14.6 Directional Derivatives and the Gradient Vector

1. We can approximate the directional derivative of the pressure function at K in the direction of S by the average rate of change of pressure between the points where the red line intersects the contour lines closest to K (extend the red line slightly at the left). In the direction of S, the pressure changes from 1000 millibars to 996 millibars and we estimate the distance between these two points to be approximately 50 km (using the fact that the distance from K to S is 300 km). Then the rate of change of pressure in the direction given is approximately $\frac{996-1000}{50} = -0.08$ millibar/km.

2. First we draw a line passing through Dubbo and Sydney. We approximate the directional derivative at Dubbo in the direction of Sydney by the average rate of change of temperature between the points where the line intersects the contour lines closest to Dubbo. In the direction of Sydney, the temperature changes from $30°\mathrm{C}$ to $27°\mathrm{C}$. We estimate the distance between these two points to be approximately 120 km, so the rate of change of maximum temperature in the direction given is approximately $\frac{27-30}{120} = -0.025°\mathrm{C/km}$.

3. $D_{\mathbf{u}}\, f(-20,30) = \nabla f\,(-20,30)\cdot \mathbf{u} = f_T(-20,30)\left(\frac{1}{\sqrt{2}}\right) + f_v(-20,30)\left(\frac{1}{\sqrt{2}}\right)$.

$f_T(-20,30) = \lim\limits_{h\to 0} \dfrac{f(-20+h,30) - f(-20,30)}{h}$, so we can approximate $f_T(-20,30)$ by considering $h = \pm 5$ and using the values given in the table: $f_T(-20,30) \approx \dfrac{f(-15,30)-f(-20,30)}{5} = \dfrac{-26-(-33)}{5} = 1.4$,

$f_T(-20,30) \approx \dfrac{f(-25,30)-f(-20,30)}{-5} = \dfrac{-39-(-33)}{-5} = 1.2$. Averaging these values gives $f_T(-20,30)\approx 1.3$.

Similarly, $f_v(-20,30) = \lim\limits_{h\to 0}\dfrac{f(-20,30+h)-f(-20,30)}{h}$, so we can approximate $f_v(-20,30)$ with $h=\pm 10$:

$f_v(-20,30) \approx \dfrac{f(-20,40)-f(-20,30)}{10} = \dfrac{-34-(-33)}{10} = -0.1$,

$f_v(-20,30) \approx \dfrac{f(-20,20)-f(-20,30)}{-10} = \dfrac{-30-(-33)}{-10} = -0.3$. Averaging these values gives $f_v(-20,30)\approx -0.2$.

Then $D_{\mathbf{u}} f(-20,30) \approx 1.3\left(\frac{1}{\sqrt{2}}\right) + (-0.2)\left(\frac{1}{\sqrt{2}}\right) \approx 0.778$.

4. $f(x,y) = xy^3 - x^2 \;\Rightarrow\; f_x(x,y) = y^3 - 2x$ and $f_y(x,y) = 3xy^2$. If $\mathbf{u}$ is a unit vector in the direction of $\theta = \pi/3$, then from Equation 6, $D_{\mathbf{u}} f(1,2) = f_x(1,2)\cos\left(\frac{\pi}{3}\right) + f_y(1,2)\sin\left(\frac{\pi}{3}\right) = 6\cdot\frac{1}{2} + 12\cdot\frac{\sqrt{3}}{2} = 3 + 6\sqrt{3}$.

5. $f(x,y) = y\cos(xy) \;\Rightarrow\; f_x(x,y) = y[-\sin(xy)](y) = -y^2\sin(xy)$ and

$f_y(x,y) = y[-\sin(xy)](x) + [\cos(xy)](1) = \cos(xy) - xy\sin(xy)$. If $\mathbf{u}$ is a unit vector in the direction of $\theta = \pi/4$, then from Equation 6, $D_{\mathbf{u}} f(0,1) = f_x(0,1)\cos\left(\frac{\pi}{4}\right) + f_y(0,1)\sin\left(\frac{\pi}{4}\right) = 0\cdot\frac{\sqrt{2}}{2} + 1\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}$.

6. $f(x,y) = \sqrt{2x+3y} \;\Rightarrow\; f_x(x,y) = \frac{1}{2}(2x+3y)^{-1/2}(2) = 1/\sqrt{2x+3y}$ and

$f_y(x,y) = \frac{1}{2}(2x+3y)^{-1/2}(3) = 3/\left(2\sqrt{2x+3y}\right)$. If $\mathbf{u}$ is a unit vector in the direction of $\theta = -\pi/6$, then from Equation 6,

$D_{\mathbf{u}} f(3,1) = f_x(3,1)\cos\left(-\frac{\pi}{6}\right) + f_y(3,1)\sin\left(-\frac{\pi}{6}\right) = \frac{1}{3}\cdot\frac{\sqrt{3}}{2} + \frac{1}{2}\cdot\left(-\frac{1}{2}\right) = \frac{\sqrt{3}}{6} - \frac{1}{4}$.

7. $f(x, y) = x/y = xy^{-1}$

(a) $\nabla f(x, y) = \dfrac{\partial f}{\partial x}\,\mathbf{i} + \dfrac{\partial f}{\partial y}\,\mathbf{j} = y^{-1}\,\mathbf{i} + (-xy^{-2})\,\mathbf{j} = \dfrac{1}{y}\,\mathbf{i} - \dfrac{x}{y^2}\,\mathbf{j}$

(b) $\nabla f(2, 1) = \dfrac{1}{1}\,\mathbf{i} - \dfrac{2}{1^2}\,\mathbf{j} = \mathbf{i} - 2\,\mathbf{j}$

(c) By Equation 9, $D_{\mathbf{u}}\, f(2, 1) = \nabla f(2, 1) \cdot \mathbf{u} = (\mathbf{i} - 2\,\mathbf{j}) \cdot \left(\frac{3}{5}\,\mathbf{i} + \frac{4}{5}\,\mathbf{j}\right) = \frac{3}{5} - \frac{8}{5} = -1.$

8. $f(x, y) = x^2 \ln y$

(a) $\nabla f(x, y) = \dfrac{\partial f}{\partial x}\,\mathbf{i} + \dfrac{\partial f}{\partial y}\,\mathbf{j} = 2x \ln y\,\mathbf{i} + (x^2/y)\,\mathbf{j}$

(b) $\nabla f(3, 1) = 0\,\mathbf{i} + (9/1)\,\mathbf{j} = 9\,\mathbf{j}$

(c) By Equation 9, $D_{\mathbf{u}}\, f(3, 1) = \nabla f(3, 1) \cdot \mathbf{u} = 9\,\mathbf{j} \cdot \left(-\frac{5}{13}\,\mathbf{i} + \frac{12}{13}\,\mathbf{j}\right) = 0 + \frac{108}{13} = \frac{108}{13}.$

9. $f(x, y, z) = xe^{2yz}$

(a) $\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)\rangle = \left\langle e^{2yz}, 2xze^{2yz}, 2xye^{2yz}\right\rangle$

(b) $\nabla f(3, 0, 2) = \langle 1, 12, 0\rangle$

(c) By Equation 14, $D_{\mathbf{u}} f(3, 0, 2) = \nabla f(3, 0, 2) \cdot \mathbf{u} = \langle 1, 12, 0\rangle \cdot \left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right\rangle = \frac{2}{3} - \frac{24}{3} + 0 = -\frac{22}{3}.$

10. $f(x, y, z) = \sqrt{x + yz} = (x + yz)^{1/2}$

(a) $\nabla f(x, y, z) = \left\langle \frac{1}{2}(x + yz)^{-1/2}(1), \frac{1}{2}(x + yz)^{-1/2}(z), \frac{1}{2}(x + yz)^{-1/2}(y)\right\rangle$

$= \left\langle 1/(2\sqrt{x + yz}\,), z/(2\sqrt{x + yz}\,), y/(2\sqrt{x + yz}\,)\right\rangle$

(b) $\nabla f(1, 3, 1) = \left\langle \frac{1}{4}, \frac{1}{4}, \frac{3}{4}\right\rangle$

(c) $D_{\mathbf{u}} f(1, 3, 1) = \nabla f(1, 3, 1) \cdot \mathbf{u} = \left\langle \frac{1}{4}, \frac{1}{4}, \frac{3}{4}\right\rangle \cdot \left\langle \frac{2}{7}, \frac{3}{7}, \frac{6}{7}\right\rangle = \frac{2}{28} + \frac{3}{28} + \frac{18}{28} = \frac{23}{28}$

11. $f(x, y) = e^x \sin y \quad\Rightarrow\quad \nabla f(x, y) = \langle e^x \sin y, e^x \cos y\rangle$, $\nabla f(0, \pi/3) = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{(-6)^2 + 8^2}}\,\langle -6, 8\rangle = \frac{1}{10}\,\langle -6, 8\rangle = \left\langle -\frac{3}{5}, \frac{4}{5}\right\rangle$, so

$D_{\mathbf{u}}\, f(0, \pi/3) = \nabla f(0, \pi/3) \cdot \mathbf{u} = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle \cdot \left\langle -\frac{3}{5}, \frac{4}{5}\right\rangle = -\frac{3\sqrt{3}}{10} + \frac{4}{10} = \frac{4 - 3\sqrt{3}}{10}.$

12. $g(r, s) = \tan^{-1}(rs) \quad\Rightarrow\quad \nabla g(r, s) = \left(\dfrac{1}{1 + (rs)^2} \cdot s\right)\mathbf{i} + \left(\dfrac{1}{1 + (rs)^2} \cdot r\right)\mathbf{j} = \dfrac{s}{1 + r^2 s^2}\,\mathbf{i} + \dfrac{r}{1 + r^2 s^2}\,\mathbf{j},$

$\nabla g(1, 2) = \frac{2}{5}\,\mathbf{i} + \frac{1}{5}\,\mathbf{j}$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{5^2 + 10^2}}(5\,\mathbf{i} + 10\,\mathbf{j}) = \frac{1}{5\sqrt{5}}(5\,\mathbf{i} + 10\,\mathbf{j}) = \frac{1}{\sqrt{5}}\,\mathbf{i} + \frac{2}{\sqrt{5}}\,\mathbf{j}$,

so $D_{\mathbf{u}}\, g(1, 2) = \nabla g(1, 2) \cdot \mathbf{u} = \left(\frac{2}{5}\,\mathbf{i} + \frac{1}{5}\,\mathbf{j}\right) \cdot \left(\frac{1}{\sqrt{5}}\,\mathbf{i} + \frac{2}{\sqrt{5}}\,\mathbf{j}\right) = \frac{2}{5\sqrt{5}} + \frac{2}{5\sqrt{5}} = \frac{4}{5\sqrt{5}}$ or $\frac{4\sqrt{5}}{25}$.

13. $g(s, t) = s\sqrt{t} \quad\Rightarrow\quad \nabla g(s, t) = \left(\sqrt{t}\,\right)\mathbf{i} + \left(s/(2\sqrt{t}\,)\right)\mathbf{j}$, $\nabla g(2, 4) = 2\,\mathbf{i} + \frac{1}{2}\,\mathbf{j}$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{2^2 + (-1)^2}}(2\,\mathbf{i} - \mathbf{j}) = \frac{1}{\sqrt{5}}(2\,\mathbf{i} - \mathbf{j})$, so $D_{\mathbf{u}}\, g(2, 4) = \nabla g(2, 4) \cdot \mathbf{u} = \left(2\,\mathbf{i} + \frac{1}{2}\,\mathbf{j}\right) \cdot \frac{1}{\sqrt{5}}(2\,\mathbf{i} - \mathbf{j}) = \frac{1}{\sqrt{5}}\left(4 - \frac{1}{2}\right) = \frac{7}{2\sqrt{5}}$ or $\frac{7\sqrt{5}}{10}$.

14. $g(u,v) = u^2 e^{-v} \quad\Rightarrow\quad \nabla g(u,v) = \left(2ue^{-v}\right)\mathbf{i} + \left(-u^2 e^{-v}\right)\mathbf{j}$, $\nabla g(3,0) = 6\,\mathbf{i} - 9\,\mathbf{j}$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{3^2+4^2}}(3\,\mathbf{i}+4\,\mathbf{j}) = \frac{1}{5}(3\,\mathbf{i}+4\,\mathbf{j})$, so $D_{\mathbf{u}}\,g(3,0) = \nabla g(3,0)\cdot\mathbf{u} = (6\,\mathbf{i}-9\,\mathbf{j})\cdot\frac{1}{5}(3\,\mathbf{i}+4\,\mathbf{j}) = \frac{1}{5}\,(18-36) = -\frac{18}{5}$.

15. $f(x,y,z) = x^2y + y^2z \quad\Rightarrow\quad \nabla f(x,y,z) = \left\langle 2xy, x^2+2yz, y^2\right\rangle$, $\nabla f(1,2,3) = \langle 4, 13, 4\rangle$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{4+1+4}}\langle 2,-1,2\rangle = \frac{1}{3}\langle 2,-1,2\rangle$, so

$D_{\mathbf{u}}\,f(1,2,3) = \nabla f(1,2,3)\cdot\mathbf{u} = \langle 4,13,4\rangle\cdot\frac{1}{3}\langle 2,-1,2\rangle = \frac{1}{3}\,(8-13+8) = \frac{3}{3} = 1.$

16. $f(x,y,z) = xy^2\tan^{-1} z \quad\Rightarrow\quad \nabla f(x,y,z) = \left\langle y^2\tan^{-1} z, 2xy\tan^{-1} z, \dfrac{xy^2}{1+z^2}\right\rangle$,

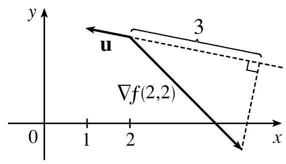
$\nabla f(2,1,1) = \left\langle 1\cdot\frac{\pi}{4}, 4\cdot\frac{\pi}{4}, \frac{2}{1+1}\right\rangle = \left\langle\frac{\pi}{4},\pi,1\right\rangle$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{1+1+1}}\langle 1,1,1\rangle = \frac{1}{\sqrt{3}}\,\langle 1,1,1\rangle$,

so $D_{\mathbf{u}}\,f(2,1,1) = \nabla f(2,1,1)\cdot\mathbf{u} = \left\langle\frac{\pi}{4},\pi,1\right\rangle\cdot\frac{1}{\sqrt{3}}\,\langle 1,1,1\rangle = \frac{1}{\sqrt{3}}\left(\frac{\pi}{4}+\pi+1\right) = \frac{1}{\sqrt{3}}\left(\frac{5\pi}{4}+1\right)$.

17. $f(x,y,z) = xe^y + ye^z + ze^x \quad\Rightarrow\quad \nabla f(x,y,z) = \langle e^y + ze^x, xe^y + e^z, ye^z + e^x\rangle$, $\nabla f(0,0,0) = \langle 1,1,1\rangle$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{25+1+4}}\langle 5,1,-2\rangle = \frac{1}{\sqrt{30}}\,\langle 5,1,-2\rangle$, so

$D_{\mathbf{u}}\,f(0,0,0) = \nabla f(0,0,0)\cdot\mathbf{u} = \langle 1,1,1\rangle\cdot\frac{1}{\sqrt{30}}\,\langle 5,1,-2\rangle = \frac{4}{\sqrt{30}}.$

18. $D_{\mathbf{u}}f(2,2) = \nabla f(2,2)\cdot\mathbf{u}$, the scalar projection of $\nabla f(2,2)$ onto $\mathbf{u}$, so we draw a perpendicular from the tip of $\nabla f(2,2)$ to the line containing $\mathbf{u}$. We can use the point $(2,2)$ to determine the scale of the axes, and we estimate the length of the projection to be approximately 3.0 units. Since the angle between $\nabla f(2,2)$ and $\mathbf{u}$ is greater than 90°, the scalar projection is negative. Thus $D_{\mathbf{u}}\,f(2,2) \approx -3$.



19. $f(x,y) = \sqrt{xy} \quad\Rightarrow\quad \nabla f(x,y) = \left\langle\frac{1}{2}(xy)^{-1/2}(y), \frac{1}{2}(xy)^{-1/2}(x)\right\rangle = \left\langle\dfrac{y}{2\sqrt{xy}}, \dfrac{x}{2\sqrt{xy}}\right\rangle$, so $\nabla f(2,8) = \left\langle 1,\frac{1}{4}\right\rangle$.

The unit vector in the direction of $\overrightarrow{PQ} = \langle 5-2, 4-8\rangle = \langle 3,-4\rangle$ is $\mathbf{u} = \left\langle\frac{3}{5},-\frac{4}{5}\right\rangle$, so

$D_{\mathbf{u}}\,f(2,8) = \nabla f(2,8)\cdot\mathbf{u} = \left\langle 1,\frac{1}{4}\right\rangle\cdot\left\langle\frac{3}{5},-\frac{4}{5}\right\rangle = \frac{2}{5}.$

20. $f(x,y,z) = xy^2z^3 \quad\Rightarrow\quad \nabla f(x,y,z) = \left\langle y^2z^3, 2xyz^3, 3xy^2z^2\right\rangle$, so $\nabla f(2,1,1) = \langle 1,4,6\rangle$. The unit vector in the direction of $\overrightarrow{PQ} = \langle -2,-4,4\rangle$ is $\mathbf{u} = \frac{1}{\sqrt{4+16+16}}\langle -2,-4,4\rangle = \frac{1}{6}\,\langle -2,-4,4\rangle$, so

$D_{\mathbf{u}}\,f(2,1,1) = \nabla f(2,1,1)\cdot\mathbf{u} = \langle 1,4,6\rangle\cdot\frac{1}{6}\,\langle -2,-4,4\rangle = \frac{1}{6}\,(-2-16+24) = 1.$

21. $f(x,y) = 4y\sqrt{x} \quad\Rightarrow\quad \nabla f(x,y) = \left\langle 4y\cdot\frac{1}{2}x^{-1/2}, 4\sqrt{x}\right\rangle = \langle 2y/\sqrt{x}, 4\sqrt{x}\,\rangle$.

$\nabla f(4,1) = \langle 1,8\rangle$ is the direction of maximum rate of change, and the maximum rate is $|\nabla f(4,1)| = \sqrt{1+64} = \sqrt{65}$.

22. $f(x,y,z) = \dfrac{x+y}{z} \quad\Rightarrow\quad \nabla f(x,y,z) = \left\langle\dfrac{1}{z},\dfrac{1}{z},-\dfrac{x+y}{z^2}\right\rangle$, $\nabla f(1,1,-1) = \langle -1,-1,-2\rangle$. Thus the maximum rate of change is $|\nabla f(1,1,-1)| = \sqrt{1+1+4} = \sqrt{6}$ in the direction $\langle -1,-1,-2\rangle$.

23. $f(x,y)=\sin(xy) \quad\Rightarrow\quad \nabla f(x,y)=\langle y\cos(xy), x\cos(xy)\rangle$, $\nabla f(1,0)=\langle 0,1\rangle$. Thus the maximum rate of change is $|\nabla f(1,0)|=1$ in the direction $\langle 0,1\rangle$.

24. $f(x,y,z)=x\ln(yz) \quad\Rightarrow\quad \nabla f(x,y,z)=\left\langle \ln(yz), x\cdot\dfrac{z}{yz}, x\cdot\dfrac{y}{yz}\right\rangle=\left\langle \ln(yz),\dfrac{x}{y},\dfrac{x}{z}\right\rangle$, $\nabla f(1,2,\frac{1}{2})=\langle 0,\frac{1}{2},2\rangle$. Thus the maximum rate of change is $\left|\nabla f(1,2,\frac{1}{2})\right|=\sqrt{0+\frac{1}{4}+4}=\sqrt{\frac{17}{4}}=\frac{\sqrt{17}}{2}$ in the direction $\langle 0,\frac{1}{2},2\rangle$ or equivalently $\langle 0,1,4\rangle$.

25. $f(x,y,z)=x/(y+z)=x(y+z)^{-1} \quad\Rightarrow$

$\nabla f(x,y,z)=\langle 1/(y+z), -x(y+z)^{-2}(1), -x(y+z)^{-2}(1)\rangle=\left\langle \dfrac{1}{y+z}, -\dfrac{x}{(y+z)^2}, -\dfrac{x}{(y+z)^2}\right\rangle$,

$\nabla f(8,1,3)=\langle \frac{1}{4}, -\frac{8}{4^2}, -\frac{8}{4^2}\rangle=\langle \frac{1}{4},-\frac{1}{2},-\frac{1}{2}\rangle$. Thus the maximum rate of change is

$|\nabla f(8,1,3)|=\sqrt{\frac{1}{16}+\frac{1}{4}+\frac{1}{4}}=\sqrt{\frac{9}{16}}=\frac{3}{4}$ in the direction $\langle \frac{1}{4},-\frac{1}{2},-\frac{1}{2}\rangle$ or equivalently $\langle 1,-2,-2\rangle$.

26. $f(p,q,r)=\arctan(pqr) \quad\Rightarrow\quad \nabla f(p,q,r)=\left\langle \dfrac{qr}{1+(pqr)^2}, \dfrac{pr}{1+(pqr)^2}, \dfrac{pq}{1+(pqr)^2}\right\rangle$, $\nabla f(1,2,1)=\langle \frac{2}{5},\frac{1}{5},\frac{2}{5}\rangle$. Thus the maximum rate of change is $|\nabla f(1,2,1)|=\sqrt{\frac{4}{25}+\frac{1}{25}+\frac{4}{25}}=\sqrt{\frac{9}{25}}=\frac{3}{5}$ in the direction $\langle \frac{2}{5},\frac{1}{5},\frac{2}{5}\rangle$ or equivalently $\langle 2,1,2\rangle$.

27. (a) As in the proof of Theorem 15, $D_{\mathbf{u}}\,f=|\nabla f|\cos\theta$. Since the minimum value of $\cos\theta$ is -1 occurring when $\theta=\pi$, the minimum value of $D_{\mathbf{u}}\,f$ is $-|\nabla f|$ occurring when $\theta=\pi$, that is when $\mathbf{u}$ is in the opposite direction of ∇f (assuming $\nabla f\neq \mathbf{0}$).

(b) $f(x,y)=x^4y-x^2y^3 \quad\Rightarrow\quad \nabla f(x,y)=\langle 4x^3y-2xy^3, x^4-3x^2y^2\rangle$, so f decreases fastest at the point $(2,-3)$ in the direction $-\nabla f(2,-3)=-\langle 12,-92\rangle=\langle -12,92\rangle$.

28. $f(x,y)=x^2+xy^3 \quad\Rightarrow\quad \nabla f(x,y)=\langle 2x+y^3, 3xy^2\rangle$ so $\nabla f(2,1)=\langle 5,6\rangle$. If $\mathbf{u}=\langle a,b\rangle$ is a unit vector in the desired direction then $D_{\mathbf{u}}f(2,1)=2 \quad\Leftrightarrow\quad \langle 5,6\rangle\cdot\langle a,b\rangle=2 \quad\Leftrightarrow\quad 5a+6b=2 \quad\Leftrightarrow\quad b=\frac{1}{3}-\frac{5}{6}a$. But $a^2+b^2=1 \quad\Leftrightarrow$ $a^2+\left(\frac{1}{3}-\frac{5}{6}a\right)^2=1 \quad\Leftrightarrow\quad \frac{61}{36}a^2-\frac{5}{9}a+\frac{1}{9}=1 \quad\Leftrightarrow\quad 61a^2-20a-32=0$. By the quadratic formula, the solutions are

$$a=\frac{-(-20)\pm\sqrt{(-20)^2-4(61)(-32)}}{2(61)}=\frac{20\pm\sqrt{8208}}{122}=\frac{10\pm 6\sqrt{57}}{61}.$$ If $a=\dfrac{10+6\sqrt{57}}{61}\approx 0.9065$ then

$b=\dfrac{1}{3}-\dfrac{5}{6}\left(\dfrac{10+6\sqrt{57}}{61}\right)\approx -0.4221$, and if $a=\dfrac{10-6\sqrt{57}}{61}\approx -0.5787$ then $b=\dfrac{1}{3}-\dfrac{5}{6}\left(\dfrac{10-6\sqrt{57}}{61}\right)\approx 0.8156$.

Thus the two directions giving a directional derivative of 2 are approximately $\langle 0.9065,-0.4221\rangle$ and $\langle -0.5787, 0.8156\rangle$.

29. The direction of fastest change is $\nabla f(x,y)=(2x-2)\,\mathbf{i}+(2y-4)\,\mathbf{j}$, so we need to find all points (x,y) where $\nabla f(x,y)$ is parallel to $\mathbf{i}+\mathbf{j} \quad\Leftrightarrow\quad (2x-2)\,\mathbf{i}+(2y-4)\,\mathbf{j}=k\,(\mathbf{i}+\mathbf{j}) \quad\Leftrightarrow\quad k=2x-2$ and $k=2y-4$. Then $2x-2=2y-4 \quad\Rightarrow$ $y=x+1$, so the direction of fastest change is $\mathbf{i}+\mathbf{j}$ at all points on the line $y=x+1$.

30. The fisherman is traveling in the direction $\langle -80, -60\rangle$. A unit vector in this direction is $\mathbf{u} = \frac{1}{100}\langle -80, -60\rangle = \left\langle -\frac{4}{5}, -\frac{3}{5}\right\rangle$, and if the depth of the lake is given by $f(x,y) = 200 + 0.02x^2 - 0.001y^3$, then $\nabla f(x,y) = \left\langle 0.04x, -0.003y^2\right\rangle$.

$D_{\mathbf{u}}\, f(80,60) = \nabla f(80,60) \cdot \mathbf{u} = \langle 3.2, -10.8\rangle \cdot \left\langle -\frac{4}{5}, -\frac{3}{5}\right\rangle = 3.92$. Since $D_{\mathbf{u}}\, f(80,60)$ is positive, the depth of the lake is increasing near $(80,60)$ in the direction toward the buoy.

31. $T = \dfrac{k}{\sqrt{x^2+y^2+z^2}}$ and $120 = T(1,2,2) = \dfrac{k}{3}$ so $k = 360$.

(a) $\mathbf{u} = \dfrac{\langle 1,-1,1\rangle}{\sqrt{3}}$,

$$D_{\mathbf{u}}T(1,2,2) = \nabla T(1,2,2)\cdot\mathbf{u} = \left[-360\left(x^2+y^2+z^2\right)^{-3/2}\langle x,y,z\rangle\right]_{(1,2,2)}\cdot\mathbf{u} = -\tfrac{40}{3}\langle 1,2,2\rangle\cdot\tfrac{1}{\sqrt{3}}\langle 1,-1,1\rangle = -\tfrac{40}{3\sqrt{3}}$$

(b) From (a), $\nabla T = -360\left(x^2+y^2+z^2\right)^{-3/2}\langle x,y,z\rangle$, and since $\langle x,y,z\rangle$ is the position vector of the point (x,y,z), the vector $-\langle x,y,z\rangle$, and thus ∇T, always points toward the origin.

32. $\nabla T = -400e^{-x^2-3y^2-9z^2}\langle x, 3y, 9z\rangle$

(a) $\mathbf{u} = \frac{1}{\sqrt{6}}\langle 1,-2,1\rangle$, $\nabla T(2,-1,2) = -400e^{-43}\langle 2,-3,18\rangle$ and

$$D_{\mathbf{u}}\,T(2,-1,2) = \left(-\frac{400e^{-43}}{\sqrt{6}}\right)(26) = -\frac{5200\sqrt{6}}{3e^{43}}\ {}^\circ\mathrm{C/m}.$$

(b) $\nabla T(2,-1,2) = 400e^{-43}\langle -2,3,-18\rangle$ or equivalently $\langle -2,3,-18\rangle$.

(c) $|\nabla T| = 400e^{-x^2-3y^2-9z^2}\sqrt{x^2+9y^2+81z^2}\ {}^\circ\mathrm{C/m}$ is the maximum rate of increase. At $(2,-1,2)$ the maximum rate of increase is $400e^{-43}\sqrt{337}\ {}^\circ\mathrm{C/m}$.

33. $\nabla V(x,y,z) = \langle 10x-3y+yz, xz-3x, xy\rangle,\ \nabla V(3,4,5) = \langle 38,6,12\rangle$

(a) $D_{\mathbf{u}}\,V(3,4,5) = \langle 38,6,12\rangle\cdot\frac{1}{\sqrt{3}}\langle 1,1,-1\rangle = \frac{32}{\sqrt{3}}$

(b) $\nabla V(3,4,5) = \langle 38,6,12\rangle$, or equivalently, $\langle 19,3,6\rangle$.

(c) $|\nabla V(3,4,5)| = \sqrt{38^2+6^2+12^2} = \sqrt{1624} = 2\sqrt{406}$

34. $z = f(x,y) = 1000 - 0.005x^2 - 0.01y^2 \;\Rightarrow\; \nabla f(x,y) = \langle -0.01x, -0.02y\rangle$ and $\nabla f(60,40) = \langle -0.6, -0.8\rangle$.

(a) Due south is in the direction of the unit vector $\mathbf{u} = -\mathbf{j}$ and

$D_{\mathbf{u}}f(60,40) = \nabla f\,(60,40)\cdot\langle 0,-1\rangle = \langle -0.6,-0.8\rangle\cdot\langle 0,-1\rangle = 0.8$. Thus, if you walk due south from $(60,40,966)$ you will ascend at a rate of 0.8 vertical meters per horizontal meter.

(b) Northwest is in the direction of the unit vector $\mathbf{u} = \frac{1}{\sqrt{2}}\langle -1,1\rangle$ and

$D_{\mathbf{u}}f(60,40) = \nabla f\,(60,40)\cdot\frac{1}{\sqrt{2}}\langle -1,1\rangle = \langle -0.6,-0.8\rangle\cdot\frac{1}{\sqrt{2}}\langle -1,1\rangle = -\frac{0.2}{\sqrt{2}} \approx -0.14$. Thus, if you walk northwest from $(60,40,966)$ you will descend at a rate of approximately 0.14 vertical meters per horizontal meter.

(c) $\nabla f(60, 40) = \langle -0.6, -0.8 \rangle$ is the direction of largest slope with a rate of ascent given by

$|\nabla f(60, 40)| = \sqrt{(-0.6)^2 + (-0.8)^2} = 1$. The angle above the horizontal in which the path begins is given by

$\tan\theta = 1 \quad \Rightarrow \quad \theta = 45°$.

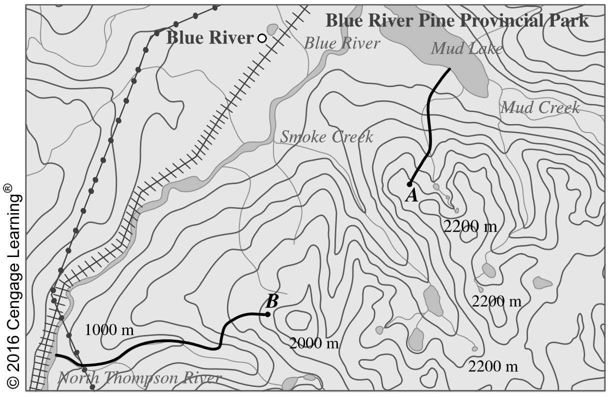
35. A unit vector in the direction of $\overrightarrow{AB}$ is $\mathbf{i}$ and a unit vector in the direction of $\overrightarrow{AC}$ is $\mathbf{j}$. Thus $D_{\overrightarrow{AB}}\, f(1,3) = f_x(1,3) = 3$ and

$D_{\overrightarrow{AC}}\, f(1,3) = f_y(1,3) = 26$. Therefore $\nabla f(1,3) = \langle f_x(1,3), f_y(1,3) \rangle = \langle 3, 26 \rangle$, and by definition,

$D_{\overrightarrow{AD}}\, f(1,3) = \nabla f \cdot \mathbf{u}$ where $\mathbf{u}$ is a unit vector in the direction of $\overrightarrow{AD}$, which is $\left\langle \frac{5}{13}, \frac{12}{13} \right\rangle$. Therefore,

$D_{\overrightarrow{AD}}\, f\,(1,3) = \langle 3, 26 \rangle \cdot \left\langle \frac{5}{13}, \frac{12}{13} \right\rangle = 3 \cdot \frac{5}{13} + 26 \cdot \frac{12}{13} = \frac{327}{13}$.

36. The curves of steepest ascent or descent are perpendicular to all of the contour lines (see Figure 12) so we sketch curves beginning at A and B that head toward lower elevations, crossing each contour line at a right angle.



37. (a) $\nabla(au + bv) = \left\langle \dfrac{\partial(au+bv)}{\partial x}, \dfrac{\partial(au+bv)}{\partial y} \right\rangle = \left\langle a\,\dfrac{\partial u}{\partial x} + b\,\dfrac{\partial v}{\partial x}, a\,\dfrac{\partial u}{\partial y} + b\,\dfrac{\partial v}{\partial y} \right\rangle = a\left\langle \dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y} \right\rangle + b\left\langle \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y} \right\rangle$

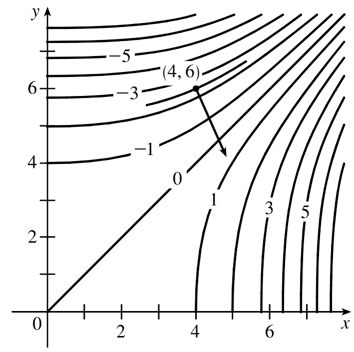
$= a\,\nabla u + b\,\nabla v$

(b) $\nabla(uv) = \left\langle v\,\dfrac{\partial u}{\partial x} + u\,\dfrac{\partial v}{\partial x}, v\,\dfrac{\partial u}{\partial y} + u\,\dfrac{\partial v}{\partial y} \right\rangle = v\left\langle \dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y} \right\rangle + u\left\langle \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y} \right\rangle = v\,\nabla u + u\,\nabla v$

(c) $\nabla\left(\dfrac{u}{v}\right) = \left\langle \dfrac{v\,\dfrac{\partial u}{\partial x} - u\,\dfrac{\partial v}{\partial x}}{v^2}, \dfrac{v\,\dfrac{\partial u}{\partial y} - u\,\dfrac{\partial v}{\partial y}}{v^2} \right\rangle = \dfrac{v\left\langle \dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y} \right\rangle - u\left\langle \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y} \right\rangle}{v^2} = \dfrac{v\,\nabla u - u\,\nabla v}{v^2}$

(d) $\nabla u^n = \left\langle \dfrac{\partial(u^n)}{\partial x}, \dfrac{\partial(u^n)}{\partial y} \right\rangle = \left\langle nu^{n-1}\,\dfrac{\partial u}{\partial x}, nu^{n-1}\,\dfrac{\partial u}{\partial y} \right\rangle = nu^{n-1}\,\nabla u$

38. If we place the initial point of the gradient vector $\nabla f(4,6)$ at $(4,6)$, the vector is perpendicular to the level curve of f that includes $(4,6)$, so we sketch a portion of the level curve through $(4,6)$ (using the nearby level curves as a guideline) and draw a line perpendicular to the curve at $(4,6)$. The gradient vector is parallel to this line, pointing in the direction of increasing function values, and with length equal to the maximum value of the directional derivative of f at $(4,6)$. We can estimate this length by finding the average rate of change in the direction of the gradient. The line intersects the contour lines corresponding to -2 and -3 with an estimated distance of 0.5 units. Thus the rate of change is approximately $\dfrac{-2-(-3)}{0.5} = 2$, and we sketch the gradient vector with length 2.



39. $f(x,y) = x^3 + 5x^2y + y^3 \quad \Rightarrow$

$D_{\mathbf{u}}f(x,y) = \nabla f(x,y) \cdot \mathbf{u} = \left\langle 3x^2 + 10xy, 5x^2 + 3y^2 \right\rangle \cdot \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle = \frac{9}{5}x^2 + 6xy + 4x^2 + \frac{12}{5}y^2 = \frac{29}{5}x^2 + 6xy + \frac{12}{5}y^2$. Then

$$\begin{aligned} D_{\mathbf{u}}^2 f(x,y) &= D_{\mathbf{u}}\left[D_{\mathbf{u}}f(x,y)\right] = \nabla\left[D_{\mathbf{u}}f(x,y)\right] \cdot \mathbf{u} = \left\langle \tfrac{58}{5}x + 6y, 6x + \tfrac{24}{5}y \right\rangle \cdot \left\langle \tfrac{3}{5}, \tfrac{4}{5} \right\rangle \\ &= \tfrac{174}{25}x + \tfrac{18}{5}y + \tfrac{24}{5}x + \tfrac{96}{25}y = \tfrac{294}{25}x + \tfrac{186}{25}y \end{aligned}$$

and $D_{\mathbf{u}}^2 f(2,1) = \frac{294}{25}(2) + \frac{186}{25}(1) = \frac{774}{25}$.

40. (a) From Equation 9 we have $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = \langle f_x, f_y \rangle \cdot \langle a, b \rangle = f_x a + f_y b$ and from Exercise 39 we have

$D_{\mathbf{u}}^2 f = D_{\mathbf{u}}\left[D_{\mathbf{u}}f\right] = \nabla\left[D_{\mathbf{u}}f\right] \cdot \mathbf{u} = \langle f_{xx}a + f_{yx}b, f_{xy}a + f_{yy}b \rangle \cdot \langle a, b \rangle = f_{xx}a^2 + f_{yx}ab + f_{xy}ab + f_{yy}b^2$.

But $f_{yx} = f_{xy}$ by Clairaut's Theorem, so $D_{\mathbf{u}}^2 f = f_{xx}a^2 + 2f_{xy}ab + f_{yy}b^2$.

(b) $f(x,y) = xe^{2y} \quad \Rightarrow \quad f_x = e^{2y}$, $f_y = 2xe^{2y}$, $f_{xx} = 0$, $f_{xy} = 2e^{2y}$, $f_{yy} = 4xe^{2y}$ and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{4^2+6^2}}\langle 4, 6 \rangle = \left\langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right\rangle = \langle a, b \rangle$. Then

$$D_{\mathbf{u}}^2 f = f_{xx}a^2 + 2f_{xy}ab + f_{yy}b^2 = 0 \cdot \left(\tfrac{2}{\sqrt{13}}\right)^2 + 2 \cdot 2e^{2y}\left(\tfrac{2}{\sqrt{13}}\right)\left(\tfrac{3}{\sqrt{13}}\right) + 4xe^{2y}\left(\tfrac{3}{\sqrt{13}}\right)^2 = \tfrac{24}{13}e^{2y} + \tfrac{36}{13}xe^{2y}.$$

41. Let $F(x,y,z) = 2(x-2)^2 + (y-1)^2 + (z-3)^2$. Then $2(x-2)^2 + (y-1)^2 + (z-3)^2 = 10$ is a level surface of F.

$F_x(x,y,z) = 4(x-2) \quad \Rightarrow \quad F_x(3,3,5) = 4$, $F_y(x,y,z) = 2(y-1) \quad \Rightarrow \quad F_y(3,3,5) = 4$, and

$F_z(x,y,z) = 2(z-3) \quad \Rightarrow \quad F_z(3,3,5) = 4$.

(a) Equation 19 gives an equation of the tangent plane at $(3,3,5)$ as $4(x-3) + 4(y-3) + 4(z-5) = 0 \quad \Leftrightarrow$

$4x + 4y + 4z = 44$ or equivalently $x + y + z = 11$.

(b) By Equation 20, the normal line has symmetric equations $\dfrac{x-3}{4} = \dfrac{y-3}{4} = \dfrac{z-5}{4}$ or equivalently

$x - 3 = y - 3 = z - 5$. Corresponding parametric equations are $x = 3 + t$, $y = 3 + t$, $z = 5 + t$.

42. Let $F(x, y, z) = y^2 + z^2 - x$. Then $x = y^2 + z^2 + 1 \quad \Leftrightarrow \quad y^2 + z^2 - x = -1$ is a level surface of F.

$F_x(x, y, z) = -1 \quad \Rightarrow \quad F_x(3, 1, -1) = -1, \quad F_y(x, y, z) = 2y \quad \Rightarrow \quad F_y(3, 1, -1) = 2,$ and $F_z(x, y, z) = 2z \quad \Rightarrow$ $F_z(3, 1, -1) = -2$.

(a) By Equation 19, an equation of the tangent plane at $(3, 1, -1)$ is $(-1)(x - 3) + 2(y - 1) + (-2)[z - (-1)] = 0$ or $-x + 2y - 2z = 1$ or $x - 2y + 2z = -1$.

(b) By Equation 20, the normal line has symmetric equations $\dfrac{x - 3}{-1} = \dfrac{y - 1}{2} = \dfrac{z - (-1)}{-2}$ or equivalently

$x - 3 = \dfrac{y - 1}{-2} = \dfrac{z + 1}{2}$ and parametric equations $x = 3 - t$, $y = 1 + 2t$, $z = -1 - 2t$.

43. Let $F(x, y, z) = xy^2z^3$. Then $xy^2z^3 = 8$ is a level surface of F and $\nabla F(x, y, z) = \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle$.

(a) $\nabla F(2, 2, 1) = \langle 4, 8, 24 \rangle$ is a normal vector for the tangent plane at $(2, 2, 1)$, so an equation of the tangent plane is $4(x - 2) + 8(y - 2) + 24(z - 1) = 0$ or $4x + 8y + 24z = 48$ or equivalently $x + 2y + 6z = 12$.

(b) The normal line has direction $\nabla F(2, 2, 1) = \langle 4, 8, 24 \rangle$ or equivalently $\langle 1, 2, 6 \rangle$, so parametric equations are $x = 2 + t$, $y = 2 + 2t$, $z = 1 + 6t$, and symmetric equations are $x - 2 = \dfrac{y - 2}{2} = \dfrac{z - 1}{6}$.

44. Let $F(x, y, z) = x^2 - z^2 - y$. Then $y = x^2 - z^2 \quad \Leftrightarrow \quad x^2 - z^2 - y = 0$ is a level surface of F. $F_x(x, y, z) = 2x \quad \Rightarrow$ $F_x(4, 7, 3) = 8$, $F_y(x, y, z) = -1 \quad \Rightarrow \quad F_y(4, 7, 3) = -1$, and $F_z(x, y, z) = -2z \quad \Rightarrow \quad F_z(4, 7, 3) = -6$.

(a) An equation of the tangent plane at $(4, 7, 3)$ is $8(x - 4) - 1(y - 7) - 6(z - 3) = 0$ or $8x - y - 6z = 7$.

(b) The normal line has symmetric equations $\dfrac{x - 4}{8} = \dfrac{y - 7}{-1} = \dfrac{z - 3}{-6}$ and parametric equations $x = 4 + 8t$, $y = 7 - t$, $z = 3 - 6t$.

45. Let $F(x, y, z) = xyz^2$. Then $xyz^2 = 6$ is a level surface of F and $\nabla F(x, y, z) = \langle yz^2, xz^2, 2xyz \rangle$.

(a) $\nabla F(3, 2, 1) = \langle 2, 3, 12 \rangle$ is a normal vector for the tangent plane at $(3, 2, 1)$, so an equation of the tangent plane is $2(x - 3) + 3(y - 2) + 12(z - 1) = 0$ or $2x + 3y + 12z = 24$.

(b) The normal line has direction $\langle 2, 3, 12 \rangle$, so parametric equations are $x = 3 + 2t$, $y = 2 + 3t$, $z = 1 + 12t$, and symmetric equations are $\dfrac{x - 3}{2} = \dfrac{y - 2}{3} = \dfrac{z - 1}{12}$.

46. Let $F(x, y, z) = x^4 + y^4 + z^4 - 3x^2y^2z^2$. Then $x^4 + y^4 + z^4 = 3x^2y^2z^2$ is the level surface $F(x, y, z) = 0$, and $\nabla F(x, y, z) = \langle 4x^3 - 6xy^2z^2, 4y^3 - 6x^2yz^2, 4z^3 - 6x^2y^2z \rangle$.

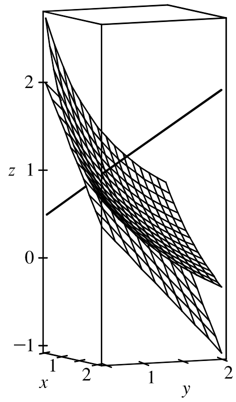
(a) $\nabla F(1, 1, 1) = \langle -2, -2, -2 \rangle$ or equivalently $\langle 1, 1, 1 \rangle$ is a normal vector for the tangent plane at $(1, 1, 1)$, so an equation of the tangent plane is $1(x - 1) + 1(y - 1) + 1(z - 1) = 0$ or $x + y + z = 3$.

(b) The normal line has direction $\langle 1, 1, 1 \rangle$, so parametric equations are $x = 1 + t$, $y = 1 + t$, $z = 1 + t$, and symmetric equations are $x - 1 = y - 1 = z - 1$ or equivalently $x = y = z$.

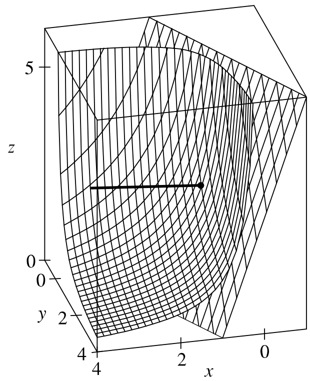
47. $F(x,y,z) = xy + yz + zx$,

$\nabla F(x,y,z) = \langle y+z, x+z, y+x \rangle$,

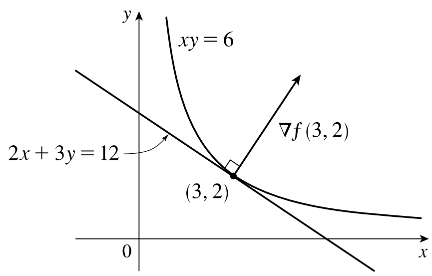
$\nabla F(1,1,1) = \langle 2,2,2 \rangle$, so an equation of the tangent plane is $2x + 2y + 2z = 6$ or $x + y + z = 3$, and the normal line is given by $x - 1 = y - 1 = z - 1$ or $x = y = z$. To graph the surface we solve for z: $z = \dfrac{3 - xy}{x + y}$.



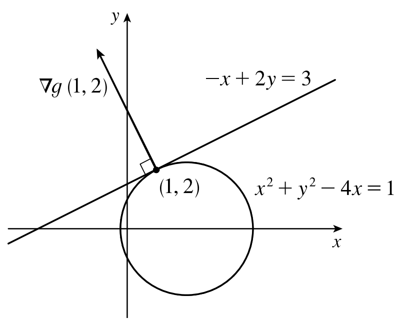
48. $F(x,y,z) = xyz$, $\nabla F(x,y,z) = \langle yz, xz, yx \rangle$, $\nabla F(1,2,3) = \langle 6,3,2 \rangle$, so an equation of the tangent plane is $6x + 3y + 2z = 18$, and the normal line is given by $\dfrac{x-1}{6} = \dfrac{y-2}{3} = \dfrac{z-3}{2}$ or $x = 1 + 6t$, $y = 2 + 3t$, $z = 3 + 2t$. To graph the surface we solve for z: $z = \dfrac{6}{xy}$.



49. $f(x,y) = xy \quad \Rightarrow \quad \nabla f(x,y) = \langle y, x \rangle$, $\nabla f(3,2) = \langle 2,3 \rangle$. $\nabla f(3,2)$ is perpendicular to the tangent line, so the tangent line has equation $\nabla f(3,2) \cdot \langle x-3, y-2 \rangle = 0 \quad \Rightarrow \quad \langle 2,3 \rangle \cdot \langle x-3, x-2 \rangle = 0 \quad \Rightarrow$ $2(x-3) + 3(y-2) = 0$ or $2x + 3y = 12$.



50. $g(x,y) = x^2 + y^2 - 4x \quad \Rightarrow \quad \nabla g(x,y) = \langle 2x-4, 2y \rangle$, $\nabla g(1,2) = \langle -2, 4 \rangle$. $\nabla g(1,2)$ is perpendicular to the tangent line, so the tangent line has equation $\nabla g(1,2) \cdot \langle x-1, y-2 \rangle = 0 \quad \Rightarrow$ $\langle -2, 4 \rangle \cdot \langle x-1, y-2 \rangle = 0 \quad \Rightarrow \quad -2(x-1) + 4(y-2) = 0 \quad \Leftrightarrow$ $-2x + 4y = 6$ or equivalently $-x + 2y = 3$.



51. $\nabla F(x_0,y_0,z_0) = \left\langle \dfrac{2x_0}{a^2}, \dfrac{2y_0}{b^2}, \dfrac{2z_0}{c^2} \right\rangle$. Thus an equation of the tangent plane at (x_0,y_0,z_0) is

$\dfrac{2x_0}{a^2}\,x + \dfrac{2y_0}{b^2}\,y + \dfrac{2z_0}{c^2}\,z = 2\left(\dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2} + \dfrac{z_0^2}{c^2}\right) = 2(1) = 2$ since (x_0,y_0,z_0) is a point on the ellipsoid. Hence

$\dfrac{x_0}{a^2}\,x + \dfrac{y_0}{b^2}\,y + \dfrac{z_0}{c^2}\,z = 1$ is an equation of the tangent plane.

52. $\nabla F(x_0, y_0, z_0) = \left\langle \dfrac{2x_0}{a^2}, \dfrac{2y_0}{b^2}, \dfrac{-2z_0}{c^2} \right\rangle$, so an equation of the tangent plane at (x_0, y_0, z_0) is

$$\frac{2x_0}{a^2}\,x + \frac{2y_0}{b^2}\,y - \frac{2z_0}{c^2}\,z = 2\left(\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} - \frac{z_0^2}{c^2}\right) = 2 \text{ or } \frac{x_0}{a^2}\,x + \frac{y_0}{b^2}\,y - \frac{z_0}{c^2}\,z = 1.$$

53. $\nabla F(x_0, y_0, z_0) = \left\langle \dfrac{2x_0}{a^2}, \dfrac{2y_0}{b^2}, \dfrac{-1}{c} \right\rangle$, so an equation of the tangent plane is $\dfrac{2x_0}{a^2}\,x + \dfrac{2y_0}{b^2}\,y - \dfrac{1}{c}\,z = \dfrac{2x_0^2}{a^2} + \dfrac{2y_0^2}{b^2} - \dfrac{z_0}{c}$

or $\dfrac{2x_0}{a^2}\,x + \dfrac{2y_0}{b^2}\,y = \dfrac{z}{c} + 2\left(\dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2}\right) - \dfrac{z_0}{c}$. But $\dfrac{z_0}{c} = \dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2}$, so the equation can be written as

$$\frac{2x_0}{a^2}\,x + \frac{2y_0}{b^2}\,y = \frac{z + z_0}{c}.$$

54. Let $F(x, y, z) = x^2 + y^2 + 2z^2$; then the ellipsoid $x^2 + y^2 + 2z^2 = 1$ is a level surface of F. $\nabla F(x, y, z) = \langle 2x, 2y, 4z \rangle$ is a normal vector to the surface at (x, y, z) and so it is a normal vector for the tangent plane there. The tangent plane is parallel to the plane $x + 2y + z = 1$ when the normal vectors of the planes are parallel, so we need a point (x_0, y_0, z_0) on the ellipsoid where $\langle 2x_0, 2y_0, 4z_0 \rangle = k\,\langle 1, 2, 1 \rangle$ for some $k \neq 0$. Comparing components we have $2x_0 = k \;\Rightarrow\; x_0 = k/2$, $2y_0 = 2k \;\Rightarrow\; y_0 = k$, $4z_0 = k \;\Rightarrow\; z_0 = k/4$. $(x_0, y_0, z_0) = (k/2, k, k/4)$ lies on the ellipsoid, so $(k/2)^2 + k^2 + 2(k/4)^2 = 1 \;\Rightarrow\; \frac{11}{8}k^2 = 1 \;\Rightarrow\; k^2 = \frac{8}{11} \;\Rightarrow\; k = \pm 2\sqrt{\frac{2}{11}}$. Thus the tangent planes at the points $\left(\sqrt{\frac{2}{11}}, 2\sqrt{\frac{2}{11}}, \frac{1}{2}\sqrt{\frac{2}{11}}\right)$ and $\left(-\sqrt{\frac{2}{11}}, -2\sqrt{\frac{2}{11}}, -\frac{1}{2}\sqrt{\frac{2}{11}}\right)$ are parallel to the given plane.

55. The hyperboloid $x^2 - y^2 - z^2 = 1$ is a level surface of $F(x, y, z) = x^2 - y^2 - z^2$ and $\nabla F\,(x, y, z) = \langle 2x, -2y, -2z \rangle$ is a normal vector to the surface and hence a normal vector for the tangent plane at (x, y, z). The tangent plane is parallel to the plane $z = x + y$ or $x + y - z = 0$ if and only if the corresponding normal vectors are parallel, so we need a point (x_0, y_0, z_0) on the hyperboloid where $\langle 2x_0, -2y_0, -2z_0 \rangle = c\,\langle 1, 1, -1 \rangle$ or equivalently $\langle x_0, -y_0, -z_0 \rangle = k\,\langle 1, 1, -1 \rangle$ for some $k \neq 0$. Then we must have $x_0 = k$, $y_0 = -k$, $z_0 = k$ and substituting into the equation of the hyperboloid gives $k^2 - (-k)^2 - k^2 = 1 \;\Leftrightarrow\; -k^2 = 1$, an impossibility. Thus there is no such point on the hyperboloid.

56. First note that the point $(1, 1, 2)$ is on both surfaces. The ellipsoid is a level surface of $F(x, y, z) = 3x^2 + 2y^2 + z^2$ and $\nabla F(x, y, z) = \langle 6x, 4y, 2z \rangle$. A normal vector to the surface at $(1, 1, 2)$ is $\nabla F(1, 1, 2) = \langle 6, 4, 4 \rangle$ and an equation of the tangent plane there is $6(x - 1) + 4(y - 1) + 4(z - 2) = 0$ or $6x + 4y + 4z = 18$ or $3x + 2y + 2z = 9$. The sphere is a level surface of $G(x, y, z) = x^2 + y^2 + z^2 - 8x - 6y - 8z + 24$ and $\nabla G(x, y, z) = \langle 2x - 8, 2y - 6, 2z - 8 \rangle$. A normal vector to the sphere at $(1, 1, 2)$ is $\nabla G(1, 1, 2) = \langle -6, -4, -4 \rangle$ and the tangent plane there is $-6(x - 1) - 4(y - 1) - 4(z - 2) = 0$ or $3x + 2y + 2z = 9$. Since these tangent planes are identical, the surfaces are tangent to each other at the point $(1, 1, 2)$.

57. Let (x_0, y_0, z_0) be a point on the cone [other than $(0, 0, 0)$]. The cone is a level surface of $F(x, y, z) = x^2 + y^2 - z^2$ and $\nabla F(x, y, z) = \langle 2x, 2y, -2z \rangle$, so $\nabla F(x_0, y_0, z_0) = \langle 2x_0, 2y_0, -2z_0 \rangle$ is a normal vector to the cone at this point and an equation of the tangent plane there is $2x_0\,(x-x_0)+2y_0\,(y-y_0)-2z_0\,(z-z_0)=0$ or $x_0x+y_0y-z_0z=x_0^2+y_0^2-z_0^2$. But $x_0^2+y_0^2=z_0^2$ so the tangent plane is given by $x_0x+y_0y-z_0z=0$, a plane which always contains the origin.

58. Let (x_0,y_0,z_0) be a point on the sphere. Then the normal line is given by $\dfrac{x-x_0}{2x_0}=\dfrac{y-y_0}{2y_0}=\dfrac{z-z_0}{2z_0}$. For the center $(0,0,0)$ to be on the line, we need $-\dfrac{x_0}{2x_0}=-\dfrac{y_0}{2y_0}=-\dfrac{z_0}{2z_0}$ or equivalently $1=1=1$, which is true.

59. Let $F(x,y,z)=x^2+y^2-z$. Then the paraboloid is the level surface $F(x,y,z)=0$ and $\nabla F(x,y,z)=\langle 2x,2y,-1\rangle$, so $\nabla F(1,1,2)=\langle 2,2,-1\rangle$ is a normal vector to the surface. Thus the normal line at $(1,1,2)$ is given by $x=1+2t$, $y=1+2t$, $z=2-t$. Substitution into the equation of the paraboloid $z=x^2+y^2$ gives $2-t=(1+2t)^2+(1+2t)^2 \;\Leftrightarrow\; 2-t=2+8t+8t^2 \;\Leftrightarrow\; 8t^2+9t=0 \;\Leftrightarrow\; t(8t+9)=0$. Thus the line intersects the paraboloid when $t=0$, corresponding to the given point $(1,1,2)$, or when $t=-\frac{9}{8}$, corresponding to the point $\left(-\frac{5}{4},-\frac{5}{4},\frac{25}{8}\right)$.

60. The ellipsoid is a level surface of $F(x,y,z)=4x^2+y^2+4z^2$ and $\nabla F(x,y,z)=\langle 8x,2y,8z\rangle$, so $\nabla F(1,2,1)=\langle 8,4,8\rangle$ or equivalently $\langle 2,1,2\rangle$ is a normal vector to the surface. Thus the normal line to the ellipsoid at $(1,2,1)$ is given by $x=1+2t$, $y=2+t$, $z=1+2t$. Substitution into the equation of the sphere gives $(1+2t)^2+(2+t)^2+(1+2t)^2=102 \;\Leftrightarrow\; 6+12t+9t^2=102 \;\Leftrightarrow\; 9t^2+12t-96=0 \;\Leftrightarrow\; 3(t+4)(3t-8)=0$. Thus the line intersects the sphere when $t=-4$, corresponding to the point $(-7,-2,-7)$, and when $t=\frac{8}{3}$, corresponding to the point $\left(\frac{19}{3},\frac{14}{3},\frac{19}{3}\right)$.

61. Let (x_0,y_0,z_0) be a point on the surface. Then an equation of the tangent plane at the point is $\dfrac{x}{2\sqrt{x_0}}+\dfrac{y}{2\sqrt{y_0}}+\dfrac{z}{2\sqrt{z_0}}=\dfrac{\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0}}{2}$. But $\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0}=\sqrt{c}$, so the equation is $\dfrac{x}{\sqrt{x_0}}+\dfrac{y}{\sqrt{y_0}}+\dfrac{z}{\sqrt{z_0}}=\sqrt{c}$. The x-, y-, and z-intercepts are $\sqrt{cx_0}$, $\sqrt{cy_0}$ and $\sqrt{cz_0}$ respectively. (The x-intercept is found by setting $y=z=0$ and solving the resulting equation for x, and the y- and z-intercepts are found similarly.) So the sum of the intercepts is $\sqrt{c}\left(\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0}\right)=c$, a constant.

62. The surface $xyz=1$ is a level surface of $F(x,y,z)=xyz$ and $\nabla F(x,y,z)=\langle yz,xz,xy\rangle$ is normal to the surface, so a normal vector for the tangent plane to the surface at (x_0,y_0,z_0) is $\langle y_0z_0,x_0z_0,x_0y_0\rangle$. An equation for the tangent plane there is $y_0z_0(x-x_0)+x_0z_0(y-y_0)+x_0y_0(z-z_0)=0 \;\Rightarrow\; y_0z_0x+x_0z_0y+x_0y_0z=3x_0y_0z_0$ or $\dfrac{x}{x_0}+\dfrac{y}{y_0}+\dfrac{z}{z_0}=3$. If (x_0,y_0,z_0) is in the first octant, then the tangent plane cuts off a pyramid in the first octant with vertices $(0,0,0)$, $(3x_0,0,0)$, $(0,3y_0,0)$, $(0,0,3z_0)$. The base in the xy-plane is a triangle with area $\frac{1}{2}\,(3x_0)\,(3y_0)$ and the height (along the z-axis) of the pyramid is $3z_0$. The volume of the pyramid for any point (x_0,y_0,z_0) on the surface $xyz=1$ in the first octant is $\frac{1}{3}\,(\text{base})\,(\text{height})=\frac{1}{3}\cdot\frac{1}{2}\,(3x_0)\,(3y_0)\cdot 3z_0=\frac{9}{2}x_0y_0z_0=\frac{9}{2}$ since $x_0y_0z_0=1$.

63. If $f(x,y,z) = z - x^2 - y^2$ and $g(x,y,z) = 4x^2 + y^2 + z^2$, then the tangent line is perpendicular to both ∇f and ∇g at $(-1,1,2)$. The vector $\mathbf{v} = \nabla f \times \nabla g$ will therefore be parallel to the tangent line.

We have $\nabla f(x,y,z) = \langle -2x, -2y, 1\rangle \quad\Rightarrow\quad \nabla f(-1,1,2) = \langle 2,-2,1\rangle$, and $\nabla g(x,y,z) = \langle 8x, 2y, 2z\rangle \quad\Rightarrow$

$\nabla g(-1,1,2) = \langle -8,2,4\rangle$. Hence $\mathbf{v} = \nabla f \times \nabla g = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & 1 \\ -8 & 2 & 4 \end{vmatrix} = -10\,\mathbf{i} - 16\,\mathbf{j} - 12\,\mathbf{k}$.

Parametric equations are: $x = -1 - 10t,\ y = 1 - 16t,\ z = 2 - 12t$.

64. (a) Let $f(x,y,z) = y + z$ and $g(x,y,z) = x^2 + y^2$. Then the required tangent line is perpendicular to both ∇f and ∇g at $(1,2,1)$ and the vector $\mathbf{v} = \nabla f \times \nabla g$ is parallel to the tangent line. We have

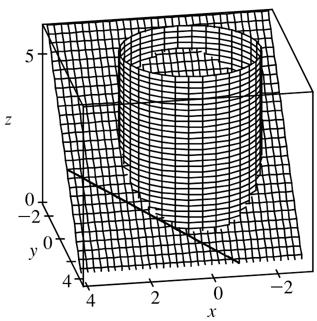
$\nabla f(x,y,z) = \langle 0,1,1\rangle \quad\Rightarrow\quad \nabla f(1,2,1) = \langle 0,1,1\rangle$, and

$\nabla g(x,y,z) = \langle 2x, 2y, 0\rangle \quad\Rightarrow\quad \nabla g(1,2,1) = \langle 2,4,0\rangle$. Hence

$\mathbf{v} = \nabla f \times \nabla g = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 1 \\ 2 & 4 & 0 \end{vmatrix} = -4\,\mathbf{i} + 2\,\mathbf{j} - 2\,\mathbf{k}$. So parametric equations

of the desired tangent line are $x = 1 - 4t,\ y = 2 + 2t,\ z = 1 - 2t$.

(b)



65. Parametric equations for the helix are $x = \cos \pi t,\ \ y = \sin \pi t,\ \ z = t$, and substituting into the equation of the paraboloid gives $t = \cos^2 \pi t + \sin^2 \pi t \quad\Rightarrow\quad t = 1$. Thus the helix intersects the surface at the point $(\cos \pi, \sin \pi, 1) = (-1,0,1)$. Here $\mathbf{r}'(t) = \langle -\pi \sin \pi t, \pi \cos \pi t, 1\rangle$, so the tangent vector to the helix at that point is $\mathbf{r}'(1) = \langle -\pi \sin \pi, \pi \cos \pi, 1\rangle = \langle 0, -\pi, 1\rangle$. The paraboloid $z = x^2 + y^2 \quad\Leftrightarrow\quad x^2 + y^2 - z = 0$ is a level surface of $F(x,y,z) = x^2 + y^2 - z$ and $\nabla F(x,y,z) = \langle 2x, 2y, -1\rangle$, so a normal vector to the tangent plane at $(-1,0,1)$ is $\nabla F(-1,0,1) = \langle -2,0,-1\rangle$. The angle θ between $\mathbf{r}'(1)$ and $\nabla F(-1,0,1)$ is given by

$$\cos\theta = \frac{\langle 0,-\pi,1\rangle \cdot \langle -2,0,-1\rangle}{|\langle 0,-\pi,1\rangle|\,|\langle -2,0,-1\rangle|} = \frac{0+0-1}{\sqrt{0+\pi^2+1}\,\sqrt{4+0+1}} = \frac{-1}{\sqrt{5(\pi^2+1)}} \quad\Rightarrow$$

$\theta = \cos^{-1}\dfrac{-1}{\sqrt{5(\pi^2+1)}} \approx 97.8^\circ$. Because $\nabla F(-1,0,1)$ is perpendicular to the tangent plane, the angle of intersection between the helix and the paraboloid is approximately $97.8^\circ - 90^\circ = 7.8^\circ$.

66. Parametric equations for the helix are $x = \cos(\pi t/2),\ \ y = \sin(\pi t/2),\ \ z = t$, and substituting into the equation of the sphere gives $\cos^2(\pi t/2) + \sin^2(\pi t/2) + t^2 = 2 \quad\Rightarrow\quad 1 + t^2 = 2 \quad\Rightarrow\quad t = \pm 1$. Thus the helix intersects the sphere at two points: $(\cos(\pi/2), \sin(\pi/2), 1) = (0,1,1)$, when $t = 1$, and $(\cos(-\pi/2), \sin(-\pi/2), -1) = (0,-1,-1)$, when $t = -1$. Here $\mathbf{r}'(t) = \left\langle -\frac{\pi}{2}\sin(\pi t/2), \frac{\pi}{2}\cos(\pi t/2), 1\right\rangle$, so the tangent vector to the helix at $(0,1,1)$ is $\mathbf{r}'(1) = \langle -\pi/2, 0, 1\rangle$. The sphere $x^2 + y^2 + z^2 = 2$ is a level surface of $F(x,y,z) = x^2 + y^2 + z^2$ and $\nabla F(x,y,z) = \langle 2x, 2y, 2z\rangle$, so a normal vector to the tangent plane at $(0,1,1)$ is $\nabla F(0,1,1) = \langle 0,2,2 \rangle$. As in Exercise 65, the angle of intersection between the helix and the sphere is the angle between the tangent vector to the helix and the tangent plane to the sphere. The angle θ between $\mathbf{r}'(1)$ and $\nabla F(0,1,1)$ is given by

$$\cos\theta = \frac{\langle -\pi/2,0,1\rangle \cdot \langle 0,2,2\rangle}{|\langle -\pi/2,0,1\rangle|\,|\langle 0,2,2\rangle|} = \frac{2}{\sqrt{(\pi^2/4)+1}\,\sqrt{8}} = \frac{2}{\sqrt{2\pi^2+8}} \quad\Rightarrow\quad \theta = \cos^{-1}\frac{2}{\sqrt{2\pi^2+8}} \approx 67.7^\circ$$

Because $\nabla F(0,1,1)$ is perpendicular to the tangent plane, the angle between $\mathbf{r}'(1)$ and the tangent plane is approximately $90^\circ - 67.7^\circ = 22.3^\circ$.

At $(0,-1,-1)$, $\mathbf{r}'(-1) = \langle \pi/2,0,1\rangle$ and $\nabla F(0,-1,-1) = \langle 0,-2,-2\rangle$, and the angle ϕ between these vectors is given by $\cos\phi = \dfrac{\langle \pi/2,0,1\rangle \cdot \langle 0,-2,-2\rangle}{|\langle \pi/2,0,1\rangle|\,|\langle 0,-2,-2\rangle|} = \dfrac{-2}{\sqrt{2\pi^2+8}} \quad\Rightarrow\quad \phi = \cos^{-1}\dfrac{-2}{\sqrt{2\pi^2+8}} \approx 112.3^\circ$. Thus the angle between the helix and the sphere at $(0,-1,-1)$ is approximately $112.3^\circ - 90^\circ = 22.3^\circ$. (By symmetry, we would expect the angles to be identical.)

67. (a) The direction of the normal line of F is given by ∇F, and that of G by ∇G. Assuming that $\nabla F \neq 0 \neq \nabla G$, the two normal lines are perpendicular at P if $\nabla F \cdot \nabla G = 0$ at P $\Leftrightarrow$ $\langle \partial F/\partial x, \partial F/\partial y, \partial F/\partial z\rangle \cdot \langle \partial G/\partial x, \partial G/\partial y, \partial G/\partial z\rangle = 0$ at P $\Leftrightarrow$ $F_xG_x + F_yG_y + F_zG_z = 0$ at P.

(b) Here $F = x^2+y^2-z^2$ and $G = x^2+y^2+z^2-r^2$, so

$\nabla F \cdot \nabla G = \langle 2x,2y,-2z\rangle \cdot \langle 2x,2y,2z\rangle = 4x^2+4y^2-4z^2 = 4F = 0$, since the point (x,y,z) lies on the graph of $F=0$. To see that this is true without using calculus, note that $G=0$ is the equation of a sphere centered at the origin and $F=0$ is the equation of a right circular cone with vertex at the origin (which is generated by lines through the origin). At any point of intersection, the sphere's normal line (which passes through the origin) lies on the cone, and thus is perpendicular to the cone's normal line. So the surfaces with equations $F=0$ and $G=0$ are everywhere orthogonal.

68. (a) The function $f(x,y) = (xy)^{1/3}$ is continuous on $\mathbb{R}^2$ since it is a composition of a polynomial and the cube root function, both of which are continuous. (See the text just after Example 14.2.8.)

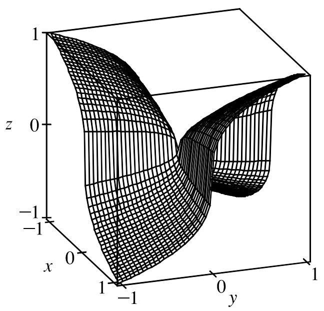
$$f_x(0,0) = \lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{(h\cdot 0)^{1/3}-0}{h} = 0,$$

$$f_y(0,0) = \lim_{h\to 0}\frac{f(0,0+h)-f(0,0)}{h} = \lim_{h\to 0}\frac{(0\cdot h)^{1/3}-0}{h} = 0.$$

Therefore, $f_x(0,0)$ and $f_y(0,0)$ do exist and are equal to 0. Now let $\mathbf{u}$ be any unit vector other than $\mathbf{i}$ and $\mathbf{j}$ (these correspond to f_x and f_y respectively.) Then $\mathbf{u} = a\,\mathbf{i} + b\,\mathbf{j}$ where $a \neq 0$ and $b \neq 0$. Thus

$D_{\mathbf{u}}\,f(0,0) = \displaystyle\lim_{h\to 0}\frac{f(0+ha,0+hb)-f(0,0)}{h} = \lim_{h\to 0}\frac{\sqrt[3]{(ha)(hb)}}{h} = \lim_{h\to 0}\frac{\sqrt[3]{ab}}{h^{1/3}}$ and this limit does not exist, so $D_{\mathbf{u}}\,f(0,0)$ does not exist.

(b)



Notice that if we start at the origin and proceed in the direction of the x- or y-axis, then the graph is flat. But if we proceed in any other direction, then the graph is extremely steep.

69. Let $\mathbf{u} = \langle a, b \rangle$ and $\mathbf{v} = \langle c, d \rangle$. Then we know that at the given point, $D_{\mathbf{u}}\, f = \nabla f \cdot \mathbf{u} = af_x + bf_y$ and $D_{\mathbf{v}}\, f = \nabla f \cdot \mathbf{v} = cf_x + df_y$. But these are just two linear equations in the two unknowns f_x and f_y, and since $\mathbf{u}$ and $\mathbf{v}$ are not parallel, we can solve the equations to find $\nabla f = \langle f_x, f_y \rangle$ at the given point. In fact,

$$\nabla f = \left\langle \frac{d\,D_{\mathbf{u}}\, f - b\,D_{\mathbf{v}}\, f}{ad - bc}, \frac{a\,D_{\mathbf{v}}\, f - c\,D_{\mathbf{u}}\, f}{ad - bc} \right\rangle.$$

70. Since $z = f(x, y)$ is differentiable at $\mathbf{x}_0 = (x_0, y_0)$, by Definition 14.4.7 we have

$\Delta z = f_x(x_0, y_0)\,\Delta x + f_y(x_0, y_0)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$ where $\varepsilon_1, \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$. Now $\Delta z = f(\mathbf{x}) - f(\mathbf{x}_0)$, $\langle \Delta x, \Delta y \rangle = \mathbf{x} - \mathbf{x}_0$ so $(\Delta x, \Delta y) \to (0, 0)$ is equivalent to $\mathbf{x} \to \mathbf{x}_0$ and $\langle f_x(x_0, y_0)\,, f_y(x_0, y_0) \rangle = \nabla f(\mathbf{x}_0)$. Substituting into 14.4.7 gives $f(\mathbf{x}) - f(\mathbf{x}_0) = \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) + \langle \varepsilon_1, \varepsilon_2 \rangle \cdot \langle \Delta x, \Delta y \rangle$ or $\langle \varepsilon_1, \varepsilon_2 \rangle \cdot (\mathbf{x} - \mathbf{x}_0) = f(\mathbf{x}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)$,

and so $\dfrac{f(\mathbf{x}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)}{|\mathbf{x} - \mathbf{x}_0|} = \dfrac{\langle \varepsilon_1, \varepsilon_2 \rangle \cdot (\mathbf{x} - \mathbf{x}_0)}{|\mathbf{x} - \mathbf{x}_0|}$. But $\dfrac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|}$ is a unit vector so

$\displaystyle\lim_{\mathbf{x} \to \mathbf{x}_0} \frac{\langle \varepsilon_1, \varepsilon_2 \rangle \cdot (\mathbf{x} - \mathbf{x}_0)}{|\mathbf{x} - \mathbf{x}_0|} = 0$ since $\varepsilon_1, \varepsilon_2 \to 0$ as $\mathbf{x} \to \mathbf{x}_0$. Hence $\displaystyle\lim_{\mathbf{x} \to \mathbf{x}_0} \frac{f(\mathbf{x}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)}{|\mathbf{x} - \mathbf{x}_0|} = 0.$

14.7 Maximum and Minimum Values

1. (a) First we compute $D(1, 1) = f_{xx}(1, 1)\, f_{yy}(1, 1) - [f_{xy}(1, 1)]^2 = (4)(2) - (1)^2 = 7$. Since $D(1, 1) > 0$ and $f_{xx}(1, 1) > 0$, f has a local minimum at $(1, 1)$ by the Second Derivatives Test.

(b) $D(1, 1) = f_{xx}(1, 1)\, f_{yy}(1, 1) - [f_{xy}(1, 1)]^2 = (4)(2) - (3)^2 = -1$. Since $D(1, 1) < 0$, f has a saddle point at $(1, 1)$ by the Second Derivatives Test.

2. (a) $D = g_{xx}(0, 2)\, g_{yy}(0, 2) - [g_{xy}(0, 2)]^2 = (-1)(1) - (6)^2 = -37$. Since $D < 0$, g has a saddle point at $(0, 2)$ by the Second Derivatives Test.

(b) $D = g_{xx}(0, 2)\, g_{yy}(0, 2) - [g_{xy}(0, 2)]^2 = (-1)(-8) - (2)^2 = 4$. Since $D > 0$ and $g_{xx}(0, 2) < 0$, g has a local maximum at $(0, 2)$ by the Second Derivatives Test.

(c) $D = g_{xx}(0, 2)\, g_{yy}(0, 2) - [g_{xy}(0, 2)]^2 = (4)(9) - (6)^2 = 0$. In this case the Second Derivatives Test gives no information about g at the point $(0, 2)$.