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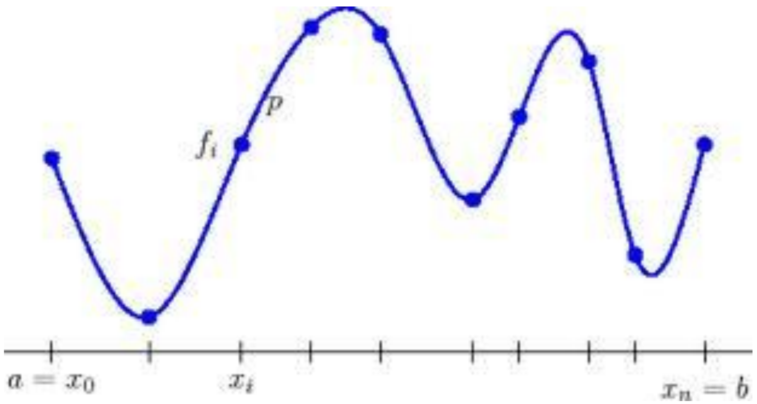
Programming Task 2: Spline Movement



Softwaredevelopment for Industrial Robotics

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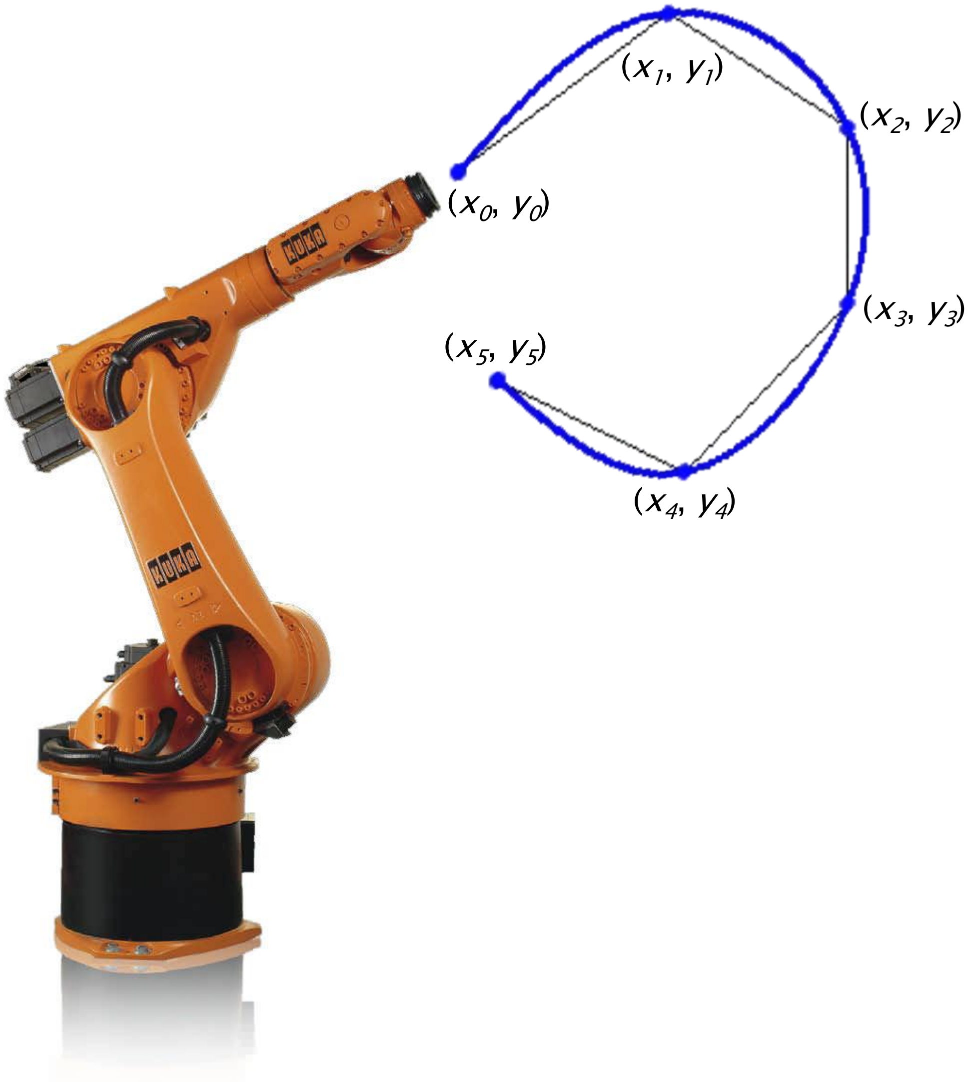

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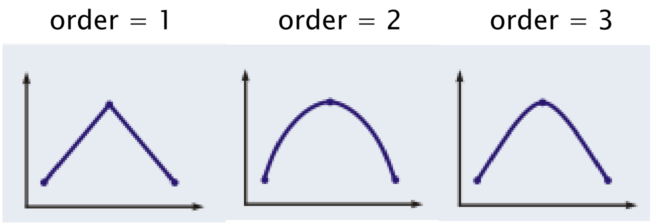
1. Fundamentals – What are splines?

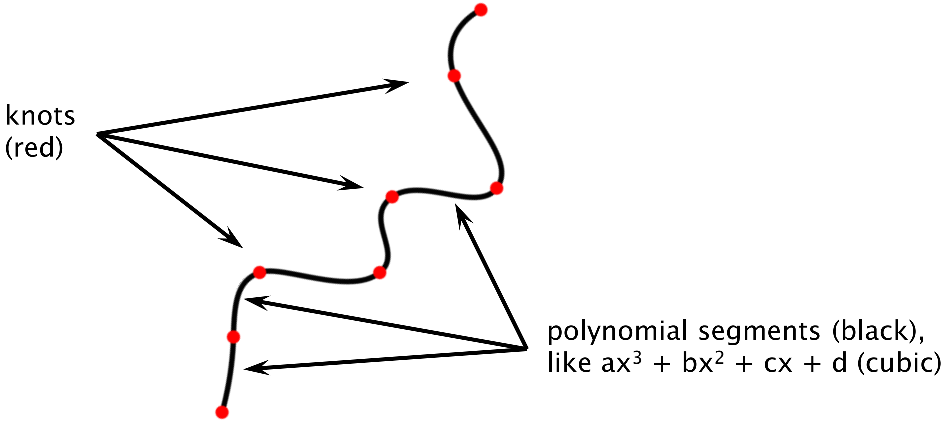
Origin:

- First reference in 1946 [1]
- Suitable to describe curved lines → CAD (curves & freeform surfaces)
- i.e. rollercoaster, automobile and shipbuilding, high-speed railways

Definition:

- *k* knots[1] as base
- *k-1* continuous, smooth[2], polynomial segments of degree[3] *n*
- commonly used splines with degree=3 (cubic) → ax^3+bx^2+cx+d





[1] knot = Knotenpunkt, o. Stützpunkt
[2] order = Grad
[3] smooth = stetig

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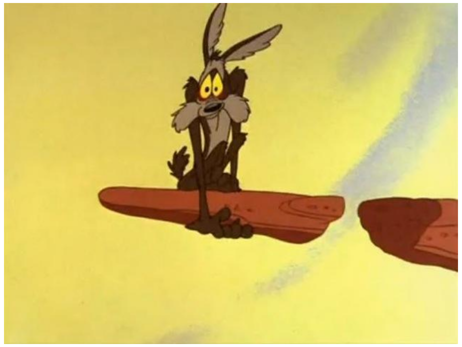
1. Fundamentals – Splines have to be…



…continuous:

- Function plot does not jump

(to get an ongoing spline)

…smooth:

- every argument is calculable

(no infinity[1] /overshooting)

…differentiable:

- need for calculating the spline conditions

[1] Unendlichkeit

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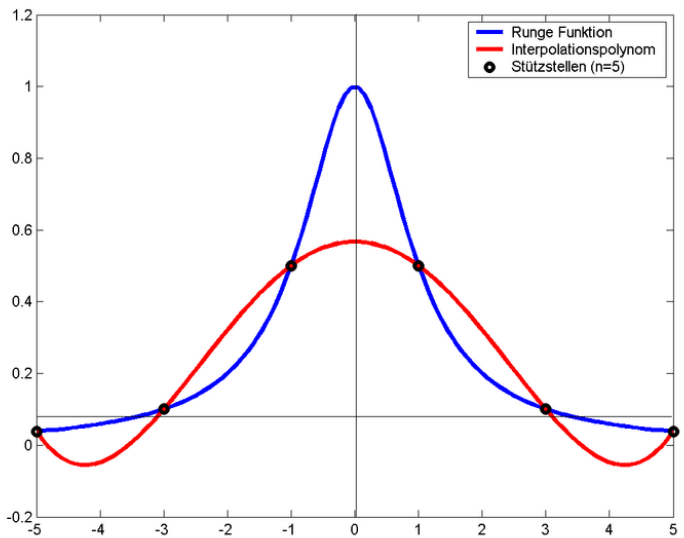
1. Fundamentals – Why Splines?

Typical Problem: Set of data points

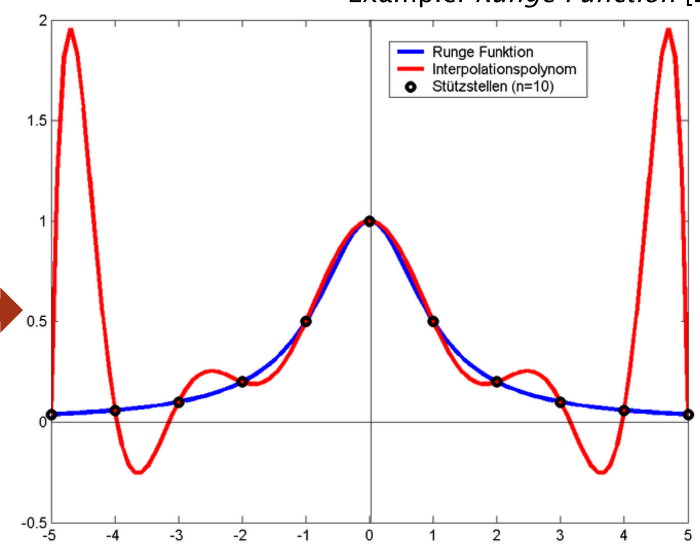
→ ad hoc approach: one (global) polynomial passing every point

But: Increasing number of points means increasing error at interval boundary, too.

Example: *Runge Function* [2]



Increased number of data points

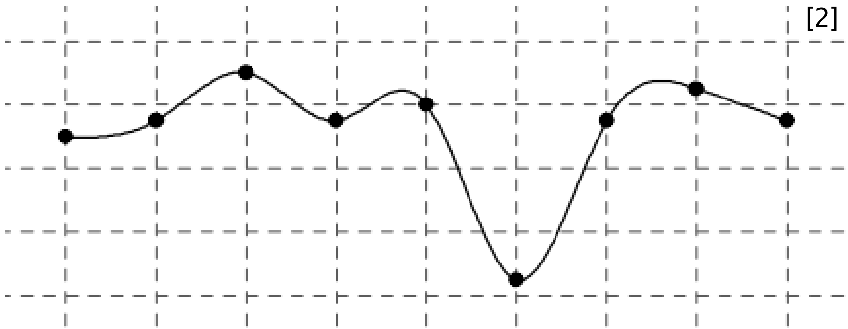


→ better use another interpolation technique, like i.e. spline interpolation

1. Fundamentals – Splines can…

<u>…interpolate the given input points:</u>

- The given input points are part of the spline.

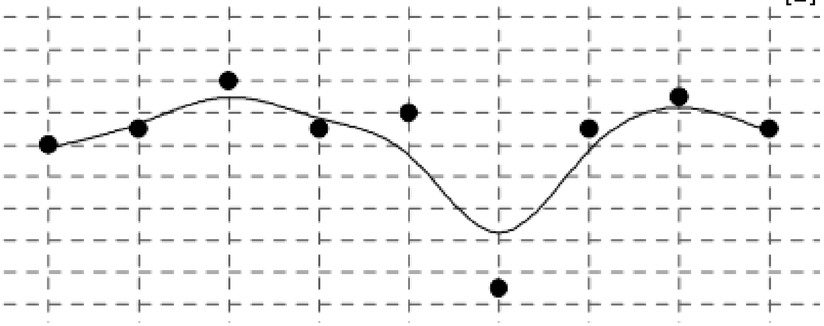


[2]

Inpolation with a cubic spline.

<u>…approximate the given input points:</u>

- The given input points are **not** part of the spline (i.e. *b-spline*).



[2]

Approximation with a cubic b-spline.

[1] unendlich

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2. Spline Approximation – B-Spline

Approach:

Linear combination of all segments:

$$\boxed{\mathbf{x}(t) = \sum_{i=0}^{n} N_{i,k}(t) \cdot \mathbf{d}_i} \tag{1}$$

With:

- Given: n+1 control points $\mathbf{D} = (d_0 \ldots d_n)$ (so called *de Boor points*)
- Basis function $\mathbf{N}_{i,k}(t)$
- Order **k** (degree = k-1)
- Time variable **t** represents the position on the spline
- Knot vector $\mathbf{t}_i = (t_0, \ldots, t_n, \ldots t_{n+k})$ (mathematical construct)

Feature:

- De Boor points influence spline´s shape

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2. Spline Approximation – B-Spline

Single polynomial:

- Calculated with recursive equation
- Order **k** is a sum from 2 splines with Order **k-1**
- recursive definition:

(2)

$$N_{i,1}(t) = \begin{cases} 1 & \text{für} \quad t_i \le t < t_{i+1} \\ 0 & \text{sonst} \end{cases}$$

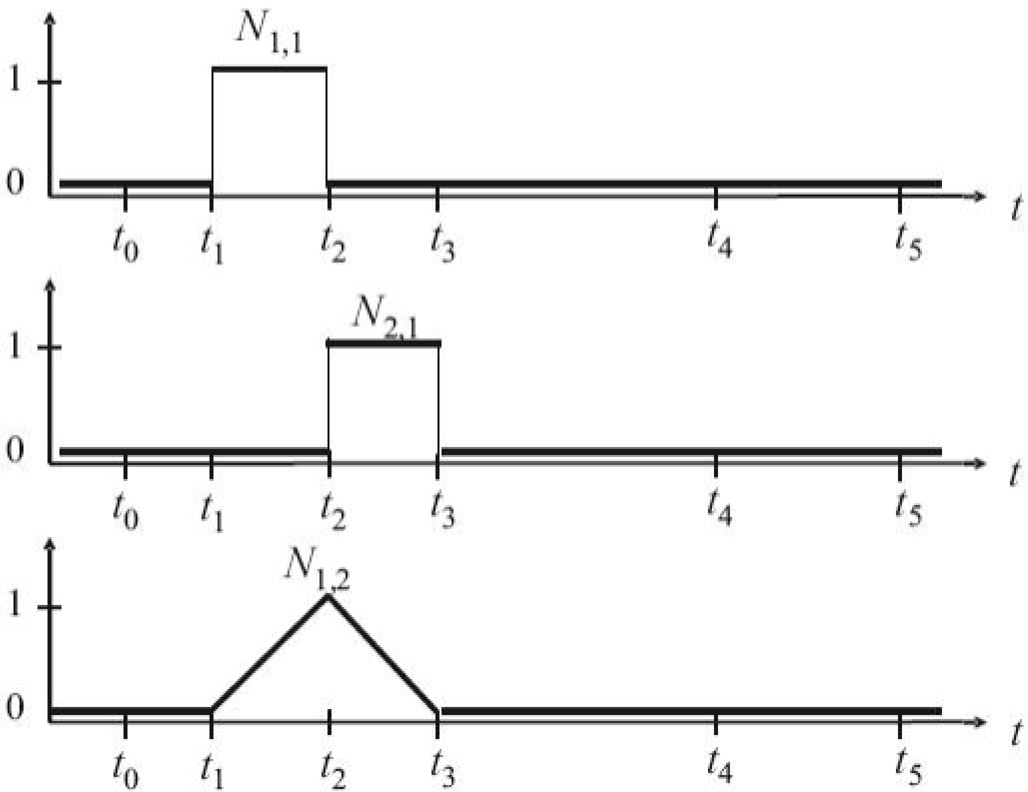
$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t)$$

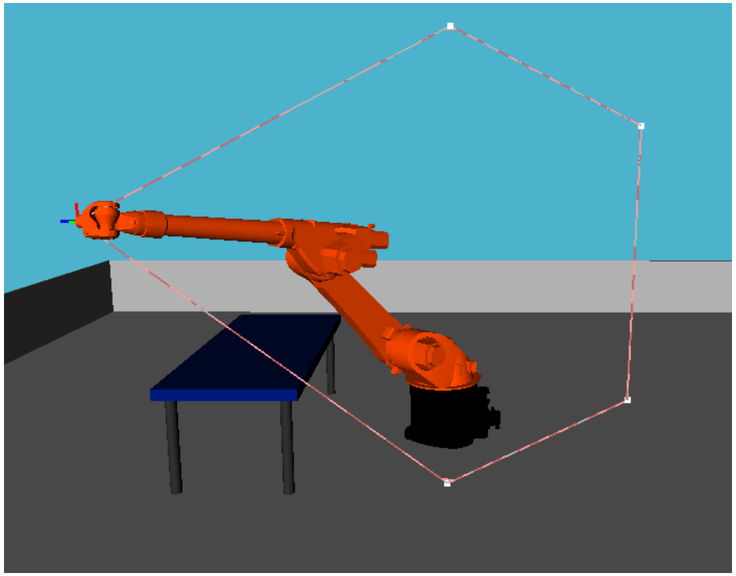
for $k>1$ and $i=0,\ldots,n$

Workaround:

- Calculate the segments considering the knot vector $t_i = (t_0,\ldots,t_n,\ldots t_{n+k})$

2. Spline Approximation – B-Spline



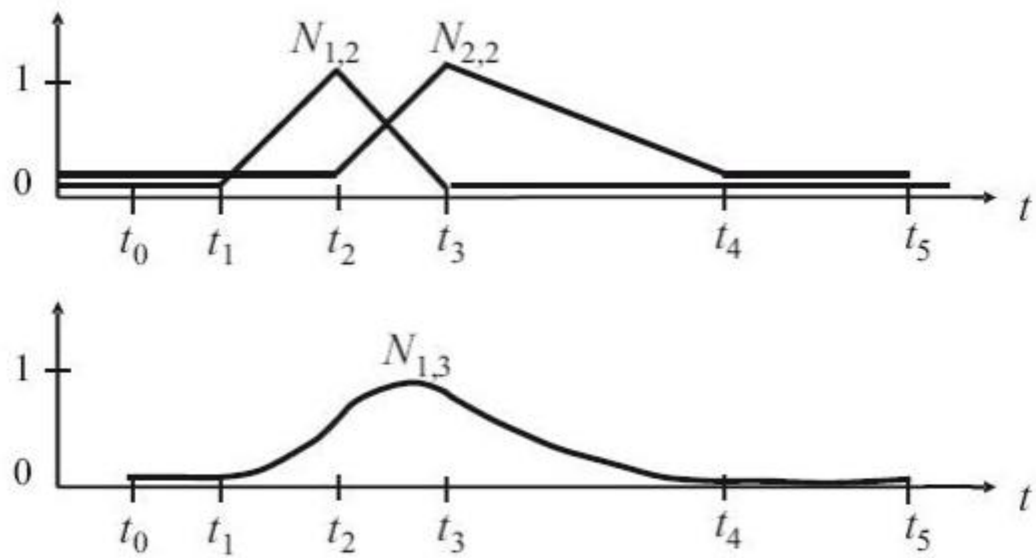


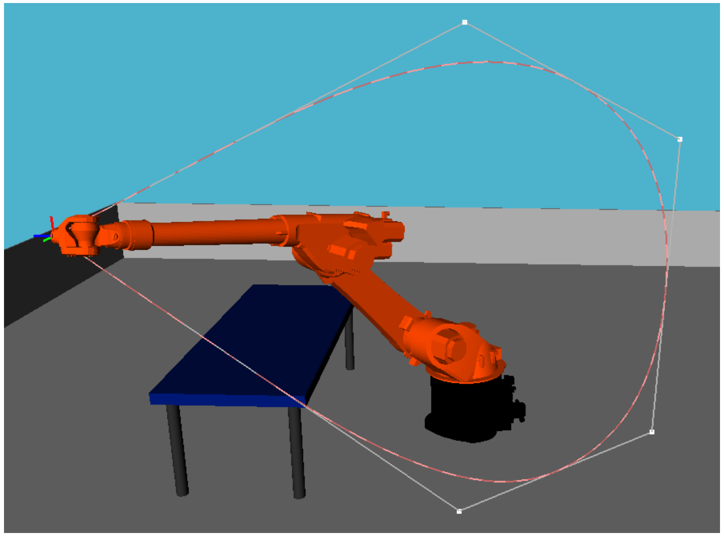
order 1 order 1

order 2

- b-spline of order k = 2
- polynom of degree 1

2. Spline Approximation – B-Spline



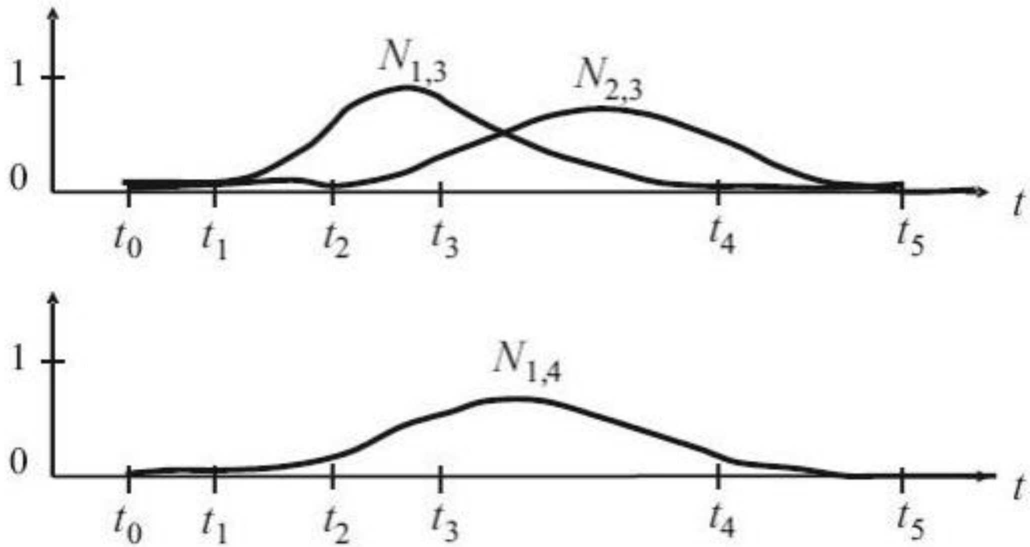


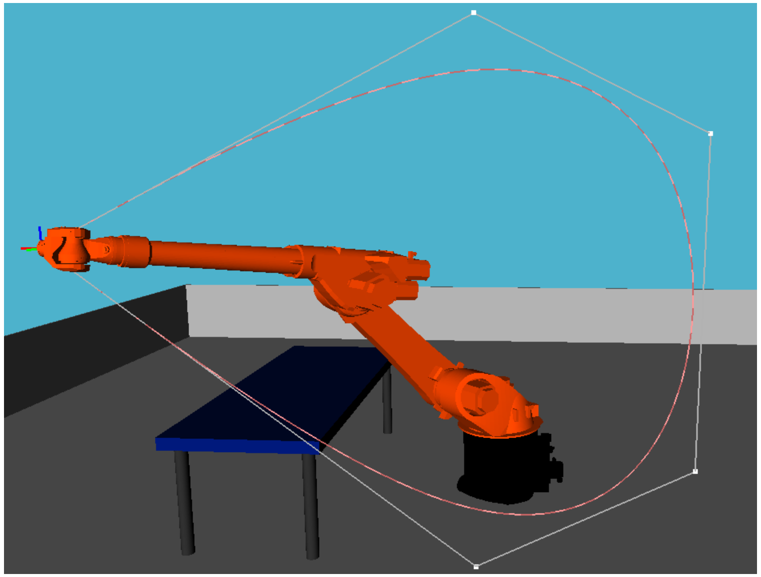
- b-spline of order k = 3
- polynom of degree 2

order 2 order 2

order 3

2. Spline Approximation – B-Spline





- b-spline of order k = 4
- polynom of degree 3

order 3 order 3

order 4

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2. Spline Approximation – B-Spline

Knot vector $\mathbf{t_i}$:

- $\mathbf{t_i} = (t_0,\ldots,t_n,\ldots t_{n+k})$ condition: $t_i <= t_{i+1}$

- Uniform knot vector: $t_i = t_{i-1} + \text{delta\_t}$ (delta_t = constant)

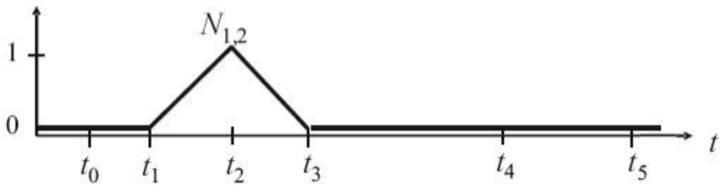
  example: delta_t = 1; $t_i = (t_0, t_1, t_2, t_3) = (0, 1, 2, 3)$

- Non-uniform: delta_t != constant

- knot vector represents a wheigt for the basis functions

$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t)$$

- Factorises the influence of the *de Boor points*



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2. Spline Approximation – B-Spline

Demonstration

of B-spline movement

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3. Spline Interpolation

System of equations: every segment has one equation!

k knots & order=n → ($k-1$) polynomial equations with ($n+1$) variables.

→ total amount of variables = $(n+1) * (k-1)$.

→ with order = 3 → ax^3+bx^2+cx+d → n=4. (our case)

→ amount of variables = 4(k−1) = 4k−4

→ need for conditions to solve system of equations.

Conditions: Are given by spline´s behavior!

- Every knot is part of one segment → $p_i(x_i)=y_i$ → k conditions
- Segment´s contact points → $p_i(x_{i+1})=p_{i+1}(x_{i+1})$ → k−2 conditions
- Same in first derivation → $p_i'(x_{i+1})=p_{i+1}'(x_{i+1})$ → k−2 conditions
- Same in second derivation → $p_i''(x_{i+1})=p_{i+1}''(x_{i+1})$ → k−2 conditions

(4k−6) < (4k−4) → we still need two conditions!

Last two conditions:

- Define the behavior at the spline´s start and end point (i.e. slope)
- Arbitrary decision → $p_0'(x_0)=0$ & $p_k'(x_k)=0$

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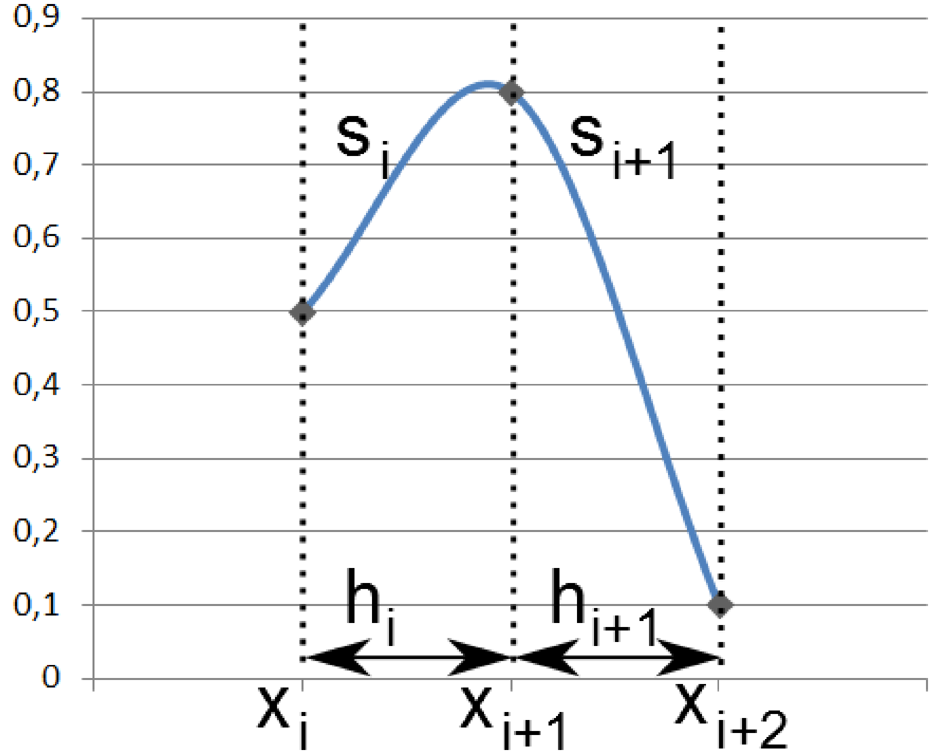
3. Spline Interpolation

- With polynomial s_i between point (x_i, y_i) and point (x_{i+1}, y_{i+1})

$$s_i(x) = a_i\left(x - x_i\right)^3 + b_i\left(x - x_i\right)^2 + c_i\left(x - x_i\right) + d_i$$

→ only one polynomial is active for one input

- Length of the interval $h_i = x_{i+1} - x_i$, where polynomial s_i is "active"



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3. Spline Interpolation

- Calculation of parameter a_i, b_i, c_i and d_i through curvature k_i
  → curvature is second derivative of a function

$$a_i = \frac{\kappa_{i+1} - \kappa_i}{6h_i} \qquad b_i = \frac{\kappa_i}{2} \qquad c_i = \frac{y_{i+1} - y_i}{h_i} - \frac{h_i}{6}\left(2\kappa_i + \kappa_{i+1}\right) \qquad d_i = y_i$$

- Through conditions for spline we can derive this function

$$\frac{2\kappa_i + \kappa_{i-1}}{6} h_{i-1} + \frac{y_i - y_{i-1}}{h_{i-1}} \stackrel{!}{=} -\frac{2\kappa_i + \kappa_{i+1}}{6} h_i + \frac{y_{i+1} - y_i}{h_i}$$

→ this leads to a tridiagonal linear equation system

$$\begin{bmatrix} 2(h_1+h_0) & h_1 & & 0 \\ & \ddots & & \\ h_{i-1} & 2(h_i+h_{i-1}) & h_i & \\ & & \ddots & \\ 0 & & h_{n-1} & 2(h_n+h_{n-1}) \end{bmatrix} \begin{bmatrix} k_0 \\ \vdots \\ k_n \end{bmatrix} = \begin{bmatrix} \frac{6}{h_1}(y_2 - y_1) - \frac{6}{h_0}(y_1 - y_0) \\ \frac{6}{h_i}(y_{i+1} - y_i) - \frac{6}{h_{i-1}}(y_i - y_{i-1}) \\ \frac{6}{h_{n-1}}(y_n - y_{n-1}) - \frac{6}{h_{n-2}}(y_{n-1} - y_{n-2}) \end{bmatrix}$$

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Many thanks for your attention!

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A. Spline Approximation – B-Spline

Another approach:

- Spline with degree=n is a linear combination from 2 splines with degree=n-1 → recursive defintion.

$$\mathbf{x}(t) = \sum_{i=0}^{n} N_{i,k}(t) \cdot \mathbf{d}_i$$

→ input: {Set of points | (x_i,y_i,z_i)}

$$C(u) = \sum_{i=1}^{n-p} P_i \, N_{i,p,\tau}(u) \qquad N_{i,0,\tau}(u) = \begin{cases} 1, & u \in [\tau_i\,, \tau_{i+1}[\\ 0, & \text{sonst} \end{cases}$$

$$N_{i,p,\tau}(u) = \frac{u - \tau_i}{\tau_{i+p} - \tau_i} N_{i,p-1,\tau}(u) \; + \; \frac{\tau_{i+p+1} - u}{\tau_{i+p+1} - \tau_{i+1}} N_{i+1,p-1,\tau}(u)$$

A. Fundamentals – Splines definition

Piecewise polynomial calculation of data set $\{(x_i, y_i), i = 0, 1, \ldots, n\}$

$$S(x) = \begin{cases} S_0(x), & \text{if } x_0 \le x < x_1 \\ S_1(x), & \text{if } x_1 \le x < x_2 \\ \vdots & \\ S_{n-1}(& \end{cases}$$