14.3 Partial Derivatives

1. (a) $\partial T/\partial x$ represents the rate of change of T when we fix y and t and consider T as a function of the single variable x, which describes how quickly the temperature changes when longitude changes but latitude and time are constant. $\partial T/\partial y$ represents the rate of change of T when we fix x and t and consider T as a function of y, which describes how quickly the temperature changes when latitude changes but longitude and time are constant. $\partial T/\partial t$ represents the rate of change of T when we fix x and y and consider T as a function of t, which describes how quickly the temperature changes over time for a constant longitude and latitude.

(b) $f_x(158, 21, 9)$ represents the rate of change of temperature at longitude 158°W, latitude 21°N at 9:00 AM when only longitude varies. Since the air is warmer to the west than to the east, increasing longitude results in an increased air temperature, so we would expect $f_x(158, 21, 9)$ to be positive. $f_y(158, 21, 9)$ represents the rate of change of temperature at the same time and location when only latitude varies. Since the air is warmer to the south and cooler to the north, increasing latitude results in a decreased air temperature, so we would expect $f_y(158, 21, 9)$ to be negative. $f_t(158, 21, 9)$ represents the rate of change of temperature at the same time and location when only time varies. Since typically air temperature increases from the morning to the afternoon as the sun warms it, we would expect $f_t(158, 21, 9)$ to be positive.

2. By Definition 4, $f_T(34, 75) = \lim\limits_{h\to 0} \dfrac{f(34+h, 75) - f(34, 75)}{h}$, which we can approximate by considering $h = 2$ and $h = -2$ and using the values given in Table 1: $f_T(34, 75) \approx \dfrac{f(36, 75) - f(34, 75)}{2} = \dfrac{54 - 51}{2} = 1.5$,

$f_T(34, 75) \approx \dfrac{f(32, 75) - f(34, 75)}{-2} = \dfrac{46 - 51}{-2} = 2.5$. Averaging these values, we estimate $f_T(34, 75)$ to be approximately 2. Thus, when the actual temperature is 34°C and the relative humidity is 75%, the apparent temperature rises by about 2°C for every degree that the actual temperature rises.

Similarly, $f_H(34, 75) = \lim\limits_{h\to 0} \dfrac{f(34, 75+h) - f(34, 75)}{h}$ which we can approximate by considering $h = 5$ and $h = -5$:

$f_H(34, 75) \approx \dfrac{f(34, 80) - f(34, 75)}{5} = \dfrac{52 - 51}{5} = 0.2$, $f_H(34, 75) \approx \dfrac{f(34, 70) - f(34, 75)}{-5} = \dfrac{49 - 51}{-5} = 0.4$.

Averaging these values, we estimate $f_H(34, 75)$ to be approximately 0.3. Thus, when the actual temperature is 34°C and the relative humidity is 75%, the apparent temperature rises by about 0.3°C for every percent that the relative humidity increases.

3. (a) By Definition 4, $f_T(-15, 30) = \lim\limits_{h\to 0} \dfrac{f(-15+h, 30) - f(-15, 30)}{h}$, which we can approximate by considering $h = 5$ and $h = -5$ and using the values given in the table:

$f_T(-15, 30) \approx \dfrac{f(-10, 30) - f(-15, 30)}{5} = \dfrac{-20 - (-26)}{5} = \dfrac{6}{5} = 1.2$,

$f_T(-15, 30) \approx \dfrac{f(-20, 30) - f(-15, 30)}{-5} = \dfrac{-33 - (-26)}{-5} = \dfrac{-7}{-5} = 1.4$. Averaging these values, we estimate

$f_T(-15, 30)$ to be approximately 1.3. Thus, when the actual temperature is $-15°$C and the wind speed is 30 km/h, the apparent temperature rises by about 1.3°C for every degree that the actual temperature rises.

Similarly, $f_v(-15, 30) = \lim\limits_{h \to 0} \dfrac{f(-15, 30+h) - f(-15, 30)}{h}$ which we can approximate by considering $h = 10$ and $h = -10$: $f_v(-15, 30) \approx \dfrac{f(-15, 40) - f(-15, 30)}{10} = \dfrac{-27 - (-26)}{10} = \dfrac{-1}{10} = -0.1$,

$f_v(-15, 30) \approx \dfrac{f(-15, 20) - f(-15, 30)}{-10} = \dfrac{-24 - (-26)}{-10} = \dfrac{2}{-10} = -0.2$. Averaging these values, we estimate $f_v(-15, 30)$ to be approximately -0.15. Thus, when the actual temperature is $-15°$C and the wind speed is 30 km/h, the apparent temperature decreases by about 0.15°C for every km/h that the wind speed increases.

(b) For a fixed wind speed v, the values of the wind-chill index W increase as temperature T increases (look at a column of the table), so $\dfrac{\partial W}{\partial T}$ is positive. For a fixed temperature T, the values of W decrease (or remain constant) as v increases (look at a row of the table), so $\dfrac{\partial W}{\partial v}$ is negative (or perhaps 0).

(c) For fixed values of T, the function values $f(T, v)$ appear to become constant (or nearly constant) as v increases, so the corresponding rate of change is 0 or near 0 as v increases. This suggests that $\lim\limits_{v \to \infty} (\partial W/\partial v) = 0$.

4. (a) $\partial h/\partial v$ represents the rate of change of h when we fix t and consider h as a function of v, which describes how quickly the wave heights change when the wind speed changes for a fixed time duration. $\partial h/\partial t$ represents the rate of change of h when we fix v and consider h as a function of t, which describes how quickly the wave heights change when the duration of time changes, but the wind speed is constant.

(b) By Definition 4, $f_v(80, 15) = \lim\limits_{h \to 0} \dfrac{f(80+h, 15) - f(80, 15)}{h}$ which we can approximate by considering $h = 20$ and $h = -20$ and using the values given in the table:

$f_v(80, 15) \approx \dfrac{f(100, 15) - f(80, 15)}{20} = \dfrac{11.0 - 7.7}{20} = 0.165$,

$f_v(80, 15) \approx \dfrac{f(60, 15) - f(80, 15)}{-20} = \dfrac{4.9 - 7.7}{-20} = 0.14$. Averaging these values, we have $f_v(80, 15) \approx 0.1525$. Thus, when an 80-km/h wind has been blowing for 15 hours, the wave heights should increase by about 0.15 meter for every km/h that the wind speed increases (with the same time duration). Similarly,

$f_t(80, 15) = \lim\limits_{h \to 0} \dfrac{f(80, 15+h) - f(80, 15)}{h}$ which we can approximate by considering $h = 5$ and $h = -5$:

$f_t(80, 15) \approx \dfrac{f(80, 20) - f(80, 15)}{5} = \dfrac{8.6 - 7.7}{5} = 0.18$, $f_t(80, 15) \approx \dfrac{f(80, 10) - f(80, 15)}{-5} = \dfrac{6.4 - 7.7}{-5} = 0.26$.

Averaging these values, we have $f_t(80, 15) \approx 0.22$. Thus, when an 80-km/h wind has been blowing for 15 hours, the wave heights increase by about 0.22 meter for every additional hour that the wind blows.

(c) For fixed values of v, the function values $f(v, t)$ appear to increase in smaller and smaller increments, becoming nearly constant as t increases. Thus, the corresponding rate of change is nearly 0 as t increases, suggesting that $\lim\limits_{t \to \infty} (\partial h/\partial t) = 0$.

5. (a) If we start at $(1,2)$ and move in the positive x-direction, the graph of f increases. Thus $f_x(1,2)$ is positive.

(b) If we start at $(1,2)$ and move in the positive y-direction, the graph of f decreases. Thus $f_y(1,2)$ is negative.

6. (a) The graph of f decreases if we start at $(-1,2)$ and move in the positive x-direction, so $f_x(-1,2)$ is negative.

(b) The graph of f decreases if we start at $(-1,2)$ and move in the positive y-direction, so $f_y(-1,2)$ is negative.

7. (a) $f_{xx} = \frac{\partial}{\partial x}(f_x)$, so f_{xx} is the rate of change of f_x in the x-direction. f_x is negative at $(-1,2)$ and if we move in the positive x-direction, the surface becomes less steep. Thus the values of f_x are increasing and $f_{xx}(-1,2)$ is positive.

(b) f_{yy} is the rate of change of f_y in the y-direction. f_y is negative at $(-1,2)$ and if we move in the positive y-direction, the surface becomes steeper. Thus the values of f_y are decreasing, and $f_{yy}(-1,2)$ is negative.

8. (a) $f_{xy} = \frac{\partial}{\partial y}(f_x)$, so f_{xy} is the rate of change of f_x in the y-direction. f_x is positive at $(1,2)$ and if we move in the positive y-direction, the surface becomes steeper, looking in the positive x-direction. Thus the values of f_x are increasing and $f_{xy}(1,2)$ is positive.

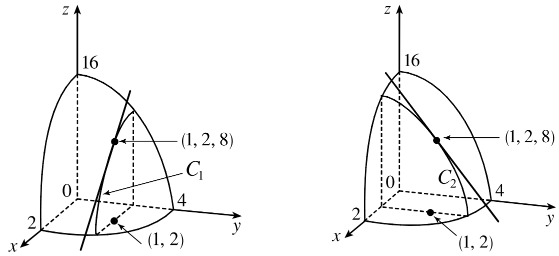
(b) f_x is negative at $(-1,2)$ and if we move in the positive y-direction, the surface gets steeper (with negative slope), looking in the positive x-direction. This means that the values of f_x are decreasing as y increases, so $f_{xy}(-1,2)$ is negative.

9. First of all, if we start at the point $(3,-3)$ and move in the positive y-direction, we see that both b and c decrease, while a increases. Both b and c have a low point at about $(3,-1.5)$, while a is 0 at this point. So a is definitely the graph of f_y, and one of b and c is the graph of f. To see which is which, we start at the point $(-3,-1.5)$ and move in the positive x-direction. b traces out a line with negative slope, while c traces out a parabola opening downward. This tells us that b is the x-derivative of c. So c is the graph of f, b is the graph of f_x, and a is the graph of f_y.

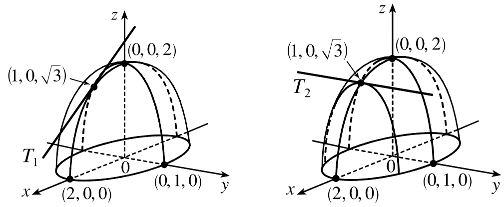
10. $f_x(2,1)$ is the rate of change of f at $(2,1)$ in the x-direction. If we start at $(2,1)$, where $f(2,1)=10$, and move in the positive x-direction, we reach the next contour line [where $f(x,y)=12$] after approximately 0.6 units. This represents an average rate of change of about $\frac{2}{0.6}$. If we approach the point $(2,1)$ from the left (moving in the positive x-direction) the output values increase from 8 to 10 with an increase in x of approximately 0.9 units, corresponding to an average rate of change of $\frac{2}{0.9}$. A good estimate for $f_x(2,1)$ would be the average of these two, so $f_x(2,1) \approx 2.8$. Similarly, $f_y(2,1)$ is the rate of change of f at $(2,1)$ in the y-direction. If we approach $(2,1)$ from below, the output values decrease from 12 to 10 with a change in y of approximately 1 unit, corresponding to an average rate of change of -2. If we start at $(2,1)$ and move in the positive y-direction, the output values decrease from 10 to 8 after approximately 0.9 units, a rate of change of $\frac{-2}{0.9}$. Averaging these two results, we estimate $f_y(2,1) \approx -2.1$.

11. $f(x,y) = 16 - 4x^2 - y^2 \quad\Rightarrow\quad f_x(x,y) = -8x$ and $f_y(x,y) = -2y \quad\Rightarrow\quad f_x(1,2) = -8$ and $f_y(1,2) = -4$. The graph of f is the paraboloid $z = 16 - 4x^2 - y^2$ and the vertical plane $y=2$ intersects it in the parabola $z = 12 - 4x^2$, $y = 2$

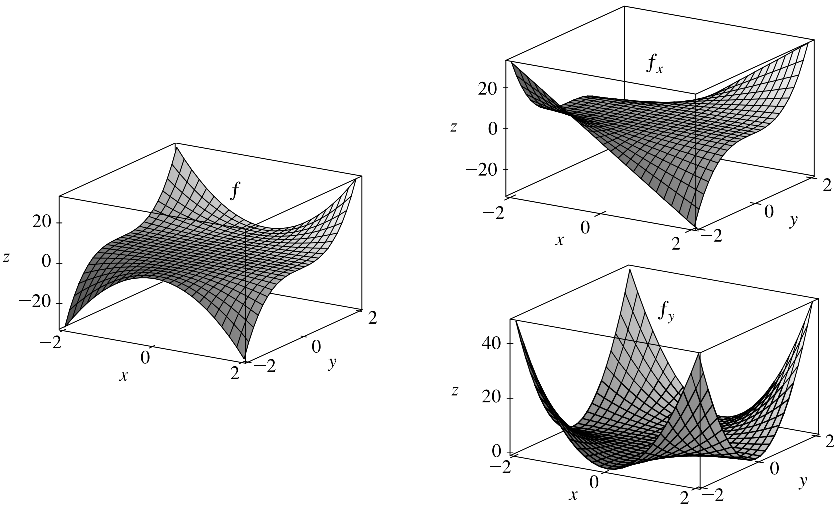
(the curve C_1 in the first figure). The slope of the tangent line to this parabola at $(1, 2, 8)$ is $f_x(1, 2) = -8$. Similarly the plane $x = 1$ intersects the paraboloid in the parabola $z = 12 - y^2$, $x = 1$ (the curve C_2 in the second figure) and the slope of the tangent line at $(1, 2, 8)$ is $f_y(1, 2) = -4$.



12. $f(x, y) = (4 - x^2 - 4y^2)^{1/2} \quad \Rightarrow \quad f_x(x, y) = -x(4 - x^2 - 4y^2)^{-1/2}$ and $f_y(x, y) = -4y(4 - x^2 - 4y^2)^{-1/2} \quad \Rightarrow$ $f_x(1, 0) = -\frac{1}{\sqrt{3}}$, $f_y(1, 0) = 0$. The graph of f is the upper half of the ellipsoid $z^2 + x^2 + 4y^2 = 4$ and the plane $y = 0$ intersects the graph in the semicircle $x^2 + z^2 = 4$, $z \geq 0$ and the slope of the tangent line T_1 to this semicircle at $\left(1, 0, \sqrt{3}\right)$ is $f_x(1, 0) = -\frac{1}{\sqrt{3}}$. Similarly the plane $x = 1$ intersects the graph in the semi-ellipse $z^2 + 4y^2 = 3$, $z \geq 0$ and the slope of the tangent line T_2 to this semi-ellipse at $\left(1, 0, \sqrt{3}\right)$ is $f_y(1, 0) = 0$.



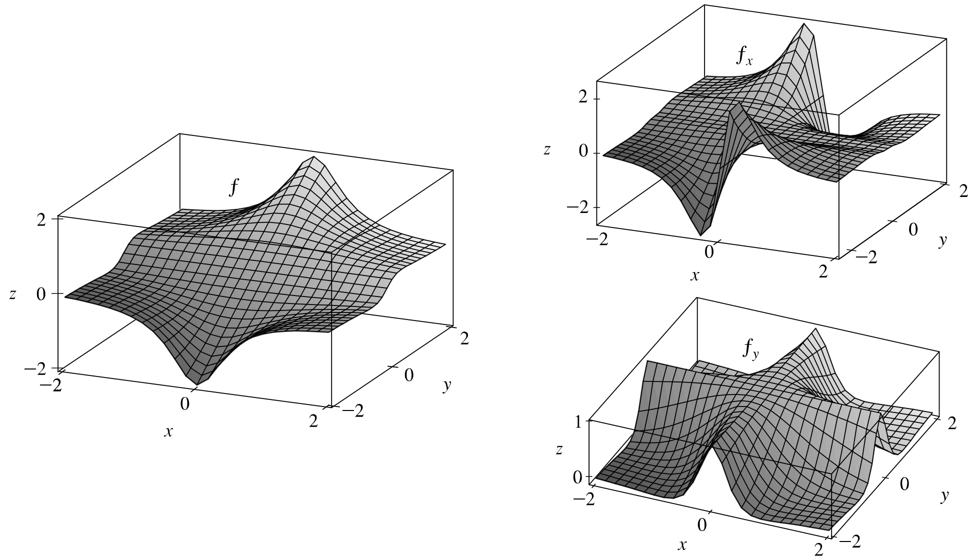
13. $f(x, y) = x^2 y^3 \quad \Rightarrow \quad f_x = 2xy^3, \quad f_y = 3x^2 y^2$



Note that traces of f in planes parallel to the xz-plane are parabolas which open downward for $y < 0$ and upward for $y > 0$, and the traces of f_x in these planes are straight lines, which have negative slopes for $y < 0$ and positive slopes for $y > 0$. The traces of f in planes parallel to the yz-plane are cubic curves, and the traces of f_y in these planes are parabolas.

14. $f(x, y) = \dfrac{y}{1 + x^2 y^2} \quad \Rightarrow \quad f_x = \dfrac{(1 + x^2 y^2)(0) - y(2xy^2)}{(1 + x^2 y^2)^2} = -\dfrac{2xy^3}{(1 + x^2 y^2)^2}$,

$$f_y = \frac{(1 + x^2 y^2)(1) - y(2x^2 y)}{(1 + x^2 y^2)^2} = \frac{1 - x^2 y^2}{(1 + x^2 y^2)^2}$$



Note that traces of f in planes parallel to the xz-plane have only one extreme value (a minimum for $y < 0$, a maximum for $y > 0$), and the traces of f_x in these planes have only one zero (going from negative to positive if $y < 0$ and from positive to negative if $y > 0$). The traces of f in planes parallel to the yz-plane have two extreme values, and the traces of f_y in these planes have two zeros.

15. $f(x,y) = x^4 + 5xy^3 \quad\Rightarrow\quad f_x(x,y) = 4x^3 + 5y^3,\ f_y(x,y) = 0 + 5x \cdot 3y^2 = 15xy^2$

16. $f(x,y) = x^2y - 3y^4 \quad\Rightarrow\quad f_x(x,y) = 2x \cdot y - 0 = 2xy,\ f_y(x,y) = x^2 \cdot 1 - 3 \cdot 4y^3 = x^2 - 12y^3$

17. $f(x,t) = t^2e^{-x} \quad\Rightarrow\quad f_x(x,t) = t^2 \cdot e^{-x}(-1) = -t^2e^{-x},\ f_t(x,t) = 2te^{-x}$

18. $f(x,t) = \sqrt{3x+4t} \quad\Rightarrow\quad f_x(x,t) = \frac{1}{2}(3x+4t)^{-1/2}(3) = \dfrac{3}{2\sqrt{3x+4t}},\ f_t(x,t) = \frac{1}{2}(3x+4t)^{-1/2}(4) = \dfrac{2}{\sqrt{3x+4t}}$

19. $z = \ln(x+t^2) \quad\Rightarrow\quad \dfrac{\partial z}{\partial x} = \dfrac{1}{x+t^2},\ \dfrac{\partial z}{\partial t} = \dfrac{2t}{x+t^2}$

20. $z = x\sin(xy) \quad\Rightarrow\quad \dfrac{\partial z}{\partial x} = x \cdot [\cos(xy)](y) + [\sin(xy)] \cdot 1 = xy\cos(xy) + \sin(xy),\ \dfrac{\partial z}{\partial y} = x\,[\cos(xy)](x) = x^2\cos(xy)$

21. $f(x,y) = x/y = xy^{-1} \quad\Rightarrow\quad f_x(x,y) = y^{-1} = 1/y,\ f_y(x,y) = -xy^{-2} = -x/y^2$

22. $f(x,t) = \sqrt{x}\,\ln t \quad\Rightarrow\quad f_x(x,t) = \frac{1}{2}x^{-1/2}\ln t = (\ln t)/(2\sqrt{x}),\ f_t(x,t) = \sqrt{x} \cdot \dfrac{1}{t} = \sqrt{x}/t$

23. $f(x,t) = e^{-t}\cos \pi x \quad\Rightarrow\quad f_x(x,t) = e^{-t}(-\sin \pi x)(\pi) = -\pi e^{-t}\sin \pi x,\ f_t(x,t) = e^{-t}(-1)\cos \pi x = -e^{-t}\cos \pi x$

24. $z = \tan xy \quad\Rightarrow\quad \dfrac{\partial z}{\partial x} = (\sec^2 xy)(y) = y\sec^2 xy,\ \dfrac{\partial z}{\partial y} = (\sec^2 xy)(x) = x\sec^2 xy$

25. $z = (2x+3y)^{10} \quad\Rightarrow\quad \dfrac{\partial z}{\partial x} = 10(2x+3y)^9 \cdot 2 = 20(2x+3y)^9,\ \dfrac{\partial z}{\partial y} = 10(2x+3y)^9 \cdot 3 = 30(2x+3y)^9$

26. $f(x,t) = \arctan\left(x\sqrt{t}\right) \quad \Rightarrow \quad f_x(x,t) = \dfrac{1}{1+\left(x\sqrt{t}\right)^2} \cdot \sqrt{t} = \dfrac{\sqrt{t}}{1+x^2t}$,

$$f_t(x,t) = \frac{1}{1+\left(x\sqrt{t}\right)^2} \cdot x\left(\frac{1}{2}t^{-1/2}\right) = \frac{x}{2\sqrt{t}\,(1+x^2t)}$$

27. $w = \sin\alpha\,\cos\beta \quad \Rightarrow \quad \dfrac{\partial w}{\partial \alpha} = \cos\alpha\,\cos\beta,\ \dfrac{\partial w}{\partial \beta} = -\sin\alpha\,\sin\beta$

28. $f(x,y) = x^y \quad \Rightarrow \quad f_x(x,y) = yx^{y-1},\ f_y(x,y) = x^y\ln x$

29. $F(x,y) = \displaystyle\int_y^x \cos(e^t)\,dt \quad \Rightarrow \quad F_x(x,y) = \dfrac{\partial}{\partial x}\displaystyle\int_y^x \cos\left(e^t\right)\,dt = \cos(e^x)$ by the Fundamental Theorem of Calculus, Part 1;

$$F_y(x,y) = \frac{\partial}{\partial y}\int_y^x \cos\left(e^t\right)\,dt = \frac{\partial}{\partial y}\left[-\int_x^y \cos\left(e^t\right)\,dt\right] = -\frac{\partial}{\partial y}\int_x^y \cos\left(e^t\right)\,dt = -\cos(e^y).$$

30. $F(\alpha,\beta) = \displaystyle\int_\alpha^\beta \sqrt{t^3+1}\,dt \quad \Rightarrow$

$F_\alpha(\alpha,\beta) = \dfrac{\partial}{\partial \alpha}\displaystyle\int_\alpha^\beta \sqrt{t^3+1}\,dt = \dfrac{\partial}{\partial \alpha}\left[-\displaystyle\int_\beta^\alpha \sqrt{t^3+1}\,dt\right] = -\dfrac{\partial}{\partial \alpha}\displaystyle\int_\beta^\alpha \sqrt{t^3+1}\,dt = -\sqrt{\alpha^3+1}$ by the Fundamental Theorem of Calculus, Part 1; $\quad F_\beta(\alpha,\beta) = \dfrac{\partial}{\partial \beta}\displaystyle\int_\alpha^\beta \sqrt{t^3+1}\,dt = \sqrt{\beta^3+1}.$

31. $f(x,y,z) = x^3yz^2 + 2yz \quad \Rightarrow \quad f_x(x,y,z) = 3x^2yz^2,\ f_y(x,y,z) = x^3z^2 + 2z,\ f_z(x,y,z) = 2x^3yz + 2y$

32. $f(x,y,z) = xy^2e^{-xz} \quad \Rightarrow \quad f_x(x,y,z) = y^2\left[x \cdot e^{-xz}(-z) + e^{-xz}\cdot 1\right] = (1-xz)y^2e^{-xz},\ \ f_y(x,y,z) = 2xye^{-xz}$,

$f_z(x,y,z) = xy^2e^{-xz}(-x) = -x^2y^2e^{-xz}$

33. $w = \ln(x+2y+3z) \quad \Rightarrow \quad \dfrac{\partial w}{\partial x} = \dfrac{1}{x+2y+3z},\ \dfrac{\partial w}{\partial y} = \dfrac{2}{x+2y+3z},\ \dfrac{\partial w}{\partial z} = \dfrac{3}{x+2y+3z}$

34. $w = y\tan(x+2z) \quad \Rightarrow \quad \dfrac{\partial w}{\partial x} = y\left[\sec^2(x+2z)\right](1) = y\sec^2(x+2z),\ \ \dfrac{\partial w}{\partial y} = \tan(x+2z)$,

$$\frac{\partial w}{\partial z} = y\left[\sec^2(x+2z)\right](2) = 2y\sec^2(x+2z)$$

35. $u = xy\sin^{-1}(yz) \quad \Rightarrow \quad \dfrac{\partial u}{\partial x} = y\sin^{-1}(yz),\ \ \dfrac{\partial u}{\partial y} = xy\cdot\dfrac{1}{\sqrt{1-(yz)^2}}\,(z) + \sin^{-1}(yz)\cdot x = \dfrac{xyz}{\sqrt{1-y^2z^2}} + x\sin^{-1}(yz)$,

$$\frac{\partial u}{\partial z} = xy\cdot\frac{1}{\sqrt{1-(yz)^2}}\,(y) = \frac{xy^2}{\sqrt{1-y^2z^2}}$$

36. $u = x^{y/z} \quad \Rightarrow \quad u_x = \dfrac{y}{z}\,x^{(y/z)-1},\ u_y = x^{y/z}\ln x\cdot\dfrac{1}{z} = \dfrac{x^{y/z}}{z}\ln x,\ \ u_z = x^{y/z}\ln x\cdot\dfrac{-y}{z^2} = -\dfrac{yx^{y/z}}{z^2}\ln x$

37. $h(x,y,z,t) = x^2 y\cos(z/t) \quad\Rightarrow\quad h_x(x,y,z,t) = 2xy\cos(z/t),\ h_y(x,y,z,t) = x^2\cos(z/t),$

$h_z(x,y,z,t) = -x^2 y\sin(z/t)(1/t) = (-x^2y/t)\sin(z/t),\ h_t(x,y,z,t) = -x^2y\sin(z/t)(-zt^{-2}) = (x^2yz/t^2)\sin(z/t)$

38. $\phi(x,y,z,t) = \dfrac{\alpha x + \beta y^2}{\gamma z + \delta t^2} \quad\Rightarrow\quad \phi_x(x,y,z,t) = \dfrac{1}{\gamma z + \delta t^2}(\alpha) = \dfrac{\alpha}{\gamma z + \delta t^2},$

$$\phi_y(x,y,z,t) = \frac{1}{\gamma z + \delta t^2}(2\beta y) = \frac{2\beta y}{\gamma z + \delta t^2},\ \phi_z(x,y,z,t) = \frac{(\gamma z + \delta t^2)(0) - (\alpha x + \beta y^2)(\gamma)}{(\gamma z + \delta t^2)^2} = \frac{-\gamma(\alpha x + \beta y^2)}{(\gamma z + \delta t^2)^2},$$

$$\phi_t(x,y,z,t) = \frac{(\gamma z + \delta t^2)(0) - (\alpha x + \beta y^2)(2\delta t)}{(\gamma z + \delta t^2)^2} = -\frac{2\delta t(\alpha x + \beta y^2)}{(\gamma z + \delta t^2)^2}$$

39. $u = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$. For each $i = 1, \ldots, n$, $u_{x_i} = \frac{1}{2}\left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{-1/2}(2x_i) = \dfrac{x_i}{\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}}$.

40. $u = \sin(x_1 + 2x_2 + \cdots + nx_n)$. For each $i = 1, \ldots, n$, $u_{x_i} = i\cos(x_1 + 2x_2 + \cdots + nx_n)$.

41. $R(s,t) = te^{s/t} \quad\Rightarrow\quad R_t(s,t) = t\cdot e^{s/t}(-s/t^2) + e^{s/t}\cdot 1 = \left(1 - \dfrac{s}{t}\right)e^{s/t}$, so $R_t(0,1) = \left(1 - \dfrac{0}{1}\right)e^{0/1} = 1$.

42. $f(x,y) = y\sin^{-1}(xy) \quad\Rightarrow\quad f_y(x,y) = y\cdot\dfrac{1}{\sqrt{1-(xy)^2}}(x) + \sin^{-1}(xy)\cdot 1 = \dfrac{xy}{\sqrt{1 - x^2y^2}} + \sin^{-1}(xy),$

so $f_y\left(1, \frac{1}{2}\right) = \dfrac{1\cdot\frac{1}{2}}{\sqrt{1 - 1^2\left(\frac{1}{2}\right)^2}} + \sin^{-1}\left(1\cdot\frac{1}{2}\right) = \dfrac{\frac{1}{2}}{\sqrt{\frac{3}{4}}} + \sin^{-1}\frac{1}{2} = \frac{1}{\sqrt{3}} + \frac{\pi}{6}$.

43. $f(x,y,z) = \ln\dfrac{1 - \sqrt{x^2+y^2+z^2}}{1 + \sqrt{x^2+y^2+z^2}} \quad\Rightarrow$

$$f_y(x,y,z) = \frac{1}{\dfrac{1 - \sqrt{x^2+y^2+z^2}}{1 + \sqrt{x^2+y^2+z^2}}}\cdot \frac{\left(1 + \sqrt{x^2+y^2+z^2}\right)\left(-\frac{1}{2}(x^2+y^2+z^2)^{-1/2}\cdot 2y\right) - \left(1 - \sqrt{x^2+y^2+z^2}\right)\left(\frac{1}{2}(x^2+y^2+z^2)^{-1/2}\cdot 2y\right)}{\left(1 + \sqrt{x^2+y^2+z^2}\right)^2}$$

$$= \frac{1 + \sqrt{x^2+y^2+z^2}}{1 - \sqrt{x^2+y^2+z^2}}\cdot\frac{-y(x^2+y^2+z^2)^{-1/2}\left(1 + \sqrt{x^2+y^2+z^2} + 1 - \sqrt{x^2+y^2+z^2}\right)}{\left(1 + \sqrt{x^2+y^2+z^2}\right)^2}$$

$$= \frac{-y(x^2+y^2+z^2)^{-1/2}\,(2)}{\left(1 - \sqrt{x^2+y^2+z^2}\right)\left(1 + \sqrt{x^2+y^2+z^2}\right)} = \frac{-2y}{\sqrt{x^2+y^2+z^2}\,[1 - (x^2+y^2+z^2)]}$$

so $f_y(1,2,2) = \dfrac{-2(2)}{\sqrt{1^2+2^2+2^2}\,[1 - (1^2+2^2+2^2)]} = \dfrac{-4}{\sqrt{9}\,(1-9)} = \dfrac{1}{6}$.

44. $f(x,y,z) = x^{yz} \quad\Rightarrow\quad f_z(x,y,z) = (x^{yz}\ln x)(y) = yx^{yz}\ln x$, so $f_z(e,1,0) = 1e^{(1)(0)}\ln e = 1$.

45. $f(x,y) = xy^2 - x^3y \quad \Rightarrow$

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} = \lim_{h\to 0} \frac{(x+h)y^2 - (x+h)^3y - (xy^2 - x^3y)}{h}$$

$$= \lim_{h\to 0} \frac{h(y^2 - 3x^2y - 3xyh - yh^2)}{h} = \lim_{h\to 0}(y^2 - 3x^2y - 3xyh - yh^2) = y^2 - 3x^2y$$

$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h} = \lim_{h\to 0} \frac{x(y+h)^2 - x^3(y+h) - (xy^2 - x^3y)}{h} = \lim_{h\to 0} \frac{h(2xy + xh - x^3)}{h}$$

$$= \lim_{h\to 0}(2xy + xh - x^3) = 2xy - x^3$$

46. $f(x,y) = \dfrac{x}{x+y^2} \quad \Rightarrow$

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} = \lim_{h\to 0} \frac{\frac{x+h}{x+h+y^2} - \frac{x}{x+y^2}}{h} \cdot \frac{(x+h+y^2)(x+y^2)}{(x+h+y^2)(x+y^2)}$$

$$= \lim_{h\to 0} \frac{(x+h)(x+y^2) - x(x+h+y^2)}{h(x+h+y^2)(x+y^2)} = \lim_{h\to 0} \frac{y^2h}{h(x+h+y^2)(x+y^2)}$$

$$= \lim_{h\to 0} \frac{y^2}{(x+h+y^2)(x+y^2)} = \frac{y^2}{(x+y^2)^2}$$

$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h} = \lim_{h\to 0} \frac{\frac{x}{x+(y+h)^2} - \frac{x}{x+y^2}}{h} \cdot \frac{\left[x+(y+h)^2\right](x+y^2)}{\left[x+(y+h)^2\right](x+y^2)}$$

$$= \lim_{h\to 0} \frac{x(x+y^2) - x\left[x+(y+h)^2\right]}{h[x+(y+h)^2](x+y^2)} = \lim_{h\to 0} \frac{h(-2xy - xh)}{h[x+(y+h)^2](x+y^2)}$$

$$= \lim_{h\to 0} \frac{-2xy - xh}{[x+(y+h)^2](x+y^2)} = \frac{-2xy}{(x+y^2)^2}$$

47. $x^2 + 2y^2 + 3z^2 = 1 \quad \Rightarrow \quad \dfrac{\partial}{\partial x}(x^2 + 2y^2 + 3z^2) = \dfrac{\partial}{\partial x}(1) \quad \Rightarrow \quad 2x + 0 + 6z\,\dfrac{\partial z}{\partial x} = 0 \quad \Rightarrow \quad 6z\,\dfrac{\partial z}{\partial x} = -2x \quad \Rightarrow$

$\dfrac{\partial z}{\partial x} = \dfrac{-2x}{6z} = -\dfrac{x}{3z}$, and $\dfrac{\partial}{\partial y}(x^2 + 2y^2 + 3z^2) = \dfrac{\partial}{\partial y}(1) \quad \Rightarrow \quad 0 + 4y + 6z\,\dfrac{\partial z}{\partial y} = 0 \quad \Rightarrow \quad 6z\,\dfrac{\partial z}{\partial y} = -4y \quad \Rightarrow$

$\dfrac{\partial z}{\partial y} = \dfrac{-4y}{6z} = -\dfrac{2y}{3z}$.

48. $x^2 - y^2 + z^2 - 2z = 4 \quad \Rightarrow \quad \dfrac{\partial}{\partial x}(x^2 - y^2 + z^2 - 2z) = \dfrac{\partial}{\partial x}(4) \quad \Rightarrow \quad 2x - 0 + 2z\,\dfrac{\partial z}{\partial x} - 2\,\dfrac{\partial z}{\partial x} = 0 \quad \Rightarrow$

$(2z-2)\,\dfrac{\partial z}{\partial x} = -2x \quad \Rightarrow \quad \dfrac{\partial z}{\partial x} = \dfrac{-2x}{2z-2} = \dfrac{x}{1-z}$, and $\dfrac{\partial}{\partial y}(x^2 - y^2 + z^2 - 2z) = \dfrac{\partial}{\partial y}(4) \quad \Rightarrow$

$0 - 2y + 2z\,\dfrac{\partial z}{\partial y} - 2\,\dfrac{\partial z}{\partial y} = 0 \quad \Rightarrow \quad (2z-2)\,\dfrac{\partial z}{\partial y} = 2y \quad \Rightarrow \quad \dfrac{\partial z}{\partial y} = \dfrac{2y}{2z-2} = \dfrac{y}{z-1}$.

49. $e^z = xyz \quad \Rightarrow \quad \dfrac{\partial}{\partial x}(e^z) = \dfrac{\partial}{\partial x}(xyz) \quad \Rightarrow \quad e^z\,\dfrac{\partial z}{\partial x} = y\left(x\,\dfrac{\partial z}{\partial x} + z \cdot 1\right) \quad \Rightarrow \quad e^z\,\dfrac{\partial z}{\partial x} - xy\,\dfrac{\partial z}{\partial x} = yz \quad \Rightarrow$

$(e^z - xy)\,\dfrac{\partial z}{\partial x} = yz$, so $\dfrac{\partial z}{\partial x} = \dfrac{yz}{e^z - xy}$.

$$\frac{\partial}{\partial y}(e^z) = \frac{\partial}{\partial y}(xyz) \quad \Rightarrow \quad e^z\,\frac{\partial z}{\partial y} = x\left(y\,\frac{\partial z}{\partial x} + z\cdot 1\right) \quad \Rightarrow \quad e^z\,\frac{\partial z}{\partial y} - xy\,\frac{\partial z}{\partial y} = xz \quad \Rightarrow \quad (e^z - xy)\,\frac{\partial z}{\partial y} = xz, \text{ so}$$

$$\frac{\partial z}{\partial y} = \frac{xz}{e^z - xy}.$$

50. $yz + x\ln y = z^2 \quad \Rightarrow \quad \dfrac{\partial}{\partial x}(yz + x\ln y) = \dfrac{\partial}{\partial x}(z^2) \quad \Rightarrow \quad y\,\dfrac{\partial z}{\partial x} + \ln y = 2z\,\dfrac{\partial z}{\partial x} \quad \Rightarrow \quad \ln y = 2z\,\dfrac{\partial z}{\partial x} - y\,\dfrac{\partial z}{\partial x} \quad \Rightarrow$

$\ln y = (2z - y)\,\dfrac{\partial z}{\partial x}$, so $\dfrac{\partial z}{\partial x} = \dfrac{\ln y}{2z - y}$.

$\dfrac{\partial}{\partial y}(yz + x\ln y) = \dfrac{\partial}{\partial y}(z^2) \quad \Rightarrow \quad y\,\dfrac{\partial z}{\partial y} + z\cdot 1 + x\cdot\dfrac{1}{y} = 2z\,\dfrac{\partial z}{\partial y} \quad \Rightarrow \quad z + \dfrac{x}{y} = 2z\,\dfrac{\partial z}{\partial y} - y\,\dfrac{\partial z}{\partial y} \quad \Rightarrow$

$z + \dfrac{x}{y} = (2z - y)\,\dfrac{\partial z}{\partial y}$, so $\dfrac{\partial z}{\partial y} = \dfrac{z + (x/y)}{2z - y} = \dfrac{x + yz}{y(2z - y)}$.

51. (a) $z = f(x) + g(y) \quad \Rightarrow \quad \dfrac{\partial z}{\partial x} = f'(x), \quad \dfrac{\partial z}{\partial y} = g'(y)$

(b) $z = f(x + y)$. Let $u = x + y$. Then $\dfrac{\partial z}{\partial x} = \dfrac{df}{du}\dfrac{\partial u}{\partial x} = \dfrac{df}{du}(1) = f'(u) = f'(x + y)$,

$\dfrac{\partial z}{\partial y} = \dfrac{df}{du}\dfrac{\partial u}{\partial y} = \dfrac{df}{du}(1) = f'(u) = f'(x + y)$.

52. (a) $z = f(x)g(y) \quad \Rightarrow \quad \dfrac{\partial z}{\partial x} = f'(x)g(y), \quad \dfrac{\partial z}{\partial y} = f(x)g'(y)$

(b) $z = f(xy)$. Let $u = xy$. Then $\dfrac{\partial u}{\partial x} = y$ and $\dfrac{\partial u}{\partial y} = x$. Hence $\dfrac{\partial z}{\partial x} = \dfrac{df}{du}\dfrac{\partial u}{\partial x} = \dfrac{df}{du}\cdot y = yf'(u) = yf'(xy)$

and $\dfrac{\partial z}{\partial y} = \dfrac{df}{du}\dfrac{\partial u}{\partial y} = \dfrac{df}{du}\cdot x = xf'(u) = xf'(xy)$.

(c) $z = f\left(\dfrac{x}{y}\right)$. Let $u = \dfrac{x}{y}$. Then $\dfrac{\partial u}{\partial x} = \dfrac{1}{y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{x}{y^2}$. Hence $\dfrac{\partial z}{\partial x} = \dfrac{df}{du}\dfrac{\partial u}{\partial x} = f'(u)\dfrac{1}{y} = \dfrac{f'(x/y)}{y}$

and $\dfrac{\partial z}{\partial y} = \dfrac{df}{du}\dfrac{\partial u}{\partial y} = f'(u)\left(-\dfrac{x}{y^2}\right) = -\dfrac{xf'(x/y)}{y^2}$.

53. $f(x,y) = x^4y - 2x^3y^2 \quad \Rightarrow \quad f_x(x,y) = 4x^3y - 6x^2y^2, \quad f_y(x,y) = x^4 - 4x^3y$. Then $f_{xx}(x,y) = 12x^2y - 12xy^2$,

$f_{xy}(x,y) = 4x^3 - 12x^2y, \quad f_{yx}(x,y) = 4x^3 - 12x^2y$, and $f_{yy}(x,y) = -4x^3$.

54. $f(x,y) = \ln(ax + by) \quad \Rightarrow \quad f_x(x,y) = \dfrac{a}{ax + by} = a(ax + by)^{-1}, \quad f_y(x,y) = \dfrac{b}{ax + by} = b(ax + by)^{-1}$. Then

$f_{xx}(x,y) = -a(ax + by)^{-2}(a) = -\dfrac{a^2}{(ax + by)^2}, \quad f_{xy}(x,y) = -a(ax + by)^{-2}(b) = -\dfrac{ab}{(ax + by)^2}$,

$f_{yx}(x,y) = -b(ax + by)^{-2}(a) = -\dfrac{ab}{(ax + by)^2}$, and $f_{yy}(x,y) = -b(ax + by)^{-2}(b) = -\dfrac{b^2}{(ax + by)^2}$.

55. $z = \dfrac{y}{2x+3y} = y(2x+3y)^{-1} \quad \Rightarrow \quad z_x = y(-1)(2x+3y)^{-2}(2) = -\dfrac{2y}{(2x+3y)^2}$,

$z_y = \dfrac{(2x+3y)\cdot 1 - y\cdot 3}{(2x+3y)^2} = \dfrac{2x}{(2x+3y)^2}$. Then $z_{xx} = -2y(-2)(2x+3y)^{-3}(2) = \dfrac{8y}{(2x+3y)^3}$,

$z_{xy} = -\dfrac{(2x+3y)^2\cdot 2 - 2y\cdot 2(2x+3y)(3)}{\left[(2x+3y)^2\right]^2} = -\dfrac{(2x+3y)\,(4x+6y-12y)}{(2x+3y)^4} = \dfrac{6y-4x}{(2x+3y)^3}$,

$z_{yx} = \dfrac{(2x+3y)^2\cdot 2 - 2x\cdot 2(2x+3y)(2)}{\left[(2x+3y)^2\right]^2} = \dfrac{6y-4x}{(2x+3y)^3}$, $z_{yy} = 2x(-2)(2x+3y)^{-3}(3) = -\dfrac{12x}{(2x+3y)^3}$.

56. $T = e^{-2r}\cos\theta \quad \Rightarrow \quad T_r = -2e^{-2r}\cos\theta$, $T_\theta = -e^{-2r}\sin\theta$. Then $T_{rr} = -2e^{-2r}(-2)\cos\theta = 4e^{-2r}\cos\theta$,

$T_{r\theta} = 2e^{-2r}\sin\theta$, $T_{\theta r} = -e^{-2r}(-2)\sin\theta = 2e^{-2r}\sin\theta$, $T_{\theta\theta} = -e^{-2r}\cos\theta$.

57. $v = \sin(s^2-t^2) \quad \Rightarrow \quad v_s = \cos(s^2-t^2)\cdot 2s = 2s\cos(s^2-t^2)$, $v_t = \cos(s^2-t^2)\cdot(-2t) = -2t\cos(s^2-t^2)$. Then

$v_{ss} = 2s\left[-\sin(s^2-t^2)\cdot 2s\right] + \cos(s^2-t^2)\cdot 2 = 2\cos(s^2-t^2) - 4s^2\sin(s^2-t^2)$,

$v_{st} = 2s\left[-\sin(s^2-t^2)\cdot(-2t)\right] = 4st\sin(s^2-t^2)$, $v_{ts} = -2t\left[-\sin(s^2-t^2)\cdot 2s\right] = 4st\sin(s^2-t^2)$,

$v_{tt} = -2t\cdot\left[-\sin(s^2-t^2)\cdot(-2t)\right] + \cos(s^2-t^2)\cdot(-2) = -2\cos(s^2-t^2) - 4t^2\sin(s^2-t^2)$.

58. $w = \sqrt{1+uv^2} \quad \Rightarrow \quad w_u = \frac{1}{2}(1+uv^2)^{-1/2}\cdot v^2 = \dfrac{v^2}{2\sqrt{1+uv^2}}$, $w_v = \frac{1}{2}(1+uv^2)^{-1/2}\cdot 2uv = \dfrac{uv}{\sqrt{1+uv^2}}$.

Then $w_{uu} = \frac{1}{2}v^2\left(-\frac{1}{2}\right)(1+uv^2)^{-3/2}(v^2) = -\dfrac{v^4}{4(1+uv^2)^{3/2}}$,

$$\begin{aligned} w_{uv} &= \frac{2\sqrt{1+uv^2}\cdot 2v - v^2\cdot 2\left(\frac{1}{2}\right)(1+uv^2)^{-1/2}(2uv)}{\left(2\sqrt{1+uv^2}\right)^2} = \frac{4v\sqrt{1+uv^2} - 2uv^3/\sqrt{1+uv^2}}{4(1+uv^2)} \\ &= \frac{4v(1+uv^2) - 2uv^3}{4\left(1+uv^2\right)^{3/2}} = \frac{2v+uv^3}{2\left(1+uv^2\right)^{3/2}} \end{aligned}$$

$$\begin{aligned} w_{vu} &= \frac{\sqrt{1+uv^2}\cdot v - uv\cdot\frac{1}{2}(1+uv^2)^{-1/2}(v^2)}{\left(\sqrt{1+uv^2}\right)^2} = \frac{v\sqrt{1+uv^2} - \frac{1}{2}uv^3/\sqrt{1+uv^2}}{(1+uv^2)} \\ &= \frac{v(1+uv^2) - \frac{1}{2}uv^3}{(1+uv^2)^{3/2}} = \frac{2v+uv^3}{2\left(1+uv^2\right)^{3/2}} \end{aligned}$$

$$\begin{aligned} w_{vv} &= \frac{\sqrt{1+uv^2}\cdot u - uv\cdot\frac{1}{2}(1+uv^2)^{-1/2}(2uv)}{\left(\sqrt{1+uv^2}\right)^2} = \frac{u\sqrt{1+uv^2} - u^2v^2/\sqrt{1+uv^2}}{(1+uv^2)} \\ &= \frac{u(1+uv^2) - u^2v^2}{(1+uv^2)^{3/2}} = \frac{u}{\left(1+uv^2\right)^{3/2}} \end{aligned}$$

59. $u = x^4y^3 - y^4 \quad \Rightarrow \quad u_x = 4x^3y^3$, $u_{xy} = 12x^3y^2$ and $u_y = 3x^4y^2 - 4y^3$, $u_{yx} = 12x^3y^2$.

Thus $u_{xy} = u_{yx}$.

60. $u = e^{xy}\sin y \;\Rightarrow\; u_x = ye^{xy}\sin y,\; u_{xy} = ye^{xy}\cos y + (\sin y)(y \cdot xe^{xy} + e^{xy} \cdot 1) = e^{xy}(y\cos y + xy\sin y + \sin y)$,

$u_y = e^{xy}\cos y + (\sin y)(xe^{xy}) = e^{xy}(\cos y + x\sin y)$,

$u_{yx} = e^{xy} \cdot \sin y + (\cos y + x\sin y) \cdot ye^{xy} = e^{xy}(\sin y + y\cos y + xy\sin y)$. Thus $u_{xy} = u_{yx}$.

61. $u = \cos(x^2y) \;\Rightarrow\; u_x = -\sin(x^2y) \cdot 2xy = -2xy\sin(x^2y)$,

$u_{xy} = -2xy \cdot \cos(x^2y) \cdot x^2 + \sin(x^2y) \cdot (-2x) = -2x^3y\cos(x^2y) - 2x\sin(x^2y)$ and

$u_y = -\sin(x^2y) \cdot x^2 = -x^2\sin(x^2y),\;\; u_{yx} = -x^2 \cdot \cos(x^2y) \cdot 2xy + \sin(x^2y) \cdot (-2x) = -2x^3y\cos(x^2y) - 2x\sin(x^2y)$.

Thus $u_{xy} = u_{yx}$.

62. $u = \ln(x+2y) \;\Rightarrow\; u_x = \dfrac{1}{x+2y} = (x+2y)^{-1},\;\; u_{xy} = (-1)(x+2y)^{-2}(2) = -\dfrac{2}{(x+2y)^2}$ and

$u_y = \dfrac{1}{x+2y} \cdot 2 = 2(x+2y)^{-1},\;\; u_{yx} = (-2)(x+2y)^{-2} = -\dfrac{2}{(x+2y)^2}$. Thus $u_{xy} = u_{yx}$.

63. $f(x,y) = x^4y^2 - x^3y \;\Rightarrow\; f_x = 4x^3y^2 - 3x^2y,\; f_{xx} = 12x^2y^2 - 6xy,\; f_{xxx} = 24xy^2 - 6y$ and

$f_{xy} = 8x^3y - 3x^2,\; f_{xyx} = 24x^2y - 6x$.

64. $f(x,y) = \sin(2x+5y) \;\Rightarrow\; f_y = \cos(2x+5y) \cdot 5 = 5\cos(2x+5y),\; f_{yx} = -5\sin(2x+5y) \cdot 2 = -10\sin(2x+5y)$,

$f_{yxy} = -10\cos(2x+5y) \cdot 5 = -50\cos(2x+5y)$

65. $f(x,y,z) = e^{xyz^2} \;\Rightarrow\; f_x = e^{xyz^2} \cdot yz^2 = yz^2e^{xyz^2},\; f_{xy} = yz^2 \cdot e^{xyz^2}(xz^2) + e^{xyz^2} \cdot z^2 = (xyz^4 + z^2)e^{xyz^2}$,

$f_{xyz} = (xyz^4 + z^2) \cdot e^{xyz^2}(2xyz) + e^{xyz^2} \cdot (4xyz^3 + 2z) = (2x^2y^2z^5 + 6xyz^3 + 2z)e^{xyz^2}$.

66. $g(r,s,t) = e^r\sin(st) \;\Rightarrow\; g_r = e^r\sin(st),\; g_{rs} = e^r\cos(st) \cdot t = te^r\cos(st)$,

$g_{rst} = te^r(-\sin(st) \cdot s) + \cos(st) \cdot e^r = e^r[\cos(st) - st\sin(st)]$.

67. $W = \sqrt{u+v^2} \;\Rightarrow\; \dfrac{\partial W}{\partial v} = \tfrac{1}{2}(u+v^2)^{-1/2}(2v) = v(u+v^2)^{-1/2}$,

$\dfrac{\partial^2 W}{\partial u\,\partial v} = v\left(-\tfrac{1}{2}\right)(u+v^2)^{-3/2}(1) = -\tfrac{1}{2}v(u+v^2)^{-3/2},\;\; \dfrac{\partial^3 W}{\partial u^2\,\partial v} = -\tfrac{1}{2}v\left(-\tfrac{3}{2}\right)(u+v^2)^{-5/2}(1) = \tfrac{3}{4}v(u+v^2)^{-5/2}$.

68. $V = \ln(r+s^2+t^3) \;\Rightarrow\; \dfrac{\partial V}{\partial t} = \dfrac{3t^2}{r+s^2+t^3} = 3t^2(r+s^2+t^3)^{-1}$,

$\dfrac{\partial^2 V}{\partial s\,\partial t} = 3t^2(-1)(r+s^2+t^3)^{-2}(2s) = -6st^2(r+s^2+t^3)^{-2}$,

$\dfrac{\partial^3 V}{\partial r\,\partial s\,\partial t} = -6st^2(-2)(r+s^2+t^3)^{-3}(1) = 12st^2(r+s^2+t^3)^{-3} = \dfrac{12st^2}{(r+s^2+t^3)^3}$.

69. $u = e^{r\theta}\sin\theta \;\Rightarrow\; \dfrac{\partial u}{\partial\theta} = e^{r\theta}\cos\theta + \sin\theta \cdot e^{r\theta}\,(r) = e^{r\theta}\,(\cos\theta + r\sin\theta)$,

$\dfrac{\partial^2 u}{\partial r\,\partial\theta} = e^{r\theta}\,(\sin\theta) + (\cos\theta + r\sin\theta)\,e^{r\theta}\,(\theta) = e^{r\theta}\,(\sin\theta + \theta\cos\theta + r\theta\sin\theta)$,

$\dfrac{\partial^3 u}{\partial r^2\,\partial\theta} = e^{r\theta}\,(\theta\sin\theta) + (\sin\theta + \theta\cos\theta + r\theta\sin\theta) \cdot e^{r\theta}\,(\theta) = \theta e^{r\theta}\,(2\sin\theta + \theta\cos\theta + r\theta\sin\theta)$.

70. $z = u\sqrt{v-w} = u(v-w)^{1/2} \Rightarrow \dfrac{\partial z}{\partial w} = u\left[\frac{1}{2}(v-w)^{-1/2}(-1)\right] = -\frac{1}{2}u(v-w)^{-1/2}$,

$\dfrac{\partial^2 z}{\partial v\,\partial w} = -\frac{1}{2}u\left(-\frac{1}{2}(v-w)^{-3/2}(1)\right) = \frac{1}{4}u(v-w)^{-3/2}$, $\dfrac{\partial^3 z}{\partial u\,\partial v\,\partial w} = \frac{1}{4}(v-w)^{-3/2}$.

71. Assuming that the third partial derivatives of f are continuous (easily verified), we can write $f_{xzy} = f_{yxz}$. Then

$f(x,y,z) = xy^2z^3 + \arcsin\left(x\sqrt{z}\right) \Rightarrow f_y = 2xyz^3 + 0$, $f_{yx} = 2yz^3$, and $f_{yxz} = 6yz^2 = f_{xzy}$.

72. Let $f(x,y,z) = \sqrt{1+xz}$ and $h(x,y,z) = \sqrt{1-xy}$ so that $g = f + h$. Then $f_y = 0 = f_{yx} = f_{yxz}$ and $h_z = 0 = h_{zx} = h_{zxy}$. But (since the partial derivatives are continous on their domains) $f_{xyz} = f_{yxz}$ and $h_{xyz} = h_{zxy}$, so $g_{xyz} = f_{xyz} + h_{xyz} = 0 + 0 = 0$.

73. By Definition 4, $f_x(3,2) = \lim\limits_{h\to 0} \dfrac{f(3+h,2) - f(3,2)}{h}$ which we can approximate by considering $h = 0.5$ and $h = -0.5$:

$f_x(3,2) \approx \dfrac{f(3.5,2) - f(3,2)}{0.5} = \dfrac{22.4 - 17.5}{0.5} = 9.8$, $f_x(3,2) \approx \dfrac{f(2.5,2) - f(3,2)}{-0.5} = \dfrac{10.2 - 17.5}{-0.5} = 14.6$. Averaging these values, we estimate $f_x(3,2)$ to be approximately 12.2. Similarly, $f_x(3,2.2) = \lim\limits_{h\to 0} \dfrac{f(3+h,2.2) - f(3,2.2)}{h}$ which we can approximate by considering $h = 0.5$ and $h = -0.5$: $f_x(3,2.2) \approx \dfrac{f(3.5,2.2) - f(3,2.2)}{0.5} = \dfrac{26.1 - 15.9}{0.5} = 20.4$,

$f_x(3,2.2) \approx \dfrac{f(2.5,2.2) - f(3,2.2)}{-0.5} = \dfrac{9.3 - 15.9}{-0.5} = 13.2$. Averaging these values, we have $f_x(3,2.2) \approx 16.8$.

To estimate $f_{xy}(3,2)$, we first need an estimate for $f_x(3,1.8)$:

$f_x(3,1.8) \approx \dfrac{f(3.5,1.8) - f(3,1.8)}{0.5} = \dfrac{20.0 - 18.1}{0.5} = 3.8$, $f_x(3,1.8) \approx \dfrac{f(2.5,1.8) - f(3,1.8)}{-0.5} = \dfrac{12.5 - 18.1}{-0.5} = 11.2$.

Averaging these values, we get $f_x(3,1.8) \approx 7.5$. Now $f_{xy}(x,y) = \dfrac{\partial}{\partial y}\left[f_x(x,y)\right]$ and $f_x(x,y)$ is itself a function of two variables, so Definition 4 says that $f_{xy}(x,y) = \dfrac{\partial}{\partial y}\left[f_x(x,y)\right] = \lim\limits_{h\to 0} \dfrac{f_x(x,y+h) - f_x(x,y)}{h} \Rightarrow$

$f_{xy}(3,2) = \lim\limits_{h\to 0} \dfrac{f_x(3,2+h) - f_x(3,2)}{h}$. We can estimate this value using our previous work with $h = 0.2$ and $h = -0.2$:

$f_{xy}(3,2) \approx \dfrac{f_x(3,2.2) - f_x(3,2)}{0.2} = \dfrac{16.8 - 12.2}{0.2} = 23$, $f_{xy}(3,2) \approx \dfrac{f_x(3,1.8) - f_x(3,2)}{-0.2} = \dfrac{7.5 - 12.2}{-0.2} = 23.5$.

Averaging these values, we estimate $f_{xy}(3,2)$ to be approximately 23.25.

74. (a) If we fix y and allow x to vary, the level curves indicate that the value of f decreases as we move through P in the positive x-direction, so f_x is negative at P.

(b) If we fix x and allow y to vary, the level curves indicate that the value of f increases as we move through P in the positive y-direction, so f_y is positive at P.

(c) $f_{xx} = \dfrac{\partial}{\partial x}(f_x)$, so if we fix y and allow x to vary, f_{xx} is the rate of change of f_x as x increases. Note that at points to the right of P the level curves are spaced farther apart (in the x-direction) than at points to the left of P, demonstrating that f decreases less quickly with respect to x to the right of P. So as we move through P in the positive x-direction the (negative) value of f_x increases, hence $\dfrac{\partial}{\partial x}(f_x) = f_{xx}$ is positive at P.

(d) $f_{xy} = \dfrac{\partial}{\partial y}(f_x)$, so if we fix x and allow y to vary, f_{xy} is the rate of change of f_x as y increases. The level curves are closer together (in the x-direction) at points above P than at those below P, demonstrating that f decreases more quickly with respect to x for y-values above P. So as we move through P in the positive y-direction, the (negative) value of f_x decreases, hence f_{xy} is negative.

(e) $f_{yy} = \dfrac{\partial}{\partial y}(f_y)$, so if we fix x and allow y to vary, f_{yy} is the rate of change of f_y as y increases. The level curves are closer together (in the y-direction) at points above P than at those below P, demonstrating that f increases more quickly with respect to y above P. So as we move through P in the positive y-direction the (positive) value of f_y increases, hence $\dfrac{\partial}{\partial y}(f_y) = f_{yy}$ is positive at P.

75. $u = e^{-\alpha^2 k^2 t}\sin kx \quad\Rightarrow\quad u_x = ke^{-\alpha^2 k^2 t}\cos kx,\;\; u_{xx} = -k^2 e^{-\alpha^2 k^2 t}\sin kx,$ and $u_t = -\alpha^2 k^2 e^{-\alpha^2 k^2 t}\sin kx$. Thus $\alpha^2 u_{xx} = u_t$.

76. (a) $u = x^2 + y^2 \quad\Rightarrow\quad u_x = 2x,\; u_{xx} = 2;\quad u_y = 2y,\; u_{yy} = 2$. Thus $u_{xx} + u_{yy} \neq 0$ and $u = x^2 + y^2$ does not satisfy Laplace's Equation.

(b) $u = x^2 - y^2$ is a solution: $u_{xx} = 2,\; u_{yy} = -2$ so $u_{xx} + u_{yy} = 0$.

(c) $u = x^3 + 3xy^2$ is not a solution: $u_x = 3x^2 + 3y^2,\; u_{xx} = 6x;\quad u_y = 6xy,\; u_{yy} = 6x$.

(d) $u = \ln\sqrt{x^2 + y^2}$ is a solution: $u_x = \dfrac{1}{\sqrt{x^2+y^2}}\left(\dfrac{1}{2}\right)(x^2+y^2)^{-1/2}(2x) = \dfrac{x}{x^2+y^2}$,

$u_{xx} = \dfrac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \dfrac{y^2 - x^2}{(x^2+y^2)^2}$. By symmetry, $u_{yy} = \dfrac{x^2 - y^2}{(x^2+y^2)^2}$, so $u_{xx} + u_{yy} = 0$.

(e) $u = \sin x\,\cosh y + \cos x\,\sinh y$ is a solution: $u_x = \cos x\,\cosh y - \sin x\,\sinh y$, $u_{xx} = -\sin x\,\cosh y - \cos x\,\sinh y$, and $u_y = \sin x\,\sinh y + \cos x\,\cosh y$, $u_{yy} = \sin x\,\cosh y + \cos x\,\sinh y$.

(f) $u = e^{-x}\cos y - e^{-y}\cos x$ is a solution: $u_x = -e^{-x}\cos y + e^{-y}\sin x,\quad u_{xx} = e^{-x}\cos y + e^{-y}\cos x$, and $u_y = -e^{-x}\sin y + e^{-y}\cos x,\quad u_{yy} = -e^{-x}\cos y - e^{-y}\cos x$.

77. $u = \dfrac{1}{\sqrt{x^2+y^2+z^2}} \quad\Rightarrow\quad u_x = \left(-\tfrac{1}{2}\right)(x^2+y^2+z^2)^{-3/2}(2x) = -x(x^2+y^2+z^2)^{-3/2}$ and

$u_{xx} = -(x^2+y^2+z^2)^{-3/2} - x\left(-\tfrac{3}{2}\right)(x^2+y^2+z^2)^{-5/2}(2x) = \dfrac{2x^2 - y^2 - z^2}{(x^2+y^2+z^2)^{5/2}}$.

By symmetry, $u_{yy} = \dfrac{2y^2 - x^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}}$ and $u_{zz} = \dfrac{2z^2 - x^2 - y^2}{(x^2 + y^2 + z^2)^{5/2}}$.

Thus $u_{xx} + u_{yy} + u_{zz} = \dfrac{2x^2 - y^2 - z^2 + 2y^2 - x^2 - z^2 + 2z^2 - x^2 - y^2}{(x^2 + y^2 + z^2)^{5/2}} = 0$.

78. (a) $u = \sin(kx)\,\sin(akt) \quad\Rightarrow\quad u_t = ak\sin(kx)\,\cos(akt)$, $u_{tt} = -a^2k^2\sin(kx)\,\sin(akt)$, $u_x = k\cos(kx)\,\sin(akt)$, $u_{xx} = -k^2\sin(kx)\,\sin(akt)$. Thus $u_{tt} = a^2u_{xx}$.

(b) $u = \dfrac{t}{a^2t^2 - x^2} \quad\Rightarrow\quad u_t = \dfrac{(a^2t^2 - x^2) - t(2a^2t)}{(a^2t^2 - x^2)^2} = -\dfrac{a^2t^2 + x^2}{(a^2t^2 - x^2)^2}$,

$$u_{tt} = \frac{-2a^2t(a^2t^2 - x^2)^2 + (a^2t^2 + x^2)(2)(a^2t^2 - x^2)(2a^2t)}{(a^2t^2 - x^2)^4} = \frac{2a^4t^3 + 6a^2tx^2}{(a^2t^2 - x^2)^3},$$

$$u_x = t(-1)(a^2t^2 - x^2)^{-2}(-2x) = \frac{2tx}{(a^2t^2 - x^2)^2},$$

$$u_{xx} = \frac{2t(a^2t^2 - x^2)^2 - 2tx\,(2)(a^2t^2 - x^2)(-2x)}{(a^2t^2 - x^2)^4} = \frac{2a^2t^3 - 2tx^2 + 8tx^2}{(a^2t^2 - x^2)^3} = \frac{2a^2t^3 + 6tx^2}{(a^2t^2 - x^2)^3}.$$

Thus $u_{tt} = a^2u_{xx}$.

(c) $u = (x - at)^6 + (x + at)^6 \quad\Rightarrow\quad u_t = -6a(x - at)^5 + 6a(x + at)^5$, $u_{tt} = 30a^2(x - at)^4 + 30a^2(x + at)^4$, $u_x = 6(x - at)^5 + 6(x + at)^5$, $u_{xx} = 30(x - at)^4 + 30(x + at)^4$. Thus $u_{tt} = a^2u_{xx}$.

(d) $u = \sin(x - at) + \ln(x + at) \quad\Rightarrow\quad u_t = -a\cos(x - at) + \dfrac{a}{x + at}$, $u_{tt} = -a^2\sin(x - at) - \dfrac{a^2}{(x + at)^2}$, $u_x = \cos(x - at) + \dfrac{1}{x + at}$, $u_{xx} = -\sin(x - at) - \dfrac{1}{(x + at)^2}$. Thus $u_{tt} = a^2u_{xx}$.

79. Let $v = x + at$, $w = x - at$. Then $u_t = \dfrac{\partial[f(v) + g(w)]}{\partial t} = \dfrac{df(v)}{dv}\dfrac{\partial v}{\partial t} + \dfrac{dg(w)}{dw}\dfrac{\partial w}{\partial t} = af'(v) - ag'(w)$ and $u_{tt} = \dfrac{\partial[af'(v) - ag'(w)]}{\partial t} = a[af''(v) + ag''(w)] = a^2[f''(v) + g''(w)]$. Similarly, by using the Chain Rule we have $u_x = f'(v) + g'(w)$ and $u_{xx} = f''(v) + g''(w)$. Thus $u_{tt} = a^2u_{xx}$.

80. For each i, $i = 1, \ldots, n$, $\partial u/\partial x_i = a_i e^{a_1x_1 + a_2x_2 + \cdots + a_nx_n}$ and $\partial^2 u/\partial x_i^2 = a_i^2 e^{a_1x_1 + a_2x_2 + \cdots + a_nx_n}$.

Then $\dfrac{\partial^2 u}{\partial x_1^2} + \dfrac{\partial^2 u}{\partial x_2^2} + \cdots + \dfrac{\partial^2 u}{\partial x_n^2} = \left(a_1^2 + a_2^2 + \cdots + a_n^2\right)e^{a_1x_1 + a_2x_2 + \cdots + a_nx_n} = e^{a_1x_1 + a_2x_2 + \cdots + a_nx_n} = u$

since $a_1^2 + a_2^2 + \cdots + a_n^2 = 1$.

81. $c(x, t) = \dfrac{1}{\sqrt{4\pi Dt}}\, e^{-x^2/(4Dt)} \quad\Rightarrow$

$$\begin{aligned}\frac{\partial c}{\partial t} &= \frac{1}{\sqrt{4\pi Dt}} \cdot e^{-x^2/(4Dt)}\left[-x^2(-1)(4Dt)^{-2}(4D)\right] + e^{-x^2/(4Dt)} \cdot \left(-\tfrac{1}{2}\right)(4\pi Dt)^{-3/2}\,(4\pi D) \\ &= (4\pi Dt)^{-3/2}\left(4\pi Dt \cdot \frac{x^2}{4Dt^2} - 2\pi D\right)e^{-x^2/(4Dt)} = \frac{2\pi D}{(4\pi Dt)^{3/2}}\left(\frac{x^2}{2Dt} - 1\right)e^{-x^2/(4Dt)},\end{aligned}$$

$\dfrac{\partial c}{\partial x} = \dfrac{1}{\sqrt{4\pi Dt}}\, e^{-x^2/(4Dt)} \cdot \dfrac{-2x}{4Dt} = \dfrac{-2\pi x}{(4\pi Dt)^{3/2}}\, e^{-x^2/(4Dt)}$, and

$$\frac{\partial^2 c}{\partial x^2} = \frac{-2\pi}{(4\pi Dt)^{3/2}}\left[x \cdot e^{-x^2/(4Dt)} \cdot \frac{-2x}{4Dt} + e^{-x^2/(4Dt)} \cdot 1\right]$$
$$= \frac{-2\pi}{(4\pi Dt)^{3/2}}\left(-\frac{x^2}{2Dt} + 1\right)e^{-x^2/(4Dt)} = \frac{2\pi}{(4\pi Dt)^{3/2}}\left(\frac{x^2}{2Dt} - 1\right)e^{-x^2/(4Dt)}.$$

Thus $\dfrac{\partial c}{\partial t} = \dfrac{2\pi D}{(4\pi Dt)^{3/2}}\left(\dfrac{x^2}{2Dt} - 1\right)e^{-x^2/(4Dt)} = D\left[\dfrac{2\pi}{(4\pi Dt)^{3/2}}\left(\dfrac{x^2}{2Dt} - 1\right)e^{-x^2/(4Dt)}\right] = D\,\dfrac{\partial^2 c}{\partial x^2}.$

82. (a) $\partial T/\partial x = -60(2x)/(1 + x^2 + y^2)^2$, so at $(2, 1)$, $T_x = -240/(1 + 4 + 1)^2 = -\frac{20}{3}$.

(b) $\partial T/\partial y = -60(2y)/(1 + x^2 + y^2)^2$, so at $(2, 1)$, $T_y = -120/36 = -\frac{10}{3}$. Thus from the point $(2, 1)$ the temperature is decreasing at a rate of $\frac{20}{3}\,^\circ\text{C/m}$ in the x-direction and is decreasing at a rate of $\frac{10}{3}\,^\circ\text{C/m}$ in the y-direction.

83. By the Chain Rule, taking the partial derivative of both sides with respect to R_1 gives

$$\frac{\partial R^{-1}}{\partial R}\frac{\partial R}{\partial R_1} = \frac{\partial\left[(1/R_1) + (1/R_2) + (1/R_3)\right]}{\partial R_1} \quad \text{or} \quad -R^{-2}\frac{\partial R}{\partial R_1} = -R_1^{-2}. \text{ Thus } \frac{\partial R}{\partial R_1} = \frac{R^2}{R_1^2}.$$

84. $P = bL^\alpha K^\beta$, so $\dfrac{\partial P}{\partial L} = \alpha bL^{\alpha-1}K^\beta$ and $\dfrac{\partial P}{\partial K} = \beta bL^\alpha K^{\beta-1}$. Then

$$L\,\frac{\partial P}{\partial L} + K\,\frac{\partial P}{\partial K} = L(\alpha bL^{\alpha-1}K^\beta) + K(\beta bL^\alpha K^{\beta-1}) = \alpha bL^{1+\alpha-1}K^\beta + \beta bL^\alpha K^{1+\beta-1} = (\alpha+\beta)bL^\alpha K^\beta = (\alpha+\beta)P$$

85. If we fix $K = K_0$, $P(L, K_0)$ is a function of a single variable L, and $\dfrac{dP}{dL} = \alpha\,\dfrac{P}{L}$ is a separable differential equation. Then $\dfrac{dP}{P} = \alpha\dfrac{dL}{L} \;\Rightarrow\; \displaystyle\int\frac{dP}{P} = \int\alpha\,\frac{dL}{L} \;\Rightarrow\; \ln|P| = \alpha\ln|L| + C(K_0)$, where $C(K_0)$ can depend on K_0. Then $|P| = e^{\alpha\ln|L| + C(K_0)}$, and since $P > 0$ and $L > 0$, we have $P = e^{\alpha\ln L}e^{C(K_0)} = e^{C(K_0)}e^{\ln L^\alpha} = C_1(K_0)L^\alpha$ where $C_1(K_0) = e^{C(K_0)}$.

86. (a) $P(L, K) = 1.01L^{0.75}K^{0.25} \quad\Rightarrow\quad P_L(L, K) = 1.01(0.75L^{-0.25})K^{0.25} = 0.7575L^{-0.25}K^{0.25}$ and $P_K(L, K) = 1.01L^{0.75}(0.25K^{-0.75}) = 0.2525L^{0.75}K^{-0.75}$.

(b) The marginal productivity of labor in 1920 is $P_L(194, 407) = 0.7575(194)^{-0.25}(407)^{0.25} \approx 0.912$. Recall that P, L, and K are expressed as percentages of the respective amounts in 1899, so this means that in 1920, if the amount of labor is increased, production increases at a rate of about 0.912 percentage points per percentage point increase in labor. The marginal productivity of capital in 1920 is $P_K(194, 407) = 0.2525(194)^{0.75}(407)^{-0.75} \approx 0.145$, so an increase in capital investment would cause production to increase at a rate of about 0.145 percentage points per percentage point increase in capital.

(c) The value of $P_L(194, 407)$ is greater than the value of $P_K(194, 407)$, suggesting that increasing labor in 1920 would have increased production more than increasing capital.

87. $\left(P + \dfrac{n^2 a}{V^2}\right)(V - nb) = nRT \quad \Rightarrow \quad T = \dfrac{1}{nR}\left(P + \dfrac{n^2 a}{V^2}\right)(V - nb)$, so $\dfrac{\partial T}{\partial P} = \dfrac{1}{nR}(1)(V - nb) = \dfrac{V - nb}{nR}$.

We can also write $P + \dfrac{n^2 a}{V^2} = \dfrac{nRT}{V - nb} \quad \Rightarrow \quad P = \dfrac{nRT}{V - nb} - \dfrac{n^2 a}{V^2} = nRT(V - nb)^{-1} - n^2 a V^{-2}$, so

$\dfrac{\partial P}{\partial V} = -nRT(V - nb)^{-2}(1) + 2n^2 a V^{-3} = \dfrac{2n^2 a}{V^3} - \dfrac{nRT}{(V - nb)^2}$.

88. $P = \dfrac{mRT}{V}$ so $\dfrac{\partial P}{\partial V} = \dfrac{-mRT}{V^2}$; $\quad V = \dfrac{mRT}{P}$, so $\dfrac{\partial V}{\partial T} = \dfrac{mR}{P}$; $\quad T = \dfrac{PV}{mR}$, so $\dfrac{\partial T}{\partial P} = \dfrac{V}{mR}$.

Thus $\dfrac{\partial P}{\partial V}\dfrac{\partial V}{\partial T}\dfrac{\partial T}{\partial P} = \dfrac{-mRT}{V^2}\dfrac{mR}{P}\dfrac{V}{mR} = \dfrac{-mRT}{PV} = -1$, since $PV = mRT$.

89. By Exercise 88, $PV = mRT \quad \Rightarrow \quad P = \dfrac{mRT}{V}$, so $\dfrac{\partial P}{\partial T} = \dfrac{mR}{V}$. Also, $PV = mRT \quad \Rightarrow \quad V = \dfrac{mRT}{P}$ and $\dfrac{\partial V}{\partial T} = \dfrac{mR}{P}$.

Since $T = \dfrac{PV}{mR}$, we have $T\dfrac{\partial P}{\partial T}\dfrac{\partial V}{\partial T} = \dfrac{PV}{mR} \cdot \dfrac{mR}{V} \cdot \dfrac{mR}{P} = mR$.

90. $\dfrac{\partial W}{\partial T} = 0.6215 + 0.3965v^{0.16}$. When $T = -15°\text{C}$ and $v = 30$ km/h, $\dfrac{\partial W}{\partial T} = 0.6215 + 0.3965(30)^{0.16} \approx 1.3048$, so we would expect the apparent temperature to drop by approximately $1.3°\text{C}$ if the actual temperature decreases by $1°\text{C}$.

$\dfrac{\partial W}{\partial v} = -11.37(0.16)v^{-0.84} + 0.3965T(0.16)v^{-0.84}$ and when $T = -15°\text{C}$ and $v = 30$ km/h,

$\dfrac{\partial W}{\partial v} = -11.37(0.16)(30)^{-0.84} + 0.3965(-15)(0.16)(30)^{-0.84} \approx -0.1592$, so we would expect the apparent temperature to drop by approximately $0.16°\text{C}$ if the wind speed increases by 1 km/h.

91. (a) $S = f(w, h) = 0.1091w^{0.425}h^{0.725} \quad \Rightarrow \quad \dfrac{\partial S}{\partial w} = 0.1091(0.425)w^{0.425-1}h^{0.725} = 0.0463675w^{-0.575}h^{0.725}$, so

$\dfrac{\partial S}{\partial w}(160, 70) = 0.0463675(160)^{-0.575}(70)^{0.725} \approx 0.0545$. This means that for a person 70 inches tall who weighs 160 pounds, an increase in weight (while height remains constant) causes the surface area to increase at a rate of about 0.0545 square feet (about 7.85 square inches) per pound.

(b) $\dfrac{\partial S}{\partial h} = 0.1091(0.725)w^{0.425}h^{0.725-1} = 0.0790975w^{0.425}h^{-0.275}$, so

$\dfrac{\partial S}{\partial h}(160, 70) = 0.0790975(160)^{0.425}(70)^{-0.275} \approx 0.213$. This means that for a person 70 inches tall who weighs 160 pounds, an increase in height (while weight remains unchanged at 160 pounds) causes the surface area to increase at a rate of about 0.213 square feet (about 30.7 square inches) per inch of height.

92. $R = C\dfrac{L}{r^4} \quad \Rightarrow \quad \dfrac{\partial R}{\partial L} = \dfrac{C}{r^4}$ and $\dfrac{\partial R}{\partial r} = CL\left(-4r^{-5}\right) = -4C\dfrac{L}{r^5}$.

$\partial R/\partial L$ is the rate at which the resistance of the flowing blood increases with respect to the length of the artery when the radius stays constant. $\partial R/\partial r$ is the rate of change of the resistance with respect to the radius of the artery when the length remains unchanged. Because $\partial R/\partial r$ is negative, the resistance decreases if the radius increases.

93. $P(v,x,m)=Av^3+\dfrac{B(mg/x)^2}{v}=Av^3+Bm^2g^2x^{-2}v^{-1}$.

$\partial P/\partial v=3Av^2-\dfrac{B(mg/x)^2}{v^2}$ is the rate of change of the power needed during flapping mode with respect to the bird's velocity when the mass and fraction of flapping time remain constant. $\partial P/\partial x=-2Bm^2g^2x^{-3}v^{-1}=-\dfrac{2Bm^2g^2}{x^3v}$ is the rate at which the power changes with respect to the fraction of time spent in flapping mode when the mass and velocity are held constant. $\partial P/\partial m=2Bmg^2x^{-2}v^{-1}=\dfrac{2Bmg^2}{x^2v}$ is the rate of change of the power with respect to mass when the velocity and fraction of flapping time remain constant.

94. $E(m,v)=2.65m^{0.66}+\dfrac{3.5m^{0.75}}{v}\quad\Rightarrow$

$E_m(m,v)=2.65(0.66)m^{0.66-1}+\dfrac{3.5(0.75)m^{0.75-1}}{v}=1.749m^{-0.34}+\dfrac{2.625m^{-0.25}}{v}$,

$E_v(m,v)=3.5m^{0.75}\left(-v^{-2}\right)=-\dfrac{3.5m^{0.75}}{v^2}$. Then $E_m(400,8)=1.749(400)^{-0.34}+\dfrac{2.625(400)^{-0.25}}{8}\approx 0.301$ which means that the average energy needed for a lizard to walk or run 1 km increases at a rate of about 0.301 kcal per gram of body mass increase from 400 g if the speed is 8 km/h. $E_v(400,8)=-\dfrac{3.5(400)^{0.75}}{8^2}\approx-4.89$, which means that the average energy needed by a lizard with body mass 400 g decreases at a rate of about 4.89 kcal per km/h when the speed increases from 8 km/h.

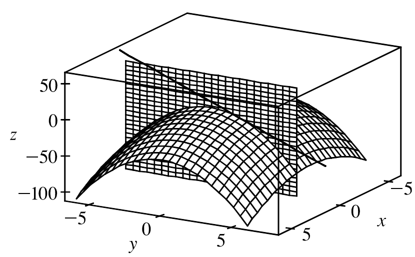
95. $\dfrac{\partial K}{\partial m}=\frac{1}{2}v^2$, $\dfrac{\partial K}{\partial v}=mv$, $\dfrac{\partial^2 K}{\partial v^2}=m$. Thus $\dfrac{\partial K}{\partial m}\cdot\dfrac{\partial^2 K}{\partial v^2}=\frac{1}{2}v^2m=K$.

96. The Law of Cosines says that $a^2=b^2+c^2-2bc\cos A$. Thus $\dfrac{\partial(a^2)}{\partial a}=\dfrac{\partial(b^2+c^2-2ab\cos A)}{\partial a}$ or

$2a=-2bc\,(-\sin A)\,\dfrac{\partial A}{\partial a}$, implying that $\dfrac{\partial A}{\partial a}=\dfrac{a}{bc\sin A}$. Taking the partial derivative of both sides with respect to b gives $0=2b-2c(\cos A)-2bc\,(-\sin A)\,\dfrac{\partial A}{\partial b}$. Thus $\dfrac{\partial A}{\partial b}=\dfrac{c\cos A-b}{bc\sin A}$. By symmetry, $\dfrac{\partial A}{\partial c}=\dfrac{b\cos A-c}{bc\sin A}$.

97. $f_x(x,y)=x+4y\quad\Rightarrow\quad f_{xy}(x,y)=4$ and $f_y(x,y)=3x-y\quad\Rightarrow\quad f_{yx}(x,y)=3$. Since f_{xy} and f_{yx} are continuous everywhere but $f_{xy}(x,y)\neq f_{yx}(x,y)$, Clairaut's Theorem implies that such a function $f(x,y)$ does not exist.

98. Setting $x=1$, the equation of the parabola of intersection is $z=6-1-1-2y^2=4-2y^2$. The slope of the tangent is $\partial z/\partial y=-4y$, so at $(1,2,-4)$ the slope is -8. Parametric equations for the line are therefore $x=1,\ y=2+t,\ z=-4-8t$.



99. By the geometry of partial derivatives, the slope of the tangent line is $f_x(1,2)$. By implicit differentiation of $4x^2 + 2y^2 + z^2 = 16$, we get $8x + 2z\,(\partial z/\partial x) = 0 \quad \Rightarrow \quad \partial z/\partial x = -4x/z$, so when $x = 1$ and $z = 2$ we have $\partial z/\partial x = -2$. So the slope is $f_x(1,2) = -2$. Thus the tangent line is given by $z - 2 = -2(x-1),\ y = 2$. Taking the parameter to be $t = x - 1$, we can write parametric equations for this line: $x = 1 + t,\ y = 2,\ z = 2 - 2t$.

100. $T(x,t) = T_0 + T_1 e^{-\lambda x} \sin(\omega t - \lambda x)$

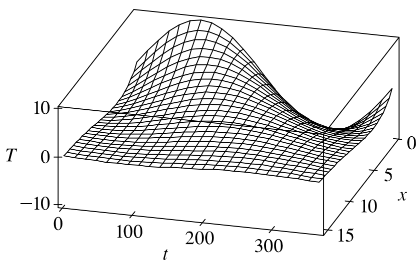
(a) $\partial T/\partial x = T_1 e^{-\lambda x}\,[\cos(\omega t - \lambda x)(-\lambda)] + T_1(-\lambda e^{-\lambda x})\sin(\omega t - \lambda x) = -\lambda T_1 e^{-\lambda x}\,[\sin(\omega t - \lambda x) + \cos(\omega t - \lambda x)]$.
This quantity represents the rate of change of temperature with respect to depth below the surface, at a given time t.

(b) $\partial T/\partial t = T_1 e^{-\lambda x}[\cos(\omega t - \lambda x)(\omega)] = \omega T_1 e^{-\lambda x}\cos(\omega t - \lambda x)$. This quantity represents the rate of change of temperature with respect to time at a fixed depth x.

(c)
$$\begin{aligned} T_{xx} &= \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) \\ &= -\lambda T_1\left(e^{-\lambda x}[\cos(\omega t - \lambda x)(-\lambda) - \sin(\omega t - \lambda x)(-\lambda)] + e^{-\lambda x}(-\lambda)[\sin(\omega t - \lambda x) + \cos(\omega t - \lambda x)]\right) \\ &= 2\lambda^2 T_1 e^{-\lambda x}\cos(\omega t - \lambda x) \end{aligned}$$

But from part (b), $T_t = \omega T_1 e^{-\lambda x}\cos(\omega t - \lambda x) = \dfrac{\omega}{2\lambda^2}\,T_{xx}$. So with $k = \dfrac{\omega}{2\lambda^2}$, the function T satisfies the heat equation.

(d)



Note that near the surface (that is, for small x) the temperature varies greatly as t changes, but deeper (for large x) the temperature is more stable.

(e) The term $-\lambda x$ is a phase shift: it represents the fact that since heat diffuses slowly through soil, it takes time for changes in the surface temperature to affect the temperature at deeper points. As x increases, the phase shift also increases. For example, at the surface the highest temperature is reached at $t \approx 100$, whereas at a depth of 5 meters the peak temperature is attained at $t \approx 150$, and at a depth of 10 meters, at $t \approx 220$.

101. By Clairaut's Theorem, $f_{xyy} = (f_{xy})_y = (f_{yx})_y = f_{yxy} = (f_y)_{xy} = (f_y)_{yx} = f_{yyx}$.

102. (a) Since we are differentiating n times, with two choices of variable at each differentiation, there are 2^n nth-order partial derivatives.

(b) If these partial derivatives are all continuous, then the order in which the partials are taken doesn't affect the value of the result, that is, all nth-order partial derivatives with p partials with respect to x and $n - p$ partials with respect to y are equal. Since the number of partials taken with respect to x for an nth-order partial derivative can range from 0 to n, a function of two variables has $n + 1$ distinct partial derivatives of order n if these partial derivatives are all continuous.

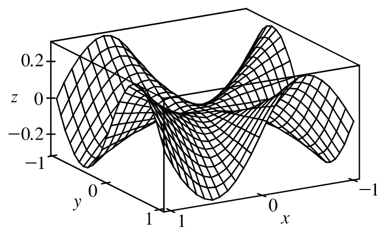
(c) Since n differentiations are to be performed with three choices of variable at each differentiation, there are 3^n nth-order partial derivatives of a function of three variables.

103. Let $g(x) = f(x, 0) = x(x^2)^{-3/2}e^0 = x\,|x|^{-3}$. But we are using the point $(1, 0)$, so near $(1, 0)$, $g(x) = x^{-2}$. Then $g'(x) = -2x^{-3}$ and $g'(1) = -2$, so using (1) we have $f_x(1, 0) = g'(1) = -2$.

104. $f_x(0,0) = \lim\limits_{h \to 0} \dfrac{f(0+h,0) - f(0,0)}{h} = \lim\limits_{h \to 0} \dfrac{(h^3+0)^{1/3} - 0}{h} = \lim\limits_{h \to 0} \dfrac{h}{h} = 1.$

Or: Let $g(x) = f(x, 0) = \sqrt[3]{x^3 + 0} = x$. Then $g'(x) = 1$ and $g'(0) = 1$ so, by (1), $f_x(0, 0) = g'(0) = 1$.

105. (a)



(b) For $(x, y) \neq (0, 0)$,

$$f_x(x,y) = \frac{(3x^2y - y^3)(x^2+y^2) - (x^3y - xy^3)(2x)}{(x^2+y^2)^2} = \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}$$

and by symmetry $f_y(x,y) = \dfrac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2}$.

(c) $f_x(0,0) = \lim\limits_{h \to 0} \dfrac{f(h,0) - f(0,0)}{h} = \lim\limits_{h \to 0} \dfrac{(0/h^2) - 0}{h} = 0$ and $f_y(0,0) = \lim\limits_{h \to 0} \dfrac{f(0,h) - f(0,0)}{h} = 0.$

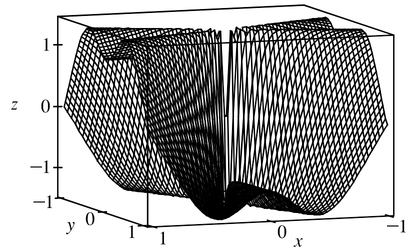
(d) By (3), $f_{xy}(0,0) = \dfrac{\partial f_x}{\partial y} = \lim\limits_{h \to 0} \dfrac{f_x(0,h) - f_x(0,0)}{h} = \lim\limits_{h \to 0} \dfrac{(-h^5 - 0)/h^4}{h} = -1$ while by (2),

$$f_{yx}(0,0) = \frac{\partial f_y}{\partial x} = \lim_{h \to 0} \frac{f_y(h,0) - f_y(0,0)}{h} = \lim_{h \to 0} \frac{h^5/h^4}{h} = 1.$$

(e) For $(x, y) \neq (0, 0)$, we use a CAS to compute

$$f_{xy}(x,y) = \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2+y^2)^3}$$

Now as $(x, y) \to (0, 0)$ along the x-axis, $f_{xy}(x, y) \to 1$ while as $(x, y) \to (0, 0)$ along the y-axis, $f_{xy}(x, y) \to -1$. Thus f_{xy} isn't continuous at $(0, 0)$ and Clairaut's Theorem doesn't apply, so there is no contradiction. The graphs of f_{xy} and f_{yx} are identical except at the origin, where we observe the discontinuity.



14.4 Tangent Planes and Linear Approximations

1. $z = f(x,y) = 2x^2 + y^2 - 5y \;\Rightarrow\; f_x(x,y) = 4x,\; f_y(x,y) = 2y - 5$, so $f_x(1,2) = 4,\; f_y(1,2) = -1$.

By Equation 2, an equation of the tangent plane is $z - (-4) = f_x(1,2)(x-1) + f_y(1,2)(y-2) \;\Rightarrow$

$z + 4 = 4(x-1) + (-1)(y-2)$ or $z = 4x - y - 6$.