DM PROJECT

Combinatorial game thoery

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- So what is combinatorial game theory?

Combinatorial Game Theory is a branch of mathematics that deals with games of perfect information and no chance elements and involves the study of Combinatorial Games.

A combinatorial game is a game in which 2 players play such that they take turns making moves, their moves are not simultaneous

They have complete information about what has happened in the game so far and what each player’s options are for any particular position

We will be dealing with a specific kind of Combinatorial Game , those which satisfy all of below conditions:

Finiteness: This implies that the game will be guaranteed to eventually end because one of the players will eventually be unable to move and lose the match regardless of what they do.

Impartiality: This indicates that every player has an identical set of movements accessible to them from every position. Chess, for instance, is not an impartial game since only one player can move the white pieces, and the other player can move the black pieces.

Standard Play: This implies that the person who runs out of moves first determines who wins and loses. In conventional play, the loser is the first player to fail to make a move during their turn. Specifically, there are none.

3 PILE NIM GAME

. Each of the two players begins with three nim heaps.

- N and P Positions
- Game Graph
- Nim Table

N and P Positions

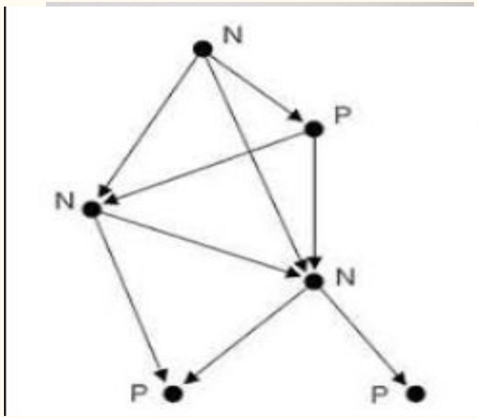
1. •**N-Position (Next Player Wins):** If the player about to make a move has a strategy that ensures they will win, regardless of what the other player does, we call that game position an "N-position." So, if you're the next player and you play smart, you're guaranteed to win.

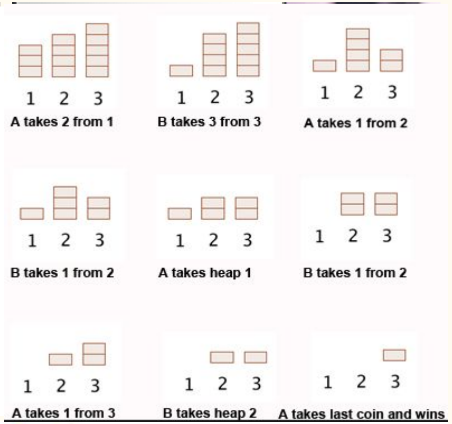
2) •**P-Position (Previous Player Wins):** If the player who just made a move has a winning strategy, meaning they can force a win no matter what their opponent does, we call that game position a "P-position." So, if you've just made a move and you have a good strategy, you're in a winning position.

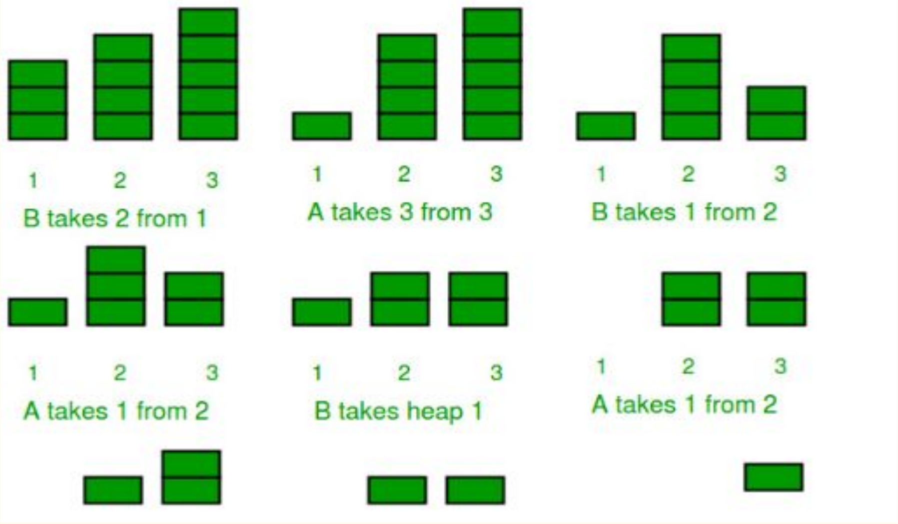
N and P Position

- In simpler terms, N-positions are good for the player about to move because they can win, and P-positions are good for the player who just moved because they have a winning strategy. The goal is to understand these positions to make the best moves and increase your chances of winning the game.

- N and P position is part of combinatorial game strategies.

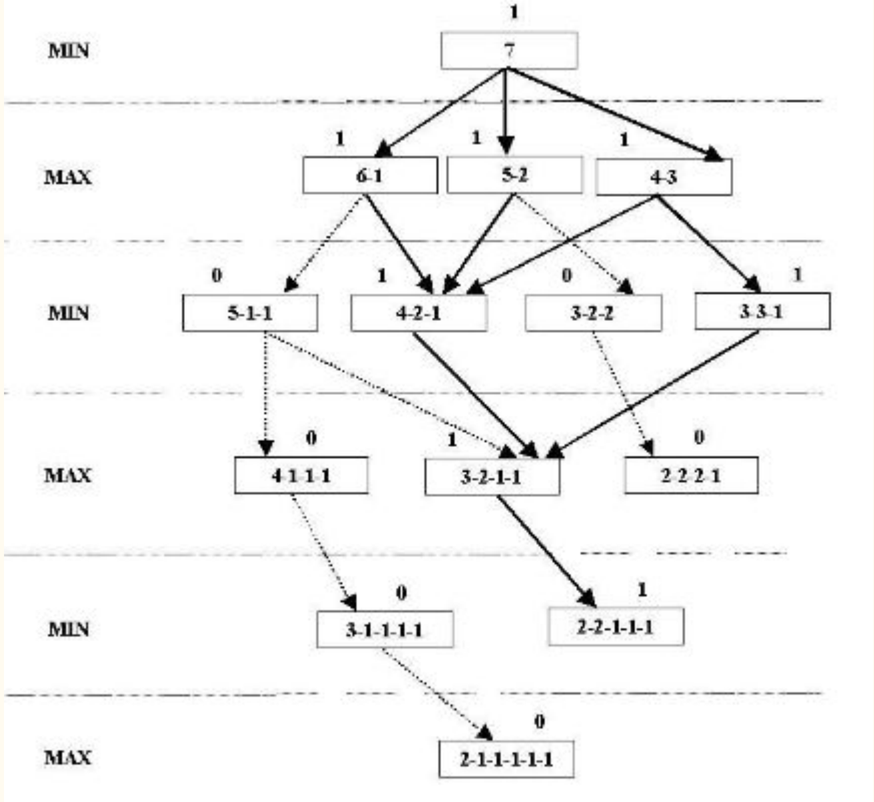






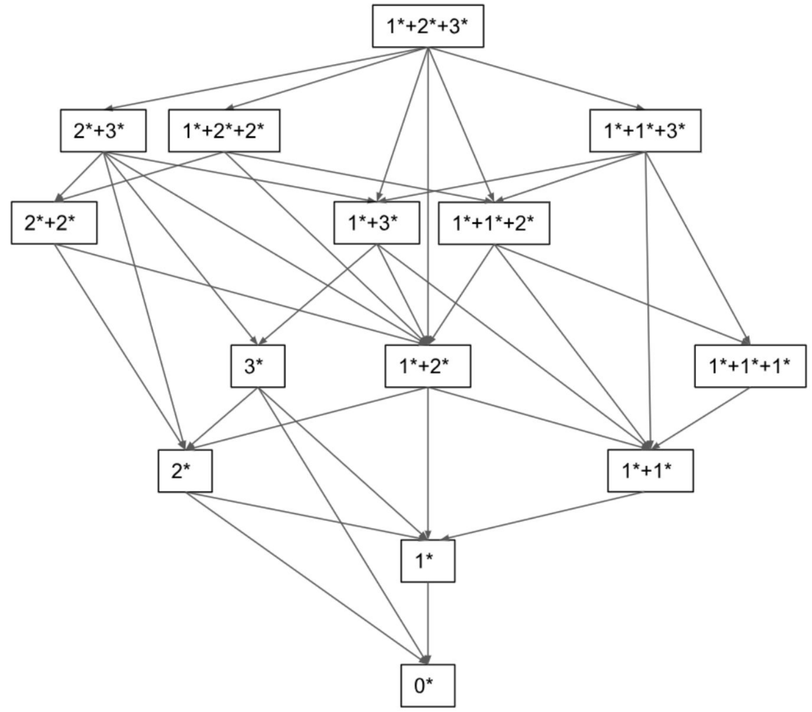
Game Graph

A game graph is a graphical depiction of a game. It is a directed graph where the game's potential positions are represented by the graph's vertices, and moves between those positions are represented by its edges.



Nim Table

The Nim table, a crucial tool in understanding Nim strategies, is constructed by analysing P-positions and N-positions. Below graph for a 3 pile Num visualizes all the P- and N-positions but can also make things complicated. Hence, to simplify the process, we define the theorem:



Mathematical work

Theorem 1.1:

For any two numbers X and Y , there exists at most one number Z such that X*+Y *+Z * is a P-position.

Proof

- Let us take two P- positions X*+Y*+Z* and X*+Y*+Z'* (Z>Z').
- If we take Z-Z' stones from the former pile, we will end up with the latter pile. However, there cannot be a move from a P- position to another P-position. This is because in a P-position, the previous player has a winning strategy. This implies that regardless of what the next player does, they can always make a move that will ultimately result in their victory.

Theorem 1.2:

For any position X in any FISP game, X+X is always a P -position.

Proof

By copying her opponent's movements, the second player can always win. For instance, if her opponent plays in the first copy of g and moves to g0 + g, she can make the same move in the second copy of g and move to g0 + g0, . With this tactic, the second player can copy her opponent's moves and always be able to make a legitimate move in return. As a result, her opponent will have to be the one to run out of moves and lose the game.

Using Theorem 1.2, we can say that X*+X* is a P -position for all X. This can also be interpreted as X* + X* + 0*. Thus, for all X,

$$[X, X] = 0, [X, 0] = [0, X] = X$$

By analyzing Figure 1.1, we can clearly show that 1* + 2* + 3* is a P-position. Thus:

$$[1, 2] = [2, 1] = 3, [2, 3] = [3, 2] = 1,$$
$$[1, 3] = [3, 1] = 2.$$

Conclusion

An interactive version of understanding the Nim Table can be found here: https://demonstrations.wolfram.com/WinningOrLosingInTheGameOfNimOnGraphs/

In order to expand our table, we will have to make use of Grundy numbers which are calculated by MEX Rule

	0	1	2	3	4	5	6	7	8	...
0	0	1	2	3	4	5	6	7	8	...
1	1	0	3	2						...
2	2	3	0	1						...
3	3	2	1	0						...
4	4				0					...
5	5					0				...
6	6						0			...
7	7							0		...
8	8								0	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Theorem 1.3:

For all integers $X, Y > 0$

$M = [X, Y] = \mathrm{MEX}(\{[X',Y] : X'<X\} \cup \{[X,Y']:Y'<Y\})$

is a P -position

Proof and delving into MEX

Definition: *MEX (or "minimum excluded") is the smallest possible non negative integer among a set of non negative integers (let it be S)*
For example, M(0, 1, 2, 3) is 4 | M(1,2,3,4,5) is 0.

Proof: To prove this, consider the player's possible moves:

a. Moves to $X + Y1 + M^*$, where $Y1<Y$. By definition of MEX, M is not equal to [X, Y1]. Hence, this is not a P but an N position. Same holds in the case move $X1 + Y + M^*$ is made where $X1<X$.
b. By travelling to $X^* + Y^* + M1$ with $M1 < M$, then by MEX definition, M1 is in $\{[X0, Y] : X0 < X\} \wedge \{[X, Y0] : Y0 < Y\}$.
   Assume that $M1 = [X1, Y]$ and that $X1 < X$ WLOG. This indicates that $X + Y^* + M^*$ must be an N-position since $X1 + Y^* + M1$ is a P-position.

Since all paths point to next position being a N position, therefore current position is a P -position. **Q.E.D .**

Contd.

Now if we go ahead constructing the table, we can easily do it with MEX rule. Like if we want to find Nim(1,4); first we see the possible moves which are [0,4], [0, 3], [0, 2], [0, 1]. Applying MEX rule to {0,1,2,3,4} we get 5. Hence, [1,4] = 5. Using this rule rest of the Nim table can be constructed as:

	0	1	2	3	4	5	6	7	8
0	0	1	2	3	4	5	6	7	8
1	1	0	3	2	5	4	7	6	9
2	2	3	0	1	6	7	4	5	10
3	3	2	1	0	7	6	5	4	11
4	4	5	6	7	0	1	2	3	12
5	5	4	7	6	1	0	3	2	13
6	6	7	4	5	2	3	0	1	14
7	7	6	5	4	3	2	1	0	15
8	8	9	10	11	12	13	14	15	0

Sprague-Grundy Theorem

The Sprague-Grundy Theorem, a cornerstone of CGT, provides a robust framework for analysing impartial games. This theorem facilitates determining winning strategies and outcomes by breaking down a game into its subgames. It provides a methodological framework for analysing impartial games, lacking chance elements and where both players have equal opportunities for moves from any given position.

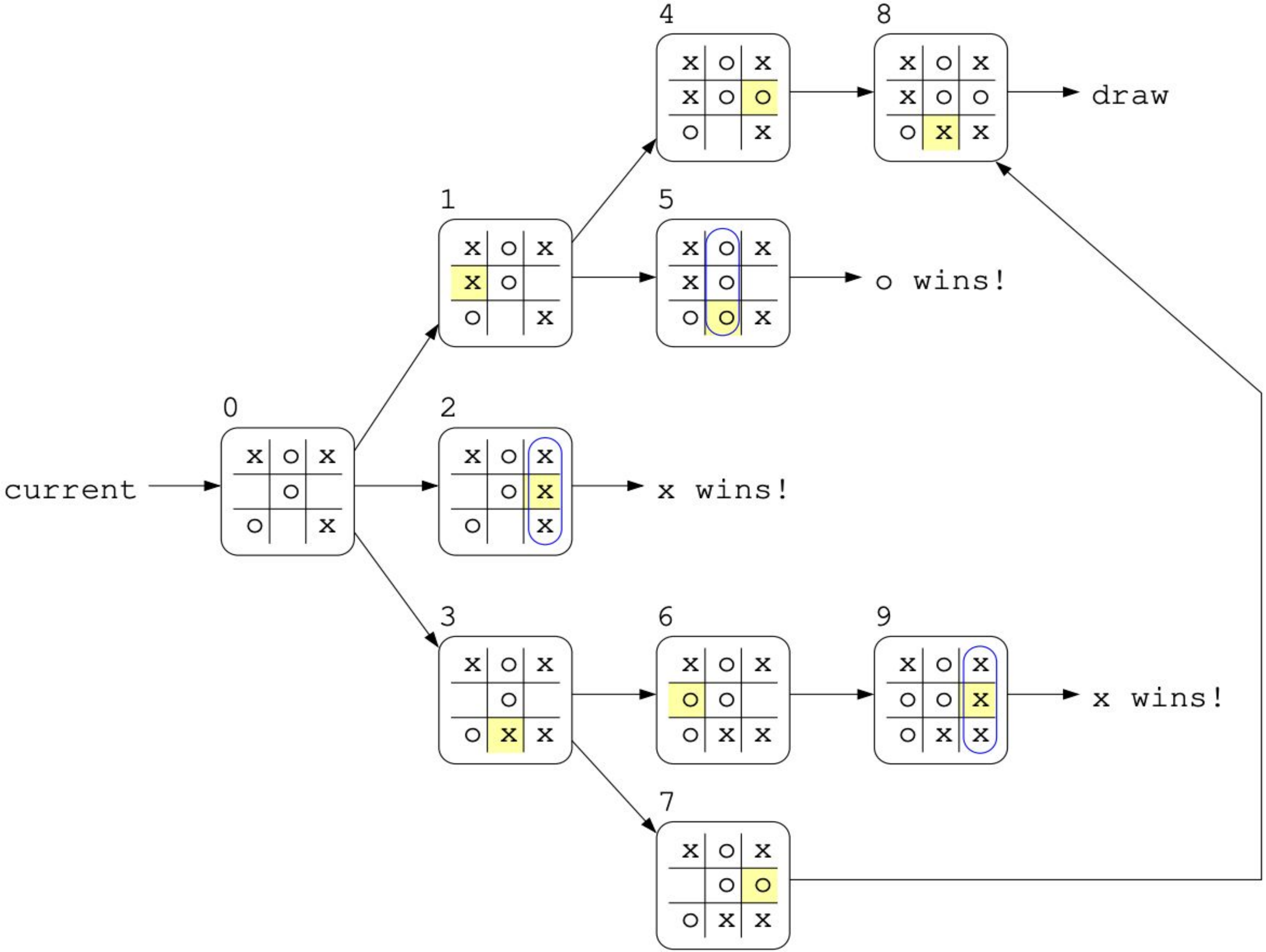
- Analysing Game Positions:
- Impartial Game Analysis:
- Subgames and Winning Strategies:
- Practical Application:
- Mathematical Rigour:

- conclusion:In conclusion, the SGT serves as a cornerstone in the study of impartial games, offering a systematic method to determine the winning strategies and outcomes

TIC -TAC -TOE

Tic-Tac-Toe is a 2-player combinatorial game played on a 3X3 grid. Each player alternately marks the grid's squares with either the X or the O symbol. The winner of the game is the first person to mark three of their symbols in a row, either horizontally, vertically, or diagonally.
Positions in the game can be represented by constructing a 3X3 matrix where each element of the matrix is either “O” or “X” or empty.

This game satisfies the conditions of Finiteness, Impartiality and Standard Play:\
Now that we have established that this game is a FISP combinatorial game, we can go ahead with applying the concepts of combinatorial game theory on it.


current
0
1
2
3
4
5
6
7
8
9
draw
o wins!
x wins!
x wins!

Practical Implementation: We have implemented the game for a practical understanding of the game and underlying concepts using C programming language. Code is attached along with the submission.

Conclusion: In summary, this research has provided an in-depth exploration of the complex field of combinatorial game theory, with a focus on impartial games and their real-world applications. The Python implementation illustrated the practical applicability of CGT, while the theoretical investigation offered a thorough grasp of basic ideas.

CODE FOR TIC-TAC-TOE

main.c

Save

Run

```c
#include <stdio.h>

// Function to print the Tic Tac Toe board
void printBoard(const char board[3][3]) {
    int i = 0;
    while (i < 3) {
        int j = 0;
        while (j < 3) {
            printf("%c", board[i][j]);
            if (j < 2) printf(" | ");  // Add vertical separators
            j++;
        }
        printf("\n");

        if (i < 2) printf("---------\n");  // Add horizontal separators
        i++;
    }
    printf("\n");
}

// Function to check if a player has won
int check(const char board[3][3], char player) {
    int i = 0;
    while (i < 3) {
    int tempp = i++;
    if ((board[tempp][1] == player && board[tempp][0] == player) ||
        (board[1][tempp] == player && board[0][tempp] == player &&
            board[2][tempp] == player)) {
```

Output

Clear

```
/tmp/2SSZQqtrxm.o
Welcome to the game! Let's begin.
Player 1 (X), enter your name: Sumit
Player 2 (O), enter your name: Aditya
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  |   | 

Player Sumit, enter your move (row and column): 0 0
X |   | 
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  |   | 
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  |   | 

Player Aditya, enter your move (row and column): 1 1
X |   | 
---------
  | O | 
---------
  |   | 

Player Sumit, enter your move (row and column): 2 0
X |   | 
---------
  | O | 
```

THANK YOU

ANY QUESTIONS?