

# Topics in Mathematical Physics

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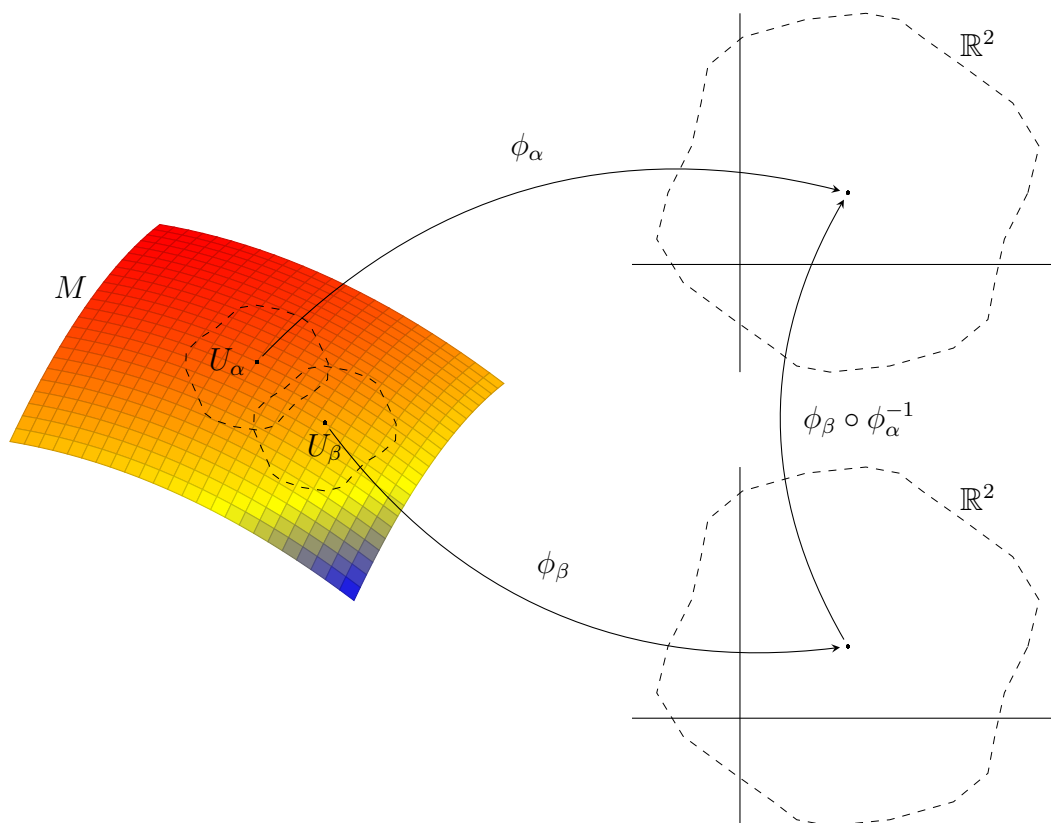
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## 1 Preface

- These notes were written up for the postgraduate masters course in mathematical physics at the University of Edinburgh
- The topic this year was 'Applications of Geometry in Physics'
- It was lectured by James Lucietti, who's notes are unfortunately only available on the Edinburgh intranet
- These notes were largely produced with those official notes as reference
- These notes were written primarily with myself in mind; it has some colloquialisms in it, and some topics are presented in the way I best understand them, and not necessarily the 'best' way
- The notes are also exclusively bullet pointed because I find prose difficult to digest
- Any errors are most certainly mine and not James's - let me know at [benkarsberg@gmail.com](mailto:benkarsberg@gmail.com) if you note any bad ones
- Generally, text in *italic* is the definition of something the first time it shows up, and text in **bold** is something I think is important/want to remember/found tricky when writing these up
- Summation convention is implicit throughout this entire set of notes, unless explicitly stated otherwise
- Euclidean spatial vectors are denoted by e.g.  $\mathbf{v}$ , and 4-vectors are denoted by just  $v = (v^0, \mathbf{v})$
- The metric signature is  $\eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$  unless stated otherwise
- $\hbar = c = 1$
- I hope these notes are helpful to someone who isn't me - enjoy!

## 2 Differential Geometry Preliminaries

- This course is all about geometry, almost entirely differential
- Differentiable geometry is a listed prerequisite for this course, but I haven't done any properly, with the exception of pseudo-Riemannian geometry in coordinate bases in a GR course last year
- Therefore, we need to set the scene
- The principal arena for differential geometry is a **differential manifold**:
  - Informally, this is a set  $M$  that locally looks like  $\mathbb{R}^n$
  - Formally, this is a topological space  $M$  equipped with a collection of charts  $\{(U_\alpha, \phi_\alpha)\}$  called an **atlas**, where  $\{U_\alpha\}$  is an open cover of  $M$  and  $\phi_\alpha : U_\alpha \rightarrow V_\alpha \subset \mathbb{R}^n$  are homeomorphisms
- The image we want in mind is something like this, generalised to  $\mathbb{R}^n$  rather than just  $\mathbb{R}^2$ :



- Essentially, a chart  $(U, \phi)$  assigns coordinates to a point  $p \in U$  given by  $\phi(p) = (x^1(p), \dots, x^n(p))$ , usually written  $\phi = (x^i)$
- Importantly, the **transition functions**  $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \rightarrow \phi_\alpha(U_\alpha \cap U_\beta)$  should be  $C^\infty(\mathbb{R}^n)$  bijections between open subsets of  $\mathbb{R}^n$  (see diagram)
- Essentially, all changes of coordinates are given by a smooth map
- The atlas allows us to define calculus on a manifold, and we can classify a hierarchy of objects on  $M$ :

- $C^\infty(M)$  is the set of all smooth functions on  $M$ , with the function of a standard commutative algebra
- The **tangent space**  $T_p M$  at  $p \in M$  is the  $n$ -dimensional vector space of directional derivatives at  $p$ ; that is, the space of all  $\mathbb{R}$ -linear maps  $X_p : C^\infty(M) \rightarrow \mathbb{R}$  obeying the Leibnitz/product rule

$$X_p(fg) = X_p(f)g(p) + f(p)X_p(g) \quad (2.1)$$

for all  $f, g \in C^\infty(M)$

- The elements  $X_p \in T_p M$  are called **tangent vectors**; if we consider a smooth curve  $\gamma : (a, a) \rightarrow M$  where  $\gamma(0) = p$ , we can define a tangent vector to the curve called the velocity  $\dot{\gamma} \in T_p M$  by  $\dot{\gamma}(f) = \frac{d}{dt}f(\gamma(t))|_{t=0}$ , and all tangent vectors arise like this
- A **vector field**  $X \in \mathfrak{X}(M)$  is a smooth assignment to each  $p \in M$  of a tangent vector  $X_p \in T_p M$  (or more formally, a section of the **tangent bundle**  $TM$ )
- A **differential  $k$ -form**  $\alpha_p$  at  $p \in M$  is a smooth alternating multilinear map  $\alpha_p : T_p M \times \dots \times T_p M \rightarrow \mathbb{R}$
- The set of all  $k$ -forms is denoted  $\Omega^k(M)$ , where  $\alpha(X_1, \dots, X_k) \in C^\infty(M)$  and  $X_i \in \mathfrak{X}(M)$
- In particular, 1-forms at  $p$  are just dual vectors  $\alpha_p \in T_p^* M$
- The **wedge produce** of  $\alpha \in \Omega^k(M)$  and  $\beta \in \Omega^l(M)$  is  $\alpha \wedge \beta \in \Omega^{k+l}(M)$  where  $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$ ; in particular, for 1-forms we have

$$(\alpha \wedge \beta)(X, Y) = \alpha(X)\beta(Y) - \beta(X)\alpha(Y) \quad (2.2)$$

- The **exterior derivative** is a  $\mathbb{R}$ -linear map  $d : \Omega^k(M) \rightarrow \Omega^{k+1}(M)$  satisfying  $d^2 = 0$  and  $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge (d\beta)$
- The **interior derivative**  $\iota_X : \Omega^k(M) \rightarrow \Omega^{k-1}(M)$  is defined by  $\iota_X \alpha = \alpha(X, \dots)$  for  $X \in \mathfrak{X}(M)$
- In particular, if we regard a smooth function  $f$  on  $M$  as a 0-form,  $(df)(X) = X(f)$
- **Tensors** and **tensor fields** of type  $(r, s)$  are very similar to vectors and vector fields, acting as multilinear maps  $T : T_p^* M \times \dots \times T_p^* M \times T_p M \times \dots \times T_p M \rightarrow \mathbb{R}$
- The derivative of a smooth map  $\phi : M \rightarrow N$  is called the **push-forward map**  $\phi_* : T_p M \rightarrow T_{\phi(p)} N$  given by

$$(\phi_* X_p)(f) = X_p(f \circ \phi) \quad (2.3)$$

where  $f \in C^\infty(N)$

- Similarly, the **pull-back** is  $\phi^* : T_{\phi(p)}^* N \rightarrow T_p^* M$  given by

$$(\phi^* \alpha_{\phi(p)})(X_p) = \alpha_{\phi(p)}(\phi_* X_p) \quad (2.4)$$

- A smooth bijection  $\phi : M \rightarrow N$  is called a **diffeomorphism**, and for a diffeomorphism we have  $(\phi^{-1})_* = \phi^*$
- The **flow** of a vector field  $X$  is a 1-parameter group of diffeomorphisms; i.e. a family  $\phi_t : M \rightarrow M$  such that  $\phi_{t+s} = \phi_t \circ \phi_s$  for all real  $t, s$
- In particular, the integral curve of  $X$  through  $p \in M$  is given by  $\phi_t(p)$  so  $X_p(f) = \frac{d}{dt}f(\phi_t(p))|_{t=0}$

- The **Lie derivative** of a tensor field  $T$  along a vector field  $X$  is

$$\mathcal{L}_X T = \left. \frac{d}{dt}(\phi_t^* T) \right|_{t=0} \quad (2.5)$$

where  $\phi_t$  is the flow of  $X \in \mathfrak{X}(M)$

- In particular, we have  $\mathcal{L}_X f = X(f)$  for  $f \in C^\infty(M)$ ;  $\mathcal{L}_X Y = [X, Y]$  for  $Y \in \mathfrak{X}(M)$ ;  $\mathcal{L}_X \alpha = (d\iota_X + \iota_X d)\alpha$  for  $\alpha \in \Omega^k(M)$
- Suppose we have a local basis  $\{e_a \mid a = 1, \dots, n\}$  of vectors; we can then expand a vector field as  $X = X^a e_a$  where  $X^a$  are the components
- The dual basis is  $\{f^a \mid a = 1, \dots, n\}$  defined by  $f^a(e_b) = \delta_b^a$ ; differential 1-forms can then be expanded as  $\alpha = \alpha_a f^a$  where  $\alpha_a = \alpha(e_a)$  are the components
- More generally, a  $k$ -form can be expanded as  $\alpha = \frac{1}{k!} \alpha_{a_1 \dots a_k} f^{a_1} \wedge \dots \wedge f^{a_k}$  where  $\alpha_{a_1 \dots a_k} = \alpha(e_{a_1}, \dots, e_{a_k})$
- This generalises to tensors in the expected way
- In particular, a chart  $(U, \phi)$  where  $\phi = x^i$  defines a basis of  $T_p M$  called the **coordinate basis**,  $(\frac{\partial}{\partial x^i})_p$
- In this basis, any vector field can be expanded as  $X = X^i \frac{\partial}{\partial x^i}$  where  $X^i = X(x^i)$  are smooth functions on  $U$
- Similarly, the dual basis of  $T_p^* M$  is  $(dx^i)_p$ , where 1-forms can be expanded as  $\alpha = \alpha_i dx^i$ , and this can be expanded to a basis for  $k$ -forms by  $dx^{i_1} \wedge \dots \wedge dx^{i_k}$  where  $i_1 < \dots < i_k$

### 3 Symplectic Geometry and Classical Mechanics

- To motivate what follows, consider a classical Hamiltonian system with phase space  $\mathbb{R}^{2n}$  and canonical coordinates  $(\mathbf{p}, \mathbf{q})$
- Hamilton's equations for the time-evolution of the system are

$$\dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}}, \quad \dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}} \quad (3.1)$$

where  $H$  is the Hamiltonian

- This can be rewritten in matrix form:

$$\begin{pmatrix} \dot{\mathbf{p}} \\ \dot{\mathbf{q}} \end{pmatrix} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial \mathbf{q}} \\ \frac{\partial H}{\partial \mathbf{p}} \end{pmatrix} \iff \dot{\mathbf{x}} = \Omega^{-1} \nabla_{\mathbf{x}} H \quad (3.2)$$

where

$$\Omega = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix} \quad (3.3)$$

is implicit

- As we shall see, this is the canonical example of a symplectic vector space

#### 3.1 Symplectic Vector Spaces

- A **symplectic vector space** is a pair  $(V, \omega)$ , where  $V$  is a  $2n$ -dimensional real vector space, and  $\omega: V \times V \rightarrow \mathbb{R}$  is a bilinear form satisfying:
  1. *Skew-symmetry*:  $\omega(v, w) = -\omega(w, v), \forall v, w \in V$
  2. *Non-degeneracy*:  $\omega(v, w) = 0 \forall v \in V \implies w = 0$
- $\omega$  is called a **symplectic structure** on  $V$
- Contrasting  $\omega$  with the standard Euclidean inner product, the only real difference is skew vs non-skew symmetry
- In physics, we usually need a basis to work in - defining the arbitrary basis  $\{e_a \mid a = 1, \dots, 2n\}$ , the components of  $\omega$  are given by the usual  $\omega_{ab} = \omega(e_a, e_b)$
- From this, it follows that  $\omega_{ab}$  is an antisymmetric matrix
- Moreover, non-degeneracy of  $\omega$  is equivalent to

$$\omega_{ab} w^b \implies w^a = 0 \quad (3.4)$$

which just means  $\omega_{ab}$  is invertible

- **Example:** the canonical example of a symplectic vector space is  $V = \mathbb{R}^{2n}$  and  $\omega(\mathbf{v}, \mathbf{w}) = \mathbf{v}^T \Omega \mathbf{w}$ , with  $\Omega$  as defined in the introduction:

$$\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \quad (3.5)$$

- We now show that, in fact, all symplectic vector spaces are isomorphic to this example

- **Theorem:** any symplectic vector space  $(V, \omega)$  has a **symplectic basis**  $\{e_i, f_i \mid i = 1, \dots, n\}$  defined by

$$\omega(e_i, e_j) = \omega(f_i, f_j) = 0 \quad \text{and} \quad \omega(e_i, f_j) = \delta_{ij} \quad (3.6)$$

- **Proof:** by induction

- For the  $n = 1$  case, pick an arbitrary vector  $0 \neq e_1 \in V$
- By non-degeneracy,  $\exists f_1 \in V$  such that  $\omega(e_1, f_1) \neq 0$ , and we can rescale so that  $\omega(e_1, f_1) = 1$
- Then, by skew-symmetry, we see  $\omega(e_1, e_1) = \omega(f_1, f_1) = 0$  as required
- Assume the hypothesis holds for  $n = k \geq 1$ , and consider the  $n = k + 1$  case
- By the same procedure that we used for the base case, we can construct  $e_1$  and  $f_1$
- Then, define the *skew-orthogonal complement*:

$$W = \{w \in V \mid \omega(w, e_1) = \omega(w, f_1) = 0\} \quad (3.7)$$

- $W$  has dimension  $2k$ , and we now wish to show that it is a symplectic vector space
- First note that any  $v \in V$  can be uniquely decomposed as  $v = u + w$ , where  $u \in \text{span}(e_1, f_1)$  and  $w \in W$
- Then consider  $\omega|_W$ , i.e. the symplectic structure restricted to  $W$ ; pick  $\tilde{w} \in W$ , then:

$$\begin{aligned} \omega(\tilde{w}, w) = 0, \forall w \in W &\implies \omega(\tilde{w}, v) = 0, \forall v \in V && \text{(by decomposition above)} \\ &\implies \tilde{w} = 0 && \text{(by non-degeneracy)} \end{aligned} \quad (3.8)$$

- $\omega|_W$  clearly inherits skew-symmetry from  $\omega$ , so it is a symplectic form, and so  $W$  is indeed a symplectic vector space
- By induction the conclusion follows ■
- Explicitly, this means in the symplectic basis  $\omega_{ab} = \Omega_{ab}$ , establishing the isomorphism
- The next result shows a radical departure from Euclidean geometry in these spaces
- **Theorem:** let  $(V, \omega)$  be a  $2n$ -dimensional symplectic vector space, and  $W \subset V$  a *null subspace* so  $\omega|_W = 0$ . Then  $\dim W \leq n$

- **Proof:**

- By the earlier theorem, we can assume  $V = \mathbb{R}^{2n}$  and  $\omega = \Omega$
- Then,  $W$  is a null space iff  $W \perp \Omega W$ , where this is with respect to the Euclidean norm  $\mathbf{w} \cdot \mathbf{v} = \mathbf{w}^T \mathbf{v}$
- Therefore, we have

$$\dim W + \dim \Omega W \leq 2n \quad (3.9)$$

- However,  $\Omega$  is invertible so  $\dim W = \dim \Omega W$ , so we are done ■
- We now want to rewrite the symplectic structure in a symplectic basis, but in a more useful way
- Define the dual symplectic basis  $\{e_*^i, f_*^i \mid i = 1, \dots, n\}$  of  $V^*$ , so specifically:

$$e_*^i(e_j) = f_*^i(f_j) = \delta_j^i, \quad e_*^i(f_j) = f_*^i(e_j) = 0 \quad (3.10)$$

- Then, the basis of 2-forms is given by the set

$$\{e_*^i \wedge e_*^j, e_*^i \wedge f_*^j, f_*^i \wedge f_*^j\} \quad (3.11)$$

as usual

- We therefore claim that

$$\omega = e_*^i \wedge f_*^i \quad (3.12)$$

with summation convention implied

- We can easily verify this by its action on basis elements:

$$\omega(e_a, e_b) = (e_*^i \wedge f_*^i)(e_a, e_b) = e_*^i(e_a)f_*^i(f_b) - e_*^i(f_b)f_*^i(e_a) = \delta_a^i \delta_b^i = \delta_{ab} \quad (3.13)$$

- More generally, consider the  $k$ -fold wedge product  $\omega^k$
- We have  $\omega^k = 0$  for  $k > n$  as  $\omega^n$  is the highest ranked form on  $V$ , which has dimension  $2n$
- **Lemma:** let  $(V, \omega)$  be a  $2n$ -dimensional symplectic vector space. Then, in a symplectic basis we have

$$\omega^n = n! e_*^1 \wedge f_*^1 \wedge \dots \wedge e_*^n \wedge f_*^n \quad (3.14)$$

- **Proof:** by induction (not given, but it's not hard - future Ben, just try and do it yourself for revision xx)

### 3.2 Symplectic Manifolds and the Cotangent Bundle

- A **symplectic manifold** is a pair  $(M, \omega)$  where  $M$  is a  $2n$ -dimensional smooth manifold, and  $\omega \in \Omega^2(M)$  is a 2-form satisfying:

1.  $d\omega = 0$  (closed)
2.  $\omega(X, Y) = 0 \ \forall X \in \mathfrak{X}(M) \implies Y = 0$  (non-degenerate)

- $\omega$  is called a **symplectic form**
- Note that if  $(M, \omega)$  is a symplectic manifold, then  $(T_x M, \omega_x)$  is a symplectic vector space at all points  $x \in M$
- **Example:** let  $M = \mathbb{R}^{2n}$  with coordinates  $(p_i, q_i)$ ,  $i = 1, \dots, n$ , and  $\omega = dp_i \wedge dq_i$  (summation implied)
  - $\omega = d(p_i dq_i)$ , so it is exact and hence closed
  - Consider the components of  $\omega$  in the coordinate basis  $(\partial/\partial p_i, \partial/\partial q_i)$ ; we find

$$\begin{aligned} \omega \left( \frac{\partial}{\partial p_j}, \frac{\partial}{\partial p_k} \right) &= \omega \left( \frac{\partial}{\partial q_j}, \frac{\partial}{\partial q_k} \right) = 0 \\ \omega \left( \frac{\partial}{\partial p_j}, \frac{\partial}{\partial q_k} \right) &= dp_i \left( \frac{\partial}{\partial p_j} \right) dq_i \left( \frac{\partial}{\partial q_k} \right) = \delta_{ij} \delta_{ik} = \delta_{jk} \end{aligned} \quad (3.15)$$

so in particular,  $\omega_{ij} = \Omega_{ij}$  in this basis and is non-degenerate

- In particular, this means  $\omega$  is a symplectic form and that  $(\partial/\partial p_i, \partial/\partial q_i)$  is a symplectic basis at every point
- **Theorem:** any symplectic manifold  $(M, \omega)$  of dimension  $2n$  is orientable with volume form  $\omega^n$

- **Proof:**

- A manifold is **orientable** if there is a consistent definition of clockwise and anti-clockwise (e.g. a Möbius strip would be non-orientable), and a **volume form** is a highest-rank differentiable form on the manifold (of rank equal to the dimension of the manifold)
- One characterisation of orientability is that a manifold is orientable if and only if there exists a nowhere vanishing top-ranked differential form/volume form
- Therefore, if we can show  $\omega^n = \omega \wedge \dots \wedge \omega$  is nowhere vanishing, we are done
- At each  $x \in M$ ,  $(T_x M, \omega_x)$  is a symplectic vector space
- But we know that in symplectic vector spaces

$$\omega_x^n = n! e_*^1 \wedge f_*^1 \wedge \dots \wedge e_*^n \wedge f_*^n \quad (3.16)$$

which is nowhere vanishing ■

- **Example:** consider the manifold given by the unit 2-sphere  $M = S^2 = \{\mathbf{x} \in \mathbb{R}^3 \mid |\mathbf{x}| = 1\}$

- The tangent space at  $\mathbf{x}$  is  $T_{\mathbf{x}} S^2 = \text{span}(\mathbf{x})^\perp$
- The volume form at this point is given by

$$\omega_{\mathbf{x}}(\mathbf{V}, \mathbf{w}) = \mathbf{x} \cdot (\mathbf{v} \times \mathbf{w}) \quad (3.17)$$

for  $\mathbf{v}, \mathbf{w} \in T_{\mathbf{x}} S^2$

### 3.2.1 The Cotangent Bundle

- The following discussion is why symplectic manifolds arise in classical mechanics
- Suppose  $Q$  is a  $2n$ -dimensional smooth manifold; its **cotangent bundle** is defined by

$$T^*Q = \{(p_x, x) \mid p_x \in T_x^*Q, x \in Q\} \quad (3.18)$$

which we think of as the set of all points  $x$  paired with all their covectors (one-forms)

- The cotangent bundle is a  $2n$ -dimensional manifold
- To show this, consider the chart on  $Q$  given by  $\phi(x) = (q_i(x))$ , containing  $x \in Q$
- We can see that  $\Phi(p_x, x) = (p_i(x), q_i(x))$  is then a chart on  $T^*Q$ , where  $p_i$  are the components of  $p_x = p_i(x)(dq_i)_x$
- To check the transition functions are smooth, consider the projection  $\pi: T^*Q \rightarrow Q$ ,  $(p_x, x) \rightarrow x$ ; this is smooth and surjective, and the inverse projection has image  $\pi^{-1}(x) = T_x^*Q$
- Therefore,  $\phi \circ \pi \circ \Phi^{-1}(p_i(x), q_i(x)) = (q_i(x))$  is smooth

- **Theorem:**  $T^*Q$  has a symplectic form  $\omega$ , given in the above chart by

$$\omega = dp_i \wedge dq_i \quad (3.19)$$

- **Proof:**

- Recall that the derivative of a smooth map  $\phi: M \rightarrow N$  is the push-forward map  $\phi_*: T_p M \rightarrow T_{\phi(p)} N$ , given by

$$(\phi_* X_p)(f) = X_p(\phi^* f) \quad (3.20)$$



- Moreover, the pull-back of  $\phi$  is  $\phi^*: T_{\phi(p)}^*N \rightarrow T_p^*M$ , given by

$$(\phi^* \alpha_{\phi(p)})(X_p) = \alpha_{\phi(p)}(\phi_* X_p) \quad (3.21)$$

- The projector above is a smooth map of this form, so its derivative is the push-forward  $\pi_*: T_{(p_x, x)}(T^*Q) \rightarrow T_x Q$
- We now define  $\theta_{(p_x, x)} \in T_{(p_x, x)}^*(T^*Q)$ , given by

$$\theta_{(p_x, x)}(\xi) = p_x(\pi_* \xi), \quad \forall \xi \in T_{(p_x, x)}(T^*Q) \quad (3.22)$$

so  $\theta_{(p_x, x)} = \pi^* p_x$  is the pull-back of  $p_x$  under  $\pi$ , meaning  $\theta \in \Omega^1(T^*Q)$  is a one-form on  $T^*Q$

- Define  $\omega = d\theta \in \Omega^2(T^*Q)$ , which is clearly closed since it is exact
- To show non-degeneracy, we check the components of  $\theta$  in the chart  $\Phi = (p_i, q_i)$  above of  $T^*Q$ , defined by chart  $\phi = (q_i)$  of  $Q$
- We find first that

$$\theta \left( \frac{\partial}{\partial q_i} \right) = p \left( \pi_* \frac{\partial}{\partial q_i} \right) \quad (3.23)$$

and note in particular that

$$\begin{aligned} \left( \pi_* \frac{\partial}{\partial q_i} \right) (f) &= \frac{\partial}{\partial q_i} (f \circ \pi) \\ &= \frac{d}{dt} f \circ \pi \circ \Phi^{-1}(p_1, \dots, p_n, q_1, \dots, q_i + t, \dots, q_n)|_{t=0} \\ &= \frac{d}{dt} f \circ \phi^{-1}(q_1, \dots, q_i + t, \dots, q_n)|_{t=0} \\ &= \frac{\partial}{\partial q_i} (f) \end{aligned} \quad (3.24)$$

for all  $f \in C^\infty(Q)$

- Therefore,  $\theta(\partial/\partial q_i) = p(\partial/\partial q_i) = p_i$
- A similar calculation finds  $\pi_* \partial/\partial p_i = 0$ , so  $\theta(\partial/\partial p_i) = 0$
- Therefore,  $\theta = p_i dq_i$  in this chart ■
- This has a very clear link to classical mechanics:  $Q$  is the configuration space, and  $T^*Q$  is the phase space
- A symplectic manifold also induces a rather natural isomorphism between the tangent and cotangent space
- Specifically, this isomorphism is between the tangent space at each point and its dual
- **Theorem:** let  $(V, \omega)$  be a symplectic vector space; then there is a ‘natural’ isomorphism called the musical isomorphism  $V \equiv V^*$
- **Proof:**

- Let  $\xi \in V$ ; we define the linear map  $\flat: V \rightarrow V^*$  by

$$\xi^\flat(\eta) = \omega(\eta, \xi), \quad \forall \eta \in V \quad (3.25)$$

- $\flat$  is injective since  $\omega$  is non-degenerate

- Instead, let  $\lambda \in V^*$ ; we define the linear map  $\sharp: V^* \rightarrow V$  by

$$\omega(\eta, \lambda^\sharp) = \lambda(\eta), \quad \forall \eta \in V \quad (3.26)$$

- This defines  $\lambda^\sharp$  uniquely by non-degeneracy of  $\omega$ , and it is again injective
- These two maps are mutual inverses:

$$\begin{aligned} \omega(\eta, (\xi^\flat)^\sharp) &= \xi^\flat(\eta) = \omega(\eta, \xi), \quad \forall \eta \in V \implies (\xi^\flat)^\sharp = \xi, \quad \forall \xi \in V \\ (\lambda^\sharp)^\flat(\eta) &= \omega(\eta, \lambda^\sharp) = \lambda(\eta), \quad \forall \eta \in V \implies (\lambda^\sharp)^\flat = \lambda, \quad \forall \lambda \in V^* \quad \blacksquare \end{aligned} \quad (3.27)$$

- Moreover, the musical isomorphism defines a (2,0) tensor over  $V$ :

$$\omega^\sharp(\eta, \lambda) = -\omega(\eta^\sharp, \lambda^\sharp) \quad (3.28)$$

for all  $\eta, \lambda \in V^*$

- This is called the **inverse symplectic form**
- Choosing a basis  $\{e_a\}$  of  $V$ , we find:

$$(\xi^\flat)_a = \omega_{ab}\xi^b, \quad \text{and} \quad \omega_{ab}(\lambda^\sharp)^b = \lambda_a \implies (\lambda^\sharp)^a = \omega^{ab}\lambda_b \quad (3.29)$$

- This is, in a sense, raising and lowering indices using the symplectic form
- This makes sense as to why  $\omega^\sharp$  is the inverse symplectic form:

$$(\omega^\sharp)^{ab}\eta_a\lambda_b = -\omega_{ab}(\eta^\sharp)^a(\lambda^\sharp)^b = -\omega_{ab}\omega^{ac}\omega^{bd}\eta_c\lambda_d = \delta_b^c\omega^{bd}\eta_c\lambda_d = \omega^{cd}\eta_c\lambda_d \quad (3.30)$$

- The analogous construction in pseudo-Riemannian geometry (c.f. GR) is defined by the metric tensor: an inner product on  $V$
- This means that for any symplectic manifold  $(M, \omega)$ , there is a musical isomorphism  $T_x M \equiv T_x^* M$  for every  $x \in M$ , which induces a musical isomorphism between vector fields and 1-forms on  $M$