Chapter 1: Theorems & Defintions

Theorem 1 (Uniqueness of the Reduced Echelon Form) *Each matrix is row equivalent to one and only one reduced echelon matrix.*

Theorem 2 (Existence and Uniqueness Theorem) *A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column; ie, if and only if an echelon form of the augmented matrix has no row of the form $[0 \cdots 0b]$ with b nonzero.*

Spans: If $v_1, \ldots, v_p$ are in $\mathbb{R}^2$ the the set of all linear combinations of $v_1, \ldots, v_p$ is denoted by **Span**$\{v_1, \ldots, v_p\}$ and is called the **subset of $\mathbb{R}^2$ spanned by** $v_1, \ldots, v_p$. That is, **Span**$\{v_1, \ldots, v_p\}$ is the collection of all vectors, with scalars $c_1, \ldots, c_p$, that can be written as

$$c_1v_1 + c_2v_2 + \cdots + c_pv_p$$

Matrix Equation: If A is an $m \times n$ matrix, with columns $a_1, \ldots, a_n$, and if x is in $\mathbb{R}^n$, then the product of $A \times x$, by Ax is the linear combination of the columns of A using the corresponding entries in x as weights:

$$Ax = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = x_1a_1 + x_2a_2 + \cdots + x_na_n$$

Theorem 3 *If A is an $m \times n$ matrix, with columns $a_1, \ldots, a_n$, and if b is in $\mathbb{R}^m$, then the matrix equation $Ax = b$ has the same solution set as the vector equation*

$$x_1a_1 + x_2a_2 + \cdots + x_na_n$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$Ax = \begin{bmatrix} a_1 & a_2 & \cdots & a_n & b \end{bmatrix}$$

Theorem 4 *Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or they are all false.*

a.) For each b in $\mathbb{R}^m$, the equation $Ax = b$ has a solution.

b.) Each b in $\mathbb{R}^m$ is a linear combination of the columns of A.

c.) The columns of A span $\mathbb{R}^m$.

d.) A has a pivot position in every row.

Theorem 5 *If A is an $m \times n$ matrix, u and v are vectors in $\mathbb{R}^n$, and c is a scalar, then*

a.) $A(u + v) = Au + Av$

b.) $A(cu) = c(Au)$

Homogeneous Systems: A system of linear equations is said to be **homogeneous** if it can be written in the form $Ax = 0$, where A is an $m \times n$ matrix and 0 is the zero vector in $\mathbb{R}^m$.

- Such a system *always* has at least one solution, namely the **trivial solution**, the zero vector.
- Such a system has a nontrivial solution if and only if the equation has at least one free variable.

Linear Independence: An indexed set of vectors $\{v_1, \ldots, v_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1v_1 + x_2v_2 + \cdots + x_pv_p$$

has only the trivial solution. The set $\{v_1, \ldots, v_p\}$ is said to be **linearly dependent** if there exist weights $\{c_1, \ldots, c_p\}$, not all zero, such that

$$c_1v_1 + c_2v_2 + \cdots + c_pv_p = 0$$

Theorem 6 (Characterization of Linearly Dependent Sets) *An indexed set $S = \{v_1, \ldots, v_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others.*

Theorem 7 *If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{v_1, \ldots, v_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.*

Theorem 8 *If a set $S = \{v_1, \ldots, v_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.*