Real Analysis

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Contents

0.1 Construction of $\mathbb{R}$

Definition 1. The *supremum* $\sup(S) = \alpha \in A$ of a set $S \subset A$ is an upper bound of S, where for $\beta < \alpha$, β is not an upper bound for S.

Example. Consider $\mathbb{Q}^-$; let $\sup \mathbb{Q}^- = \gamma < 0$ But there exists $\frac{\gamma}{2} < 0$, so $\gamma \neq \sup \mathbb{Q}^-$. It follows that $\sup \mathbb{Q}^- = 0$.

Definition 2. A set S has the *least upper bound property*, or *satisfies the Completeness Axiom*, if every non-empty subset of S with an upper-bound also has a supremum.

Definition 3. A *Dedekind Cut* α is a subset of $\mathbb{Q}$ satisfying:

1. $\alpha \neq \emptyset, \mathbb{Q}$.
2. If $p \in \alpha$, and $p > q \in \mathbb{Q}$, then $q \in \alpha$ (closed downwards).
3. If $p \in \alpha$ then $p < r$ for some $r \in \alpha$.

These cuts are essentially *rays* on the rationals pointing towards the negative direction, $(-\infty, p)$ for $p \in \mathbb{Q}$.

Example. The set $\mathbb{Q}^-$ is trivially closed downwards, and has no largest member, so it is therefore a Dedekind cut.

Definition 4. The *set of all real numbers*, $\mathbb{R}$, is defined as the set $\mathbb{R} := \{\alpha \subset \mathbb{Q} : \alpha \text{ is a Dedekind cut}\}$, where:

1. $\alpha < \beta$ if $\alpha \subset \beta$.
2. $\alpha + \beta := \{r + s : r \in \alpha \text{ and } s \in \beta\}$
3. If $\alpha, \beta \in \mathbb{R}^+$, then $\alpha\beta := \{p : p < rs \text{ for } r, s > 0, r \in \alpha, s \in \beta\}$.

Example. The real number $\sqrt{2}$ is defined as $\alpha = \{x \in \mathbb{Q} : x^2 < 2\}$.

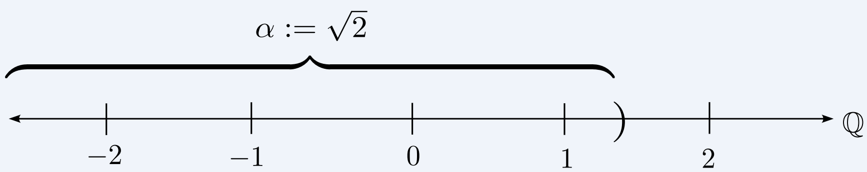

Figure 1: The Dedekind Cut corresponding to $\sqrt{2}$

Theorem 1. $\mathbb{R}$ is an ordered field.

Proof.
1. $\alpha + \beta := \{r + s : r \in \alpha \text{ and } s \in \beta\} = \beta + \alpha := \{s + r : r \in \alpha \text{ and } s \in \beta\}$, because $\mathbb{Q}$ is a field (and therefore commutative). So, addition in $\mathbb{R}$ commutes.
2. For γ a Dedekind cut, $(\alpha+\beta)+\gamma = \{(s+r)+t : r \in \alpha, s \in \beta \text{ and } t \in \gamma\} = \{s + (r + t) : r \in \alpha, s \in \beta \text{ and } t \in \gamma\} = \alpha + (\beta + \gamma)$ because $\mathbb{Q}$ is a field (and therefore associative). So, addition in $\mathbb{R}$ is associative.
3. Let $\mathbb{Q}^- =: 0_{\mathbb{R}} \in \mathbb{R}$. $0_{\mathbb{R}} + \alpha = \{s + 0 : s \in \alpha\} = \{s : s \in \alpha\} = \alpha$, so $0_{\mathbb{R}}$ is an additive identity.

The remainder of the proof can be found in Baby Rudin. □

0.2 The Axiom of Completeness

Theorem 2. $\mathbb{R}$ contains $\mathbb{Q}$ as a subfield.

Proof. Associate to each $q \in \mathbb{Q}$ the cut $q^* := \{r \in \mathbb{Q} : r < q\}$. Then the map $f : \mathbb{Q} \to \mathbb{R}$ with rule $q \mapsto q^*$ is an injection which preserves the field operations $+, \times$ and the order $<$. Then $\mathbb{Q}^* = \{q^* : q \in \mathbb{Q}\}$ is a subfield of $\mathbb{R}$. □

Theorem 3. $\mathbb{R}$ satisfies the Axiom of Completeness.

Proof. Let $A \subset \mathbb{R}$ be a collection of cuts with upper bound β. Let $\gamma = \bigcup_{\alpha \in A} A \subset \mathbb{Q}$. We first prove that γ is a cut. γ is trivially non-empty and not equal to $\mathbb{Q}$, because it must contain at least one cut, and because it is bounded by β (and therefore properly contained in $\mathbb{R}$). Now, select some $p, q \in \alpha$, where $\alpha \in A$. If $q < p$, then $q \in \alpha$, and therefore $q \in \gamma$, so γ is closed downwards. Now select $r \in \alpha$ such that $r > p$; it then follows that $r \in \gamma$ (since each α can not have a largest element). Hence, γ is a Dedekind cut.

Now, we show that $\sup A = \gamma$. Assume $\delta < \gamma$. Then there is an $s \in \gamma$ such that $s \notin \delta$. Then s must be contained in some α, and thus $\delta < \alpha$. So, δ can not be an upper bound of A, and $\gamma = \sup A$ as a result. $\square$

Definition 5. The *infimum* $\inf(S) = \alpha \in A$ of a set $S \subset A$ is a lower bound of S, where for $\beta > \alpha$, β is not a lower bound for S.

Theorem 4. *The Archimedean Property:* If $x, y \in \mathbb{R}$, with $x > 0$, then there exists some $n \in \mathbb{Z}^+$ such that $nx > y$.

Proof. Consider $A = \{nx : n \in \mathbb{Z}^+\}$. Let y bound A. Then A has a least upper bound $\sup A = \alpha$. It is clear that $\alpha - x$ is not an upper bound for A; so for some $m \in \mathbb{Z}^+$, $\alpha - x < mx$. But then $\alpha < mx + x = (m+1)x$, so α can not be an upper bound for A.

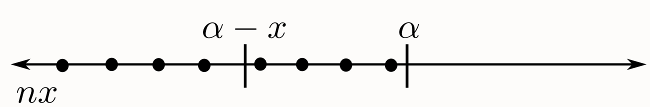

Figure 2: A's boundings.

$\square$

Theorem 5. $\mathbb{Q}$ *is dense in* $\mathbb{R}$: For each $x, y \in \mathbb{R}$, there is some $q \in \mathbb{Q}$ such that $x < q < y$.

Proof. Choose n such that $\frac{1}{n} < y - x$ (Archimedean property); consider multiples of $\frac{1}{n}$, $\frac{m}{n}$, which grow without bound. Choose the least $\frac{m}{n}$ such that $\frac{m}{n} > x$. We want to show that $\frac{m}{n} < y$; claim the contrary. Then $\frac{m}{n} \geq y$ and $\frac{m-1}{x} < x$. But then

$$-\frac{m-1}{n} > -x \Rightarrow -\frac{m-1+m}{n} = \frac{1}{n} > y - x,$$

a contradiction. □

Theorem 6. *Properties of suprema:* For $\gamma = \sup A$,

1. γ is an upper bound for some set A if and only if $\sup A \leq \gamma$.
2. For all $a \in A$, $a \leq \gamma$, $\sup A \leq \gamma$.
3. For all $a \in A$, $a < \gamma$, $\sup A \leq \gamma$.
4. If $\gamma < \sup A$, then there exists an $a \in A$ such that $\gamma < a \leq \sup A$.
5. If $A \subset B$, then $\sup A \leq \sup B$ (i.e., for all $a \in A$, $a \in B$, so $a \leq \sup B$).
6. To show that $\sup A = \sup B$, show that for all $a \in A$, there exists a $b \in B$ such that $a \leq b$, so $\sup A \leq \sup B$. A similar process applies to showing that $\sup B \in \sup A$, which proves equivalence.

0.3 Extensions of $\mathbb{R}$: the Complex Numbers $\mathbb{C}$

Definition 6. *The Extended Reals*: $\overline{\mathbb{R}} := \{-\infty, \infty\} \cup \mathbb{R}$, with the usual order on $\mathbb{R}$, with $-\infty < x < \infty$, for all $x \in \mathbb{R}$, such that:

1. $x \pm \infty = \pm\infty$
2. $x \cdot \infty = \pm\infty$
3. $\frac{x}{\infty} = 0$

And so forth.

This set shares many features with the real numbers, but it is important to note that it itself is not an ordered field. The reason that we care about the extended reals, then, is that *every* $A \subset \overline{\mathbb{R}}$ *has a supremum.*

Definition 7. *Euclidean Space:* $\mathbb{R}^k := \{(x_1, x_2, ..., x_n) : x_i \in \mathbb{R}\}$, such that $(x_1, ..., x_n) + (y_1, ..., y_n) = (x_1 + y_1, ..., x_n + y_n)$.

This set also is often useful, despite not being an ordered field. A special case of Euclidean space is the vector space, which we develop in the following:

Definition 8. *Vector Space Properties*:

1. *Scalar Multiplication*: $\alpha(x_1, ..., x_n) = \alpha\mathbf{x} = (\alpha x_1, ..., \alpha x_n)$

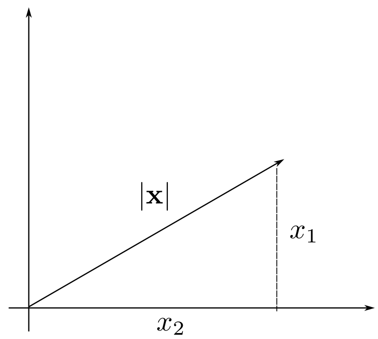

Figure 3: A 2-tuple in Euclidean 2-space represented on the plane.

2. *Dot Product*: $\mathbf{x} \cdot \mathbf{y} = \sum_i^k x_i y_i$

3. *Norms*: $|\mathbf{x}| := \sqrt{x_1 + ... + x_n}$

One can see that the concept of a norm is analogous to that of Euclidean distance, with the norm acting as a generalization of the Pythagorean theorem.

Definition 9. *The Complex Number Field* $\mathbb{C}$: For $(a, b) \in \mathbb{R}^2$, let:

1. $(a, b) + (c, d) = (a + b, c + d)$

2. $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$

3. $(0, 0)$ and $(1, 0)$ are the additive and multiplicative identities, respectively

4. $(0, 1) = i$

5. For $z = a + bi \in \mathbb{C}$, $\overline{z} = a - bi$

$\mathbb{C}$ is a natural extension of $\mathbb{R}$ (i.e., $\{a + 0i : a \in \mathbb{R}\}$ is isomorphic to $\mathbb{R}^2$). Note: $(0, 1) \cdot (0, 1) = (-1, 0)$, so $i^2 = -1$.

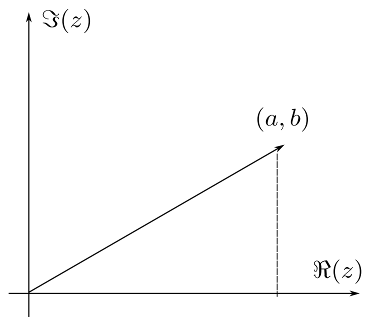

Figure 4: A planar representation of $z = (a, b) \in \mathbb{C}$.

Theorem 7. *The Cauchy-Shawrz Inequality*: If $a_1, ..., a_n$ and $b_1, ..., b_n$ are complex numbers, then

$$\left| \sum_{i=0}^{n} a_i \overline{b_i} \right|^2 \leq \sum_{i=0}^{n} |a_i|^2 \cdot \sum_{i=0}^{n} |b_i|^2$$

will hold.

0.4 Cantor Diagonalization and Metric Spaces

Theorem 8. The countable union of countable sets is countable.

Proof. Say each set within the collection $\{A_\alpha\}_{\alpha \in J}$ is countable; then we may use a 'zig-zag' argument to place the elements of each respective set into a sequence:

$$A_1 = \{a_{1,1}, a_{1,2}, a_{1,3}, ...\},$$
$$A_2 = \{a_{2,1}, a_{2,2}, a_{2,3}, ...\},$$
$$A_3 = \{a_{3,1}, a_{3,2}, a_{3,3}, ...\},$$

Which produces a sequence

$$A_{n,m} = \{a_{1,1}, a_{1,2}, a_{2,1}, a_{3,1}, ...\}$$

and so on. Hence, $\bigcup_{\alpha \in J} A_\alpha$ is countable. $\square$

Definition 10. Given a set A, the *power set* of A, denoted $\mathscr{P}(A)$ or 2^A, is the set of all subsets of A.

It is a standard combinatorial fact that the cardinality of 2^A is equal to 2^k, for $|A| = k$.

Example. Let $A = \{a_1, a_2, a_3\}$. Let $D = \{a_1, a_3\}$ and $E = \{a_2\}$, both subsets of A. Then we can define a mapping such that each subset is given by a binary value:

$$A = \{a_1, a_2, a_3\}$$

$$D \mapsto (1, 0, 1)$$

$$E \mapsto (0, 1, 0)$$

$$\emptyset \mapsto (0, 0, 0)$$

and so on.

This idea is essential to the following theorem:

Theorem 9. *Cantor's Theorem* For any set A, we have $A \not\sim 2^A$.

Proof. Suppose that there exists a bijection $f : A \hookrightarrow 2^A$; then $a \mapsto f(a) \subset A$. Then we desire to construct a subset $B \subset A$ such that $B \notin f(A)$. In particular, let us say that

$$B = \{a : a \notin f(a)\}.$$

But if $b \in B$, we have $b \notin f(b) = B$, a contradiction; then $b \notin B$. But then $b \notin f(b) = B$; this implies $b \in B$, yet another contradiction. So $B \neq f(b)$ for any $b \in A$, as desired.

□

Example. By membership, we have $2^A \sim \{f \mid f : A \to \{0, 1\}$; therefore, there exists a cardinality greater than that of $\mathbb{R}$; namely, the cardinality of $2^{\mathbb{R}}$.

The list of cardinalities is $0, 1, 2, 3, 4, ..., \aleph_0, \aleph_1, \aleph_3, ..., \aleph_\alpha, ...$ for α an infinite ordinal. Let us say that $|\mathbb{Z}| = \aleph_0$, and that $|\mathbb{R}| = \mathfrak{c}$; the Continuum Hypothesis claims that $\mathfrak{c} = \aleph_1$, a fact which is provably undecidable under ZF+C.

Definition 11. A set X is a *metric space* if it is endowed with a *metric* $d : X \times X \to \mathbb{R}$, such that for each $a, b \in X$:

1. $d(a, b) \geq 0$

2. $d(a,b) = d(b,a)$

3. *Triangle Inequality* $d(a,b) \leq d(a,r) + d(r,b)$.

Example. $\mathbb{R}$ has the standard metric of $d(x,y) = |x-y|$; more generally, $\mathbb{R}^n$ has the Euclidean metric of

$$d(\mathbf{x},\mathbf{y}) = \sqrt{(x_1-y_1)^2 + ... + (x_n-y_n)^2}.$$

Example. Let $\mathbb{R}^2$ have the metric

$$d(\mathbf{x},\mathbf{y}) = \sum_{i=0}^{n} |\mathbf{x}-\mathbf{y}|;$$

this would be a completely valid metric on $\mathbb{R}^2$, or even $\mathbb{R}^n$.

Example. Let us consider the *space of functions*; there are many ways to define a metric on such a space. One way to do so is to define a metric d on the space of continuous functions on the interval $[a,b]$, $\mathscr{C}[a,b]$:

$$\int_a^b |f(x) - g(x)| dx.$$

A good way to get a sense for what a metric space is actually *doing* is to use *balls*:

Definition 12. An *open ball* is a set $N_r(x) = \{y : d(x,y) < r\}$. A *closed ball* is a set $\overline{N_r(x)} = \{y : d(x,y) \leq r\}$.

Definition 13. A *limit point* of a set E is a point $p \in X$ such that every neighborhood of p contains some $p \neq q \in E$.