MODELLING AND SIMULATIONS

CASE STUDY: BALANCING SHAFT ON WHEELS

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Agenda

- Develop a model of a mechatronic system
  - Include models of mechanics
  - Include models of electronics
- Combine the different models
  - Get a feeling for the different modeling aspects
- Identify some unknown parameters

System Description

The system can be seen as a:

Balancing Shaft on Wheels

I.e. Model the **inverted pendulum** on a moving set of **wheels,** driven by a motor

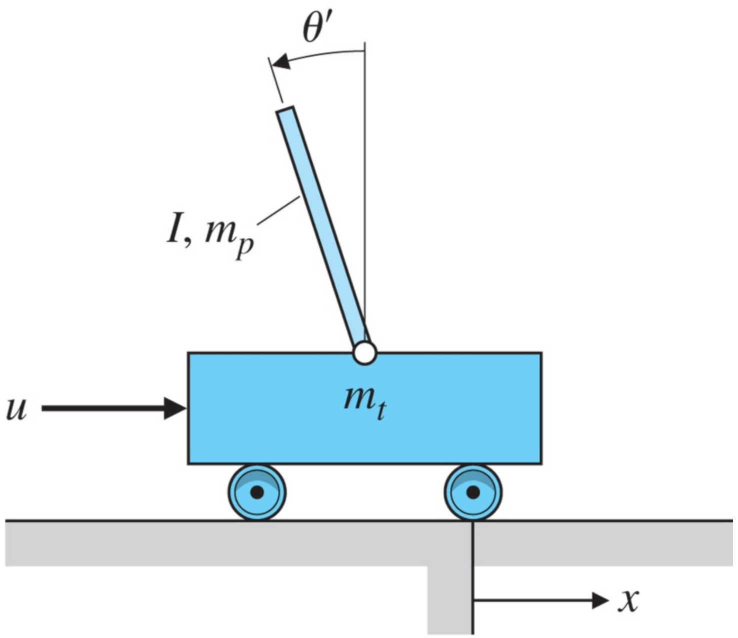
Where the motor and body are one part

And

Wheel rotates with an applied torque τ from the motor

System Description

Simplified problem, notice that we here do no apply any force to the wheels

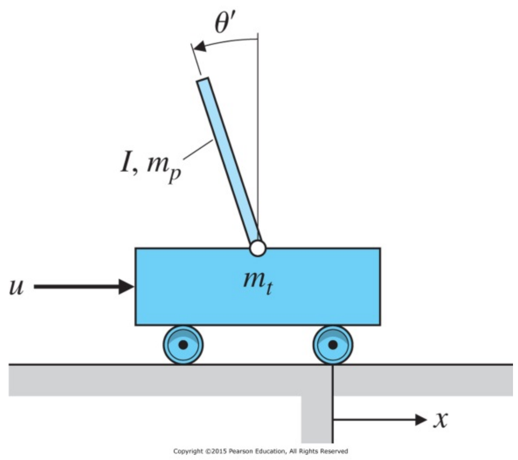


System Description

Deconstructing the problem:

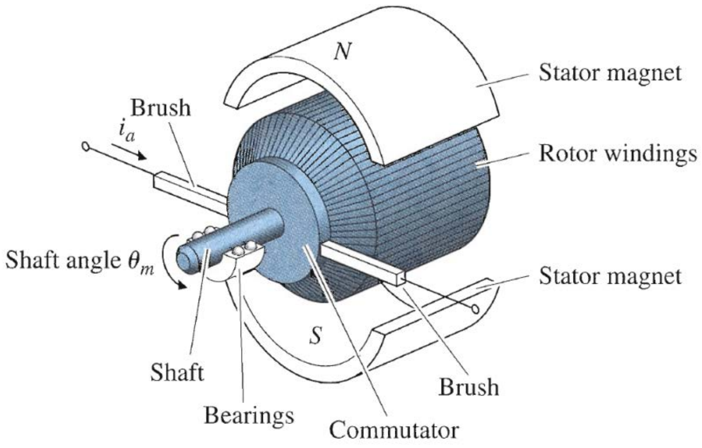
- The pendulum transforms the gravitational acceleration into a horizontal force, F_{pH}
- This force influences the wheels torque τ_w negativly
- Which further is transmitted into the motor as a negative torque τ_w
- The motor generates a torque τ_m when a voltage v_a is applied
- Which counteracts τ_w and rotates the wheels to effect the wheels position and consequently the pendulums angle

- The pendulum which is moved by the two above in the lateral direction, produces an angle around the pivot point Θ_p



Objective

Motor Equations



- There are two equations which are used to describe a DC motor.
- The following equation is the mechanical description where motor torque τ_m is proportional to the armature current i_a

$$\tau_m = K_t \cdot i_a$$

- The second equation is the electrical description, and it describes the relationship between the shaft rotational velocity $\dot{\Theta}_m$ and the back EMF(Electromotive force) voltage e.

$$e = K_e \cdot \dot{\Theta}_m$$

Where K_e is the back EMF constant.

- The voltage is produced due to the coil moving through a magnetic field created by the

stator magnets

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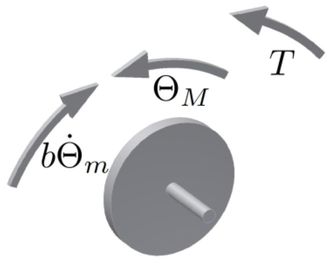
Motor's Mechanic Equation

Free body diagram of the mechanical part (motors rotor)



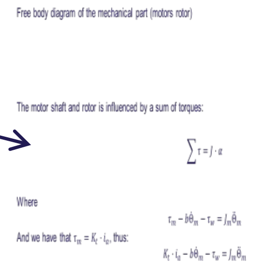
The motor shaft and rotor is influenced by a sum of torques:

$$\sum \tau = J \cdot \alpha$$

Motor’s Mechanic Equation

Free body diagram of the mechanical part (motors rotor)

$$\boxed{b = friction}$$


Free body diagram of the mechanical part (motors rotor)
The motor shaft and rotor is influenced by a sum of torques:
$\sum \tau = J \cdot \alpha$
Where
$\tau_m - b\dot{\Theta}_m - \tau_w = J_m\ddot{\Theta}_m$
And we have that $\tau_m = K_t \cdot i_a$, thus:
$K_t \cdot i_a - b\dot{\Theta}_m - \tau_w = J_m\ddot{\Theta}_m$

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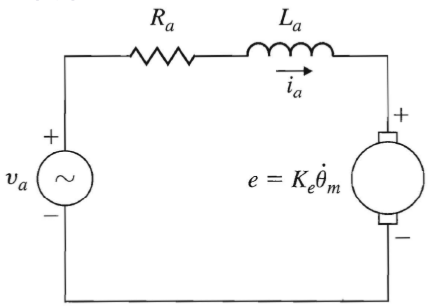
Motor’s Electric Equation

The following diagram is a electric circuit of the motor.



W can set up an equation of this circuit following the Kirchoff’s voltage law, which states:

For a closed loop series path **the algebraic sum of all the voltages around any closed loop in a circuit is equal to zero**, as a circuit loop is a *closed conducting path,* so no energy is lost, i.e.

$$\boldsymbol{\sum V = 0}$$

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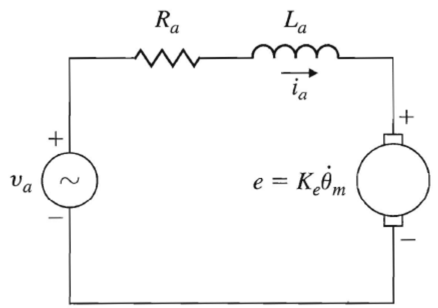
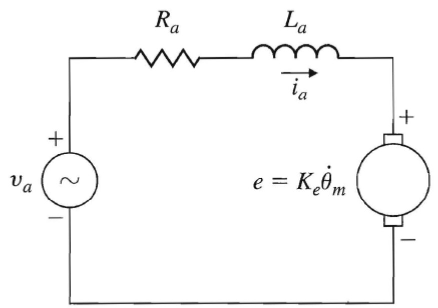
Motor’s Electric Equation



Folliwng the Kirchhoff's voltage law $(\sum \boldsymbol{V} = \boldsymbol{0})$ we obtain the following equation:

$$v_a = v_{R_a} + L_a \frac{di_a}{dt} + V_e$$

Motor’s Electric Equation



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$$v_a = \boxed{v_{R_a}} + L_a \frac{di_a}{dt} + \boxed{e}$$

Where we have $(v = R \cdot\ i)$, and $e = K_e \cdot \dot{\Theta}_m$, and if we insert this we have:

$$v_a = R_a i_a + L_a \frac{di_a}{dt} + K_e \cdot \dot{\Theta}_m$$

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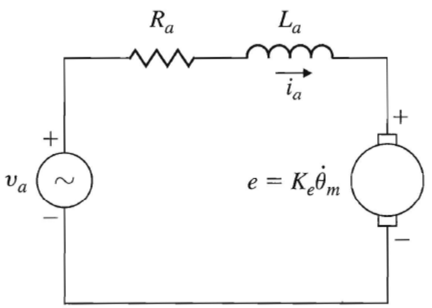
Simplification of the Motor’s Electric Equation



Inductance of the rotor winding

$$v_a = R_a i_a + \boxed{L_a \frac{di_a}{dt}} + K_e \cdot \dot{\Theta}_m$$

From experience we know that the inductance of a DC motor’s rotor winding has a relatively fast dynamic compared to the rest of the system.

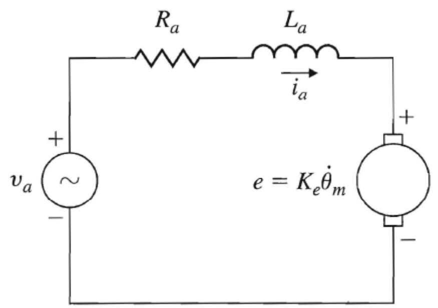
From experimental data the inductances’ time constant was identified to:

$$1.67\ ms$$

Since its dynamic is much faster then the dynamic of the rest of the system it will be omitted.

Identification of parameters will be discussed later.

Simplification of the Motor’s Electric Equation



$$v_a = R_a i_a + L_a \frac{di_a}{dt} + K_e \cdot \dot{\Theta}_m$$

Isolating the current i_a and omitting the inductance L_a leads to the following equation of the DC motor’s electrical part:

$$i_a = \frac{v_a - K_e \dot{\Theta}_m}{R_a}$$

Motor Equation (Final)

- Combination of the mechanical part and the electrical part yields:

$$K_t \cdot \frac{v_a - K_e \dot{\Theta}_m}{R_a} - b\dot{\Theta}_m - \tau_w = J_m \ddot{\Theta}_m$$

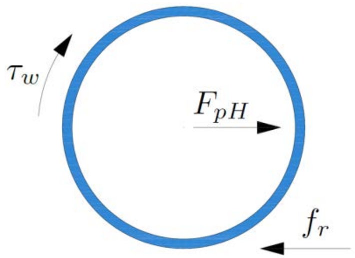
Wheel Equation

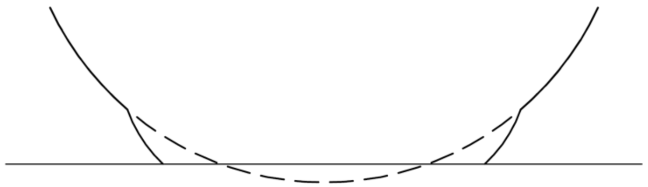
Sum of forces acting on the wheel

$$\sum \tau = J\alpha$$

Where

$$\tau_w = J_w \ddot{\Theta}_w - f_r r$$





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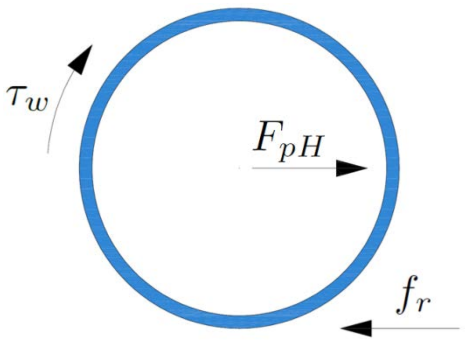
Wheel Equation



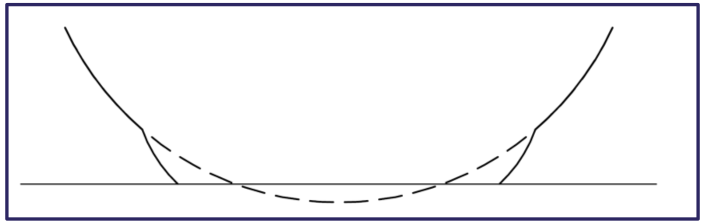
Sum of torques acting on the wheel (Angular)

$$\sum \tau = J\alpha$$

Where

radius

$$\tau_w = J_w \ddot{\Theta}_w - f_r r$$



Sum of torques acting on the wheel (Longitudinal)

$$\sum \vec{F} = m\vec{a}$$

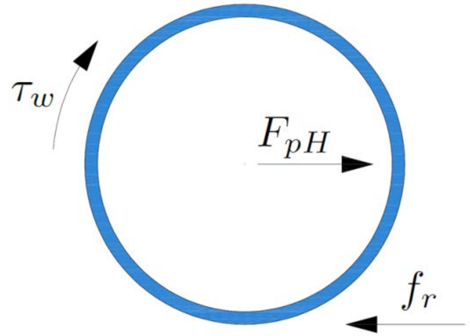
Where

$$F_{pH} - f_r = m_w \cdot \ddot{x}$$

Isolating for the friction and combing the angular and the longitudinal we get

$$\tau_w = J_w \cdot \ddot{\Theta}_w - F_{pH} \cdot r_w + m_w \cdot \ddot{x} \cdot r_w$$

Wheel Equation (final)



$$\tau_w = J_w \cdot \ddot{\Theta}_w - F_{pH} \cdot r_w + m_w \cdot \ddot{x} \cdot r_w$$

The wheel we wish to be represented by an angle thus we substitute $\ddot{x}$ with $\ddot{x} = \ddot{\Theta}_w \cdot r_w$, and get:

$$\tau_w = J_w\ddot{\Theta}_w - F_{pH} \cdot r_w + m_w\ddot{\Theta}_w \cdot {r_w}^2$$

Body Equation *(Inverted Pendulum)*

- Unlike a normal inverted pendulum, **no** stationary pivot point
  - Wheels can move freely. This
- The forces which effect the pendulum are the gravity pulling the mass of the pendulum down, and the motor torque
- The motor does not have full control over the body’s angle
  - Wheels are able to move freely

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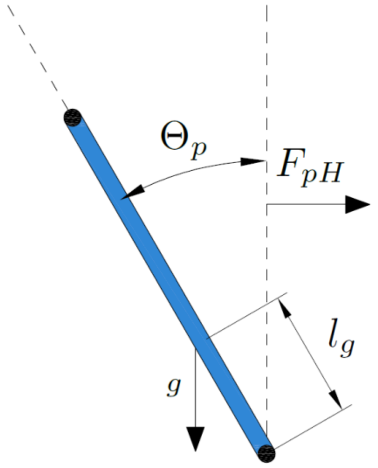
Body Equation *(Horizontal Force)*



Horizontal Force Applied To The Wheel

- The sum of forces: mass of the pendulum and the horizontal and gravitational acceleration

$$\sum \vec{F} = m_p \cdot \overrightarrow{a_{x,g}}$$

Which yields

$$F_{pH} = m_p \cdot \overrightarrow{a_{x,g}}$$

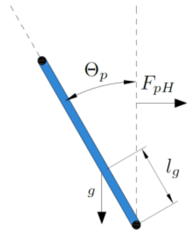
Where

m_p = Pendulum mass, without the wheels

F_{pH} = Horizontal force created by the pendulum

$a_{x,g}$ = Longitudinal acceleration of the pendulum

Body Equation *(Longitudinal acceleration)*



We wish to find $\overrightarrow{a_{x,g}}$

Which represents the longitudinal acceleration, driven by the gravitational acceleration

To models this we use the kinematic equations
Where the **position vector** of a **particle** is a **vector** drawn from the **origin** of the **reference frame** to the **particle**

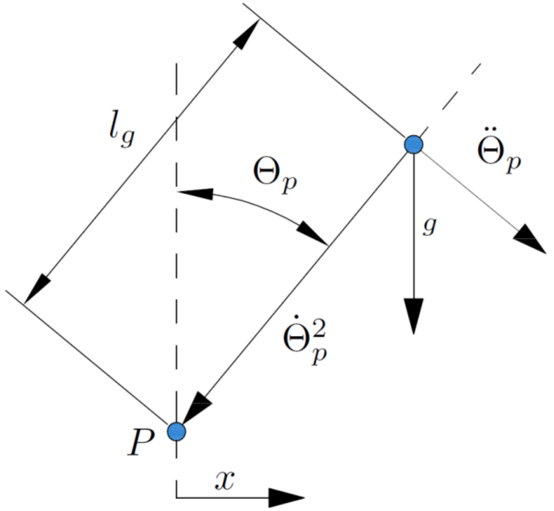
$$\vec{r} = x\vec{i} + u\vec{j}$$

Replacing $\vec{r}$ with $\overrightarrow{a_{x,g}}$ we get:

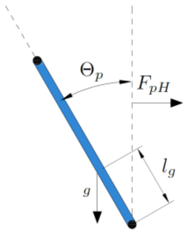
$$a_{x,g} = a_{x,P} + a_{x,g/P}$$

Where:

- $a_{x,P}$ is the vector from the point P to a point on the x axis
- $a_{x,g/P}$ is the vector from the center of gravity to the point P



Body Equation *(Inverted Pendulum)*



To find $a_{x,P}$ *and* $a_{x,g/P}$, we use trigonometric functions *(Read about trigonometric functions on your own)*:

First we have the vector that describes the rotational acceleration of the pendulum:

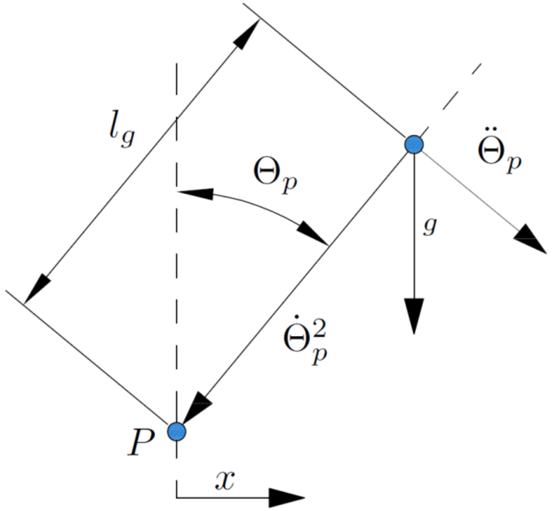
$$a_{x,P} = cos\Theta_P l_g \ddot{\Theta}_P$$

Next we find the vector that described the centrifugal acceleration:

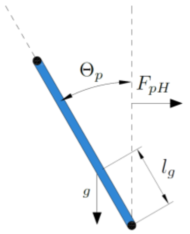
$$a_{x,g/P} = sin\Theta_P l_g \dot{\Theta}^2$$

We then have the following final equation that describes the longitudinal acceleration, where we add the longitudinal acceleration added by the wheels:

$$a_{x,g} = \ddot{x} + cos\Theta_P l_g \ddot{\Theta}_P + sin\Theta_P l_g \dot{\Theta}^2$$



Body Equation *(Linearisation)*

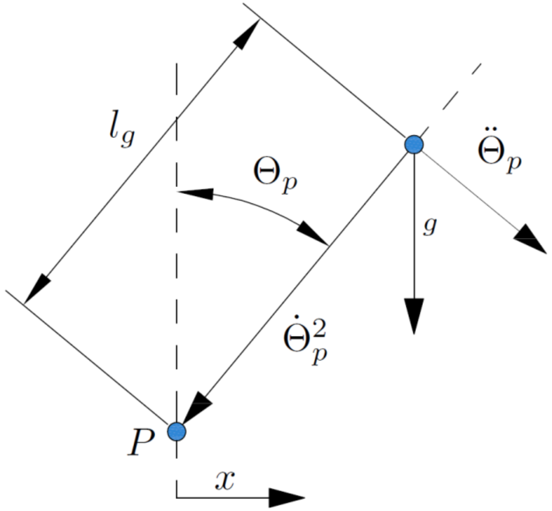


We have now introduced a nonlinear term:

$$a_{x,g} = cos\Theta_P l_g \ddot{\Theta}_P + sin\Theta_P l_g \dot{\Theta}^2$$

After linearization we get to the following equation (Linearization is left to you):

$$a_{x,g} = \ddot{x} + l_g \ddot{\Theta}_P$$



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Body Equation

- Now we can insert the longitudinal acceleration equation ($\overrightarrow{a_{x,g}} = \ddot{x} + l_g \ddot{\Theta}_P$) into the equation of the horizontal force crated by the pendulum ($F_{pH} = m_p \cdot \overrightarrow{a_{x,g}}$), where we get:

$$F_{pH} = m_p \cdot \ddot{x} + m_p \cdot l_g \ddot{\Theta}_P$$

As we wish to represent the systems acceleration solely by angular motion (the motion of the pendulum), we can replace $\ddot{x}$ with $\ddot{\Theta}_w \cdot r$, which yields:

$$F_{pH} = m_p \cdot \ddot{\Theta}_w \cdot r + m_p \cdot l_g \ddot{\Theta}_P$$

COMBINING THE EQUATIONS

Combining the equations

Wheel:

$$\tau_w = J_w\ddot{\Theta}_w - F_{pH} \cdot r_w + m_w\ddot{\Theta}_w \cdot {r_w}^2$$

Horizontal force created by the pendulum:

$$F_{pH} = m_p \cdot \ddot{\Theta}_w \cdot r + m_p \cdot l_g\ddot{\Theta}_P$$

Motor equation:

$$K_t \cdot \frac{v_a - K_e\dot{\Theta}_m}{R_a} - b\dot{\Theta}_m - \tau_w = J_m\ddot{\Theta}_m$$

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Combining the equations

Through various isolations we get the following final differential equation:

$$\ddot{\Theta}_p = \dot{\Theta}_p \left(\frac{K_t K_e J_p + K_t K_e l_g^2 m_p - J_p b\, R_a - l_g^2 b\, m_p R_a}{R_a (J_m J_p - J_m l_g^2 m_p + J_w J_p G_r - J_w l_g^2 G_r m_p + r_w^2 J_p G_r m_p + r_w^2 m_w J_p G_r - r_w^2 m_w l_g^2 G_r m_p)} \right) v_a \left(\frac{m_p\, r_w\, l_g}{R_a (J_m J_p - J_m l_g^2 m_p + J_w J_p G_r - J_w l_g^2 G_r m_p + r_w^2 J_p G_r m_p + r_w^2 m_w J_p G_r - r_w^2 m_w l_g^2 G_r m_p)} \right)$$

PARAMETER IDENTIFICATION

Parameter Identification

There exists many ways of identifying parameters

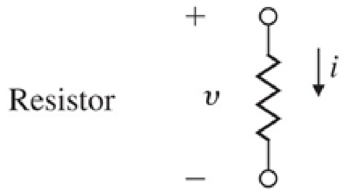
- Start by investigating the model of your system
- Which parameters are unknown
- Can we isolate them and find them based on some known parameters or collected data?

Parameter Identification

Can we isolate them and find them based on some known parameters or collected data?

An example with resistance, where the model of a resistor is:

$$v = Ri$$



Isolating for the Resistance we have

$$\frac{v}{i} = R$$

Thus if we measure the voltage and the current in a simple circuit including the resistor we can find the resistance R

Parameter Identification

EXAMPLES

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Parameter Identification (Resistance of the DC motor)

A DC motor can have multiple windings

Thus the resistance can be found by measuring the resistance at 4 different axle positions.

And example is shown below:

$$\frac{3\Omega + 2.8\Omega + 3.8\Omega + 4.2\Omega}{4} = 3.45\,\Omega$$

For this motor

Parameter Identification (Inductance)

The inductance of the inductor is represented by the following equation

$$L = \frac{v(t)}{\frac{di(t)}{dt}}$$

i.e. the *time varying voltage* is dependent on the *time-varying current* passing through the inductor or a sinusoidal alternating current (AC) through an inductor, induces a sinusoidal voltage.

The inductance can be found using a low AC source, where the current and the voltage are measured and the inductance is calculated

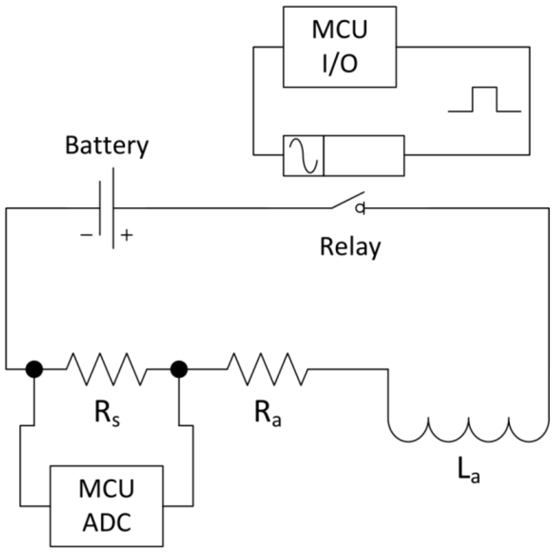
Instruments that do this exist as well

Parameter Identification (Inductance)

The inductance of the motor can be found from experimental analysis.

The following circuit is designed

Which allows for measurement of voltage over the sense resistor, allowing calculation of the current running through the system.



Parameter Identification (Inductance)

The inductance of the motor can be found from the following electrical circuit:

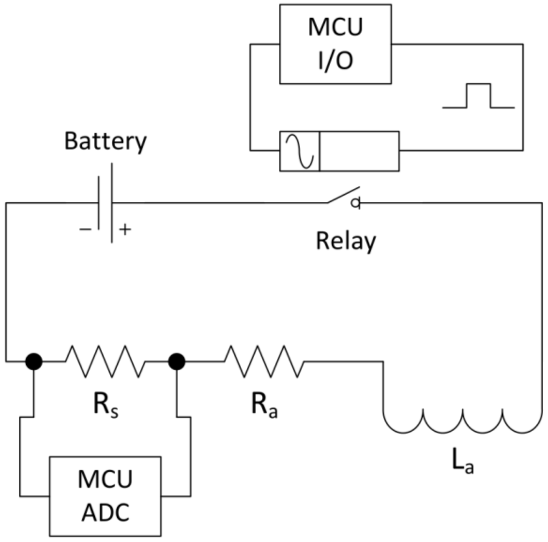
$$V_a = i(R_a + R_s) + L_a \frac{di}{dt}$$

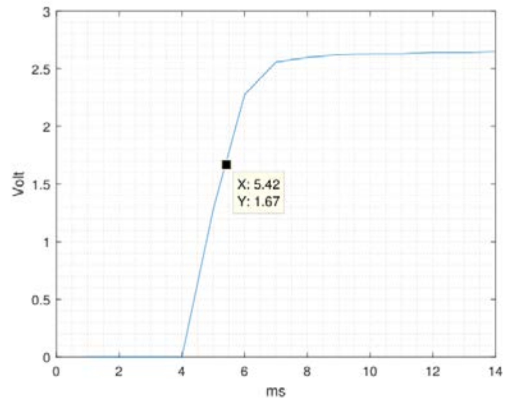
Isolating for the inductance

$$L_a = \frac{V_a - i(R_a + R_s)}{\frac{di}{dt}}$$

Through measurements and some calculation we get:

$$L_a = 0.604$$





Parameter Identification (Bandwidth of the Inductor)

Earlier it was beneficial for us to know something about the dynamics of the inductor

I.e. if the bandwidth of the inductor is much slower than the rest of the system, **we can omit it** in the equations.

The Bandwidth can be obtained from in the following way.

Parameter Identification (Bandwidth of the Inductor)

Now that we have the inductance we can calculate the dynamics of the inductor

By taking the Laplace transform of the electrical circuit and converting into the standard transfer function form, we get:

Where
$R_s = 0.5\Omega$ and $R_a = 3.45\Omega$

$$V_a(s) = i(s)(R_a + R_s) + L_a I(s)s \rightarrow \frac{I(s)}{V_a(s)} = \frac{1}{(R_a + R_s) + L_a s} = \frac{1}{3.95 + 0.604e - 03s}$$

Now that we have the inductance we can calculate the dynamics of the inductor

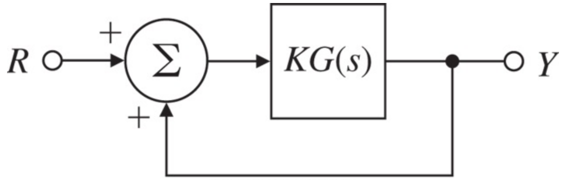
By taking the Laplace transform of the electrical circuit and converting into the standard transfer function form, we get:

$$V_a(s) = i(s)(R_a + R_s) + L_a I(s)s \rightarrow \frac{I(s)}{V_a(s)} = \frac{1}{(R_a + R_s) + L_a s} = \frac{1}{3.95 + 0.604e\text{-}03s}$$

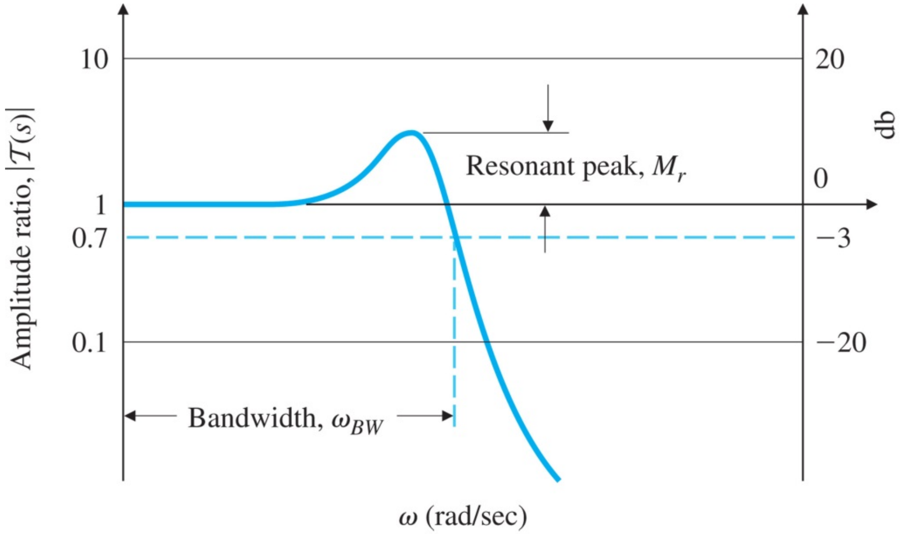
Parameter Identification (Bandwidth of the Inductor)

The bandwidth of a system is defined as the maximum frequency at which the output of a system will track an input sinusoid (r) in a satisfactory manner.

i.e. for the presented system, the bandwidth is the frequency of r at which the output y is attenuated to a factor of 0.707 times the input



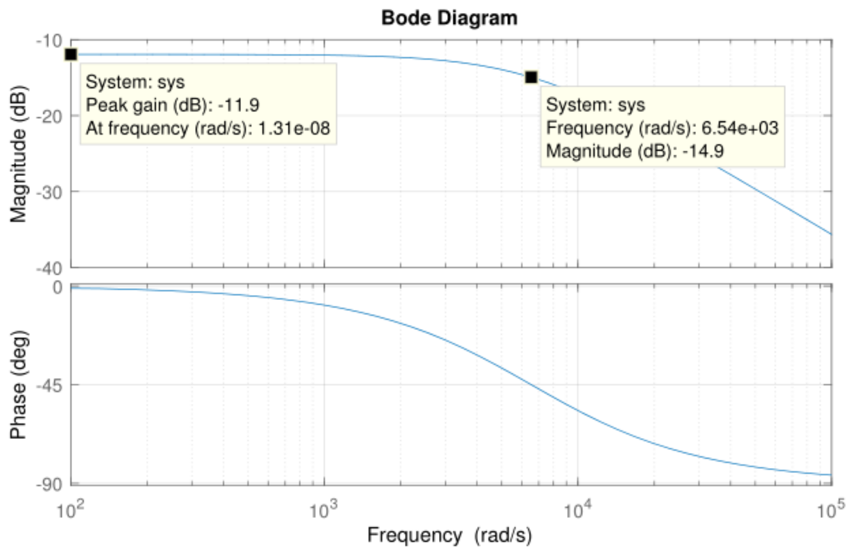
Where the presented system's transfer function $\frac{Y(s)}{R(s)} = \frac{KG(s)}{1+KG(s)}$ has the following response:



Parameter Identification (Bandwidth of the Inductor)

$$\frac{I(s)}{V_a(s)} = \frac{1}{3.95 + 0.604e - 03s}$$

We can now use Bode plot to analyze the dynamics of the system:



Where we find that the bandwidth is

$$6.54e + 3\,\frac{rad}{s}$$

Or in Hz

$$\approx 1041Hz$$

Parameter Identification (Bandwidth of the Inductor)

Now if your systems bandwidth is considerable slower than the motors inductance, it can be omitted.

Motor Back EFM

We have the Back EMF relationship defined by the following equation:

$$e = K_e \cdot \dot{\Theta}_m \rightarrow \frac{e}{\dot{\Theta}_m} = K_e$$

Through a test where the motor is rotated by an other motor the voltage and the rotational velocity of the motor shaft are measured.

The found coefficient are: $e = 10.85V$ and $418.5\frac{rad}{s}$

Which gives:

$$K_e \ = 0.026$$

Law of Generators

If a conducting wire of length l is moving in a magnetic field of B Teslas* with a velocity of $v = \frac{m}{s}$

The voltage across the conductor can be expressed with the following equation:

$$e(t) = B \cdot l \cdot v \; [volt]$$

Where Tesla has the following unit:

$$T = \frac{V \cdot s}{m^2} = \frac{N}{A \cdot m} = \frac{kg}{A \cdot s^2}$$

*Unit used to defined a magnetic flux density or magnetic field strength

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Motor Torque Constant

The motor torque constant equals the back EMF constant if the units are correct, if not propper unit conversions are required,

Motor Static Friction

The static friction introduces a saturation (dead zone)
i.e. a dead zone where the input i_a produces no output Θ_m

Can be found by experiment

Apply an increasing voltage to the motor, until the shaft moves.

The voltage which moves the shaft is the threshold voltage

The static friction can be calculated based on this voltage in the following way:

Motor Static Friction

The static friction can be calculated based on this voltage in the following way:

$$\tau_m = K_t \cdot i_a \rightarrow \tau_m = K_t \cdot \frac{v_a}{r_a}$$

Lets assume that the threshold voltage has been found to be $1.208v$ and that $K_t = 0.026$

We can now find the static friction

$$0.0091 = 0.026 \cdot \frac{1.208}{3.45}$$

EXERCISE

Exercise 1

- The following is a hypothetical differential equation of the system, with identified parameters:

$$\ddot{\Theta}_{\mathrm{p}} = -1.95 \cdot \dot{\Theta}_p - 0.11 \cdot v_a$$

Convert this system into a state space form, where we wish to have v_a as the input and Θ as the output