MODELING & SIMULATIONS

TRANSFER FUNCTIONS AND STATE SPACE MODELING 2ND PART

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TRANSFER FUNCTIONS RECAP

Transfer Function

The function $H(s)$ is the transfer gain from $U(s)\ to\ Y(s)$ or the (input to output), thus:

$$\frac{Y(s)}{U(s)} = H(s)$$

Assuming that all initial conditions of $H(s)$ are 0,

Transfer Function

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$$\frac{Y(s)}{U(s)} = H(s)$$

Assuming that all initial conditions of $H(s)$ are 0,

If the *unit impulse* $\delta(t)$ is the input $u(t)$, than $y(t)$ is the *unit impulse response:*

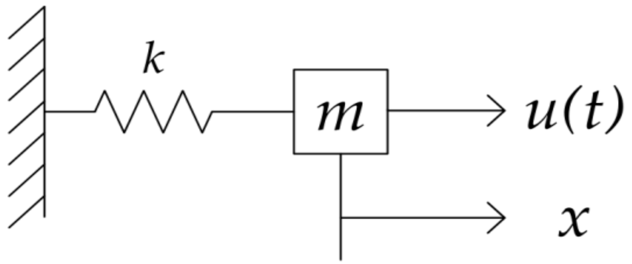
Then we have that $\mathcal{L}\big(u(t)\big) = 1$, and $\mathcal{L}\big(y(t)\big) = H(s)$, as:

$$Y(s) = H(s)$$

i.e. The transfer function $H(s)$ is the Laplace transform of the unit impulse response $h(s)$

Example: Differential equation to Transfer function

Differential equation of a harmonic oscillator from slide 22, i.e. equation of motion:



$$u(t) = m\ddot{x}(t) + kx(t)$$

The Laplace transform of the differential equation:

$$ms^2X(s) + kX(s) = U(s)$$

Set up in transfer fraction construction from previous slide, output over input, $\frac{Y(s)}{U(s)} = H(s)$:

$$\frac{X(s)}{U(s)} = \frac{1}{ms^2 + k}$$

First Order Systems

- Linear time-invariant (LTI) systems can be expressed as a combination of first order transfer functions (as shown with partial fraction decomposition):

$$H(s) = \frac{K}{\tau s + 1} = \frac{K/\tau}{s + 1/\tau} = \frac{K/\tau}{s - \left(-\frac{1}{\tau}\right)}$$

With the inpulse response represented by:

$$h(t) = \frac{K}{\tau} e^{-t/\tau}$$

- Where the DC gain is (K) at $H(0)$
- The time constant $\boldsymbol{\tau}$**,** represents the convergence speed of the system

First Order Systems

- Linear time-invariant (LTI) systems can be expressed as a combination of first order transfer functions (as shown with partial fraction decomposition):

$$H(s) = \frac{K}{\tau s + 1} = \frac{K/\tau}{s + 1/\tau} = \frac{K/\tau}{s - \left(-\frac{1}{\tau}\right)}$$

With the inpulse response represented by the exponential function:

$$h(t) = \frac{K}{\tau} e^{-t/\tau}$$

- Where the DC gain is (K) at $H(0)$
- The time constant $\boldsymbol{\tau}$**,** represents the convergence speed of the system

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First Order Systems: Poles

For a trasnfer function

$$H(s) = \frac{1}{s + \sigma}$$

The poles are equal to the roots of the denominator:

$$s + \sigma = 0$$

Where $\sigma > 0$ results in $s < 0$ and vice versa.

First Order Systems: Poles, Stability

For a trasnfer function

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The poles are equal to the roots of the denominator:

$$s + \sigma = 0$$

Where $\sigma > 0$ results in $s < 0$ and vice versa.

Impulse response of the transfer fucntion is:

$$h(t) = e^{-\sigma t} 1(t)$$

Thus as an example if
$\sigma > 0$ the exponential expression decays (i.e. the system is stable)
And if
$\sigma < 0$ the exponential expression grows (i.e. the system is unstable)

Second Order Systems

Second order transfer function is defined as:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Where:

$$\zeta = Damping\ ratio$$
$$\omega_n = natural\ frequency$$

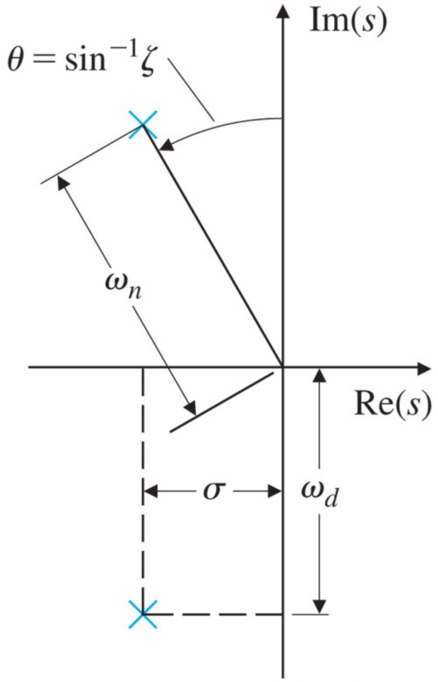
From here we can find the complex poles of the system to be:

$$s = -\sigma \pm j\omega_d$$

Where σ is the real part and $j\omega_d$ is the imaginary part

The angle of the poles is:

$$\theta = \sin^{-1}\zeta$$



Second Order Systems

Second order transfer function is defined in the canonical form as:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Where:

$$\zeta = Damping\ ratio = 'zeta'$$
$$\omega_n = natural\ frequency =' omega'$$

From here we can find the complex poles of the system to be:

$$s = -\sigma \pm j\omega_d$$

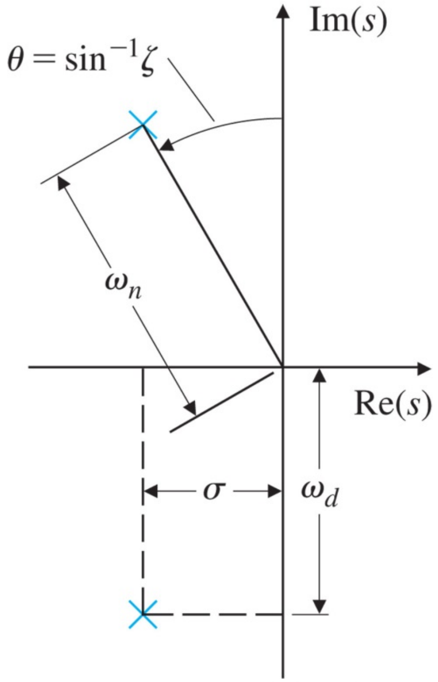
The angle of the poles is:

$$\theta = \sin^{-1}\zeta$$

And the damping frequency is:

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

i.e. the damping frequency is determined by the damping ratio



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Second Order Systems

$$\theta = \sin^{-1} \zeta$$

If we have that:

$$\theta = 0$$

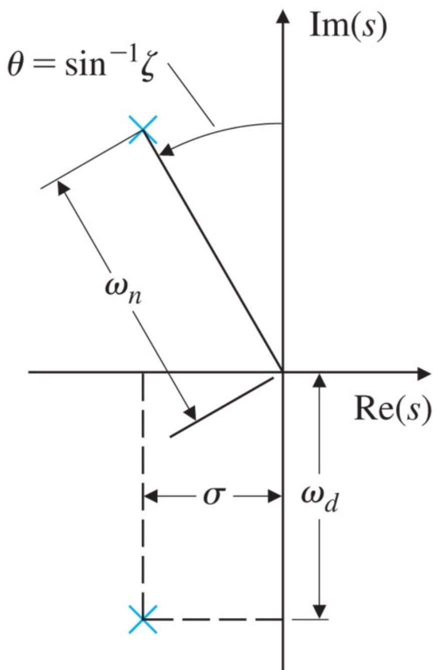
Then:

$$\omega_d = \omega_n$$

i.e. the dampened natural frequency will be equal the natural frequency.

We also have the relationship:

$$\sigma = \zeta\omega_n \quad and \quad \omega_d = \omega_n\sqrt{1-\zeta^2}$$



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Examples of Damping ratios

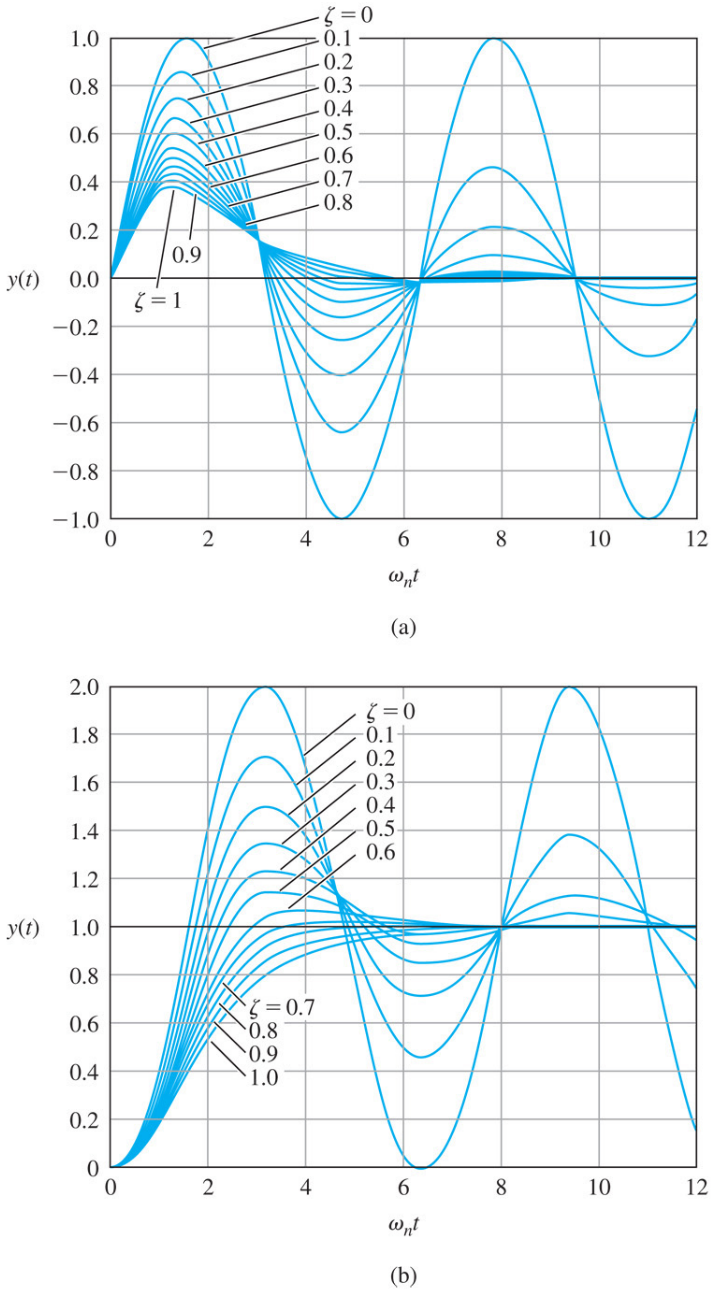
Reponeses of second-order systems with different ζ values:

a) Impulse response $h(t)$

```
a) Impulse(sys)
```

b) Step response $y(t)$

```
a) Step(sys)
```



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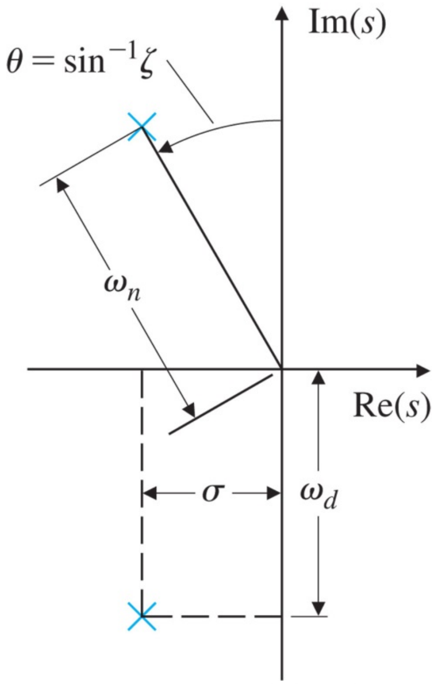
Example: Second order system analysis

$$H(s) = \frac{1}{s^2 + 2s + 2}$$

Analyze the system from its:

Natural frequency ω_n
Damping ratio ζ

And find its poles.



Time domain specifications

- Rise time

$$t_r = 1.8/\omega_n$$

- Peak time

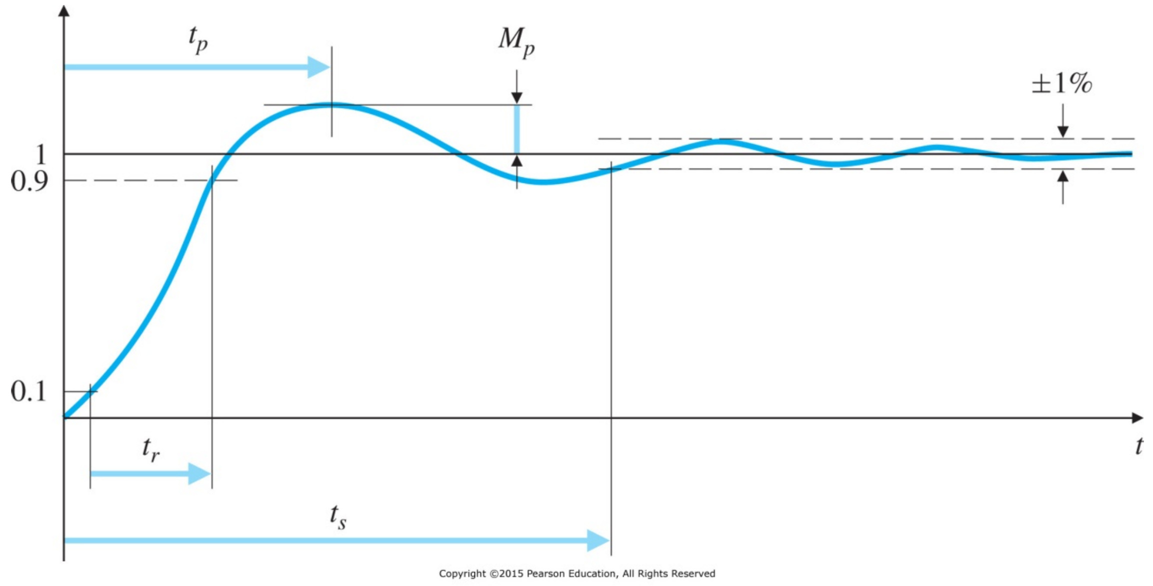
$$t_p = \frac{\pi}{\omega_d}$$

- Overshoot

$$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$$

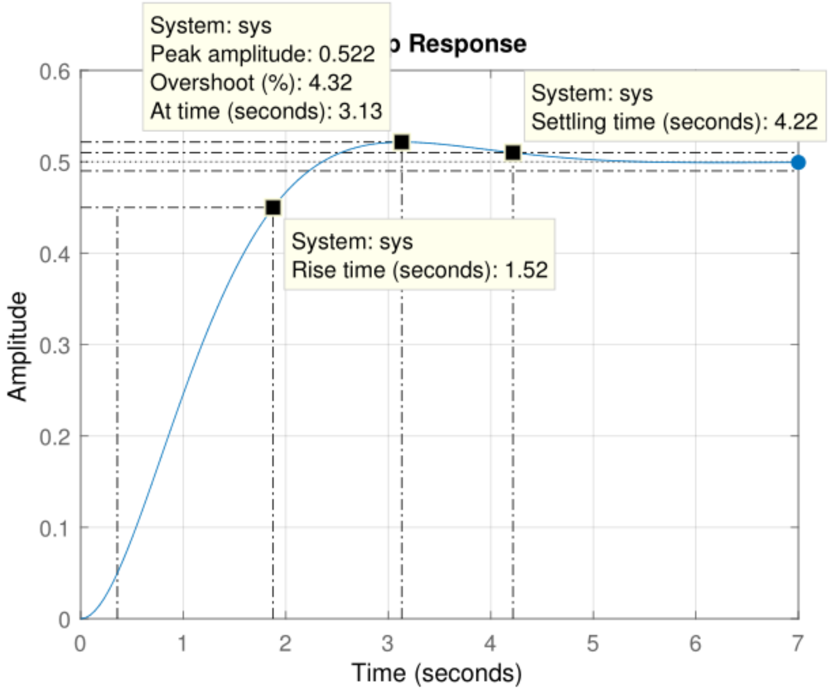
- Settling time (decaying exponential reaches 1%)

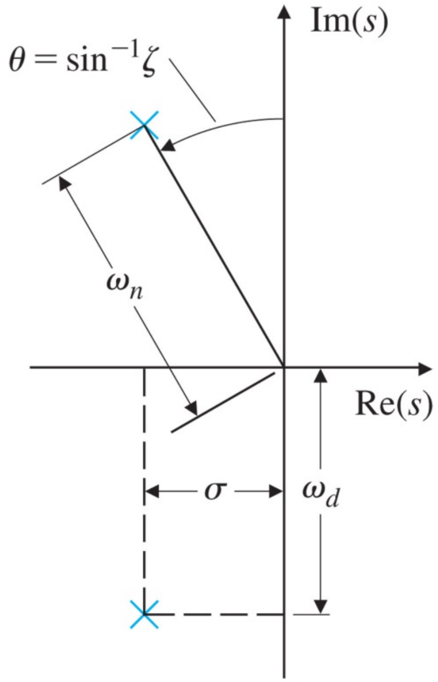
$$t_s = \frac{4.6}{\sigma}$$

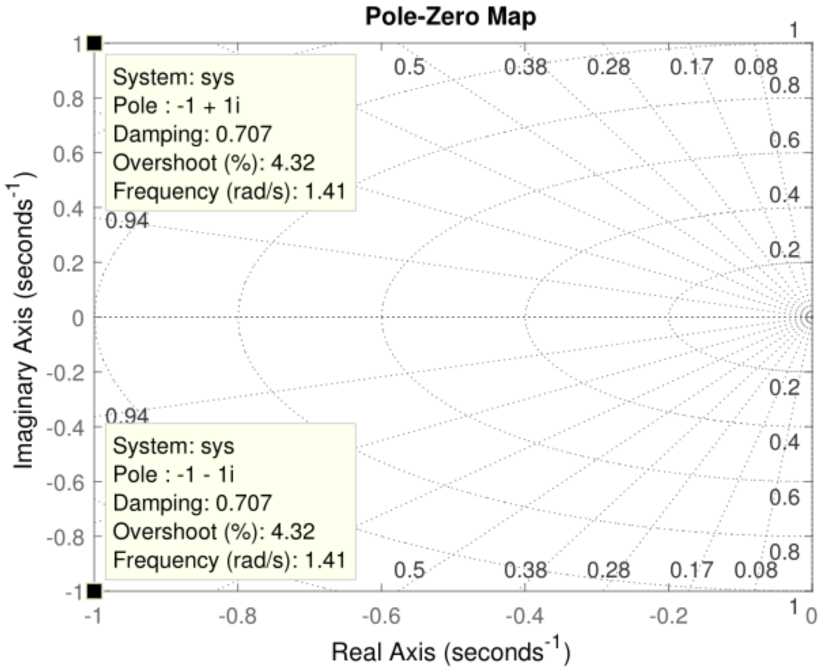


Example: Second order system analysis

Time response and poles location

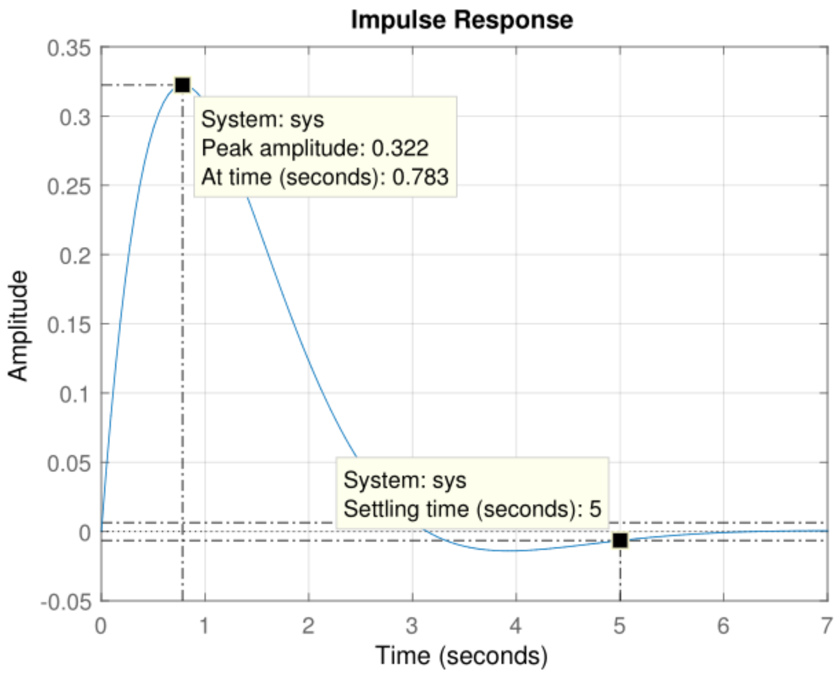


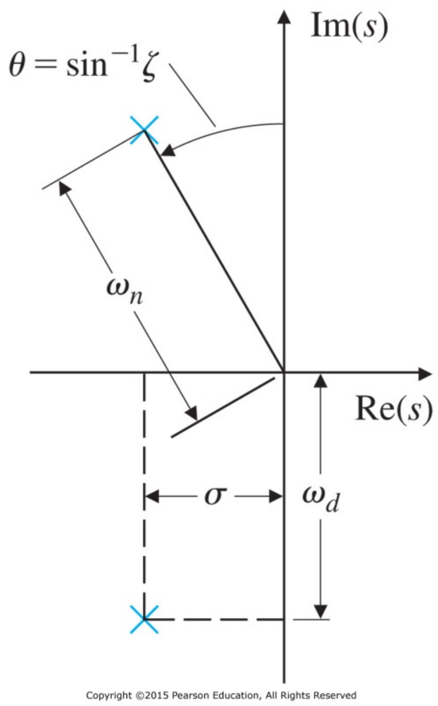


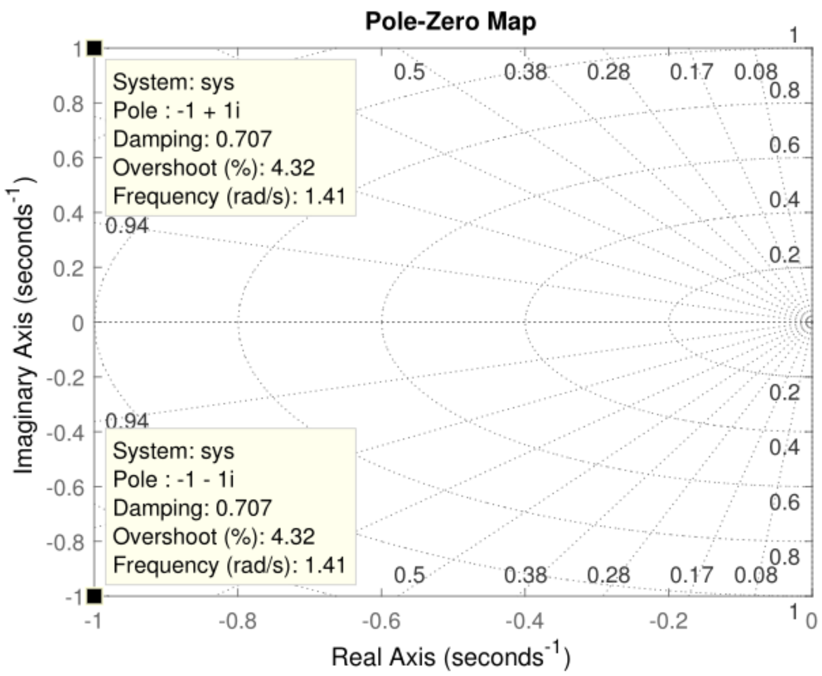


Example: Second order system analysis

Time response and poles location







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Poles and zeros

- The complex poles can be written as following

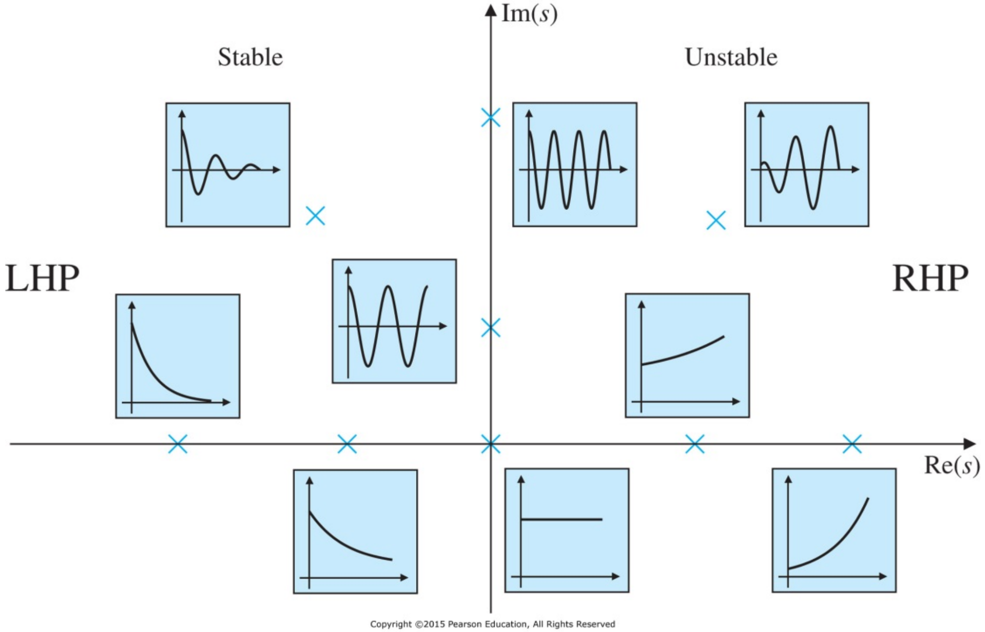
$$p_{i+1} = -\sigma_i \pm j\omega_i$$

- The poles of a system describe the **stability** of the system

$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}$$

$$H(s) = \frac{N(s)}{D(s)} = K \frac{(s - z_1)(s - z_2)\ldots(s - z_{m-1})(s - z_m)}{(s - p_1)(s - p_2)\ldots(s - p_{n-1})(s - p_n)}$$

Location of poles

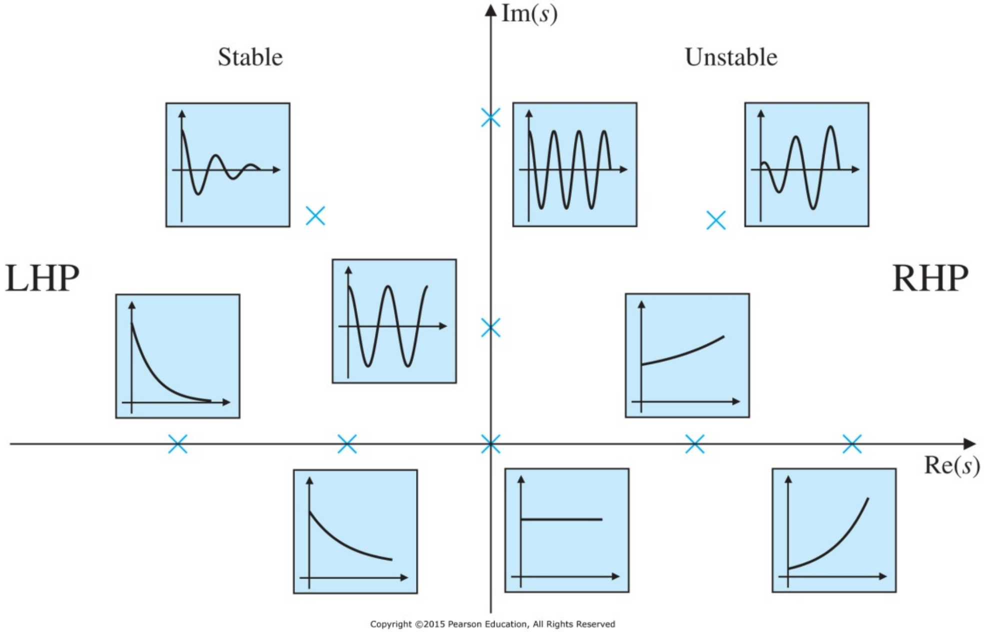


Franklin, Gene F., et al. *Feedback control of dynamic systems*. Vol. 3. Reading, MA: Addison-Wesley, 1994.

Stability

- The poles of a system describe the **stability** of the system
- When **all** the poles are located on the **left half plane (LHP)** of the real axis the system is stable
- If one or more poles are located in the right half plane (RHP) the system is unstable.

Location of poles



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STATE SPACE MODELS RECAP

State-space Models

- In the state-space method the a dynamic system is described by a set of first-order differential equations in the vector-valued state, and the solution is visualized as a trajectory of this state vector in space.
- General class of ODE models

- Advantages:
  - To study more general models
  - To deal with multiple input and multiple output systems
    - Without having extra TF's.
  - To connect internal and external descriptions

State-space definition (non-linear, time invariant)

State-space equations

State equation:

$$\frac{d\boldsymbol{x}(t)}{dt} = \boldsymbol{f}\big(\boldsymbol{x}(t), \boldsymbol{u}(t)\big)$$

Output equation:

$$\boldsymbol{y}(t) = \boldsymbol{h}(\boldsymbol{x}(t), \boldsymbol{u}(t))$$

Explanation

- $\boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t))$ is a nonlinear vector function of state and input
- $\boldsymbol{h}(\boldsymbol{x}(t), \boldsymbol{u}(t))$ is a nonlinear vector function of state and input

- $\boldsymbol{x}(t)$ is the state
- $\boldsymbol{u}(t)$ is the input

State-space definition (linear, time invariant)

State-space equations

State equation:

$$\frac{d\boldsymbol{x}(t)}{dt} = A \cdot \boldsymbol{x}(t) + B \cdot \boldsymbol{u}(t)$$

Output equation:

$$\boldsymbol{y}(t) = C \cdot \boldsymbol{x}(t) + D \cdot \boldsymbol{u}(t)$$

Explanation

- **x**(t) is an $n \times 1$ vector representing the state (e.g., position and velocity variables in mechanical systems)
- **u**(t) is a $m \times 1$ vector representing the input
- **y**(t) is a $p \times 1$ vector representing the output

- The matrices $A(n \times n)$, $B(n \times m)$, and $C(p \times n)$, determine the relationships between the state and input and output variables. These are constant matrices.

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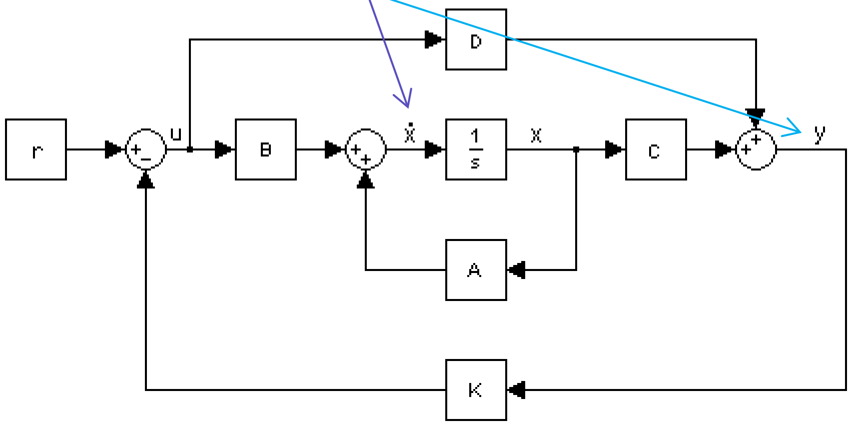
State Space Block Diagram

The state space representation has the following block diagram.

$$\frac{d\boldsymbol{x}(t)}{dt} = A \cdot \boldsymbol{x}(t) + B \cdot \mathbf{u}(t)$$

$$\boldsymbol{y}(t) = C \cdot \boldsymbol{x}(t) + D \cdot \mathbf{u}(t)$$



Linearization of Non-linear models

- All systems are nonlinear (Large signals)
  - I.e. its dynamics depends on the process variables in a non linear fashion
- Prefer linear models for control design
  - Control + analysis is easier with linear models
- Use small signals to linearize non-linear systems
- For control linear systems OK, control keeps system close to equilibrium

- Three ways of linearization:
  - **Linearization of the model (focus of this course)**
  - Linearization by feedback
  - Inverse linearization

Stability of nonlinear systems

- If the linearized system around a specific equilibrium point is strictly stable, (roots in LHP)
- The non-linear system is stable around the equilibrium point of the linearization
- If it has at least 1 root in RHP, then the nonlinear system is unstable around the equilibrium point

Linearization: Small Signal Analysis

Linear:

$$\dot{x} = Ax + Bu$$

Nonlinear:

$$\dot{x} = f(x, u)$$

1st Define equilibrium values: u_0, x_0, such that:

$$\dot{x}_0 = 0 = f(x_o, u_0)$$

And the state and input vectors are:

$$x = x_0 + \delta x, u = u_0 + \delta u$$

Linearization: Small Signal Analysis

The **nonlinear function** $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$, can now be represented with perturbations from the equilibrium points:

$$\dot{x}_0 + \delta\dot{x} \cong f(x_0, u_0) + A\delta x + B\delta u$$

A and B can be computed as following:

$$A = \left[\frac{\delta f}{\delta x}\right]_{x_0, u_0}, \qquad B = \left[\frac{\delta f}{\delta u}\right]_{x_0, u_0}$$

Where A and B are the best linear fits to the nonlinear function,

The linear diff. eq. approximating the dynamics of the state around the equilibrium points (x_0, u_0):

$$\delta\dot{x} = A\delta x + B\delta u$$

Linearization with different variables

Taylor series expansion of a single variable function, presented with $\boldsymbol{x_s}$ as the steady state operation point:

$$f(x) \approx f(x_s) + \frac{f'(x_s)}{1!} \cdot (x - x_s) + \frac{f''(x_s)}{2!} \cdot (x - x_s)^2 + \cdots$$

We truncate after the first order term (linear term) and set the deviation variables (from steady state) equal $x' = x - x_s$

$$f'(x_s) = \frac{\delta\, f(x)}{dx}\bigg|_{x=x_s}$$

Taylor series expansion with the output and input variable:

$$f(y,u) \cong f(y_s, u_s) + \frac{\delta f}{\delta y}\bigg|_{y_s,u_s} (y, y_s) + \frac{\delta f}{\delta u}\bigg|_{y_s,u_s} (u, u_s)$$

Where in steady state we have that $\boldsymbol{f(y_s, u_s)}$**=0,** where, the deviation variables (from steady state) equal $y' = y - y_s \;\; and \;\; u' = u - u_s$. Then we have:

$$\frac{dy'}{dt} = \frac{\delta f}{\delta y}\bigg|_{y_s,u_s} y' + \frac{\delta f}{\delta u}\bigg|_{y_s,u_s} u'$$

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EXAMPLES

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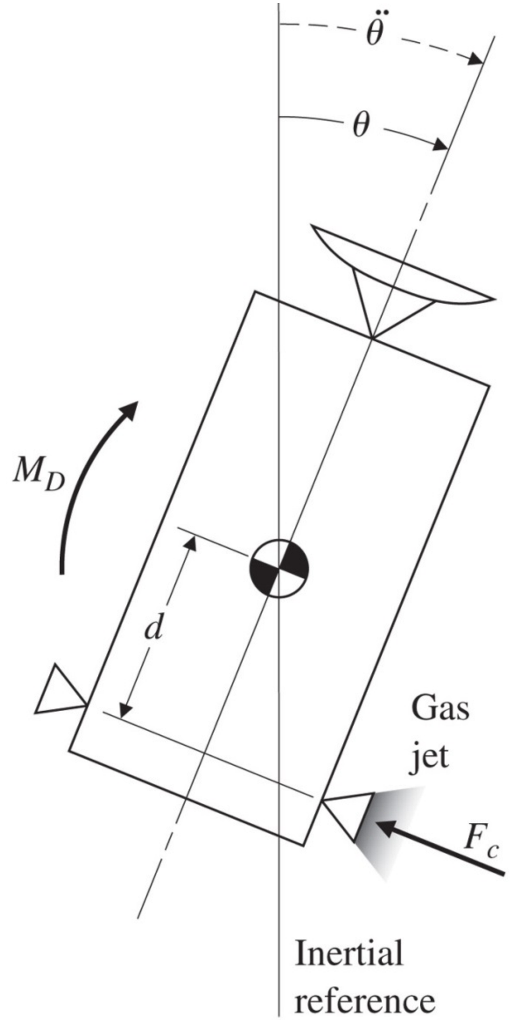
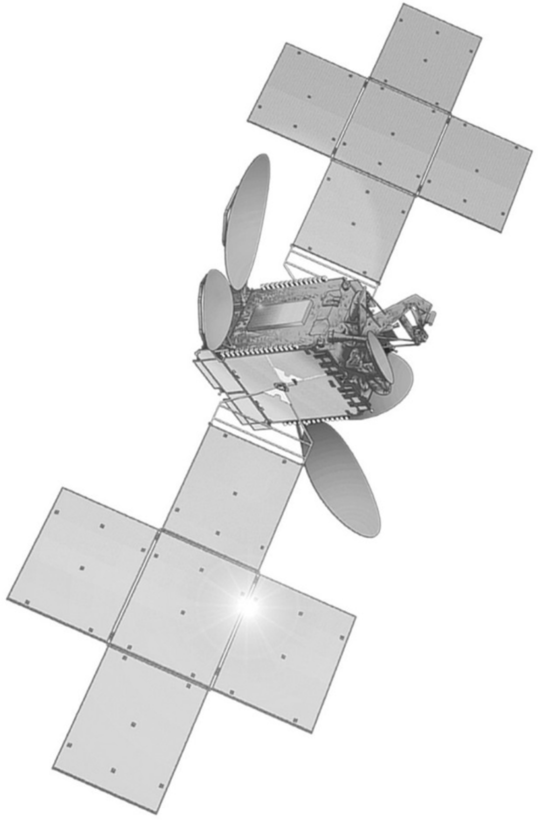
Satellite Model Example

Satellite model



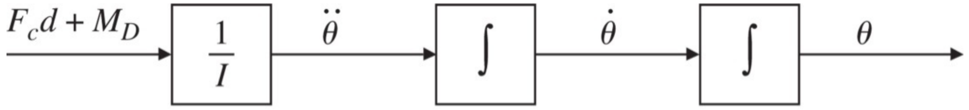


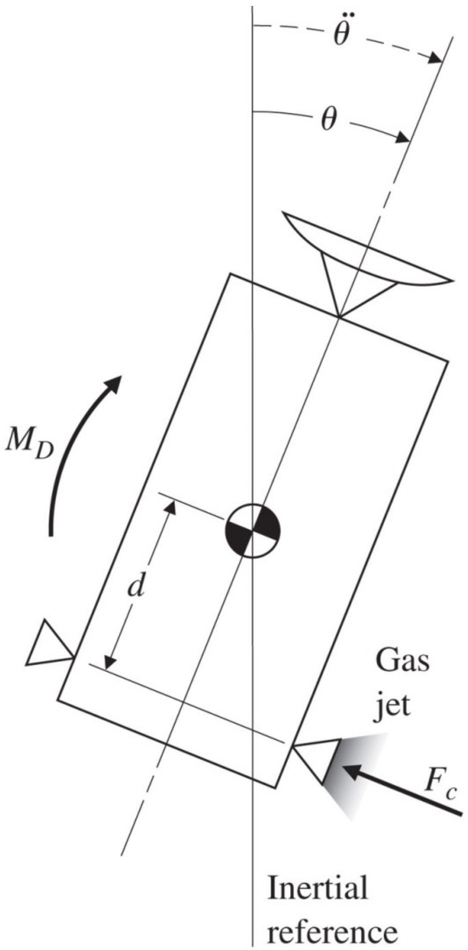
- We wish to model the angular motion of the satellite around its **center of mass** with the **angular position** Θ
- Then the motion is described by the **angular acceleration** $\ddot{\Theta}$

The **angular acceleration** can be modeled by the following diff. equation:

$$I\ddot{\Theta} = F_c d + M_D$$

And can be represented by the following block diagram:





Conversion into State Space

We have the 2nd order diff. equation

$$I\ddot{\Theta} = F_c d + M_D$$

We regard $M_D = 0$ for simplicity

We can define the State Variables $x = [\ddot{\Theta}\ \dot{\Theta}]^T$

Rocket thrust as the input: $u = F_c$

Satellite attitude as the output: $y = \Theta$

Where:

$$\ddot{\Theta} = \frac{d}{I} F_c$$

We can arrange the states following the state-variable form:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

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Conversion into State Space

$$I\ddot{\Theta} = F_c d + M_D$$
$$x = [\ddot{\Theta}\ \dot{\Theta}]^T$$
$$u = F_c$$
$$y = \Theta$$

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Which becomes:

$$\begin{bmatrix} \dot{\Theta} \\ \ddot{\Theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Theta \\ \dot{\Theta} \end{bmatrix} + \begin{bmatrix} 0 \\ d/I \end{bmatrix} F_c$$

$$y = [1\ 0] \begin{bmatrix} \Theta \\ \dot{\Theta} \end{bmatrix}$$

Giving us the state variables: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ d/I \end{bmatrix},\ C = [1\ 0],\ D = 0$

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Simulation in Matlab

$$\begin{bmatrix}\dot{\Theta}\\ \ddot{\Theta}\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}\Theta\\ \dot{\Theta}\end{bmatrix} + \begin{bmatrix}0\\ d/I\end{bmatrix} F_c$$

$$y = [1\ 0]\begin{bmatrix}\Theta\\ \dot{\Theta}\end{bmatrix}$$

We find that the parameters of the model are: $d = 0.5m, I = m \cdot r^2 = 2.5$

And insert into our State Space model : $A = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix},\ B = \begin{bmatrix}0\\ 0.5/2.5\end{bmatrix},\ C = [1\ 0],\ D = 0$

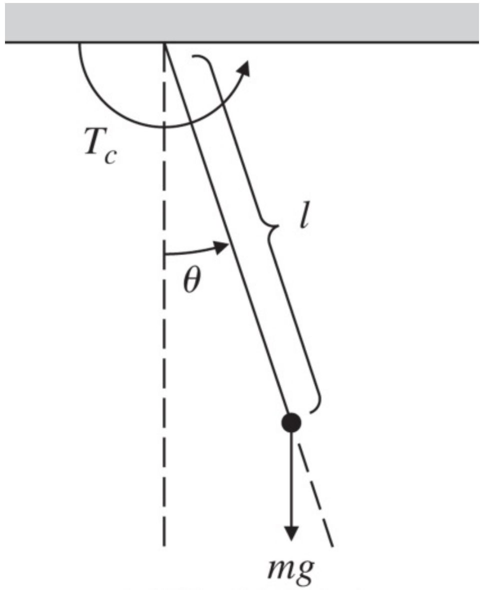
BLACK BOARD EXAMPLE

PENDULUM EXAMPLE

Example on the black board with linearization and State space construction

- Using the Pendulum as an example with the following model:

$$\ddot{\theta} + \frac{g}{l} sin\theta = \frac{T_c}{ml^2}$$



How to get SS representations

- Via modelling techniques
  - Differential equations (+ potential linearization)
  - `sys = ss(A,B,C,D)`
- Via transfer functions
  - MATLAB: `[A,B,C,D] = tf2ss(NUM,DEN)`
- Via other SS descriptions
  - MATLAB: `sysT = ss2ss(sys,T)`
- Via system identification
  - MATLAB: `ident`

Transfer Function to State Space

Transforming the state space into frequency domain:

$$\dot{x} = Ax + Bu \rightarrow (sI - A)\hat{x} = B\hat{u} + x(0)$$

If we solve for $\hat{x}$ and use $y = Cx + Du$, we get:

$$y = c(sI - A)^{-1}B\hat{u} + D\hat{u} + C(sI - A)^{-1}x(0)$$

For $x(0) = 0$, we can simplify the relationship between $\hat{u}$ and $\hat{x}$ and y to:

$$\begin{cases} \hat{x} = (sI - A)^{-1}B\hat{u} \\ \hat{y} = (C(sI - A)^{-1}B + D)\hat{u} \end{cases}$$

Where the transfer matrix relating the input to the state vector is:

$$(sI - A)^{-1}B$$

And the transfer matrix relating the input to the output is:

$$C(sI - A)^{-1}B + D$$

Transfer Function to State Space

That is:

$$G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

Transfer Function to State Space

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \rightarrow G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

In matlab you can use the functions

```
[A,B,C,D]=tf2ss(NUM,DEN)
[NUM,DEN]=ss2tf(A,B,C,D)
```

Poles and Zeros of: Transfer Functions and State Space

That is:

$$G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

Poles and zeros:

Eigenvalues of A	⇔	Poles of G(s)
`Eig(A)`	⇔	`roots(DEN)`
Transmission zero	⇔	Zeros of G(s)
`Tzero(A,B,C,D)`	⇔	`roots(NUM)`

State Space Analysis

- Stability
- Controllability
  - Full state feedback control design
- Observability
  - Observer & Estimator design

Stability of State Space Models

- The continuous time LTI system is stable iff. the eigenvalues of the system matrix A all lie in the left-half s-plane
- The poles are found by finding the Eigenvalues of the A matrix

$$Au = \lambda u$$

Controllability

- Controllability is concerned with the question whether it is at all possible to control all states disregarding how this might be done
- Controllability: for any given initial state, there always exists a piecewise continuous control input such that within a finite period the LTI system will reach the original point from the initial state.
- A LTI system is controllable if and only if the **controllability matrix** is full row rank

$$T_C = [B \; AB \; A^{2B} \ldots A^{N-1}B]$$

Where T_c is the **controllability matrix**

Controllability

In Matlab the **Controllability** matrix can be computed using:

```
CO = ctrb(A,B)
```

Then to find the rank of the controllability matrix use:

```
Rank(CO)
```

Observability

- Observability is concerned with the question whether it is possible to find all states from the measured outputs, disregarding how this might be done.
- **Observability:** any given initial state can be determined from the knowledge of input U and output Y over a finite time interval
- The considered system is called observable if and only if the **observability matrix** is full column rank:
- $O = \begin{bmatrix} C \\ CA \\ \dots \\ CA^{n-1} \end{bmatrix}$
- Where n is the dimension of matrix A

Observability

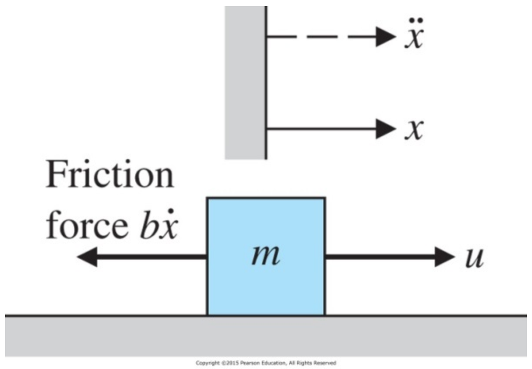
- In matlab the observability matrix can be found using:

`obsv(A,C)`

Then to find the rank of the observability matrix use:

`Rank(CO)`

Exercises



1. Linearize sin(x), x^2, cos(x), Hint: use Matlab to solve the Taylor series
   1. Set $x_0 = 0$
   2. Change $x_0 = 0.1, 0.5, 10$
   3. Plot the different functions alongside the non-linear functions
2. Given the transfer function

$$\frac{1}{0.5s^2 + 0.5s + 1}$$

   1. Find the system poles and zeros, by hand or 'hint use roots()'
   2. Plot the poles and zeros by hand
   3. What can you say about the system from its pole and zero locations
   4. Find the Damping and natural frequency of the system by hand
   5. Plot the system response in Matlab and see the response to a step

1. Given the state space model of the pendulum:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$$

   c) Where m = 100, l = 10, g=9.82

3. Is the model stable
   3. Hint) Use matlab
4. Is the model controllable
   3. Hint) Use matlab
5. Is the model observable
   3. Hint) Use matlab