Section 1.4: The Matrix Equation

- If A is an $m \times n$, with columns $a_1, \ldots, a_n$, and if x is in $\mathbb{R}^n$, then the product of A and x, denoted by Ax, is the linear combination of the columns of A using the corresponding entries in x as weights; that is:

$$Ax = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = x_1a_1 + x_2a_2 + \cdots + x_na_n$$

- Ax is defined only if the number of columns of A equals the number of entries in x.

Theorem 1 *If A is an $m \times n$ matrix, with columns $a_1, \ldots, a_n$, and if b is in $\mathbb{R}^m$, then the matrix equation $Ax = b$ has the same solution set as the vector equation*

$$x_1a_1 + x_2a_2 + \cdots + x_na_n$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$Ax = \begin{bmatrix} a_1 & a_2 & \ldots & a_n & b \end{bmatrix}$$

- The equation $Ax = b$ has a solution if and only if b is a linear combination of the columns of A.

Theorem 2 *Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or they are all false.*

a.) For each b in $\mathbb{R}^m$, the equation $Ax = b$ has a solution.

b.) Each b in $\mathbb{R}^m$ is a linear combination of the columns of A.

c.) The columns of A span $\mathbb{R}^m$.

d.) A has a pivot position in every row.

- The matrix with 1s on the diagonal and 0s elsewhere is called the **identity matrix** and is denoted by I:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Theorem 3 *If A is an $m \times n$ matrix, u and v are vectors in $\mathbb{R}^n$, and c is a scalar, then*

a.) $A(u + v) = Au + Av$

b.) $A(cu) = c(Au)$