Computer Systems in Engineering

Software Develpoment for Industrial Robots

Underlying mathematics

1. Vector spaces and matrices
2. Coordinate systems
3. Homogenouos coordinates
4. Robot kinematics
5. Inverse kinematics

1. Vector spaces

- Definition: Vector space
  A vector space V (over a field K) is a tripel (V,+,.), where
  - V and "+" is an Abelian group and
  - "." is an associative, distributive Operation with
    $$K \times V \rightarrow V$$

- „+" is called vector addition and „." is called multiplication with a scalar.

- Definition: Vector
  Elments of V are called vectors.

Examples

- Simple example:
  - $V=R^n$ with
    - "+" = addition by component-wise addition
    - " ." = multiplication of all components with a scalar

- Vectors are often interpreted in one of the two following, <u>different</u> geometric meanings:
  - as descriptor of a location (ger. Ortsvektor)

    OR

  - as descriptor of a translation

Linear combination

- Definition: Linear combination

  For a set of vectors v_i, all vectors of the form $w_1 v_1 + \ldots + w_n v_n$ with scalars (of K) w_i are called linear combinations (of the set of vectors v_i).

- Definition: Lineare (in-)dependence

  A set of vectors v_i is called linear dependent, if there exist scalars w_i (not ALL equal to 0) such that $w_1 v_1 + \ldots + w_n v_n = 0$.

Basis

- Definition: Generating system

  A subset E of a vector space V is called generating system, if every v of V is a linear combination of E.

- Definition: Basis

  A linear independent, generating system E is called a basis of the vector space.

Inner product / Scalar product

- Definition: Inner product
  The mapping

$$V \times V \to R$$

$$\begin{pmatrix} v_1 \\ \\ v_n \end{pmatrix} \bullet \begin{pmatrix} w_1 \\ \\ w_n \end{pmatrix} = v_1 w_1 + \ldots + v_n w_n$$

  is called inner product.

- Definition: Absolute value

  The term $\sqrt{v \bullet v}$ is called (absolute) value of the vector v.

2. Coordinate systems

- Definition: Coordinate system

  A tupel CS=(o, E) of some vector o and a basis E is called a coordinate system. The vectors of E are called basis vectors.

- Definition: Coordinate vector

  The representation of a vector v by the vector of it‘s linear combination (according to E)

  $$v = o + w_1 e_1 + \ldots + w_n e_n$$

  is called coordinate vector (of v according to CS).

Orthonormal coordinate system

- Definition: Orthogonal coordinate system

- A coordinate system is called orthogonal, iff the inner product of each pair of basis vectors equals 0.

- Geometrical meaning Euclidian space:
  90° angles between basis vectors

- Definition: Orthonormal coordinate system

- An orthogonal CS is called orthonormal, if the absolute value of all basis vectors equals 1.

Transformations of coordinate systems

- In general:

  Any coordinate system CS�much may be defined wrt. a coordinate system CS by definition of the origin o‵ and each basis vector of CS‵ as coordinate vectors of CS.

- From now on:
  We will only consider orthonormal, right-handed coordinate systems in Euclidian space.

- Geometric possibilities:
  - Translations
    2D: in two directions
    3D: in three directions
  - Rotations
    2D: around one axis
    3D: around three axis

- Important observation:
  - Any combination of two transformations may be expressed as a single transformation!

Mathematical model

- Translation by v:

$$T_o : V \to V$$

$$v \mapsto o + v$$

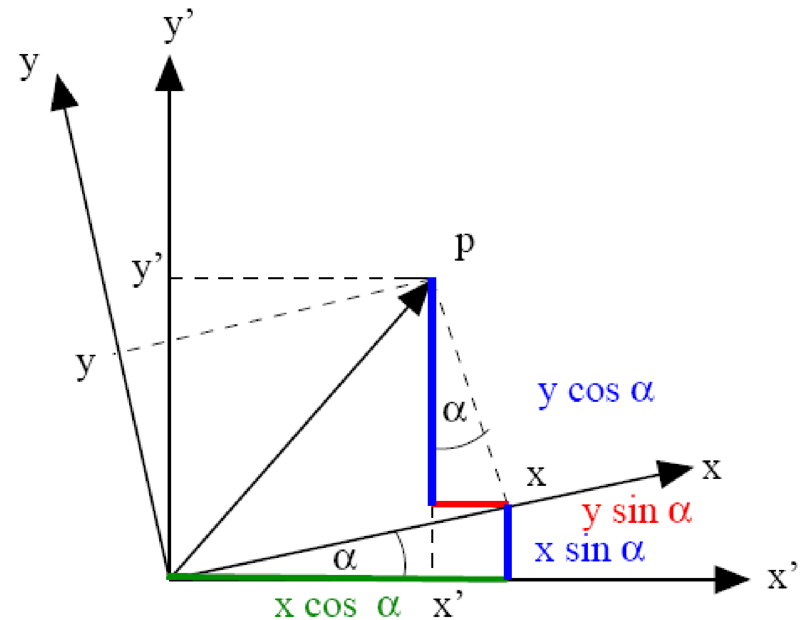
Mathematical model

- Rotation by α (around origin of CS):

$$R_\alpha : V \to V$$

$$v = \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x * \cos\alpha - y * \sin\alpha \\ x * \sin\alpha + y * \cos\alpha \end{pmatrix}$$

- Proof:
  Making use of addition theorems of triangular functions



Remember: matrices

- Definition: Matrix

  A n x m matrix is a n-vector of m (row-) vectors.

$$A = \begin{pmatrix} a_1 \\ \dots \\ a_n \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} a_{11} & \dots & a_{1n} \end{pmatrix} \\ \dots \\ \begin{pmatrix} a_{n1} & \dots & a_{nn} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & & \dots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$$

Multiplying matrices

- Definition: Multiplication of a matrices

  Two matrices A and B are multiplied by all inner products between all (row-)vectors of A and all (column) vectors of B.

$$A \circ B = \begin{pmatrix} \vec{a}_1{}^T \\ \dots \\ \vec{a}_n{}^T \end{pmatrix} \begin{pmatrix} \vec{b}_1 & \dots & \vec{b}_n \end{pmatrix} = \begin{pmatrix} \vec{a}_1 \bullet \vec{b}_1 & \dots & \vec{a}_1 \bullet \vec{b}_n \\ \dots & & \dots \\ \vec{a}_n \bullet \vec{b}_1 & \dots & \vec{a}_n \bullet \vec{b}_n \end{pmatrix}$$

Inverse matrix

- Definition: Inverse matrix
  For some matrix A the matrix A^{-1} is called inverse (matrix) of A, iff for all v in V:

$$A^{-1}\left(Av\right)=v$$

- Remark:

  The matrix multiplication $A^{-1}A$ is diagonal matrix (with only „1“ on the diagonal). This matrix is called identity or 1-matrix.

Some things to know

- Matrix multiplication is (usually) not commutative

$$A \circ B \neq B \circ A$$

- In general, matrices may not be inverted

- Examples: YOUR turn!

Orthonormale matrices

- Definition: Orthonormale matrix
  A matrix whose row- or column vectors are pairwise orthonormal is called orthonormal.

- Theorem: Inverse of an orthonormale matrix
  The inverse of an orthonormal matrix is the transponed matrix.

- **Remark:**
  The transponed matrix is generated by swapping row- and column indices.

Corresponding mapping

- Definition: Corresponding mapping
  For some (n x n) matrix and some n-dimensional vector space V

$$f_A : V \to V$$

$$v \mapsto Av = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & & \dots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} v_1 \\ \dots \\ v_n \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{n} a_{1i} v_i \\ \dots \\ \sum_{i=1}^{n} a_{ni} v_i \end{pmatrix}$$

defines a mapping from V to V.

Rotation by matrix operations

- Rotations may be easyly described by matrix mappings

- Matrix for a rotation:

$$A_\alpha := \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

- Corresponding mapping:

$$R_\alpha : V \to V$$

$$v \mapsto A_\alpha v = \begin{pmatrix} v_1 * \cos\alpha - v_2 * \sin\alpha \\ v_1 * \sin\alpha + v_2 * \cos\alpha \end{pmatrix}$$

Matrices for rotation in 3D

- Rotation x-axis:

$$A_{\alpha,x} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}$$

- Rotation y-axis:

$$A_{\alpha,y} := \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix}$$

- Rotation z-axis:

$$A_{\alpha,z} := \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Translation as a matrix operation

- Can you describe a translation as a matrix operation?
- Answer: No
- Proof by contradiction:
  For all matrix mapping, the following holds
  $$A(v+w)=Av + Aw$$

  However, for translation this does never hold (except for translation by 0 vector)

Summary

- Situation:
  Orthogonal matrices allows very easy representation of rotations.

- Problem:
  Translation may not be represented by matrix operations!

- Solution:
  Embedded vector space in high-dimensional vector space.