MODELLING AND SIMULATIONS

RECAP OF LAPLACE TRANSFORMS, TRANSFER FUNCTIONS, STATE SPACE

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TRANSFER FUNCTION

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Transfer Function recap

The transfer function $\boldsymbol{H(s)}$ is the Laplace transform of the unit impulse response $\boldsymbol{h(t)}$

Assuming that all initial condition of the system are 0

Where $H(s)$ is the transfer gain from $U(S)$ to $Y(s)$

i.e.

$$\frac{Y(s)}{U(s)} = H(s)$$

From the transfer function we can find the frequency response.

Transfer Function

We use Laplace transforms to go from time domain (t) to s domain (s)

To transform differential equations into ‘easier-to-manipulate’ algebraic from [1]

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

i.e.

$$Time - domain \overset{\mathcal{L}}{\Leftrightarrow} s - domain$$

[1] Franklin, Gene F., et al. *Feedback control of dynamic systems*. Vol. 3. Reading, MA: Addison-Wesley, 1994.

Transfer Function

In practice we do not have to solve the integral to get to the Laplace transform

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

We can find the most common Laplace transform from tables such as the following:

[1] Franklin, Gene F., et al. *Feedback control of dynamic systems*. Vol. 3. Reading, MA: Addison-Wesley, 1994.

Common Laplace Transforms

Table of Laplace Transforms

Number	$F(s)$	$f(t), t \geq 0$
1	1	$\delta(t)$
2	$1/s$	$1(t)$
3	$1/s^2$	t
4	$2!/s^3$	t^2
5	$3!/s^4$	t^3
6	$m!/s^{m+1}$	t^m
7	$\dfrac{1}{s+a}$	e^{-at}
8	$\dfrac{1}{(s+a)^2}$	te^{-at}
9	$\dfrac{1}{(s+a)^3}$	$\dfrac{1}{2!}t^2e^{-at}$
10	$\dfrac{1}{(s+a)^m}$	$\dfrac{1}{(m-1)!}t^{m-1}e^{-at}$
11	$\dfrac{a}{s(s+a)}$	$1-e^{-at}$

Example of Laplace transform

Find the Laplace Transform of:

$$f(t) = 1 + 2\sin(\omega t)$$

We use the tables to find the most common Laplace transforms where:

$$\mathcal{L}\{1(t)\} = \frac{1}{s}, \qquad \mathcal{L}\{k \cdot \sin(at)\} = \frac{k \cdot a}{s^2 + a^2}$$

And thus we have:

$$\mathcal{L}\{f(t)\} = \frac{1}{s} + 2 \cdot \frac{\omega}{s^2 + \omega^2}$$

Example: Differential Equation to Transfer Function

Transfrom the following differential equation into a transfer function

$$\ddot{y} + \dot{y}a_1 + ya_2 = bu$$

Show this on the black board

STATE SPACE

State Space

- Selection of states and organization into state form
- Choice of state is not unique

State-space definition (non-linear, time invariant)

State-space equations

State equation:

$$\frac{d\boldsymbol{x}(t)}{dt} = \boldsymbol{f}\big(\boldsymbol{x}(t), \boldsymbol{u}(t)\big)$$

Output equation:

$$\boldsymbol{y}(t) = \boldsymbol{h}(\boldsymbol{x}(t), \boldsymbol{u}(t))$$

Explanation

- $\boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t))$ is a nonlinear vector function of state and input
- $\boldsymbol{h}(\boldsymbol{x}(t), \boldsymbol{u}(t))$ is a nonlinear vector function of state and input

- $\boldsymbol{x}(t)$ is the state
- $\boldsymbol{u}(t)$ is the input

State-space definition (linear, time invariant)

State-space equations

State equation:

$$\frac{d\boldsymbol{x}(t)}{dt} = A \cdot \boldsymbol{x}(t) + B \cdot \boldsymbol{u}(t)$$

Output equation:

$$\boldsymbol{y}(t) = C \cdot \boldsymbol{x}(t) + D \cdot \boldsymbol{u}(t)$$

Explanation

- **x**(t) is an $n \times 1$ vector representing the state (e.g., position and velocity variables in mechanical systems)
- **u**(t) is a $m \times 1$ vector representing the input
- **y**(t) is a $p \times 1$ vector representing the output

- The matrices $A(n \times n)$, $B(n \times m)$, and $\mathrm{C}(p \times n)$, determine the relationships between the state and input and output variables. These are constant matrices.

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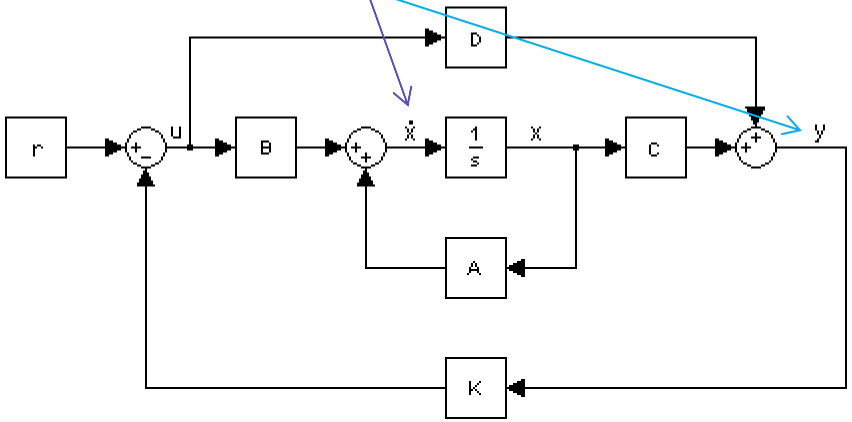
State Space Block Diagram

The state space representation has the following block diagram.

$$\frac{d\boldsymbol{x}(t)}{dt} = A \cdot \boldsymbol{x}(t) + B \cdot \mathbf{u}(t)$$

$$\boldsymbol{y}(t) = C \cdot \boldsymbol{x}(t) + D \cdot \mathbf{u}(t)$$



Stability

Identify the location of the system poles

Transfer functions

The **poles** are the **roots** of the **denominator,** or in can be found through **factored form** of the **transfer function**:

$$G(s) = \frac{Y(s)}{U(s)} = K\frac{(s - z_1)(s - z_2)\dots(s - z_m)}{(s - p_1)(s - p_2)\dots(s - p_n)}$$

State Space

The eigenvalues λ of the matrix A are the poles of the system.

Note that the denominator of the transfer function is the characteristic polynomial and can be found by the:

$$\lambda(s) = |sI - A|$$

i.e. the determinant of $sI - A$

Example: Differential Equation to State Space

Convert the following differential equation into state space:

$$\ddot{y} + \dot{y}a_1 + ya_2 = bu$$

This is done on the blackboard

Conversion from state space to transfer function

The transfer function from state space equations is given by:

$$G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

Show how this is obtained on the **black board**

Note: see Franklin p. 454

Example: State Space to Transfer Function

1. **Show on black board how it is done by hand**
2. **Show Result in Matlab**

Example: State Space to Transfer Function

1. **Show on black board how it is done by hand**
2. **Show Result in Matlab**

```
sys=b/(s^2 + a1*s + a2)
```

Laplace Transform of State Space

If we have a state space model:

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$
$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u}$$

And we take the Laplace transform, (with the assumption that all initial conditions on the system are zero):

$$\boldsymbol{s}\boldsymbol{X}(\boldsymbol{s}) = \boldsymbol{A}\boldsymbol{x}(\boldsymbol{s}) + \boldsymbol{B}\boldsymbol{U}(\boldsymbol{s})$$
$$\boldsymbol{Y}(\boldsymbol{s}) = \boldsymbol{C}\boldsymbol{X}(\boldsymbol{s}) + \boldsymbol{D}\boldsymbol{U}(\boldsymbol{s})$$

State Space froms

- There are many ways to represent a system using state space
- One of the is the controllable Canonical form
- This form is very common
  - As it is useful for the pole placement controller design technique.
- This will be our focus

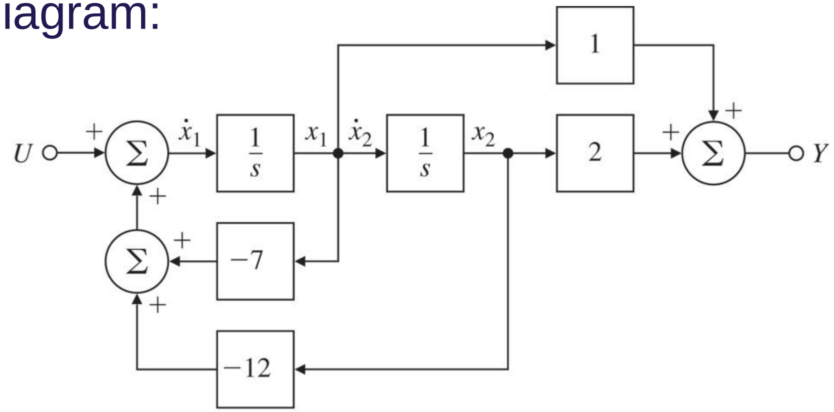
Transfer Function into State Space, Control Canonical Form

Given the transfer function:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{s+2}{s^2+7s+12}$$

We look at the transfer function as ratio of polynomials

Now we look at the transfer function using only isolated integrators as the dynamic elements, and construct the following block diagram:



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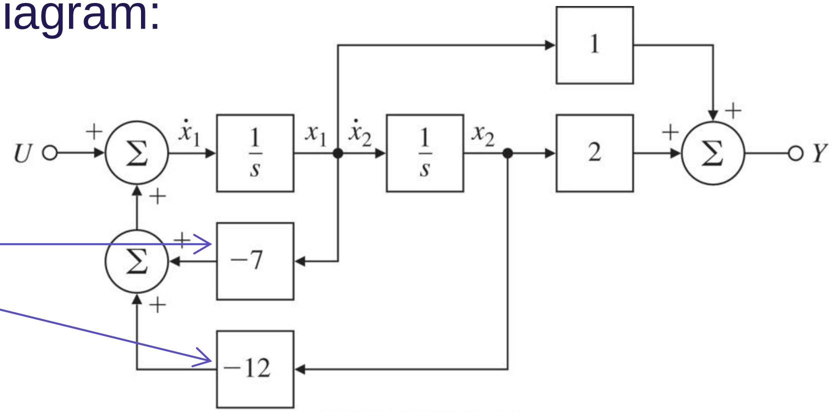
Each state variable $[p_n]$ is connected by the feedback to the control input U



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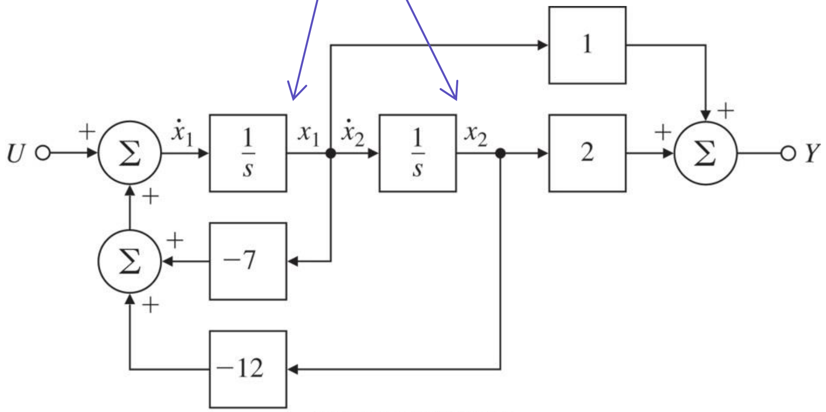
W can now identify the state variables $[x]$ and their derivatives $[\dot{x_n}]$



Transfer Function into State Space, Control Canonical Form

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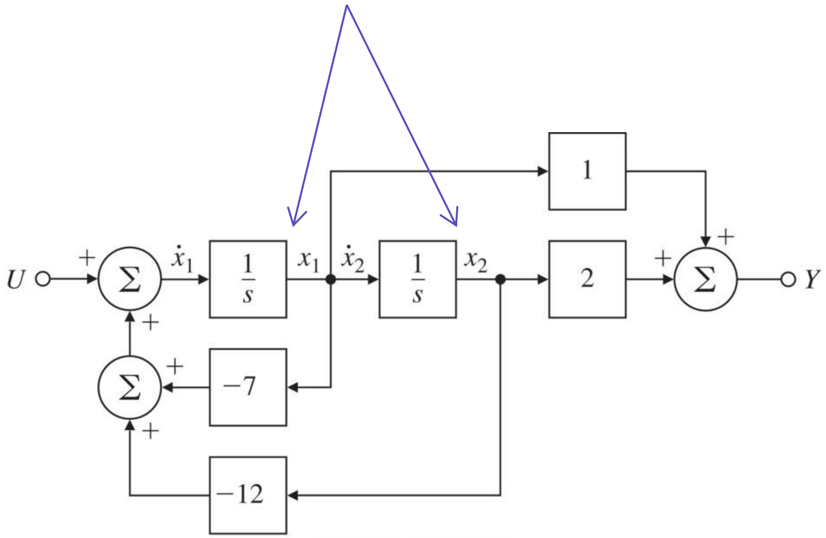
The rest is shown on the black board



TF to SS in Control Canonical Form (General Case)

Each state variable is connected by the feedback to the control input.

$$G(s) = \frac{Y(s)}{U(s)} = \frac{z_1 s^{n-1} + z_2 s^{n-2} + \cdots + z_n}{s^n + p_1 s^{n-1} + p_2 s^{n-2} + \cdots + p_n}$$

We construct this into the controllable canonical form:

$$\dot{x} = Ax + Bu = \begin{bmatrix} -p_1 & -p_2 & \cdots \ldots & -p_n \\ 1 & 0 & \cdots \ldots & 0 \\ 0 & 1 & 0 \; \cdots & 0 \\ \vdots & \vdots & \ddots \; 0 & \vdots \\ 0 & \cdots & \ldots \; 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} u$$

$$y = Cx + Du = \begin{bmatrix} z_1 & z_2 & \cdots & \cdots & z_n \end{bmatrix} x + 0$$

This can also be fulfilled using Matlab function: `TF2SS`

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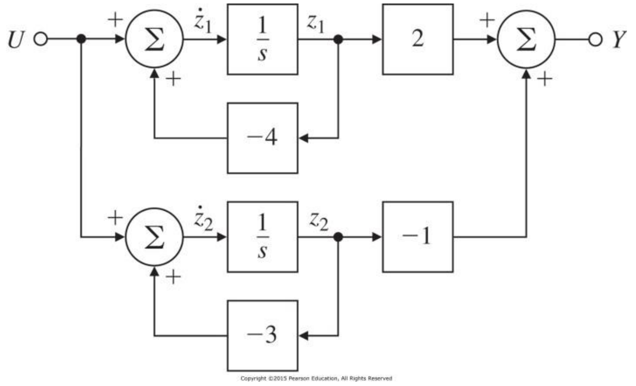
Additional forms, not discussed in detail.



Modal Canonical Form

- We can also represent the transfer function in modal canonical form
- This is done through partial fraction expansion of the transfer function, i.e.

$$G(s) = \frac{Y(s)}{U(s)} = \frac{s+2}{s^2+7s+12} = \frac{2}{s+4} + \frac{-1}{s+3}$$

For more information refer to. Franklin cha. 7.4

Observable Canonical Form

We also have the observable canonical form which has the following relationship to the control canonical form

$$A_{obs} = A_{cont}^T$$
$$B_{obs} = B_{cont}^T$$
$$C_{obs} = C_{cont}^T$$
$$D_{obs} = D_{cont}^T$$

Where all the feedback is from the output to the state variables

For more information refer to. Franklin cha. 7.7.1

Where to find more information

- **State Space**
  - [1] {7},[2] {3.4,A}
- **Transfer Functions**
  - [1] {3},[2]{3,A}
- **Differential equations**
  - [1] {2}, [2] {3.3}
- **General modeling**
  - [2]{1-7}
- **Model analysis, stability**
  - [2]{3}
- **Simulations**
  - [2]{11}
- **Validation**
  - [2]{12}

- *Filtering and Frequency Analysis*
  - *[2] {8.4}[3]{7,B} and [3]{8, '8.1'}*
- *System Identification*
  - *[2] {8}*

[1] Franklin, Gene F., et al. *Feedback control of dynamic systems*. Vol. 3., MA: Addison-Wesley, 1994.
[2] Ljung, Lennart, and Torkel Glad. "Modeling of dynamic systems." (1994).
[3] Alan V. Oppenheim and Ronald W. Schafer. "Discrete-time Signal Processing." , Vol. 3., (2010,1989)

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Exercise

Given the differential equation

$$\dddot{y} + \ddot{y}k_1 + \dot{y}k_2 + yk_3 = bu$$

1. Transform the differential equation into a state space:
2. Transform the differential equation into transfer function:

Given the Transfer Function:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{s^2 + s + 2}{s^3 + 2s^2 + 3s + 3}$$

3. Now go from transfer function into state space *(Hint: use the General Case Control Canonical Form)*
   a) Compare your results to `tf2ss`
4. Now go from state space to transfer function *(Hint: you can use Matlab to help with calculations)*
   a) Compare your results to `ss2tf`