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OPTIMIZATION OF PUMP CONFIGURATIONS AS A MINLP PROBLEM

T. Westerlund,[1] F. Pettersson[1] and I. E. Grossmann[2]

[1] Process Design Laboratory, Åbo Akademi University, Biskopsgatan 8, FIN-20500 Åbo, Finland and
[2] Carnegie Mellon University, Pittsburgh, PA 15213-3890, U.S.A.

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Abstract—This paper introduces a method for solving optimal pump configurations as a MINLP (mixed integer non-linear programming) problem. The pump configurations considered consist of an arbitrary number of centrifugal pumps, of different sizes, coupled in series and/or parallel. Given the total required pressure rise and the required flow, the optimal pump configuration is sought. The pump configuration is optimized with respect to the minimum total cost including the investment and the running costs. The optimization problem consists of a non-linear objective function subject to linear and non-linear equality and inequality constraints including real, integer and binary variables. The MINLP problem is solved by a proposed ECP (extended cutting plane) method combined with a general (integer) branch and bound method, as well as by the DICOPT++ software in which an outer approximation method is used coupled with a zero–one branch and bound method. Simple linear transformations illustrate that integer problems are also efficiently solved by zero–one programming. Different problem formulations are also given to improve the solution by the methods. Results obtained by solving the optimization problem as a min–min problem are also given. Some problems arise from the fact that the objective function may have several local optima and is generally non-convex. Examples are given to illustrated the procedures.

INTRODUCTION

The minimization of energy costs, as well as the total cost of pumping systems, is an important issue in industries with high energy consumption for pumping. In the pulp and paper industry, for example, approx. 10–20% of the electrical energy requirement is consumed by pumps.

Much effort has been invested in optimizing the design of single pumps as well as in obtaining optimal working conditions for single pumps. However, less attention has been given the selection of optimal pump configurations.

In recent years, a lot of work has been done on developing algorithms for mixed-integer optimization (Geoffrion, 1972; Duran and Grossmann, 1986; Yuan *et al.*, 1989) and solution strategies for structural optimization problems in process synthesis (Papoulias and Grossmann, 1983; Kocis and Grossmann, 1987; Floudas *et al.*, 1989; Viswanathan and Grossmann, 1990a; Kravanja and Grossmann, 1990; Achenie and Biegler, 1990).

One of the most popular approaches has been the MINLP approach. According to this approach, the structural process optimization problem is formulated as a non-linear optimization problem in terms of continuous and discrete variables where the discrete variables are usually restricted to binary variables. Efficient algorithms have also been developed for zero–one programming problems (Crowder *et al.*, 1983; Van Roy and Wolsey, 1987). From the examples in this paper it is found that a zero–one algorithm can also be used in a most efficient way in solving the actual integer problems.

Many practical MINLP problems are non-convex. Algorithms solving certain classes of non-convex MINLP problems have also become an important research issue (Floudas *et al.*, 1989; Viswanathan and Grossmann, 1990a; Floudas and Visweswaran, 1990). The multi-level pump configuration problem, considered in this paper, is non-convex and it is found that the algorithms used in the paper are able to solve, but have some problems when applying them to, the actual multi-level problem. The single-level pump optimization problem was efficiently solved by all of the algorithms considered.

PUMP CONFIGURATIONS AND FORMULATION OF THE PROBLEM

The problem we would like to solve in this paper can be worded as follows: select the best pump or configuration of pumps coupled in series and/or parallel, given the pressure rise (total head) and power data as a function of the capacity of a set of centrifugal pumps, as well as the total required pressure rise and total flow for the configuration to be selected. For simplicity we will restrict the total

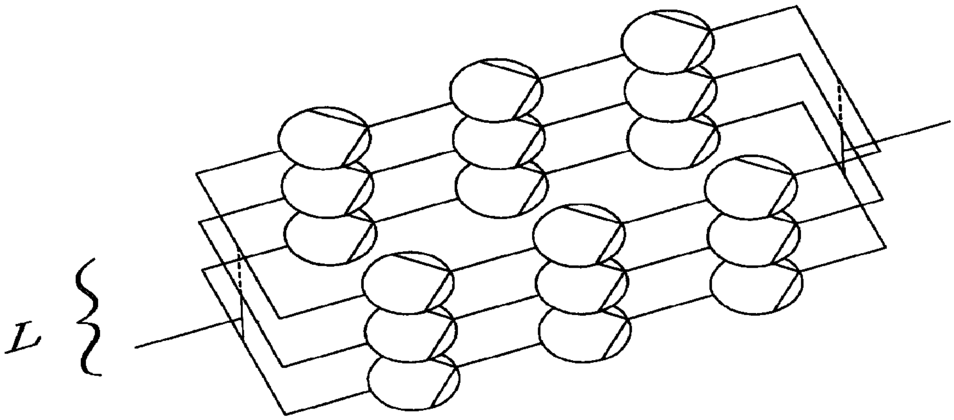

Fig. 1. An L level pump configuration.

pump configuration to an L "level" pump configuration with $N_{p,i}$ parallel pumping lines of $N_{s,i}$ pumps in series in every line. The pumps on the same "level", i, are centrifugal pumps of the same size and use the same (but variable) rotation speed. For each "level", the pressure rise is the same as the total required pressure rise, while the total flow is the sum of flows through all L "levels". The required data is restricted, for each level, to two real variables (the rotation speed, ω_i, and the fraction, x_i, of the total flow, $\dot{V}_{\text{tot}}$) and two integer variables (the number of parallel pumping lines, $N_{p,i}$, of $N_{s,i}$ pumps in series). In Fig. 1 an L level pump configuration is illustrated.

The total cost of the pump configuration can now be expressed as follows,

$$J=\sum_{i=1}^{L}(C_i+C_i'\cdot P_i)\cdot N_{p,i}\cdot N_{s,i} \tag{1}$$

where L is the number of "levels", C_i the (yearly) instalment of the capital costs for one pump, P_i and C_i' the power and the energy cost for every single pump on each level, respectively.

The required power, P_i, for a single pump is, typically, a function of the flow, $\dot{V}_i$, through the pump, the rotation speed, ω_i, of the centrifugal pump and the density of the fluid to be pumped,

$$P_i=f_{1,i}(\dot{V}_i,\omega_i,\rho). \tag{2}$$

The flow through a single pump on "level" i will be given by

$$\dot{V}_i=\frac{x_i}{N_{p,i}}\cdot\dot{V}_{\text{tot}}. \tag{3}$$

The pressure rise Δp_i is also a function of the flow, $\dot{V}_i$, through the pump and the rotation speed, ω_i, as well as the density of the fluid,

$$\Delta p_i=f_{2,i}(\dot{V}_i,\omega_i,\rho). \tag{4}$$

The pressure rise over a single pump on each "level" will be given by,

$$\Delta p_i=\frac{1}{N_{s,i}}\cdot\Delta p_{\text{tot}}. \tag{5}$$

The rotation speed ω_i can thus be implicitly solved from equations (4)–(5) for a given flow, number of pumps in series and the total pressure rise.

The functions, $f_1(\cdot)$ and $f_2(\cdot)$, are usually not known explicitly. However, pressure rise (total head) and power data, at a constant rotation speed and a specific fluid (usually water) are often given by the manufacturer as a function of the capacity for a specified pump.

The pressure rise data, provided by the pump manufacturer, given at a constant rotation speed, ω_{m}, and for a fluid with the density, ρ_{m}, may be expressed by the relation,

$$\Delta p_{\text{m}}=g_1(\dot{V}_{\text{m}},\omega_{\text{m}},\rho_{\text{m}}) \tag{6}$$

and the corresponding power data by the relation,

$$P_{\text{m}}=g_2(\dot{V}_{\text{m}},\omega_{\text{m}},\rho_{\text{m}}). \tag{7}$$

Now using the proportionality relation as defined for example in Coulson and Richardsson (1985) we find,

$$\dot{V}_i=\left(\frac{\omega_i}{\omega_{\text{m}}}\right)\cdot\dot{V}_{\text{m}} \tag{8}$$

$$\Delta p_i=\left(\frac{\omega_i}{\omega_{\text{m}}}\right)^2\cdot\left(\frac{\rho}{\rho_{\text{m}}}\right)\cdot\Delta p_{\text{m}} \tag{9}$$

$$P_i=\left(\frac{\omega_i}{\omega_{\text{m}}}\right)^3\cdot\left(\frac{\rho}{\rho_{\text{m}}}\right)\cdot P_{\text{m}}. \tag{10}$$

Thus, the relations, $f_{1,i}(\cdot)$ and $f_{2,i}(\cdot)$, can be obtained by combining equations (8)–(10) with the relations, $g_{1,i}(\cdot)$ and $g_{2,i}(\cdot)$. $g_{1,i}(\cdot)$ and $g_{2,i}(\cdot)$ refer to the "manufacturers" total head curve and the power curve for the specific pump type used at level "i".

THE MINLP PROBLEM

The MINLP problem, for the solution of the optimal pump configurations, can now be expressed as follows,

$$\min_{N_{p,i}, N_{s,i}, x_i, \omega_i, i=1,\ldots,L}\left\{\sum_{i=1}^{L}(C_i+C_i'\cdot P_i)\cdot N_{p,i}\cdot N_{s,i}\right\} \quad (11)$$

subject to,

$$\Delta p_i = f_{2,i}(\dot{V}_i, \omega_i) \quad (12)$$

$$\sum_{i=1}^{L} x_i = 1 \quad (13)$$

$$\omega_i - \omega_{i,\max} \leq 0 \quad (14)$$

where

$$P_i = f_{1,i}(\dot{V}_i, \omega_i) \quad (15)$$

$$\dot{V}_i = \frac{x_i}{N_{p,i}}\cdot \dot{V}_{\text{tot}} \quad (16)$$

$$\Delta p_i = \frac{1}{N_{s,i}}\cdot \Delta_{p\text{tot}}. \quad (17)$$

ω_i and x_i are non-negative real variables and the variables, $N_{p,i}$ and $N_{s,i}$, are non-negative integers. Appendix C presents the explicit MINLP formulation for a three-level pump configuration problem which is reported later on in the paper as a numerical example.

Some algorithms have difficulties in using non-linear equality constraints in the problem formulation. The non-linear equality constraints can, however, be eliminated from the problem formulation if an implicit formulation of the objective function is used.

In order to calculate the power, P_i, the required rotation speed, ω_i, should be calculated. The rotation speed, ω_i, can, however, be implicitly obtained from $N_{p,i}$, $N_{s,i}$ and x_i. $\dot{V}_i$ is a function of x_i and $N_{p,i}$ [equation (16)]. Δp_i is a function of $N_{s,i}$ [equation (17)]. ω_i is then given by the root of equation (12), the root being a function of $N_{p,i}$, $N_{s,i}$ and x_i,

$$\omega_i = r_i(N_{p,i}, N_{s,i}, x_i) \quad (18)$$

$r_i(\cdot)$ represents the implicit solution of ω_i given the total flow and the total pressure rise. The power, P_i, (15), as well as the objective function, (11), can, thus, be expressed as a function of the variables, $N_{p,i}$, $N_{s,i}$ and x_i, only.

Analyzing the non-linear equality constraints, (12), (16) and (17) as well as the linear inequality constraint, (14), it can be found that the lower limit of $N_{s,i}$, is obtained at $\omega_{i,\max}$ (and as a function of $N_{p,i}$, and x_i),

$$q_i(N_{p,i}, x_i, \omega_{i,\max}) - N_{s,i} \leq 0 \quad (19)$$

where

$$q_i(N_{p,i}, x_i, \omega_{i,\max}) = \frac{\Delta p_{\text{tot}}}{f_{2,i}(\dot{V}_i, \omega_{i,\max})}. \quad (20)$$

Thus, the variables, ω_i, can be eliminated from the optimization problem and the non-linear equality constraints, (12) and (17), as well as the linear inequality, (14), can be replaced by the non-linear inequality constraint (19) and an implicit objective function in $N_{p,i}$, $N_{s,i}$ and x_i where the power is obtained from equation (15) combined with equations (16) and (18).

In Figure 2, an example of a single "level" objective function (11) and the constraint, (19), are illustrated. From equations (16) and (17) one can observe that N_p is proportional to $\dot{V}_{\text{tot}}$ and N_s proportional to Δp_{tot}. Thus, the objective function (11) is proportional to $\dot{V}_{\text{tot}}\cdot \Delta p_{\text{tot}}$.

In the figure both the objective function and the variables (number of parallel pumping lines, N_p, of N_s pumps in series) are, therefore, normalized with respect to the required pressure rise and total flow. The normalized figure can, thus be used for illustrating different pumping conditions.

The minimization of the corresponding continuous relaxation problem, equation (11), can under certain conditions be solved numerically as a NLP problem using an LP cutting plane algorithm, Kelley (1960), for the transformed problem,

$$\min_{N_{p,i}, N_{s,i}, x_i, Q, i=1,\ldots,L}\{Q\} \quad (21)$$

subject to the constraints (13) and (19) as well as the new (implicit) constraint in $N_{p,i}$, $N_{s,i}$ and x_i,

$$\left(\sum_{i=1}^{L}(C_i+C_i'\cdot P_i)\cdot N_{p,i}\cdot N_{s,i}\right) - Q \leq 0 \quad (22)$$

where the power is given by equation (15) combined with equations (16) and (18).

In the cutting plane algorithm, new linearized constraints are added to the old linearized constraints in each iteration. The total number of constraints can thus be very large if fast convergence is not achieved if inactive constraints are not removed during the iteration procedure. The inclusion of asymptotic properties of the non-linear constraints can also improve the properties of the method. In the present case it can be shown that the number of parallel pumping lines and the number of pumps in series are limited (lower bounds) by the maximum rotation speed, $\omega_{i,\max}$, and asymptotically approaches the values, $N_{p,i,\min}$ and $N_{s,i,\min}$ (see Fig. 2). $N_{p,i,\min}$ and $N_{s,i,\min}$ are given by,

$$N_{p,i,\min} = N'_{p,i,\min}\cdot x_i \quad (23)$$

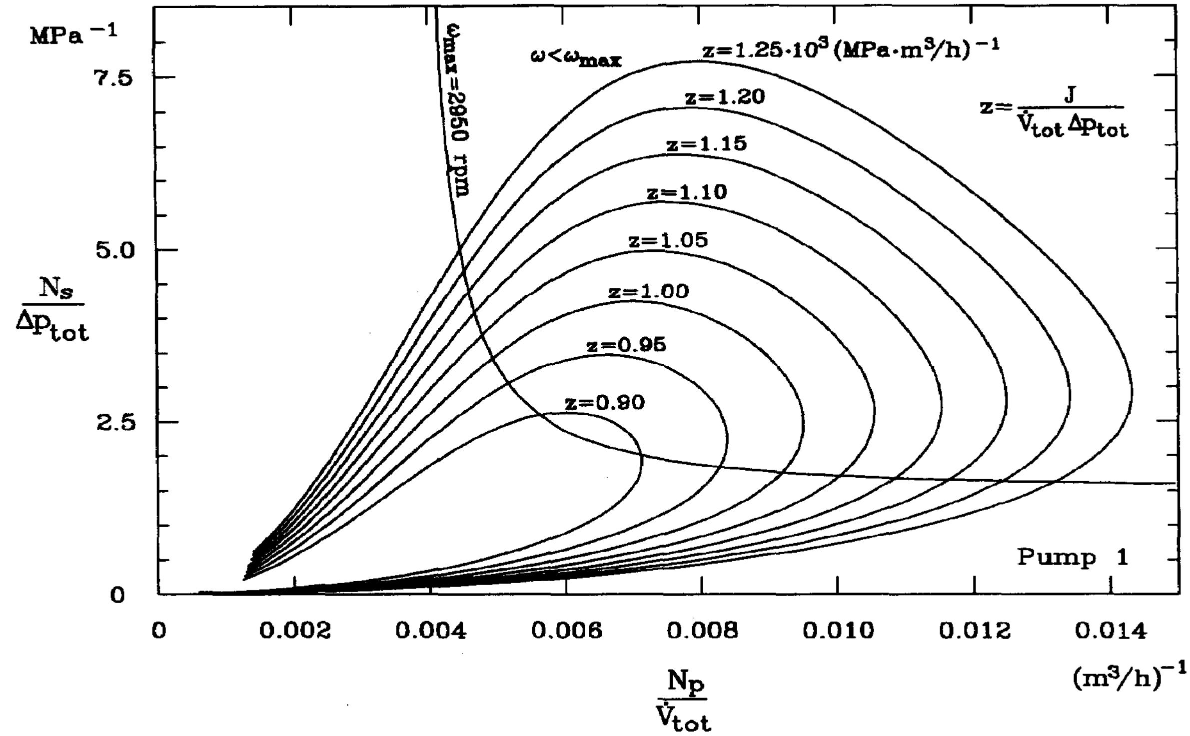

Fig. 2. Normalized objective function and constraints for a single level optimization problem.

where

$$N'_{p,i,\min}=\left(\frac{\omega_m}{\omega_{i,\max}}\right)\cdot\left(\frac{\dot{V}_{tot}}{\dot{V}^0_{m,i}}\right) \tag{24}$$

and

$$N_{s,i,\min}=\left(\frac{\omega_m}{\omega_{i,\max}}\right)^2\cdot\left(\frac{\rho_m}{\rho}\right)\cdot\left(\frac{\Delta p_{tot}}{\Delta p^0_{m,i}}\right) \tag{25}$$

where $\dot{V}^0_{m,i}$ is the flow where the manufacturer's pressure rise curve approaches zero and $\Delta p^0_{m,i}$ is the pressure rise at zero flow on the manufacturer's pressure rise curve. Expressions (23)–(25) thus give rise to two linear inequality constraints:

$$N'_{p,i,\min}\cdot x_i-N_{p,i}\leqslant 0 \tag{26}$$

$$N_{s,i,\min}-N_{s,i}\leqslant 0. \tag{27}$$

Since both $N_{p,i}$ and $N_{s,i}$ are integer variables, the lower limits should also be integers. For inequality (27) there is no problem and we obtain,

$$\lceil N_{s,i,\min}\rceil-N_{s,i}\leqslant 0. \tag{28}$$

Equation (23) allows $N_{p,i,\min}$ to be equal to zero when x_i is equal to zero. However, $N_{p,i}=0$ will give rise to numerical difficulties [in equation (16)] in the calculations. Thus it is more convenient to define a lower limit, $N_{p,i,\min}=1$, for an existing level according to,

$$1-N_{p,i}\leqslant 0. \tag{29}$$

In this case we need to define binary variables, y_i, for each level, defining the existence or non-existence of the level.

It is clear that practical pump configurations must be limited by some upper values of the number of parallel pumping lines and pumps in series. $N_{p,i}$ and $N_{s,i}$ may, therefore, also be restricted by some prespecified upper limits, $N_{p,i,\max}$ and $N_{s,i,\max}$. $N_{p,i}$ and $N_{s,i}$ should further be specified whether the level exists or not. We have specified $N_{p,i}$ and $N_{s,i}$ to their lower bounds if the level does not exist. The latter specification is, especially, important in order to avoid numerical difficulties with lower bounds equal to zero on $N_{p,i}$ and $N_{s,i}$ when the level does not exist.

The inequalities can be written as follows,

$$N_{p,i}-(N_{p,i,\max}-1)\cdot y_i-1\leqslant 0 \tag{30}$$

$$N_{s,i}-(N_{s,i,\max}-\lceil N_{s,i,\min}\rceil)\cdot y_i-\lceil N_{s,i,\min}\rceil\leqslant 0. \tag{31}$$

The variables, x_i, can, further, be defined to be greater than or equal to a prespecified lower limit, $x_{\min}$, if the level exists. If the level does not exist, x_i, should be equal to zero. Thus, the following inequalities can be applied for x_i,

$$x_{\min}\cdot y_i-x_i\leqslant 0 \tag{32}$$

$$x_i-y_i\leqslant 0. \tag{33}$$

Applying inequalities (26), (28)–(33) together with inequality (22) as well as equality (13), the variables will obtain proper values if the level exists. If the level does not exist, the variables, $N_{p,i}$ and $N_{s,i}$, obtain the minimum values for an existing level. In this case, the objective function portion of inequality (22) can be multiplied by the binary variables, y_i, in order to obtain a proper value. The variables, thus, have a physical interpretation if $y_i=1$. If $y_i=0$, the variables obtain values restricted by a possible existence of the level.

Inequalities (26) and (28)–(33) can, thus, be included, in advance, when applying, for example, the cutting plane method.

NUMERICAL SOLUTION OF THE MINLP PROBLEM BY THE ECP METHOD

Summing up the aspects considered above, the MINLP problem can now be formulated as follows,

$$\min_{N_{p,i}, N_{s,i}, x_i, y_i, Q, i=1,\ldots,L} \{Q\} \tag{34}$$

subject to the following linear inequality and equality constraints,

$$1-N_{p,i}\leqslant 0 \tag{35}$$

$$N'_{p,i,\min}\cdot x_i - N_{p,i}\leqslant 0 \tag{36}$$

$$N_{p,i}-(N_{p,i,\max}-1)\cdot y_i - 1\leqslant 0 \tag{37}$$

$$\lceil N_{s,i,\min}\rceil - N_{s,i}\leqslant 0 \tag{38}$$

$$N_{s,i}-(N_{s,i,\max}-\lceil N_{s,i,\min}\rceil)\cdot y_i - \lceil N_{s,i,\min}\rceil\leqslant 0 \tag{39}$$

$$x_{\min}\cdot y_i - x_i\leqslant 0 \tag{40}$$

$$x_i - y_i\leqslant 0 \tag{41}$$

$$\left(\sum_{i=1}^{L} x_i\right)-1=0 \tag{42}$$

and subject to the non-linear inequality constraints,

$$\left(\sum_{i=1}^{L}(C_i+C'_i\cdot P_i)\cdot N_{p,i}\cdot N_{s,i}\cdot y_i\right)-Q\leqslant 0 \tag{43}$$

$$q_i(N_{p,i}, x_i, \omega_{i,\max})-N_{s,i}\leqslant 0 \tag{44}$$

x_i and Q are non-negative real variables, $N_{p,i}$ and $N_{s,i}$ non-negative integers and y_i binary variables. $P_i(\cdot)$ and $q_i(\cdot)$ are given by equations (15), (16), (18) and (20), respectively.

By using the standard cutting plane method (Kelley, 1960), only the continuous optimal solution is solved. In order to obtain the MI solution, the linearized LP problem can, however, be solved as an MILP problem by a branch and bound method in each iteration. With this modification the algorithm showed good convergence properties for convex MINLP problems (the one-level optimization problems). Typically the algorithm converged within five to ten iterations for the problems considered.

For multi-level problems ($L>1$), the objective function was found to be non-convex (see Appendix A) and a satisfactory solution was not always obtained. Since the problem is non-convex Kelleys cutting plane algorithm (Kelley, 1960), cannot be applied directly. We have, therefore, used an extended procedure where linearized constraints are both removed, replaced and added in each iteration. First, old linearized constraints are removed or replaced from the set of linear constraints if they do not satisfy a local convexity and gradient test. Thereafter new linearized constraints are added to the set of linear constraints according to the standard cutting plane procedure. With this extension the algorithm showed good convergence properties also for non-convex problems. Typically the algorithm converged within ten to twenty five iterations for the non-convex problems considered. Global convergence can only be ensured for convex problems. It should be pointed out that the removal and replacement procedures are heuristic for non-convex problems. But as can be found from the example section, the extended algorithm converged even to the global optimal solution in most of the non-convex examples considered. The ECP algorithm is given in more detail in Appendix B and in Westerlund and Pettersson (1992).

NUMERICAL SOLUTION WITH DICOPT++

DICOPT++ is a software package for solving MINLP problems that involve linear zero-one variables and linear and non-linear continuous variables (Viswanathan and Grossmann, 1990b).

DICOPT++ uses only binary and continuous variables and the objective function as well as the constraints must be linear in the binary variables but may be non-linear in the continuous variables. The program is based on the augmented penalty function version of the OA/ER (outer approximation/equality relaxation) algorithm, and also allows for non-linear equality constraints. The algorithm does not have any strict convexity assumptions. The algorithm gave good results for the single-level problem. However, for multi-level problems a satisfactory solution was not always obtained. This is illustrated in the examples below. DICOPT++ needs an explicit problem formulation. The general multi-level problem is thus formulated for DICOPT++ as follows,

$$\min_{\omega_i, x_i, y'_i, y_i, y_{i,p,k}, y_{i,s,k}, i=1,\ldots,L, k=1,\ldots,K} \times\left\{\sum_{i=1}^{L}(C_i+C'_i\cdot P_i)\cdot N_{p,i}\cdot N_{s,i}\cdot y'_i\right\} \tag{45}$$

subject to the linear equality and inequality constraints (13) and (14) as well as the non-linear equality constraints (12) and (15) combined with the relations (16) and (17). Also the constraints (35) and (38) defining the lower limits on $N_{p,i}$ and $N_{s,i}$ are considered in this case. Equations (40)–(41) are used to define the lower and upper limits for x_i. In its standard version, DICOPT++ uses binary variables instead of general integer variables and the objective function as well as the constraints must be linear in the binary variables but may be non-linear in the continuous variables. Thus a continuous variable, y_i', equal to the binary variable, y_i, has been introduced in objective function (45).

Both $N_{p,i}$ and $N_{s,i}$ are obtained by linear expressions in binary variables and considered continuous variables with integer values in the objective function. The following new linear equality and inequality constraints are thus further added in order to obtain a proper problem formulation in terms of DICOPT++,

$$N_{p,i}-\left(\sum_{k=1}^{K}2^{k-1}\cdot y_{i,p,k}\right)=0 \tag{46}$$

$$N_{s,i}-\left(\sum_{k=1}^{K}2^{k-1}\cdot y_{i,s,k}\right)=0 \tag{47}$$

$$y_i'-y_i=0 \tag{48}$$

$$y_i'-1\leq 0 \tag{49}$$

$$-y_i'\leq 0 \tag{50}$$

$\omega_i, x_i, y_i', N_{p,i}$ and $N_{s,i}$ are all continuous variables and $y_i, y_{i,p,k}$ as well as $y_{i,s,k}$ are binary variables in DICOPT++. In this case, the variables, $y_i', y_{i,p,k}$ and $y_{i,s,k}$, have no direct physical interpretation. These definitions of the variables were only used in order to obtain a proper problem formulation in terms of DICOPT++, Viswanathan and Grossmann (1990b).

NUMERICAL SOLUTION AS A MIN–MIN PROBLEM

Since the single-level problem is convex (and we can obtain the global optimal solution for those problems by the given methods) and the only variable that affects the variables in each pumping level is x_i the optimization problem was also formulated as a min–min problem according to,

$$\min_{x_i, i=1,\dots,L}\left\{\sum_{i=1}^{L}(C_i+C_i'\cdot P_i)\cdot N_{p,i}\cdot N_{s,i}\cdot y_i\right\} \tag{51}$$

subject to the linear inequality and equality constraints (40)–(42) as well as the non-linear (observe the round up operator) equality constraint,

$$y_i-\lceil x_i\rceil=0 \tag{52}$$

and subject to

$$\min_{N_{p,1},N_{s,1}}\{(C_1+C_1'\cdot P_1)\cdot N_{p,1}\cdot N_{s,1}\} \tag{53}$$

$$\min_{N_{p,2},N_{s,2}}\{(C_2+C_2'\cdot P_2)\cdot N_{p,2}\cdot N_{s,2}\} \tag{54}$$

$$\dots$$

$$\min_{N_{p,L},N_{s,L}}\{(C_L+C_L'\cdot P_L)\cdot N_{p,L}\cdot N_{s,L}\}. \tag{55}$$

Since objective function (51) is generally non-convex (see Appendix A) and we do not have explicit gradients information we have solved the upper minimization problem of the min–min problem by using a non-gradient based optimizer, the flexible geometric simplex method (Nelder and Mead, 1965). Both the ECP method as well as the DICOPT++ software can then be used for the minimization of (53)–(55). In the case when DICOPT++ is used, $N_{p,i}$ and $N_{s,i}$ are replaced by $\omega_i, y_{i,p,k}$ and $y_{i,s,k}$. Constraints (12), (14)–(17) as well as constraints (35), (38) and (46)–(47) are all considered while minimizing (53)–(55) in this case.

For the ECP method objective functions (53)–(55) are replaced by the variables $Q_1, Q_2, \dots, Q_L$ and a non-linear inequality constraint equal to equation (43) is added in each case. Equations (53)–(55) are all minimized subject to linear inequality constraints (35)–(39) and non-linear inequality constraint (44) in this case.

In solving the min–min problem we have used a two-step procedure. First, (53)–(55) have been solved as continuous problems [$N_{p,i}$ and $N_{s,i}$ are considered continuous variables in (53)–(55)]. Thereafter, the mixed integer min–min problem, (51)–(55), is solved. The two-step procedure is chosen because objective function (51) has several local minima for integer solutions of (53)–(55) while objective function (51) is found to be optimal for one of the levels for continuous solutions of (53)–(55). Solving (53)–(55) first as continuous problems thus generates good starting points for the MI problems (51)–(55).

Global optimality cannot be expected. However, as can be seen from the examples, this method also converged to the global solution from the given starting points, $x_i=1/L$, in some of the non-convex examples.

AN ALTERNATIVE IMPROVED FORMULATION OF THE PROBLEM

Generally, bilinear expressions as well as non-linear functions multiplied by binary variables should be avoided when formulating the MINLP problem. It may, thus, be expected that the result with DICOPT++ and the ECP method could be improved by rewriting the objective function as well as the corresponding constraint such that multiplication by the binary variables is avoided.

An alternative formulation is obtained by rewriting objective function (34) and the corresponding constraint, (43), as follows,

$$\min_{N_{p,i},N_{s,i},x_i,y_i,\mu_i,i=1,\ldots,L}\left\{\sum_{i=1}^{L}\mu_i\right\} \tag{56}$$

subject to

$$(C_i+C_i'\cdot P_i)\cdot N_{p,i}\cdot N_{s,i}-\mu_i-U_i\cdot(1-y_i)\le 0 \tag{57}$$

$$-\mu_i\le 0. \tag{58}$$

The objective function, (56), and the corresponding constraints, (57)–(58), may also be used for DICOPT++. In that case, constraints (48)–(50) can be removed, since the binary variables appear linearly in equation (57). In the above formulation, μ_i will be the cost terms if $y_i=1$ while μ_i will be zero for $y_i=0$, if a sufficiently large value is used for U_i.

A proper selection of the U_i values will be the most critical step in this alternative formulation. From equations (56)–(57) it can be noted that

$$U_i>\min\{((C_i+C_i'\cdot P_i)\cdot N_{p,i}\cdot N_{s,i})\} \tag{59}$$

in order to obtain $\mu_i=0$ for $y_i=0$. The minimum is obtained at $x_i=0$. Thus we obtain the condition,

$$U_i>C_i\cdot N_{p,i,\min}\cdot N_{s,i,\min}. \tag{60}$$

Equations (56)–(58) are satisfied with all U_is satisfying equation (60). The numerical accuracy will, however, give an upper bound for the U_is to which the alternative formulation is still applicable.

Generally, only local optimality can be expected even in the case of the alternative formulation. However, it was found that it was possible to obtain global optimal result for all problems considered by using the above formulation with the ECP method. The results obtained with the methods with different values of the U_is are given in the following example.

AN EXAMPLE

In this section an example of the solution of the optimal pump configuration problem is given.

In the example we have specified the required flow (of water) through the pump or configuration of pumps at 350 m^3/h, while the required pressure rise over the pump or configuration of pumps should be 400 kPa. We have three different pumps with total head and power data given for water at 20°C and for a rotation speed of 2950 rpm. The maximum rotation speed for the pumps is 2950 rpm. The total head and power curves as given by the manufacturer can, with good accuracy, be approximated by the following models,

$$\left(\frac{\Delta p_{m,i}}{\text{kPa}}\right)=a+b\cdot\left(\frac{\dot V_{m,i}}{\text{m}^3/\text{h}}\right)+c\cdot\left(\frac{\dot V_{m,i}}{\text{m}^3/\text{h}}\right)^2 \tag{61}$$

$$\left(\frac{P_{m,i}}{\text{kW}}\right)=\alpha+\beta\cdot\left(\frac{\dot V_{m,i}}{\text{m}^3/\text{h}}\right)+\gamma\cdot\left(\frac{\dot V_{m,i}}{\text{m}^3/\text{h}}\right)^2. \tag{62}$$

The prices as well as the parameters for the total head and power curves for the pumps are given in Table 1.

Table 1. Parameters for the considered pumps

	Pump 1	Pump 2	Pump 3
Price (FIM)	38900	15300	20700
a	629.0	215.0	361.0
b	0.696	2.95	0.530
c	−0.0116	−0.115	−0.00946
α	19.9	1.21	6.52
β	0.161	0.0644	0.102
γ	−0.000561	−0.000564	−0.000232

By using an interest rate of 10% and an economic life of 10 years, the pump prices must be multiplied by a factor of 0.1627 in order to get the yearly instalment of the capital costs. The running costs are obtained from a running time of 6000 h/year and a price of electricity of 0.3 FIM/kWh.

Using these data we can solve optimal pump configurations for seven different cases. We can solve a single-level pump configuration problem for each pump separately. Furthermore, we can solve a two-level pump configuration problem by using P1 and P2, using P1 and P3 or P2 and P3 as well as a three-level pump configuration problem by using all pumps.

Global optimal results as well as optimal parameters for the different problems are given in Tables 2 and 3. The results obtained with both the ECP algorithm and the DICOPT++ software as well as results obtained with the min–min formulation are given in Tables 4–7. The results in Tables 4–6 for the ECP method and the DICOPT++ software correspond to the results obtained using the first problem formulation. When using the ECP algor-

Table 2. Global optimal results of the pump configuration problems

Problem type	Global optimum (FIM/year)	Number of local optima in (51)
P1	134.264	1
P2	170.580	1
P3	135.057	1
P1&P2	128.894	9
P1&P3	131.514	4
P2&P3	135.057	13
P1&P2&P3	128.894	37

Table 3. Global optimal parameters for the pump configuration problems

Problem	$N_{p,1}$	$N_{s,1}$	x_1	ω_1 (rpm)	$N_{p,2}$	$N_{s,2}$	x_2	ω_2 (rpm)	$N_{p,3}$	$N_{s,3}$	x_3	ω_3 (rpm)
P1	3	1	1	2594								
P2					12	2	1	2932				
P3									3	2	1	2574
P1&P2	2	1	0.914	2855	1	2	0.086	2950				
P1&P3	1	1	0.449	2836					2	2	0.551	2434
P2&P3					—	—	—	—	3	2	1	2574
P1&P2&P3	2	1	0.914	2855	1	2	0.086	2950	—	—	—	—

Table 4. Values of the objective function obtained with different algorithms for the pump configuration problems while using the first problem formulation. The asterisk indicates a global optimal result

Problem type	ECP algorithm (FIM/year)	DICOPT++ (FIM/year)	Min–min formulation (FIM/year)
P1	*	*	*
P2	*	*	*
P3	*	*	*
P1&P2	*	134.264	*
P1&P3	*	134.264	131.760
P2&P3	*	*	*
P1&P2&P3	131.514	135.057	131.760

ithm, the initial values of $N_{p,i}$ and $N_{s,i}$ have been selected at $N_{p,i,\min}$ and $N_{s,i,\min}$, respectively, and the initial values of the fractions of the total flow have been selected according to $x_i = 1/L$. Step lengths equal to $\Delta x_i^0 = 0.25$ (in the multi-level problems) and $\Delta N_{p,i}^0 = \Delta N_{s,i} = 1$ have been used in the numerical calculations. In each iteration new linearized constraints have been added for all non-linear constraints that are not satisfied at the actual solution. The number of iterations (number of MILP problems) for the seven problems were 4, 6, 6, 11, 13, 9 and 17, respectively, for the ECP method.

Table 5. Results of the pump configuration problems obtained with the ECP algorithm

Problem	$N_{p,1}$	$N_{s,1}$	x_1	ω_1 (rpm)	$N_{p,2}$	$N_{s,2}$	x_2	ω_2 (rpm)	$N_{p,3}$	$N_{s,3}$	x_3	ω_3 (rpm)
P1	*	*	*	*								
P2					*	*	*	*				
P3									*	*	*	*
P1&P2	*	*	*	*	*	*	*	*				
P1&P3	*	*	*	*					*	*	*	*
P2&P3					*	*	*	*	*	*	*	*
P1&P2&P3	1	1	0.449	2836	—	—	—	—	2	2	0.551	2434

Table 6. Results of the pump configuration problems obtained with DICOPT++

Problem	$N_{p,1}$	$N_{s,1}$	x_1	ω_1 (rpm)	$N_{p,2}$	$N_{s,2}$	x_2	ω_2 (rpm)	$N_{p,3}$	$N_{s,3}$	x_3	ω_3 (rpm)
P1	*	*	*	*								
P2					*	*	*	*				
P3									*	*	*	*
P1&P2	3	1	1	2594	—	—	—	—				
P1&P3	3	1	1	2594					—	—	—	—
P2&P3					*	*	*	*	*	*	*	*
P1&P2&P3	—	—	—	—	—	—	—	—	3	2	1	2574

Table 7. Results of the pump configuration problems obtained with the min–min formulation

Problem	$N_{p,1}$	$N_{s,1}$	x_1	ω_1 (rpm)	$N_{p,2}$	$N_{s,2}$	x_2	ω_2 (rpm)	$N_{p,3}$	$N_{s,3}$	x_3	ω_3 (rpm)
P1	*	*	*	*								
P2					*	*	*	*				
P3									*	*	*	*
P1&P2	*	*	*	*	*	*	*	*				
P1&P3	2	1	0.769	2694					1	2	0.231	2344
P2&P3					*	*	*	*	*	*	*	*
P1&P2&P3	2	1	0.769	2694	—	—	—	—	1	2	0.231	2344

Table 8. Values of the objective function obtained with the ECP method for the alternative problem formulation and for different values of λ. The asterisk indicates a global optimal result

λ	P1&P2 (FIM/year)	P1&P3 (FIM/year)	P2&P3 (FIM/year)	P1&P2&P3 (FIM/year)
0	*	*	138.884	135.451
0.5	*	*	138.888	135.451
1	*	*	138.858	135.429
10	*	131.767	*	*
100	*	131.769	*	*
500	*	*	*	*
1000	*	*	*	*
10000	*	134.284	*	*
100000	Infeasible	Infeasible	Infeasible	Infeasible

For the min–min formulation the ECP method has been used when solving the lower level MINLP problem. The results are given in Tables 4 and 7. In Table 8 results with the ECP algorithm are given when the alternative formulation (56)–(58) of the problem was used. The numerical values of the parameters, U_i, in equation (57) have been defined according to,

$$U_i = \lambda \cdot C_i \cdot N_{p,i,\min} \cdot N_{s,i,\min} \tag{63}$$

where λ has been given different values. In Tables 4–8 only results that differ from the global optimal results are indicated numerically. Global optimal results are indicated with an asterisk.

In Fig. 3, objective function (51) is illustrated for the three-level pump configuration optimization problem vs x_1 and x_2. The corners correspond to the single-level pump optimization problems, the borders correspond to the two-level optimization problems and the figure itself to the entire three-level pump optimization problem.

The obtained pump configurations as well as the fraction of the total flow through the levels and the rotation speed for the pumps as obtained by the ECP algorithm are given in Table 5. In Tables 6 and 7 the corresponding results obtained with DICOPT++ and with the min–min formulation are given. The lower-level optimization problem is, for the min–min formulation, solved with the ECP method.

From Tables 4–7 it can be found that all considered algorithms were able to solve the convex single-level problems while, as expected, the algorithms had more problems solving the non-convex multi-level problems by the first problem formulation. In Table 8 the results obtained with the ECP method are presented for the alternative problem formulation, when solving the non-convex multi-level problems. From Table 8 it can be found that it was possible to obtain the global optimal result, for all problems, by the ECP algorithm when using the alternative problem formulation. When solving the

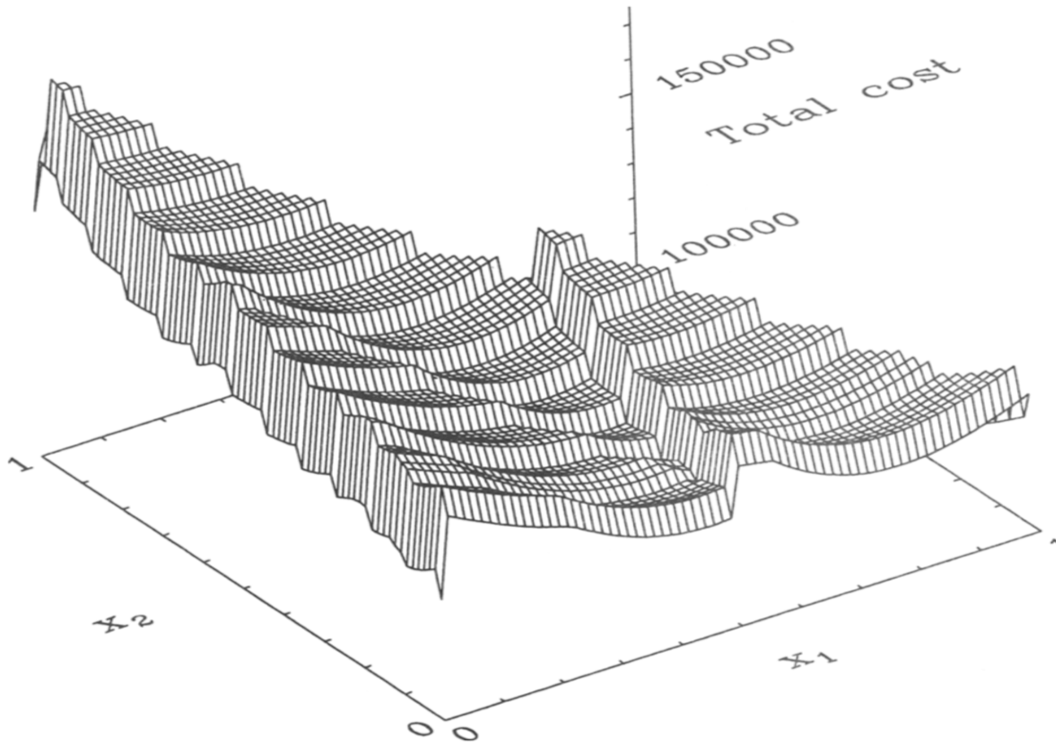

Fig. 3. Objective function (51) for the three-level pump optimization problem.

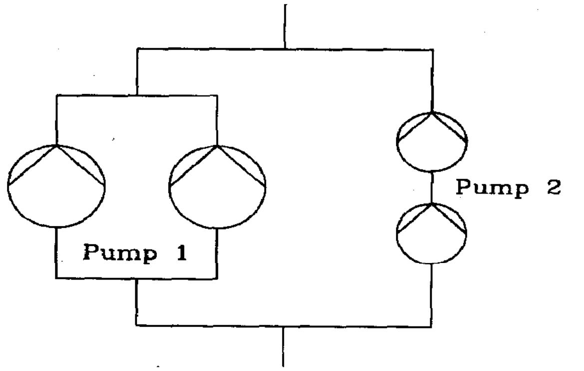

Fig. 4. Global optimal pump configuration for the desired pressure rise and flow.

pump configuration problem by using the alternative problem formulation, it was found that a proper selection of the U_i values was more critical for DICOPT++. The first problem (P1&P2), was efficiently (and correctly) solved by DICOPT++ for all λ values in Table 8. However, for the other problems some difficulties were encountered with infeasible solutions, not because of the OA/ER method but rather in conjunction with the nonlinear programming subproblems in GAMS/MINOS.

Finally, an illustration of the global optimal pump configuration for the desired pressure rise, 400 kPa, and the flow, 350 m^3/h, is given in Fig. 4.

FLEXIBILITY IN THE DESIGN

The above example gave us the optimal pump configuration for a prespecified pressure rise and flow. In most engineering design problems a degree of flexibility in the design is desired. The above solution may, thus, be called the point optimal solution to the design problem. An optimal flexible design solution may, simply, be found by further adding inequality restrictions to the design variables when formulating the MINLP problem. If a feasible solution exists, then this solution is the optimal flexible solution. Also, different types of flexibility indices may be used in the formulation of the problem. However, most optimal flexible design problems are very difficult to solve numerically, since the point optimal problem is usually a subproblem in the solution of the optimal flexible design problem. In order to illustrate flexibility in the design of optimal pump configurations, we have calculated the regions for different point optimal solutions to the actual problem, while changing the pressure rise and flow. The point optimal solution solved previously (for 400 kPa and 350 m^3/h) is represented by an asterisk in Fig. 5. In Fig. 5, each row, with numbers for different regions, represents the numbers of parallel pumps and pumps in series of each pump type. The dashed regions represent regions where more than one pump type is included in the optimal solution. In Westerlund and Pettersson (1993) optimal flexible pump configurations have been solved.

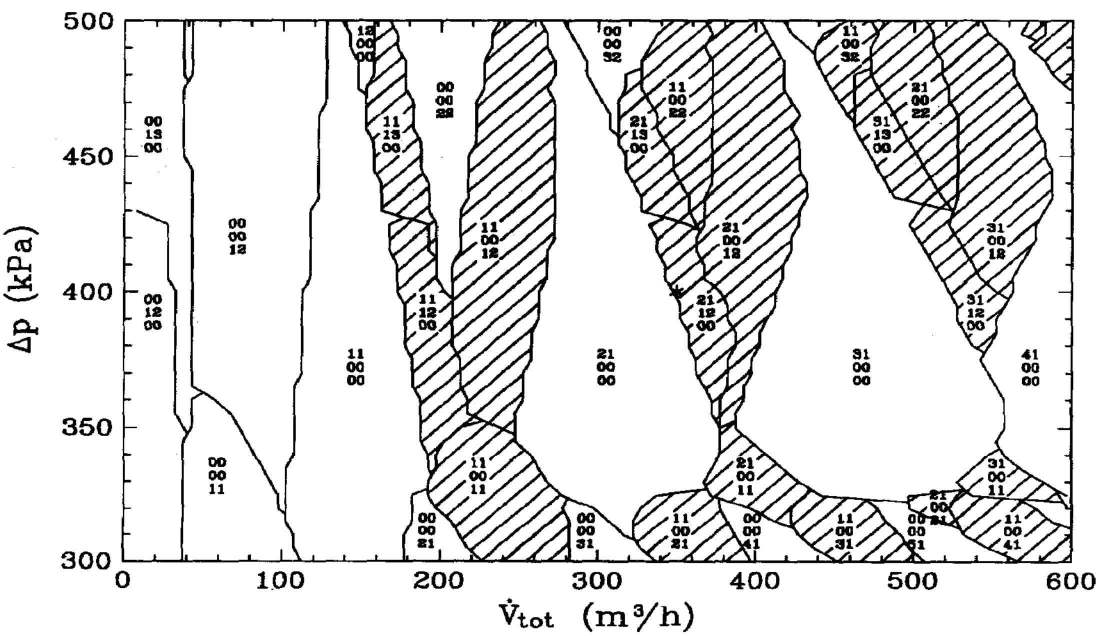

Fig. 5. Regions for different global optimal pump configurations.

DISCUSSION

Optimal pump configurations were solved as a MINLP problem in the present paper. The procedure can be used in order to select the optimal pump or the optimal configuration of pumps in series and/or parallel given the total flow and the required total pressure rise. For the numerical solution of the MINLP problem, an extended cutting plane (ECP) method was applied. Also, the DICOPT++ software solving mixed binary nonlinear programming problems was applied to the present MINLP problem. It was found that a single-level pump configuration optimization problem could be efficiently solved by the methods. For multi-level problems the objective function is generally non-convex and it was found that the methods did not always give global optimal results. By using an alternative problem formulation, the results could be significantly improved.

The multi-level pump configuration problem, as given in the paper, is non trivial and may, thus, also be used as a test example when developing new algorithms for non-convex MINLP problems.

REFERENCES

Achenie L. K. E. and L. T. Biegler, A superstructure-based approach to chemical reactor network synthesis. *Computers chem. Engng* **14**, 23–40 (1990).

Coulson J. M. and J. F. Richardsson, *Chemical Engineering,* Vol. 1. Pergamon Press, Oxford (1985).

Crowder H., E. L. Johnson and M. Padberg, Solving large-scale zero-one linear programming problems. *Opns Res.* **31**, 803–834 (1983).

Duran M. A. and I. E. Grossmann, An outer-approximation algorithm for a class of mixed-integer nonlinear programs. *Math. Prog.* **36**, 307–339 (1986).

Floudas C. A. and V. Visweswaran, A global optimization algorithm (GOP) for certain classes of nonconvex NLPs—I. Theory. *Computers chem. Engng* **14**, 1397–1417 (1990).

Floudas C. A., A. Aggarwal and A. R. Ciric, Global optimum search for nonconvex NLP and MINLP problems. *Computers chem. Engng* **13**, 1117–1132 (1989).

Geoffrion A. M., Generalized benders decomposition. *J. Optim. Theory Applic.* **10**, 237–260 (1972).

Kelley Jr J. E., The cutting-plane method for solving convex programs. *J. Soc. Indust. Appl. Math.* **8**, 703–712 (1960).

Kocis G. R. and I. E. Grossmann, Relaxation strategy for the structural optimization of process flow sheets. *Ind. Engng Chem. Res.* **26**, 1869–1880 (1987).

Kravanja Z. and I. E. Grossmann, PROSYN—a MINLP process synthesizer. *Computers chem. Engng* **14**, 1363–1373 (1990).

Nelder J. A. and R. Mead, A simplex method for function minimization. *Computer J.* **7**, 308–313 (1965).

Papoulias S. A. and I. E. Grossmann, A structural optimization approach in process synthesis. Heat recovery networks. *Computers chem. Engng* **7**, 707–721 (1983).

Van Roy T. J. and L. A. Wolsey, Solving mixed 0–1 programs by automatic reformulation. *Opns Res.* **35**, 45–57 (1987).

Viswanathan J. and I. E. Grossmann, A combined penalty function and outer-approximation method for MINLP optimization. *Computers chem. Engng* **14**, 769–782 (1990a).

Viswanathan J. and I. E. Grossmann, DICOPT++: a program for mixed integer non-linear optimization. *Users Manual.* Carnegie Mellon University (1990b).

Westerlund T. and F. Pettersson, An extended cutting plane (ECP) method for the solution of MINLP problems. Report 92-124-A, Process Design Laboratory, Åbo Akademi University (1992).

Westerlund T. and F. Pettersson, Solving optimal flexible pump configuration design problems by MINLP. *Proceedings of the Twelfth IASTED MIC Conference,* pp. 172–174. Acta Press (1993).

Yuan X., L. Pibouleau and S. Domenech, Experiments in process synthesis via mixed-integer programming. *Chem. Engng Process* **25**, 99–116 (1989).

APPENDIX A

Convexity Analysis

The single-level problem

Consider the relaxed one-level ($L=1$) problem (P_1)

$$\min\{(C+C'\cdot P)\cdot N_p\cdot N_s\} \tag{A1}$$

subject to

$$\dot V \geq \frac{\dot V_{\mathrm{tot}}}{N_p} \tag{A2}$$

$$\Delta p \geq \frac{\Delta p_{\mathrm{tot}}}{N_s} \tag{A3}$$

$$\dot V \geq \left(\frac{\omega}{\omega_m}\right)\cdot \dot V_m \tag{A4}$$

$$P \geq \left(\frac{\omega}{\omega_m}\right)^3\cdot\left(\frac{\rho}{\rho_m}\right)\cdot P_m \tag{A5}$$

$$\Delta p \geq \left(\frac{\omega}{\omega_m}\right)^2\cdot\left(\frac{\rho}{\rho_m}\right)\cdot \Delta p_m \tag{A6}$$

$$\omega \leq \omega_{\max} \tag{A7}$$

$$P_m \geq g_1(\dot V_m) \tag{A8}$$

$$\Delta p_m \geq g_2(\dot V_m) \tag{A9}$$

where N_p and N_s are positive integers and $\dot V$, Δp, P, P_m, Δp_m, $\dot V_m$ and ω positive real variables and C, C', $\dot V_{\mathrm{tot}}$, Δp_{tot}, ω_m, ρ_m, ρ_m, ρ, $\omega_{\max}$ given parameters and $g_1(\cdot\, V_m)$ as well as $g_2(\cdot\, V_m)$ given functions.

The problem, (P_1), can be convexified by replacing $1/P_m$ and $1/\Delta p_m$ with two new variables with the corresponding new functions,

$$\frac{1}{P_m} \equiv \mathcal{P}_m = \frac{1}{g_1(\dot V_m)} \tag{A10}$$

$$\frac{1}{\Delta p_m} \equiv D_m = \frac{1}{g_2(\dot V_m)} \tag{A11}$$

and letting any variable X be replaced by $X=e^x$ (for ω we have used the symbol $\omega = e^{\omega}$). The problem, (P_1), can now be rewritten in the convexified form,

$$\min\{(C+C'\cdot e^{p})\cdot e^{n_p}\cdot e^{n_s}\} \tag{A12}$$

subject to

$$\dot v \geq \ln(\dot V_{\mathrm{tot}}) - n_p \tag{A13}$$

$$\delta p \geq \ln(\Delta p_{\mathrm{tot}}) - n_s \tag{A14}$$

$$\dot v \geq \omega - \ln(\omega_m) + \dot v_m \tag{A15}$$

$$p \geq 3 \cdot w + \ln\left(\frac{\rho}{\omega_m^3 \cdot \rho_m}\right) - r_m \tag{A16}$$

$$\delta p \geq 2 \cdot w + \ln\left(\frac{\rho}{\omega_m^2 \cdot \rho_m}\right) - d_m \tag{A17}$$

$$w \leq \ln(\omega_{max}) \tag{A18}$$

$$-r_m - \ln(g_1(\dot{v}_m)) \leq 0 \tag{A19}$$

$$-d_m - \ln(g_2(\dot{v}_m)) \leq 0. \tag{A20}$$

Equation (A12) is now cleary a convex function and the inequalities, (A13)–(A18), linear constraints. The inequalities, (A19) and (A20), are cleary convex constraints if the functions, $g_i(\dot{V}_m)$, are positive functions that satisfy the general condition,

$$g_i''(\dot{V}_m) \cdot \dot{V}_m + g_i'(\dot{V}_m) \cdot \left(1 - \frac{g'(\dot{V}_m)}{g(\dot{V}_m)} \cdot \dot{V}_m\right) \leq 0. \tag{A21}$$

The power curve, $P_m = g_1(\dot{V}_m)$, is generally a positive, increasing–decreasing, concave function. Thus the condition, (A21), can be applied to the model, in order to ensure convexity. Using an approximative model of the type,

$$P_m = \alpha + \beta \cdot \dot{V}_m + \gamma \cdot \dot{V}_m^2 \tag{A22}$$

we have the conditions $\alpha > 0, \beta > 0$ and $\gamma < 0$. Furthermore, when applying the inequality, (A21), we obtain a lower limit for $\dot{V}_m$, in order to ensure the convexity of the problem. The lower limit is given by,

$$\dot{V}_m \geq -\frac{2 \cdot \alpha}{\beta} + \sqrt{\left(\frac{2 \cdot \alpha}{\beta}\right)^2 - \frac{\alpha}{\gamma}} \tag{A23}$$

and thus the lower limit of the flow through the pump (in order to ensure convexity) is given by,

$$\dot{V}_{min} = \left(\frac{\omega}{\omega_m}\right) \cdot \left(-\frac{2 \cdot \alpha}{\beta} + \sqrt{\left(\frac{2 \cdot \alpha}{\beta}\right)^2 - \frac{\alpha}{\gamma}}\right). \tag{A24}$$

For the data given in Table 1, we obtain $\dot{V}_{min} = (\omega/\omega_m) \cdot \dot{V}_{m,min}$, where $\dot{V}_{m,min} = 63.6$, 22.1 and 83.0 m^3/h, respectively, for the pumps in the example.

The total head curve, $\Delta p_m = g_1(\dot{V}_m)$, is generally a positive, decreasing, concave function. Thus, condition (A21) will generally hold. Using an approximative model of the type,

$$\Delta p_m = a + b \cdot \dot{V}_m^c \tag{A25}$$

we obtain the general conditions, $a > 0$, $b < 0$ and $c > 1$. A model of the type (A22) may also be used for approximating the total head curve.

The multi-level problem

For a multi-level problem, (P_L), each variable will obtain an index i corresponding to the level, and the objective function will be replaced with the sum of objective functions for each level respectively. Furthermore, the inequality, (A2), will be replaced by,

$$\dot{V}_i \geq \frac{\dot{V}_{tot}}{N_p} \cdot X_i \tag{A26}$$

where X_i is the fraction of the total flow through the corresponding level. Furthermore, the sum of all fractions of the total flow, X_i, must be equal to one. The corresponding relaxed inequality is thus given by,

$$\sum_{i=1}^{L} X_i \geq 1. \tag{A27}$$

Inequality (A26) can in the convexified form be replaced by the linear inequality,

$$\dot{v} \geq \ln(\dot{V}_{tot}) - n_p + x_i \tag{A28}$$

while inequality (A27) can in the convexified form be replaced by

$$-\left(\sum_{i=1}^{L} e^{X_i}\right) + 1 \leq 0. \tag{A29}$$

The only significant diffrence between the single-level and the multi-level problem is inequality (A29). Since exponential terms are non-convex due to the negative sign, constraint (A29) is non-convex and so is the multi-level MINLP problem.

Comments on mixed zero–one non-linear programming

If a mixed zero–one non-linear programming algorithm such as that of DICOPT++ is used, the following equalities can further be added to represent the variables, $N_{p,i}$ and $N_{s,i}$, in terms of binary variables,

$$N_{p,i} = \sum_{k=1}^{K} 2^{k-1} \cdot y_{i,p,k} \tag{A30}$$

$$N_{s,i} = \sum_{k=1}^{K} s^{k-1} \cdot y_{i,s,k} \tag{A31}$$

In equations (A30) and (A31), $y_{i,p,k}$ and $y_{i,s,k}$ are binary variables.

An alternative representation of (A30)–(A31), which may also be written in linear form in the convexified problem, is given by,

$$N_{p,i} = \sum_{k=1}^{K} k \cdot y_{i,p,k} \tag{A32}$$

$$N_{s,i} = \sum_{k=1}^{K} k \cdot y_{i,s,k}. \tag{A33}$$

In this case the equalities (A32) and (A33) must be combined with the following equalities,

$$\sum_{k=1}^{K} y_{i,p,k} = 1 \tag{A34}$$

$$\sum_{k=1}^{k} y_{i,s,k} = 1. \tag{A35}$$

Equalities (A32) and (A33) can now be written as follows for the convexified problem,

$$n_{p,i} = \sum_{k=1}^{K} \ln(k) \cdot y_{i,p,k} \tag{A36}$$

$$n_{s,i} = \sum_{k=1}^{K} \ln(k) \cdot y_{i,x,j}. \tag{A37}$$

Equalities (A36) and (A37) can then be combined with equalities (A34) and (A35).

APPENDIX B

The Extended Cutting Plane (ECP) Algorithm

The main structure of the MINLP problem for the ECP algorithm is given by,

$$\min_{x,y} \{\mathbf{c}_1^T \mathbf{x} + \mathbf{c}_2^T \mathbf{y}\} \tag{B1}$$

subject to

$$\mathbf{Ax}+\mathbf{By}+\mathbf{d}\leq\mathbf{0} \quad (B2)$$

$$\mathbf{Ex}+\mathbf{Fy}+\mathbf{h}=\mathbf{0} \quad (B3)$$

$$\mathbf{g}(\mathbf{x},\mathbf{y})\leq\mathbf{0} \quad (B4)$$

$\mathbf{x}_{min}\leq\mathbf{x}\leq\mathbf{x}_{max}$ and $\mathbf{y}_{min}\leq\mathbf{y}\leq\mathbf{y}_{max}$ where $\mathbf{x}\in R^n$ and $y_i\in Z^+$.

The extended cutting plane (ECP) algorithm is given by four steps as follows,

Step A (initialization)

Let $k=0$

Given step lengths, $\Delta\mathbf{x}^k$, $\Delta\mathbf{y}^k$, and termination criteria, ε_1 and ε_2, an initial point,

$$\mathbf{x}^0, \mathbf{y}^0$$

must be given. The procedure is thereafter initialized by adding $m_0=\dim(g)$ linearized constraints to the set of linear inequality constraints according to,

$$l_j(\mathbf{x},\mathbf{y})=L_i^0(\mathbf{x},\mathbf{y}) \quad (B5)$$

where $j=1,\ldots,m_0$, $j=i$ and

$$L_i^k(\mathbf{x},\mathbf{y})=g_i(\mathbf{x}^k,\mathbf{y}^k)+\left(\frac{\partial g_i}{\partial\mathbf{x}}\right)_{\mathbf{x}^k,\mathbf{y}^k}(\mathbf{x}-\mathbf{x}^k)+\left(\frac{\partial g_i}{\partial\mathbf{y}}\right)_{\mathbf{x}^k,\mathbf{y}^k}(\mathbf{y}-\mathbf{y}^k). \quad (B6)$$

Step B (solve the kth MILP problem)

Let $k=k+1$ and solve,

$$\min_{\mathbf{x}^k,\mathbf{y}^k}\{\mathbf{c}_1^T\mathbf{x}^k+\mathbf{c}_2^T\mathbf{y}^k\} \quad (B7)$$

subject to

$$\mathbf{Ax}^k+\mathbf{By}^k+\mathbf{d}\leq\mathbf{0} \quad (B8)$$

$$\mathbf{Ex}^k+\mathbf{Fy}^k+\mathbf{h}=\mathbf{0} \quad (B9)$$

$$l_j(\mathbf{x}^k,\mathbf{y}^k)\leq 0;\, j=1,\ldots,m_{k-1} \quad (B10)$$

$$\max\{\mathbf{x}^{k-1}+\Delta\mathbf{x}^k,\mathbf{x}_{min}\}\leq\mathbf{x}^k\leq\min\{\mathbf{x}^{k-1}+\Delta\mathbf{x}^k,\mathbf{x}_{max}\} \quad (B11)$$

$$\max\{\mathbf{y}^{k-1}-\Delta\mathbf{y}^k,\mathbf{y}_{min}\}\leq\mathbf{y}^k\leq\min\{\mathbf{y}^{k-1}+\Delta\mathbf{y}^k,\mathbf{y}_{max}\}. \quad (B12)$$

In this specific application we have used $\Delta\mathbf{x}^k=\Delta\mathbf{x}^0$ and $\Delta\mathbf{y}^k=\Delta\mathbf{y}^0$ [except when solving the final MILP problem, where $\Delta\mathbf{x}^k$ and $\Delta\mathbf{y}^k$ are defined by (B18) and (B19)].

Step C (test for termination)

If the following criteria are not satisfied,

$$(\mathbf{x}^k-\mathbf{x}^{k-1})^T(\mathbf{x}^k-\mathbf{x}^{k-1})\leq\varepsilon_1 \quad (B13)$$

$$\mathbf{y}^k-\mathbf{y}^{k-1}=0 \quad (B14)$$

$$g_i(\mathbf{x}^k,\mathbf{y}^k)\leq\varepsilon_2 \quad (B15)$$

then the procedure continues at step D. Otherwise if

$$\Delta\mathbf{x}^k<\mathbf{x}_{max}-\mathbf{x}_{min} \quad (B16)$$

$$\Delta\mathbf{y}^k<\mathbf{y}_{max}-\mathbf{y}_{min} \quad (B17)$$

solve a final MILP problem (B7)–(B12) with

$$\Delta\mathbf{x}^k=\mathbf{x}_{max}-\mathbf{x}_{min} \quad (B18)$$

$$\Delta\mathbf{y}^k=\mathbf{y}_{max}-\mathbf{y}_{min}. \quad (B19)$$

If the solution satisfies (B13)–(B15), the procedure is terminated, $\mathbf{x}^*=\mathbf{x}^k$ and $\mathbf{y}^*=\mathbf{y}^k$, otherwise the procedure continues with $\Delta\mathbf{x}^k=\Delta x^0$ and $\Delta\mathbf{y}^k=\Delta\mathbf{y}^0$ at step D.

Step D (set up new linearized constraints)

Remove all linearized constraints that satisfy,

$$l_j(\mathbf{x}^k,\mathbf{y}^k)>g_{i(j)}(\mathbf{x}^k,\mathbf{y}^k) \quad (B20)$$

for $j=1,\ldots,m_{k-1}$.

Replace linearized constraints that satisfy the condition

$$l_j(\mathbf{x}^k,\mathbf{y}^k)=g_{i(j)}(\mathbf{x}^k,\mathbf{y}^k)=0 \quad (B21)$$

If any of the non-linear constraints satisfy,

$$g_i(\mathbf{x}^k,\mathbf{y}^k)>0. \quad (B22)$$

Then *add* new linearized constraints, corresponding to the non-linear constraints satisfying (B22). Also (as in the standard cutting plane method) only one new linearized constraint corresponding to the non-linear constraint that achieved the maximal value can be added. In this case,

$$l_{m_k}(\mathbf{x}^k,\mathbf{y}^k)=L_a^k(\mathbf{x}^k,\mathbf{y}^k) \quad (B23)$$

$$g_a(x^k,\mathbf{y}^k)=\max\{g_i(\mathbf{x}^k,\mathbf{y}^k)\}. \quad (B24)$$

Then $m_k=m_{k-1}-m_r+m_a$ where m_r is the number of linear constraints that was removed by (B20) and m_a is the number of linear constraints that was added because of (B22).

Then go to step B.

Convergence Properties of the ECP Method for Convex Problems

For convex functions (B4), the algorithm can be slightly simplified. In this case inequality (B20) never holds. Also, both $\Delta\mathbf{x}^k$ and Δy^k can be given the values $\mathbf{x}_{max}-\mathbf{x}_{min}$ and $\mathbf{y}_{max}-\mathbf{y}_{min}$, respectively, during the iteration phase.

Let then the feasible set, $\{\mathbf{x},\mathbf{y}\}$, for the convex problem, (B1)–(B4), be Ω and the feasible set, $\{\mathbf{x}^k,\mathbf{y}^k\}$, for the kth problem, (B7)–(B12), be Ω^k.

Because (B4) is assumed to be convex the resulting tangent hyperplanes, (B10), will result in $\Omega\subset\Omega^k$. As long as the solutions $(\mathbf{x}^k,\mathbf{y}^k)\notin\Omega$ the set $\Omega^k\subset\Omega^{k-1}$. Because of the increasing number of tangent hyperplanes during the iteration procedure we thus obtain,

$$\Omega\subset\ldots\Omega^k\subset\Omega^{k-1}\subset\Omega^{k-2}\ldots\subset\Omega^0. \quad (B25)$$

If J^* represents the optimal solution in Ω it follows from (B25) that,

$$J^*\geq\ldots J^k\geq J^{k-1}\geq J^{k-2}\ldots\geq J^0. \quad (B26)$$

Thus the procedure will converge monotonically to the optimal solution which is the global optimal solution for a convex problem. The procedure is terminated when the solution $(\mathbf{x}^k,\mathbf{y}^k)\in\Omega$ with a predifined accuracy.

APPENDIX C

Explicit Formulation for a Three Level Pump Configuration Problem

$$\min_{N_{p,i},N_{s,i},x_i,\omega_i,i=1,\ldots,3}\left\{\sum_{i=1}^{3}(C_i+C_i'\cdot P_i)\cdot N_{p,i}\cdot N_{s,i}\right\} \quad (C1)$$

subject to

$$\frac{\Delta p_1}{\text{kPa}}=629.0\left(\frac{\omega_1}{2950}\right)^2+0.696\left(\frac{\omega_1}{2950}\right)\dot{V}_1-0.0116\dot{V}_1^2 \quad (C2)$$

$$\frac{\Delta p_2}{\text{kPa}}=215.0\left(\frac{\omega_2}{2950}\right)^2+2.95\left(\frac{\omega_2}{2950}\right)\dot{V}_2-0.115\dot{V}_2^2 \quad (C3)$$

$$\frac{\Delta p_3}{\text{kPa}}=361.0\left(\frac{\omega_3}{2950}\right)^2+0.530\left(\frac{\omega_3}{2950}\right)\dot{V}_3-0.00946\dot{V}_3^2 \quad (C4)$$

$$\sum_{i=1}^{3}x_i=1 \quad (C5)$$

$$\omega_i - 2950\,\text{rpm} \leq 0 \tag{C6}$$

where

$$\frac{P_1}{\text{kW}} = \left(\frac{\omega_1}{2950}\right)\left(19.9\left(\frac{\omega_1}{2950}\right)^2 + 0.161\left(\frac{\omega_1}{2950}\right)\dot{V}_1 - 0.000561\dot{V}_1^2\right) \tag{C7}$$

$$\frac{P_2}{\text{kW}} = \left(\frac{\omega_2}{2950}\right)\left(1.21\left(\frac{\omega_2}{2950}\right)^2 + 0.0644\left(\frac{\omega_2}{2950}\right)\dot{V}_2 - 0.000564\dot{V}_2^2\right) \tag{C8}$$

$$\frac{P_3}{\text{kW}} = \left(\frac{\omega_3}{2950}\right)\left(6.52\left(\frac{\omega_3}{2950}\right)^2 + 0.102\left(\frac{\omega_3}{2950}\right)\dot{V}_3 - 0.000232\dot{V}_3^2\right) \tag{C9}$$

$$\dot{V}_i = \frac{x_i}{N_{p,i}} \cdot \dot{V}_{\text{tot}} \tag{C10}$$

$$\Delta p_i = \frac{1}{N_{s,i}} \cdot\cdot\, \Delta p_{\text{tot}}. \tag{C11}$$

With the following values on respective constant, $C_i' = 0.30 \cdot 6000$ FIM/kW//year, $C_1 = 0.1627 \cdot 38900$ FIM/year, $C_2 = 0.1627 \cdot 15300$ FIM/year, $C_3 = 0.1627 \cdot 20700$ FIM/year, $\dot{V}_{\text{tot}} = 350\ \text{m}^3/\text{h}$, $\Delta p_{\text{tot}} = 400$ kPa.