# Real Analysis

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# Contents



## 0.1 Construction of $\mathbb{R}$

**Definition 1.** The *supremum* $\sup(S) = \alpha \in A$ of a set $S \subset A$ is an upper bound of $S$, and for $\beta < \alpha$, $\beta$ is not an upper bound for $S$.

**Example.** Consider $\mathbb{Q}^-$; let $\sup \mathbb{Q}^- = \gamma < 0$ But there exists $\frac{\gamma}{2} < 0$, so $\gamma \neq \sup \mathbb{Q}^-$. It follows that $\sup \mathbb{Q}^- = 0$.

**Definition 2.** A set $S$ has the *least upper bound property*, or *satisfies the Completeness Axiom*, if every non-empty subset of $S$ with an upper-bound also has a supremum.

**Definition 3.** A *Dedekind Cut* $\alpha$ is a subset of $\mathbb{Q}$ satisfying:

1. $\alpha \neq \emptyset, \mathbb{Q}$.
2. If $p \in \alpha$, and $p > q \in \mathbb{Q}$, then $q \in \alpha$ (closed downwards).
3. If $p \in \alpha$ then $p < r$ for some $r \in \alpha$.

These cuts are essentially *rays* on the rationals pointing towards the negative direction, $(-\infty, p)$ for $p \in \mathbb{Q}$.

**Example.** The set $\mathbb{Q}^-$ is trivially closed downwards, and has no largest member, so it is therefore a Dedekind cut.

**Definition 4.** The *set of all real numbers*, $\mathbb{R}$, is defined as the set $\mathbb{R} := \{\alpha \subset \mathbb{Q} : \alpha \text{ is a Dedekind cut}\}$, where:

1. $\alpha < \beta$ if $\alpha \subset \beta$.
2. $\alpha + \beta := \{r + s : r \in \alpha \text{ and } s \in \beta\}$
3. If $\alpha, \beta \in \mathbb{R}^+$, then $\alpha\beta := \{p : p < rs \text{ for } r, s > 0, r \in \alpha, s \in \beta\}$.

**Example.** The real number $\sqrt{2}$ is defined as $\alpha = \{x \in \mathbb{Q} : x^2 < 2\}$.

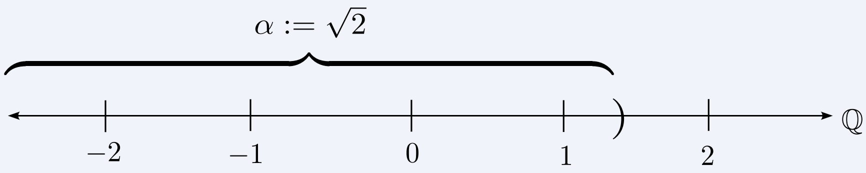


Figure 1: The Dedekind Cut corresponding to $\sqrt{2}$

**Theorem 1.** $\mathbb{R}$ is an ordered field.

**Proof.**

1. $\alpha + \beta := \{r + s : r \in \alpha \text{ and } s \in \beta\} = \beta + \alpha := \{s + r : r \in \alpha \text{ and } s \in \beta\}$, because $\mathbb{Q}$ is a field (and therefore commutative). So, addition in $\mathbb{R}$ commutes.

2. For $\gamma$ a Dedekind cut, $(\alpha+\beta)+\gamma = \{(s+r)+t : r \in \alpha, s \in \beta \text{ and } t \in \gamma\} = \{s + (r + t) : r \in \alpha, s \in \beta \text{ and } t \in \gamma\} = \alpha + (\beta + \gamma)$ because $\mathbb{Q}$ is a field (and therefore associative). So, addition in $\mathbb{R}$ is associative.

3. Let $\mathbb{Q}^- =: 0_{\mathbb{R}} \in \mathbb{R}$. $0_{\mathbb{R}} + \alpha = \{s + 0 : s \in \alpha\} = \{s : s \in \alpha\} = \alpha$, so $0_{\mathbb{R}}$ is an additive identity.

The remainder of the proof can be found in Baby Rudin. $\square$

**Theorem 2.** $\mathbb{R}$ satisfies the Axiom of Completeness.