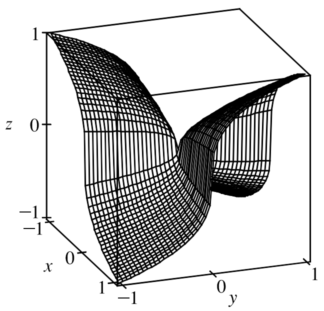
(b)



Notice that if we start at the origin and proceed in the direction of the x- or y-axis, then the graph is flat. But if we proceed in any other direction, then the graph is extremely steep.

69. Let $\mathbf{u} = \langle a, b\rangle$ and $\mathbf{v} = \langle c, d\rangle$. Then we know that at the given point, $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = af_x + bf_y$ and $D_{\mathbf{v}} f = \nabla f \cdot \mathbf{v} = cf_x + df_y$. But these are just two linear equations in the two unknowns f_x and f_y, and since $\mathbf{u}$ and $\mathbf{v}$ are not parallel, we can solve the equations to find $\nabla f = \langle f_x, f_y\rangle$ at the given point. In fact,

$$\nabla f = \left\langle \frac{d\,D_{\mathbf{u}} f - b\,D_{\mathbf{v}} f}{ad - bc}, \frac{a\,D_{\mathbf{v}} f - c\,D_{\mathbf{u}} f}{ad - bc} \right\rangle.$$

70. Since $z = f(x, y)$ is differentiable at $\mathbf{x}_0 = (x_0, y_0)$, by Definition 14.4.7 we have $\Delta z = f_x(x_0, y_0)\,\Delta x + f_y(x_0, y_0)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$ where $\varepsilon_1, \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$. Now $\Delta z = f(\mathbf{x}) - f(\mathbf{x}_0)$, $\langle \Delta x, \Delta y\rangle = \mathbf{x} - \mathbf{x}_0$ so $(\Delta x, \Delta y) \to (0, 0)$ is equivalent to $\mathbf{x} \to \mathbf{x}_0$ and $\langle f_x(x_0, y_0), f_y(x_0, y_0)\rangle = \nabla f(\mathbf{x}_0)$. Substituting into 14.4.7 gives $f(\mathbf{x}) - f(\mathbf{x}_0) = \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) + \langle \varepsilon_1, \varepsilon_2\rangle \cdot \langle \Delta x, \Delta y\rangle$ or $\langle \varepsilon_1, \varepsilon_2\rangle \cdot (\mathbf{x} - \mathbf{x}_0) = f(\mathbf{x}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)$,

and so $\dfrac{f(\mathbf{x}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)}{|\mathbf{x} - \mathbf{x}_0|} = \dfrac{\langle \varepsilon_1, \varepsilon_2\rangle \cdot (\mathbf{x} - \mathbf{x}_0)}{|\mathbf{x} - \mathbf{x}_0|}$. But $\dfrac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|}$ is a unit vector so

$\displaystyle\lim_{\mathbf{x} \to \mathbf{x}_0} \frac{\langle \varepsilon_1, \varepsilon_2\rangle \cdot (\mathbf{x} - \mathbf{x}_0)}{|\mathbf{x} - \mathbf{x}_0|} = 0$ since $\varepsilon_1, \varepsilon_2 \to 0$ as $\mathbf{x} \to \mathbf{x}_0$. Hence $\displaystyle\lim_{\mathbf{x} \to \mathbf{x}_0} \frac{f(\mathbf{x}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)}{|\mathbf{x} - \mathbf{x}_0|} = 0.$

14.7 Maximum and Minimum Values

1. (a) First we compute $D(1, 1) = f_{xx}(1, 1)\,f_{yy}(1, 1) - [f_{xy}(1, 1)]^2 = (4)(2) - (1)^2 = 7$. Since $D(1, 1) > 0$ and $f_{xx}(1, 1) > 0$, f has a local minimum at $(1, 1)$ by the Second Derivatives Test.

(b) $D(1, 1) = f_{xx}(1, 1)\,f_{yy}(1, 1) - [f_{xy}(1, 1)]^2 = (4)(2) - (3)^2 = -1$. Since $D(1, 1) < 0$, f has a saddle point at $(1, 1)$ by the Second Derivatives Test.

2. (a) $D = g_{xx}(0, 2)\,g_{yy}(0, 2) - [g_{xy}(0, 2)]^2 = (-1)(1) - (6)^2 = -37$. Since $D < 0$, g has a saddle point at $(0, 2)$ by the Second Derivatives Test.

(b) $D = g_{xx}(0, 2)\,g_{yy}(0, 2) - [g_{xy}(0, 2)]^2 = (-1)(-8) - (2)^2 = 4$. Since $D > 0$ and $g_{xx}(0, 2) < 0$, g has a local maximum at $(0, 2)$ by the Second Derivatives Test.

(c) $D = g_{xx}(0, 2)\,g_{yy}(0, 2) - [g_{xy}(0, 2)]^2 = (4)(9) - (6)^2 = 0$. In this case the Second Derivatives Test gives no information about g at the point $(0, 2)$.

3. In the figure, a point at approximately $(1, 1)$ is enclosed by level curves which are oval in shape and indicate that as we move away from the point in any direction the values of f are increasing. Hence we would expect a local minimum at or near $(1, 1)$. The level curves near $(0, 0)$ resemble hyperbolas, and as we move away from the origin, the values of f increase in some directions and decrease in others, so we would expect to find a saddle point there.

To verify our predictions, we have $f(x, y) = 4 + x^3 + y^3 - 3xy \quad \Rightarrow \quad f_x(x, y) = 3x^2 - 3y$, $f_y(x, y) = 3y^2 - 3x$. We have critical points where these partial derivatives are equal to 0: $3x^2 - 3y = 0$, $3y^2 - 3x = 0$. Substituting $y = x^2$ from the first equation into the second equation gives $3(x^2)^2 - 3x = 0 \quad \Rightarrow \quad 3x(x^3 - 1) = 0 \quad \Rightarrow \quad x = 0$ or $x = 1$. Then we have two critical points, $(0, 0)$ and $(1, 1)$. The second partial derivatives are $f_{xx}(x, y) = 6x$, $f_{xy}(x, y) = -3$, and $f_{yy}(x, y) = 6y$, so $D(x, y) = f_{xx}(x, y)\, f_{yy}(x, y) - [f_{xy}(x, y)]^2 = (6x)(6y) - (-3)^2 = 36xy - 9$. Then $D(0, 0) = 36(0)(0) - 9 = -9$, and $D(1, 1) = 36(1)(1) - 9 = 27$. Since $D(0, 0) < 0$, f has a saddle point at $(0, 0)$ by the Second Derivatives Test. Since $D(1, 1) > 0$ and $f_{xx}(1, 1) > 0$, f has a local minimum at $(1, 1)$.

4. In the figure, points at approximately $(-1, 1)$ and $(-1, -1)$ are enclosed by oval-shaped level curves which indicate that as we move away from either point in any direction, the values of f are increasing. Hence we would expect local minima at or near $(-1, \pm 1)$. Similarly, the point $(1, 0)$ appears to be enclosed by oval-shaped level curves which indicate that as we move away from the point in any direction the values of f are decreasing, so we should have a local maximum there. We also show hyperbola-shaped level curves near the points $(-1, 0)$, $(1, 1)$, and $(1, -1)$. The values of f increase along some paths leaving these points and decrease in others, so we should have a saddle point at each of these points.

To confirm our predictions, we have $f(x, y) = 3x - x^3 - 2y^2 + y^4 \quad \Rightarrow \quad f_x(x, y) = 3 - 3x^2, \quad f_y(x, y) = -4y + 4y^3$. Setting these partial derivatives equal to 0, we have $3 - 3x^2 = 0 \quad \Rightarrow \quad x = \pm 1$ and $-4y + 4y^3 = 0 \quad \Rightarrow$ $y\left(y^2 - 1\right) = 0 \quad \Rightarrow \quad y = 0, \pm 1$. So our critical points are $(\pm 1, 0)$, $(\pm 1, 1)$, $(\pm 1, -1)$.

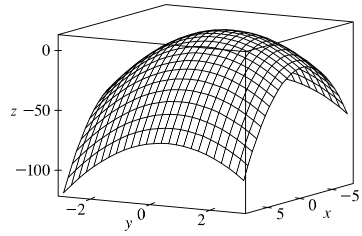
The second partial derivatives are $f_{xx}(x, y) = -6x$, $f_{xy}(x, y) = 0$, and $f_{yy}(x, y) = 12y^2 - 4$, so

$D(x, y) = f_{xx}(x, y)\, f_{yy}(x, y) - [f_{xy}(x, y)]^2 = (-6x)(12y^2 - 4) - (0)^2 = -72xy^2 + 24x$.

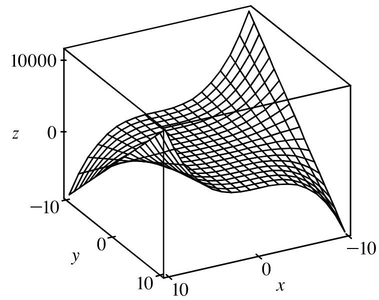
We use the Second Derivatives Test to classify the 6 critical points:

Critical Point	D	f_{xx}	Conclusion
$(1, 0)$	24	-6	$D > 0$, $f_{xx} < 0 \quad \Rightarrow \quad f$ has a local maximum at $(1, 0)$
$(1, 1)$	-48		$D < 0 \quad \Rightarrow \quad f$ has a saddle point at $(1, 1)$
$(1, -1)$	-48		$D < 0 \quad \Rightarrow \quad f$ has a saddle point at $(1, -1)$
$(-1, 0)$	-24		$D < 0 \quad \Rightarrow \quad f$ has a saddle point at $(-1, 0)$
$(-1, 1)$	48	6	$D > 0$, $f_{xx} > 0 \quad \Rightarrow \quad f$ has a local minimum at $(-1, 1)$
$(-1, -1)$	48	6	$D > 0$, $f_{xx} > 0 \quad \Rightarrow \quad f$ has a local minimum at $(-1, -1)$

5. $f(x,y) = 9 - 2x + 4y - x^2 - 4y^2 \quad \Rightarrow \quad f_x = -2 - 2x$, $f_y = 4 - 8y$, $f_{xx} = -2$, $f_{xy} = 0$, $f_{yy} = -8$. Then $f_x = 0$ and $f_y = 0$ imply $x = -1$ and $y = \frac{1}{2}$, and the only critical point is $\left(-1, \frac{1}{2}\right)$. $D(x,y) = f_{xx}f_{yy} - (f_{xy})^2 = (-2)(-8) - 0^2 = 16$, and since $D\left(-1, \frac{1}{2}\right) = 16 > 0$ and $f_{xx}\left(-1, \frac{1}{2}\right) = -2 < 0$, $f\left(-1, \frac{1}{2}\right) = 11$ is a local maximum by the Second Derivatives Test.

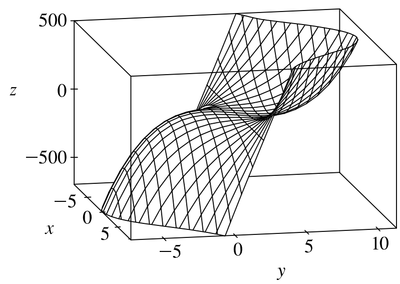


6. $f(x,y) = x^3y + 12x^2 - 8y \quad \Rightarrow \quad f_x = 3x^2y + 24x$, $f_y = x^3 - 8$, $f_{xx} = 6xy + 24$, $f_{xy} = 3x^2$, $f_{yy} = 0$. Then $f_y = 0$ implies $x = 2$, and substitution into $f_x = 0$ gives $12y + 48 = 0 \quad \Rightarrow \quad y = -4$. Thus, the only critical point is $(2, -4)$. $D(2,-4) = (-24)(0) - 12^2 = -144 < 0$, so $(2,-4)$ is a saddle point.

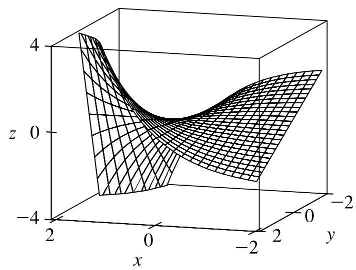


7. $f(x,y) = y^3 + 3x^2y - 6x^2 - 6y^2 + 2 \quad \Rightarrow \quad f_x = 6xy - 12x$, $f_y = 3y^2 + 3x^2 - 12y$, $f_{xx} = 6y - 12$, $f_{xy} = 6x$, $f_{yy} = 6y - 12$. Then $f_x = 0$ implies $6x(y-2) = 0$, so $x = 0$ or $y = 2$. If $x = 0$ then substitution into $f_y = 0$ gives $3y^2 - 12y = 0 \quad \Rightarrow \quad 3y(y-4) = 0 \quad \Rightarrow \quad y = 0$ or $y = 4$, so we have critical points $(0,0)$ and $(0,4)$. If $y = 2$, substitution into $f_y = 0$ gives $12 + 3x^2 - 24 = 0 \quad \Rightarrow \quad x^2 = 4 \quad \Rightarrow$ $x = \pm 2$, so we have critical points $(\pm 2, 2)$.

$D(0,0) = (-12)(-12) - 0^2 = 144 > 0$ and $f_{xx}(0,0) = -12 < 0$, so $f(0,0) = 2$ is a local maximum. $D(0,4) = (12)(12) - 0^2 = 144 > 0$ and $f_{xx}(0,4) = 12 > 0$, so $f(0,4) = -30$ is a local minimum. $D(\pm 2, 2) = (0)(0) - (\pm 12)^2 = -144 < 0$, so $(\pm 2, 2)$ are saddle points.



8. $f(x,y) = y(e^x - 1) \quad \Rightarrow \quad f_x = ye^x$, $f_y = e^x - 1$, $f_{xx} = ye^x$, $f_{xy} = e^x$, $f_{yy} = 0$. Because e^x is never zero, $f_x = 0$ only when $y = 0$, and $f_y = 0$ when $e^x = 1 \quad \Rightarrow \quad x = 0$, so the only critical point is $(0,0)$. $D(x,y) = f_{xx}f_{yy} - (f_{xy})^2 = (ye^x)(0) - (e^x)^2 = -e^{2x}$, and since $D(0,0) = -1 < 0$, $(0,0)$ is a saddle point.



9. $f(x,y) = x^2 + y^4 + 2xy \quad \Rightarrow \quad f_x = 2x + 2y$, $f_y = 4y^3 + 2x$, $f_{xx} = 2$, $f_{xy} = 2$, $f_{yy} = 12y^2$. Then $f_x = 0$ implies $y = -x$, and substitution into $f_y = 4y^3 + 2x = 0$ gives $-4x^3 + 2x = 0 \quad \Rightarrow \quad 2x\left(1 - 2x^2\right) = 0 \quad \Rightarrow \quad x = 0$ or $x = \pm\frac{1}{\sqrt{2}}$. Thus the critical points are $(0,0)$, $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, and $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Now

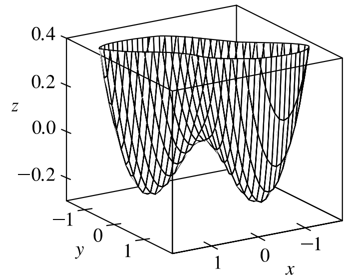
$D(x,y) = f_{xx}f_{yy} - (f_{xy})^2 = (2)(12y^2) - (2)^2 = 24y^2 - 4$,

so $D(0,0) = -4 < 0$ and $(0,0)$ is a saddle point.

$D\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = D\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 24\left(\frac{1}{2}\right) - 4 = 8 > 0$ and

$f_{xx}\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = f_{xx}\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 2 > 0$, so $f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = -\frac{1}{4}$

and $f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = -\frac{1}{4}$ are local minima.



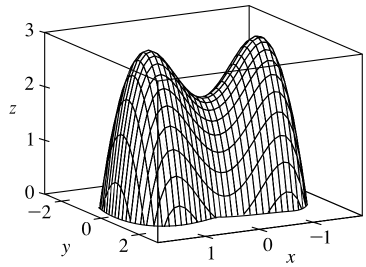
10. $f(x,y) = 2 - x^4 + 2x^2 - y^2 \quad \Rightarrow \quad f_x = -4x^3 + 4x,\ f_y = -2y,\ f_{xx} = -12x^2 + 4,\ f_{xy} = 0,\ f_{yy} = -2$. Then $f_x = 0$ implies $-4x(x^2 - 1) = 0$, so $x = 0$ or $x = \pm 1$, and $f_y = 0$ implies $y = 0$. Thus the critical points are $(0,0)$, $(\pm 1, 0)$.

$D(0,0) = (4)(-2) - 0^2 = -8 < 0$, so $(0,0)$ is a saddle point.

$D(1,0) = D(-1,0) = (-8)(-2) - (0)^2 = 16 > 0$, and

$f_{xx}(1,0) = f_{xx}(-1,0) = -8 < 0$, so $f(1,0) = 3$ and $f(-1,0) = 3$

are local maxima.

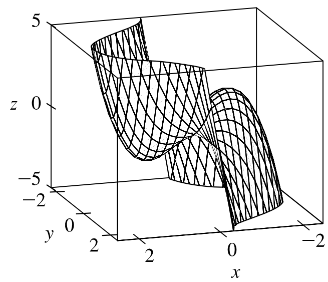


11. $f(x,y) = x^3 - 3x + 3xy^2 \quad \Rightarrow \quad f_x = 3x^2 - 3 + 3y^2,\ f_y = 6xy,\ f_{xx} = 6x,\ f_{xy} = 6y,\ f_{yy} = 6x$. Then $f_y = 0$ implies $x = 0$ or $y = 0$. If $x = 0$, substitution into $f_x = 0$ gives $3y^2 = 3 \quad \Rightarrow \quad y = \pm 1$, and if $y = 0$, substitution into $f_x = 0$ gives $x = \pm 1$. Thus the critical points are $(0, \pm 1)$ and $(\pm 1, 0)$.

$D(0, \pm 1) = 0 - 36 < 0$, so $(0, \pm 1)$ are saddle points.

$D(\pm 1, 0) = 36 - 0 > 0$, $f_{xx}(1,0) = 6 > 0$, and $f_{xx}(-1,0) = -6 < 0$,

so $f(1,0) = -2$ is a local minimum and $f(-1,0) = 2$ is a local maximum.



12. $f(x,y) = x^3 + y^3 - 3x^2 - 3y^2 - 9x \quad \Rightarrow \quad f_x = 3x^2 - 6x - 9,\ f_y = 3y^2 - 6y,\ f_{xx} = 6x - 6,\ f_{xy} = 0,\ f_{yy} = 6y - 6$. Then $f_x = 0$ implies $3(x+1)(x-3) = 0 \quad \Rightarrow \quad x = -1$ or $x = 3$, and $f_y = 0$ implies $3y(y-2) = 0 \quad \Rightarrow \quad y = 0$ or $y = 2$. Thus the critical points are $(-1,0)$, $(-1,2)$, $(3,0)$, and $(3,2)$. $D(-1,2) = (-12)(6) - (0)^2 = -72 < 0$ and

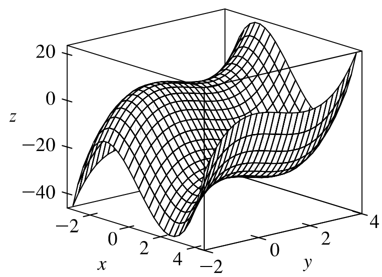
$D(3,0) = (12)(-6) - (0)^2 = -72 < 0$, so $(-1,2)$ and $(3,0)$ are

saddle points. $D(-1,0) = (-12)(-6) - (0)^2 = 72 > 0$ and

$f_{xx}(-1,0) = -12 < 0$, so $f(-1,0) = 5$ is a local maximum.

$D(3,2) = (12)(6) - (0)^2 = 72 > 0$ and $f_{xx}(3,2) = 12 > 0$, so

$f(3,2) = -31$ is a local minimum.



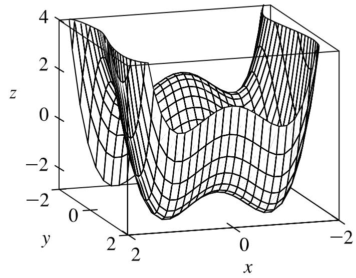
13. $f(x,y) = x^4 - 2x^2 + y^3 - 3y \quad \Rightarrow \quad f_x = 4x^3 - 4x,\ f_y = 3y^2 - 3,\ f_{xx} = 12x^2 - 4,\ f_{xy} = 0,\ f_{yy} = 6y.$

Then $f_x = 0$ implies $4x(x^2 - 1) = 0 \quad \Rightarrow \quad x = 0$ or $x = \pm 1$, and $f_y = 0$ implies $3(y^2 - 1) = 0 \quad \Rightarrow \quad y = \pm 1$.

Thus there are six critical points: $(0, \pm 1)$, $(\pm 1, 1)$, and $(\pm 1, -1)$.

$D(0,1) = (-4)(6) - (0)^2 = -24 < 0$ and

$D(\pm 1, -1) = (8)(-6) = -48 < 0$, so $(0,1)$ and $(\pm 1, -1)$ are saddle points. $D(0,-1) = (-4)(-6) = 24 > 0$ and $f_{xx}(0,-1) = -4 < 0$, so $f(0,-1) = 2$ is a local maximum. $D(\pm 1, 1) = (8)(6) = 48 > 0$ and $f_{xx}(\pm 1, 1) = 8 > 0$, so $f(\pm 1, 1) = -3$ are local minima.

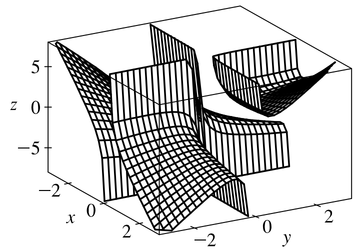


14. $f(x,y) = xy + \dfrac{1}{x} + \dfrac{1}{y} \quad \Rightarrow \quad f_x = y - \dfrac{1}{x^2},\ f_y = x - \dfrac{1}{y^2},\ f_{xx} = \dfrac{2}{x^3},$

$f_{xy} = 1,\ f_{yy} = \dfrac{2}{y^3}$. Then $f_x = 0$ implies $y = \dfrac{1}{x^2}$ and $f_y = 0$ implies

$x = \dfrac{1}{y^2}$. Substituting the first equation into the second gives

$x = \dfrac{1}{(1/x^2)^2} \quad \Rightarrow \quad x = x^4 \quad \Rightarrow \quad x(x^3 - 1) = 0 \quad \Rightarrow \quad x = 0$ or $x = 1$.



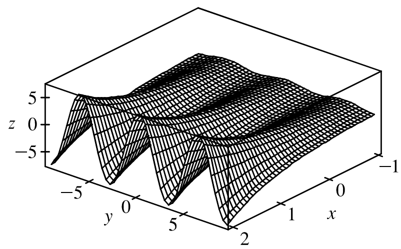
f is not defined when $x = 0$, and when $x = 1$ we have $y = 1$, so the only critical point is $(1,1)$.

$D(1,1) = (2)(2) - 1^2 = 3 > 0$ and $f_{xx}(1,1) = 2 > 0$, so $f(1,1) = 3$ is a local minimum.

15. $f(x,y) = e^x \cos y \quad \Rightarrow \quad f_x = e^x \cos y,\ f_y = -e^x \sin y.$

Now $f_x = 0$ implies $\cos y = 0$ or $y = \frac{\pi}{2} + n\pi$ for n an integer.

But $\sin\left(\frac{\pi}{2} + n\pi\right) \neq 0$, so there are no critical points.



16. $f(x,y) = xye^{-(x^2+y^2)/2} \quad \Rightarrow \quad f_x = xy \cdot e^{-(x^2+y^2)/2}(-x) + e^{-(x^2+y^2)/2} \cdot y = y(1 - x^2)e^{-(x^2+y^2)/2},$

$f_y = xy \cdot e^{-(x^2+y^2)/2}(-y) + e^{-(x^2+y^2)/2} \cdot x = x(1 - y^2)e^{-(x^2+y^2)/2},$

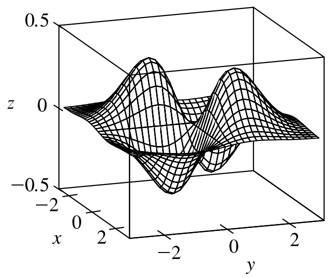
$f_{xx} = y\left[(1 - x^2) \cdot e^{-(x^2+y^2)/2}(-x) + e^{-(x^2+y^2)/2}(-2x)\right] = xy(x^2 - 3)e^{-(x^2+y^2)/2},$

$f_{xy} = (1 - x^2)\left[y \cdot e^{-(x^2+y^2)/2}(-y) + e^{-(x^2+y^2)/2}(1)\right] = (1 - x^2)(1 - y^2)e^{-(x^2+y^2)/2},$

$f_{yy} = x\left[(1 - y^2) \cdot e^{-(x^2+y^2)/2}(-y) + e^{-(x^2+y^2)/2}(-2y)\right] = xy(y^2 - 3)e^{-(x^2+y^2)/2}.$

Then $f_x = 0$ implies $y(1 - x^2) = 0 \quad \Rightarrow \quad y = 0$ or $x = \pm 1$. Substituting $y = 0$ into $f_y = 0$ gives $xe^{-x^2/2} = 0 \quad \Rightarrow$ $x = 0$, and substituting $x = \pm 1$ into $f_y = 0$ gives $\pm(1 - y^2)e^{-(1+y^2)/2} = 0 \quad \Rightarrow \quad y = \pm 1$, so the critical points are $(0,0)$, $(1, \pm 1)$, and $(-1, \pm 1)$. $D(0,0) = (0)(0) - (1)^2 = -1 < 0$, so $(0,0)$ is a saddle point.

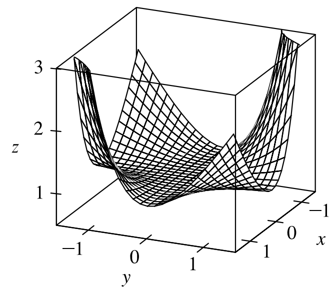
[continued]



$D(1,1) = D(-1,-1) = (-2e^{-1})(-2e^{-1}) - (0)^2 = 4e^{-2} > 0$ and $f_{xx}(1,1) = f_{xx}(-1,-1) = -2e^{-1} < 0$, so $f(1,1) = f(-1,-1) = e^{-1}$ are local maxima.

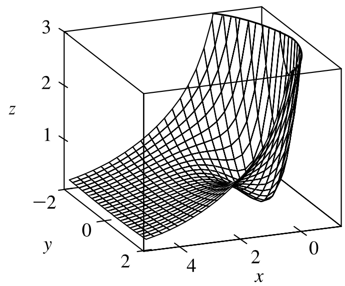
$D(1,-1) = D(-1,1) = (2e^{-1})(2e^{-1}) - (0)^2 = 4e^{-2} > 0$ and $f_{xx}(1,-1) = f_{xx}(-1,1) = 2e^{-1} > 0$, so $f(1,-1) = f(-1,1) = -e^{-1}$ are local minima.

17. $f(x,y) = xy + e^{-xy} \quad \Rightarrow \quad f_x = y - ye^{-xy},\ f_y = x - xe^{-xy},\ f_{xx} = y^2e^{-xy}$, $f_{xy} = 1 - \left[y(-xe^{-xy}) + e^{-xy}(1)\right] = 1 + (xy-1)e^{-xy},\ f_{yy} = x^2e^{-xy}$. Then $f_x = 0$ implies $y(1 - e^{-xy}) = 0 \quad \Rightarrow$ $y = 0$ or $e^{-xy} = 1 \quad \Rightarrow \quad x = 0$ or $y = 0$. If $x = 0$ then $f_y = 0$ for any y-value, so all points of the form $(0, y_0)$ are critical points. If $y = 0$, then $f_y = x - xe^0 = 0$ for any x-value, so all points of the form $(x_0, 0)$ are critical points. We have $D(x_0, 0) = (0)(x_0^2) - (0)^2 = 0$ and $D(0, y_0) = (y_0^2)(0) - (0)^2 = 0$, so the Second Derivatives Test gives no information.



Notice that if we let $t = xy$, then $f(x,y) = g(t) = t + e^{-t} \quad \Rightarrow$ $g'(t) = 1 - e^{-t}$. Now $g'(t) = 0$ only for $t = 0$, and $g'(t) < 0$ for $t < 0$, $g'(t) > 0$ for $t > 0$. Thus $g(0) = 1$ is a local and absolute minimum, so $f(x,y) = xy + e^{-xy} \geq 1$ for all (x,y) with equality if and only if $x = 0$ or $y = 0$. Hence all points on the x- and y-axes are local (and absolute) minima, where $f(x,y) = 1$.

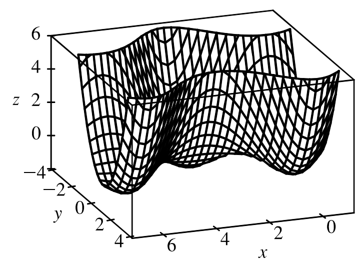
18. $f(x,y) = (x^2 + y^2)e^{-x} \quad \Rightarrow \quad f_x = (x^2 + y^2)(-e^{-x}) + e^{-x}(2x) = (2x - x^2 - y^2)e^{-x},\quad f_y = 2ye^{-x}$, $f_{xx} = (2x - x^2 - y^2)(-e^{-x}) + e^{-x}(2 - 2x) = (x^2 + y^2 - 4x + 2)e^{-x},\quad f_{xy} = -2ye^{-x},\quad f_{yy} = 2e^{-x}$. Then $f_y = 0$



implies $y = 0$ and substituting into $f_x = 0$ gives $(2x - x^2)e^{-x} = 0 \quad \Rightarrow$ $x(2 - x) = 0 \quad \Rightarrow \quad x = 0$ or $x = 2$, so the critical points are $(0,0)$ and $(2,0)$. $\quad D(0,0) = (2)(2) - (0)^2 = 4 > 0$ and $f_{xx}(0,0) = 2 > 0$, so $f(0,0) = 0$ is a local minimum.

$D(2,0) = (-2e^{-2})(2e^{-2}) - (0)^2 = -4e^{-4} < 0$ so $(2,0)$ is a saddle point.

19. $f(x,y) = y^2 - 2y\cos x \quad \Rightarrow \quad f_x = 2y\sin x,\ f_y = 2y - 2\cos x$, $f_{xx} = 2y\cos x,\ f_{xy} = 2\sin x,\ f_{yy} = 2$. Then $f_x = 0$ implies $y = 0$ or $\sin x = 0 \quad \Rightarrow \quad x = 0, \pi$, or 2π for $-1 \leq x \leq 7$. Substituting $y = 0$ into $f_y = 0$ gives $\cos x = 0 \quad \Rightarrow \quad x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$, substituting $x = 0$ or $x = 2\pi$ into $f_y = 0$ gives $y = 1$, and substituting $x = \pi$ into $f_y = 0$ gives $y = -1$. Thus the critical points are $(0,1)$, $\left(\frac{\pi}{2}, 0\right)$, $(\pi, -1)$, $\left(\frac{3\pi}{2}, 0\right)$, and $(2\pi, 1)$.



$D\left(\frac{\pi}{2},0\right) = D\left(\frac{3\pi}{2},0\right) = -4 < 0$ so $\left(\frac{\pi}{2},0\right)$ and $\left(\frac{3\pi}{2},0\right)$ are saddle points. $D(0,1) = D(\pi,-1) = D(2\pi,1) = 4 > 0$ and $f_{xx}(0,1) = f_{xx}(\pi,-1) = f_{xx}(2\pi,1) = 2 > 0$, so $f(0,1) = f(\pi,-1) = f(2\pi,1) = -1$ are local minima.

20. $f(x,y) = \sin x\,\sin y \quad\Rightarrow\quad f_x = \cos x\,\sin y$, $f_y = \sin x\,\cos y$, $f_{xx} = -\sin x\,\sin y$, $f_{xy} = \cos x\,\cos y$, $f_{yy} = -\sin x\,\sin y$. Here we have $-\pi < x < \pi$ and $-\pi < y < \pi$, so $f_x = 0$ implies $\cos x = 0$ or $\sin y = 0$. If $\cos x = 0$ then $x = -\frac{\pi}{2}$ or $\frac{\pi}{2}$, and if $\sin y = 0$ then $y = 0$. Substituting $x = \pm\frac{\pi}{2}$ into $f_y = 0$ gives $\cos y = 0 \quad\Rightarrow\quad y = -\frac{\pi}{2}$ or $\frac{\pi}{2}$, and substituting $y = 0$ into $f_y = 0$ gives $\sin x = 0 \quad\Rightarrow\quad x = 0$. Thus the critical points are $\left(-\frac{\pi}{2},\pm\frac{\pi}{2}\right)$, $\left(\frac{\pi}{2},\pm\frac{\pi}{2}\right)$, and $(0,0)$.

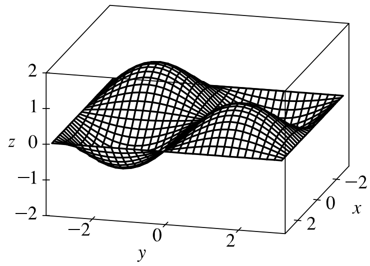
$D(0,0) = -1 < 0$ so $(0,0)$ is a saddle point.

$D\left(-\frac{\pi}{2},\pm\frac{\pi}{2}\right) = D\left(\frac{\pi}{2},\pm\frac{\pi}{2}\right) = 1 > 0$ and

$f_{xx}\left(-\frac{\pi}{2},-\frac{\pi}{2}\right) = f_{xx}\left(\frac{\pi}{2},\frac{\pi}{2}\right) = -1 < 0$ while

$f_{xx}\left(-\frac{\pi}{2},\frac{\pi}{2}\right) = f_{xx}\left(\frac{\pi}{2},-\frac{\pi}{2}\right) = 1 > 0$, so $f\left(-\frac{\pi}{2},-\frac{\pi}{2}\right) = f\left(\frac{\pi}{2},\frac{\pi}{2}\right) = 1$

are local maxima and $f\left(-\frac{\pi}{2},\frac{\pi}{2}\right) = f\left(\frac{\pi}{2},-\frac{\pi}{2}\right) = 1$ are local minima.



21. $f(x,y) = x^2 + 4y^2 - 4xy + 2 \quad\Rightarrow\quad f_x = 2x - 4y$, $f_y = 8y - 4x$, $f_{xx} = 2$, $f_{xy} = -4$, $f_{yy} = 8$. Then $f_x = 0$ and $f_y = 0$ each implies $y = \frac{1}{2}x$, so all points of the form $\left(x_0, \frac{1}{2}x_0\right)$ are critical points and for each of these we have $D\left(x_0, \frac{1}{2}x_0\right) = (2)(8) - (-4)^2 = 0$. The Second Derivatives Test gives no information, but $f(x,y) = x^2 + 4y^2 - 4xy + 2 = (x-2y)^2 + 2 \geq 2$ with equality if and only if $y = \frac{1}{2}x$. Thus $f\left(x_0, \frac{1}{2}x_0\right) = 2$ are all local (and absolute) minima.

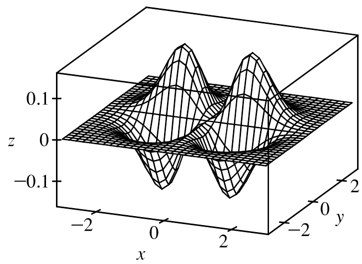
22. $f(x,y) = x^2ye^{-x^2-y^2} \quad\Rightarrow$

$f_x = x^2ye^{-x^2-y^2}(-2x) + 2xye^{-x^2-y^2} = 2xy(1-x^2)e^{-x^2-y^2}$,

$f_y = x^2ye^{-x^2-y^2}(-2y) + x^2e^{-x^2-y^2} = x^2(1-2y^2)e^{-x^2-y^2}$,

$f_{xx} = 2y(2x^4 - 5x^2 + 1)e^{-x^2-y^2}$,

$f_{xy} = 2x(1-x^2)(1-2y^2)e^{-x^2-y^2}$, $f_{yy} = 2x^2y(2y^2-3)e^{-x^2-y^2}$.



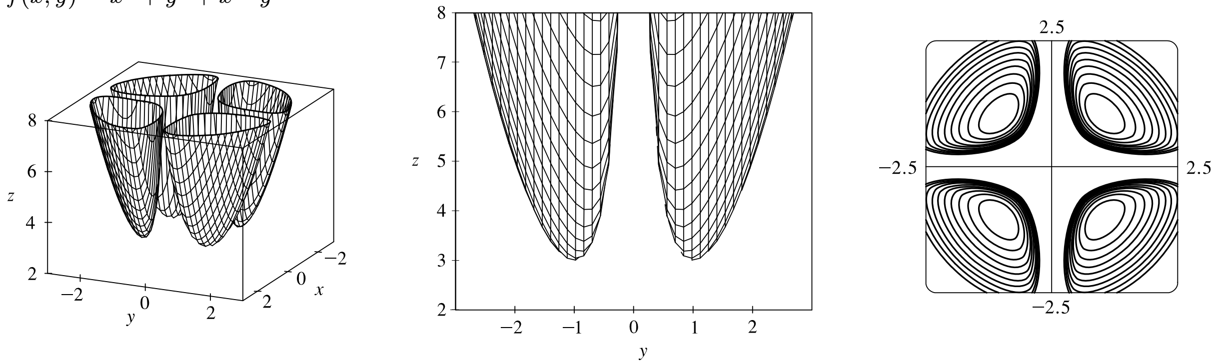
$f_x = 0$ implies $x = 0$, $y = 0$, or $x = \pm 1$. If $x = 0$ then $f_y = 0$ for any y-value, so all points of the form $(0,y)$ are critical points. If $y = 0$ then $f_y = 0 \quad\Rightarrow\quad x^2e^{-x^2} = 0 \quad\Rightarrow\quad x = 0$, so $(0,0)$ (already included above) is a critical point. If $x = \pm 1$ then $(1-2y^2)e^{-1-y^2} = 0 \quad\Rightarrow\quad y = \pm\frac{1}{\sqrt{2}}$, so $\left(\pm 1, \frac{1}{\sqrt{2}}\right)$ and $\left(\pm 1, -\frac{1}{\sqrt{2}}\right)$ are critical points. Now

$D\left(\pm 1, \frac{1}{\sqrt{2}}\right) = 8e^{-3} > 0$, $f_{xx}\left(\pm 1, \frac{1}{\sqrt{2}}\right) = -2\sqrt{2}\,e^{-3/2} < 0$ and $D\left(\pm 1, -\frac{1}{\sqrt{2}}\right) = 8e^{-3} > 0$,

$f_{xx}\left(\pm 1, -\frac{1}{\sqrt{2}}\right) = 2\sqrt{2}\,e^{-3/2} > 0$, so $f\left(\pm 1, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}e^{-3/2}$ are local maximum points while

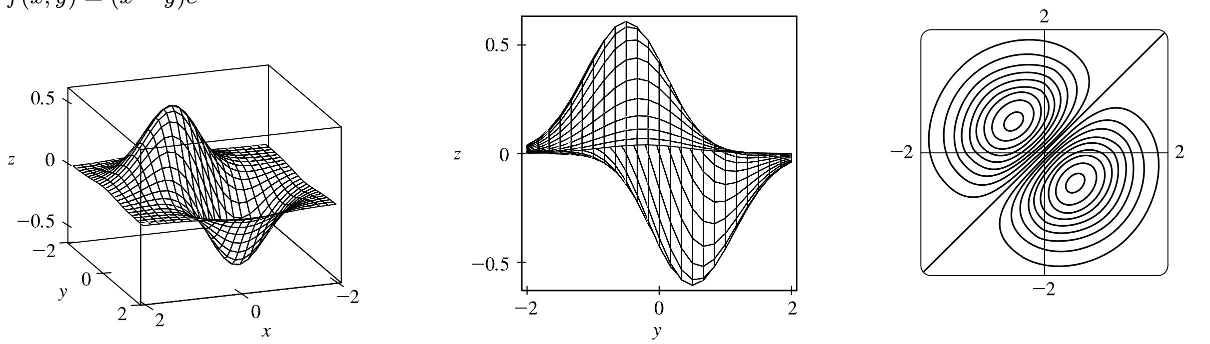
$f\left(\pm 1, -\frac{1}{\sqrt{2}}\right) = -\frac{1}{\sqrt{2}}e^{-3/2}$ are local minimum points. At all critical points $(0,y)$ we have $D(0,y) = 0$, so the Second Derivatives Test gives no information. However, if $y > 0$ then $x^2ye^{-x^2-y^2} \geq 0$ with equality only when $x = 0$, so we have local minimum values $f(0,y) = 0$, $y > 0$. Similarly, if $y < 0$ then $x^2ye^{-x^2-y^2} \leq 0$ with equality when $x = 0$ so $f(0,y) = 0$, $y < 0$ are local maximum values, and $(0,0)$ is a saddle point.

23. $f(x,y) = x^2 + y^2 + x^{-2}y^{-2}$



From the graphs, there appear to be local minima of about $f(1,\pm 1) = f(-1,\pm 1) \approx 3$ (and no local maxima or saddle points). $f_x = 2x - 2x^{-3}y^{-2}$, $f_y = 2y - 2x^{-2}y^{-3}$, $f_{xx} = 2 + 6x^{-4}y^{-2}$, $f_{xy} = 4x^{-3}y^{-3}$, $f_{yy} = 2 + 6x^{-2}y^{-4}$. Then $f_x = 0$ implies $2x^4y^2 - 2 = 0$ or $x^4y^2 = 1$ or $y^2 = x^{-4}$. Note that neither x nor y can be zero. Now $f_y = 0$ implies $2x^2y^4 - 2 = 0$, and with $y^2 = x^{-4}$ this implies $2x^{-6} - 2 = 0$ or $x^6 = 1$. Thus $x = \pm 1$ and if $x = 1$, $y = \pm 1$; if $x = -1$, $y = \pm 1$. So the critical points are $(1,1)$, $(1,-1)$,$(-1,1)$ and $(-1,-1)$. Now $D(1,\pm 1) = D(-1,\pm 1) = 64 - 16 > 0$ and $f_{xx} > 0$ always, so $f(1,\pm 1) = f(-1,\pm 1) = 3$ are local minima.

24. $f(x,y) = (x-y)e^{-x^2-y^2}$



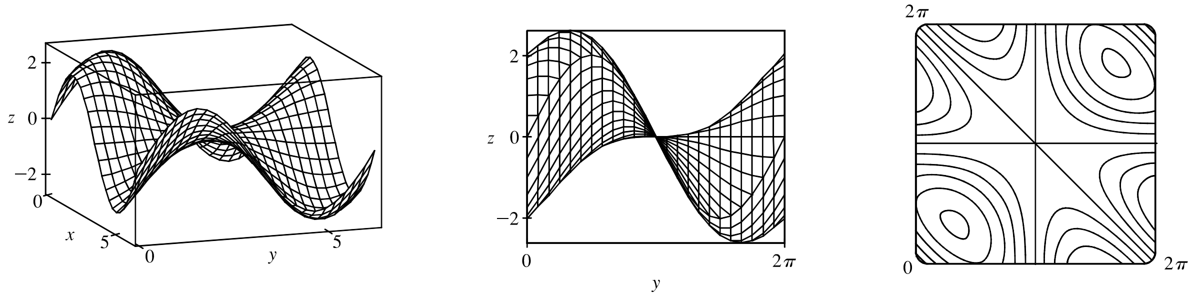
From the graphs, there appears to be a local maximum of about $f(0.5,-0.5) \approx 0.6$ and a local minimum of about $f(-0.5, 0.5) \approx -0.6$.

$f_x = (x-y)e^{-x^2-y^2}(-2x) + e^{-x^2-y^2}(1) = e^{-x^2-y^2}(1 - 2x^2 + 2xy)$,

$f_y = (x-y)e^{-x^2-y^2}(-2y) + e^{-x^2-y^2}(-1) = -e^{-x^2-y^2}(1 - 2y^2 + 2xy)$, $\quad f_{xx} = 2e^{-x^2-y^2}(2x^3 - 3x + y - 2x^2y)$,

$f_{xy} = 2e^{-x^2-y^2}(x - y + 2x^2y - 2xy^2)$, $\quad f_{yy} = -2e^{-x^2-y^2}(2y^3 - 3y + x - 2xy^2)$. Then $f_x = 0$ implies $1 - 2x^2 + 2xy = 0$ and $f_y = 0$ implies $1 - 2y^2 + 2xy = 0$. Subtracting these two equations gives $-2x^2 + 2y^2 = 0 \quad \Rightarrow \quad y = \pm x$. If $y = x$ then substituting into $f_x = 0$ gives $1 - 2x^2 + 2x^2 = 0$, an impossibility. Substituting $y = -x$ gives $1 - 2x^2 - 2x^2 = 0 \quad \Rightarrow \quad x^2 = \frac{1}{4} \quad \Rightarrow \quad x = \pm\frac{1}{2}$. Thus the critical points are $\left(\frac{1}{2}, -\frac{1}{2}\right)$ and $\left(-\frac{1}{2}, \frac{1}{2}\right)$. Now $D\left(\frac{1}{2}, -\frac{1}{2}\right) = (-3e^{-1/2})(-3e^{-1/2}) - (e^{-1/2})^2 = 8e^{-1} > 0$ with $f_{xx}\left(\frac{1}{2}, -\frac{1}{2}\right) = -3e^{-1/2} < 0$, so $f\left(\frac{1}{2}, -\frac{1}{2}\right) = e^{-1/2} \approx 0.607$ is a local maximum, and $D\left(-\frac{1}{2}, \frac{1}{2}\right) = (3e^{-1/2})(3e^{-1/2}) - (-e^{-1/2})^2 = 8e^{-1} > 0$ with $f_{xx}\left(-\frac{1}{2}, \frac{1}{2}\right) = 3e^{-1/2} > 0$, so $f\left(-\frac{1}{2}, \frac{1}{2}\right) = -e^{-1/2} \approx -0.607$ is a local minimum.

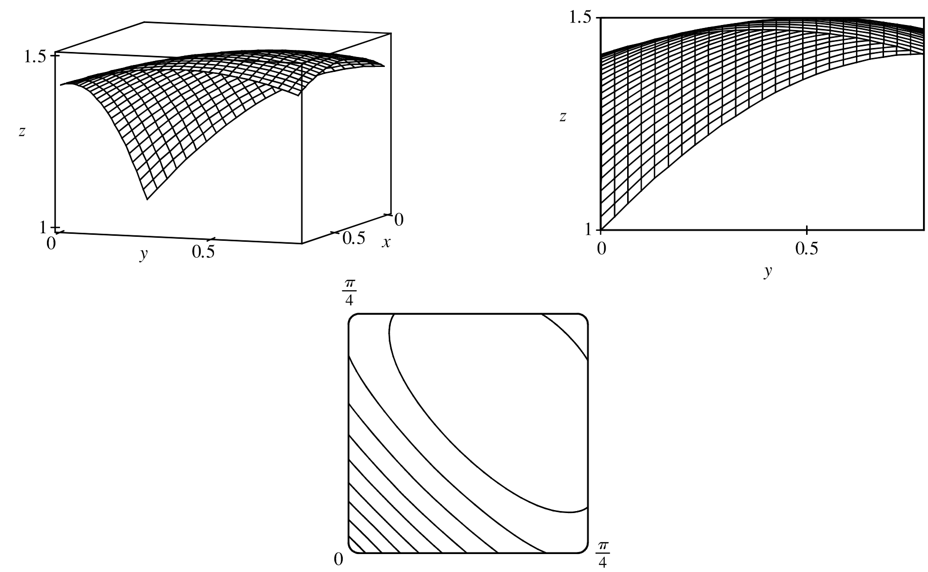
25. $f(x, y) = \sin x + \sin y + \sin(x + y),\ 0 \le x \le 2\pi,\ 0 \le y \le 2\pi$



From the graphs it appears that f has a local maximum at about $(1, 1)$ with value approximately 2.6, a local minimum at about $(5, 5)$ with value approximately -2.6, and a saddle point at about $(3, 3)$.

$f_x = \cos x + \cos(x + y),\ f_y = \cos y + \cos(x + y),\ f_{xx} = -\sin x - \sin(x + y),\ f_{yy} = -\sin y - \sin(x + y),$ $f_{xy} = -\sin(x + y)$. Setting $f_x = 0$ and $f_y = 0$ and subtracting gives $\cos x - \cos y = 0$ or $\cos x = \cos y$. Thus $x = y$ or $x = 2\pi - y$. If $x = y$, $f_x = 0$ becomes $\cos x + \cos 2x = 0$ or $2\cos^2 x + \cos x - 1 = 0$, a quadratic in $\cos x$. Thus $\cos x = -1$ or $\frac{1}{2}$ and $x = \pi$, $\frac{\pi}{3}$, or $\frac{5\pi}{3}$, giving the critical points (π, π), $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ and $\left(\frac{5\pi}{3}, \frac{5\pi}{3}\right)$. Similarly if $x = 2\pi - y$, $f_x = 0$ becomes $(\cos x) + 1 = 0$ and the resulting critical point is (π, π). Now $D(x, y) = \sin x\ \sin y + \sin x\ \sin(x + y) + \sin y\ \sin(x + y)$. So $D(\pi, \pi) = 0$ and the Second Derivatives Test doesn't apply. However, along the line $y = x$ we have $f(x, x) = 2\sin x + \sin 2x = 2\sin x + 2\sin x\ \cos x = 2\sin x(1 + \cos x)$, and $f(x, x) > 0$ for $0 < x < \pi$ while $f(x, x) < 0$ for $\pi < x < 2\pi$. Thus every disk with center (π, π) contains points where f is positive as well as points where f is negative, so the graph crosses its tangent plane ($z = 0$) there and (π, π) is a saddle point. $D\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \frac{9}{4} > 0$ and $f_{xx}\left(\frac{\pi}{3}, \frac{\pi}{3}\right) < 0$ so $f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \frac{3\sqrt{3}}{2}$ is a local maximum while $D\left(\frac{5\pi}{3}, \frac{5\pi}{3}\right) = \frac{9}{4} > 0$ and $f_{xx}\left(\frac{5\pi}{3}, \frac{5\pi}{3}\right) > 0$, so $f\left(\frac{5\pi}{3}, \frac{5\pi}{3}\right) = -\frac{3\sqrt{3}}{2}$ is a local minimum.

26. $f(x, y) = \sin x + \sin y + \cos(x + y),\ 0 \le x \le \frac{\pi}{4},\ 0 \le y \le \frac{\pi}{4}$

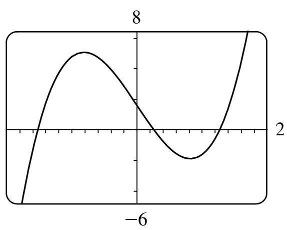


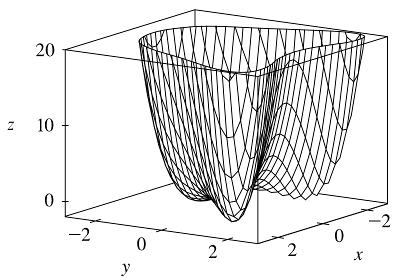
[continued]

From the graphs, it seems that f has a local maximum at about $(0.5, 0.5)$.

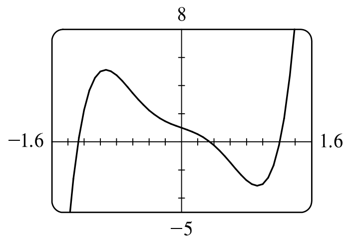
$f_x = \cos x - \sin(x+y)$, $f_y = \cos y - \sin(x+y)$, $f_{xx} = -\sin x - \cos(x+y)$, $f_{yy} = -\sin y - \cos(x+y)$, $f_{xy} = -\cos(x+y)$. Setting $f_x = 0$ and $f_y = 0$ and subtracting gives $\cos x = \cos y$. Thus $x = y$. Substituting $x = y$ into $f_x = 0$ gives $\cos x - \sin 2x = 0$ or $\cos x(1 - 2\sin x) = 0$. But $\cos x \neq 0$ for $0 \le x \le \frac{\pi}{4}$ and $1 - 2\sin x = 0$ implies $x = \frac{\pi}{6}$, so the only critical point is $\left(\frac{\pi}{6}, \frac{\pi}{6}\right)$. Here $f_{xx}\left(\frac{\pi}{6}, \frac{\pi}{6}\right) = -1 < 0$ and $D\left(\frac{\pi}{6}, \frac{\pi}{6}\right) = (-1)^2 - \frac{1}{4} > 0$. Thus $f\left(\frac{\pi}{6}, \frac{\pi}{6}\right) = \frac{3}{2}$ is a local maximum.

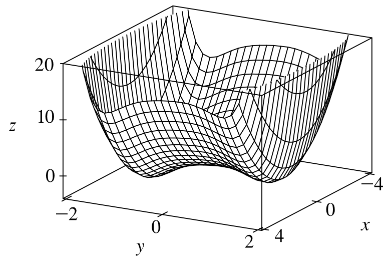
27. $f(x,y) = x^4 + y^4 - 4x^2y + 2y \quad \Rightarrow \quad f_x(x,y) = 4x^3 - 8xy$ and $f_y(x,y) = 4y^3 - 4x^2 + 2$. $f_x = 0 \quad \Rightarrow$ $4x(x^2 - 2y) = 0$, so $x = 0$ or $x^2 = 2y$. If $x = 0$ then substitution into $f_y = 0$ gives $4y^3 = -2 \quad \Rightarrow \quad y = -\frac{1}{\sqrt[3]{2}}$, so $\left(0, -\frac{1}{\sqrt[3]{2}}\right)$ is a critical point. Substituting $x^2 = 2y$ into $f_y = 0$ gives $4y^3 - 8y + 2 = 0$. Using a graph, solutions are approximately $y = -1.526$, 0.259, and 1.267. (Alternatively, we could have used a calculator or a CAS to find these roots.) We have $x^2 = 2y \quad \Rightarrow \quad x = \pm\sqrt{2y}$, so $y = -1.526$ gives no real-valued solution for x, but $y = 0.259 \quad \Rightarrow \quad x \approx \pm 0.720$ and $y = 1.267 \quad \Rightarrow \quad x \approx \pm 1.592$. Thus to three decimal places, the critical points are $\left(0, -\frac{1}{\sqrt[3]{2}}\right) \approx (0, -0.794)$, $(\pm 0.720, 0.259)$, and $(\pm 1.592, 1.267)$. Now since $f_{xx} = 12x^2 - 8y$, $f_{xy} = -8x$, $f_{yy} = 12y^2$, and $D = (12x^2 - 8y)(12y^2) - 64x^2$, we have $D(0, -0.794) > 0$, $f_{xx}(0, -0.794) > 0$, $D(\pm 0.720, 0.259) < 0$, $D(\pm 1.592, 1.267) > 0$, and $f_{xx}(\pm 1.592, 1.267) > 0$. Therefore $f(0, -0.794) \approx -1.191$ and $f(\pm 1.592, 1.267) \approx -1.310$ are local minima, and $(\pm 0.720, 0.259)$ are saddle points. There is no highest point on the graph, but the lowest points are approximately $(\pm 1.592, 1.267, -1.310)$.



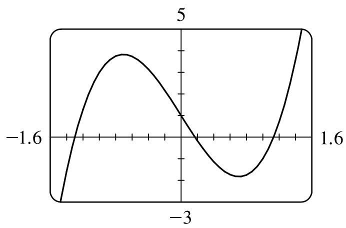


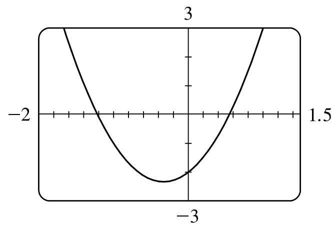
28. $f(x,y) = y^6 - 2y^4 + x^2 - y^2 + y \quad \Rightarrow \quad f_x(x,y) = 2x$ and $f_y(x,y) = 6y^5 - 8y^3 - 2y + 1$. $f_x = 0$ implies $x = 0$, and the graph of f_y shows that the roots of $f_y = 0$ are approximately $y = -1.273$, 0.347, and 1.211. (Alternatively, we could have found the roots of $f_y = 0$ directly, using a calculator or CAS.) So to three decimal places, the critical points are $(0, -1.273)$, $(0, 0.347)$, and $(0, 1.211)$. Now since $f_{xx} = 2$, $f_{xy} = 0$, $f_{yy} = 30y^4 - 24y^2 - 2$, and $D = 60y^4 - 48y^2 - 4$, we have $D(0, -1.273) > 0$, $f_{xx}(0, -1.273) > 0$, $D(0, 0.347) < 0$, $D(0, 1.211) > 0$, and $f_{xx}(0, 1.211) > 0$, so $f(0, -1.273) \approx -3.890$ and $f(0, 1.211) \approx -1.403$ are local minima, and $(0, 0.347)$ is a saddle point. The lowest point on the graph is approximately $(0, -1.273, -3.890)$.

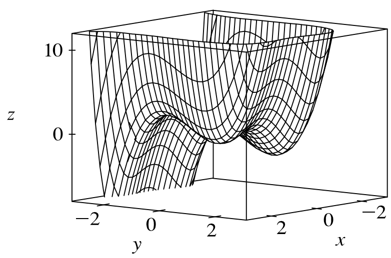




29. $f(x,y) = x^4 + y^3 - 3x^2 + y^2 + x - 2y + 1 \quad \Rightarrow \quad f_x(x,y) = 4x^3 - 6x + 1$ and $f_y(x,y) = 3y^2 + 2y - 2$. From the graphs, we see that to three decimal places, $f_x = 0$ when $x \approx -1.301$, 0.170, or 1.131, and $f_y = 0$ when $y \approx -1.215$ or 0.549. (Alternatively, we could have used a calculator or a CAS to find these roots. We could also use the quadratic formula to find the solutions of $f_y = 0$.) So, to three decimal places, f has critical points at $(-1.301, -1.215)$, $(-1.301, 0.549)$, $(0.170, -1.215)$, $(0.170, 0.549)$, $(1.131, -1.215)$, and $(1.131, 0.549)$. Now since $f_{xx} = 12x^2 - 6$, $f_{xy} = 0$, $f_{yy} = 6y + 2$, and $D = (12x^2 - 6)(6y + 2)$, we have $D(-1.301, -1.215) < 0$, $D(-1.301, 0.549) > 0$, $f_{xx}(-1.301, 0.549) > 0$, $D(0.170, -1.215) > 0$, $f_{xx}(0.170, -1.215) < 0$, $D(0.170, 0.549) < 0$, $D(1.131, -1.215) < 0$, $D(1.131, 0.549) > 0$, and $f_{xx}(1.131, 0.549) > 0$. Therefore, to three decimal places, $f(-1.301, 0.549) \approx -3.145$ and $f(1.131, 0.549) \approx -0.701$ are local minima, $f(0.170, -1.215) \approx 3.197$ is a local maximum, and $(-1.301, -1.215)$, $(0.170, 0.549)$, and $(1.131, -1.215)$ are saddle points. There is no highest or lowest point on the graph.







30. $f(x,y) = 20e^{-x^2-y^2} \sin 3x \cos 3y \quad \Rightarrow$

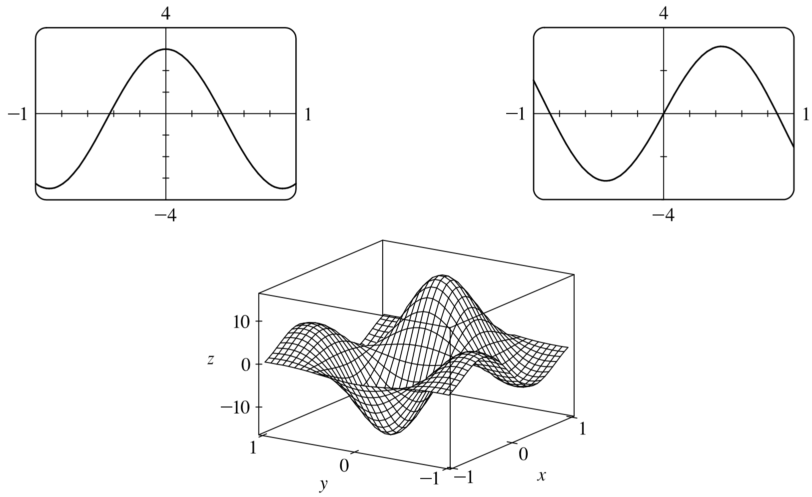
$$f_x(x,y) = 20\cos 3y \left[e^{-x^2-y^2}(3\cos 3x) + (\sin 3x)e^{-x^2-y^2}(-2x)\right]$$
$$= 20e^{-x^2-y^2}\cos 3y\,(3\cos 3x - 2x\sin 3x)$$
$$f_y(x,y) = 20\sin 3x\left[e^{-x^2-y^2}(-3\sin 3y) + (\cos 3y)e^{-x^2-y^2}(-2y)\right]$$
$$= -20e^{-x^2-y^2}\sin 3x\,(3\sin 3y + 2y\cos 3y)$$

Now $f_x = 0$ implies $\cos 3y = 0$ or $3\cos 3x - 2x\sin 3x = 0$. For $|y| \le 1$, the solutions to $\cos 3y = 0$ are $y = \pm\frac{\pi}{6} \approx \pm 0.524$. Using a graph (or a calculator or CAS), we estimate the roots of $3\cos 3x - 2x\sin 3x$ for $|x| \le 1$ to be

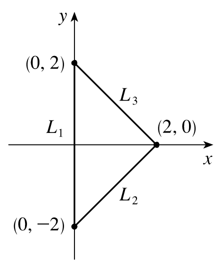
$x \approx \pm 0.430$. $f_y = 0$ implies $\sin 3x = 0$, so $x = 0$, or $3 \sin 3y + 2y \cos 3y = 0$. From a graph (or calculator or CAS), the roots of $3 \sin 3y + 2y \cos 3y$ between -1 and 1 are approximately 0 and ± 0.872. So to three decimal places, f has critical points at $(\pm 0.430, 0)$, $(0.430, \pm 0.872)$, $(-0.430, \pm 0.872)$, and $(0, \pm 0.524)$. Now

$$f_{xx} = 20e^{-x^2-y^2} \cos 3y[(4x^2 - 11) \sin 3x - 12x \cos 3x]$$
$$f_{xy} = -20e^{-x^2-y^2} (3 \cos 3x - 2x \sin 3x)(3 \sin 3y + 2y \cos 3y)$$
$$f_{yy} = 20e^{-x^2-y^2} \sin 3x[(4y^2 - 11) \cos 3y - 12y \sin 3y]$$

and $D = f_{xx} f_{yy} - f_{xy}^2$. Then $D(\pm 0.430, 0) > 0$, $f_{xx}(0.430, 0) < 0$, $f_{xx}(-0.430, 0) > 0$, $D(0.430, \pm 0.872) > 0$, $f_{xx}(0.430, \pm 0.872) > 0$, $D(-0.430, \pm 0.872) > 0$, $f_{xx}(-0.430, \pm 0.872) < 0$, and $D(0, \pm 0.524) < 0$, so $f(0.430, 0) \approx 15.973$ and $f(-0.430, \pm 0.872) \approx 6.459$ are local maxima, $f(-0.430, 0) \approx -15.973$ and $f(0.430, \pm 0.872) \approx -6.459$ are local minima, and $(0, \pm 0.524)$ are saddle points. The highest point on the graph is approximately $(0.430, 0, 15.973)$ and the lowest point is approximately $(-0.430, 0, -15.973)$.

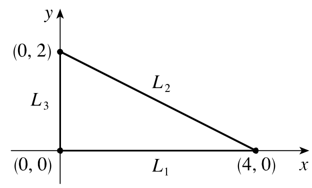


31. Since f is a polynomial it is continuous on D, so an absolute maximum and minimum exist. Here $f_x = 2x - 2$, $f_y = 2y$, and setting $f_x = f_y = 0$ gives $(1, 0)$ as the only critical point (which is inside D), where $f(1, 0) = -1$. Along L_1: $x = 0$ and $f(0, y) = y^2$ for $-2 \le y \le 2$, a quadratic function which attains its minimum at $y = 0$, where $f(0, 0) = 0$, and its maximum at $y = \pm 2$, where $f(0, \pm 2) = 4$. Along L_2: $y = x - 2$ for $0 \le x \le 2$, and $f(x, x - 2) = 2x^2 - 6x + 4 = 2\left(x - \frac{3}{2}\right)^2 - \frac{1}{2}$, a quadratic which attains its minimum at $x = \frac{3}{2}$, where $f\left(\frac{3}{2}, -\frac{1}{2}\right) = -\frac{1}{2}$, and its maximum at $x = 0$, where $f(0, -2) = 4$.



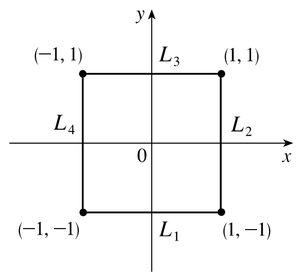
Along L_3: $y = 2 - x$ for $0 \le x \le 2$, and $f(x, 2 - x) = 2x^2 - 6x + 4 = 2\left(x - \frac{3}{2}\right)^2 - \frac{1}{2}$, a quadratic which attains its minimum at $x = \frac{3}{2}$, where $f\left(\frac{3}{2}, \frac{1}{2}\right) = -\frac{1}{2}$, and its maximum at $x = 0$, where $f(0, 2) = 4$. Thus the absolute maximum of f on D is $f(0, \pm 2) = 4$ and the absolute minimum is $f(1, 0) = -1$.

32. Since f is a polynomial it is continuous on D, so an absolute maximum and minimum exist. $f_x = 1 - y$, $f_y = 1 - x$, and setting $f_x = f_y = 0$ gives $(1, 1)$ as the only critical point (which is inside D), where $f(1, 1) = 1$. Along L_1: $y = 0$ and $f(x, 0) = x$ for $0 \le x \le 4$, an increasing function in x, so the maximum value is $f(4, 0) = 4$ and the minimum value is $f(0, 0) = 0$. Along L_2: $y = 2 - \frac{1}{2}x$ and $f\left(x, 2 - \frac{1}{2}x\right) = \frac{1}{2}x^2 - \frac{3}{2}x + 2 = \frac{1}{2}\left(x - \frac{3}{2}\right)^2 + \frac{7}{8}$ for $0 \le x \le 4$, a quadratic function which has a minimum at $x = \frac{3}{2}$, where $f\left(\frac{3}{2}, \frac{5}{4}\right) = \frac{7}{8}$, and a maximum at $x = 4$, where $f(4, 0) = 4$.



Along L_3: $x = 0$ and $f(0, y) = y$ for $0 \le y \le 2$, an increasing function in y, so the maximum value is $f(0, 2) = 2$ and the minimum value is $f(0, 0) = 0$. Thus the absolute maximum of f on D is $f(4, 0) = 4$ and the absolute minimum is $f(0, 0) = 0$.

33. $f_x(x, y) = 2x + 2xy$, $f_y(x, y) = 2y + x^2$, and setting $f_x = f_y = 0$ gives $(0, 0)$ as the only critical point in D, with $f(0, 0) = 4$.



On L_1: $y = -1$, $f(x, -1) = 5$, a constant.

On L_2: $x = 1$, $f(1, y) = y^2 + y + 5$, a quadratic in y which attains its maximum at $(1, 1)$, $f(1, 1) = 7$ and its minimum at $\left(1, -\frac{1}{2}\right)$, $f\left(1, -\frac{1}{2}\right) = \frac{19}{4}$.

On L_3: $f(x, 1) = 2x^2 + 5$ which attains its maximum at $(-1, 1)$ and $(1, 1)$ with $f(\pm 1, 1) = 7$ and its minimum at $(0, 1)$, $f(0, 1) = 5$.

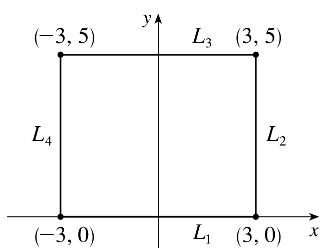
On L_4: $f(-1, y) = y^2 + y + 5$ with maximum at $(-1, 1)$, $f(-1, 1) = 7$ and minimum at $\left(-1, -\frac{1}{2}\right)$, $f\left(-1, -\frac{1}{2}\right) = \frac{19}{4}$.

Thus the absolute maximum is attained at both $(\pm 1, 1)$ with $f(\pm 1, 1) = 7$ and the absolute minimum on D is attained at $(0, 0)$ with $f(0, 0) = 4$.

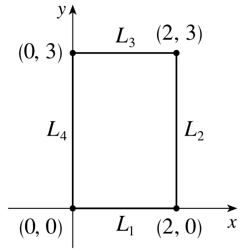
34. $f(x, y) = x^2 + xy + y^2 - 6y \quad \Rightarrow \quad f_x = 2x + y$, $f_y = x + 2y - 6$. Then $f_x = 0$ implies $y = -2x$, and substituting into $f_y = 0$ gives $x - 4x - 6 = 0 \quad \Rightarrow \quad x = -2$, so the only critical point is $(-2, 4)$ (which is in D) where $f(-2, 4) = -12$.

Along L_1: $y = 0$, so $f(x, 0) = x^2$, $-3 \le x \le 3$, which has a maximum value at $x = \pm 3$ where $f(\pm 3, 0) = 9$ and a minimum value at $x = 0$, where $f(0, 0) = 0$. Along L_2: $x = 3$, so $f(3, y) = 9 - 3y + y^2 = \left(y - \frac{3}{2}\right)^2 + \frac{27}{4}$, $0 \le y \le 5$, which has a maximum value at $y = 5$ where $f(3, 5) = 19$ and a minimum value at $y = \frac{3}{2}$ where $f(3, \frac{3}{2}) = \frac{27}{4}$.

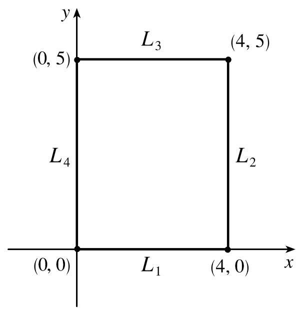
Along L_3: $y = 5$, so $f(x, 5) = x^2 + 5x - 5 = \left(x + \frac{5}{2}\right)^2 - \frac{45}{4}$, $-3 \le x \le 3$, which has a maximum value at $x = 3$ where $f(3, 5) = 19$ and a minimum value at $x = -\frac{5}{2}$, where $f\left(-\frac{5}{2}, 5\right) = -\frac{45}{4}$. Along L_4: $x = -3$, so $f(-3, y) = 9 - 9y + y^2 = \left(y - \frac{9}{2}\right)^2 - \frac{45}{4}$, $0 \le y \le 5$, which has a maximum value at $y = 0$ where $f(-3, 0) = 9$ and a minimum value at $y = \frac{9}{2}$ where $f\left(-3, \frac{9}{2}\right) = -\frac{45}{4}$. Thus the absolute maximum of f on D is $f(3, 5) = 19$ and the absolute minimum is $f(-2, 4) = -12$.



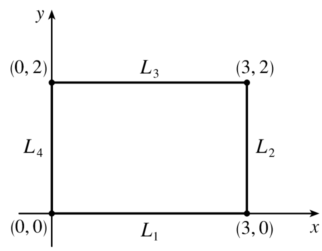
35. $f(x,y) = x^2 + 2y^2 - 2x - 4y + 1 \quad \Rightarrow \quad f_x = 2x - 2,\ f_y = 4y - 4$. Setting $f_x = 0$ and $f_y = 0$ gives $(1,1)$ as the only critical point (which is inside D), where $f(1,1) = -2$. Along L_1: $y = 0$, so $f(x,0) = x^2 - 2x + 1 = (x-1)^2$, $0 \le x \le 2$, which has a maximum value both at $x = 0$ and $x = 2$ where $f(0,0) = f(2,0) = 1$ and a minimum value at $x = 1$, where $f(1,0) = 0$. Along L_2: $x = 2$, so $f(2,y) = 2y^2 - 4y + 1 = 2(y-1)^2 - 1$, $0 \le y \le 3$, which has a maximum value at $y = 3$ where $f(2,3) = 7$ and a minimum value at $y = 1$ where $f(2,1) = -1$. Along L_3: $y = 3$, so $f(x,3) = x^2 - 2x + 7 = (x-1)^2 + 6$, $0 \le x \le 2$, which has a maximum value both at $x = 0$ and $x = 2$ where $f(0,3) = f(2,3) = 7$ and a minimum value at $x = 1$, where $f(1,3) = 6$. Along L_4: $x = 0$, so $f(0,y) = 2y^2 - 4y + 1 = 2(y-1)^2 - 1$, $0 \le y \le 3$, which has a maximum value at $y = 3$ where $f(0,3) = 7$ and a minimum value at $y = 1$ where $f(0,1) = -1$. Thus the absolute maximum is attained at both $(0,3)$ and $(2,3)$, where $f(0,3) = f(2,3) = 7$, and the absolute minimum is $f(1,1) = -2$.



36. $f_x(x,y) = 4 - 2x$ and $f_y(x,y) = 6 - 2y$, so the only critical point is $(2,3)$ (which is in D) where $f(2,3) = 13$. Along L_1: $y = 0$, so $f(x,0) = 4x - x^2 = -(x-2)^2 + 4$, $0 \le x \le 4$, which has a maximum value when $x = 2$ where $f(2,0) = 4$ and a minimum value both when $x = 0$ and $x = 4$, where $f(0,0) = f(4,0) = 0$. Along L_2: $x = 4$, so $f(4,y) = 6y - y^2 = -(y-3)^2 + 9$, $0 \le y \le 5$, which has a maximum value when $y = 3$ where $f(4,3) = 9$ and a minimum value when $y = 0$ where $f(4,0) = 0$. Along L_3: $y = 5$, so $f(x,5) = -x^2 + 4x + 5 = -(x-2)^2 + 9$, $0 \le x \le 4$, which has a maximum value when $x = 2$ where $f(2,5) = 9$ and a minimum value both when $x = 0$ and $x = 4$, where $f(0,5) = f(4,5) = 5$. Along L_4: $x = 0$, so $f(0,y) = 6y - y^2 = -(y-3)^2 + 9$, $0 \le y \le 5$, which has a maximum value when $y = 3$ where $f(0,3) = 9$ and a minimum value when $y = 0$ where $f(0,0) = 0$. Thus the absolute maximum is $f(2,3) = 13$ and the absolute minimum is attained at both $(0,0)$ and $(4,0)$, where $f(0,0) = f(4,0) = 0$.



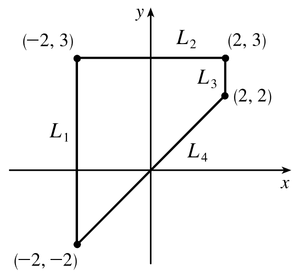
37. $f(x,y) = x^4 + y^4 - 4xy + 2$ is a polynomial and hence continuous on D, so it has an absolute maximum and minimum on D. $f_x(x,y) = 4x^3 - 4y$ and $f_y(x,y) = 4y^3 - 4x$; then $f_x = 0$ implies $y = x^3$, and substitution into $f_y = 0 \quad \Rightarrow \quad x = y^3$ gives $x^9 - x = 0 \quad \Rightarrow \quad x(x^8 - 1) = 0 \quad \Rightarrow \quad x = 0$ or $x = \pm 1$. Thus the critical points are $(0,0)$, $(1,1)$, and $(-1,-1)$, but only $(1,1)$ with $f(1,1) = 0$ is inside D. On L_1: $y = 0$, $f(x,0) = x^4 + 2$, $0 \le x \le 3$, a polynomial in x which attains its maximum at $x = 3$, $f(3,0) = 83$, and its minimum at $x = 0$, $f(0,0) = 2$. On L_2: $x = 3$, $f(3,y) = y^4 - 12y + 83$, $0 \le y \le 2$, a polynomial in y which attains its minimum at $y = \sqrt[3]{3}$, $f(3, \sqrt[3]{3}) = 83 - 9\sqrt[3]{3} \approx 70.0$, and its maximum at $y = 0$, $f(3,0) = 83$.



[continued]

On L_3: $y = 2$, $f(x, 2) = x^4 - 8x + 18$, $0 \le x \le 3$, a polynomial in x which attains its minimum at $x = \sqrt[3]{2}$, $f(\sqrt[3]{2}, 2) = 18 - 6\sqrt[3]{2} \approx 10.4$, and its maximum at $x = 3$, $f(3, 2) = 75$. On L_4: $x = 0$, $f(0, y) = y^4 + 2$, $0 \le y \le 2$, a polynomial in y which attains its maximum at $y = 2$, $f(0, 2) = 18$, and its minimum at $y = 0$, $f(0, 0) = 2$. Thus the absolute maximum of f on D is $f(3, 0) = 83$ and the absolute minimum is $f(1, 1) = 0$.

38. $f_x(x, y) = 3x^2 - 3$ and $f_y(x, y) = -3y^2 + 12$ and the critical points are $(1, 2)$, $(1, -2)$, $(-1, 2)$, and $(-1, -2)$. But only $(1, 2)$ and $(-1, 2)$ are in D and $f(1, 2) = 14$, $f(-1, 2) = 18$. Along L_1: $x = -2$ and $f(-2, y) = -2 - y^3 + 12y$, $-2 \le y \le 3$, which has a maximum at $y = 2$ where $f(-2, 2) = 14$ and a minimum at $y = -2$ where $f(-2, -2) = -18$. Along L_2: $x = 2$ and $f(2, y) = 2 - y^3 + 12y$, $2 \le y \le 3$, which has a maximum at $y = 2$ where $f(2, 2) = 18$ and a minimum at $y = 3$ where $f(2, 3) = 11$. Along L_3: $y = 3$ and $f(x, 3) = x^3 - 3x + 9$, $-2 \le x \le 2$, which has a maximum at $x = -1$ and $x = 2$ where $f(-1, 3) = f(2, 3) = 11$ and a minimum at $x = 1$ and $x = -2$ where $f(1, 3) = f(-2, 3) = 7$. Along L_4: $y = x$ and $f(x, x) = 9x$, $-2 \le x \le 2$, which has a maximum at $x = 2$ where $f(2, 2) = 18$ and a minimum at $x = -2$ where $f(-2, -2) = -18$. So the absolute maximum value of f on D is $f(2, 2) = 18$ and the minimum is $f(-2, -2) = -18$.



39. $f(x, y) = -(x^2 - 1)^2 - (x^2y - x - 1)^2 \quad\Rightarrow\quad f_x(x, y) = -2(x^2 - 1)(2x) - 2(x^2y - x - 1)(2xy - 1)$ and $f_y(x, y) = -2(x^2y - x - 1)x^2$. Setting $f_y(x, y) = 0$ gives either $x = 0$ or $x^2y - x - 1 = 0$.

There are no critical points for $x = 0$, since $f_x(0, y) = -2$, so we set $x^2y - x - 1 = 0 \quad\Leftrightarrow\quad y = \dfrac{x+1}{x^2} \quad [x \ne 0]$,

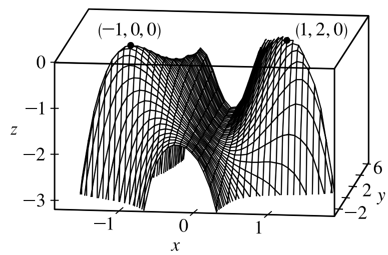
so $f_x\left(x, \dfrac{x+1}{x^2}\right) = -2(x^2 - 1)(2x) - 2\left(x^2\,\dfrac{x+1}{x^2} - x - 1\right)\left(2x\,\dfrac{x+1}{x^2} - 1\right) = -4x(x^2 - 1)$. Therefore $f_x(x, y) = f_y(x, y) = 0$ at the points $(1, 2)$ and $(-1, 0)$. To classify these critical points, we calculate

$f_{xx}(x, y) = -12x^2 - 12x^2y^2 + 12xy + 4y + 2$, $f_{yy}(x, y) = -2x^4$,

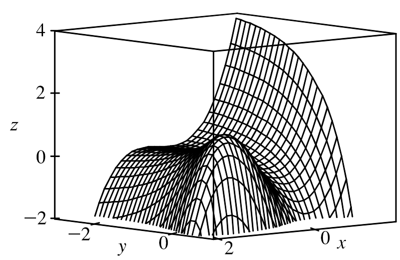
and $f_{xy}(x, y) = -8x^3y + 6x^2 + 4x$. In order to use the Second Derivatives Test we calculate

$D(-1, 0) = f_{xx}(-1, 0)\, f_{yy}(-1, 0) - [f_{xy}(-1, 0)]^2 = 16 > 0$,

$f_{xx}(-1, 0) = -10 < 0$, $D(1, 2) = 16 > 0$, and $f_{xx}(1, 2) = -26 < 0$, so both $(-1, 0)$ and $(1, 2)$ give local maxima.



40. $f(x, y) = 3xe^y - x^3 - e^{3y}$ is differentiable everywhere, so the requirement for critical points is that $f_x = 3e^y - 3x^2 = 0$ **(1)** and $f_y = 3xe^y - 3e^{3y} = 0$ **(2)**. From **(1)** we obtain $e^y = x^2$, and then **(2)** gives $3x^3 - 3x^6 = 0 \quad\Rightarrow\quad x = 1$ or 0, but only $x = 1$ is valid, since $x = 0$ makes **(1)** impossible. So substituting $x = 1$ into **(1)** gives $y = 0$, and the only critical point is $(1, 0)$.



[continued]

The Second Derivatives Test shows that this gives a local maximum, since $D(1,0) = \left[-6x(3xe^y - 9e^{3y}) - (3e^y)^2\right]_{(1,0)} = 27 > 0$ and $f_{xx}(1,0) = [-6x]_{(1,0)} = -6 < 0$. But $f(1,0) = 1$ is not an absolute maximum because, for instance, $f(-3,0) = 17$. This can also be seen from the graph.

41. Let d be the distance from $(2,0,-3)$ to any point (x,y,z) on the plane $x+y+z=1$, so $d = \sqrt{(x-2)^2 + y^2 + (z+3)^2}$ where $z = 1-x-y$, and we minimize $d^2 = f(x,y) = (x-2)^2 + y^2 + (4-x-y)^2$. Then $f_x(x,y) = 2(x-2) + 2(4-x-y)(-1) = 4x + 2y - 12$, $f_y(x,y) = 2y + 2(4-x-y)(-1) = 2x + 4y - 8$. Solving $4x + 2y - 12 = 0$ and $2x + 4y - 8 = 0$ simultaneously gives $x = \frac{8}{3}$, $y = \frac{2}{3}$, so the only critical point is $\left(\frac{8}{3}, \frac{2}{3}\right)$. An absolute minimum exists (since there is a minimum distance from the point to the plane) and it must occur at a critical point, so the shortest distance occurs for $x = \frac{8}{3}$, $y = \frac{2}{3}$ for which $d = \sqrt{\left(\frac{8}{3}-2\right)^2 + \left(\frac{2}{3}\right)^2 + \left(4 - \frac{8}{3} - \frac{2}{3}\right)^2} = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}}$.

42. Here the distance d from a point on the plane to the point $(0,1,1)$ is $d = \sqrt{x^2 + (y-1)^2 + (z-1)^2}$, where $z = 2 - \frac{1}{3}x + \frac{2}{3}y$. We can minimize $d^2 = f(x,y) = x^2 + (y-1)^2 + \left(1 - \frac{1}{3}x + \frac{2}{3}y\right)^2$, so

$f_x(x,y) = 2x + 2\left(1 - \frac{1}{3}x + \frac{2}{3}y\right)\left(-\frac{1}{3}\right) = \frac{20}{9}x - \frac{4}{9}y - \frac{2}{3}$ and

$f_y(x,y) = 2(y-1) + 2\left(1 - \frac{1}{3}x + \frac{2}{3}y\right)\left(\frac{2}{3}\right) = -\frac{4}{9}x + \frac{26}{9}y - \frac{2}{3}$. Solving $\frac{20}{9}x - \frac{4}{9}y - \frac{2}{3} = 0$ and $-\frac{4}{9}x + \frac{26}{9}y - \frac{2}{3} = 0$ simultaneously gives $x = \frac{5}{14}$ and $y = \frac{2}{7}$, so the only critical point is $\left(\frac{5}{14}, \frac{2}{7}\right)$.

This point must correspond to the minimum distance, so the point on the plane closest to $(0,1,1)$ is $\left(\frac{5}{14}, \frac{2}{7}, \frac{29}{14}\right)$.

43. Let d be the distance from the point $(4,2,0)$ to any point (x,y,z) on the cone, so $d = \sqrt{(x-4)^2 + (y-2)^2 + z^2}$ where $z^2 = x^2 + y^2$, and we minimize $d^2 = (x-4)^2 + (y-2)^2 + x^2 + y^2 = f(x,y)$. Then $f_x(x,y) = 2(x-4) + 2x = 4x - 8$, $f_y(x,y) = 2(y-2) + 2y = 4y - 4$, and the critical points occur when $f_x = 0 \;\Rightarrow\; x = 2$, $f_y = 0 \;\Rightarrow\; y = 1$. Thus the only critical point is $(2,1)$. An absolute minimum exists (since there is a minimum distance from the cone to the point) which must occur at a critical point, so the points on the cone closest to $(4,2,0)$ are $\left(2, 1, \pm\sqrt{5}\right)$.

44. The distance from the origin to a point (x,y,z) on the surface is $d = \sqrt{x^2 + y^2 + z^2}$ where $y^2 = 9 + xz$, so we minimize $d^2 = x^2 + 9 + xz + z^2 = f(x,z)$. Then $f_x = 2x + z$, $f_z = x + 2z$, and $f_x = 0$, $f_z = 0 \;\Rightarrow\; x = 0$, $z = 0$, so the only critical point is $(0,0)$. $D(0,0) = (2)(2) - 1 = 3 > 0$ with $f_{xx}(0,0) = 2 > 0$, so this is a minimum. Thus $y^2 = 9 + 0 \;\Rightarrow\; y = \pm 3$ and the points on the surface closest to the origin are $(0, \pm 3, 0)$.

45. Let x, y, z be the positive numbers. Then $x + y + z = 100 \;\Rightarrow\; z = 100 - x - y$, and we want to maximize $xyz = xy(100 - x - y) = 100xy - x^2y - xy^2 = f(x,y)$ for $0 < x, y, z < 100$. $f_x = 100y - 2xy - y^2$, $f_y = 100x - x^2 - 2xy$, $f_{xx} = -2y$, $f_{yy} = -2x$, $f_{xy} = 100 - 2x - 2y$. Then $f_x = 0$ implies $y(100 - 2x - y) = 0 \;\Rightarrow\; y = 100 - 2x$ (since $y > 0$). Substituting into $f_y = 0$ gives $x[100 - x - 2(100 - 2x)] = 0 \;\Rightarrow\; 3x - 100 = 0$ (since $x > 0$) $\;\Rightarrow\; x = \frac{100}{3}$. Then $y = 100 - 2\left(\frac{100}{3}\right) = \frac{100}{3}$, and the only critical point is $\left(\frac{100}{3}, \frac{100}{3}\right)$.

$D\left(\frac{100}{3}, \frac{100}{3}\right) = \left(-\frac{200}{3}\right)\left(-\frac{200}{3}\right) - \left(-\frac{100}{3}\right)^2 = \frac{10{,}000}{3} > 0$ and $f_{xx}\left(\frac{100}{3}, \frac{100}{3}\right) = -\frac{200}{3} < 0$. Thus $f\left(\frac{100}{3}, \frac{100}{3}\right)$ is a local maximum. It is also the absolute maximum (compare to the values of f as x, y, or $z \to 0$ or 100), so the numbers are $x = y = z = \frac{100}{3}$.

46. Let x, y, z, be the positive numbers. Then $x+y+z=12$ and we want to minimize $x^2+y^2+z^2=x^2+y^2+(12-x-y)^2=f(x,y)$ for $0<x,y<12$. $f_x=2x+2(12-x-y)(-1)=4x+2y-24$, $f_y=2y+2(12-x-y)(-1)=2x+4y-24$, $f_{xx}=4$, $f_{xy}=2$, $f_{yy}=4$. Then $f_x=0$ implies $4x+2y=24$ or $y=12-2x$ and substituting into $f_y=0$ gives $2x+4(12-2x)=24 \quad\Rightarrow\quad 6x=24 \quad\Rightarrow\quad x=4$ and then $y=4$, so the only critical point is $(4,4)$. $D(4,4)=16-4>0$ and $f_{xx}(4,4)=4>0$, so $f(4,4)$ is a local minimum. $f(4,4)$ is also the absolute minimum [compare to the values of f as $x,y\to 0$ or 12] so the numbers are $x=y=z=4$.

47. Center the sphere at the origin so that its equation is $x^2+y^2+z^2=r^2$, and orient the inscribed rectangular box so that its edges are parallel to the coordinate axes. Any vertex of the box satisfies $x^2+y^2+z^2=r^2$, so take (x,y,z) to be the vertex in the first octant. Then the box has length $2x$, width $2y$, and height $2z=2\sqrt{r^2-x^2-y^2}$ with volume given by

$V(x,y)=(2x)(2y)\left(2\sqrt{r^2-x^2-y^2}\right)=8xy\sqrt{r^2-x^2-y^2}$ for $0<x<r$, $0<y<r$. Then

$$V_x=(8xy)\cdot\tfrac{1}{2}(r^2-x^2-y^2)^{-1/2}(-2x)+\sqrt{r^2-x^2-y^2}\cdot 8y=\frac{8y(r^2-2x^2-y^2)}{\sqrt{r^2-x^2-y^2}} \text{ and } V_y=\frac{8x(r^2-x^2-2y^2)}{\sqrt{r^2-x^2-y^2}}.$$

Setting $V_x=0$ gives $y=0$ or $2x^2+y^2=r^2$, but $y>0$ so only the latter solution applies. Similarly, $V_y=0$ with $x>0$ implies $x^2+2y^2=r^2$. Substituting, we have $2x^2+y^2=x^2+2y^2 \quad\Rightarrow\quad x^2=y^2 \quad\Rightarrow\quad y=x$. Then $x^2+2y^2=r^2 \quad\Rightarrow\quad 3x^2=r^2 \quad\Rightarrow\quad x=\sqrt{r^2/3}=r/\sqrt{3}=y$. Thus the only critical point is $\left(r/\sqrt{3},r/\sqrt{3}\right)$. There must be a maximum volume and here it must occur at a critical point, so the maximum volume occurs when $x=y=r/\sqrt{3}$ and the maximum volume is $V\left(\frac{r}{\sqrt{3}},\frac{r}{\sqrt{3}}\right)=8\left(\frac{r}{\sqrt{3}}\right)\left(\frac{r}{\sqrt{3}}\right)\sqrt{r^2-\left(\frac{r}{\sqrt{3}}\right)^2-\left(\frac{r}{\sqrt{3}}\right)^2}=\dfrac{8}{3\sqrt{3}}\,r^3$.

48. Let x, y, and z be the dimensions of the box. We wish to minimize surface area $=2xy+2xz+2yz$, but we have volume $=xyz=1000 \quad\Rightarrow\quad z=\dfrac{1000}{xy}$ so we minimize

$f(x,y)=2xy+2x\left(\dfrac{1000}{xy}\right)+2y\left(\dfrac{1000}{xy}\right)=2xy+\dfrac{2000}{y}+\dfrac{2000}{x}$. Then $f_x=2y-\dfrac{2000}{x^2}$ and $f_y=2x-\dfrac{2000}{y^2}$. Setting $f_x=0$ implies $y=\dfrac{1000}{x^2}$ and substituting into $f_y=0$ gives $x-\dfrac{x^4}{1000}=0 \quad\Rightarrow\quad x^3=1000$ [since $x\neq 0$] $\quad\Rightarrow\quad x=10$. The surface area has a minimum but no maximum and it must occur at a critical point, so the minimal surface area occurs for a box with dimensions $x=10$ cm, $y=1000/10^2=10$ cm, $z=1000/10^2=10$ cm.

49. Maximize $f(x,y)=\dfrac{xy}{3}(6-x-2y)$, then the maximum volume is $V=xyz$.

$f_x=\frac{1}{3}(6y-2xy-y^2)=\frac{1}{3}y(6-2x-2y)$ and $f_y=\frac{1}{3}x(6-x-4y)$. Setting $f_x=0$ and $f_y=0$ gives the critical point $(2,1)$ which geometrically must give a maximum. Thus the volume of the largest such box is $V=(2)(1)\left(\frac{2}{3}\right)=\frac{4}{3}$.

50. Surface area $=2(xy+xz+yz)=64\text{ cm}^2$, so $xy+xz+yz=32$ or $z=\dfrac{32-xy}{x+y}$. Maximize the volume

$f(x,y)=xy\,\dfrac{32-xy}{x+y}$. Then $f_x=\dfrac{32y^2-2xy^3-x^2y^2}{(x+y)^2}=y^2\,\dfrac{32-2xy-x^2}{(x+y)^2}$ and $f_y=x^2\,\dfrac{32-2xy-y^2}{(x+y)^2}$. Setting

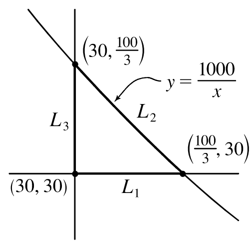
$f_x = 0$ implies $y = \dfrac{32 - x^2}{2x}$ and substituting into $f_y = 0$ gives $32(4x^2) - (32 - x^2)(4x^2) - (32 - x^2)^2 = 0$ or $3x^4 + 64x^2 - (32)^2 = 0$. Thus $x^2 = \frac{64}{6}$ or $x = \frac{8}{\sqrt{6}}$, $y = \frac{64/3}{16/\sqrt{6}} = \frac{8}{\sqrt{6}}$ and $z = \frac{8}{\sqrt{6}}$. Thus the box is a cube with edge length $\frac{8}{\sqrt{6}}$ cm.

51. Let the dimensions be x, y, and z; then $4x + 4y + 4z = c$ and the volume is $V = xyz = xy\left(\frac{1}{4}c - x - y\right) = \frac{1}{4}cxy - x^2y - xy^2$, $x > 0$, $y > 0$. Then $V_x = \frac{1}{4}cy - 2xy - y^2$ and $V_y = \frac{1}{4}cx - x^2 - 2xy$, so $V_x = 0 = V_y$ when $2x + y = \frac{1}{4}c$ and $x + 2y = \frac{1}{4}c$. Solving, we get $x = \frac{1}{12}c$, $y = \frac{1}{12}c$ and $z = \frac{1}{4}c - x - y = \frac{1}{12}c$. From the geometrical nature of the problem, this critical point must give an absolute maximum. Thus the box is a cube with edge length $\frac{1}{12}c$.

52. The cost equals $5xy + 2(xz + yz)$ and $xyz = V$, so $C(x, y) = 5xy + 2V(x + y)/(xy) = 5xy + 2V(x^{-1} + y^{-1})$. Then $C_x = 5y - 2Vx^{-2}$, $C_y = 5x - 2Vy^{-2}$, $C_x = 0$ implies $y = 2V/(5x^2)$, $C_y = 0$ implies $x = \sqrt[3]{\frac{2}{5}V} = y$. Thus the dimensions of the aquarium which minimize the cost are $x = y = \sqrt[3]{\frac{2}{5}V}$ units, $z = V^{1/3}\left(\frac{5}{2}\right)^{2/3}$.

53. Let the dimensions be x, y and z, then minimize $xy + 2(xz + yz)$ if $xyz = 32{,}000\text{ cm}^3$. Then $f(x, y) = xy + [64{,}000(x + y)/xy] = xy + 64{,}000(x^{-1} + y^{-1})$, $f_x = y - 64{,}000x^{-2}$, $f_y = x - 64{,}000y^{-2}$. And $f_x = 0$ implies $y = 64{,}000/x^2$; substituting into $f_y = 0$ implies $x^3 = 64{,}000$ or $x = 40$ and then $y = 40$. Now $D(x, y) = [(2)(64{,}000)]^2x^{-3}y^{-3} - 1 > 0$ for $(40, 40)$ and $f_{xx}(40, 40) > 0$ so this is indeed a minimum. Thus the dimensions of the box are $x = y = 40$ cm, $z = 20$ cm.

54. Let x be the length of the north and south walls, y the length of the east and west walls, and z the height of the building. The heat loss is given by $h = 10(2yz) + 8(2xz) + 1(xy) + 5(xy) = 6xy + 16xz + 20yz$. The volume is 4000 m^3, so $xyz = 4000$, and we substitute $z = \frac{4000}{xy}$ to obtain the heat loss function $h(x, y) = 6xy + 80{,}000/x + 64{,}000/y$.



(a) Since $z = \frac{4000}{xy} \geq 4$, $xy \leq 1000 \quad\Rightarrow\quad y \leq 1000/x$. Also $x \geq 30$ and $y \geq 30$, so the domain of h is $D = \{(x, y) \mid x \geq 30, 30 \leq y \leq 1000/x\}$.

(b) $h(x, y) = 6xy + 80{,}000x^{-1} + 64{,}000y^{-1} \quad\Rightarrow$

$h_x = 6y - 80{,}000x^{-2}$, $h_y = 6x - 64{,}000y^{-2}$.

$h_x = 0$ implies $6x^2y = 80{,}000 \quad\Rightarrow\quad y = \dfrac{80{,}000}{6x^2}$ and substituting into

$h_y = 0$ gives $6x = 64{,}000\left(\dfrac{6x^2}{80{,}000}\right)^2 \quad\Rightarrow\quad x^3 = \dfrac{80{,}000^2}{6 \cdot 64{,}000} = \dfrac{50{,}000}{3}$, so

$x = \sqrt[3]{\dfrac{50{,}000}{3}} = 10\sqrt[3]{\dfrac{50}{3}} \quad\Rightarrow\quad y = \dfrac{80}{\sqrt[3]{60}}$, and the only critical point of h is $\left(10\sqrt[3]{\dfrac{50}{3}}, \dfrac{80}{\sqrt[3]{60}}\right) \approx (25.54, 20.43)$

which is not in D. Next we check the boundary of D.

On L_1: $y = 30$, $h(x, 30) = 180x + 80{,}000/x + 6400/3$, $30 \le x \le \frac{100}{3}$. Since $h'(x, 30) = 180 - 80{,}000/x^2 > 0$ for $30 \le x \le \frac{100}{3}$, $h(x, 30)$ is an increasing function with minimum $h(30, 30) = 10{,}200$ and maximum $h\left(\frac{100}{3}, 30\right) \approx 10{,}533$.

On L_2: $y = 1000/x$, $h(x, 1000/x) = 6000 + 64x + 80{,}000/x$, $30 \le x \le \frac{100}{3}$.

Since $h'(x, 1000/x) = 64 - 80{,}000/x^2 < 0$ for $30 \le x \le \frac{100}{3}$, $h(x, 1000/x)$ is a decreasing function with minimum $h\left(\frac{100}{3}, 30\right) \approx 10{,}533$ and maximum $h\left(30, \frac{100}{3}\right) \approx 10{,}587$.

On L_3: $x = 30$, $h(30, y) = 180y + 64{,}000/y + 8000/3$, $30 \le y \le \frac{100}{3}$. $h'(30, y) = 180 - 64{,}000/y^2 > 0$ for $30 \le y \le \frac{100}{3}$, so $h(30, y)$ is an increasing function of y with minimum $h(30, 30) = 10{,}200$ and maximum $h\left(30, \frac{100}{3}\right) \approx 10{,}587$.

Thus the absolute minimum of h is $h(30, 30) = 10{,}200$, and the dimensions of the building that minimize heat loss are walls 30 m in length and height $\frac{4000}{30^2} = \frac{40}{9} \approx 4.44$ m.

(c) From part (b), the only critical point of h, which gives a local (and absolute) minimum, is approximately $h(25.54, 20.43) \approx 9396$. So a building of volume 4000 m^2 with dimensions $x \approx 25.54$ m, $y \approx 20.43$ m, $z \approx \frac{4000}{(25.54)(20.43)} \approx 7.67$ m has the least amount of heat loss.

55. Let x, y, z be the dimensions of the rectangular box. Then the volume of the box is xyz and

$L = \sqrt{x^2 + y^2 + z^2} \quad \Rightarrow \quad L^2 = x^2 + y^2 + z^2 \quad \Rightarrow \quad z = \sqrt{L^2 - x^2 - y^2}$.

Substituting, we have volume $V(x, y) = xy\sqrt{L^2 - x^2 - y^2}$ $(x, y > 0)$.

$$V_x = xy \cdot \tfrac{1}{2}(L^2 - x^2 - y^2)^{-1/2}(-2x) + y\sqrt{L^2 - x^2 - y^2} = y\sqrt{L^2 - x^2 - y^2} - \frac{x^2y}{\sqrt{L^2 - x^2 - y^2}},$$

$V_y = x\sqrt{L^2 - x^2 - y^2} - \dfrac{xy^2}{\sqrt{L^2 - x^2 - y^2}}$. $V_x = 0$ implies $y(L^2 - x^2 - y^2) = x^2y \quad \Rightarrow \quad y(L^2 - 2x^2 - y^2) = 0 \quad \Rightarrow$ $2x^2 + y^2 = L^2$ (since $y > 0$), and $V_y = 0$ implies $x(L^2 - x^2 - y^2) = xy^2 \quad \Rightarrow \quad x(L^2 - x^2 - 2y^2) = 0 \quad \Rightarrow$ $x^2 + 2y^2 = L^2$ (since $x > 0$). Substituting $y^2 = L^2 - 2x^2$ into $x^2 + 2y^2 = L^2$ gives $x^2 + 2L^2 - 4x^2 = L^2 \quad \Rightarrow$ $3x^2 = L^2 \quad \Rightarrow \quad x = L/\sqrt{3}$ (since $x > 0$) and then $y = \sqrt{L^2 - 2\left(L/\sqrt{3}\right)^2} = L/\sqrt{3}$.

So the only critical point is $\left(L/\sqrt{3}, L/\sqrt{3}\right)$ which, from the geometrical nature of the problem, must give an absolute maximum. Thus the maximum volume is $V\left(L/\sqrt{3}, L/\sqrt{3}\right) = \left(L/\sqrt{3}\right)^2\sqrt{L^2 - \left(L/\sqrt{3}\right)^2 - \left(L/\sqrt{3}\right)^2} = L^3/\left(3\sqrt{3}\right)$ cubic units.

56. $Y(N, P) = kNPe^{-N-P} \quad \Rightarrow \quad Y_N = kP\left[N\left(-e^{-N-P}\right) + e^{-N-P}(1)\right] = kP(1 - N)e^{-N-P}$,

$Y_P = kN\left[P\left(-e^{-N-P}\right) + e^{-N-P}(1)\right] = kN(1 - P)e^{-N-P}$. Here $N \ge 0$ and $P \ge 0$, but if either $N = 0$ or $P = 0$ then the yield is zero. Assuming that $N > 0$ and $P > 0$, $Y_N = 0$ implies $N = 1$ and $Y_P = 0$ implies $P = 1$, so the only critical point in $\{(N, P) \mid N > 0, P > 0\}$ is $(1, 1)$ where $Y(1, 1) = ke^{-2}$.

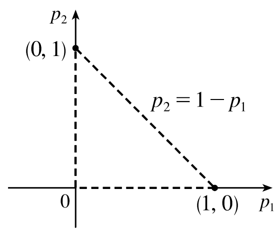
$D(N,P) = Y_{NN}Y_{PP} - (Y_{NP})^2 = \left[kP(N-2)e^{-N-P}\right]\left[kN(P-2)e^{-N-P}\right] - \left[k(1-N)(1-P)e^{-N-P}\right]^2 \quad \Rightarrow$

$D(1,1) = \left(-ke^{-2}\right)\left(-ke^{-2}\right) - (0)^2 = k^2e^{-4} > 0$ and $Y_{NN}(1,1) = -ke^{-2} < 0$, so $Y(1,1) = ke^{-2}$ is a local maximum. $Y(1,1)$ is also the absolute maximum (we have only one critical point, and $Y \to 0$ as $N \to 0$ or $P \to 0$ and $Y \to 0$ as N or P grow large) so the best yield is achieved when both the nitrogen and phosphorus levels are 1 (measured in appropriate units).

57. (a) We are given that $p_1 + p_2 + p_3 = 1 \quad \Rightarrow \quad p_3 = 1 - p_1 - p_2$, so

$H = -p_1 \ln p_1 - p_2 \ln p_2 - p_3 \ln p_3 = -p_1 \ln p_1 - p_2 \ln p_2 - (1 - p_1 - p_2) \ln (1 - p_1 - p_2)$.

(b) Because p_i is a proportion we have $0 \le p_i \le 1$, but H is undefined unless $p_1 > 0$, $p_2 > 0$, and $1 - p_1 - p_2 > 0 \quad \Leftrightarrow \quad p_1 + p_2 < 1$. This last restriction forces $p_1 < 1$ and $p_2 < 1$, so the domain of H is $\{(p_1, p_2) \mid 0 < p_1 < 1,\ p_2 < 1 - p_1\}$. It is the interior of the triangle drawn in the figure.



(c)
$$\begin{aligned} H_{p_1} &= -\left[p_1 \cdot (1/p_1) + (\ln p_1) \cdot 1\right] - \left[(1 - p_1 - p_2) \cdot (-1)/(1 - p_1 - p_2) + \ln (1 - p_1 - p_2) \cdot (-1)\right] \\ &= -1 - \ln p_1 + 1 + \ln (1 - p_1 - p_2) = \ln (1 - p_1 - p_2) - \ln p_1 \end{aligned}$$

Similarly $H_{p_2} = \ln (1 - p_1 - p_2) - \ln p_2$. Then $H_{p_1} = 0$ implies $\ln (1 - p_1 - p_2) = \ln p_1 \quad \Rightarrow \quad 1 - p_1 - p_2 = p_1 \quad \Rightarrow \quad p_2 = 1 - 2p_1$, and $H_{p_2} = 0$ implies $\ln (1 - p_1 - p_2) = \ln p_2 \quad \Rightarrow \quad p_1 = 1 - 2p_2$. Substituting, we have $p_1 = 1 - 2(1 - 2p_1) \quad \Rightarrow$ $3p_1 = 1 \quad \Rightarrow \quad p_1 = \frac{1}{3}$, and then $p_2 = 1 - 2\left(\frac{1}{3}\right) = \frac{1}{3}$. Thus the only critical point is $\left(\frac{1}{3}, \frac{1}{3}\right)$.

$D(p_1, p_2) = H_{p_1p_1}H_{p_2p_2} - (H_{p_1p_2})^2 = \left(\dfrac{-1}{1 - p_1 - p_2} - \dfrac{1}{p_1}\right)\left(\dfrac{-1}{1 - p_1 - p_2} - \dfrac{1}{p_2}\right) - \left(\dfrac{-1}{1 - p_1 - p_2}\right)^2$, so

$D\left(\frac{1}{3}, \frac{1}{3}\right) = (-6)(-6) - (-3)^2 = 27 > 0$ and $H_{p_1p_1}\left(\frac{1}{3}, \frac{1}{3}\right) = -6 < 0$. Thus $H\left(\frac{1}{3}, \frac{1}{3}\right) = -\frac{1}{3} \ln \frac{1}{3} - \frac{1}{3} \ln \frac{1}{3} - \frac{1}{3} \ln \frac{1}{3} = -\ln \frac{1}{3} = \ln 3$ is a local maximum. Here it is also the absolute maximum, so the maximum value of H is $\ln 3$, which occurs for $p_1 = p_2 = p_3 = \frac{1}{3}$ (all three species have equal proportion in the ecosystem).

58. Since $p + q + r = 1$ we can substitute $p = 1 - r - q$ into P giving $P = P(q, r) = 2(1 - r - q)q + 2(1 - r - q)r + 2rq = 2q - 2q^2 + 2r - 2r^2 - 2rq$. Since p, q and r represent proportions and $p + q + r = 1$, we know $q \ge 0$, $r \ge 0$, and $q + r \le 1$. Thus, we want to find the absolute maximum of the continuous function $P(q, r)$ on the closed set D enclosed by the lines $q = 0$, $r = 0$, and $q + r = 1$. To find any critical points, we set the partial derivatives equal to zero: $P_q(q,r) = 2 - 4q - 2r = 0$ and $P_r(q,r) = 2 - 4r - 2q = 0$. The first equation gives $r = 1 - 2q$, and substituting into the second equation we have $2 - 4(1-2q) - 2q = 0 \quad \Rightarrow \quad q = \frac{1}{3}$. Then we have one critical point, $\left(\frac{1}{3}, \frac{1}{3}\right)$, where $P\left(\frac{1}{3}, \frac{1}{3}\right) = \frac{2}{3}$. Next we find the maximum values of P on the boundary of D which consists of three line segments. For the segment given by $r = 0$, $0 \le q \le 1$, $P(q,r) = P(q,0) = 2q - 2q^2$, $0 \le q \le 1$. This represents a parabola with maximum value $P\left(\frac{1}{2}, 0\right) = \frac{1}{2}$. On the segment $q = 0$, $0 \le r \le 1$ we have $P(0,r) = 2r - 2r^2$, $0 \le r \le 1$. This represents a parabola with maximum value $P\left(0, \frac{1}{2}\right) = \frac{1}{2}$. Finally, on the segment $q + r = 1$, $0 \le q \le 1$, $P(q,r) = P(q, 1-q) = 2q - 2q^2$, $0 \le q \le 1$ which has a maximum value of $P\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}$. Comparing these values with the value of P at the critical point, we see that the absolute maximum value of $P(q,r)$ on D is $\frac{2}{3}$.

59. Note that here the variables are m and b, and $f(m,b) = \sum\limits_{i=1}^{n} [y_i - (mx_i + b)]^2$. Then $f_m = \sum\limits_{i=1}^{n} -2x_i[y_i - (mx_i + b)] = 0$ implies $\sum\limits_{i=1}^{n} \left(x_i y_i - m x_i^2 - b x_i\right) = 0$ or $\sum\limits_{i=1}^{n} x_i y_i = m \sum\limits_{i=1}^{n} x_i^2 + b \sum\limits_{i=1}^{n} x_i$ and $f_b = \sum\limits_{i=1}^{n} -2[y_i - (mx_i + b)] = 0$ implies $\sum\limits_{i=1}^{n} y_i = m \sum\limits_{i=1}^{n} x_i + \sum\limits_{i=1}^{n} b = m\left(\sum\limits_{i=1}^{n} x_i\right) + nb$. Thus we have the two desired equations.

Now $f_{mm} = \sum\limits_{i=1}^{n} 2x_i^2$, $f_{bb} = \sum\limits_{i=1}^{n} 2 = 2n$ and $f_{mb} = \sum\limits_{i=1}^{n} 2x_i$. And $f_{mm}(m,b) > 0$ always and $D(m,b) = 4n\left(\sum\limits_{i=1}^{n} x_i^2\right) - 4\left(\sum\limits_{i=1}^{n} x_i\right)^2 = 4\left[n\left(\sum\limits_{i=1}^{n} x_i^2\right) - \left(\sum\limits_{i=1}^{n} x_i\right)^2\right] > 0$ always so the solutions of these two equations do indeed minimize $\sum\limits_{i=1}^{n} d_i^2$.

60. Any such plane must cut out a tetrahedron in the first octant. We need to minimize the volume of the tetrahedron that passes through the point $(1,2,3)$. Writing the equation of the plane as $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$, the volume of the tetrahedron is given by $V = \dfrac{abc}{6}$. But $(1,2,3)$ must lie on the plane, so we need $\dfrac{1}{a} + \dfrac{2}{b} + \dfrac{3}{c} = 1 \ (\star)$ and thus can think of c as a function of a and b. Then $V_a = \dfrac{b}{6}\left(c + a\dfrac{\partial c}{\partial a}\right)$ and $V_b = \dfrac{a}{6}\left(c + b\dfrac{\partial c}{\partial b}\right)$. Differentiating $(\star)$ with respect to a we get $-a^{-2} - 3c^{-2}\dfrac{\partial c}{\partial a} = 0 \quad \Rightarrow$ $\dfrac{\partial c}{\partial a} = \dfrac{-c^2}{3a^2}$, and differentiating $(\star)$ with respect to b gives $-2b^{-2} - 3c^{-2}\dfrac{\partial c}{\partial b} = 0 \quad \Rightarrow \quad \dfrac{\partial c}{\partial b} = \dfrac{-2c^2}{3b^2}$. Then $V_a = \dfrac{b}{6}\left(c + a\dfrac{-c^2}{3a^2}\right) = 0 \quad \Rightarrow \quad c = 3a$, and $V_b = \dfrac{a}{6}\left(c + b\dfrac{-2c^2}{3b^2}\right) = 0 \quad \Rightarrow \quad c = \frac{3}{2}b$. Thus $3a = \frac{3}{2}b$ or $b = 2a$. Putting these into $(\star)$ gives $\frac{3}{a} = 1$ or $a = 3$ and then $b = 6$, $c = 9$. Thus the equation of the required plane is $\dfrac{x}{3} + \dfrac{y}{6} + \dfrac{z}{9} = 1$ or $6x + 3y + 2z = 18$.