<u>Combinatorial Game Theory</u>

Course Project - Discrete Mathematics

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Abstract
Combinatorial Game Theory (CGT) is a field that blends math and logical thinking to analyze and solve a number of combinatorial games. This project aims to explore the theoretical underpinnings of CGT, with a particular emphasis on impartial games and their connection to the Sprague-Grundy theorem. Additionally, the project involves the practical implementation of these theoretical insights through C programs, simulating and solving various impartial games. This not only enhances our understanding of CGT but also shows applications of discrete mathematics in the realm of strategic decision-making.

Introduction
Combinatorial Game Theory is a branch of mathematics that deals with games of perfect information and no chance elements and involves the study of Combinatorial Games.
A combinatorial game is a game in which 2 players play such that

- They take turns making moves, their moves are not simultaneous
- They have complete information about what has happened in the game so far and what each player's options are for any particular position.

Some examples of popular combinatorial games that come to mind are Tic-Tac-Toe, Chess, and Go. All these games have 2 players that have their own set of moves, and the game ends when either one of the players has run out of successful moves to run.
Rock-Paper-Scissors is, however, NOT a combinatorial game since the 2 players are moving simultaneously.

Literature Review:
The existing literature on CGT provides a comprehensive foundation for our project. Numerous scholars have explored impartial games, defining their characteristics and establishing the mathematical frameworks for analyzing their outcomes. The concept of Grundy numbers, or numbers, has been extensively studied, offering insights into the determination of winning strategies. The Sprague-Grundy theorem, a cornerstone of CGT, has been a focal point in understanding the structure of impartial games. Classic examples like Nim serve as benchmarks for illustrating the practical applications of CGT principles.

The Topic of Study

Theoretical Explanation:

In this report, we will focus on a specific type of combinatorial game, having the following characteristics

1. **Finiteness:** This implies that the game will be guaranteed to eventually end because one of the players will eventually be unable to move and lose the match regardless of what they do.
2. **Impartiality:** This indicates that every player has an identical set of movements accessible to them from every position. Chess, for instance, is not an impartial game since only one player can move the white pieces, and the other player can move the black pieces.
3. **Standard Play:** This implies that the person who runs out of moves first determines who wins and loses. In conventional play, the loser is the first player to fail to make a move during their turn. Specifically, there are none.

We will call such combinatorial games satisfying the above 3 conditions as "FISP" games.

I. NIM

3-pile Nim is the most widely used version of Nim. Each of the two players begins with three stone piles(also called Nim Heaps). A player must remove a nonzero number of stones from any one pile throughout her turn. (Specifically, she is permitted to take every stone out of a pile.) As usual, the last stone to fall wins; the person who is unable to move loses.

Nim meets all three requirements for a combinatorial game:

- Standard play: Whoever runs out of moves first determines who wins and who loses.
- Finiteness: Since there are only a certain number of things left to eliminate, the game will always come to an end.
- Impartiality: Every player has the same set of moves at their disposal from every position.

N and P Positions

- The position is regarded as an N-position (or "has status N") if the player with the next move (the "Next" player) has a winning strategy. The next player always has higher chances of winning no matter what.
- A position is said to be in a P-position (or to "have status P") when the player whose turn finished just now (the Previous player) has got a winning strategy. (If there hasn't been a prior move and the game is just getting started, the person who goes second is the prior player.)

Game Graph: A game graph is a graphical depiction of a game. It is a directed graph where the game's potential positions are represented by the graph's vertices, and moves between those positions are represented by its edges.

Below is an implementation of the game graph for Nim, also called a Nim table.

Nim Table

The Nim table, a crucial tool in understanding Nim strategies, is constructed by analysing P-positions and N-positions. Below graph for a 3 pile Num visualizes all the P- and N-positions but can also make things complicated. Hence, to simplify the process, we define the theorem:

<u>Theorem 1.1:</u> For any two numbers X and Y , there exists at most one number Z such that $X*+Y*+Z*$ is a P-position.

<u>Proof:</u>

- Let us take two P- positions $X*+Y*+Z*$ and $X*+Y*+Z'*$ ($Z>Z'$).
- If we take Z-Z’ stones from the former pile, we will end up with the latter pile. However, there cannot be a move from a P- position to another P-position. This is because in a P-position, the previous player has a winning strategy. This implies that regardless of what the next player does, they can always make a move that will ultimately result in their victory.

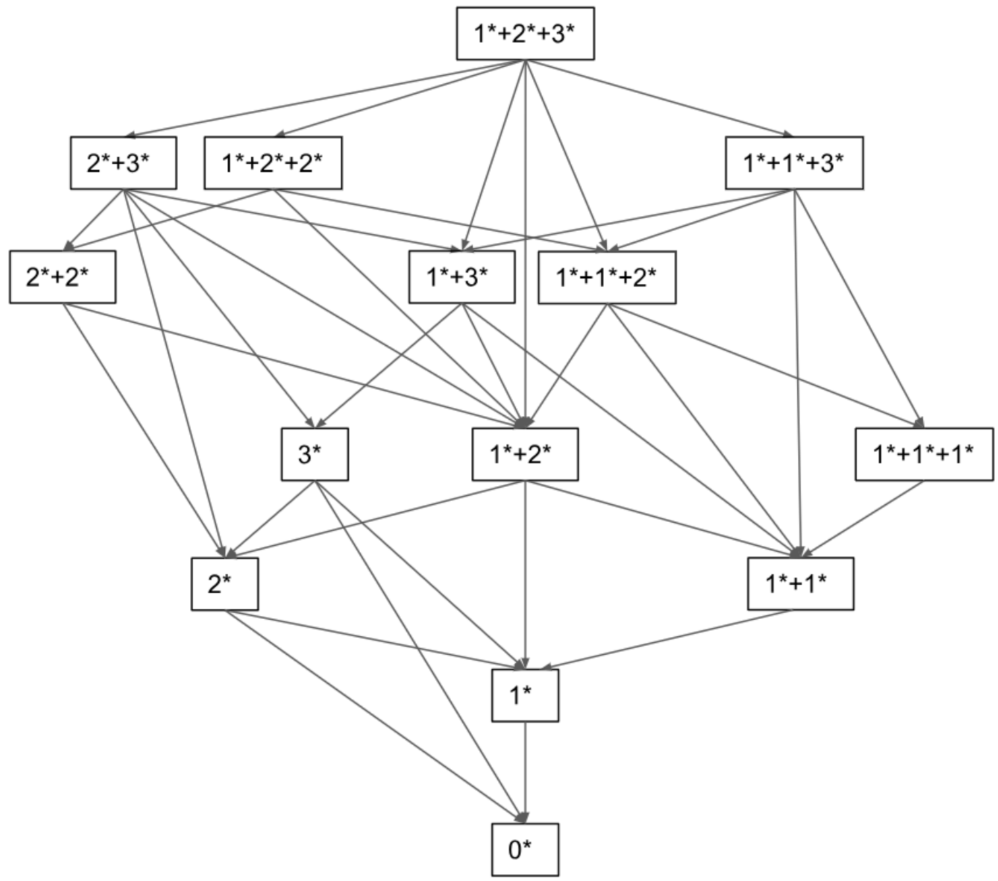

Fig 1.1: Nim Table for 1 + 2* + 3**

Now that we have established that we can have only one unique Z for a given X and Y. Hence, we cen try constructing a table where the entry in row X and column Y is the unique value of Z such that X* + Y* + Z* is a P- position.

Theorem 1.2: For any position X in any FISP game, X+X is always a P -position.
Proof: By copying her opponent's movements, the second player can always win. For instance, if her opponent plays in the first copy of g and moves to g0 + g, she can make the same move in the second copy of g and move to g0 + g0, . With this tactic, the second player can copy her opponent's moves and always be able to make a legitimate move in return. As a result, her opponent will have to be the one to run out of moves and lose the game.

Using Theorem 1.2, we can say that X*+X* is a P -position for all X. This can also be interpreted as X* + X* + 0*. Thus, for all X,

$$[X, X] = 0, [X, 0] = [0, X] = X$$

By analyzing Figure 1.1, we can clearly show that 1∗ + 2∗ + 3∗ is a P-position. Thus:

[1, 2] = [2, 1] = 3, [2, 3] = [3, 2] = 1, [1, 3] = [3, 1] = 2.

From the given knowledge, we can start constructing the table:

	0	1	2	3	4	5	6	7	8	...
0	0	1	2	3	4	5	6	7	8	...
1	1	0	3	2						...
2	2	3	0	1						...
3	3	2	1	0						...
4	4				0					...
5	5					0				...
6	6						0			...
7	7							0		...
8	8								0	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Fig 1.2: Table derived from the Nim table denoting P -positions

Essentially, value of any row and column along with their value mapped on the table collectively represent a P value.

An interactive version of understanding the Nim Table can be found here:

https://demonstrations.wolfram.com/WinningOrLosingInTheGameOfNimOnGraphs/

In order to expand our table, we will have to make use of Grundy numbers which are calculated by MEX Rule:

Definition: *MEX (or "minimum excluded") is the smallest possible non negative integer among a set of non negative integers (let it be S)*

For example, M(0, 1, 2, 3) is 4 | M(1,2,3,4,5) is 0.

Theorem 1.3 (MEX Rule): For all integers X, Y > 0

$$M = [X, Y] = MEX(\{[X',Y] : X'<X\} \cup \{[X,Y']:Y'<Y\})$$

is a P -position

Proof: To prove this, consider the player's possible moves:

a. Moves to X + Y1 + M*, where Y1<Y. By definition of MEX, M is not equal to [X, Y1]. Hence, this is not a P but an N position. Same holds in the case move X1 + Y + M* is made where X1<X.

b. By travelling to X* + Y* + M1 with M1 < M, then by MEX definition, M1 is in {[X0, Y] : X0 < X} ∧ {[X, Y0] : Y0 < Y}.
Assume that M1 = [X1, Y] and that X1 < X WLOG. This indicates that X + Y* + M* must be an N-position since X1 + Y* + M1 is a P-position.

Since all paths point to next position being a N position, therefore current position is a P -position. **Q.E.D .**

Now if we go ahead constructing the table, we can easily do it with MEX rule. Like if we want to find Nim(1,4); first we see the possible moves which are [0,4], [0, 3], [0, 2], [0, 1]. Applying MEX rule to {0,1,2,3,4} we get 5. Hence, [1,4] = 5.
Using this rule rest of the Nim table can be constructed as:

	0	1	2	3	4	5	6	7	8
0	0	1	2	3	4	5	6	7	8
1	1	0	3	2	5	4	7	6	9
2	2	3	0	1	6	7	4	5	10
3	3	2	1	0	7	6	5	4	11
4	4	5	6	7	0	1	2	3	12
5	5	4	7	6	1	0	3	2	13
6	6	7	4	5	2	3	0	1	14
7	7	6	5	4	3	2	1	0	15
8	8	9	10	11	12	13	14	15	0

II. Tic-Tac-Toe

Tic-Tac-Toe is a 2-player combinatorial game played on a 3X3 grid. Each player alternately marks the grid's squares with either the X or the O symbol. The winner of the game is the first person to mark three of their symbols in a row, either horizontally, vertically, or diagonally. Positions in the game can be represented by constructing a 3X3 matrix where each element of the matrix is either "O" or "X" or empty.

This game satisfies the conditions of Finiteness, Impartiality and Standard Play:

1. Finiteness: Since the board is played on a 3X3 grid, there are only a finite number of valid positions. Also, since each player only haw a finite number of moves on them (can only play either an X or O), hence, the game must eventually end.
2. Impartiality: At any given positions, each player has the same set of moves with them, rendering them moot on any inherent advantage, making this game completely impartial.
3. Standard Play: The game has a well defined instructions for playing as well as for identifying the winner, with no room for any ambiguity. Hence, this game is an example of Standard Play.

Now that we have established that this game is a FISP combinatorial game, we can go ahead with applying the concepts of combinatorial game theory on it.

Below position denotes a starting position in the game, when not moves have taken place:

```
[   [``   ``   ``],
    [``   ``   ``],
    [``   ``   ``]   ]
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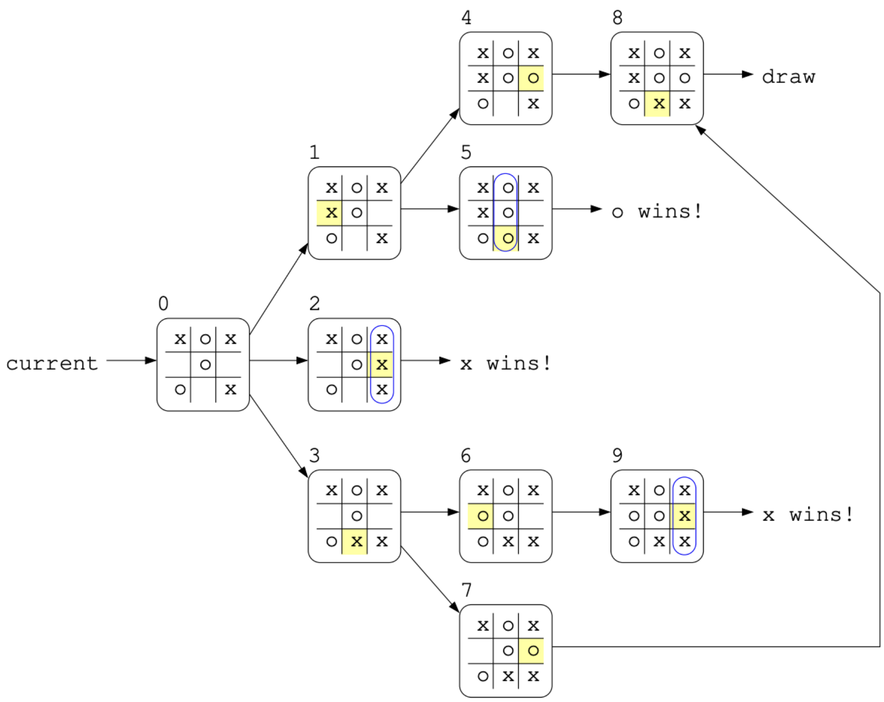

Fig 1.3: The game graph for Tic-Tac-Toe, with P and N -positions indicated.

Practical Implementation: We have implemented the game for a more practical understanding of the game and its workings using C programming language. Code is attached along with the submission.

Conclusion: To sum up, this study has been a thorough exploration of the complex field of combinatorial game theory, with a focus on impartial games and their real-world applications. The C implementation illustrated the practical applicability of CGT, while the theoretical investigation offered a thorough grasp of basic ideas. CGT offers a strong toolkit for making strategic decisions in a variety of industries. Subsequent research endeavours may broaden the scope of this study to encompass intricate games, investigate modifications of current ones, or investigate the use of CGT in nascent technologies like artificial intelligence.

References:

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