Unit 9

• Weightage = 7%

CONGRUENT TRIANGLES

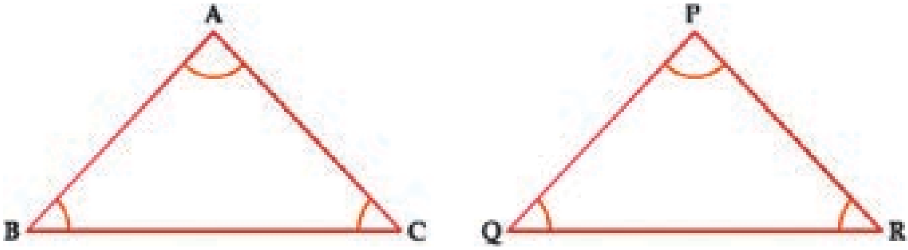
Student Learning Outcomes (SLOs)

After completing this unit, students will be able to:

- Understand the following theorems along with their corollaries and apply them to solve allied problems.
- In any correspondence of two triangles, if one side and any two angles of one triangle are congruent to the corresponding sides and angles of the other, the two triangles are congruent.
- If two angles of a triangle are congruent then the sides opposite to them are also congruent.
- In the correspondence of the two triangles, if three sides of one triangle are congruent to the corresponding three sides of the other, the two triangles are congruent or similar triangles.
- If in the correspondence of two right angled triangles, the hypotenuse and one side of one are congruent to the hypotenuse and the corresponding side of the other, then the triangles are congruent.

Introduction

A triangle has six elements, three sides and three angles. If we are given two triangles ABC and PQR, we can associate their vertices to establish a (1-1) correspondence between the sides and angles of these triangles in six different ways given as under:



In the correspondence $\Delta ABC \leftrightarrow \Delta PQR$ it means

(i)	$\angle A \leftrightarrow \angle P$	($\angle A$ corresponds to $\angle P$).
(ii)	$\angle B \leftrightarrow \angle Q$	($\angle B$ corresponds to $\angle Q$).
(iii)	$\angle C \leftrightarrow \angle R$	($\angle C$ corresponds to $\angle R$).
(iv)	$\overline{AB} \leftrightarrow \overline{PQ}$	($\overline{AB}$ corresponds to $\overline{PQ}$).
(v)	$\overline{BC} \leftrightarrow \overline{QR}$	($\overline{BC}$ corresponds to $\overline{QR}$).
(vi)	$\overline{CA} \leftrightarrow \overline{RP}$	($\overline{CA}$ corresponds to $\overline{RP}$).

9.1 Congruent triangles

"Sameness of size and shape" in the mathematics called congruence.

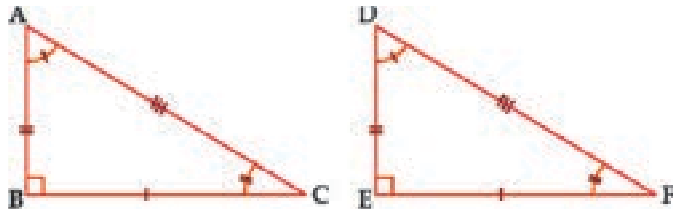
Consider two cars having different colours and positions as shown in the adjacent figure. But they have same size and shape. These two cars are said to be congruent. If we keep the picture of one car on the other car then they will overlap with each other.

ACTIVITY

Exploration

Can you identify any congruent figures or objects in your classroom or school? Make a list of these congruent figures by drawing or taking photos.

Two triangles are said to be congruent if their corresponding angles and sides are congruent.
Let's see the figures.



These two triangles ABC and DEF are congruent and written as:

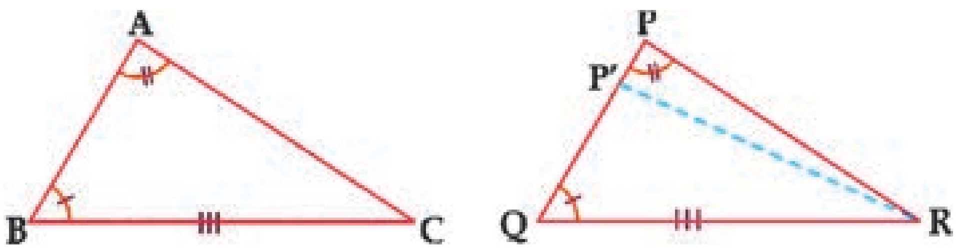
$$\Delta ABC \cong \Delta DEF$$

The ΔABC and ΔDEF having their corresponding sides and angles are equal in measure.
Note: Following results are useful.

(i) Identity congruence i.e $\Delta ABC \cong \Delta ABC$.
(ii) Symmetric property i.e $\Delta ABC \cong \Delta PQR$ then $\Delta PQR \cong \Delta ABC$.
(iii) Transitive property of congruence, if $\Delta ABC \cong \Delta PQR$ and $\Delta PQR \cong \Delta DEF$, then $\Delta ABC \cong \Delta DEF$.

Theorem 9.1.1 (A.S.A. $\cong$ A.S.A.)

In any correspondence of two triangles, if one side and any two angles of one triangle are congruent to the corresponding side and angles of the other, the two triangles are congruent.



Given:

In $\Delta ABC \leftrightarrow \Delta PQR$, then
$\angle B \cong \angle Q$, $m\overline{BC} \cong m\overline{QR}$,
and $\angle A \cong \angle P$.

To prove:

$\Delta ABC \cong \Delta PQR$

Construction:

Suppose, $\overline{AB} \ncong \overline{PQ}$ then take a point P′ on $\overline{PQ}$ such that $\overline{AB} \cong \overline{P'Q}$.
Join P′ to R.

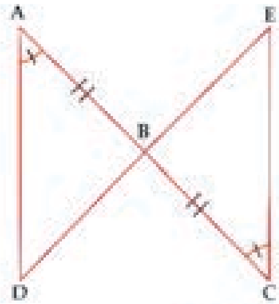
Proof:

Statements	Reasons
1. In $\Delta ABC \leftrightarrow \Delta PQR$	**1.** Correspondence of two Δs
i. $\angle A \cong \angle P$	i. Given
ii. $\angle B \cong \angle Q$	ii. Given
2. $\therefore$ $\angle C \cong \angle R$	**2.** Two corresponding angles of both triangles are congruent.
3. If $\overline{BA} \ncong \overline{QP}$, take a point P′ on $\overline{QP}$ (or $\overline{QP}$ produced) such that: $\overline{QP'} \cong \overline{BA}$	**3.** Assumption
4. In $\Delta ABC \leftrightarrow \Delta P'QR$	**4.** Correspondence of two Δs
i. $\overline{BC} \cong \overline{QR}$	i. Given
ii. $\angle B \cong \angle Q$	ii. Given
iii. $\overline{BA} \cong \overline{QP'}$	iii. By supposition
5. $\therefore$ $\Delta ABC \cong \Delta P'QR$	**5.** S.A.S postulate
6. $\therefore$ $\angle C \cong \angle QRP'$	**6.** Corresponding $\angle$s of congruent Δs.
7. But $\angle C \cong \angle QRP$	**7.** Proved in 2 (above).
8. $\therefore$ $\angle QRP' \cong \angle QRP$	**8.** Transitive property of congruence
9. This is possible only when points P′ and P coincide and $\overline{RP'} \cong \overline{RP}$	**9.** By angle construction postulate
10. Hence $\overline{BA} \cong \overline{QP}$	**10.** As P and P′ coincide.
11. In $\Delta ABC \leftrightarrow \Delta PQR$	**11.** Correspondence of two Δs
i. $\overline{BC} \cong \overline{QR}$	i. Given
ii. $\angle B \cong \angle Q$	ii. Given
iii. $\overline{BA} \cong \overline{QP}$	iii. Proved above
12. $\therefore$ $\Delta ABC \cong \Delta PQR$	**12.** S.A.S Postulate.

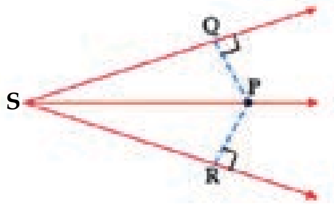
Q.E.D.

Exercise 9.1

1. In the adjacent figure, $m\overline{AB} = m\overline{CB}$ and $\angle A \cong \angle C$ prove that $\Delta ABD \cong \Delta CBE$

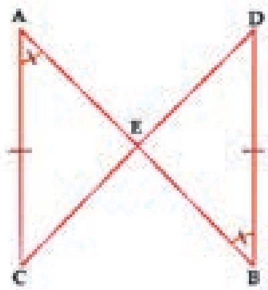


2. From a point on the line bisector of an angle, perpendiculars are drawn to the arms of the angle. Prove that these perpendiculars are equal in measure.

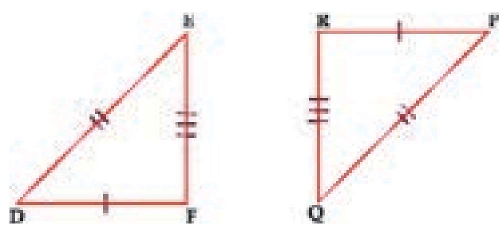


3. In the given figure, we have, $\Delta ACE \cong \Delta BDE$, such that $m\angle A = (3x + 1)°$, $m\angle B = (x + 35)°$, $m\angle AEC = (3y - 2)°$ and $m\angle DEB = (y + 8)°$.

Find the values of x and y.

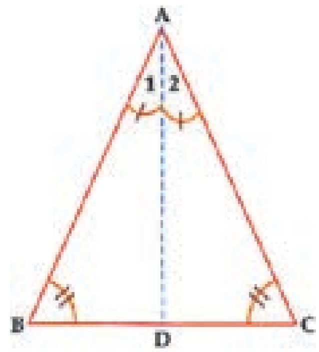


4. In the given figure,
$\Delta DEF \cong \Delta PQR$, such that:
$m\overline{DE} = (6x + 1)$cm, $m\overline{EF} = 8$cm,
and $m\,\overline{RQ} = (5y - 7)$cm
and $m\overline{PQ} = (10x - 19)$cm.
Find the values of x and y.



Theorem 9.1.2

If two angles of a triangle are congruent, then the sides opposite to them are also congruent.



Given:

In ΔABC,

We have, $\angle B \cong \angle C$

To prove:

$\overline{AC} \cong \overline{AB}$

Construction: Draw $\overline{AD}$ the bisector of $\angle A$, meeting $\overline{BC}$ at point D.

Proof:

Statement	Reason
In $\Delta ADB \leftrightarrow \Delta ADC$	
i. $\angle B \cong \angle C$	i. Given
ii. $\angle 1 \cong \angle 2$	ii. Construction
iii. $\overline{AD} \cong \overline{AD}$	iii. Common side of both Δs (Identity congruence)
$\therefore$ $\Delta ADB \cong \Delta ADC$	A.S.A $\cong$ A.S.A
$\therefore$ $\overline{AB} \cong \overline{AC}$	Corresponding sides of congruent Δs

Q.E.D

Exercise 9.2

1. ABC is a triangle in which $m\angle A = 35°$ and $m\angle B = 100°$, $\overline{BD} \perp \overline{AC}$. Prove that ΔBDC is an isosceles triangle.

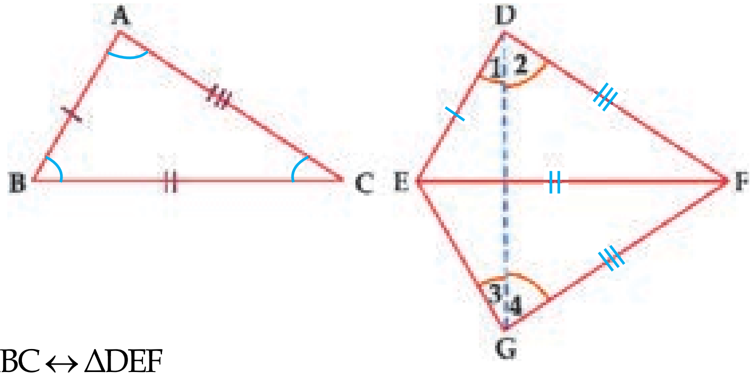
2. If the bisector of an angle of a triangle is perpendicular to its opposite side, then prove that triangle is an isosceles triangle.

3. ABC is a triangle in which $m\angle B = 45°$ and $\overline{CD} \perp \overline{AB}$. Prove that ΔDBC is an isosceles Δ.

Theorem 9.1.3

In a correspondence of two triangles, if three sides of one triangle are congruent to the corresponding three sides of the other, then the two triangles are congruent.

Proof:



Given:

In $\Delta ABC \leftrightarrow \Delta DEF$

$\overline{AB} \cong \overline{DE}, \overline{BC} \cong \overline{EF}$ and $\overline{CA} \cong \overline{FD}$

To prove that: $\Delta ABC \cong \Delta DEF$

Construction: Suppose $\overline{BC}$ is the greatest of all the three sides of ΔABC. Construct ΔGEF such that:

i. Point G is on the opposite side of point D.

ii. $\angle FEG \cong \angle B$

iii. $\overline{EG} \cong \overline{BA}$

Join D and G.

Proof:

Statements	Reasons
1. In $\Delta ABC \leftrightarrow \Delta GEF$	**1.** Correspondence of two Δs
i. $\overline{BC} \cong \overline{EF}$	i. Given
ii. $\angle B \cong \angle GEF$	ii. Construction
iii. $\overline{BA} \cong \overline{GE}$	iii. Construction
2. $\therefore$ $\Delta ABC \cong \Delta GEF$	**2.** S.A.S. postulate.
3. $\therefore$ $\overline{AC} \cong \overline{GF}$ and $\angle A \cong \angle G$	**3.** By the congruence of triangles.
4. But $\overline{DF} \cong \overline{AC}$	**4.** Given
5. $\therefore$ $\overline{GF} \cong \overline{DF}$	**5.** Transitive property.
6. $\therefore$ In ΔDEG, $m\angle 1 = m\angle 3$	**6.** Opposite sides congruent $\overline{EG} \cong \overline{BA} \cong \overline{ED}$

7. Similarly, in ΔGFD, $m\angle 2 = m\angle 4$	7. $\overline{DF} \cong \overline{GF}$
8. $\therefore$ $m\angle 1 + m\angle 2 = m\angle 3 + m\angle 4$	8. Addition property of equation
9. or $m\angle D = m\angle G$	9. $m\angle 1 + m\angle 2 = m\angle D$ $m\angle 3 + m\angle 4 = m\angle G$
10. But $m\angle G = m\angle A$	10. Proved in (3) above
11. $\therefore$ $m\angle A = m\angle D$	11. Transitive property in (3)
12. In $\Delta ABC \leftrightarrow \Delta DEF$	12. Correspondence of two Δs
i. $\overline{AB} \cong \overline{DE}$	i. Given
ii. $\angle A \cong \angle D$	ii. Proved above
iii. $\overline{AC} \cong \overline{DF}$	iii. Given
13. $\therefore$ $\Delta ABC \cong \Delta DEF$	13. S.A.S Postulate

Q.E.D

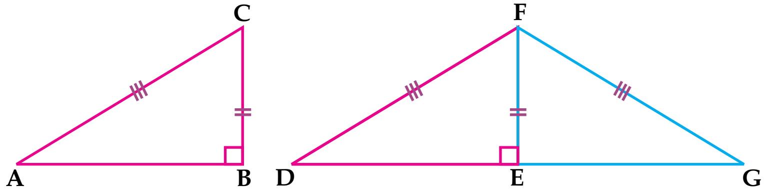
Corollary: The angles of an equilateral triangle are also equal in measurement.

Exercise 9.3

1. ABC is an isosceles triangle. D is the mid-point of base $\overline{BC}$. Prove that $\overline{AD}$ bisects $\angle A$ and $\overline{AD} \perp \overline{BC}$.
2. ABC and DBC are two isosceles triangles on the same side of a common base $\overline{BC}$. Prove that $\overline{AD}$ is the right bisector of $\overline{BC}$.
3. PQRS is a square. X,Y and Z are the mid-points of $\overline{PQ}$, $\overline{QR}$ and $\overline{RS}$ respectively. Prove that $\Delta PXY \cong \Delta SZY$.
4. Prove that, in an equilateral triangle any two median are congruent.

Theorem 9.1.4

If in the correspondence of two right-angled triangles, the hypotenuse and one side of one triangle are congruent to the hypotenuse and the corresponding side of the other, then the triangles are congruent (H.S≅H.S).



Given: In correspondence

$\Delta ABC \leftrightarrow \Delta DEF$

$\angle B \cong \angle E$ (rt $\angle$s) $\overline{AC} \cong \overline{DF}$ (Hyp) and $\overline{BC} \cong \overline{EF}$

To Prove: $\Delta ABC \cong \Delta DEF$

Construction: Produce $\overline{DE}$ to point G such that $\overline{EG} \cong \overline{AB}$. Then join F and G.

Proof:

Statements	Reasons
$m\angle DEF + m\angle GEF = 180^\circ$	Supplement postulate
But $m\angle DEF = 90^\circ$	Given
$\therefore$ $m\angle GEF = 90^\circ$	$\therefore$ $180^\circ - 90^\circ = 90^\circ$
In $\Delta GEF \leftrightarrow \Delta ABC$	
i. $\overline{GE} \cong \overline{AB}$	Construction
ii. $\angle GEF \cong \angle ABC$	Each is right angle
iii. $\overline{EF} \cong \overline{BC}$	Given
$\therefore$ $\Delta GEF \cong \Delta ABC$	$S.A.S. \cong S.A.S.$
$\therefore$ $\overline{FG} \cong \overline{AC}$ and $\angle G \cong \angle A$	By the congruence of Δs.
$\therefore$ $\overline{AC} \cong \overline{DF}$	$\therefore$ $\overline{AC} \cong \overline{DF}$ (Given)
In ΔDFG, $\angle D \cong \angle G$	Opposite sides congruents
$\therefore$ $\angle D \cong \angle A$	Each is congruent to ΔG
In $\Delta ABC \leftrightarrow \Delta DEF$	
i. $\angle A \cong \angle D$	i. Proved
ii. $\angle ABC \cong \angle DEF$	ii. rt Δs
iii. $\overline{AC} \cong \overline{DF}$	iii. Given
$\therefore$ $\Delta ABC \cong \Delta DEF$	$A.A.S \cong A.A.S$

Q.E.D

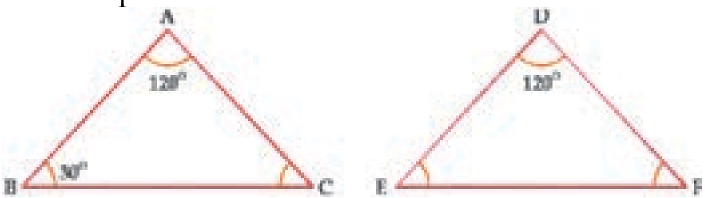
Note: Theorem 9.1.4 can be proved by S.A.S postulate.

Exercise 9.4

1. Prove that:
   The perpendiculars from the vertices of the base to opposite sides of an isoceles triangle are congruent. (Hint: Medians and altitudes of the triangle are congruent)
2. Prove that, if the bisector of an angle of a triangle bisects its opposite side, then the triangle will be an isosceles triangle.
3. Prove that the median bisecting the base of an isosceles triangle bisects the vertical angle and is perpendicular to the base.
4. Prove that if three altitudes of a triangle are congruent, then the triangle is equilateral.

Review Exercise 9

1. If $\Delta ABC \cong \Delta DEF$, $m\angle F$ is equal to
   A. 90°
   B. 60°
   C. 30°
   D. 20°



2. Identify true and false statement in the following:
   (i) The sum of the measure of all angles in an quadrilateral is 360°.
   (ii) The sum of the measure of all angles in a triangle is 270°.
   (iii) In an equilateral triangle, angles are of the same measurement.
   (iv) There are two right angles in a traiangle.
   (v) In an isosceles triangles, corresponding angles and correspoinding sides are equal in measure.
3. Fill in the blanks to make the sentences true sentences:
   (i) In $\Delta ABC \leftrightarrow \Delta DEF$, then $\overline{AC}$ corresponds to _____________.
   (ii) In $\Delta KLM \leftrightarrow \Delta PQR$, then $\angle MKL$ corresponds to _____________
   (iii) In an isosceles triangle, the base angle are _____________.
   (iv) If the mesure of each of the angles of a triangle is 60°, then the triangle is_____________.
   (v) In a right-angled triangle, side opposite to right angle is called _____________.
   (vi) The sum of the measures of acute angle of a right triangle is_____________.

4. Encircle the corresponding letters a,b,c or d for correct answer:

(i) Which of the following is not a sufficent condition for congurence of two triangles?

(a) A.S.A ≅A.S.A (b) H.S ≅ H.S

(c) S.A.A≅ S. A. A (d) A.A.A≅ A. A. A

(ii) In ΔABC, if ∠A ≅ ∠B, then the bisector of _____ angle divides the triangle into congruent triangles:

(a) ∠A (b) ∠B

(c) ∠C (d) any one of its angles.

(iii) The diagonal of ____ does not divide it into two congruent triangles:

(a) Rectangle (b) Trapezium

(c) Parallogram (d) Square

(iv) How many acute angles are there in an acute angled triangle?

(a) 1 (b) 2

(c) 3 (d) not more than 2.

Summary

In this unit we stated and proved the following theorem:

- In any correspondence of two traingles, if one side and any two angles of one traingle are congruent to the corresponding side and angles of the other, the two taringles are congruent. (A.S.A. ≅ A.S.A)
- If two angles of a traingles are congruent, then the side opposite to them are also congruent.
- In the correspondence of two traingles, if three sides of two traingles are congruent to the corresponding three sides of other, then the two traingles are congruent (S.S.S ≅ S.S.S).
- If in the correspondence of the two right-angled traingles the hypotenuse and one side of one traingles are congruent to the hypotenuse and the corresponding side of other, then the traingles are congruent. (H.S ≅ H.S).
- Two traingles are said to be congruent, if there exists a correspondence between them such that all the corresponding sides and traingles are congruent.(S.S.S).