Topics in Mathematical Physics

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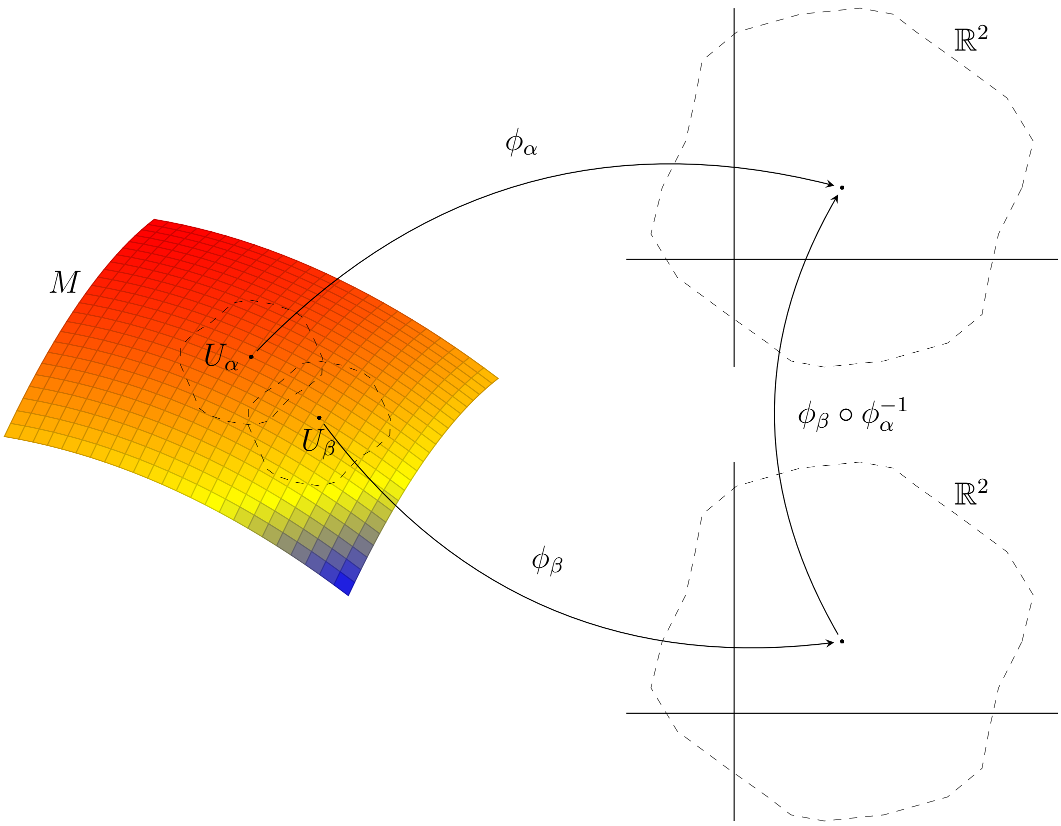
1 Preface

- These notes were written up for the postgraduate masters course in mathematical physics at the University of Edinburgh
- The topic this year was 'Applications of Geometry in Physics'
- It was lectured by James Lucietti, who's notes are unfortunately only available on the Edinburgh intranet
- These notes were largely produced with those official notes as reference
- These notes were written primarily with myself in mind; it has some colloquialisms in it, and some topics are presented in the way I best understand them, and not necessarily the 'best' way
- The notes are also exclusively bullet pointed because I find prose difficult to digest
- Any errors are most certainly mine and not James's - let me know at benkarsberg@gmail.com if you note any bad ones
- Generally, text in *italic* is the definition of something the first time it shows up, and text in **bold** is something I think is important/want to remember/found tricky when writing these up
- Summation convention is implicit throughout this entire set of notes, unless explicitly stated otherwise
- Euclidean spatial vectors are denoted by e.g. $\mathbf{v}$, and 4-vectors are denoted by just $v = (v^0, \mathbf{v})$
- The metric signature is $\eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$ unless stated otherwise
- $\hbar = c = 1$
- I hope these notes are helpful to someone who isn't me - enjoy!

2 Differential Geometry Preliminaries

- This course is all about geometry, almost entirely differential
- Differentiable geometry is a listed prerequisite for this course, but I haven't done any properly, with the exception of pseudo-Riemannian geometry in coordinate bases in a GR course last year
- Therefore, we need to set the scene
- The principal arena for differential geometry is a **differential manifold**:
  - Informally, this is a set M that locally looks like $\mathbb{R}^n$
  - Formally, this is a topological space M equipped with a collection of charts $\{(U_\alpha, \phi_\alpha)\}$ called an **atlas**, where $\{U_\alpha\}$ is an open cover of M and $\phi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^n$ are homeomorphisms
- The image we want in mind is something like this, generalised to $\mathbb{R}^n$ rather than just $\mathbb{R}^2$:



- Essentially, a chart (U, ϕ) assigns coordinates to a point $p \in U$ given by $\phi(p) = (x^1(p), \ldots, x^n(p))$, usually written $\phi = (x^i)$
- Importantly, the **transition functions** $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$ should be $C^\infty(\mathbb{R}^n)$ bijections between open subsets of $\mathbb{R}^n$ (see diagram)
- Essentially, all changes of coordinates are given by a smooth map
- The atlas allows us to define calculus on a manifold, and we can classify a hierarchy of objects on M:

- $C^\infty(M)$ is the set of all smooth functions on M, with the function of a standard commutative algebra
- The **tangent space** T_pM at $p \in M$ is the n-dimensional vector space of directional derivatives at p; that is, the space of all $\mathbb{R}$-linear maps $X_p : C^\infty(M) \to \mathbb{R}$ obeying the Leibnitz/product rule

$$X_p(fg) = X_p(f)g(p) + f(p)X_p(g) \tag{2.1}$$

  for all $f, g \in C^\infty(M)$
- The elements $X_p \in T_pM$ are called **tangent vectors**; if we consider a smooth curve $\gamma : (a, a) \to M$ where $\gamma(0) = p$, we can define a tangent vector to the curve called the velocity $\dot{\gamma} \in T_pM$ by $\dot{\gamma}(f) = \frac{d}{dt} f(\gamma(t))\big|_{t=0}$, and all tangent vectors arise like this
- A **vector field** $X \in \mathfrak{X}(M)$ is a smooth assignment to each $p \in M$ of a tangent vector $X_p \in T_pM$ (or more formally, a section of the **tangent bundle** TM)
- A **differential** k**-form** α_p at $p \in M$ is a smooth alternating multilinear map $\alpha_p : T_pM \times \ldots \times T_pM \to \mathbb{R}$
- The set of all k-forms is denoted $\Omega^k(M)$, where $\alpha(X_1, \ldots, X_k) \in C^\infty(M)$ and $X_i \in \mathfrak{X}(M)$
- In particular, 1-forms at p are just dual vectors $\alpha_p \in T_p^*M$
- The **wedge produce** of $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^l(M)$ is $\alpha \wedge \beta \in \Omega^{k+l}(M)$ where $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$; in particular, for 1-forms we have

$$(\alpha \wedge \beta)(X, Y) = \alpha(X)\beta(Y) - \beta(X)\alpha(Y) \tag{2.2}$$

- The **exterior derivative** is a $\mathbb{R}$-linear map $d : \Omega^k(M) \to \Omega^{k+1}(M)$ satisfying $d^2 = 0$ and $d(\alpha \wedge \beta = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge (d\beta)$
- The **interior derivative** $\iota_X : \Omega^k(M) \to \Omega^{k-1}(M)$ is defined by $\iota_X \alpha = \alpha(X, \ldots)$ for $X \in \mathfrak{X}(M)$
- In particular, if we regard a smooth function f on M as a 0-form, $(df)(X) = X(f)$
- **Tensors** and **tensor fields** of type (r, s) are very similar to vectors and vector fields, acting as multilinear maps $T : T_p^*M \times \ldots T_p^*M \times T_pM \times \ldots T_pM \to \mathbb{R}$
- The derivative of a smooth map $\phi : M \to N$ is called the **push-forward map** $\phi_* : T_pM \to T_{\phi(p)}N$ given by

$$(\phi_* X_p)(f) = X_p(f \circ \phi) \tag{2.3}$$

  where $f \in C^\infty(N)$
- Similarly, the **pull-back** is $\phi^* : T^*_{\phi(p)}N \to T_p^*M$ given by

$$(\phi^* \alpha_{\phi(p)})(X_p) = \alpha_{\phi(p)}(\phi_* X_p) \tag{2.4}$$

- A smooth bijection $\phi : M \to N$ is called a **diffeomorphism**, and for a diffeomorphism we have $(\phi^{-1})_* = \phi^*$
- The **flow** of a vector field X is a 1-parameter group of diffeomorphisms; i.e. a family $\phi_t : M \to M$ such that $\phi_{t+s} = \phi_t \circ \phi_s$ for all real t, s
- In particular, the integral curve of X through $p \in M$ is given by $\phi_t(p)$ so $X_p(f) = \frac{d}{dt} f(\phi_t(p))|_{t=0}$

- The **Lie derivative** of a tensor field T along a vector field X is

$$\mathcal{L}_X T = \frac{d}{dt}(\phi_t^* T)\bigg|_{t=0} \tag{2.5}$$

  where ϕ_t is the flow of $X \in \mathfrak{X}(M)$

- In particular, we have $\mathcal{L}_X f = X(f)$ for $f \in C^\infty(M)$; $\mathcal{L}_X Y = [X, Y]$ for $Y \in \mathfrak{X}(M)$; $\mathcal{L}_X \alpha = (d\iota_X + \iota_X d)\alpha$ for $\alpha \in \Omega^k(M)$

- Suppose we have a local basis $\{e_a \mid a = 1, \ldots, n\}$ of vectors; we can then expand a vector field as $X = X^a e_a$ where X^a are the components

- The dual basis is $\{f^a \mid a = 1, \ldots, n\}$ defined by $f^a(e_b) = \delta^a_b$; differential 1-forms can then be expanded as $\alpha = \alpha_a f^a$ where $\alpha_a = \alpha(e_a)$ are the components

- More generally, a k-form can be expanded as $\alpha = \frac{1}{k!}\alpha_{a_1 \ldots a_k} f^{a_1} \wedge \ldots \wedge f^{a_k}$ where $\alpha_{a_1 \ldots a_k} = \alpha(e_{a_1}, \ldots, e_{a_k})$

- This generalises to tensors in the expected way

- In particular, a chart (U, ϕ) where $\phi = x^i$ defines a basis of T_pM called the **coordinate basis**, $\left(\frac{\partial}{\partial x^i}\right)_p$

- In this basis, any vector field can be expanded as $X = X^i \frac{\partial}{\partial x^i}$ where $X^i = X(x^i)$ are smooth functions on U

- Similarly, the dual basis of T_p^*M is $(dx^i)_p$, where 1-forms can be expanded as $\alpha = \alpha_i dx^i$, and this can be expanded to a basis for k-forms by $dx^{i_1} \wedge \ldots \wedge dx^{i_k}$ where $i_1 < \ldots < i_k$

3 Symplectic Geometry and Classical Mechanics

- To motivate what follows, consider a classical Hamiltonian system with phase space $\mathbb{R}^{2n}$ and canonical coordinates $(\mathbf{p}, \mathbf{q})$
- Hamilton's equations for the time-evolution of the system are

$$\dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}}, \quad \dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}} \tag{3.1}$$

where H is the Hamiltonian

- This can be rewritten in matrix form:

$$\begin{pmatrix} \dot{\mathbf{p}} \\ \dot{\mathbf{q}} \end{pmatrix} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \begin{pmatrix} \dfrac{\partial H}{\partial \mathbf{q}} \\ \dfrac{\partial H}{\partial \mathbf{p}} \end{pmatrix} \iff \dot{\mathbf{x}} = \Omega^{-1} \nabla_{\mathbf{x}} H \tag{3.2}$$

where

$$\Omega = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix} \tag{3.3}$$

is implicit

- As we shall see, this is the canonical example of a symplectic vector space

3.1 Symplectic Vector Spaces

- A **symplectic vector space** is a pair (V, ω), where V is a $2n$-dimensional real vector space, and $\omega : V \times V \to \mathbb{R}$ is a bilinear form satisfying:

  1. *Skew-symmetry*: $\omega(v, w) = -\omega(w, v)$, $\forall v, w \in V$
  2. *Non-degeneracy*: $\omega(v, w) = 0 \, \forall v \in V \implies w = 0$

- ω is called a **symplectic structure** on V
- Contrasting ω with the standard Euclidean inner product, the only real difference is skew vs non-skew symmetry
- In physics, we usually need a basis to work in - defining the arbitrary basis $\{e_a \,|\, a = 1, \ldots, 2n\}$, the components of ω are given by the usual $\omega_{ab} = \omega(e_a, e_b)$
- From this, it follows that ω_{ab} is an antisymmetric matrix
- Moreover, non-degeneracy of ω is equivalent to

$$\omega_{ab} w^b \implies w^a = 0 \tag{3.4}$$

which just means ω_{ab} is invertible

- **Example:** the canonical example of a symplectic vector space is $V = \mathbb{R}^{2n}$ and $\omega(\mathbf{v}, \mathbf{w}) = \mathbf{v}^T \Omega \mathbf{w}$, with Ω as defined in the introduction:

$$\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \tag{3.5}$$

- We now show that, in fact, all symplectic vector spaces are isomorphic to this example

- **Theorem:** any symplectic vector space (V, ω) has a **symplectic basis** $\{e_i, f_i \,|\, i = 1, \ldots, n\}$ defined by

$$\omega(e_i, e_j) = \omega(f_i, f_j) = 0 \quad \text{and} \quad \omega(e_i, f_j) = \delta_{ij} \tag{3.6}$$

- **Proof:** by induction
  - For the $n = 1$ case, pick an arbitrary vector $0 \neq e_1 \in V$
  - By non-degeneracy, $\exists f_1 \in V$ such that $\omega(e_1, f_1) \neq 0$, and we can rescale so that $\omega(e_1, f_1) = 1$
  - Then, by skew-symmetry, we see $\omega(e_1, e_1) = \omega(f_1, f_1) = 0$ as required
  - Assume the hypothesis holds for $n = k \geq 1$, and consider the $n = k + 1$ case
  - By the same procedure that we used for the base case, we can construct e_1 and f_1
  - Then, define the *skew-orthogonal complement*:

$$W = \{w \in V \,|\, \omega(w, e_1) = \omega(w, f_1) = 0\} \tag{3.7}$$

  - W has dimension $2k$, and we now wish to show that it is a symplectic vector space
  - First note that any $v \in V$ can be uniquely decomposed as $v = u + w$, where $u \in \text{span}(e_1, f_1)$ and $w \in W$
  - Then consider $\omega|_W$, i.e. the symplectic structure restricted to W; pick $\tilde{w} \in W$, then:

$$\begin{aligned} \omega(\tilde{w}, w) = 0, \ \forall w \in W &\implies \omega(\tilde{w}, v) = 0, \ \forall v \in V && \text{(by decomposition above)} \\ &\implies \tilde{w} = 0 && \text{(by non-degeneracy)} \end{aligned} \tag{3.8}$$

  - $\omega|_W$ clearly inherits skew-symmetry from ω, so it is a symplectic form, and so W is indeed a symplectic vector space
  - By induction the conclusion follows ■

- Explicitly, this means in the symplectic basis $\omega_{ab} = \Omega_{ab}$, establishing the isomorphism

- The next result shows a radical departure from Euclidean geometry in these spaces

- **Theorem:** let (V, ω) be a $2n$-dimensional symplectic vector space, and $W \subset V$ a *null subspace* so $\omega|_W = 0$. Then $\dim W \leq n$

- **Proof:**
  - By the earlier theorem, we can assume $V = \mathbb{R}^{2n}$ and $\omega = \Omega$
  - Then, W is a null space iff $W \perp \Omega W$, where this is with respect to the Euclidean norm $\mathbf{w} \cdot \mathbf{v} = \mathbf{w}^T \mathbf{v}$
  - Therefore, we have

$$\dim W + \dim \Omega W \leq 2n \tag{3.9}$$

  - However, Ω is invertible so $\dim W = \dim \Omega W$, so we are done ■

- We now want to rewrite the symplectic structure in a symplectic basis, but in a more useful way

- Define the dual symplectic basis $\{e^i_*, f^i_* \,|\, i = 1, \ldots, n\}$ of V^*, so specifically:

$$e^i_*(e_j) = f^i_*(f_j) = \delta^i_j, \quad e^i_*(f_j) = f^i_*(e_j) = 0 \tag{3.10}$$

- Then, the basis of 2-forms is given by the set

$$\{e_*^i \wedge e_*^j,\ e_*^i \wedge f_*^j,\ f_*^i \wedge f_*^j\} \tag{3.11}$$

as usual

- We therefore claim that

$$\omega = e_*^i \wedge f_*^i \tag{3.12}$$

with summation convention implied

- We can easily verify this by its action on basis elements:

$$\omega(e_a, e_b) = (e_*^i \wedge f_*^i)(e_a, e_b) = e_*^i(e_a) f_*^i(f_b) - e_*^i(f_b) f_*^i(e_a) = \delta_a^i \delta_b^i = \delta_{ab} \tag{3.13}$$

- More generally, consider the k-fold wedge product ω^k
- We have $\omega^k = 0$ for $k > n$ as ω^n is the highest ranked form on V, which has dimension $2n$
- **Lemma:** let (V, ω) be a $2n$-dimensional symplectic vector space. Then, in a symplectic basis we have

$$\omega^n = n! e_*^1 \wedge f_*^1 \wedge \ldots \wedge e_*^n \wedge f_*^n \tag{3.14}$$

- **Proof:** by induction (not given, but it's not hard - future Ben, just try and do it yourself for revision xx)

3.2 Symplectic Manifolds and the Cotangent Bundle

- A **symplectic manifold** is a pair (M, ω) where M is a 2-dimensional smooth manifold, and $\omega \in \Omega^2(M)$ is a 2-form satisfying:
  1. $d\omega = 0$ (closed)
  2. $\omega(X, Y) = 0\ \forall X \in \mathfrak{X}(M) \implies Y = 0$ (non-degenerate)
- ω is called a **symplectic form**
- Note that if (M, ω) is a symplectic manifold, then $(T_x M, \omega_x)$ is a symplectic vector space at all points $x \in M$
- **Example:** let $M = \mathbb{R}^{2n}$ with coordinates (p_i, q_i), $i = 1, \ldots, n$, and $\omega = dp_i \wedge dq_i$ (summation implied)
  - $\omega = d(p_i dq_i)$, so it is exact and hence closed
  - Consider the components of ω in the coordinate basis $(\partial/\partial p_i, \partial/\partial q_i)$; we find

$$\begin{aligned} \omega\left(\frac{\partial}{\partial p_j}, \frac{\partial}{\partial p_k}\right) &= \omega\left(\frac{\partial}{\partial q_j}, \frac{\partial}{\partial q_k}\right) = 0 \\ \omega\left(\frac{\partial}{\partial p_j}, \frac{\partial}{\partial q_k}\right) &= dp_i\left(\frac{\partial}{\partial p_j}\right) dq_i\left(\frac{\partial}{\partial q_k}\right) = \delta_{ij}\delta_{ik} = \delta_{jk} \end{aligned} \tag{3.15}$$

    so in particular, $\omega_{ij} = \Omega_{ij}$ in this basis and is non-degenerate
  - In particular, this means ω is a symplectic form and that $(\partial/\partial p_i, \partial/\partial q_i)$ is a symplectic basis at every point
- **Theorem:** any symplectic manifold (M, ω) of dimension $2n$ is orientable with volume form ω^n

- **Proof:**
  - A manifold is **orientable** if there is a consistent definition of clockwise and anti-clockwise (e.g. a Mobius strip would be non-orientable), and a **volume form** is a highest-rank differentiable form on the manifold (of rank equal to the dimension of the manifold)
  - One characterisation of orientability is that a manifold is orientable if and only if there exists a nowhere vanishing top-ranked differential form/volume form
  - Therefore, if we can show $\omega^n = \omega \wedge \ldots \wedge \omega$ is nowhere vanishing, we are done
  - At each $x \in M$, $(T_x M, \omega_x)$ is a symplectic vector space
  - But we know that in symplectic vector spaces

    $$\omega_x^n = n! e_*^1 \wedge f_*^1 \wedge \ldots \wedge e_*^n \wedge f_*^n \tag{3.16}$$

    which is nowhere vanishing ■

- **Example:** consider the manifold given by the unit 2-sphere $M = S^2 = \{\mathbf{x} \in \mathbb{R}^3 \,|\, |\mathbf{x}| = 1\}$
  - The tangent space at $\mathbf{x}$ is $T_{\mathbf{x}} S^2 = span(\mathbf{x})^{\perp}$
  - The volume form at this point is given by

    $$\omega_{\mathbf{x}}(\mathbf{V}, \mathbf{w}) = \mathbf{x} \cdot (\mathbf{v} \times \mathbf{w}) \tag{3.17}$$

    for $\mathbf{v}, \mathbf{w} \in T_{\mathbf{x}} S^2$

3.2.1 The Cotangent Bundle

- The following discussion is why symplectic manifolds arise in classical mechanics
- Suppose Q is a $2n$-dimensional smooth manifold; its **cotangent bundle** is defined by

  $$T^*Q = \{(p_x, x) \,|\, p_x \in T_x^* Q,\ x \in Q\} \tag{3.18}$$

  which we think of as the set of all points x paired with all their covectors (one-forms)
  - The cotangent bundle is a $2n$-dimensional manifold
  - To show this, consider the chart on Q given by $\phi(x) = (q_i(x))$, containing $x \in Q$
  - We can see that $\Phi(p_x, x) = (p_i(x), q_i(x))$ is then a chart on T^*Q, where p_i are the components of $p_x = p_i(x)(dq_i)_x$
  - To check the transition functions are smooth, consider the projection $\pi : T^*Q \to Q,\ (p_x, x) \to x$; this is smooth and surjective, and the inverse projection has image $\pi^{-1}(x) = T_x^* Q$
  - Therefore, $\phi \circ \pi \circ \Phi^{-1}(p_i(x), q_i(x)) = (q_i(x))$ is smooth
- **Theorem:** T^*Q has a symplectic form ω, given in the above chart by

  $$\omega = dp_i \wedge dq_i \tag{3.19}$$

- **Proof:**
  - Recall that the derivative of a smooth map $\phi : M \to N$ is the push-forward map $\phi_* : T_p M \to T_{\phi(p)} N$, given by

    $$(\phi_* X_p)(f) = X_p(f\phi) \tag{3.20}$$

- Moreover, the pull-back of ϕ is $\phi^* : T^*_{\phi(p)}N \to T^*_p M$, given by

$$(\phi^* \alpha_{\phi(p)})(X_p) = \alpha_{\phi(p)}(\phi_* X_p) \tag{3.21}$$

- The projector above is a smooth map of this form, so its derivative is the push-forward $\pi_* : T_{(p_x,x)}(T^*Q) \to T_x Q$
- We now define $\theta_{(p_x,x)} \in T^*_{(p_x,x)}(T^*Q)$, given by

$$\theta_{(p_x,x)}(\xi) = p_x(\pi_* \xi), \quad \forall \xi \in T_{(p_x,x)}(T^*Q) \tag{3.22}$$

  so $\theta_{(p_x,x)} = \pi^* p_x$ is the pull-back of p_x under π, meaning $\theta \in \Omega^1(T^*Q)$ is a one-form on T^*Q
- Define $\omega = d\theta \in \Omega^2(T^*Q)$, which is clearly closed since it is exact
- To show non-degeneracy, we check the components of θ in the chart $\Phi = (p_i, q_i)$ above of T^*Q, defined by chart $\phi = (q_i)$ of Q
- We find first that

$$\theta\left(\frac{\partial}{\partial q_i}\right) = p\left(\pi_* \frac{\partial}{\partial q_i}\right) \tag{3.23}$$

  and note in particular that

$$\begin{aligned}
\left(\pi_* \frac{\partial}{\partial q_i}\right)(f) &= \frac{\partial}{\partial q_i}(f \circ \pi) \\
&= \frac{d}{dt} f \circ \pi \circ \Phi^{-1}(p_1, \ldots, p_n, q_1, \ldots, q_i + t, \ldots, q_n)|_{t=0} \\
&= \frac{d}{dt} f \circ \phi^{-1}(q_1, \ldots, q_i + t, \ldots, q_n)|_{t=0} \\
&= \frac{\partial}{\partial q_i}(f)
\end{aligned} \tag{3.24}$$

  for all $f \in C^\infty(Q)$
- Therefore, $\theta(\partial/\partial q_i) = p(\partial/\partial q_i) = p_i$
- A similar calculation finds $\pi_* \partial/\partial p_i = 0$, so $\theta(\partial/\partial p_i) = 0$
- Therefore, $\theta = p_i dq_i$ in this chart ■

- This has a very clear link to classical mechanics: Q is the configuration space, and T^*Q is the phase space
- A symplectic manifold also induces a rather natural isomorphism between the tangent and cotangent space
- Specifically, this isomorphism is between the tangent space at each point and its dual
- **Theorem:** let (V, ω) be a symplectic vector space; then there is a 'natural' isomorphism called the musical isomorphism $V \equiv V^*$
- **Proof:**
  - Let $\xi \in V$; we define the linear map $\flat : V \to V^*$ by

$$\xi^\flat(\eta) = \omega(\eta, \xi), \quad \forall \eta \in V \tag{3.25}$$

  - $\flat$ is injective since ω is non-degenerate

– Instead, let $\lambda \in V^*$; we define the linear map $\sharp : V^* \to V$ by

$$\omega(\eta, \lambda^\sharp) = \lambda(\eta), \quad \forall \eta \in V \tag{3.26}$$

– This defines $\lambda^\sharp$ uniquely by non-degeneracy of ω, and it is again injective

– These two maps are mutual inverses:

$$\begin{aligned} \omega(\eta, (\xi^\flat)^\sharp) = \xi^\flat(\eta) = \omega(\eta, \xi), \quad \forall \eta \in V \implies (\xi^\flat)^\sharp = \xi, \ \forall \xi \in V \\ (\lambda^\sharp)^\flat(\eta) = \omega(\eta, \lambda^\sharp) = \lambda(\eta), \quad \forall \eta \in V \implies (\lambda^\sharp)^\flat = \lambda, \ \forall \lambda \in V^* \ \blacksquare \end{aligned} \tag{3.27}$$

- Moreover, the musical isomorphism defines a (2,0) tensor over V:

$$\omega^\sharp(\eta, \lambda) = -\omega(\eta^\sharp, \lambda^\sharp) \tag{3.28}$$

for all $\eta, \lambda \in V^*$

- This is called the **inverse symplectic form**
- Choosing a basis $\{e_a\}$ of V, we find:

$$(\xi^\flat)_a = \omega_{ab}\xi^b, \quad \text{and} \quad \omega_{ab}(\lambda^\sharp)^b = \lambda_a \implies (\lambda^\sharp)^a = \omega^{ab}\lambda_b \tag{3.29}$$

- This is, in a sense, raising and lowering indices using the symplectic form
- This makes sense as to why $\omega^\sharp$ is the inverse symplectic form:

$$(\omega^\sharp)^{ab}\eta_a\lambda_b = -\omega_{ab}(\eta^\sharp)^a(\lambda^\sharp)^b = -\omega_{ab}\omega^{ac}\omega^{bd}\eta_c\lambda_d = \delta^c_b\omega^{bd}\eta_c\lambda_d = \omega^{cd}\eta_c\lambda_d \tag{3.30}$$

- The analogous construction in pseudo-Riemannian geometry (c.f. GR) is defined by the metric tensor: an inner product on V
- This means that for any symplectic manifold (M, ω), there is a musical isomorphism $T_xM \equiv T^*_xM$ for every $x \in M$, which induces a musical isomorphism between vector fields and 1-forms on M

3.3 Hamiltonian Vector Fields and Flows

- Given arbitrary **Hamiltonian function** $H \in C^\infty(M)$, the vector field $X_H = (dH)^\sharp \in \mathfrak{X}(M)$ is the **Hamiltonian vector field**
- The set of all such vector fields is denoted $Ham(M)$
- Equivalently, by the musical isomorphism, we have

$$X_H^\flat = dH \implies X_H^\flat(\eta) = \omega(\eta, X_H) = -\omega(X_H, \eta) = -(\iota_{X_H}\omega)(\eta) \tag{3.31}$$

- Therefore, X_H is Hamiltonian if and only if

$$\iota_{X_H}\omega = -dH \tag{3.32}$$

- Just as a reminder, ι_X denotes the interior derivative of a k-form with respect to X, defined as contraction with X in the first argument
- In a coordinate basis $\{\partial_a : a = 1, \ldots, 2n\}$, we find components

$$(\iota_{X_H}\omega)_a = \omega_{ab}X^a_H = -\omega_{ab}X^b_H = (-dH)_a = -\partial_aH \implies \omega_{ab}X^b_H = \partial_aH \tag{3.33}$$

- Equivalently:

$$X_H^a = \omega^{ab}\partial_b H \tag{3.34}$$

- **Example:** consider symplectic manifold $(\mathbb{R}^{2n}, \omega)$ with coordinates (p_i, q_i) and symplectic form $\omega = dp_i \wedge dq_i$
  - Pick some $H \in C^\infty(\mathbb{R}^{2n})$, and we compute the integral curves $\gamma(t)$ of X_H
  - Recall integral curves are defined by

$$\dot{\gamma}(t) = X_H(\gamma(t)) = X_H|_{\gamma(t)} \tag{3.35}$$

  - In our defining chart (p_i, q_i), we have

$$\dot{\gamma}(t) = \dot{p}_i \frac{\partial}{\partial p_i} + \dot{q}_i \frac{\partial}{\partial q_i} \tag{3.36}$$

    along the curve, where we slightly abuse notation to write $\dot{p}_i = \frac{d}{dt} p_i(\gamma(t))$ etc.
  - Therefore, along the curve we must have

$$\iota_{X_H}\omega|_{\gamma(t)} = (dp_i \wedge dq_i)(X_H(\gamma(t))) = dp_i(\dot{\gamma})dq_i - dq_i(\dot{\gamma})dp_i = \dot{p}_i dq_i - \dot{q}_i dp_i \tag{3.37}$$

  - We also have by definition:

$$dH = \frac{dH}{dp_i} dp_i + \frac{dH}{dq_i} dq_i \tag{3.38}$$

  - So by comparison, we deduce Hamilton's equations from purely geometric principles:

$$\dot{p}_i = -\frac{dH}{dq_i}, \quad \dot{q}_i = \frac{dH}{dp_i} \tag{3.39}$$

- This is a new way of looking at Hamilton's equations: solving them is equivalent to finding the integral curves of a certain vector field in phase space
- Now recall: integral curves of a complete vector field $X \in \mathfrak{X}(M)$ are in bijection with a one-parameter family of diffeomorphisms $\phi_t : M \to M$, $t \in \mathbb{R}$ satisfying composition law $\phi_{t+s} = \phi_t \circ \phi_s$ called **flows**
- Given flow ϕ_t, the corresponding vector field X can be reconstructed:

$$X_x(f) = \frac{d}{dt} f(\phi_t(x))\bigg|_{t=0} \quad \forall x \in M,\ f \in C^\infty(M) \tag{3.40}$$

- A **Hamiltonian flow** is the flow of a complete Hamiltonian vector field $X_H \in Ham(M)$
- A **Hamiltonian system** $(M, \omega; H)$ is a symplectic manifold (M, ω) with a chosen function $H \in C^\infty(M)$
- Time-evolution of a Hamiltonian system is defined by the Hamiltonian flow of X_H
- **Lemma:** for $H \in C^\infty(M)$ and $X_H \in Ham(M)$, $\mathcal{L}_{X_H}\omega = 0$ so Hamiltonian vector fields preserve the symplectic form
- **Proof:**
  - Recall Cartan's magic identity for the Lie derivative of k-forms:

$$\mathcal{L}_{X_H}\omega = d\iota_{X_H}\omega + \iota_{X_H} d\omega = -d^2 H = 0 \quad \blacksquare \tag{3.41}$$

- **Theorem:** let ϕ_t be a Hamiltonian flow on symplectic manifold (M, ω); then the symplectic form is preserved by the flow, $\phi_t^*\omega = \omega$
  - At arbitrary $x \in M$, we have

$$\begin{aligned}(\phi_t^*\omega - \omega)_x &= \int_0^t \frac{d}{du}(\phi_u^*\omega)_x \, du \\ &= \int_0^t \frac{d}{ds}\left(\phi_{u+s}^*\omega\right)_x\bigg|_{s=0} du \quad \text{(chain rule)} \\ &= \int_0^t \left(\phi_u^* \frac{d}{ds}\phi_s^*\omega\bigg|_{s=0}\right)_x du \\ &= \int_0^t \phi_u^*(\mathcal{L}_{X_H}\omega)_{\phi_u(x)} \, du \\ &= 0\end{aligned} \tag{3.42}$$

  - The second last line comes from definition of the Lie derivative of a k-form:

$$\mathcal{L}_{X_H}\omega = \frac{d}{dt}\phi_t^*\omega\bigg|_{t=0} \quad \blacksquare \tag{3.43}$$

- **Corollary:** a Hamiltonian flow ϕ_t preserves any exterior power of the symplectic form; that is, $\phi_t^*\omega^k = \omega^k$
- **Corollary:** (Liouville's theorem) a Hamiltonian flow ϕ_t preserves the volume form $\phi_t^*\omega^n = \omega^n$
- This means that Hamiltonian flows preserve volumes in phase space
- **Theorem:** the Hamiltonian function $H \in C^\infty(M)$ is preserved by Hamiltonian flow associated to $X_H \in Ham(M)$
- **Proof:**
  - First, note

$$\mathcal{L}_{X_H}H = X_H(H) = dH(X_H) = -(\iota_{X_H}\omega)(X_H)) = -\omega(X_H, X_H) = 0 \tag{3.44}$$

  - Then, by a similar argument to the last theorem, we are done $\blacksquare$
- This result is essentially conservation of energy

3.4 The Poisson Bracket

- Given symplectic manifold (M, ω), we can define the **Poisson bracket** - a binary operation on $C^\infty(M)$
- Specifically, given $f, g \in C^\infty(M)$, the Poisson bracket is

$$\{f, g\} = X_g(f) \tag{3.45}$$

  where X_g is the Hamiltonian vector field associated with g
- Alternatively, we can think of this as the rate of change of f along the Hamiltonian flow ϕ_t of g:

$$\{f, g\}_x = \frac{d}{dt}f(\phi_t(x))\bigg|_{t=0}, \ \forall x \in M \tag{3.46}$$

- This definition is appealing geometrically, but in practice it is more useful to rewrite by noticing the following:

$$X_g(f) = df(X_g) = \iota_{X_g} df = -\iota_{X_g}\iota_{X_f}\omega = -\omega(X_f, X_g) \tag{3.47}$$

  and so

$$\{f, g\} = -\omega(X_f, X_g) \tag{3.48}$$

- This immediately implies that the Poisson bracket acts as $\{\cdot,\cdot\} : C^\infty(M) \to C^\infty(M) \to C^\infty(M)$, is $\mathbb{R}$-*bilinear*, and is *skew-symmetric*

- Moreover, using the musical isomorphism, we notice

$$\omega^\sharp(df, dg) = -\omega\left((df)^\sharp, (dg)^\sharp\right) = -\omega(X_f, X_g) = \{f, g\} \tag{3.49}$$

  which implies the following lemma

- **Lemma:** the Poisson bracket satisfies the *Leibnitz rule*

$$\{f_1 f_2, g\} = f_1\{f_2, g\} + \{f_1, g\}f_2 \tag{3.50}$$

  for all $f_1, f_2, g \in C^\infty(M)$

- As always, to do more useful things with the Poisson bracket, we need to express it in a coordinate basis

- Consider basis $e_a = \partial_a$ defined by chart (x^a), and the dual basis $f^a = dx^a$; then we find

$$\{f, g\} = \omega^\sharp\left(\partial_a f\, dx^a, \partial_b g\, dx^b\right) = \omega^{ab}\partial_a f \partial_b g \tag{3.51}$$

- **Example:** we look again at $(M, \omega) = (\mathbb{R}^{2n}, \omega)$ with coordinates (p_i, q_i) so $\omega = dp_i \wedge dq_i$
  - We know that in the $(\partial/\partial p_i, \partial/\partial q_i)$ basis

$$\omega_{ab} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \implies \omega^{ab} = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \tag{3.52}$$

  - Then, we find

$$\{f, g\} = \omega^{ab}\partial_a f \partial_b g = \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \tag{3.53}$$

    which is the usual definition of it from classical dynamics

- We also find that we can define the Poisson bracket for locally defined functions by just restricting the domain

- **Example:** consider the example from a while back, where Q is an n-dimensional smooth manifold with chart $(q_i(x))$ and the cotangent bundle $T^*Q = \{(p_x, x) : p_x \in T^*_x Q,\ x \in Q\}$ is a $2n$-dimensional symplectic manifold with chart $(p_i(x), q_i(x))$ and $\omega = dp_i \wedge dq_i$
  - The coordinates are functions defined only on the chart $U \subset M$
  - Then by the earlier example, we find the *fundamental Poisson brackets*:

$$\{q_i, q_j\} = \{p_i, p_j\} = 0, \quad \{q_i, p_j\} = \delta_{ij} \tag{3.54}$$

- Now recall that the time-evolution of a Hamiltonian system $(M, \omega; H)$ is given by the flow ϕ_t of X_H

- Therefore for a smooth function $f \in C^\infty(M)$, along the flow we find

$$\frac{d}{dt}\phi_t^* f = X_H(f) = \{f, H\} \tag{3.55}$$

- We usually write this in the shorthand

$$\frac{d}{dt} f = \{f, H\} \tag{3.56}$$

- We can define a **first integral/constant of motion** of $(M, \omega; H)$ is any function $f \in C^\infty(M)$ which is invariant under the flow of X_H; that is, f is a constant of motion iff it Poisson commutes with H: $\{f, H\} = 0$
- **Theorem:** the Hamiltonian H is preserved by the flow of $X_f \in Ham(M)$ if and only if $f \in C^\infty(M)$ is a constant of motion
- **Proof:**
  - Note that
$$X_f(H) = \{H, f\} = -\{f, H\} \tag{3.57}$$
    and so $X_f(H) = 0$ iff $\{f, H\} = 0$ as required ■
- This is just Noether's theorem in phase space
- It turns out the Poisson bracket has a *Lie algebra* structure; before we get to this, we need to first relate the Lie bracket of two Hamiltonian vector fields to their Poisson bracket
- Recall the definition of the Lie bracket of two vector fields:

$$[X, Y](f) = X(Y(f)) - Y(X(f)), \quad \forall f \in C^\infty(M) \tag{3.58}$$

- **Theorem:** let X_f and X_g be two Hamiltonian vector fields; then

$$[X_f, X_g] = -X_{\{f,g\}} \tag{3.59}$$

  for $f, g \in C^\infty(M)$
- **Proof:**
  - Recall that X_H is Hamiltonian if and only if there exists $H \in C^\infty(M)$ such that $\iota_{X_H}\omega = -dH$
  - Therefore, we compute

$$\begin{aligned}
\iota_{[X_f, X_g]}\omega &= \left(\mathcal{L}_{X_f}\iota_{X_g} - \iota_{X_g}\mathcal{L}_{X_f}\right)\omega && \text{(Cartan identity)} \\
&= -\mathcal{L}_{X_f} dg && (\iota_{X_g}\omega = -dg \text{ and } \mathcal{L}_{X_f}\omega = 0) \\
&= -d\iota_{X_f} dg && \text{(Cartan's magic identity)} \\
&= -d\left(X_f(g)\right) && \text{(by definition)} \\
&= d\{f, g\} && \text{(by definition)}
\end{aligned} \tag{3.60}$$

  - The proof then follows from the recalled definition of Hamiltonian vector fields and non-degeneracy of ω ■