Section 1.1: Systems of Linear Equations

- A **linear equation** in the variables $x_1, \ldots, x_n$ is an equation that can be written in the form $a_1x_1 + \cdots + a_nx_n = b$ where b and coefficients $a_1, \ldots, a_n$ are usually known in advance.
- A **system of linear equations** is a collection of one or more linear equations using the same variables, $x_1, \ldots, x_n$.
- A **solution** of the system is a list $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation a true statement when the values $s_1, \ldots, s_n$ are substituted for $x_1, \ldots, x_n$ respectively.
- The set of all possible solutions is called the **solution set** of the linear system.
- Two linear systems are called **equivalent** if they have the same solution set.
- A system of linear equations has
  1. no solution, or
  2. exactly one solution, or
  3. infinitely many solutions
- A system of linear equations is said to be **consistent** if it has either one solution or infinity many solutions.
- A system of linear equations is said to be **inconsistent** if it has no solution.
- The essential information of a linear system can be recorded compactly in a rectangular array called a **matrix**.
- The **coefficient matrix** of a system of equations is a matrix with the coefficients of each variable, written as such:

$$\left.\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}\right\} \rightarrow \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}$$

- The **augmented matrix** of a system of equations is the coefficient matrix with an additional column for the constants on the right side of the equation, written as such:

$$\left.\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}\right\} \rightarrow \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

- Elementary row operations include the following:
  1. **Replacement** Replace one row by the sum of itself and a multiple of another row
  2. **Interchange** Interchange two rows
  3. **Scaling** Multiply all entires in a row by a nonzero constant
- Two matrices are called **row equivalent** if there is a sequence of elementary row operations that transforms one matrix into the other.
- It is important to note that row operations are reversible. If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.
- Two fundamental questions about a linear system are as follows:
  1. Is the system consistent; that is, does at least one solution *exist*?
  2. If a solution exists, is it the *only* one; that is, is the solution *unique*?

Example 1

Solve the given system of equations:

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

1. Determine the augmented matrix of the initial system.

$$\left.\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \quad (1) \\ 2x_2 - 8x_3 &= 8 \quad (2) \\ -4x_1 + 5x_2 + 9x_3 &= -9 \quad (3) \end{aligned}\right\} \to \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

2. Keep x_1 in the first equation and eliminate it from the other equations. To do so, add $4 \times (1)$ to (3):

$$\begin{array}{ccccccc} 4x_1 & - & 8x_2 & + & 4x_3 & = & 0 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \\ \hline & - & 3x_2 & + & 13x_3 & = & -9 \end{array}$$

3. The result of this calculation is written in place of the original third equation.

$$\left.\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \quad (1) \\ 2x_2 - 8x_3 &= 8 \quad (2) \\ -3x_2 + 13x_3 &= -9 \quad (3) \end{aligned}\right\} \to \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

4. Now, multiply (2) by 1/2 in order to obtain 1 as the coefficient for x_2.

$$\left.\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \quad (1) \\ x_2 - 4x_3 &= 4 \quad (2) \\ -3x_2 + 13x_3 &= -9 \quad (3) \end{aligned}\right\} \to \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

5. Using the x_2 in (2), we can eliminate the x_2 in (3) by using $3 \times (2)$.

$$\begin{array}{ccccc} 3x_2 & - & 12x_3 & = & 12 \\ -3x_2 & + & 13x_3 & = & -9 \\ \hline & & x_3 & = & 3 \end{array}$$

6. The new system takes a triangular form.

$$\left.\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \quad (1) \\ x_2 - 4x_3 &= 4 \quad (2) \\ -3x_2 + 13x_3 &= -9 \quad (3) \end{aligned}\right\} \to \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

7. Using solution in row 3, it's possible to solve for rows 1 and 2 and get a solution of $(x_1, x_2, x_3) = (29, 16, 3)$.