Unit 15

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PROJECTION OF A SIDE OF A TRIANGLE

Student Learning Outcomes (SLOs)

After completing this unit, students will be able to:

- In an obtuse–angled triangle, the square on the side opposite to the obtuse angle is equal to the sum of the squares on the sides containing the obtuse angle together with twice the rectangle contained by one of the sides, and the projection on it of the other.
- In any triangle, the square on the side opposite to an acute angle is equal to the sum of the squares on the sides containing that acute angle diminished by twice the rectangle contained by one of those sides and the projection on it of the other.
- In any triangle, the sum of the squares on any two sides is equal to twice the square on half of the third side together with twice the square on the median which bisects the third side, (Apollonius' theorem).

15.1 Projection of a side of a Triangle

Understand the following theorems along with their corollaries and their applications to solve the allied problems.

Theorem 15.1.1

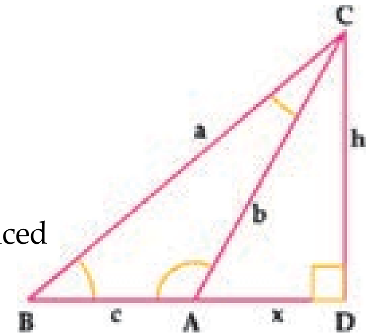
In an obtuse–angled triangle, the square on the side opposite to the obtuse angle is equal to the sum of the squares on the sides containing the obtuse angle together with twice the rectangle contained by one of the sides, and the projection on it of the other.

Given:

ΔABC with an obtuse angle at vertex A.

To prove:

i.e. $a^2 = b^2 + c^2 + 2cx$



Construction:

Draw perpendicular $\overline{CD}$ on $\overline{BA}$ produced meeting at point D, so that $\overline{AD}$ is the projection of $\overline{AC}$ on $\overline{BA}$ produced.

Taking, $m\overline{BC} = a$, $m\overline{CA} = b$, and $m\overline{AB} = c$ also $m\overline{AD} = x$ and $m\overline{CD} = h$.

Proof:

Statements	Reasons
In right angled ΔCDA	
$m\angle CDA = 90^0$	Construction
so, $(m\overline{AC})^2 = (m\overline{AD})^2 + (m\overline{DC})^2$	By Pythagoras theorem.
i.e $b^2 = x^2 + h^2$ … (i) ,	By supposition
In right angled ΔCDB	
$m\angle CDA = 90^0$	Construction
so, $(m\overline{BC})^2 = (m\overline{BD})^2 + (m\overline{CD})^2$	By Pythagoras theorem.
i.e. $a^2 = (c+x)^2 + h^2$	$m\overline{BD} = m\overline{BA} + m\overline{AD}$
or $a^2 = c^2 + 2cx + x^2 + h^2$ … (ii)	
$a^2 = c^2 + 2cx + b^2$	$b^2 = x^2 + h^2$
Thus $(m\overline{BC})^2 = (m\overline{AB})^2 + 2(m\overline{AB})(m\overline{AD}) + (m\overline{AC})^2$	

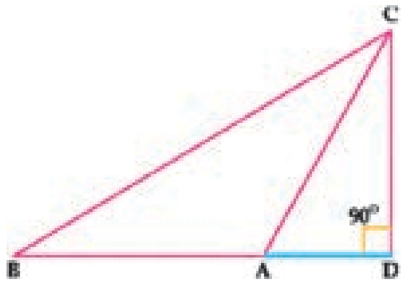
Q.E.D

Example: In a ΔABC with obtuse angle is at vertex A, if $\overline{CD}$ is an altitude on $\overline{BA}$ produced, meeting at point D, and $m\overline{AC}=m\overline{AB}$

Then, $\left(m\overline{BC}\right)^2 = 2\left(m\overline{AB}\right)\left(m\overline{BD}\right)$.

Given:

In a ΔABC, $m\angle A$ is an obtuse, $m\overline{AC} = m\overline{AB}$ and $\overline{CD}$ being altitude on $\overline{BA}$ produced, meeting at point D.



To prove:

$$\left(m\overline{BC}\right)^2 = 2\left(m\overline{AB}\right)\left(m\overline{BD}\right).$$

Proof:

Statements	Reasons
In a ΔABC	
$\left(m\overline{BC}\right)^2 = \left(m\overline{AB}\right)^2 + 2\left(m\overline{AB}\right)\left(m\overline{AD}\right) + \left(m\overline{AC}\right)^2$	Theorem 15.1.1
$\left(m\overline{BC}\right)^2 = \left(m\overline{AB}\right)^2 + 2\left(m\overline{AB}\right)\left(m\overline{AD}\right) + \left(m\overline{AB}\right)^2$	Given that $m\overline{AC} = m\overline{AB}$
$\left(m\overline{BC}\right)^2 = 2\left(m\overline{AB}\right)^2 + 2\left(m\overline{AB}\right)\left(m\overline{AD}\right)$	Taking $2m\overline{AB}$ as common
$\left(m\overline{BC}\right)^2 = 2m\overline{AB}\left(m\overline{AB} + m\overline{AD}\right)$	$m\overline{BD} = m\overline{AD} + m\overline{AB}$
$\left(m\overline{BC}\right)^2 = 2\left(m\overline{AB}\right)\left(m\overline{BD}\right)$	

Q.E.D

Exercise 15.1

1. Find the length of $\overline{AB}$ and area of the triangle ABC, when

(i) $m\overline{AC} = 3cm$, $m\overline{BC} = 6\ cm$ and $m\angle C = 120°$, where $m\overline{CD} = m\overline{BC}\ \cos\left(180° - m\angle C\right)$

(ii) $m\overline{AC} = 40\,mm$, $m\overline{BC} = 80\ mm$ and $m\angle C = 120°$, where $m\overline{CD} = m\overline{BC}\ \cos\left(180° - m\angle C\right)$

Hint: $\left(m\overline{BC}\right)^2 = \left(m\overline{AC}\right)^2 + \left(m\overline{AB}\right)^2 + 2\left(m\overline{AB}\right)\left(m\overline{AD}\right)$

2. Find the length of $m\overline{AC}$ in the ΔABC when $m\overline{BC} = 6cm$, $m\overline{AB} = 4\sqrt{2}\ cm$ and $m\angle ABC = 135°$,. If possible, find the area of the ΔABC.

3. Find the length of $m\overline{AC}$ in the ΔABC when $m\overline{BC} = 6\sqrt{2}\,cm$, $m\overline{AB} = 8\ cm$ and $m\angle ABC = 135°$,. If possible, find the area of the ΔABC.

Theorem 15.1.2

In any triangle, the square on the side opposite to an acute angle is equal to sum of the squares on the sides containing that acute angle diminished by twice the rectangle contained by one of those sides and the projection on it of the other.

Given:

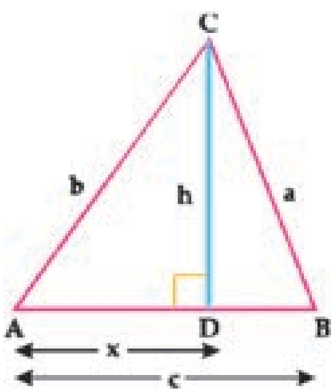
ΔABC with an acute $\angle CAB$ at vertex A.

Taking, $m\overline{BC} = a$, $m\overline{AC} = b$ and $m\overline{AB} = c$

Construction:

Draw $\overline{CD} \perp \overline{AB}$ so that $\overline{AD}$ is projection of $\overline{AC}$ on $\overline{AB}$

Also $m\overline{AD} = x$ and $m\overline{CD} = h$



To prove:

$$\left(m\overline{BC}\right)^2 = \left(m\overline{AC}\right)^2 + \left(m\overline{AB}\right)^2 - 2\left(m\overline{AB}\right)\left(m\overline{AD}\right)$$

i.e. $a^2 = b^2 + c^2 - 2cx$

Proof:

Statements	Reasons
In right angled ΔCDA,	
$m\angle CDA = 90^0$,	Construction
so, $\left(m\overline{AC}\right)^2 = \left(m\overline{AD}\right)^2 + \left(m\overline{CD}\right)^2$	Using Pythagoras Theorem.
i.e. $b^2 = x^2 + h^2$ … (i)	By supposition
In right angled ΔCDA	
$m\angle CDB = 90^0$	Construction
$\left(m\overline{BC}\right)^2 = \left(m\overline{BD}\right)^2 + \left(m\overline{CD}\right)^2$	By Pythagoras Theorem.
so, i.e.	From the figure.
$a^2 = (c - x)^2 + h^2$	$\therefore\ m\overline{BD} = m\overline{AB} - m\overline{AD}$
$a^2 = c^2 - 2cx + x^2 + h^2$	
or $a^2 = c^2 - 2cx + b^2$ … (ii)	Using equation… (i)
$a^2 = c^2 + b^2 - 2cx$	
Hence	
$\left(m\overline{BC}\right)^2 = \left(m\overline{AC}\right)^2 + \left(m\overline{AB}\right)^2 - 2\left(m\overline{AB}\right)\left(m\overline{AD}\right)$	

Q.E.D

Apollonius and the theorem of Apollonius:

Apollonius was a great geometer and astronomer.

Now we state and prove one of his well-known theorem "the Apollonius theorem".

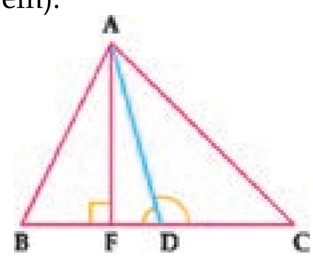
Theorem 15.1.3 (Apollonius theorem)

In any triangle, the sum of the squares on any two sides is equal to twice the square on half of the third side together with twice the square on the median which bisects the third side, (Apollonius' theorem).

Given:

In ΔABC, the median $\overline{AD}$ bisects $\overline{BC}$ at point D.

such that $m\overline{BD} = m\overline{CD}$.



To prove:

$$\left(m\overline{AB}\right)^2+\left(m\overline{AC}\right)^2=2\left(m\overline{BD}\right)^2+2\left(m\overline{AD}\right)^2$$

Construction:

Draw $\overline{AF}\perp\overline{BC}$.

Proof:

Statements	Reasons
In ΔADB Since, $\angle ADB$ is acute	ΔADF is right angled triangle with right angled at F (construction)
So, $\left(m\overline{AB}\right)^2=\left(m\overline{BD}\right)^2+\left(m\overline{AD}\right)^2-2\left(m\overline{BD}\right)\left(m\overline{FD}\right)$...(i)	Using theorem 15.1.2
Now, In ΔADC We $\angle ADC$ is an obtuse angle at point D.	Supplement of in $\angle ADB$
So, $\left(m\overline{AC}\right)^2=\left(m\overline{CD}\right)^2+\left(m\overline{AD}\right)^2+2\left(m\overline{CD}\right)\left(m\overline{FD}\right)$	Theorem 15.1.1
$\left(m\overline{AC}\right)^2=\left(m\overline{BD}\right)^2+\left(m\overline{AD}\right)^2+2\left(m\overline{BD}\right)\left(m\overline{FD}\right)$...(ii)	$m\overline{CD}=m\overline{BD}$
Thus, $\left(m\overline{AB}\right)^2+\left(m\overline{AC}\right)^2=2\left(m\overline{BD}\right)^2+2\left(m\overline{AD}\right)^2$	Adding eqs. (i) and (ii)

Q.E.D

Exercise 15.2

1. In $\Delta ABC, m\angle A = 30^0$ m, prove that
$\left(m\overline{BC}\right)^2 = \left(m\overline{AC}\right)^2 + \left(m\overline{AB}\right)^2 - \sqrt{3}\left(m\overline{AB}\right)\left(m\overline{AC}\right)$.
2. In a ΔABC, calculate $m\overline{BC}$ when $m\overline{AB} = 6cm, m\overline{AC} = 5cm$, and $m\angle A = 60^0$
3. Whether the triangle with sides 3cm, 4cm and 5cm is acute, obtuse or right angled.
4. Find the length of the median of side $\overline{BC}$ of a ΔABC where $m\overline{AB}$ = 4cm, $m\overline{AC}$ = 3cm and $m\overline{BC}$ = 6cm.

Review Exercise 15

1. **Fill in the blanks.**
   (i) In rt. $\Delta ABC, \left(m\overline{AB}\right)^2 +$ ___________ $= \left(m\overline{AC}\right)^2$,
   (ii) In ΔABC, two sides are equal to 4cm it is called__________ triangle.
   (iii) In ΔABC, with $m\angle B = 90^0$ then
   $\left(m\overline{AB}\right)^2 + \left(m\overline{BC}\right)^2 =$ __________
   (iv) 8cm, 15cm and 17 cm are the sides of ____________
2. In ΔABC, $m\overline{AC} = 3cm, m\overline{BC} = 6cm$ and $m\angle C = 120^0$. Compute $m\overline{AB}$

Summary

- In an obtuse–angled triangle, the square on the side opposite to the obtuse angle is equal to the sum of the squares on the sides containing the obtuse angle together with twice the rectangle contained by one of the sides, and the projection on it of the other.
- In any triangle, the square on the side opposite to an acute angle is equal to the sum of the squares on the sides containing that acute angle diminished by twice the rectangle contained by one of those sides and the projection on it of the other.
- In any triangle, the sum of the squares on any two sides is equal to twice the square on half the third side together with twice the square on the median which bisects the third side, (Apollonius theorem).