{t}}(p)$ (i.e. p moves by t_i along the flow of X_i)
- The derivative of this map at the origin is the pull-back $\phi_* : T_0\mathbb{R}^n \to T_pN$, where $\phi_* \left(\frac{\partial}{\partial t_i}\right)_0 = X_i(p)$
- Since the $X_i(p)$ are linearly independent (as they form a frame), this derivative has full rank
- So by the inverse function theorem, ϕ defines a local diffeomorphism in a neighbourhood of the origin
- It can also be shown that ϕ is onto
- Now consider the set $\Gamma = \{\mathbf{t} \in \mathbb{R}^n \,|\, \phi(\mathbf{t}) = p\}$, which is a subgroup of $\mathbb{R}^n$ under addition and independent of p
- Moreover, $\mathbf{0} \in \Gamma$ and since ϕ is a diffeomorphism in a neighbourhood of $\mathbf{0}$, there can be no other elements of Γ in this neighbourhood
- Similarly, for any $\mathbf{t} \in \Gamma$ there is a neighbourhood containing it and no other elements of Γ, and so Γ is a discrete subgroup
- Any discrete subgroup of $\mathbb{R}^n$ has the form

$$\Gamma = \{n_1\mathbf{t}_1 + \ldots + n_k\mathbf{t}_k \,|\, (n_1, \ldots, n_k) \in \mathbb{Z}^k\} \tag{3.78}$$

  where $\mathbf{t}_i$ is a set of $0 \leq k \leq n$ linearly independent vectors on $\mathbb{R}^n$
- Now consider an isomorphism $A : \mathbb{R}^n \to \mathbb{R}^n$ such that $A(2\pi\mathbf{e}_i) = \mathbf{t}_i$ for $i = 1, \ldots, k$ and $\mathbf{e}_i$, $i = 1, \ldots, n$ is the standard orthonormal basis of $\mathbb{R}^n$
- Define the projection $\pi : \mathbb{R}^n \to T^k \times \mathbb{R}^{n-k}$ by

$$\pi(\theta_1, \ldots, \theta_k, x_{k+1}, \ldots, x_n) = (\theta_1 \bmod 2\pi, \ldots, \theta_k \bmod 2\pi, x_{k+1}, \ldots, x_n) \tag{3.79}$$

- Then, $\pi^{-1}(0) = \{2\pi(n_1\mathbf{e}_1 + \ldots + n_k\mathbf{e}_k) \,|\, (n_1, \ldots, n_k) \in \mathbb{Z}^k\}$, and hence $A \circ \pi^{-1}(0) = \Gamma$ and $\phi \circ A \circ \pi^{-1}(0) = p$
- Similarly, $\tilde{A} = \phi \circ A \circ \pi^{-1}$ maps any other point in $T^k \times \mathbb{R}^{n-k}$ to a unique point in N, and defines a diffeomorphism $\tilde{A} : T^k \times \mathbb{R}^{n-k} \to N$
- Since N is compact, we must have $k = n$, and hence $N \cong T^n$ ■

- **Proof (2):**
  - We know already that $M_{\mathbf{c}}$ admits a commuting frame of vector fields
  - So by the lemma, we are done ■
- **Proof (3):**
  - By the second lemma, $M_{\mathbf{c}} \cong T^n = \{(\theta_1, \ldots, \theta_n) \bmod 2\pi\}$, and $A\theta = \mathbf{t}$ where $\mathbf{t} = (t_1, \ldots, t_n)$ are the flow parameters of X_{f_i} and A is an invertible matrix
  - The diffeomorphism depends on $\mathbf{c} \in \mathbb{R}^n$, and so too must A then
  - To determine the dependence of the torus coordinates on the flow, we invert so $\theta = A^{-1}\mathbf{t}$
  - $H = f_1$, so the evolution under flow ϕ^t of X_H is obtained by setting $t = t_1$
  - Therefore, θ_i is linearly dependent on t, so $\dot{\theta}_i = \omega_i(\mathbf{c})$ ■

- We don't prove (4)
- In the angular coordinates on the torus, the Hamiltonian flow takes the simple form $\theta_i(t) = \theta_i(0) + \omega_i(\mathbf{c})$
- The functions f_i together with the torus coordinates θi define a chart for M in the neighbourhood of $M_{\mathbf{c}}$
- The flow in the coordinates (f_i, θ_i) is simply $\dot{f}_i = 0$ and $\dot{\theta}_i = \omega_i(f)$, which integrates as above
- In general, (f_i, θ_i) are not symplectic, but we can show there are functions $I_i(f)$ called actions which make (I_i, θ_i) symplectic
- The Hamiltonian flow defined by $H = f_1 = f_1(I)$ is therefore just given by Hamilton's equations
$$\dot{I}_i = 0, \quad \dot{\theta}_i = \frac{\partial H}{\partial I_i} \tag{3.80}$$

4 Lie Groups and Lie Algebras

- Motivation: Lie groups and their corresponding Lie algebras arise in studying symmetries in phsyics/geometry
- A *Lie group* is a smooth manifold G which has a group structure; that is, there are group operations at points $g, h \in G$:
  - *Multiplication:* $\mu : G \times G \to G$, with $\mu(g, h) = g \cdot h$
  - *Inversion:* $\iota : G \to G$, with $\iota(g) = g^{-1}$

  which are smooth
- We typically denote the identity element as $e \in G$
- The most important examples of these are matrix groups, the largest of which is the *general linear groups* $GL(n, \mathbb{R} \text{ or } \mathbb{C})$, consisting of the invertible $n \times n$ matrices over $\mathbb{R}$ or $\mathbb{C}$, with group multiplication just matrix multiplication
- The following theorem is stated without proof
- **Theorem:** any closed subgroup H of a Lie group G is itself a Lie group
- Given this theorem, we have some common Lie groups which are closed subgroups of the general linear groups:
  - *Special linear groups:* $SL(n, \mathbb{R} \text{ or } \mathbb{C}) = \{M \in GL(n, \mathbb{R} \text{ or } \mathbb{C}) \mid \det M = 1\}$
  - *Orthogonal groups:* $O(n) = \{M \in GL(n, \mathbb{R}) \mid M^T M = I_n\}$
  - *Special orthogonal groups:* $SO(n) = O(n) \cap SL(n, \mathbb{R})$
  - *Unitary groups:* $U(n) = \{M \in GL(n, \mathbb{C}) \mid M^\dagger M = I_n\}$
  - *Special unitary groups:* $SU(n) = U(n) \cap SL(n, \mathbb{C})$
  - *Symplectic groups:* $Sp(2n) = \{M \in GL(2n, \mathbb{R}) \mid M^T \Omega M = \Omega\}$ where Ω is the usual canonical symplectic form
  - *Lorentz groups:* $O(1, 4) = \{M \in GL(4, \mathbb{R}) \mid M^T \eta M = \eta\}$ where $\eta = \text{diag}(-1, 1, 1, 1)$
- Now, recall the definition of a *Lie algebra* as a vector space equipped with a Lie bracket, which is a skew-symmetric bilinear form satisfying the Jacobi identity
- An easy (yet important) example of a Lie algebra is the set of all $n \times n$ matrices $Mat(n)$, equipped with the standard matrix commutator as a Lie bracket
- We now show that given a Lie group G, there is a 'natural' associated Lie algebra
- First, we define *left and right multiplication*, which are maps defined for each $g, h \in G$:

$$\begin{aligned} L_g : G \to G \quad &, \quad L_g h \mapsto gh \\ R_g : G \to G \quad &, \quad R_g h \mapsto hg \end{aligned} \tag{4.1}$$

- We immediately note that these are both smooth and bijections, as $L_g^{-1} = L_{g^{-1}}$ and $R_g^{-1} = R_{g^{-1}}$, and so these are therefore diffeomorphisms for each $g \in G$
- We say a vector field $X \in \mathfrak{X}(G)$ is *left-invariant* if $(L_g)_* X_h = X_{gh}$ for all $g, h \in G$, where $(L_g)_* : T_h G \to T_{gh} G$ is the derivative of L_g
- We denote the set of all such left-invariant vector fields as $L(G)$

- There's also a nice equivalent definition: $X \in L(G)$ iff $X_g = (L_g)_* X_e$ (stated without proof)
- Thus left invariant vector fields are uniquely determined by their values at the identity e
- **Lemma:** $L(G)$ is a Lie algebra under the Lie bracket of vector fields
- **Proof:**
  - Recall the definition of the Lie bracket of vector fields $X, Y \in \mathfrak{X}(G)$ acting on $f \in C^\infty(G)$:
  $$[X,Y](f) = X(Y(f)) - Y(X(f)) \tag{4.2}$$
  - We need to show two things: $L(G)$ is a vector space, and it is closed under the Lie bracket
  - It is clearly a vector space: if $X, Y \in L(G)$, then $X + Y \in L(G)$ and $\alpha X \in L(G)$ where $\alpha \in \mathbb{R}$, since $(L_g)_*$ is $\mathbb{R}$-linear
  - It is also closed: if $X, Y \in L(G)$, since $L_g : G \to G$ is a smooth bijection, $(L_g)_*[X,Y] = [(L_g)_* X, (L_g)_* Y] = [X,Y]$ ■
- In general, there is no obvious natural Lie bracket on a tangent space T_pM to some manifold M at p
- For Lie groups G, the tangent space at the identity e inherits a Lie bracket from the Lie bracket on $L(G)$; specifically, for $X_e, Y_e \in T_eG$, we have:
  $$[X_e, Y_e] = [X,Y]_e \tag{4.3}$$
  where $X|_g = (L_g)_* X_e$ etc.
- **Theorem:** $L(G) \cong T_eG$ is a Lie algebra isomorphism
- **Proof:**
  - Let $X \in L(G)$, and define $\phi : L(G) \to T_eG$ by $\phi(X) = X_e$
  - This establishes a vector space homomorphism in one direction
  - Conversely, let $V \in T_eG$, and define
  $$(X_V)_g = (L_g)_* V \tag{4.4}$$
  - Then:
  $$(L_g)_*(X_V)_h = (L_g)_*(L_h)_* V = (L_{gh})_* V = (X_V)_{gh} \tag{4.5}$$
  and so $X_V \in L(G)$
  - We now claim that X_V is inverse to ϕ
  - Note that $\phi(X_V) = (X_V)_e = V$ since $(L_e)_* = id$
  - Therefore ϕ is a vector space isomorphism
  - Now note that since $[X_e, Y_e] = [X,Y]_e$ for all $X_e, Y_e \in T_e(G)$, we have
  $$\phi([X,Y]) = [\phi(X), \phi(Y)] \tag{4.6}$$
  which is just the statement that ϕ is a Lie algebra isomorphism ■
- We refer to T_eG as **the** Lie algebra $\mathfrak{g}$ of the Lie group G

- Note that $\dim \mathfrak{g} = \dim(G)$
- As a simple example, consider the (additive) Lie group $G = (\mathbb{R}, +)$; any left invariant vector field looks like $X = c\frac{\partial}{\partial x}$ where $c \in \mathbb{R}$ and x is the defining chart. Therefore, $\left(\frac{\partial}{\partial x}\right)_a = (L_a)_* \left(\frac{\partial}{\partial x}\right)_0$, establishing the correspondence between $L(\mathbb{R})$ and $T_0\mathbb{R}$
- For matrix Lie groups, consider $G = GL(n, \mathbb{R})$
- The entries of a matrix $a = (a_{ij}) \in G$ define a global chart via $g_{ij} : G \to \mathbb{R}$ where $g_{ij}(a) = a_{ij}$
- Then any vector field on G can be written as

$$X = X_{ij} \frac{\partial}{\partial g_{ij}} \tag{4.7}$$

  where X_{ij} are smooth functions
- Now, consider $V \in T_eG$, which is $V = V_{ij} \left(\partial/\partial g_{ij}\right)_e$, where $V_{ij} \in Mat(n)$; we then have the following theorem
- **Theorem:** for Lie group $G = GL(n, \mathbb{R})$, for all $a \in G$ and $V, W \in T_eG$, we have:
  1. $X_{V_{ij}}(a) = (aV)_{ij}$
  2. $[X_V, X_W]_{ij}(a) = (a(VW - WV))_{ij}$
  3. $[V, W]_{ij} = (VW - WV)_{ij}$
- **Proof:** see tutorial sheet
- This theorem shows that for matrix Lie groups, left translation corresponds to left matrix multiplication
- We also state the following without proof
- **Theorem:** every finite dimensional Lie algebra is the Lie algebra of a Lie group

4.1 Frames and Structure Equation

- A **frame** of vector fields on an n-dimensional manifold M is a set of n linearly independent vector fields $\forall p \in M$
- Typically, a manifold does not admit a global frame, but usually does locally
- Lie groups however do admit a global frame
- **Theorem:** any lie group G is **parallelisable**; there exists a globally defined frame of vector fields
- **Proof:**
  - Let $V_1, \ldots, V_n$ be a basis of T_eG
  - Then the left-invariant vector fields $X_{V_1}, \ldots, X_{V_n}$ give a global frame ■
- We say a form $\omega \in \Omega^k(G)$ is **left-invariant** if $L_g^* \omega_{gh} = \omega_h$ for all $g, h \in G$, where L_g^* is the pull-back of L_g
- As for left-invariant vector fields, this has an equivalent characterisation in terms of the form at the identity only

- Specifically, $\omega_g = L^*_{g^{-1}}\omega_e$
- Now, consider an arbitrary basis $V_1, \dots, V_n$ of $\mathfrak{g} \cong T_eG$; since it is a Lie algebra, we must have
$$[V_i, V_j] = c_{ij}{}^k V_k \tag{4.8}$$
where $c_{ij}{}^k \in \mathbb{R}$ are known as the **structure constants** of $\mathfrak{g}$
- Now recall the Lie algebra isomorphism $L(G) \cong T_eG$; this means that the left-invariant vector fields also satisfy
$$[X_i, X_j] = c_{ij}{}^k X_k \tag{4.9}$$
where we abbreviate $X_i = X_{V_i}$
- Consider the dual one-forms to the X_i given by $\theta^i \in \Omega^1(G)$, $i = 1, \dots, n$, so $\theta^i(X_j) = \delta^i_j$
- These θ^i are clearly left-invariant:
$$(L^*_g\theta^i_g)((X_j)_e) = \theta^i_g((L_g)_*(X_j)_e) = \theta^i_g((X_i)_g) = \delta^i_j \implies L^*_g\theta^i_g = \theta^i_e \tag{4.10}$$
- Moreover, these forms satisfy the *Maurer-Cartan structure equation*
- **Theorem:** the left-invariant one-forms θ^i satisfy
$$d\theta^i = -\frac{1}{2}c_{jk}{}^i\theta^j \wedge \theta^k \tag{4.11}$$
- **Proof:**
  - Act on the dual frame of vector fields:
  $$d\theta^i(X_j, X_k) = X_j(\theta^i(X_k)) - X_k(\theta^i(X_j)) - \theta^i([X_j, X_k]) = -c_{jk}{}^i$$
  where we used a general identity for exterior derivatives of one-forms
  - Moreover, note that
  $$(\theta^l \wedge \theta^m)(X_j, X_k) = \delta^l_j\delta^m_k - \delta^l_k\delta^m_j$$
  and so we are done ■
- With this in mind, we define the *Maurer-Cartan one-form* on G, which is a **Lie algebra valued** one-form $\theta \in \Omega^1(G; \mathfrak{g})$ given by
$$\theta_g : T_gG \to \mathfrak{g}, \quad \theta_g(X_g) = (L_{g^{-1}})_*X_g \in \mathfrak{g} \tag{4.12}$$
for all $X_g \in T_gG$
- Note that $\theta_e(X_e) = X_e$, so $\theta_e = id_e$ on $\mathfrak{g}$
- Moreover:
$$(L^*_g\theta_g)V = \theta_g((L_g)_*V) = \theta_g((X_V)_g) = (L_{g^{-1}})_*(X_V)_g = V$$
for $V \in T_eG$, and so $L^*_g\theta_g = id_e$ on $\mathfrak{g}$, meaning θ is left-invariant
- **Lemma:** the Maurer-Cartan form θ on G can be written as $\theta = \theta^i \otimes V_i$ where θ^i is the dual frame of one-forms to $X_{V_i} \in L(G)$
- **Proof:**
  - We act on an arbitrary $Y_g \in T_gG$, and expand $Y = Y^iX_i$ in terms of left-invariant frame $\{X_i\}_{i=1}^n$

- The LHS gives us:

$$\theta_g(Y_g) = Y^i\theta_g(X_i) = Y^i(L_{g^{-1}})_*(L_g)_*V_i = Y^iV_i$$

  and the RHS give us

$$(\theta^i_g \otimes V_i)(Y_g) = \theta^i_g(Y_g)V_i = Y^iV_i$$

  and we are done ■

- **Theorem:** the Maurer-Cartan form on G satisfies the structure equation

$$d\theta + \frac{1}{2}[\theta,\theta] = 0 \tag{4.13}$$

where $[\cdot,\cdot]$ denotes **both** the Lie bracket on $\mathfrak{g}$ and the wedge product on $\Omega^1(G)$

- **Proof:**
  - By the previous lemma, we can write:

$$\begin{aligned} d\theta = d\theta^i \otimes V_i &= -\frac{1}{2}c_{jk}{}^i\theta^j \wedge \theta^k \otimes V_i && \text{(by (4.11))} \\ &= -\frac{1}{2}\theta^j \wedge \theta^k \otimes [V_k, V_l] && \text{(by (4.8))} \\ &= -\frac{1}{2}[\theta,\theta] && \text{(by def.)} \end{aligned}$$

  and we are done ■

- For matrix groups, the Maurer-Cartan form has an explicit matrix representation
- **Theorem:** for $G = GL(n,\mathbb{R})$, the Maurer-Cartan form is

$$\theta_{ij} = (g^{-1}dg)_{ij} \tag{4.14}$$

where $g = (g_{ij})$ is the global chart such that for $A \in G$, $g_{ij}(A) = A_{ij}$, and θ_{ij} are the components of $\theta \in \Omega^1(G,\mathfrak{g})$ in the corresponding basis of $\mathfrak{g}$; specifically, we mean that $(\theta_a)_{ij} = (g(a)^{-1})_{ik}(dg_{kj})_a$

- **Proof:**
  - In the coordinate basis, the components of $\theta_a = \theta^{ij}_a(dg_{ij})_a$ are

$$\theta^{ij}_a = \theta_a\left(\frac{\partial}{\partial g_{ij}}\right)_a = (L_{a^{-1}})_*\left(\frac{\partial}{\partial g_{ij}}\right)_a$$

  by definition of θ_a
  - Moreover, define a basis for T_eG by $E^{ij} = (\partial/\partial g_{ij})_e$; then

$$(L_a)_*E^{pq} = (aE^{pq})_{ij}\left(\frac{\partial}{\partial g_{ij}}\right)_a = a_{ip}\left(\frac{\partial}{\partial g_{iq}}\right)_a$$

  - Multiplying both sides by $a^{-1}_{pk}(L_{a^{-1}})$ gives

$$(:_{a^{-1}})_*\left(\frac{\partial}{\partial g_{kq}}\right)_a = a^{-1}_{pk}E^{pq} = g^{-1}_{pk}(a)E^{pq}$$

  - Relabelling indices as needed, we therefore obtain

$$\theta^{ij}_a = g^{-1}_{pi}(a)E^{pj} \implies \theta_a = g^{-1}_{pi}(a)(dg_{ij})_aE^{pj}$$

  and so the matrix components relative to the basis E^{pj} are

$$(\theta_a)_{pj} = g^{-1}_{pi}(a)(dg_{ij})_a$$

  for any $a \in G$, and we are done ■

- This proof works for any Lie subgroup of $GL(n,\mathbb{R})$ with the subtlety that g_{ij} is no longer a chart

4.2 The Exponential Map and Adjoint Representation

- A curve $c : \mathbb{R} \to G$ where G is a Lie group is a *one-dimensional subgroup of G* if

$$c(t)c(s) = c(t+s),\ \forall s, t \in \mathbb{R} \tag{4.15}$$

- Alternatively, c is a group homomorphism from $(\mathbb{R}, +)$ to G

- Note the basic properties that $c(0) = e$ and $c(-t) = c(t)^{-1}$, and that the subgroup is Abelian in that $c(t)c(s) = c(s)c(t)$

- **Theorem:** the one-parameter subgroups $c_V(t)$ which have $c'(0) = V \in T_eG$ are in one-to-one correspondence with left-invariant vector fields $X_V \in L(G)$

- **Proof:**
  - A one-parameter subgroup $c(t)$ defines a vector field $X \in TG$ along $c(t)$ by $c'(t) = X(c(t))$
  - We now claim that X is left-invariant along $c(t)$
  - Consider the pushforward $c_* : T_t\mathbb{R} \to T_{c(t)}G$ defined by $c'(t) = c_* \left(\frac{d}{dt}\right)_t$
  - From tutorial sheet, we know that $\frac{d}{dt} \in L(\mathbb{R})$, and so $\left(\frac{d}{dt}\right)_t = L_{t*}\left(\frac{d}{dt}\right)_0$, which gives $c'(t) = (cL_t)_* \left(\frac{d}{dt}\right)_0$
  - Now, consider $(cL_t)_*$ acting on arbitrary $s \in \mathbb{R}$; we have

    $$(cL_t)(s) = c(t+s) = c(t)c(s) = (L_{c(t)}c)(s) \implies cL_t = L_{c(t)}c \implies (cL_t)_* = (L_{c(t)})_* c_*$$

  - Putting this all together, we have

    $$c'(t) = (L_{c(t)})_* c'(0)$$

    that is, $X_{c(t)} = (L_{c(t)})_* X_e$, which implies X is left-invariant along $c(t)$
  - For the converse, let $X \in L(G)$ be a left-invariant vector field
  - It has a unique integral curve $c : \mathbb{R} \to G$ through the identity, so $c(0) = e$ and $c'(t) = X(c(t))$
  - This curve in fact is a one-parameter subgroup
  - To show this, consider the integral curves $\gamma_1(t) = c(t+s)$ and $\gamma_2(t) = c(s)c(t) = L_{c(s)}c(t)$; we show these are equivalent
  - From the definition of the curves, we have immediately that $\gamma_1'(t) = X(\gamma_1(t))$
  - Moreover:

    $$\begin{aligned}\gamma_2'(t) = \gamma_{2*}\left(\frac{d}{dt}\right)_t = (L_{c(s)}c)_*\left(\frac{d}{dt}\right)_t = L_{c(s)*}c'(t) \\ = L_{c(s)*}X(c(t)) = X(c(s)c(t)) = X(\gamma_2(t))\end{aligned}$$

    and so uniqueness of integral curves, $\gamma_1 = \gamma_2$ and we are done ■

- Note the expression $c'(t) = (L_{c(t)})_* c'(0)$ - for matrix groups, this is just the equation $c'(t) = c(t)c'(0)$, and so $c(t) = \exp(tc'(0))$ where exp is the matrix exponential

- This motivates the *exponential map*, given by $\exp : T_eG \to G$ where $\exp(V) = c_V(1)$, where c_V is the one-parameter subgroup generated by $V \in T_eG$

- **Theorem:** the one-parameter subgroup generated by $V \in T_eG$ is $C_V(t) = \exp(tV)$

- **Proof:**
  - Let $s \in \mathbb{R}$, $s \neq 0$; the curve $\tilde{c}(t) = c_V(st)$ is a one-parameter subgroup
  - It has tangent vector $\tilde{c}'(0) = sc'_V(0) = sV$
  - Therefore $\tilde{c}(t)$ is the one-parameter subgroup generated by sV, and by uniqueness $\tilde{c}(t) = c_{sV}(t)$
  - Then by definition, $\exp(sV) = c_{sV}(1) = \tilde{c}(1) = c_V(s)$ ■
- Note the notational fudging here: when we write the tangent vector $\gamma'(0) \in T_pM$ to a curve γ on manifold M, we often use shorthand $\gamma'(0) = \frac{d}{dt}\gamma(t)\big|_{t=0}$ to mean $\gamma'(0)(f) = \frac{d}{dt}f(\gamma(t))\big|_{t=0}$ for any $f \in C^\infty(M)$
- This means that if $\phi : M \to N$ is a smooth map, we can write $\phi_*\gamma'(0) = \frac{d}{dt}\phi \circ \gamma(t)\big|_{t=0}$
- The following corollaries (left unproven as they're quite easy lol) show why the map exp is called the exponential map
- **Corollary:** for $g \in G$, $V \in T_eG$, and $t, s \in \mathbb{R}$, we have:
  1. $\exp(tV)\exp(sV) = \exp((t+s)V)$
  2. $\frac{d}{dt}\exp(tV)\big|_{t=0} = V$
  3. $(X_V)_g = \frac{d}{dt}g\exp(tV)\big|_{t=0}$
  4. For matrix groups, exp coincides with the matrix exponential $\exp(V) = \sum_{k=0}^{\infty}\frac{1}{k!}V^k$
- We now discuss adjoint-adjacent things
- Given $g \in G$, we define the smooth map $Ad_g : G \to G$ by $Ad_g h = ghg^{-1}$ for all $h \in G$
- This clearly obeys $Ad_{gh} = Ad_g \circ Ad_h$, and $Ad_g(e) = e$ for any $g \in G$
- Therefore, the derivative of Ad_g at the identity is $ad_g = (Ad_g)_* : T_eG \to T_eG$
- Now for any $V \in T_eG$, we have:

$$(ad_g \circ ad_h)(V) = (Ad_g)_*(Ad_h)_*V = (Ad_g \circ Ad_h)_*V = (Ad_{gh})_*V = ad_{gh}V \tag{4.16}$$

- This means that $ad_g \circ ad_h = ad_{gh}$ for all $g, h, \in G$
- Now, recall that a *representation* of a group G on a vector space V is a homomorphism $\rho : G \to GL(V)$ where $GL(V)$ is the group of all invertible linear maps $V \to V$; in particular, $\rho(gh) = \rho(g) \circ \rho(h)$ for all $h, g, \in G$
- We therefore have that ad_g defines a representation of G on its Lie algebra $\mathfrak{g} = T_eG$
- This motivates the definition of the *adjoint representation* of G on its Lie algebra $\mathfrak{g}$, given by the representation $ad : G \to GL(\mathfrak{g})$ defined by $ad(g) = ad_g$
- Explicitly, for any $V \in T_eG$, the derivative ad_g is given by

$$ad_g V = \frac{d}{dt}(Ad_g c_V(t))\bigg|_{t=0} = \frac{d}{dt}(g\exp(tV)g^{-1})\bigg|_{t=0} \tag{4.17}$$

  where $c_V(t) = \exp(tV)$ is the one-parameter subgroup generated by V
- For matrix groups, the adjoint representation is very useful as it reduces to matrix multiplication

- Explicitly, we mean that

$$Ad_g c_V(t) = g \exp(tV)g^{-1} = \exp(tgVg^{-1}) \tag{4.18}$$

which implies that

$$ad_g V = \frac{d}{dt}\exp(tgVg^{-1})\bigg|_{t=0} = gVg^{-1} \tag{4.19}$$

where the RHS is just ordinary matrix multiplication

- Now consider taking a further derivative of the adjoint representation at the identity: $ad_* : \mathfrak{g} \to End(\mathfrak{g})$ (where we identify the tangent space of $GL(\mathfrak{g})$ at the identity with the endomorphisms of V)

- **Lemma:** for any $X, Y \in \mathfrak{g}$, we have $ad_{*X}Y = [X,Y]$ and $[ad_{*X}, ad_{*Y}] = ad_{*[X,Y]}$ where $ad_{*X} = ad_*(X)$

- **Proof:**
  - For any $X, Y \in \mathfrak{g}$, the derivative is given by

  $$ad_{*X}(Y) = \frac{d}{dt}ad_{\exp(tX)}Y\bigg|_{t=0}$$

  - For a matrix group, this is easy to evaluate using the above corollary, whereupon we find

  $$ad_{*X}(Y) = \frac{d}{dt}\exp(tX)Y\exp(-tX)\bigg|_{t=0} = [X,Y]$$

  where the RHS is the matrix commutator
  - This actually holds for all groups (not proved)
  - Finally, the Jacobi identity implies

  $$[ad_{*X}, ad_{*Y}]Z = [X,[Y,Z]] - [Y,[X,Z]] = [[X,Y],Z] = ad_{*[X,Y]}Z$$

  and we are done ■

- Now, a *representation of a Lie algebra* $\mathfrak{g}$ on vector space V is a Lie algebra homomorphism $\phi : \mathfrak{g} \to End(V)$; that is, a linear map such that $\phi([X,Y]) = [\phi(X), \phi(Y)]$

- The above lemma shows that ad_* is a Lie algebra homomorphism on $V = \mathfrak{g}$ - a representation of $\mathfrak{g}$ on itself

- This defines the *adjoint representation of a Lie algebra* $\mathfrak{g}$ on itself, given by $ad_* : \mathfrak{g} \to End(\mathfrak{g})$