Unit 3

• Weightage = 9%

ALGEBRAIC EXPRESSION AND FORMULAS

Student Learning Outcomes (SLOs)

After completing this unit, students will be able to:

- Know that a rational expressions behaves like a rational numbers.
- Define a rational expression as a quotient $\frac{p(x)}{q(x)}$ of two polynomials $p(x)$ and $q(x)$, where $q(x)$, is not the zero polynomial.
- Examine whether a given algebraic expression is a
  - Polynomial or not,
  - Rational expression or not.
- Define $\frac{p(x)}{q(x)}$ as a rational expression in its lowest form, if $p(x)$ and $q(x)$ are polynomials with integral coefficients and having no common factor.
- Examine whether a given rational algebraic expression is in its lowest form or not.
- Reduce a given rational expression to its lowest form.
- Find the sum, difference and the product of rational expressions.
- Divide a rational expression by another rational expression and express the result in its lowest form.
- Find the values of the algebraic expressions at some particular real numbers.
- Know the formulas
  - $(a + b)^2 + (a - b)^2 = 2(a^2 + b^2)$ and $(a + b)^2 - (a - b)^2 = 4ab$.
  - Find the values of $a^2 + b^2$ and of ab when the values of $a + b$ and $a - b$ are known.
- Know the formula
  $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca.$
  - Find the value of $a^2 + b^2 + c^2$ when the values of $a + b + c$ and $ab + bc + ca$ are given.
  - Find the value of $a + b + c$ when the values of $a^2 + b^2 + c^2$ and $ab + bc + ca$ are given.
  - Find the value of $ab + bc + ca$ when the values of $a^2 + b^2 + c^2$ and $a + b + c$ are given.

- Know the formulas
    - $(a \pm b)^3 = a^3 \pm 3a^2b \pm 3ab^2 \pm b^3$ or
    - $(a \pm b)^3 = a^3 \pm b^3 \pm 3ab(a \pm b)$.
  - Find the values of $a^3 \pm b^3$, when the values of $a \pm b$ and ab are given.
  - Find the values of $x^3 \pm \frac{1}{x^3}$ when the values of $x \pm \frac{1}{x}$ is given.
- Know the formula
  $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$.
  - Find the product of $x + \frac{1}{x}$ and $x^2 + \frac{1}{x^2} - 1$.
  - Find the product of $x - \frac{1}{x}$ and $x^2 + \frac{1}{x^2} + 1$.
  - Find the continued product of
    $(x + y)(x - y)(x^2 + xy + y^2))(x^2 - xy + y^2)$.
- Recognize the surds and their applications.
- Explain the surds of the second order.
- Use basic operations on surds of second order to rationalize the denominators and to evaluate them.
- Explain rationalization (with precise meaning) of real numbers of the types $\frac{1}{a + b\sqrt{x}}$, $\frac{1}{\sqrt{x} + \sqrt{y}}$ and their combinations, where x and y are natural numbers and a and b are integers.

3.1 Algebraic Expressions

We have already studied about Algebraic expression in previous classes. Let's discuss its types.

Following are the three types of algebraic expressions.

(*a*) Polynomial Expression or polynomial,
(*b*) Rational Expression,
(*c*) Irrational Expression.

(*a*) Polynomial Expression or polynomial.

A polynomial expression (simply say polynomial) in one variable x can be written as:

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + a_3x^{n-3} \ldots + a_{n-1}x^1 + a_n$$

Where 'n' is a non–negative integer and the coefficients; $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$ are real numbers. Usually, a polynomial is denoted by $p(x)$, so the above polynomial can be expressed as:

$$P(x)=a_0x^n + a_1x^{n-1} + a_2x^{n-2} + a_3x^{n-3} \ldots + a_{n-1}x^1 + a_n$$

If $a_0 \neq 0$, then the polynomial is said to be a polynomial of degree n, and a_0 is called the **leading coefficient** of the polynomial.

Some examples of polynomials and their degrees are given below.

(i) $8x-5$, degree 1 (ii) $x^4-2x^3+5x^2+1$, degree 4

(iii) $6x^{31}+3$, degree 31 (iv) $12x^4-x^3+\frac{2}{3}x^2-3x+1$, degree 4

(v) 4, degree zero. (vi) $\sqrt{10}x^{12}+2x^6-x^5-18x+1$ degree 12

The algebraic expression $x^3-x^3y^2+x^2y^2-10$ is a polynomial with two variables x and y and its degree is 5.

Similarly, the algebraic expression $x^3y^5x^2-x^3y^2z^3+x^2yz-34$ is a polynomial with three variables x, y and z and having degree 10(highest sum of powers) and so on.

Remember

- A polynomial consisting of only single term is called monomial. $3x$, $7xy$, $6xy^2z^5$ etc. are some examples of monomials.
- A polynomial consisting of two terms is called binomial e.g. $x+4$, $5x+y$, $7x-3$ etc. are some examples of binomials.
- A polynomial consisting of three terms is called trinomial e.g. x^2-2x+1

$\frac{1}{\sqrt{3}}x^2y^2-5xy+3$, etc. are some examples of trinomials.

- Other polynomial which consisting of four or more terms, called multinomial.
- The highest power (sum of powers in case when more variables are multiplied) on the variable in a polynomial is called the degree of the polynomial.

(b) Rational Expression·

An algebraic expression which can be written in the form $\frac{p(x)}{q(x)}$ where $q(x) \neq 0$, and $p(x)$ and $q(x)$ are polynomials, called a **rational expression** in x.

For example, $\frac{x+1}{x}, \frac{x^2-x+1}{x-5}, \frac{\sqrt{3}x^2-5x+4}{x^2+6x-5}$ etc. are some examples of rational expressions.

Note: Every polynomial is a rational expression but its converse is not true.

(c) Irrational Expression.

An algebraic expression which cannot be written in the form of $\frac{p(x)}{q(x)}$, where $q(x) \neq 0$, when $p(x)$ and $q(x)$ are polynomials is called **irrational expression** in x.

For example, $\frac{1}{\sqrt{x}}, \frac{\sqrt{x}+1}{x}, \frac{\sqrt{x^3}+2x+3}{\sqrt{x}-9}, \sqrt{x}+\frac{5}{\sqrt{x}}$ etc. are some examples of irrational expressions.

3.1.1 Know that a rational expressions behaves like a rational numbers

Let p and q be two integers, then $\frac{p}{q}$ may be an integer or not. Therefore, the number system is extended and $\frac{p}{q}$ is defined as a rational number, where $p, q \in Z$ provided that $q \neq 0$.

Similarly, if $p(x)$ and $q(x)$ are two polynomials, then $\frac{p(x)}{q(x)}$ is not necessarily a polynomial, where $q(x) \neq 0$. Therefore, it is similar to the idea of rational numbers; the concept of rational expressions is developed.

3.1.2 Define a Rational expression as a quotient $\frac{p(x)}{q(x)}$ of two polynomials $p(x)$ and $q(x)$, where $q(x)$ is not the zero polynomial.

As we know that the expression in the form $\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are two polynomials provided $q(x)$ is non–zero polynomial; called a rational expression.

For examples $\frac{x^2-5}{3x^2+4}, 3x^2+4 \neq 0$ are rational expressions.

3.1.3 Examine whether a given algebraic expression is a,

(i) Polynomial or not (ii) Rational expression or not

The following examples will help to identify polynomial and rational expressions.

Example 01 Examine whether the following are the polynomials or not?

(i) $2x^2 - \frac{1}{\sqrt{x}}$ (ii) $6x^3 - 4x^2 - 5x$

Solution(i): $2x^2 - \frac{1}{\sqrt{x}}$

It's not a polynomial because the second term does not have positive integral exponent.

Solution(ii): $6x^3 - 4x^2 - 5x$,

It is a polynomial, because each term has positive integral exponent.

Example 02 Examine whether the following are rational expressions or not?

(i) $\frac{x-2}{3x^2+1}$ (ii) $6x^3 - \frac{1}{\sqrt{x+4}}$

Solution (i): $\dfrac{x-2}{3x^2+1}$

The numerator and denominator both are polynomials, so it is a rational expression.

Solution (ii): $6x^3 - \dfrac{1}{\sqrt{x+4}}$

It is not a rational expression, because the denominator of the second term is not a polynomial.

3.1.4 Define $\dfrac{p(x)}{q(x)}$ as a rational expression in its lowest form, if $p(x)$ and $q(x)$ are polynomials with integral coefficients and having no common factor

The rational expression $\dfrac{p(x)}{q(x)}$ is said to be in its lowest form, if $p(x)$ and $q(x)$ are polynomials with integral coefficients and have no common factor.

For example $\dfrac{x+1}{x-1}$ is the lowest form of $\dfrac{x^2-1}{(x-1)^2}$

3.1.5 Examine whether a given rational algebraic expression is in its lowest form or not

To examine the rational expression $\dfrac{p(x)}{q(x)}$, find common factor(s) of $p(x)$ and $q(x)$. If common factor is 1, then the rational expression is in the lowest form.

For example $\dfrac{x+1}{x-1}$ is in its lowest form because, the common factor of $(x+1)$ and $(x-1)$ is 1.

3.1.6 Reduce a rational expression to its lowest form

Let $\dfrac{p(x)}{q(x)}$ be the rational expression, where $q(x) \neq 0$.

Step–1: Find the factors of polynomials $p(x)$ and $q(x)$ if possible.

Step–2: Find the common factors of $p(x)$ and $q(x)$.

Step–3: Cancel the common factors of $p(x)$ and $q(x)$.

Example 01 Reduce the following rational expression to their lowest form.

(i) $\dfrac{(x^2-x)(x^2-5x+6)}{2x(x^2-3x+2)}$ (ii) $\dfrac{5(x^2-4)}{(3x+6)\,(x-3)}$

Solution (i):

$$\frac{(x^2-x)(x^2-5x+6)}{2x(x^2-3x+2)}$$

$$=\frac{x(x-1)}{2x}\cdot\frac{x^2-3x-2x+6}{x^2-2x-x+2} \qquad \text{(Provided } x\neq 0\text{)}$$

$$=\left(\frac{x-1}{2}\right)\cdot\frac{\{x(x-3)-2(x-3)\}}{\{x(x-2)-1(x-2)\}}$$

$$=\frac{(x-1)(x-3)(x-2)}{2(x-2)(x-1)} \qquad \text{(Provided } x\neq 1 \text{ and } x\neq 2\text{)}$$

$$=\frac{(x-3)}{2}$$

$$=\frac{1}{2}(x-3) \text{ which is the required lowest form}$$

Solution (ii):

$$\frac{5(x^2-4)}{(3x+6)\,(x-3)}$$

$$=\frac{x^2-4}{x-3}\cdot\frac{5}{3x+6}$$

$$=\frac{x^2-2^2}{x-3}\cdot\frac{5}{3(x+2)}$$

$$= \frac{(x+2)(x-2)}{x-3} \cdot \frac{5}{3(x+2)} \qquad \text{(provided } x \neq -2\text{)}$$

$$= \frac{5(x-2)}{3(x-3)} \quad \text{which is the required lowest form.}$$

3.1.7 Find the sum, difference and product of rational expressions.

The sum, difference and product of rational expression is explained with the help of the following examples.

Example 01 Simplify $\frac{3}{x+1} + \frac{4x}{x^2-1}$

Solution:

$$\frac{3}{x+1} + \frac{4x}{x^2-1}$$

$$= \frac{3}{x+1} + \frac{4x}{(x-1)(x+1)} \quad \text{(Factorization)}$$

$$= \frac{3(x-1)+4x}{(x-1)(x+1)}$$

$$= \frac{3x-3+4x}{(x-1)(x+1)}$$

$$= \frac{7x-3}{(x-1)(x+1)}$$

$$= \frac{7x-3}{x^2-1} \qquad \text{Hence simplified in the lowest form.}$$

Example 02 Simplify $\frac{1}{x^2-1}-\frac{1}{x^3-1}$

Solution:

$$\frac{1}{x^2-1}-\frac{1}{x^3-1}$$

$$=\frac{1}{x^2-1}-\frac{1}{x^3-1}$$

$$=\frac{1}{(x-1)(x+1)}-\frac{1}{(x-1)(x^2+x+1)}$$

$$=\frac{(x^2+x+1)-(x+1)}{(x+1)(x-1)(x^2+x+1)}$$

$$=\frac{x^2+x+1-x-1}{(x+1)(x-1)(x^2+x+1)}$$

$$=\frac{x^2}{(x+1)(x^3-1)}$$

Hence simplified in the lowest form.

Example 03 Simplify $\frac{x^2}{x^2+x-12}\cdot\frac{x^2-9}{2x^2}$

Solution: Simplification

$$\frac{x^2}{x^2+x-12}\cdot\frac{x^2-9}{2x^2}$$

$$=\frac{x^2}{x^2+4x-3x-12}\cdot\frac{(x-3)(x+3)}{2x^2}$$

$$=\frac{1}{x(x+4)-3(x+4)}\cdot\frac{(x-3)(x+3)}{2} \quad \text{(factorization)}$$

$$=\frac{1}{(x+4)(x-3)}\cdot\frac{(x-3)(x+3)}{2} \quad \text{provided } x\neq 3$$

$$=\frac{(x+3)}{2(x+4)}$$

Hence simplified in the lowest form.

3.1.8 Divide a rational expression by another rational expression and express the result in its lowest form.

In order to divide one rational expression by another, we first convert division into multiplication and then simplify the resulting product to lowest form.

Example 01 Simplify $\frac{3x-9y}{2x+10y} \div \frac{x^2-3xy}{4x+20y}$

Solution: Simplification

$$\frac{3x-9y}{2x+10y} \div \frac{x^2-3xy}{4x+20y}$$

$$= \frac{3x-9y}{2x+10y} \times \frac{4x+20y}{x^2-3xy} \qquad \text{(Taking Reciprocal)}$$

$$= \frac{3(x-3y)}{2(x+5y)} \times \frac{4(x+5y)}{x(x-3y)}$$

$$= \frac{6}{x}$$

3.1.9 Find the values of algebraic expression at some particular real numbers.

Finding the values of algebraic expressions at some particular real numbers is explained in the following example.

Example 01 Find the value of $\frac{x^2+yz}{x^3+y^2-7yz^4}$ when $x = 3$, $y = 2$ and $z = -1$.

Given $x=3, y=2, z=-1$

$$= \frac{x^2+yz}{x^3+y^2-7yz^4}$$

$$= \frac{(3)^2+(2)(-1)}{(3)^3+(2)^2-7(2)(-1)^4}$$

$$= \frac{9-2}{27+4-14}$$

$$= \frac{7}{17}$$

Exercise 3.1

1. Examine whether the following algebraic expressions are polynomials or not.

(i) $2xy^2 - 3x^2 + 5y^3 - 6$ (ii) $3xy^{-2}$

(iii) $6x^2 - 10x + 7 - \sqrt{45}$ (iv) $5\sqrt{x} - x + 5x^2$

(v) $\dfrac{2}{x+2}$ (vi) $\dfrac{2}{x} + x^3 - 2$

2. Examine whether the following algebraic expressions are rational or not

(i) $\dfrac{x^2+2x+3}{x-4}$ (ii) $\dfrac{x^2+5\sqrt{x}-2x}{3x^2+5x+4}$ (iii) $\dfrac{13x^2-9x+4}{x^2+5x+\sqrt{7}}$

(iv) $\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$ (v) $\dfrac{7}{x+7}$ (vi) $5\sqrt{x} - x + 5x^2$

3. Reduce the following into their lowest form.

(i) $\dfrac{p^2-100}{p+10}$ (ii) $\dfrac{3a^2+3ab}{3a^2+6ab+3b^2}$ (iii) $\dfrac{(a-b)}{(a+b)} \times \dfrac{(a^2+ab)}{(2a^2-2b^2)}$

(iv) $\dfrac{(x+y)^2-z^2}{x+y+z}$ (v) $\dfrac{(m^2-6m)(3m+15)}{2m-12}$ (vi) $\dfrac{x^2-2x-3}{x^2-x-2}$

4. Simplify:

(i) $\dfrac{4x-1}{2x-2} + \dfrac{4x+1}{2x+2}$ (ii) $\dfrac{1}{x+2} + \dfrac{2}{x+3}$ (iii) $\dfrac{xy}{xy+1} + \dfrac{xy+1}{xy-1}$

(iv) $\dfrac{x-2}{x+3} - \dfrac{x+1}{x+6}$ (v) $\dfrac{1}{a+b} - \dfrac{1}{a-b}$ (vi) $\dfrac{4y}{y^2-1} - \dfrac{y+1}{y-1}$

5. Simplify into lowest form.

(i) $\left(\dfrac{x^2}{4y^2-x^2}+1\right) \div \left(1-\dfrac{x}{2y}\right)$ (ii) $\dfrac{x+3}{3y-2x} \cdot \dfrac{4x^2-9y^2}{xy+3y}$

(iii) $\left(\dfrac{x^2-1}{x^2+2x+1}\times\dfrac{x+1}{x-1}\right)$ (iv) $\dfrac{8(y+3)}{9}\times\dfrac{12(y+1)}{4(y+3)}\div\dfrac{8(y+1)}{5}$

(v) $\dfrac{q^2-25}{q^2-3q}\div\dfrac{q^2+5q}{q^2-9}$ (vi) $\dfrac{4}{z^2-4z-5}\div\dfrac{2}{4z^2-4}$

6. Find the value of $t+\dfrac{1}{t}$, when $t=\dfrac{x-y}{x+y}$

7. Find the values of

(i) $\dfrac{5(x+y)}{3x^2\sqrt{y}+6}$, if $x=-4$, $y=9$

(ii) $\dfrac{42ab^2c^3}{3a^2b+1}$, if $a=3$, $b=2$ and $c=1$

(iii) $\dfrac{(x+y)^3-z^2}{x^2y^2+z^2}$, if $x=2$, $y=-4$ and $z=3$,

(iv) $\dfrac{3x^2y}{z}-\dfrac{bc}{x+1}$, if $x=2$, $y=-1$, $z=3$, b= 4, $c=\dfrac{1}{3}$

(v) $\dfrac{(ab^2-c)}{(a+cd^2)}\times\dfrac{(c+d)}{(a^2b-d)}$, if $a=1$, $b=3$, $c=-3$ and $d=2$.

3.2 Algebraic Formulas

We have already studied and used some algebraic formulas in previous classes. In this section we will learn some more formulas and their applications.

Recall that

- $(a+b)^2=a^2+2ab+b^2$
- $(a-b)^2=a^2-2ab+b^2$
- $(a+b)(a-b)=a^2-b^2$

3.2.1 Know the Formulas

(i) $(a+b)^2+(a-b)^2=2(a^2+b^2)$

Proof:

$$\begin{aligned}\text{L.H.S} &= (a+b)^2+(a-b)^2\\ &= a^2+2ab+b^2+a^2-2ab+b^2 \qquad \because (a+b)^2=a^2+2ab+b^2\\ &= 2a^2+2b^2 \qquad \text{and } (a-b)^2=a^2-2ab+b^2\\ &= 2(a^2+b^2)=\text{R.H.S}\end{aligned}$$

Hence proved

(ii) $(a+b)^2 - (a-b)^2 = 4ab$

Proof:

$$\begin{aligned} \text{L.H.S} &= (a+b)^2 - (a-b)^2 \\ &= a^2 + 2ab + b^2 - (a^2 - 2ab + b^2) \quad \left[\because (a+b)^2 = a^2 + 2ab + b^2 \right.\\ &= a^2 + 2ab + b^2 - a^2 + 2ab - b^2 \quad \left.\text{and} (a-b)^2 = a^2 - 2ab + b^2\right] \\ &= 4ab = \text{R.H.S} \end{aligned}$$

Hence proved

The use of above formulae are explained in the following examples.

Example 01 Find the values of (i) $a^2 + b^2$ (ii) ab (iii) $8ab(a^2 + b^2)$

when $a + b = 6$ and $a - b = 4$.

Solution: Given that,

$a + b = 6$, $a - b = 4$.

(i) $a^2 + b^2 = ?$

We know that, $(a + b)^2 + (a - b)^2 = 2(a^2 + b^2)$

By substituting the values of $a+b = 6$ and $a - b = 4$, we get

$(6)^2 + (4)^2 = 2(a^2 + b^2)$,

$\Rightarrow$ $36 + 16 = 2(a^2 + b^2)$

$\Rightarrow$ $52 = 2(a^2 + b^2)$

$\Rightarrow$ $26 = a^2 + b^2$

$\Rightarrow$ $\boxed{a^2 + b^2 = 26}$

(ii) $ab = ?$

We also know that $(a + b)^2 - (a - b)^2 = 4ab$.

By substituting the values of $a + b = 6$ and $a - b = 4$, we get

$\therefore$ $(6)^2 - (4)^2 = 4ab$

$\Rightarrow$ $36 - 16 = 4ab$

$\Rightarrow$ $20 = 4ab$

$\Rightarrow$ $5 = ab$

or $\boxed{ab = 5}$

(iii) Now $8ab(a^2 + b^2) = 4ab \times 2(a^2 + b^2)$

$= 4(5) \times 2(26)$

$= 20 \times 52$

$\boxed{8ab(a^2 + b^2) = 1040}$

3.3.2 Know the formula

$(a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca$

Proof: L.H.S $= (a+b+c)^2 = (a+b+c)(a+b+c)$

$$= a(a+b+c)+b(a+b+c)+c(a+b+c)$$

$$= a^2+ab+ac+ab+b^2+bc+ac+bc+c^2$$

$$= a^2+b^2+c^2+2ab+2bc+2ca = \text{R.H.S} \quad \text{Hence proved}$$

The use of this formula is explained in the following examples.

Example 01 Find the value of $a^2+b^2+c^2$, when $a+b+c=7$ and $ab+bc+ca=15$

Solution: Given that,

$a+b+c=7$ and $ab+bc+ca=15$

$a^2+b^2+c^2=?$

We know that $(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)$

Now, substituting the values of $a+b+c=7$ and $ab+bc+ca=15$, in the above formula we get,

$\therefore \quad (7)^2 = a^2+b^2+c^2+2(15)$

$\Rightarrow \quad 49 = a^2+b^2+c^2+30$

$\Rightarrow \quad 49-30 = a^2+b^2+c^2$

$\Rightarrow \quad 19 = a^2+b^2+c^2$

$\Rightarrow \quad \boxed{a^2+b^2+c^2 = 19}$

Hence, the value of $(a^2+b^2+c^2)$ is 19.

Example 02 Find the value of $(a+b+c)$, when $a^2+b^2+c^2=38$ and $ab+bc+ac=31$

Solution: Given that,

$a^2+b^2+c^2=38$ and $ab+bc+ac=31$,

We know that $(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)$

Now, substituting the values of $a^2+b^2+c^2=38$ and $ab+bc+ac=31$, in the above formula, we get,

$(a+b+c)^2 = 38+2(31)$

$\Rightarrow \quad (a+b+c)^2 = 38+62$

$\Rightarrow \quad (a+b+c)^2 = 100$

$\Rightarrow \quad \sqrt{(a+b+c)^2} = \pm\sqrt{100}$

$\Rightarrow \quad \boxed{(a+b+c) = \pm 10}$

Hence, the value of $(a+b+c)$ is ± 10.