# MAT341 - Project 1 - Matrix Theory

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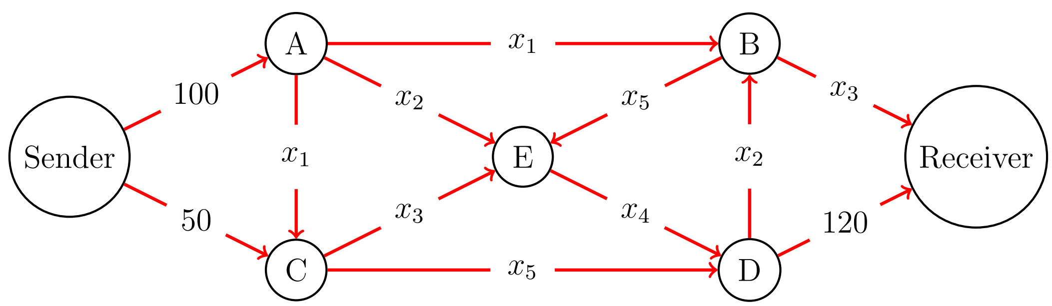


Figure 1: The Network Graph

## 1 Introduction

This project demonstrates Matrix Theory when applied to a network throughput graph. The scenario to be solved is as follows.

A sender is transmitting data with a total rate of 150 megabits per second along the network graph in Figure 1. The data is transmitted from the sender to the receiver over a network of five different routers. These routers are labeled A, B, C, D, and E. The connections and data rates between the routers are labeled as $x_1, x_2, x_3, x_4, x_5$. This project will find a solution to the data transfer through the specific routers in this network. Two methods will be used to find the solution: $LU$ Factorization and Cramer's Rule. At the end will be a recommendation on network changes.

## 2 A System of Linear Equations

This problem can be solved by utilizing a system of linear equations. Each equation represents the input and output of a specific router node. First we must construct a table of the input and output flow of each node. The input and output must be equivalent for each node. Solving for variables for each node gives us a system of linear equations.

| **Node** | **Input** | **Output** |
|---|---|---|
| A | 100 | $2x_1 + x_2$ |
| B | $x_1 + x_2$ | $x_3 + x_5$ |
| C | $50 + x_1$ | $x_3 + x_5$ |
| D | $x_4 + x_5$ | $x_2 + 120$ |
| E | $x_2 + x_3 + x_5$ | $x_4$ |

| Node | Equations |
|---|---|
| A | $2x_1 + x_2 = 100$ |
| B | $x_1 + x_2 - x_3 - x_5 = 0$ |
| C | $x_1 - x_3 - x_5 = -50$ |
| D | $-x_2 + x_4 + x_5 = 120$ |
| E | $x_2 + x_3 - x_4 + x_5 = 0$ |

Utilizing this system of equations, we can construct a matrix equation of the form $Ax = b$ representing it. This becomes equation (1) below:

$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & -1 \\ 1 & 0 & -1 & 0 & -1 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \\ -50 \\ 120 \\ 0 \end{bmatrix} \tag{1}$$

## 3 $LU$ Factorization

The first method to solve this system of equations will be $LU$ Factorization. This involves decomposing the coefficient matrix $A$ into two factors, $L$ and $U$, resulting in an equation of the form $A = LU$. Doing so gives:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 & 0 \\ \frac{1}{2} & -1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 1 & 0 \\ 0 & 2 & -\frac{3}{2} & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -1 & 0 & -1 \\ 0 & 0 & -2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Using the $LU$ decomposition it's possible to decompose the original matrix equation into two separate equations:

$$Ux = y \tag{2}$$

$$Ly = b \tag{3}$$

We must first solve for $y$ in equation (3).

$$y = L^{-1}b \tag{4}$$

This requires the inverse of $L$, which is:

$$L^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 1 & 1 \end{bmatrix}$$

Plugging in values for equation (4) gives:

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 1 & 1 \end{bmatrix} \begin{bmatrix} 100 \\ 0 \\ -50 \\ 120 \\ 0 \end{bmatrix} = \begin{bmatrix} 100 \\ -50 \\ -150 \\ 170 \\ 45 \end{bmatrix}$$

In order to compute for $x$, we rearrange equation (2) into:

$$x = U^{-1}y \tag{5}$$

Calculating the inverse of $U$ gives:

$$U^{-1} = \begin{bmatrix} \frac{1}{2} & -1 & \frac{1}{2} & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

As given in equation (5), the product of the values of $U^{-1}$ and $y$ gives the solution $x$:

$$x = \begin{bmatrix} 25 \\ 50 \\ 30 \\ 125 \\ 45 \end{bmatrix}$$

# 4 Cramer's Rule

The second method used to solve this is the use of Cramer's Rule. Let us first recap the original matrix equation.

$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & -1 \\ 1 & 0 & -1 & 0 & -1 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \\ -50 \\ 120 \\ 0 \end{bmatrix}$$

We must first find the determinant of $A$, as noted below:

$$\left|A\right| = \begin{vmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & -1 \\ 1 & 0 & -1 & 0 & -1 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 & 1 \end{vmatrix} = -2$$

Following Cramer's rule then leads to finding the determinant of each $A_i(b)$, which indicates the matrix formed when replacing the $i^{th}$ column of $A$ with $b$.

$$\left|A_1\right| = \begin{vmatrix} 100 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 \\ -50 & 0 & -1 & 0 & -1 \\ 120 & -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 & 1 \end{vmatrix} = -50$$

$$\left|A_2\right| = -100$$

$$\left|A_3\right| = -60$$

$$\left|A_4\right| = -250$$

$$\left|A_5\right| = -90$$

Then for each determinant, division by the original determinant provides the solution to that particular $x$ value.

$$\frac{1}{-2} \begin{bmatrix} -50 \\ -100 \\ -60 \\ -250 \\ -90 \end{bmatrix}$$

The result is then the answer:

$$x = \begin{bmatrix} 25 \\ 50 \\ 30 \\ 125 \\ 45 \end{bmatrix}$$

This answer is identical to the answer obtained through $LU$ Factorization.

# 5 Conclusion

Through two methods, *LU* Factorization and Cramer's Rule, the network flow has been determined. Solutions and recommendations for network changes are listed in Figure 2.

| **Network Link** | **Recommended Capacity (Mbps)** | **Solution** | **Recommendation** | **Explanation** |
|---|---|---|---|---|
| $x_1$ | 60 | 25 | No Change | The link can handle the traffic. |
| $x_2$ | 50 | 50 | No Change | The link can handle the traffic. |
| $x_3$ | 100 | 30 | No Change | The link can handle the traffic. |
| $x_4$ | 100 | 125 | Upgrade | The traffic is higher than the link can handle. |
| $x_5$ | 50 | 45 | No Change | The link can handle the traffic. |

Figure 2: Network Link Recommendations