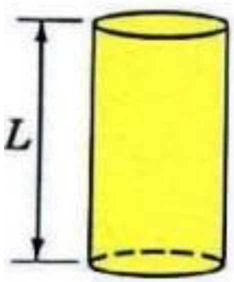
Example 37.6: One arm of a Michelson interferometer contains a transparent (透明的) cylinder of length $L = 1.5\ \text{cm}$ (figure). The cylinder is evacuated (抽空), and the cross hairs (十字準線) of the telescope (望遠鏡) are centered on a particular bright fringe with light of wavelength $600\ \text{nm}$ (in vacuum). When a gas is introduced into it, fourteen fringes move past the cross hairs. What is the refractive index of the gas?

Solution: The light travels through the cylinder twice.

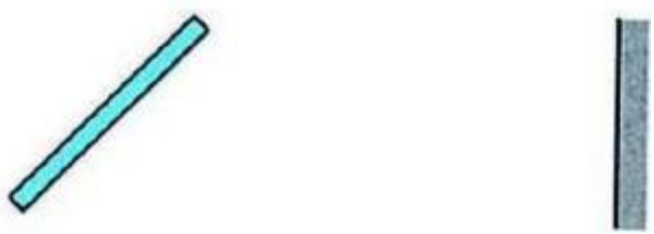
The number of wavelengths within the distance $2L$ is $2L/\lambda_0$, where λ_0 is the wavelength in vacuum.



When the gas is introduced, the wavelength changes to $\lambda = \lambda_0/n$, where n is the refractive index.

The number of wavelengths in the same distance is $2L/\lambda = 2nL/\lambda_0$.

A shift from one fringe to the next implies that the path difference has changed by one wavelength.



$$\frac{2(n-1)L}{\lambda_0} = \Delta m$$

$$n = \frac{\lambda_0 \Delta m}{2L} + 1 = \frac{(14)(6\times10^{-7}\ \text{m})}{0.03\ \text{m}} + 1 = 1.00028$$

Example 38.1: Light of wavelength $600\ \text{nm}$ is incident normally on a slit (狹縫) of width $0.1\ \text{mm}$. What is the position of the second-order (第二階) minimum on a screen $3\ \text{m}$ from the slit?

Solution:

From (Minima) $a\sin\theta = m\lambda \quad m = 1, 2, 3, \ldots$

the angular position (角(度)位置) of the second-order ($m = 2$) minimum is given by

$$\sin\theta = \frac{2\lambda}{a}$$

If y is the distance from the center of the screen, then

$$\sin\theta \approx \tan\theta = \frac{y}{L}$$

where L is the distance from the slit to the screen.

Thus, for second order ($m = 2$) we have

$$y \approx L\sin\theta = L\left(\frac{2\lambda}{a}\right)$$

$$= \frac{(3\ \text{m})(2)(600\times 10^{-9}\ \text{m})}{0.1\times 10^{-3}\ \text{m}} = 3.6\times 10^{-2}\ \text{m} = 3.6\ \text{cm}$$

Example 38.2: In a double-slit experiment the slits are $0.25\ \mathrm{mm}$ wide and their centers are separated by $1\ \mathrm{mm}$. Which interference maxima are missing?

Solution:

The missing orders in the interference pattern occur when an interference maximum, given by

(Interference maxima) $d\sin\theta = m\lambda \quad m = 0, \pm 1, \pm 2, \ldots$ (i)

coincides with a diffraction minimum given by (with a temporary change in notation)

(Diffraction minima) $a\sin\theta = M\lambda \quad M = 1,\ 2,\ 3, \ldots$ (ii)

Note that a is the width of each slit and d is the separation between the slits.

Dividing (i) by (ii) we obtain $d/a = m/M$.

When the ratio $d/a = k$, an integer, the interference peaks given by $m = kM$ are missing.

In the present example, $d = 4a$, and so the interference orders $m = 4, 8, 12, \ldots$ are missing.

Example 39.4: The figure shows a train (frame S') of proper length $L_0 = 9$ km, moving at velocity $v = 0.8c$ relative to a platform (frame S). At $t' = 0$, observers A' and B' at the ends of the train fire bullets that make marks at A and B on the platform. According to observers in S, what is the time interval between the shots (射擊, 發射)?

Solution:

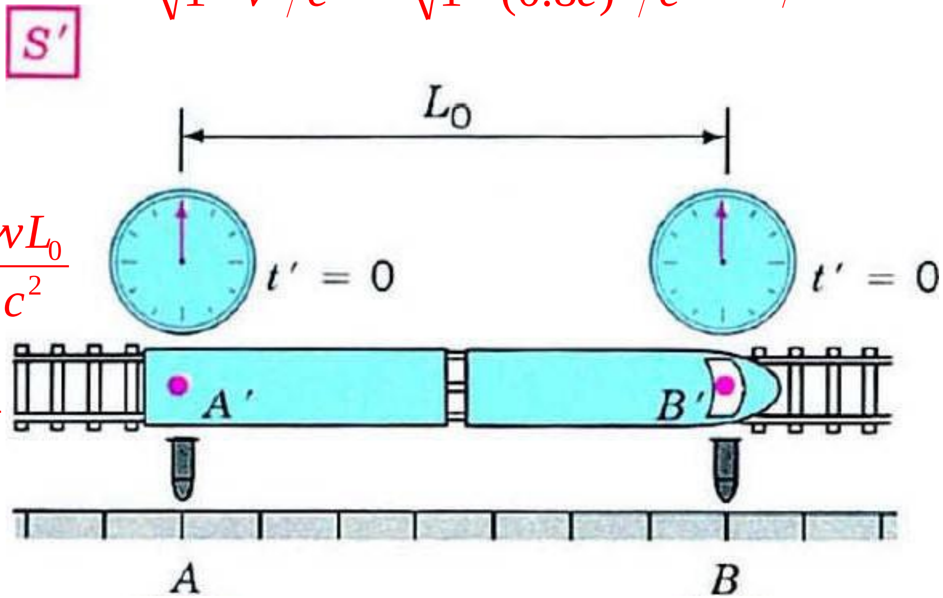
With $\Delta t' = 0$ and $\Delta x' = x'_B - x'_A = L_0$, the interval between the shots in frame S is given by $t = \gamma\left(t' + \frac{vx'}{c^2}\right)$, where $\gamma = \frac{1}{\sqrt{1-v^2/c^2}} = \frac{1}{\sqrt{1-(0.8c)^2/c^2}} = \frac{1}{3/5} = \frac{5}{3}$:

$$t_A = \gamma\left(t'_A + \frac{vx'_A}{c^2}\right) \qquad t_B = \gamma\left(t'_B + \frac{vx'_B}{c^2}\right)$$

$$t_B - t_A = \gamma\left[(t'_B - t'_A) + \frac{v(x'_B - x'_A)}{c^2}\right] = +\frac{\gamma v L_0}{c^2}$$

$$= \frac{(5/3)(2.4\times10^8\ \text{m/s})(9\times10^3\ \text{m})}{(3\times10^8\ \text{m/s})^2}$$

$$= 4\times10^{-5}\ \text{s} = 40\ \mu\text{s}$$



Example 39.7: An electron has a kinetic energy of $2\ \mathrm{MeV}$. Find its speed.

Solution:

The rest energy of the electron is

$$E_0 = m_0 c^2 = (9.11\times10^{-31}\ \mathrm{kg})(3\times10^8\ \mathrm{m/s})^2$$
$$= 8.20\times10^{-14}\ \mathrm{J} = 0.512\ \mathrm{MeV}$$

The total energy is

$$E = K + m_0 c^2 = 2 + 0.512 = 2.51\ \mathrm{MeV}$$

On comparing $E = \gamma m_0 c^2 = 2.51\ \mathrm{MeV}$ with $E_0 = m_0 c^2 = 0.512\ \mathrm{MeV}$ we see that

$$\gamma = \frac{E}{E_0} = 4.91$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \quad \rightarrow \quad \gamma^2 = \frac{1}{1 - v^2/c^2}$$

We rewrite $\dfrac{1}{\gamma^2} = 1 - \dfrac{v^2}{c^2}$ in the form $\dfrac{v^2}{c^2} = 1 - \dfrac{1}{\gamma^2} = 0.96$

to find $v/c = 0.98$, or

$$v = 0.98c = (0.98)(3\times10^8) = 2.94\times10^8\ \mathrm{m/s}$$

Example 40.2: A block of mass 0.2 kg oscillates (振盪) at the end of a spring (k = 5 N/m) with an amplitude of 10 cm. What is its “quantum number (量子數)” n?

Solution:

In order to apply Einstein’s hypothesis, $E_n = nhf$, we must first calculate the energy.

The energy of a simple harmonic oscillator is

$$E = \frac{1}{2}kA^2 = \frac{1}{2}(5\ \mathrm{N/m})(0.1\ \mathrm{m})^2 = 0.025\ \mathrm{J}$$

From $\omega = \sqrt{\frac{k}{m}}$, the frequency of the oscillation is

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = 0.80\ \mathrm{Hz}$$

From $E_n = nhf$ we find

$$n = \frac{1}{2}\frac{kA^2}{hf} = \frac{(0.025\ \mathrm{J})}{(6.63\times 10^{-34}\ \mathrm{J\cdot s})(0.80\ \mathrm{Hz})} \approx 10^{32}$$

Example 40.7: An electron is in an excited state (激發態) for which $n = 3$. What are the frequencies that can be radiated?

Solution:

From the second postulate we have $f = \Delta E/h$.

From $E_n = -\dfrac{13.6 Z^2}{n^2}\,\mathrm{eV}$

$$E_1 = -\frac{13.6}{1^2} = -13.6\,\mathrm{eV} \qquad E_2 = -\frac{13.6}{2^2} = -3.4\,\mathrm{eV} \qquad E_3 = -\frac{13.6}{3^2} = -1.51\,\mathrm{eV}$$

$$h = 6.63\times10^{-34}\ \mathrm{J\cdot s} = \frac{6.63\times10^{-34}\ \mathrm{J\cdot s}}{1.6\times10^{-19}\ \mathrm{J/eV}} = 4.14\times10^{-15}\ \mathrm{eV\cdot s}$$

$$f_{31} = \frac{E_3 - E_1}{h} = \frac{(-1.51\,\mathrm{eV}) - (-13.6\,\mathrm{eV})}{4.14\times10^{-15}\ \mathrm{eV\cdot s}} = \frac{+12.1\,\mathrm{eV}}{4.14\times10^{-15}\ \mathrm{eV\cdot s}} = 2.92\times10^{15}\ \mathrm{Hz}$$

$$f_{32} = \frac{E_3 - E_2}{h} = \frac{(-1.51\,\mathrm{eV}) - (-3.4\,\mathrm{eV})}{4.14\times10^{-15}\ \mathrm{eV\cdot s}} = \frac{+1.89\,\mathrm{eV}}{4.14\times10^{-15}\ \mathrm{eV\cdot s}} = 4.57\times10^{14}\ \mathrm{Hz}$$

$$f_{21} = \frac{E_2 - E_1}{h} = \frac{(-3.4\ \mathrm{eV}) - (-13.6\ \mathrm{eV})}{4.14\times10^{-15}\ \mathrm{eV\cdot s}} = \frac{+10.2\ \mathrm{eV}}{4.14\times10^{-15}\ \mathrm{eV\cdot s}} = 2.46\times10^{15}\ \mathrm{Hz}$$

Example 41.4: What is the minimum uncertainty in the position of a 10-g bullet moving at 400 m/s if the speed is measured to an uncertainty of 0.1%?

Solution:

The uncertainty in momentum is

$$\Delta p = m\Delta v = (10\times10^{-3}\ \text{kg})(400\times10^{-3}\ \text{m/s})$$
$$= 4\times10^{-3}\ \text{kg}\cdot\text{m/s}$$

From the Heisenberg uncertainty relation (海森堡測不準關係), the uncertainty in the position is

$$\Delta x \approx \frac{h}{\Delta p} = \frac{6.63\times10^{-34}\ \text{J}\cdot\text{s}}{4\times10^{-3}\ \text{kg}\cdot\text{m/s}}$$
$$= 1.66\times10^{-31}\ \text{m}$$

This value is less than the diameter of a single proton.

The uncertainty principle (測不準原理) presents no practical restriction (限制) on determining the position of the bullet.

Example 42.1: What are the allowed values of θ for $\ell = 2$?

Solution:

With $\ell = 2$, we have

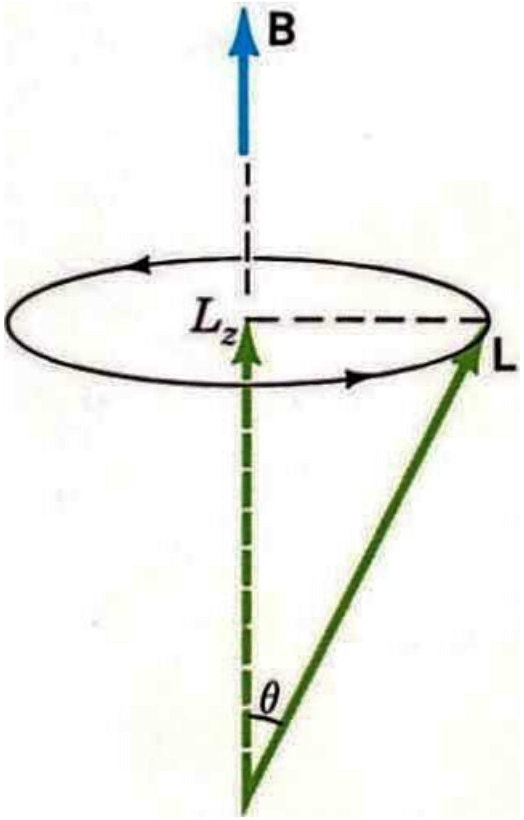
$$L = \sqrt{\ell(\ell+1)}\hbar = \sqrt{2(2+1)}\hbar = \sqrt{6}\hbar$$

and $m_\ell = 0, \pm 1, \pm 2$.

Therefore, from

$$\cos\theta = \frac{L_z}{L} = \frac{m_\ell}{\sqrt{\ell(\ell+1)}}$$

$$\cos\theta = \frac{m_\ell}{\sqrt{6}} = 0; \pm\frac{1}{\sqrt{6}}; \pm\frac{2}{\sqrt{6}}$$



from which we find $\theta = 90^\circ$, 65.9°, 35.3°, 114.1°, and 144.7°.

Example 43.5: A sample with 10 g of carbon registers a decay rate of 30 decays/min. How old is it?

Solution:

The number of ^{12}C atoms in each gram is

$$N_0 = \frac{mN_A}{M} = \frac{(1\text{ g})(6.02\times10^{23}\text{ atoms/mol})}{12\text{ g/mol}} = 5.02\times10^{22}\text{ atoms}$$

From the relative abundance (相對豐度, 相對含量) quoted (引用, 引述) above, we find the number of ^{14}C atoms is $(1.3\times10^{-12})N_0 = 6.5\times10^{10}$.

The decay constant $\lambda = \dfrac{\ln 2}{T_{1/2}} = \dfrac{0.693}{5730\text{ y}} = \dfrac{0.693}{(5730)(365)(24)(60)(60)\text{ s}} = 3.835\times10^{-12}$

The initial decay rate for each gram is

$$R_0 = \lambda N_0 = (3.835\times10^{-12})(6.5\times10^{10}) = 0.25\text{ Bq}$$

For the 10 g sample there are 30 decays/min = 0.5 decay/s, which means R = 0.05 Bq for each gram.

$$R = \lambda N = R_0 e^{-\lambda t} \quad \rightarrow \quad \frac{R}{R_0} = e^{-\lambda t}$$

$$\rightarrow\ t = \frac{1}{\lambda}\ln\left(\frac{R_0}{R}\right) = \frac{1}{3.835\times10^{-12}}\ln\left(\frac{0.25}{0.05}\right) = 4.2\times10^{11}\text{ s} = 13{,}300\text{ y}$$