**Exercise 1.** Consider the portion of the rectangular hyperbola defined by $y = \sqrt{x^2+2x}$, $x \geq 0$. Show by Leibniz' transmutation method, that is, by a computation analoguous to his arithmetical quadrature of the circle, that the area of shaded region in the figure below is

$$\int_0^x \sqrt{x^2+2x}\,dx = \left(3 - \frac{1}{3}\right)z^3 + \left(5 - \frac{1}{5}\right)z^5 + \cdots.$$

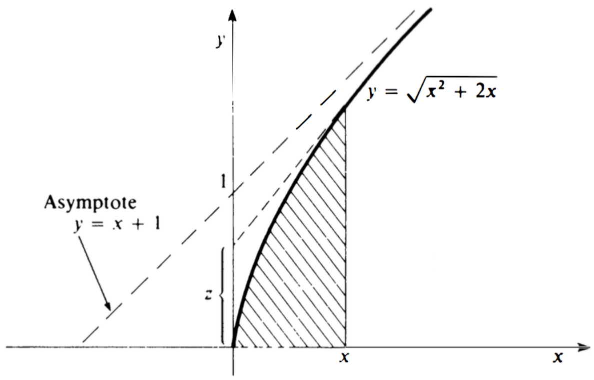


Figure 1.

**Answer:** Let $P = (x, y)$ be a a point along the curve of the hyperbola defined above. Now, consider the the tangent line tangent to this curve at the point $P$. Then the equation of the tangent line is given by

$$y = \frac{dy}{dx}x + z. \tag{1}$$

Now, we shall provide a definition for point of the transmutation curve.

**Definition.** Given any curve and a system of perpendicular axes (abscissas and ordinates), for each point $(x, y)$ in the locus of the curve, a ***point on the transmutation curve*** $(x, z)$ is that which has abscissa $x$, and has at its ordinate $z$, the $y$-intercept of the line tangent to the curve at the point $(x, y)$.

Thus, the equation of the transmutation curve is derived from (1) as

$$z = -\frac{dy}{dx}x + y. \tag{2}$$

Hence, we obtain $dy/dx$ as

$$\frac{dy}{dx} = \frac{1}{2}\left(x^2+2x\right)^{-1/2}(2x+2) = \frac{x+1}{\sqrt{x^2+2x}} = \frac{x+1}{y}.$$

Therefore,

$$z = \frac{x}{\sqrt{x^2+2x}} \quad \text{or} \quad x = \frac{2z^2}{1-z^2}.$$

Now, by Leibniz's transmutation method, it follows that the area under the curve from $A = (0,0)$ to $B = (0,x)$ is the half the area of the triangle $\triangle ABC$, where $C = (x, y)$ plus half the area under the transmutation curve $z$ from $0$ to $x$, that is,

$$\int_0^x y\,dx = \frac{1}{2}xy + \frac{1}{2}\int_0^x z\,dx. \tag{3}$$

Hence, for an arbitrary point $(x_0, y_0)$ on the locus of the curve, it follows by (3) that,

$$\begin{aligned}
\int_0^{x_0} y\,dx &= \frac{1}{2}x_0y_0 + \frac{1}{2}\int_0^{x_0} z\,dx = \frac{1}{2}x_0y_0 + \frac{1}{2}\left(x_0z_0 - \int_0^{z_0} x\,dz\right) && (4)\\
&= \frac{1}{2}\left[x_0\left(y_0+z_0\right) - \int_0^{z_0} x\,dz\right] = \frac{1}{2}\left[\frac{2z_0^3\left(3-z_0^2\right)}{\left(1-z_0^2\right)^2} - 2\int_0^{z_0}\frac{z^2}{1-z^2}\,dz\right] && (5)\\
&= \frac{z_0^3\left(3-z_0^2\right)}{\left(1-z_0^2\right)^2} - \int_0^{z_0}\frac{z^2}{1-z^2}\,dz = \sum_{n=1}^{\infty}(2n+1)\,z_0^{2n+1} - \int_0^{z_0} z^2\sum_{n=0}^{\infty} z^{2n}\,dz && (6)\\
&= \sum_{n=1}^{\infty}(2n+1)\,z_0^{2n+1} - \int_0^{z_0}\sum_{n=1}^{\infty} z^{2n}\,dz = \sum_{n=1}^{\infty}(2n+1)\,z_0^{2n+1} - \sum_{n=1}^{\infty}\frac{z_0^{2n+1}}{2n+1} && (7)\\
&= \sum_{n=1}^{\infty}\left(2n+1-\frac{1}{2n+1}\right)z_0^{2n+1} = \left(3-\frac{1}{3}\right)z_0^3 + \left(5-\frac{1}{5}\right)z_0^5 + \cdots, && (8)
\end{aligned}$$

☜ Be aware of the index shifting! Expand the series if need be to facilitate the process.

as desired.

*Remark.* A few observations are key to understanding this solution. First, observe that in (4), the area under the transmutation curve can also be expressed by substracting its complementary area from the area of the rectangular region bounding this curve, thus, if $ABCD$ is the bounding rectangle for the transmutation curve

$$z = \frac{x}{\sqrt{x^2+2x}}, \quad x \in [0, x_0],$$

where $A = (0,0)$, $B = (x_0, 0)$, $C = (x_0, z_0)$, $D = (0, z_0)$. Then,

$$\int_0^{x_0} z\,dx = x_0z_0 - \int_0^{z_0} x\,dz.$$

Secondly, the computation of $x_0\left(y_0+z_0\right)$ in (5) is rather tedious, but you should obtain

$$x_0\left(y_0+z_0\right) = \frac{2z_0^3\left(3-z_0^2\right)}{\left(1-z_0^2\right)^2}.$$

Now, to see why in (6), we have

$$\frac{z_0^3\left(3-z_0^2\right)}{\left(1-z_0^2\right)^2} = \sum_{n=1}^{\infty}(2n+1)\,z_0^{2n+1},$$

observe that for $|z| < 1$, we have

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}. \tag{9}$$

Thus, from (9) we deduce that

$$\frac{1}{1-z^2} = \frac{1}{1-\left(z^2\right)} = \sum_{n=0}^{\infty}\left(z^2\right)^n = \sum_{n=0}^{\infty} z^{2n} \quad |z| < 1. \tag{10}$$

Thus, taking the derivative of both sides, we have

$$\frac{2z}{\left(1-z^2\right)^2} = \sum_{n=0}^{\infty} 2nz^{2n-1}.$$

Then dividing both sides by $2z$, we have

$$\frac{1}{\left(1-z^2\right)^2} = \sum_{n=0}^{\infty} nz^{2n-2} = 1 + 2z^2 + 3z^4 + \cdots$$

And so,

$$\frac{z^3}{\left(1-z^2\right)^2} = z^3 + 2z^5 + 3z^7 + \cdots \Rightarrow \frac{\left(3-z^2\right)z^3}{\left(1-z^2\right)^2} = \left(3-z^2\right)z^3 + \left(3-z^2\right)2z^5 + \left(3-z^2\right)3z^7 + \cdots.$$

Then by multiplying and combining like terms, we finally obtain

$$\frac{\left(3-z^2\right)z^3}{\left(1-z^2\right)^2} = 3z^3 + 5z^5 + 7z^7 + \cdots = \sum_{n=1}^{\infty}(2n+1)\,z^{2n+1},$$

as desired. The remaining work in (6) should follow easily by applying (10) and performing term-wise integration as demonstrated in (7).

**Exercise 2.** The upper half of the unit circle tangent to the $y$-axis at the origin is defined by $y = \sqrt{2x - x^2}$, $x \geq 0$. Apply Leibniz's transmutation method to show that

$$\int_0^1 \sqrt{2x - x^2}\,dx = \frac{\pi}{4}.$$

If $x \in [0,1)$, what is $\int_0^x y\,dx$? **Answer:** if $P = (x, y)$ is a point along the given curve, then the equation of the line tangent to the curve at $P$ is

$$y = \frac{dy}{dx}x + z$$

Hence, from this we obtain the transmutation curve

$$z = -\frac{dy}{dx}x + y$$

where

$$\frac{dy}{dx} = \frac{1}{2}\left(2x - x^2\right)^{-1/2}(2 - 2x) = \frac{1-x}{\sqrt{2x - x^2}} = \frac{1-x}{y}.$$

Thus,

$$z = y - \left(\frac{1-x}{y}\right)x = \sqrt{\frac{x}{2-x}} \quad \text{or} \quad x = \frac{2z^2}{1+z^2}.$$

By the Transmutation method it follows that

$$\int_0^1 y\,dx = \frac{1}{2}\left[xy\right]_0^1 + \frac{1}{2}\int_0^1 z\,dx$$

which is half the area of the triangle $\triangle ABC$, where $A = (0,0)$, $B = (0,1)$, $C = (1,1)$, plus half the area under the transmutation curve where $0 \leq x \leq 1$. Thus,

$$\begin{aligned}
\int_0^1 y &= \frac{1}{2}(1) + \frac{1}{2}\int_0^1 z\,dx = \frac{1}{2}\left[1 + \left(1 - \int_0^1 x\,dz\right)\right] \\
&= 1 - \int_0^1 \frac{z^2}{1+z^2}\,dz = 1 - \int_0^1 z^2\left(\sum_{n=0}^{\infty}(-1)^n z^{2n}\right)dz \\
&= 1 + \int_0^1 \sum_{n=1}^{\infty}(-1)^n z^{2n}\,dz = 1 + \sum_{n=1}^{\infty}\frac{(-1)^n}{2n+1} \\
&= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.
\end{aligned}$$

Now, notice that the transmutation curve $z = g(x)$ is concave on the interval $[0,1]$, and therefore its global maximum on this interval is 1, attained at $x = 1$. Therefore $|z| < 1$ for

$x \in [0,1)$. Thus, for $x_0 \in [0,1)$, we have

$$\begin{aligned}
\int_0^{x_0} y\,dx &= \frac{1}{2}x_0 y_0 + \frac{1}{2}\int_0^{x_0} z\,dx = \frac{1}{2}x_0 y_0 + \frac{1}{2}\left(x_0 z_0 - \int_0^{z_0} x\,dz\right) \\
&= \frac{1}{2}\left[x_0\left(y_0 + z_0\right) - \int_0^{z_0} x\,dz\right] = \frac{1}{2}\left[\frac{2z_0^3\left(3 + z_0^2\right)}{\left(1 + z_0^2\right)^2} - 2\int_0^{z_0} \frac{z^2}{1+z^2}\,dz\right] \\
&= \frac{z_0^3\left(3 + z_0^2\right)}{\left(1 + z_0^2\right)^2} - \int_0^{z_0} \frac{z^2}{1+z^2}\,dz \\
&= \sum_{n=1}^{\infty} (-1)^{n+1}(2n+1)\,z_0^{2n+1} - \int_0^{z_0} z^2 \sum_{n=0}^{\infty} (-1)^n z_0^{2n}\,dz \\
&= \sum_{n=1}^{\infty} (-1)^{n+1}(2n+1)\,z_0^{2n+1} + \int_0^{z_0} \sum_{n=1}^{\infty} (-1)^n z^{2n}\,dz \\
&= \sum_{n=1}^{\infty} (-1)^{n+1}(2n+1)\,z_0^{2n+1} + \sum_{n=1}^{\infty} \frac{(-1)^n z_0^{2n+1}}{2n+1} \\
&= \sum_{n=1}^{\infty} (-1)^{n+1}\left(2n+1-\frac{1}{2n+1}\right) z_0^{2n+1} = \left(3 - \frac{1}{3}\right) z_0^3 - \left(5 - \frac{1}{5}\right) z_0^5 + \cdots,
\end{aligned}$$

Which converges for $|z| < 1$.

**Exercise 3.** Consider the "higher parabola" $y^q = x^p$, $q > p > 0$. Show that

$$\frac{q}{y}\frac{dy}{dx} = \frac{p}{x}, \quad \text{so } z = \frac{q-p}{q}y.$$

Conclude from the transmutation formula that

$$\int_a^b x^{p/q}\,dx = \frac{q}{p+q}\left[xy\right]_a^b = \frac{q}{p+q}\left[x^{(p+q)/q}\right]_a^b.$$

Observe that

$$y^q = x^p \Leftrightarrow y = x^{p/q}.$$

Thus,

$$\frac{dy}{dx} = \frac{p}{q}x^{(p/q)-1} = \frac{p}{q}\frac{x^{p/q}}{x} = \frac{p}{q}\cdot\frac{y}{x}.$$

Thus, it follows immediately, that

$$\frac{q}{y}\frac{dy}{dx} = \frac{p}{x}.$$

Now, since the equation of the line tangent to the curve $y = x^{p/q}$ at $(x, y)$ is,

$$y = \frac{dy}{dx}x + z,$$

it then follows that

$$\begin{aligned}
z &= y - \frac{dy}{dx}x = y - \frac{p}{q}\cdot\frac{y}{x}\cdot x \\
&= y - \frac{p}{q}y = y\left(1 - \frac{p}{q}\right) = \left(\frac{q-p}{q}\right)y.
\end{aligned}$$

Applying Leibniz' transmutation formula, it then follows that

$$\begin{aligned}
\int_a^b y\,dx &= \frac{1}{2}\left[xy\right]_a^b + \frac{1}{2}\int_a^b z\,dx = \frac{1}{2}\left[x \cdot x^{p/q}\right]_a^b + \frac{1}{2}\int_a^b \left(\frac{q-p}{q}\right) y\,dx \\
&= \frac{1}{2}\left(\left[x^{(p+q)/q}\right]_a^b + \frac{q-p}{q}\int_a^b x^{p/q}\,dx\right) = \frac{1}{2}\left(\left[x^{(p+q)/q}\right]_a^b + \frac{q-p}{q}\left[\frac{x^{(p/q)+1}}{(p/q)+1}\right]_a^b\right) \\
&= \frac{1}{2}\left(\left[x^{(p+q)/q}\right]_a^b + \frac{q-p}{q}\cdot\frac{q}{p+q}\left[x^{(p+q)/q}\right]_a^b\right) = \frac{1}{2}\left(\left[x^{(p+q)/q}\right]_a^b + \frac{q-p}{p+q}\left[x^{(p+q)/q}\right]_a^b\right) \\
&= \frac{1}{2}\left(\frac{2q}{p+q}\left[x^{(p+q)/q}\right]_a^b\right) = \frac{q}{p+q}\left[x^{(p+q)/q}\right]_a^b .
\end{aligned}$$

Furthermore,

$$\int_a^b y\,dx = \frac{q}{p+q}\left[x^{(p+q)/q}\right]_a^b = \frac{q}{p+q}\left[x^{(p/q)+1}\right]_a^b = \frac{q}{p+q}\left[x \cdot x^{p/q}\right]_a^b = \frac{q}{p+q}\left[xy\right]_a^b .$$