Unit 8

• Weightage = 8%

QUADRATIC EQUATIONS

Student Learning Outcomes (SLOs)

After completing this unit, students will be able to:

- Solve a quadratic equation in one variable by
  - Factorization,
  - Completing the squares.
- Use method of completing the squares to drive the quadratic formula.
- Use quadratic formula to solve quadratic equations.
- Solve equations, reducible to quadratic form, of the type $ax^4 + bx^2 + c = 0$, Quartic or Bi-quadratic equations.
- Solve the equations of the type $ap(x) + \frac{c}{p(x)} = b$, where a, b and c are rational numbers.
- Solve the reciprocal equations of the type $a\left(x^2 + \frac{1}{x^2}\right) + b\left(x + \frac{1}{x}\right) + c = 0$, where a , b and c are rational numbers.
- Solve the exponential equations in which the variable occurs in exponents.
- Solve the equations of the type $(x+a)(x+b)(x+c)(x+d) = k$, where $a + b = c + d$ k $\neq 0$.
- Solve the equations of the type:
  - $\sqrt{(ax + b)} = cx + d$.
  - $\sqrt{(x + a)} + \sqrt{(x + b)} = \sqrt{(x + c)}$
  - $\sqrt{(x^2 + px + m)} + \sqrt{(x^2 + px + n)} = q$.

8.1 Quadratic Equations and their Solutions

8.1.1 Elucidate, then define Quadratic Equation in it Standard Form

A polynomial equation with degree 2 is called a quadratic equation.

The standard form of quadratic equation is $ax^2+bx+c=0$, where $a \neq 0$ and $a, b, c \in \mathbb{R}$. In this form a is the coefficient of x^2, b is the coefficient of x and c is the constant term.

In $ax^2+bx+c=0$, if $a=0$, then it reduces to linear equation i.e., $bx+c=0$ and if $b=0$ then it reduces to the pure quadratic form i.e., $ax^2+c=0$

Following are the examples of quadratic equations.

(i) $4x^2+4x+1=0$, (Quadratic equation is in the standard form)

(ii) $x^2-4=0$, (Pure quadratic equation)

8.1.2 Solve a quadratic equation in one variable by

- **Factorization**
- **Completing the square**

Here we consider two methods, for the solution of the quadratic equation.

(a) Method of factorization (b) Method of completing the square.

(a) Method of Factorization

Example 01 Solve: (i) $x^2+2x-15=0$ (ii) $2x^2-5x=12$

Solution (i): $x^2+2x-15=0$

$\Rightarrow \quad x^2+5x-3x-15=0$

$\Rightarrow \quad x(x+5)-3(x+5)=0$

$\Rightarrow \quad (x-3)(x+5)=0$

$\Rightarrow \quad x-3=0$ or $x+5=0$

$\Rightarrow \quad x=3$ or $x=-5$

Thus, S.S = $\{-5,3\}$

Solution (ii): $2x^2-5x=12$

$$\Rightarrow \quad 2x^2-5x-12=0$$
$$\Rightarrow \quad 2x^2-8x+3x-12=0$$
$$\Rightarrow \quad 2x(x-4)+3(x-4)=0$$
$$\Rightarrow \quad (x-4)(2x+3)=0$$
$$\Rightarrow \quad x-4=0 \quad \text{or} \quad 2x+3=0$$
$$\Rightarrow \quad x=4 \quad \text{or} \quad x=\frac{-3}{2}$$

Thus, the solution set is $\left\{-\frac{3}{2},\ 4\right\}$

Example 02 Solve the pure quadratic equation $4m^2-1=0$ for m by factorization method:

Solution: $4m^2-1=0$

$$\Rightarrow \quad (2m)^2-(1)^2=0$$
$$\Rightarrow \quad (2m-1)(2m-1)=0 \qquad \left[\ a^2-b^2=(a-b)(a+b)\right]$$
$$\Rightarrow \quad 2m-1=0 \quad \text{or} \quad 2m+1=0$$

i.e $\Rightarrow \quad 2m=1 \quad \text{or} \quad 2m=-1$

$$\Rightarrow \quad m=\frac{1}{2} \quad \text{or} \quad m=-\frac{1}{2}$$

Thus, s.s = $\left\{-\frac{1}{2},\ \frac{1}{2}\right\}$.

(b) Method of Completing Square.

Method is explained as under:

(i) Write the equation in the standard form i.e. $ax^2+bx+c=0$

(ii) Divide both the sides of the equation by leading coefficient of x^2 in order to make it 1.

(iii) Shift the constant term to the R.H.S.

(iv) For completing the square add $\left(\frac{\text{coefficient of } x}{2}\right)^2$ on both sides

(v) Write the L.H.S. of the equations as a perfect square and then simplify the R.H.S.

(vi) Take the square root of both the sides of the given equation. Solve the resulting equation to find the solution of the equation and then write the solution set.

Example 01 Solve $2x^2+8x-1=0$

Solution: $2x^2+8x-1=0$

$\Rightarrow \quad 2x^2+8x=1 \quad ...(i)$

$\Rightarrow \quad x^2+4x=\frac{1}{2} \quad ...(ii) \qquad$ [By dividing equation (i) by 2]

By adding $\left[\frac{1}{2}\times 4\right]^2=4$ on both the sides in equation (ii)

we get,

$$x^2+4x+4=\frac{1}{2}+4$$

$$\Rightarrow \quad x^2+2(2)x+(2)^2=\frac{1}{2}+(2)^2$$

$$\Rightarrow \quad (x+2)^2=\frac{1}{2}+4$$

$$\Rightarrow \quad (x+2)^2=\frac{9}{2}$$

$$\Rightarrow \quad x+2=\pm\frac{3}{\sqrt{2}}$$

$\Rightarrow \quad x+2=\frac{3}{\sqrt{2}}$	$x+2=-\frac{3}{\sqrt{2}}$
$\Rightarrow \quad x+2=\frac{3\sqrt{2}}{2}$	$x+2=\frac{-3\sqrt{2}}{2}$
$\Rightarrow \quad x=-2+\frac{3\sqrt{2}}{2}$	$x=-2-\frac{3\sqrt{2}}{2}$
$\Rightarrow \quad x=\frac{-4+3\sqrt{2}}{2}$	$x=\frac{-4-3\sqrt{2}}{2}$

Thus, s.s is $\left\{\frac{-4+3\sqrt{2}}{2}, \frac{-4-3\sqrt{2}}{2}\right\}$.

Exercise 8.1

1. Solve the following quadratic equations by factorization method:

(i) $x^2+5x+6=0$ (ii) $6x^2-x-1=0$ (iii) $x^2-11x+30=0$

(iv) $x^2-2x=0$ (v) $x^2-2x-15=0$ (vi) $12x^2-41x+24=0$

(vii) $(x-5)^2-9=0$ (viii) $(3x+4)^2-16=0$

2. Solve each of the following by completing the square method:

(i) $x^2+6x+1=0$ (ii) $(3x+2)(x+2)=6-2(x+1)$. (iii) $3x^2-8x=-1$

(iv) $24x^2=-10x+21$ (v) $2(x^2-3)-3x=2(x+3)$ (vi) $2x^2+4x-1=0$

3. The equation $3x^2+bx-8=0$ has 2 as one of its roots.

(i) What is the value of b?

(ii) What is the other root of the equation?

8.2 Quadratic Formula

For equation $ax^2+bx+c=0, a\neq 0$ we use the following formula to solve it i.e. $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ the formula is known as Quadratic formula.

8.2.1 Use method of completing the square to derive the Quadratic Formula.

The standard form of a quadratic equation is given by

$$ax^2+bx+c=0 \quad \text{where } a\neq 0 \quad \text{..........(i)}$$

By dividing a on both sides of equation (i), we get

$$\frac{ax^2}{a}+\frac{bx}{a}+\frac{c}{a}=\frac{0}{a}$$

$$\therefore \quad x^2+\frac{bx}{a}+\frac{c}{a}=0$$

By shifting constant term $\frac{c}{a}$ to R.H.S

$$\therefore \quad x^2+\frac{bx}{a}=-\frac{c}{a} \quad ...\text{(ii)}$$

By adding $\left(\frac{b}{2a}\right)^2$ on both sides of equation (ii)

$$\therefore \quad x^2+\frac{bx}{a}+\left(\frac{b}{2a}\right)^2=-\frac{c}{a}+\left(\frac{b}{2a}\right)^2$$

$$\Rightarrow \quad x^2+2(x)\left(\frac{b}{2a}\right)+\left(\frac{b}{2a}\right)^2=-\frac{c}{a}+\frac{b^2}{4a^2}$$

$$\Rightarrow \quad \left(x+\frac{b}{2a}\right)^2=\frac{-4ac+b^2}{4a^2}$$

$$\Rightarrow \quad \left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

$$\Rightarrow \quad x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$\Rightarrow \quad x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$\Rightarrow \quad \boxed{x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}}$$ This is known as Quadratic Formula.

8.2.2 Use of Quadratic Formula to solve Quadratic Equations.

Example 01 Solve by using quadratic formula

(i) $2x^2-5x-3=0$ (ii) $x^2+x+1=0$

Solution (i): $2x^2-5x-3=0$

Here, $a=2, b=-5$ and $c=-3$

By using quadratic formula

i.e $$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\therefore \quad x=\frac{-(-5)\pm\sqrt{(-5)^2-4(2)(-3)}}{4}$$

$$\Rightarrow \quad x=\frac{5\pm\sqrt{25-(-24)}}{4}$$

$$\Rightarrow \quad x=\frac{5\pm\sqrt{49}}{4}$$

$$\Rightarrow \quad x=\frac{5\pm7}{4}$$

$$\Rightarrow \quad x = \frac{5+7}{4} \qquad\qquad x = \frac{5-7}{4}$$

$$\Rightarrow \quad x = \frac{12}{4} \qquad\qquad x = \frac{-2}{4}$$

$$\Rightarrow \quad x = 3 \qquad\qquad x = -\frac{1}{2}$$

so, the roots are 3 and $-\frac{1}{2}$

Thus, $S.S = \left\{3, -\frac{1}{2}\right\}$.

Solution (ii): $x^2 + x + 1 = 0$

Here, $a = 1$, $b = 1$ and $c = 1$

By using quadratic formula

$$\because x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore \quad x = \frac{-1 \pm \sqrt{(1)^2 - 4(1)}}{2(1)}$$

$$\Rightarrow \quad x = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$\Rightarrow \quad x = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2},$$

Thus, $S.S = \left\{\frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}\right\}$

Exercise 8.2

Solve the following equations by using the Quadratic Formula:

(i) $x^2 - 2x = 15$ (ii) $10x^2 + 19x - 15 = 0$ (iii) $x^2 = -x + 1$

(iv) $2x = 9 - 3x^2$ (v) $9x^2 = 12x - 49$ (vi) $\frac{1}{2}x^2 + \frac{3}{4}x - 1 = 0$

(vii) $3x^2 - 2x + 2 = 0$ (viii) $6x^2 - x - 1 = 0$ (ix) $4x^2 - 10x = 0$

(x) $x^2 - 1 = 0$ (xi) $x^2 - 6x + 9 = 0$ (xii) $\frac{1}{x+4} - \frac{1}{x-4} = 4$

8.3 Equations Reducible to Quadratic Form

There are various types of equations which are not quadratic, but can be reduced into the quadratic form by taking suitable substitution.

8.3.1 Solve equation reducible to quadratic form of the type $ax^4 + bx^2 + c = 0, a \neq 0$ i.e., quartic or bi-quadratic equation.

Consider the equation $ax^4 + bx^2 + c = 0$, it is quartic or bi-quadratic equation as it has degree 4, and can be reduced into the quadratic equation having form $ay^2 + by + c = 0$, where $y=x^2$. The method is explained by the following example.

Example Solve the quartic equation $4x^4 - 25x^2 + 36 = 0$

Solution: $4x^4 - 25x^2 + 36 = 0$ . . . (i)

This equation can be written as:

$4(x^2)^2 - 25(x^2) + 36 = 0$ . . . (ii)

By putting $y = x^2$ in equation (ii), we have,

$$4y^2 - 25y + 36 = 0$$

Here, $a = 4, b = -25$ and $c = 36$

$$\because \quad y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore \quad y = \frac{-(-25) \pm \sqrt{(-25)^2 - 4(4)(36)}}{2(4)}$$

$$y = \frac{25 \pm \sqrt{625 - 576}}{2(4)} = \frac{25 \pm \sqrt{49}}{8} = \frac{25 \pm 7}{8}$$

i.e., $y = \frac{25+7}{8}$	$y = \frac{25-7}{8}$
$y = \frac{32}{8} = 4$	$y = \frac{18}{8} = \frac{9}{4}$
but, $y = x^2$, then	
$x^2 = 4$	$x^2 = \frac{9}{4}$

$x = \pm 2$ | $x = \pm \dfrac{3}{2}$

Thus, $S.S = \left\{ \pm 2,\ \pm \dfrac{3}{2} \right\}$

8.3.2 Solve equation of the type $ap(x) + \dfrac{b}{p(x)} = c$ where a, b and c are real numbers, $a \neq 0$, where $p(x)$ is an algebraic expression

Example 01 Solve $8\sqrt{x+3} - \dfrac{1}{\sqrt{x+3}} = 2$

Solution:

$$8\sqrt{x+3} - \frac{1}{\sqrt{x+3}} = 2 \qquad \ldots \text{(i)}$$

Let $y = \sqrt{x+3} \Rightarrow \dfrac{1}{y} = \dfrac{1}{\sqrt{x+3}}$, so (i) becomes

$$\Rightarrow \quad 8y - \frac{1}{y} = 2$$

$$\Rightarrow \quad 8y^2 - 1 = 2y$$

$$\Rightarrow \quad 8y^2 - 2y - 1 = 0$$

$$\Rightarrow \quad 8y^2 - 4y + 2y - 1 = 0$$

$$\Rightarrow \quad 4y(2y-1) + 1(2y-1) = 0$$

$$\Rightarrow \quad (4y+1)(2y-1) = 0$$

$\Rightarrow \quad 4y + 1 = 0$ | $2y - 1 = 0$

$\Rightarrow \quad y = \dfrac{-1}{4}$ | $y = \dfrac{1}{2}$

when $y = -\dfrac{1}{4}$, | when $y = \dfrac{1}{2}$

$\therefore \quad \sqrt{x+3} = -\dfrac{1}{4}$ | $\therefore \quad \sqrt{x+3} = \dfrac{1}{2}$

i.e Squaring on both sides, we get | i.e Squaring on both sides, we get

$\Rightarrow \quad x + 3 = \dfrac{1}{16}$ | $x + 3 = \dfrac{1}{4}$

$\Rightarrow \quad x = \dfrac{1}{16} - 3$	$x = \dfrac{1}{4} - 3$
$\Rightarrow \quad x = \dfrac{1-48}{16}$	$x = \dfrac{1-12}{4}$
$\Rightarrow \quad x = -\dfrac{47}{16}$	$x = -\dfrac{11}{4}$

As eq-(i) is a radical equation. So verification of roots of (i) is essential.

Verification: $$8\sqrt{x+3} - \frac{1}{\sqrt{x+3}} = 2$$

By putting $x = -\dfrac{47}{16}$ in eq....(i)	By putting $x = -\dfrac{11}{4}$ in eq....(i)
$8\sqrt{\dfrac{-47}{16} + 3} - \dfrac{1}{\sqrt{\dfrac{-47}{16} + 3}} = 2$	$8\sqrt{\dfrac{-11}{4} + 3} - \dfrac{1}{\sqrt{\dfrac{-11}{4} + 3}} = 2$
$8\sqrt{\dfrac{-47+48}{16}} - \dfrac{1}{\sqrt{\dfrac{-47+48}{16}}} = 2$	$8\sqrt{\dfrac{-11+12}{4}} - \dfrac{1}{\sqrt{\dfrac{-11+12}{4}}} = 2$
$8\sqrt{\dfrac{1}{16}} - \dfrac{1}{\sqrt{\dfrac{1}{16}}} = 2$	$8\sqrt{\dfrac{1}{4}} - \dfrac{1}{\sqrt{\dfrac{1}{4}}} = 2$
$8\left(\dfrac{1}{4}\right) - \dfrac{1}{\left(\dfrac{1}{4}\right)} = 2$	$8\left(\dfrac{1}{2}\right) - \dfrac{1}{\left(\dfrac{1}{2}\right)} = 2$
$2 - 4 = 2$	$4 - 2 = 2$
$-2 \neq 2$	$2 = 2$
Not verified	Verified

On verification it is found that $x = -\dfrac{47}{16}$ does not satisfy the original equation.

Hence, it is an extraneous root, and cannot be included in the solution set.

Thus, S.S $= \left\{-\dfrac{11}{4}\right\}$.

8.3.3 Solve the Reciprocal Equation of the type $a\left(x^2+\frac{1}{x^2}\right)+b\left(x+\frac{1}{x}\right)+c=0$, where a,b and c are rational numbers.

Definition:

An equation in x is said to be a reciprocal equation, if it remains un-changed when x is replaced by $\frac{1}{x}$.

The method for solving reciprocal equation of the type $a\left(x^2+\frac{1}{x^2}\right)+b\left(x+\frac{1}{x}\right)+c=0$, where a, b, c are rational numbers, explained through an example.

Example 01 Solve: $2\left(x^2+\frac{1}{x^2}\right)-9\left(x+\frac{1}{x}\right)+14=0$

Solution: $2\left(x^2+\frac{1}{x^2}\right)-9\left(x+\frac{1}{x}\right)+14=0 \quad . . . \text{(i)}$

Let $x+\frac{1}{x}=y$ then $x^2+\frac{1}{x^2}=y^2-2$, so, equation (i) becomes

$$2\left(y^2-2\right)-9y+14=0$$

$$\Rightarrow \quad 2y^2-9y+10=0$$

$$\Rightarrow \quad 2y^2-4y-5y+10=0$$

$$\Rightarrow \quad 2y(y-2)-5(y-2)=0$$

$$\Rightarrow \quad (y-2)(2y-5)=0$$

i.e. $y-2=0$

$\Rightarrow \quad y=2$

when $y=2$

$\therefore \quad x+\frac{1}{x}=2$

$\Rightarrow \quad x^2+1=2x$

$\Rightarrow \quad x^2-2x+1=0$

$2y-5=0$

$y=\frac{5}{2}$

when $y=\frac{5}{2}$

$\therefore \quad x+\frac{1}{x}=\frac{5}{2}$

$\Rightarrow \quad \frac{x^2+1}{x}=\frac{5}{2}$

$\Rightarrow \quad 2x^2+2=5x$

$\Rightarrow \quad (x-1)^2 = 0$

$\Rightarrow \quad x-1 = 0$

$\Rightarrow \quad x = 1$

$\Rightarrow \quad 2x^2 - 5x + 2 = 0$

$\Rightarrow \quad 2x^2 - 4x - x + 2 = 0$

$\Rightarrow \quad 2x(x-2) - 1(x-2) = 0$

$\Rightarrow \quad (x-2)(2x-1) = 0$

Either $x - 2 = 0$

$\Rightarrow \quad x = 2$

or $2x - 1 = 0$

$\Rightarrow \quad x = \frac{1}{2}$

Thus, S.S $= \left\{\frac{1}{2}, 2, 1\right\}$.

8.3.4 Solve the Exponential Equations

Definition:

An equation in which the variable appears as an exponent, is called an exponential equation. Solution of such type of equation is explained through an example.

Example 01 solve $7^{1+x} + 7^{1-x} = 50$

Solution: $7^{1+x} + 7^{1-x} = 50 \quad \ldots$ (i)

$7.7^x + 7.7^{-x} - 50 = 0 \quad$ (Splitting power)

$\Rightarrow \quad 7.7^x + \frac{7}{7^x} = 50 \quad \ldots$ (ii)

Let $\quad y = 7^x$

$\therefore$ above equation (ii) reduces as under,

$7y + \frac{7}{y} - 50 = 0$,

$\Rightarrow \quad 7y^2 + 7 - 50y = 0$

$\Rightarrow \quad 7y^2 - 50y + 7 = 0$

$\Rightarrow \quad 7y^2 - 49y - y + 7 = 0$ (Factorizing)

$\Rightarrow \quad 7y(y-7) - 1(y-7) = 0$

$\Rightarrow \quad (7y-1)(y-7) = 0$

Either,

$$7y-1=0 \quad \text{or} \quad y-7=0$$

$$\Rightarrow \quad y=\frac{1}{7} \quad \Big| \quad \Rightarrow \quad y=7$$

when $y=\frac{1}{7} \Rightarrow 7^x=\frac{1}{7}=7^{-1}$ | when $y=7 \Rightarrow 7^x=7^1$

$\Rightarrow \quad x=-1$ | $x=1$

Thus, S.S $=\{-1,1\}$.

8.3.5 Solve the Equations of the type $(x+a)(x+b)(x+c)(x+d)=k$, where, $a+b=c+d$ and the constant $k\neq 0$.

Example 01 $(x+1)(x+2)(x+3)(x+4)=48$

Solution: $(x+1)(x+2)(x+3)(x+4)=48$

By re-arranging the factors, we have

$$(x+1)(x+4)(x+2)(x+3)=48$$

$$\left(x^2+4x+x+4\right)\left(x^2+2x+3x+6\right)=48$$

$$\left(x^2+5x+4\right)\left(x^2+5x+6\right)=48\ldots \text{(i)}$$

Let $x^2+5x=t$ ...(ii)

By substituting in equation (i)

$$\Rightarrow \quad (t+4)(t+6)=48$$

$$\Rightarrow \quad t^2+4t+6t+24=48$$

$$\Rightarrow \quad t^2+10t-24=0$$

$$\Rightarrow \quad t^2+12t-2t-24=0$$

$$\Rightarrow \quad t(t+12)-2(t+12)=0$$

$$(t+12)(t-2)=0$$

Either,

$$t+12=0 \quad \text{or} \quad t-2=0$$

$$\Rightarrow \quad t=-12 \quad \Big| \quad t=2$$

Substituting in equation (ii) | Substituting in equation (ii)

$\Rightarrow \quad x^2+5x=-12$ | $x^2+5x=2$

$\Rightarrow \quad x^2+5x+12=0$ | $x^2+5x-2=0$

Here: $a=1$, $b=5$ and $c=12$,

$$\Rightarrow \quad x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\Rightarrow \quad x=\frac{-5\pm\sqrt{25-48}}{2}$$

$$\Rightarrow \quad x=\frac{-5\pm\sqrt{-23}}{2}$$

$$\Rightarrow \quad x=\frac{-5\pm i\sqrt{23}}{2}$$

Here: $a=1$, $b=5$ and $c=-2$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-5\pm\sqrt{25+8}}{2}$$

$$x=\frac{-5\pm\sqrt{33}}{2}$$

$$\text{S.S}=\left\{\frac{-5\pm i\sqrt{23}}{2},\frac{-5\pm\sqrt{33}}{2}\right\}$$

Exercise 8.3

Solve the following equations:

1. $x^4-8x^2-9=0$
2. $x^4-3x^2-4=0$
3. $12x^4-11x^2+2=0$
4. $\frac{2x+3}{x+1}+6\left(\frac{x+1}{2x+3}\right)=7$
5. $\sqrt{\frac{2x^2+1}{x^2+1}}+6\sqrt{\frac{x^2+1}{2x^2+1}}=5$
6. $5^{x+1}+5^{2-x}=5^3+1$
7. $2^x+\frac{16}{2^x}=8$
8. $2\left(\frac{x}{x+1}\right)^2-5\left(\frac{x}{x+1}\right)+2=0$
9. $4\left(\frac{x}{x-1}\right)^2-4\left(\frac{x}{x-1}\right)+1=0$
10. $9^{x+2}-6\cdot 3^{x+1}+1=0$
11. $2^x+2^{-x+6}-20=0$
12. $(x-1)(x+5)(x+8)(x+2)=880$
13. $(x+1)(x+2)(x+3)(x+4)=120$
14. $(x-2)(x+1)(x+3)(x-4)=24$

8.4 Radical Equations

Definition

An equation in which the variable appears under the radical sign, is called a radical equation.

Solution of the radical equations must be verified as it may have extraneous root.

8.4.1 Solution of the equations of the type:

Type (i) $\sqrt{ax+b} = cx+d$ Type (ii) $\sqrt{x+a} + \sqrt{x+b} = \sqrt{x+c}$

Type (iii) $\sqrt{x^2+px+m} + \sqrt{x^2+px+n} = q$

Type (i): $\sqrt{ax+b} = cx+d$

Example 01 **Solve** $\sqrt{217-x} = x-7$

Solution: $\sqrt{217-x} = x-7$

Squaring on both sides, we have,

$$\left(\sqrt{217-x}\right)^2 = (x-7)^2$$

$\Rightarrow$ $217-x=(x-7)^2$

$\Rightarrow$ $217-x=x^2-14x+49$

$\Rightarrow$ $x^2-13x-168=0$

$\Rightarrow$ $x^2-21x+8x-168=0$

$\Rightarrow$ $x(x-21)+8(x-21)=0$

$\Rightarrow$ $(x-21)(x+8)=0$

Either, $x=21$ *or* $x=-8$

verification : when $x=21$	verification : when $x=-8$
$\sqrt{217-x} = x-7$	$\sqrt{217-x} = x-7$
$\therefore$ $\sqrt{217-21} = 21-7$	$\therefore$ $\sqrt{217-(-8)} = -8-7$
$\Rightarrow$ $\sqrt{196} = 14$	$\Rightarrow$ $\sqrt{225} = -15$
$\Rightarrow$ $14 = 14$	$\Rightarrow$ $15 \neq -15$
verified	not verified

On verification it is found that $x=-8$ does not satisfy the original equation, hence it is an extraneous root, and can't be included in the solution set. Therefore, S.S = {21}

Type (ii) : $\sqrt{x+a} + \sqrt{x+b} = \sqrt{x+c}$

Example 01 **Solve:** $\sqrt{x+7} + \sqrt{x+2} = \sqrt{6x+13}$

Solution: $\sqrt{x+7} + \sqrt{x+2} = \sqrt{6x+13}$

Squaring on both sides, we get,

$(\sqrt{x+7} + \sqrt{x+2})^2 = (\sqrt{6x+13})^2$

$\Rightarrow \quad (\sqrt{x+7})^2 + 2(\sqrt{x+7})(\sqrt{x+2}) + (\sqrt{x+2})^2 = 6x+13$

$\Rightarrow \quad x+7+2\sqrt{(x+7)(x+2x)} + x+2 = 6x+13,$

$\Rightarrow \quad 2x+9+2\sqrt{x^2+7x+2x+14} = 6x+13$

$\Rightarrow \quad 2\sqrt{x^2+9x+14} = 4x+4$

$\Rightarrow \quad \sqrt{x^2+9x+14} = 2x+2$

$\Rightarrow \quad \sqrt{x^2+9x+14} = 2(x+1)$

Again Squaring on both sides

$\left(\sqrt{x^2+9x+14}\right)^2 = [2(x+1)]^2$

$\Rightarrow \quad x^2+9x+14 = 4(x+1)^2$

$\Rightarrow \quad x^2+9x+14 = 4(x^2+2x+1)$

$\Rightarrow \quad x^2+9x+14 = 4x^2+8x+4$

$\Rightarrow \quad 3x^2-x-10 = 0$

$\Rightarrow \quad 3x^2+5x-6x-10 = 0$

$\Rightarrow \quad x(3x+5)-2(3x+5) = 0$

$\Rightarrow \quad (x-2)(3x+5) = 0$

Either

$3x+5 = 0$

$\Rightarrow \quad x = -\frac{5}{3}$

Verification:

when $x = -\frac{5}{3}$

$\sqrt{x+7} + \sqrt{x+2} = \sqrt{6x+13}$

or $\quad x-2 = 0$

$\Rightarrow \quad x = 2$

Verification

when $\quad x=2$

$\sqrt{x+7} + \sqrt{x+2} = \sqrt{6x+13}$

$\sqrt{-\frac{5}{3}+7}+\sqrt{-\frac{5}{3}+2}=\sqrt{6\left(-\frac{5}{3}\right)+13}$	$\sqrt{2+7}+\sqrt{2+2}=\sqrt{6(2)+13},$
$\Rightarrow \sqrt{\frac{16}{3}}+\sqrt{\frac{1}{3}}=\sqrt{3}$	$\Rightarrow \sqrt{9}+\sqrt{4}=\sqrt{25}$
$\Rightarrow \frac{4}{\sqrt{3}}+\frac{1}{\sqrt{3}}=\sqrt{3}$	$\Rightarrow 3+2=5$
$\Rightarrow \frac{5}{\sqrt{3}}\neq\sqrt{3}$	$\Rightarrow 5=5$
Not verified.	Verified.

Since $-\frac{5}{3}$ is an extraneous root, therefore the solution is $x=2$

Thus, the solution set = {2}.

Type (iii): $\sqrt{x^2+px+m}+\sqrt{x^2+px+n}=q$

Example 01 Solve: $\sqrt{x^2-3x+21}-\sqrt{x^2-3x+5}=2$

Solution: Put $y=x^2-3x$ in the given equation, we have,

$$\sqrt{x^2-3x+21}-\sqrt{x^2-3x+5}=2$$

$$\sqrt{y+21}-\sqrt{y+5}=2$$

$$\sqrt{y+21}=2+\sqrt{y+5}$$

Squaring on both sides, we get

$$(\sqrt{y+21})^2=(2+\sqrt{y+5})^2$$

$$\Rightarrow \quad y+21=(2)^2+4\sqrt{y+5}+\left(\sqrt{y+5}\right)^2$$

$$\Rightarrow \quad y+21=4+4\sqrt{y+5}+y+5$$

$$\Rightarrow \quad 4\sqrt{y+5}=\cancel{y}+21-4-\cancel{y}-5$$

$$\Rightarrow \quad 4\sqrt{y+5}=12$$

$$\Rightarrow \quad \sqrt{y+5}=3$$

Again squaring on both the sides, we have,

$$\Rightarrow \quad y+5=9$$

$$\Rightarrow \quad y=4$$

Put $y=4$ in the substitution $y=x^2-3x$, we have,

$$4 = x^2 - 3x$$
$$\Rightarrow \quad x^2 - 3x - 4 = 0$$
$$\Rightarrow \quad x^2 - 4x + x - 4 = 0$$
$$\Rightarrow \quad x(x-4) + 1(x-4) = 0$$
$$\Rightarrow \quad (x-4)(x+1) = 0$$

Either $x - 4 = 0$
$$\Rightarrow \quad x = 4$$

For $x = 4$

$$\sqrt{x^2 - 3x + 21} - \sqrt{x^2 - 3x + 5} = 2$$
$$\sqrt{(4)^2 - 3(4) + 21} - \sqrt{(4)^2 - 3(4) + 5} = 2$$
$$\sqrt{16 - 12 + 21} - \sqrt{16 - 12 + 5} = 2$$
$$\sqrt{25} - \sqrt{9} = 2$$
$$5 - 3 = 2$$
$$2 = 2$$

or $\quad x + 1 = 0$
$$x = -1$$

Verification :

For $x = -1$

$$\sqrt{x^2 - 3x + 21} - \sqrt{x^2 - 3x + 5} = 2$$
$$\sqrt{(-1)^2 - 3(-1) + 21} - \sqrt{(-1)^2 - 3(-1) + 5} = 2$$
$$\sqrt{1 + 3 + 21} - \sqrt{1 + 3 + 5} = 2$$
$$\sqrt{25} - \sqrt{9} = 2$$
$$5 - 3 = 2$$
$$2 = 2$$

Hence both roots are satisfied by the given equation. Thus, the solution set is $\{-1, 4\}$.

Exercise 8.4

Solve the following equations :

1. $x + \sqrt{x+5} = 7$
2. $\sqrt{x-2} = 8 - x$
3. $\sqrt{7-5x} + \sqrt{13-5x} = 3\sqrt{4-2x}$
4. $\sqrt{x+2} + \sqrt{x+7} = \sqrt{6x+13}$
5. $\sqrt{2x^2 + 3x + 4} + \sqrt{2x^2 + 3x + 9} = 5$
6. $\sqrt{y^2 - 3y + 9} - \sqrt{y^2 - 3y + 36} + 3 = 0$

Review Exercise 8

1. Fill in the blanks

(i) A polynomial equation in which degree of variable is ______________ called quadratic equation.

(ii) Standard form of quadratic equation is ______________ .

(iii) $3^x + 3^{2x} = 1$ is called ______________ equation.

(iv) Solution of $3^x = 9$ is______________.

(v) Solution of $ax^2 + bx + c = 0$ is ______________ .

2. Tick (✓) the correct answer

(i) Degree of quadratic equation is

(a) 1 (b) 2 (c) 3 (d) 4

(ii) Standard form of quadratic equation is

(a) $ax^2 + bx + c = 0, a \neq 0$ (b) $ax^2 + c = 0, a \neq 0$

(c) $ax^2 + bx = 0, a \neq 0$ (d) $ax^3 + bx^2 + c = 0, a \neq 0$

(iii) The Quadratic Formula for $ax^2 + bx + c = 0, a \neq 0$ is

(a) $x = \dfrac{-b - \sqrt{b^2 - 4ac}}{2}$ (b) $x = \dfrac{b \pm \sqrt{b^2 - 4ac}}{2a}$

(c) $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ (d) $x = \dfrac{-b \pm \sqrt{b^2 + 4ac}}{2a}$

(iv) Solution set of $x^2 + 10x + 24 = 0$ is

(a) $\{-6, -4\}$ (b) $\{-6, 4\}$

(c) $\{6, 4\}$ (d) $\{6, -4\}$

(v) How many maximum roots of quadratic equation are

(a) 2 (b) 3 (c) 1 (d) 4

(vi) Two linear factors of $x^2 - 15x + 56$ are

(a) $(x-7)$ and $(x + 8)$ (b) $(x+7)$ and $(x-8)$

(c) $(x-7)$ and $(x-8)$ (d) $(x+7)$ and $(x + 8)$

(vii) Polynomial equation, which remains unchanged when x is replaced by $\dfrac{1}{x}$ is called a/ an

(a) Exponential equation (b) Reciprocal equation

(c) Radical equation (d) none of these

(viii) An equation of the type of $3^x + 3^{2-x} + 6 = 0$ is a/an

(a) Exponential equation (b) Radical equation
(c) Reciprocal equation (d) none of these

(ix) The solution set of equation $4x^2 - 16 = 0$ is
(a) $\{\pm 4\}$ (b) $\{4\}$ (c) $\{\pm 2\}$ (d) none of these

(x) An equation of the form $2x^4 - 3x^3 + 7x^2 - 3x + 2 = 0$ is called a/an
(a) Reciprocal equation (b) Radical equation
(c) Exponential equation (d) none of these

3. **True and false questions**

Read the following sentences carefully and en-circle 'T' in case of true and 'F' in case of false statement.

(i) Every quadratic equation can be solved by factorization. T/F
(ii) Every Quartic equation has two roots. T/F
(iii) Every Quadratic equation can have no solution. T/F
(iv) $ax^2 + bx + c = 0$ is called the quadratic equation in x if a=0 and b, c are real numbers. T/F
(v) Extraneous root satisfy the equation. T/F
(vi) Extraneous roots do not satisfy the equation. T/F
(vii) In the quadratic equation the highest exponent of the variable is two. T/F

Summary

- A Polynomial equation in which degree of a variable is 2, called quadratic equation.
- $ax^2 + bx + c = 0$, $a \neq 0$, a, b, c are real numbers is called standard form of a quadratic equation.
- Formula for quadratic equation $ax^2 + bx + c = 0, a \neq 0$ is $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- In exponential equations, variables occur in exponents.
- An equation in which the variable appears under the radical sign is called a radical equation.