Unit 10

• Weightage = 7%

PARALLELOGRAM AND TRIANGLES

Student Learning Outcomes (SLOs)

After completing this unit, students will be able to:

- Understand the following theorems along with their corollaries and apply them to solve allied problems.

  a) In a parallelogram:
    - The opposite sides are congruent,
    - The opposites angles are congruent,
    - The diagonals bisect each other.

  b) If two opposite sides of a quadrilateral are congruent and parallel, it is a parallelogram.

  c) The line segments joining the midpoints of two sides of a triangle, is parallel to the third side and it is equal to one half of its length.

  d) The medians of a triangle are concurrent and their point of concurrency is the point of trisection of each median.

  e) If three or more parallel lines make congruent intercepts on the transversal, they also intercept congruent segments on any other line that cuts them.

Introduction

In the previous classes students learned and constructed many kinds of polygons like triangles, parallelogram, square, rectangle, rhombus, trapezium etc. Also observe the congruency related to their sides and angles. In this unit, we will discuss and understand the theorems related to parallelograms and Triangles.

10.1 Parallelograms and Triangles

Theorem 10.1.1

In a parallelogram:

- **The opposite sides are congruent,**
- **The opposites angles are congruent,**
- **The diagonals bisect each other.**

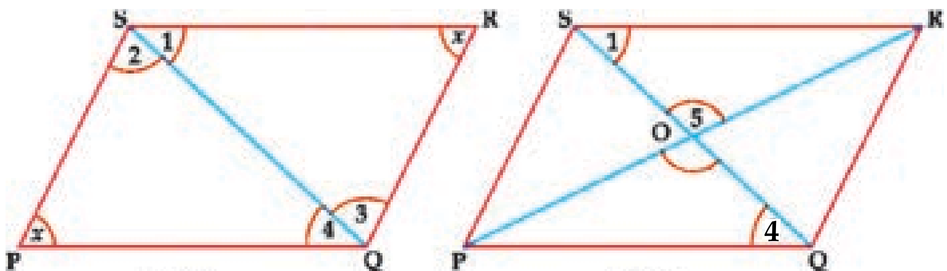


Fig (i) Fig (ii)

Given:

$\|^{m}$ PQRS

To Prove:

1. $\overline{PQ} \cong \overline{RS}$; $\overline{PS} \cong \overline{QR}$
2. $\angle P \cong \angle R$; $\angle S \cong \angle Q$
3. Diagonals $\overline{PR}$ and $\overline{SQ}$ bisect each other at point O. [fig (ii)]

Construction:

In figure (i) join points S and Q.

Proof:

Statements	Reasons
In figure (i)	
(1)	
$\overline{SR} \parallel \overline{PQ}$, $\overline{SQ}$ is transversal,	Alternate angles of $\parallel$ lines
$m\angle 1 = m\angle 4$	
Similarly, $m\angle 2 = m\angle 3$	
$\therefore\ m\angle 1 + m\angle 2 = m\angle 3 + m\angle 4$	Angle addition postulate
or $\angle S \cong \angle Q$	
Similarly, $\angle P \cong \angle R$	
i.e opposite angles are congruent	By above the same process
(2)	
$\Delta SPQ \leftrightarrow \Delta QRS$	
$\angle 1 \cong \angle 4$ and $\angle 2 \cong \angle 3$	Proved in (1) above
$\overline{SQ} \cong \overline{SQ}$	Common
$\therefore\ \Delta SPQ \cong \Delta QRS$	$A.S.A \cong A.S.A$
$\therefore\ \overline{PQ} \cong \overline{RS}$ and $\overline{PS} \cong \overline{QR}$	By the congruence of Δs
i.e opposite sides are congruent	
In figure (ii)	
(3)	
$\Delta SOQ \leftrightarrow \Delta ROS$	
$\angle 1 \cong \angle 4$	Proved in (1) above
$\angle POQ \cong \angle SOR$	Vertically opposite angles
$\overline{PQ} \cong \overline{SR}$	Proved in (2) above
$\therefore\ \Delta POQ \cong \Delta ROS$	$A.A.S \cong A.A.S$
$\therefore\ \overline{PO} \cong \overline{OR}$ and $\overline{OQ} \cong \overline{OS}$	By the congruence of Δs
$\therefore\ \overline{PR}$ and $\overline{RS}$ diagonals bisect each other	

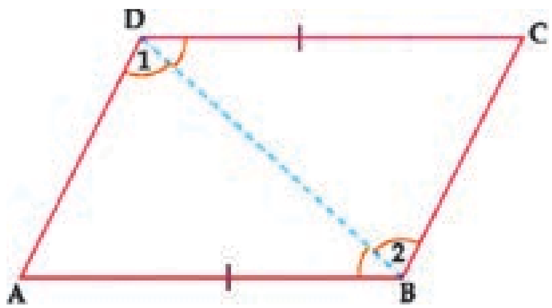
Q.E.D

Exercise 10.1

1. The line joining the mid-points of two opposite sides of parallelogram is parallel to the other sides.
2. Interior angles on any side of a parallelogram are supplementary.
3. Prove that the bisectors of two angles on same side of a parallelogram cut each other at right angle.
4. If the diagonals of a quadrilateral bisect each other, then it is a parallelogram.
5. In parallelogram opposite angles are congruent.

Theorem 10.1.2

If two opposite sides of a quadrilateral are congruent and parallel, it is a parallelogram.



Given:

In a quadrilateral ABCD,

$\overline{AB} \cong \overline{CD}$ and $\overline{AB} \parallel \overline{CD}$

To prove:

Quadrilateral ABCD is a $\|^{m}$

Construction: Join B and D.

Proof:

Statements	Reasons
1. $\overline{AB} \parallel \overline{CD}$, $\overline{BD}$ is transversal $\therefore$ $\angle ABD \cong \angle CDB$	1. Alternate angles of $\parallel$ lines
2. In $\Delta ADB \leftrightarrow \Delta CBD$	2. Correspondence of two Δs.
i. $\overline{AB} \cong \overline{CD}$	i. Given
ii. $\angle ABD \cong \angle CDB$	ii. Proved above
iii. $\overline{BD} \cong \overline{BD}$	iii. Common
3. $\therefore$ $\Delta ADB \cong \Delta CBD$	3. S.A.S $\cong$ S.A.S
4. $\therefore$ $\angle 1 \cong \angle 2$	4. By the congruence of triangles.
5. But these are alternate angles	5. By definition of alternate angles.
6. $\overline{AD} \parallel \overline{BC}$	6. Alternate angles are congruent
7. $\overline{AB} \parallel \overline{CD}$	7. Given
8. ABCD is a $\|^{m}$	8. Opposite sides are parallel

Q.E.D

Exercise 10.2

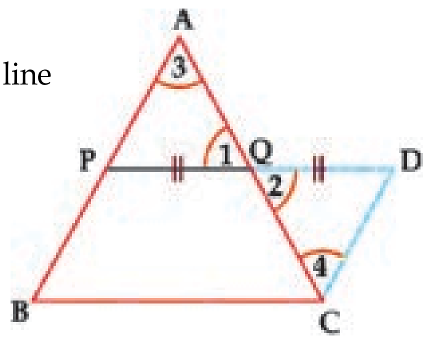
1. Prove that a quadrilateral is a parallelogram, if its opposite angles are congruent
2. Prove that a quadrilateral is a parallelogram, if its diagonals bisect each other.
3. If both pairs of opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.
4. If a quadrilateral is a parallelogram, it has diagonals which form two congruent triangles.
5. If the angles formed with every side of a quadrilateral are supplementary, it is a parallelogram.

Theorem 10.1.3

The line segment joining the mid-points of two sides of a triangle, is parallel to the third side and it is equal to one half of its length.

Given:

P and Q are midpoints of $\overline{AB}$ and $\overline{AC}$ in ΔABC, respectively and $\overline{PQ}$ is the line segment joining them.



To prove:

$$\overline{PQ} \parallel \overline{BC} \text{ and } m\overline{PQ} = \frac{1}{2} m\overline{BC}$$

Construction:

Produce $\overrightarrow{PQ}$ to D such that $\overline{QD} \cong \overline{PQ}$.
Join D and C.

Proof:

Statements	Reasons
In $\Delta APQ \leftrightarrow \Delta CDQ$	
i. $\overline{PQ} \cong \overline{QD}$	i. Construction
ii. $\angle 1 \cong \angle 2$	ii. Vertical angles
iii. $\overline{AQ} \cong \overline{QC}$	iii. Given

$\therefore$	$\Delta APQ \cong \Delta CDQ$	S.A.S postulate
$\therefore$	$\overline{AP} \cong \overline{CD}$ and $\angle 3 \cong \angle 4$	By the congruence of Δs
But	$\overline{PB} \cong \overline{AP}$	Given
$\therefore$	$\overline{PB} \cong \overline{CD}$	Each is congruent to $\overline{AP}$
$\angle 3$ and $\angle 4$ are alternate $\angle$s		By definition of alternate $\angle$s
$\therefore$	$\overline{AB} \parallel \overline{CD}$ i.e. $\overline{PB} \parallel \overline{CD}$	Alternate $\angle$s are congruent
$\therefore$	PBCD is a $\parallel^{m}$	A pair of opposite sides $\parallel$ and $\cong$
$\therefore$	$\overline{PD} \parallel \overline{BC}$ and $\overline{PD} \cong \overline{BC}$	Opposite sides of $\parallel^{m}$ are parallel and congruent.
$\therefore$	$\overline{PQ} \parallel \overline{BC}$	$\overleftrightarrow{PD}$ and $\overleftrightarrow{PQ}$ are same line and
	and $m\overline{PQ} = \frac{1}{2} m\overline{BC}$	$m\overline{PQ} = m\overline{QD} = \frac{1}{2} m\overline{PD}$

Q.E.D

Exercise 10.3

1. If the line segments joining the mid-points of two sides of a triangle, is parallel to the third side and its length is 4 cm. What is the length of third side.
2. Prove that the line-segment joining the mid-points of the opposite sides of a quadrilateral bisect each other.
3. The line segments, joining the mid-points of the sides of a quadrilateral, taken in order, form a parallelogram.
4. Prove that the line segment passing through the mid-point of one side and parallel to another side of a triangle also bisect the third side.
5. Prove that four triangles obtained by joining the mid-points of the three sides of a triangle are all congruent to each other.

Theorem 10.1.4

The medians of a triangle are concurrent and their point of concurrency is the point of trisection of each median.

Given:

ΔABC, in which medians $\overline{BE}$ and $\overline{CF}$ meet in G.

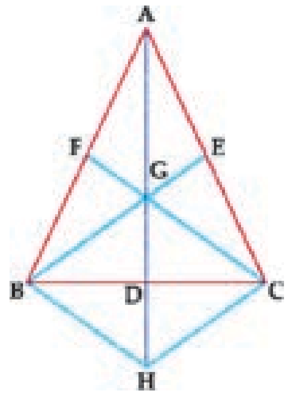
To prove:

i. $\overline{AG}$ bisects $\overline{BC}$ in D, and

ii. G is the point of trisection of each median.

Construction:

Draw $\overline{CH} \parallel \overline{EB}$ meeting $\overline{AD}$ produced in H. Join points B and H.



Proof:

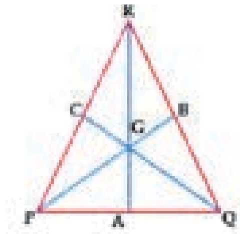
Statements	Reasons
In ΔACH, $\overline{AE} \cong \overline{EC}$	Given
and $\overline{EG} \parallel \overline{CH}$	Construction
$\therefore$ $\overline{AG} \cong \overline{GH}$	By converse of theorem 10.1.3
Further in ΔABH	
$\overline{AG} \cong \overline{GH}$	Proved above
$\overline{AF} \cong \overline{FB}$	Given
$\overline{FG} \parallel \overline{BH}$	By theorem 10.1.3
Hence BGCH is a $\parallel^m$	Opposite side are parallel
Diagonals $\overline{BC}$ and $\overline{GH}$ bisect each other	By theorem 10.1.1
i.e. $\overline{GD} \cong \overline{DH}$, $\overline{BD} \cong \overline{DC}$	
$\overline{AD}$ median of ΔABC	$\overline{BD} \cong \overline{DC}$ (Proved above)
Also $m\overline{AG} = m\overline{GH} = 2m\overline{GD}$	$\overline{GD} \cong \overline{DH} \Rightarrow m\overline{GH} = 2m\overline{GD}$
G is a point of trisection of $\overline{AD}$	As $\overline{AG}$ is twice of $\overline{GD}$
Similarly we can prove that G is a point of trisection of $\overline{BE}$ and $\overline{CF}$ as well	By the above process

Q.E.D

Exercise 10.4

1. If three medians of a triangle are congruent, prove that the triangle is an equilateral.

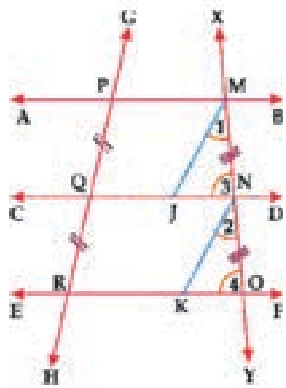
2. The medians $\overline{PB}, \overline{QC}$ and $\overline{RA}$ of ΔPQR meet in point G, show that G is the centroid of ΔPQR.



3. In the given figure, the length $\overline{GR}$ is of 2cm then find the length of $\overline{AG}$.

Theorem 10.1.5

If three or more parallel lines make congruent intercepts on the transversal, they also intercept congruent segments on any other line that cuts them.



Given:

$\overleftrightarrow{AB}, \overleftrightarrow{CD}$ and $\overleftrightarrow{EF}$ cut transversal $\overleftrightarrow{GH}$ at points P, Q and R respectively such that:

$\overline{PQ} \cong \overline{QR}$

$\overleftrightarrow{XY}$ is another transversal cutting $\overleftrightarrow{AB}$, $\overleftrightarrow{CD}$, $\overleftrightarrow{EF}$ in points M, N, O respectively.

To prove: $\overline{NM} \cong \overline{NO}$

Construction:

Draw $\overline{MJ}$ and $\overline{NK}$ each parallel to $\overleftrightarrow{GH}$ meeting $\overleftrightarrow{CD}$ and $\overleftrightarrow{EF}$ respectively in points J and K.

Proof:

Statements	Reasons
$\overline{PM} \parallel \overline{QJ}$	$\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ (Given)
and $\overline{PQ} \parallel \overline{MJ}$	Construction
$\therefore$ PMJQ is a $\parallel^m$	Opposite sides parallel
$\therefore$ $\overline{PQ} \cong \overline{MJ}$	Opposite sides of a $\parallel^m$
Similarly, QRKN is a $\parallel^m$	$\overline{QR} \parallel \overline{NK}$ and $\overline{QN} \parallel \overline{RK}$

$\therefore$ $\overline{QR} \cong \overline{NK}$	By above reason
But $\overline{PQ} \cong \overline{QR}$	Given
$\therefore$ $\overline{MJ} \cong \overline{NK}$	Transitive property of equality
Now $\overline{MJ} \parallel \overline{NK}$	Each paralle to $\overline{GH}$
$\therefore$ $\angle 1 \cong \angle 2$	Corresponding $\angle$s of $\parallel$ lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$
In $\Delta MNJ \leftrightarrow \Delta NOK$	
i. $\angle 1 \cong \angle 2$	i. Proved above
ii. $\angle 3 \cong \angle 4$	ii. Corresponding $\angle$s of $\parallel$ lines $\overleftrightarrow{MJ} \cong \overleftrightarrow{NK}$
iii. $\overline{MJ} \cong \overline{NK}$	iii. Proved above
$\therefore$ $\Delta MNJ \cong \Delta NOK$	A.A.S $\cong$ A.A.S
$\therefore$ $\overline{MN} \cong \overline{NO}$	By the congruence of Δs.

Q.E.D

Exercise 10.5

1. The triangle formed by joining the mid-points of the sides of a triangle is equivalent to the original triangle.
2. The line segment joining the mid-points of the non-parallel sides of a trapezium is parallel to the parallel sides and is equal to half their sum.
3. Every line segment drawn from the vertex to the base of a triangle is bisected by the line joining the mid-point of the other two sides.

Review Exercise 10

1. **Fill in the blanks:**

(i) In a parallelogram, opposite sides are ______________.

(ii) In a parallelogram, opposite angles are ______________.

(iii) In a triangle, medians are ______________.

(iv) In a parallelogram, the diagonals ______________ each other.

(v) In a parallelogram, corresponding angles are ______________.

(vi) Sum of the measures of interior angles of a quadrilateral is equal to __.

2. Tick (✓) the correct answer.

(i) Diagonals of a square are _______ to each other.
- a) Perpendicular   b) Non congruent
- c) Congruent   d) Both 'a' and 'c'

(ii) Sum of the measures of interior angles of a quadrilateral is
- a) 2 right angles   b) 4 right angles
- c) 3 right angles   d) none of these

(iii) Measure of a line segment joining the mid points of $\overline{AB}$ and $\overline{AC}$ of ΔABC is 3.5cm, then $m\overline{BC} =$
- a) 4.5cm   b) 5.5cm
- c) 6cm   d) 7cm

(iv) Two medians $\overline{AD}$ and $\overline{BE}$ of ΔABC intersect each other at G. If $m\overline{GD} = 1.7$cm, then $m\overline{AG} =$
- a) 2.7cm   b) 8.85cm
- c) 3.4cm   d) 5.1cm

(v) If sum of the measures of $\angle A$ and $\angle C$ of a parallelogram ABCD is 130°, then $m\angle B =$
- a) 25°   b) 115°   c) 65°   d) none of these

(vi) If opposite angles of a quadrilateral are equal in measures and none of them is a right angle, then the quadrilateral is a
- a) Square   b) Parallelogram
- c) Trapezium   d) Rectangle

(vii) Centroid is the common point of intersection of
- a) Medians of a triangle
- b) Diagonals of a parallelogram
- c) Angle bisectors of a triangle
- d) Perpendicular bisectors of a triangle

(viii) A point on median of a triangle is
- a) equidistant from its vertices
- b) equidistant from the mid points of its sides
- c) equidistant from its altitudes
- d) none of these

Summary

- Opposite sides of parallelogram are congruent.
- Opposite angles of parallelogram are congruent.
- Supplementary angles property holds for consecutive angles.
- Diagonals of a parallelogram bisect each other and each diagonal separates it into two congruent triangles
- If one angle of a parallelogram is right angle, then all the angles are right angles.
- Diagonals of a parallelogram divide it into four congruent triangles.
- Sum of the angles of a parallelogram is 360°.
- Sum of the interior angles of a triangle is 180°.
- Sum of the exterior angles of a triangle is 360°.
- If three or more parallel lines make congruent segments on a transversal they also intercept congruent segments on any other lines that cuts them.
- The medians of a triangle are concurrent and their point of concurrency is the point of trisection of each median.
- The line segment joining the mid-points of two sides of a triangle, is parallel to the third side and is equal to one half of its length.