# Real Analysis

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## 0.1 Construction of $\mathbb{R}$

**Definition 1.** The *supremum* $\sup(S) = \alpha \in A$ of a set $S \subset A$ is an upper bound of $S$, where for $\beta < \alpha$, $\beta$ is not an upper bound for $S$.

**Example.** Consider $\mathbb{Q}^-$; let $\sup \mathbb{Q}^- = \gamma < 0$ But there exists $\frac{\gamma}{2} < 0$, so $\gamma \neq \sup \mathbb{Q}^-$. It follows that $\sup \mathbb{Q}^- = 0$.

**Definition 2.** A set $S$ has the *least upper bound property*, or *satisfies the Completeness Axiom*, if every non-empty subset of $S$ with an upper-bound also has a supremum.

**Definition 3.** A *Dedekind Cut* $\alpha$ is a subset of $\mathbb{Q}$ satisfying:

1. $\alpha \neq \emptyset, \mathbb{Q}$.
2. If $p \in \alpha$, and $p > q \in \mathbb{Q}$, then $q \in \alpha$ (closed downwards).
3. If $p \in \alpha$ then $p < r$ for some $r \in \alpha$.

These cuts are essentially *rays* on the rationals pointing towards the negative direction, $(-\infty, p)$ for $p \in \mathbb{Q}$.

**Example.** The set $\mathbb{Q}^-$ is trivially closed downwards, and has no largest member, so it is therefore a Dedekind cut.

**Definition 4.** The *set of all real numbers*, $\mathbb{R}$, is defined as the set $\mathbb{R} := \{\alpha \subset \mathbb{Q} : \alpha \text{ is a Dedekind cut}\}$, where:

1. $\alpha < \beta$ if $\alpha \subset \beta$.
2. $\alpha + \beta := \{r + s : r \in \alpha \text{ and } s \in \beta\}$
3. If $\alpha, \beta \in \mathbb{R}^+$, then $\alpha\beta := \{p : p < rs \text{ for } r, s > 0, r \in \alpha, s \in \beta\}$.

**Example.** The real number $\sqrt{2}$ is defined as $\alpha = \{x \in \mathbb{Q} : x^2 < 2\}$.

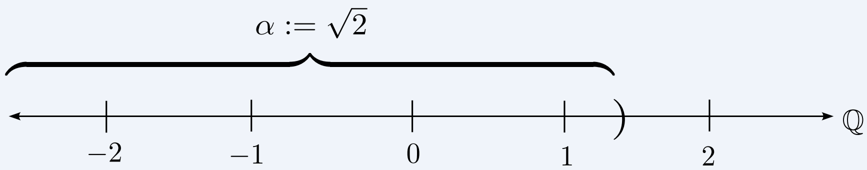


Figure 1: The Dedekind Cut corresponding to $\sqrt{2}$

**Theorem 1.** $\mathbb{R}$ is an ordered field.

**Proof.**

1. $\alpha + \beta := \{r + s : r \in \alpha \text{ and } s \in \beta\} = \beta + \alpha := \{s + r : r \in \alpha \text{ and } s \in \beta\}$, because $\mathbb{Q}$ is a field (and therefore commutative). So, addition in $\mathbb{R}$ commutes.
2. For $\gamma$ a Dedekind cut, $(\alpha+\beta)+\gamma = \{(s+r)+t : r \in \alpha, s \in \beta \text{ and } t \in \gamma\} = \{s + (r + t) : r \in \alpha, s \in \beta \text{ and } t \in \gamma\} = \alpha + (\beta + \gamma)$ because $\mathbb{Q}$ is a field (and therefore associative). So, addition in $\mathbb{R}$ is associative.
3. Let $\mathbb{Q}^- =: 0_{\mathbb{R}} \in \mathbb{R}$. $0_{\mathbb{R}} + \alpha = \{s + 0 : s \in \alpha\} = \{s : s \in \alpha\} = \alpha$, so $0_{\mathbb{R}}$ is an additive identity.

The remainder of the proof can be found in Baby Rudin. □

## 0.2 The Axiom of Completeness

**Theorem 2.** $\mathbb{R}$ contains $\mathbb{Q}$ as a subfield.

**Proof.** Associate to each $q \in \mathbb{Q}$ the cut $q^* := \{r \in \mathbb{Q} : r < q\}$. Then the map $f : \mathbb{Q} \to \mathbb{R}$ with rule $q \mapsto q^*$ is an injection which preserves the field operations $+, \times$ and the order $<$. Then $\mathbb{Q}^* = \{q^* : q \in \mathbb{Q}\}$ is a subfield of $\mathbb{R}$. □

**Theorem 3.** $\mathbb{R}$ satisfies the Axiom of Completeness.

**Proof.** Let $A \subset \mathbb{R}$ be a collection of cuts with upper bound $\beta$. Let $\gamma =$

$\bigcup_{\alpha \in A} A \subset \mathbb{Q}$. We first prove that $\gamma$ is a cut. $\gamma$ is trivially non-empty and not equal to $\mathbb{Q}$, because it must contain at least one cut, and because it is bounded by $\beta$ (and therefore properly contained in $\mathbb{R}$). Now, select some $p, q \in \alpha$, where $\alpha \in A$. If $q < p$, then $q \in \alpha$, and therefore $q \in \gamma$, so $\gamma$ is closed downwards. Now select $r \in \alpha$ such that $r > p$; it then follows that $r \in \gamma$ (since each $\alpha$ can not have a largest element). Hence, $\gamma$ is a Dedekind cut.

Now, we show that $\sup A = \gamma$. Assume $\delta < \gamma$. Then there is an $s \in \gamma$ such that $s \notin \delta$. Then $s$ must be contained in some $\alpha$, and thus $\delta < \alpha$. So, $\delta$ can not be an upper bound of $A$, and $\gamma = \sup A$ as a result. $\square$

**Definition 5.** The *infimum* $\inf(S) = \alpha \in A$ of a set $S \subset A$ is a lower bound of $S$, where for $\beta > \alpha$, $\beta$ is not a lower bound for $S$.

**Theorem 4.** *The Archimedean Property:* If $x, y \in \mathbb{R}$, with $x > 0$, then there exists some $n \in \mathbb{Z}^+$ such that $nx > y$.

**Proof.** Consider $A = \{nx : n \in \mathbb{Z}^+\}$. Let $y$ bound $A$. Then $A$ has a least upper bound $\sup A = \alpha$. It is clear that $\alpha - x$ is not an upper bound for $A$; so for some $m \in \mathbb{Z}^+$, $\alpha - x < mx$. But then $\alpha < mx + x = (m+1)x$, so $\alpha$ can not be an upper bound for $A$.

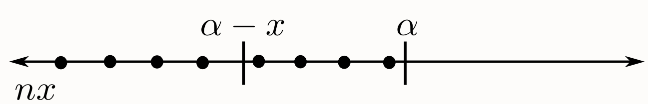


Figure 2: $A$'s boundings.

$\square$

**Theorem 5.** $\mathbb{Q}$ *is dense in* $\mathbb{R}$: For each $x, y \in \mathbb{R}$, there is some $q \in \mathbb{Q}$ such that $x < q < y$.

**Proof.** Choose $n$ such that $\frac{1}{n} < y - x$ (Archimedean property); consider multiples of $\frac{1}{n}$, $\frac{m}{n}$, which grow without bound. Choose the least $\frac{m}{n}$ such that $\frac{m}{n} > x$. We want to show that $\frac{m}{n} < y$; claim the contrary. Then $\frac{m}{n} \geq y$ and $\frac{m-1}{x} < x$. But then

$$-\frac{m-1}{n} > -x \Rightarrow -\frac{m-1+m}{n} = \frac{1}{n} > y - x,$$

a contradiction. $\square$

**Theorem 6.** *Properties of suprema:* For $\gamma = \sup A$,

1. $\gamma$ is an upper bound for some set $A$ if and only if $\sup A \leq \gamma$.
2. For all $a \in A$, $a \leq \gamma$, $\sup A \leq \gamma$.
3. For all $a \in A$, $a < \gamma$, $\sup A \leq \gamma$.
4. If $\gamma < \sup A$, then there exists an $a \in A$ such that $\gamma < a \leq \sup A$.
5. If $A \subset B$, then $\sup A \leq \sup B$ (i.e., for all $a \in A$, $a \in B$, so $a \leq \sup B$).
6. To show that $\sup A = \sup B$, show that for all $a \in A$, there exists a $b \in B$ such that $a \leq b$, so $\sup A \leq \sup B$. A similar process applies to showing that $\sup B \in \sup A$, which proves equivalence.