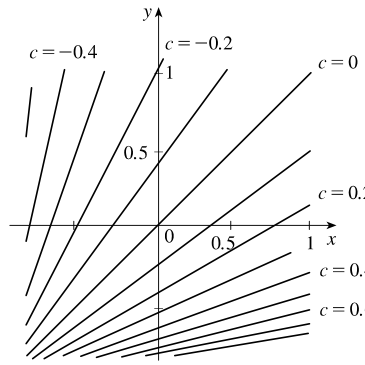
4. The level curves of $f(x,y) = \left(\dfrac{1+x}{1+y}\right)^{1/3} - 1$ are

$$\left(\frac{1+x}{1+y}\right)^{1/3} - 1 = c \quad \Rightarrow \quad \frac{1+x}{1+y} = (1+c)^3 \quad \Rightarrow$$

$$y = \frac{1+x}{(1+c)^3} - 1.$$

From the level curves, we see that increasing x (from 0) by a small amount has a similar effect on the value of f as decreasing y by a small amount. However, for larger changes, a decrease in y gives greater values of f than a similar increase in x.



14.5 The Chain Rule

1. $z = xy^3 - x^2y, \quad x = t^2+1, \quad y = t^2 - 1 \quad \Rightarrow$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (y^3 - 2xy)(2t) + (3xy^2 - x^2)(2t) = 2t(y^3 - 2xy + 3xy^2 - x^2)$$

2. $z = \dfrac{x-y}{x+2y}, \quad x = e^{\pi t}, \quad y = e^{-\pi t} \quad \Rightarrow$

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \frac{(x+2y)(1)-(x-y)(1)}{(x+2y)^2}(\pi e^{\pi t}) + \frac{(x+2y)(-1)-(x-y)(2)}{(x+2y)^2}(-\pi e^{-\pi t})\\ &= \frac{3y}{(x+2y)^2}\left(\pi e^{\pi t}\right) + \frac{-3x}{(x+2y)^2}\left(-\pi e^{-\pi t}\right) = \frac{3\pi}{(x+2y)^2}\left(ye^{\pi t} + xe^{-\pi t}\right)\end{aligned}$$

3. $z = \sin x \cos y, \quad x = \sqrt{t}, \quad y = 1/t \quad \Rightarrow$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (\cos x \cos y)\left(\tfrac{1}{2}t^{-1/2}\right) + (-\sin x \sin y)\left(-t^{-2}\right) = \frac{1}{2\sqrt{t}}\cos x \cos y + \frac{1}{t^2}\sin x \sin y$$

4. $z = \sqrt{1+xy}, \quad x = \tan t, \quad y = \arctan t \quad \Rightarrow$

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \tfrac{1}{2}(1+xy)^{-1/2}(y)\cdot \sec^2 t + \tfrac{1}{2}(1+xy)^{-1/2}(x)\cdot \frac{1}{1+t^2}\\ &= \frac{1}{2\sqrt{1+xy}}\left(y\sec^2 t + \frac{x}{1+t^2}\right)\end{aligned}$$

5. $w = xe^{y/z}, \ x = t^2, \ y = 1-t, \ z = 1+2t \quad \Rightarrow$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} = e^{y/z}\cdot 2t + xe^{y/z}\left(\frac{1}{z}\right)\cdot(-1) + xe^{y/z}\left(-\frac{y}{z^2}\right)\cdot 2 = e^{y/z}\left(2t - \frac{x}{z} - \frac{2xy}{z^2}\right)$$

6. $z = \tan^{-1}(y/x), x = e^t, y = 1 - e^{-t} \quad \Rightarrow$

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \frac{1}{1+(y/x)^2}(-yx^{-2})\cdot e^t + \frac{1}{1+(y/x)^2}(1/x)\cdot(-e^{-t})(-1)\\ &= -\frac{y}{x^2+y^2}\cdot e^t + \frac{1}{x+y^2/x}\cdot e^{-t} = \frac{xe^{-t} - ye^t}{x^2+y^2}\end{aligned}$$

7. $z = (x - y)^5, \quad x = s^2 t, \quad y = st^2 \quad \Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = 5(x-y)^4(1)\cdot 2st + 5(x-y)^4(-1)\cdot t^2 = 5(x-y)^4\left(2st - t^2\right)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = 5(x-y)^4(1)\cdot s^2 + 5(x-y)^4(-1)\cdot 2st = 5(x-y)^4\left(s^2 - 2st\right)$$

8. $z = \tan^{-1}(x^2 + y^2), \quad x = s\ln t, \quad y = te^s \quad \Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \frac{2x}{1 + (x^2+y^2)^2}\cdot \ln t + \frac{2y}{1+(x^2+y^2)^2}\cdot te^s$$

$$= \frac{2}{1+(x^2+y^2)^2}\left(x\ln t + yte^s\right)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \frac{2x}{1 + (x^2+y^2)^2}\cdot \frac{s}{t} + \frac{2y}{1+(x^2+y^2)^2}\cdot e^s$$

$$= \frac{2}{1+(x^2+y^2)^2}\left(\frac{xs}{t} + ye^s\right)$$

9. $z = \ln(3x + 2y), \quad x = s\sin t, \quad y = t\cos s \quad \Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \frac{3}{3x+2y}(\sin t) + \frac{2}{3x+2y}(-t\sin s) = \frac{3\sin t - 2t\sin s}{3x+2y}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \frac{3}{3x+2y}(s\cos t) + \frac{2}{3x+2y}(\cos s) = \frac{3s\cos t + 2\cos s}{3x+2y}$$

10. $z = \sqrt{x}\, e^{xy}, \quad x = 1 + st, \quad y = s^2 - t^2 \quad \Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \left(\sqrt{x}\cdot e^{xy}(y) + e^{xy}\cdot \tfrac{1}{2}x^{-1/2}\right)(t) + \sqrt{x}\,e^{xy}(x)\,(2s) = \left(yt\sqrt{x} + \frac{t}{2\sqrt{x}} + 2x^{3/2}s\right)e^{xy}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \left(\sqrt{x}\cdot e^{xy}(y) + e^{xy}\cdot \tfrac{1}{2}x^{-1/2}\right)(s) + \sqrt{x}\,e^{xy}(x)\,(-2t) = \left(ys\sqrt{x} + \frac{s}{2\sqrt{x}} - 2x^{3/2}t\right)e^{xy}$$

11. $z = e^r\cos\theta, \; r = st, \; \theta = \sqrt{s^2+t^2} \quad \Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial s} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial s} = e^r\cos\theta\cdot t + e^r(-\sin\theta)\cdot\tfrac{1}{2}(s^2+t^2)^{-1/2}(2s) = te^r\cos\theta - e^r\sin\theta\cdot\frac{s}{\sqrt{s^2+t^2}}$$

$$= e^r\left(t\cos\theta - \frac{s}{\sqrt{s^2+t^2}}\sin\theta\right)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial t} = e^r\cos\theta\cdot s + e^r(-\sin\theta)\cdot\tfrac{1}{2}(s^2+t^2)^{-1/2}(2t) = se^r\cos\theta - e^r\sin\theta\cdot\frac{t}{\sqrt{s^2+t^2}}$$

$$= e^r\left(s\cos\theta - \frac{t}{\sqrt{s^2+t^2}}\sin\theta\right)$$

12. $z = \arcsin(x - y), \quad x = s^2 + t^2, \quad y = 1 - 2st \quad \Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \frac{1}{\sqrt{1-(x-y)^2}}(1)\cdot 2s + \frac{1}{\sqrt{1-(x-y)^2}}(-1)\cdot(-2t) = \frac{2s+2t}{\sqrt{1-(x-y)^2}}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \frac{1}{\sqrt{1-(x-y)^2}}(1)\cdot 2t + \frac{1}{\sqrt{1-(x-y)^2}}(-1)\cdot(-2s) = \frac{2s+2t}{\sqrt{1-(x-y)^2}}$$

13. Let $x = g(t)$ and $y = h(t)$. Then $p(t) = f(x, y)$ and the Chain Rule (2) gives $\dfrac{dp}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$. When $t = 2$,

$x = g(2) = 4$ and $y = h(2) = 5$, so $p'(2) = f_x(4, 5)\, g'(2) + f_y(4, 5)\, h'(2) = (2)(-3) + (8)(6) = 42$.

14. $R(s, t) = G\left(u(s, t), v(s, t)\right) \quad \Rightarrow \quad \dfrac{\partial R}{\partial s} = \dfrac{\partial R}{\partial u}\dfrac{\partial u}{\partial s} + \dfrac{\partial R}{\partial v}\dfrac{\partial v}{\partial s}$ and $\dfrac{\partial R}{\partial t} = \dfrac{\partial R}{\partial u}\dfrac{\partial u}{\partial t} + \dfrac{\partial R}{\partial v}\dfrac{\partial v}{\partial t}$ by the Chain Rule (3). When $s = 1$ and $t = 2$, $u(1, 2) = 5$ and $v(1, 2) = 7$.

Thus $R_s(1, 2) = G_u(5, 7)\, u_s(1, 2) + G_v(5, 7)\, v_s(1, 2) = (9)(4) + (-2)(2) = 32$ and

$R_t(1, 2) = G_u(5, 7)\, u_t(1, 2) + G_v(5, 7)\, v_t(1, 2) = (9)(-3) + (-2)(6) = -39$.

15. $g(u, v) = f(x(u, v), y(u, v))$ where $x = e^u + \sin v$, $y = e^u + \cos v \quad \Rightarrow$

$\dfrac{\partial x}{\partial u} = e^u$, $\dfrac{\partial x}{\partial v} = \cos v$, $\dfrac{\partial y}{\partial u} = e^u$, $\dfrac{\partial y}{\partial v} = -\sin v$. By the Chain Rule (3), $\dfrac{\partial g}{\partial u} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial u}$. Then

$g_u(0, 0) = f_x(x(0, 0), y(0, 0))\, x_u(0, 0) + f_y(x(0, 0), y(0, 0))\, y_u(0, 0) = f_x(1, 2)(e^0) + f_y(1, 2)(e^0) = 2(1) + 5(1) = 7$.

Similarly, $\dfrac{\partial g}{\partial v} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial v}$. Then

$$\begin{aligned} g_v(0, 0) &= f_x(x(0, 0), y(0, 0))\, x_v(0, 0) + f_y(x(0, 0), y(0, 0))\, y_v(0, 0) = f_x(1, 2)(\cos 0) + f_y(1, 2)(-\sin 0) \\ &= 2(1) + 5(0) = 2 \end{aligned}$$

16. $g(r, s) = f(x(r, s), y(r, s))$ where $x = 2r - s$, $y = s^2 - 4r \quad \Rightarrow \quad \dfrac{\partial x}{\partial r} = 2$, $\dfrac{\partial x}{\partial s} = -1$, $\dfrac{\partial y}{\partial r} = -4$, $\dfrac{\partial y}{\partial s} = 2s$.

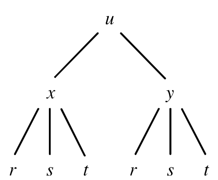
By the Chain Rule (3) $\dfrac{\partial g}{\partial r} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial r}$. Then

$$\begin{aligned} g_r(1, 2) &= f_x(x(1, 2), y(1, 2))\, x_r(1, 2) + f_y(x(1, 2), y(1, 2))\, y_r(1, 2) = f_x(0, 0)(2) + f_y(0, 0)(-4) \\ &= 4(2) + 8(-4) = -24 \end{aligned}$$

Similarly, $\dfrac{\partial g}{\partial s} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial s}$. Then

$$\begin{aligned} g_s(1, 2) &= f_x(x(1, 2), y(1, 2))\, x_s(1, 2) + f_y(x(1, 2), y(1, 2))\, y_s(1, 2) = f_x(0, 0)(-1) + f_y(0, 0)(4) \\ &= 4(-1) + 8(4) = 28 \end{aligned}$$

17.

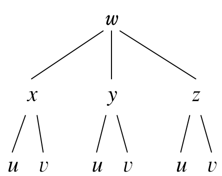


$u = f(x, y)$, $x = x(r, s, t)$, $y = y(r, s, t) \quad \Rightarrow$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}, \quad \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s},$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$

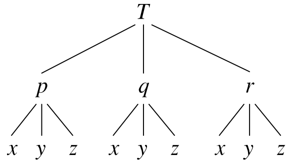
18.



$w = f(x, y, z)$, $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v) \quad \Rightarrow$

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u},$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$$

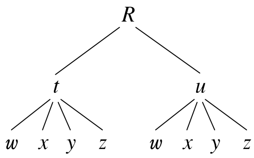
19. 

$T = F(p, q, r),\ p = p(x, y, z),\ \ q = q(x, y, z),\ r = r(x, y, z) \quad \Rightarrow$

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial T}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial T}{\partial r}\frac{\partial r}{\partial x},$$

$$\frac{\partial T}{\partial y} = \frac{\partial T}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial T}{\partial q}\frac{\partial q}{\partial y} + \frac{\partial T}{\partial r}\frac{\partial r}{\partial y},$$

$$\frac{\partial T}{\partial z} = \frac{\partial T}{\partial p}\frac{\partial p}{\partial z} + \frac{\partial T}{\partial q}\frac{\partial q}{\partial z} + \frac{\partial T}{\partial r}\frac{\partial r}{\partial z}$$

20. 

$R = F(t, u),\ \ t = t(w, x, y, z),\ \ u = u(w, x, y, z) \quad \Rightarrow$

$$\frac{\partial R}{\partial w} = \frac{\partial R}{\partial t}\frac{\partial t}{\partial w} + \frac{\partial R}{\partial u}\frac{\partial u}{\partial w},\quad \frac{\partial R}{\partial x} = \frac{\partial R}{\partial t}\frac{\partial t}{\partial x} + \frac{\partial R}{\partial u}\frac{\partial u}{\partial x},$$

$$\frac{\partial R}{\partial y} = \frac{\partial R}{\partial t}\frac{\partial t}{\partial y} + \frac{\partial R}{\partial u}\frac{\partial u}{\partial y},\quad \frac{\partial R}{\partial z} = \frac{\partial R}{\partial t}\frac{\partial t}{\partial z} + \frac{\partial R}{\partial u}\frac{\partial u}{\partial z}$$

21. $z = x^2 + xy^3,\ x = uv^2 + w^3,\ y = u + ve^w \quad \Rightarrow$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = (2x + y^3)(v^2) + (3xy^2)(1),$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = (2x + y^3)(2uv) + (3xy^2)(e^w),$$

$$\frac{\partial z}{\partial w} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial w} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial w} = (2x + y^3)(3w^2) + (3xy^2)(ve^w).$$

When $u = 2$, $v = 1$, and $w = 0$, we have $x = 2$, $y = 3$,

so $\dfrac{\partial z}{\partial u} = (31)(1) + (54)(1) = 85,\ \dfrac{\partial z}{\partial v} = (31)\,(4) + (54)(1) = 178,\ \dfrac{\partial z}{\partial w} = (31)(0) + (54)(1) = 54.$

22. $u = (r^2 + s^2)^{1/2},\ r = y + x\cos t,\ s = x + y\sin t \quad \Rightarrow$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} = \tfrac{1}{2}(r^2 + s^2)^{-1/2}(2r)(\cos t) + \tfrac{1}{2}(r^2 + s^2)^{-1/2}(2s)(1) = (r\cos t + s)/\sqrt{r^2 + s^2},$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} = \tfrac{1}{2}(r^2 + s^2)^{-1/2}(2r)(1) + \tfrac{1}{2}(r^2 + s^2)^{-1/2}(2s)(\sin t) = (r + s\sin t)/\sqrt{r^2 + s^2},$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial t} = \tfrac{1}{2}(r^2 + s^2)^{-1/2}(2r)(-x\sin t) + \tfrac{1}{2}(r^2 + s^2)^{-1/2}(2s)(y\cos t) = \frac{-rx\sin t + sy\cos t}{\sqrt{r^2 + s^2}}.$$

When $x = 1$, $y = 2$, and $t = 0$ we have $r = 3$ and $s = 1$, so $\dfrac{\partial u}{\partial x} = \dfrac{4}{\sqrt{10}},\ \dfrac{\partial u}{\partial y} = \dfrac{3}{\sqrt{10}}$, and $\dfrac{\partial u}{\partial t} = \dfrac{2}{\sqrt{10}}$.

23. $w = xy + yz + zx,\ x = r\cos\theta,\ y = r\sin\theta,\ z = r\theta \quad \Rightarrow$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} = (y + z)(\cos\theta) + (x + z)(\sin\theta) + (y + x)(\theta),$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial \theta} = (y + z)(-r\sin\theta) + (x + z)(r\cos\theta) + (y + x)(r).$$

When $r = 2$ and $\theta = \pi/2$ we have $x = 0$, $y = 2$, and $z = \pi$, so $\dfrac{\partial w}{\partial r} = (2 + \pi)(0) + (0 + \pi)(1) + (2 + 0)(\pi/2) = 2\pi$ and

$$\frac{\partial w}{\partial \theta} = (2 + \pi)(-2) + (0 + \pi)(0) + (2 + 0)(2) = -2\pi.$$

24. $P=\sqrt{u^2+v^2+w^2}=(u^2+v^2+w^2)^{1/2},\ u=xe^y,\ v=ye^x,\ w=e^{xy} \quad\Rightarrow$

$$\begin{aligned}\frac{\partial P}{\partial x}&=\frac{\partial P}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial P}{\partial v}\frac{\partial v}{\partial x}+\frac{\partial P}{\partial w}\frac{\partial w}{\partial x}\\&=\tfrac{1}{2}(u^2+v^2+w^2)^{-1/2}(2u)(e^y)+\tfrac{1}{2}(u^2+v^2+w^2)^{-1/2}(2v)(ye^x)+\tfrac{1}{2}(u^2+v^2+w^2)^{-1/2}(2w)(ye^{xy})\\&=\frac{ue^y+vye^x+wye^{xy}}{\sqrt{u^2+v^2+w^2}},\end{aligned}$$

$$\begin{aligned}\frac{\partial P}{\partial y}&=\frac{\partial P}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial P}{\partial v}\frac{\partial v}{\partial y}+\frac{\partial P}{\partial w}\frac{\partial w}{\partial y}=\frac{u}{\sqrt{u^2+v^2+w^2}}(xe^y)+\frac{v}{\sqrt{u^2+v^2+w^2}}(e^x)+\frac{w}{\sqrt{u^2+v^2+w^2}}(xe^{xy})\\&=\frac{uxe^y+ve^x+wxe^{xy}}{\sqrt{u^2+v^2+w^2}}.\end{aligned}$$

When $x=0$ and $y=2$ we have $u=0$, $v=2$, and $w=1$, so $\dfrac{\partial P}{\partial x}=\dfrac{0+4+2}{\sqrt{5}}=\dfrac{6}{\sqrt{5}}$ and $\dfrac{\partial P}{\partial y}=\dfrac{0+2+0}{\sqrt{5}}=\dfrac{2}{\sqrt{5}}$.

25. $N=\dfrac{p+q}{p+r},\ p=u+vw,\ q=v+uw,\ r=w+uv \quad\Rightarrow$

$$\begin{aligned}\frac{\partial N}{\partial u}&=\frac{\partial N}{\partial p}\frac{\partial p}{\partial u}+\frac{\partial N}{\partial q}\frac{\partial q}{\partial u}+\frac{\partial N}{\partial r}\frac{\partial r}{\partial u}\\&=\frac{(p+r)(1)-(p+q)(1)}{(p+r)^2}(1)+\frac{(p+r)(1)-(p+q)(0)}{(p+r)^2}(w)+\frac{(p+r)(0)-(p+q)(1)}{(p+r)^2}(v)\\&=\frac{(r-q)+(p+r)w-(p+q)v}{(p+r)^2},\end{aligned}$$

$$\frac{\partial N}{\partial v}=\frac{\partial N}{\partial p}\frac{\partial p}{\partial v}+\frac{\partial N}{\partial q}\frac{\partial q}{\partial v}+\frac{\partial N}{\partial r}\frac{\partial r}{\partial v}=\frac{r-q}{(p+r)^2}(w)+\frac{p+r}{(p+r)^2}(1)+\frac{-(p+q)}{(p+r)^2}(u)=\frac{(r-q)w+(p+r)-(p+q)u}{(p+r)^2},$$

$$\frac{\partial N}{\partial w}=\frac{\partial N}{\partial p}\frac{\partial p}{\partial w}+\frac{\partial N}{\partial q}\frac{\partial q}{\partial w}+\frac{\partial N}{\partial r}\frac{\partial r}{\partial w}=\frac{r-q}{(p+r)^2}(v)+\frac{p+r}{(p+r)^2}(u)+\frac{-(p+q)}{(p+r)^2}(1)=\frac{(r-q)v+(p+r)u-(p+q)}{(p+r)^2}.$$

When $u=2$, $v=3$, and $w=4$ we have $p=14$, $q=11$, and $r=10$, so $\dfrac{\partial N}{\partial u}=\dfrac{-1+(24)(4)-(25)(3)}{(24)^2}=\dfrac{20}{576}=\dfrac{5}{144}$,

$\dfrac{\partial N}{\partial v}=\dfrac{(-1)(4)+24-(25)(2)}{(24)^2}=\dfrac{-30}{576}=-\dfrac{5}{96}$, and $\dfrac{\partial N}{\partial w}=\dfrac{(-1)(3)+(24)(2)-25}{(24)^2}=\dfrac{20}{576}=\dfrac{5}{144}$.

26. $u=xe^{ty},\ x=\alpha^2\beta,\ y=\beta^2\gamma,\ t=\gamma^2\alpha \quad\Rightarrow$

$$\frac{\partial u}{\partial\alpha}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial\alpha}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial\alpha}+\frac{\partial u}{\partial t}\frac{\partial t}{\partial\alpha}=e^{ty}(2\alpha\beta)+xte^{ty}(0)+xye^{ty}(\gamma^2)=e^{ty}(2\alpha\beta+xy\gamma^2),$$

$$\frac{\partial u}{\partial\beta}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial\beta}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial\beta}+\frac{\partial u}{\partial t}\frac{\partial t}{\partial\beta}=e^{ty}(\alpha^2)+xte^{ty}(2\beta\gamma)+xye^{ty}(0)=e^{ty}(\alpha^2+2xt\beta\gamma),$$

$$\frac{\partial u}{\partial\gamma}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial\gamma}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial\gamma}+\frac{\partial u}{\partial t}\frac{\partial t}{\partial\gamma}=e^{ty}(0)+xte^{ty}(\beta^2)+xye^{ty}(2\gamma\alpha)=e^{ty}(xt\beta^2+2xy\alpha\gamma).$$

[continued]

When $\alpha = -1$, $\beta = 2$, and $\gamma = 1$ we have $x = 2$, $y = 4$, and $t = -1$, so $\dfrac{\partial u}{\partial \alpha} = e^{-4}(-4+8) = 4e^{-4}$,

$\dfrac{\partial u}{\partial \beta} = e^{-4}(1-8) = -7e^{-4}$, and $\dfrac{\partial u}{\partial \gamma} = e^{-4}(-8-16) = -24e^{-4}$.

27. $y\cos x = x^2 + y^2$, so let $F(x,y) = y\cos x - x^2 - y^2 = 0$. Then by Equation 6

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-y\sin x - 2x}{\cos x - 2y} = \frac{2x + y\sin x}{\cos x - 2y}.$$

28. $\cos(xy) = 1 + \sin y$, so let $F(x,y) = \cos(xy) - 1 - \sin y = 0$. Then by Equation 6

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-\sin(xy)(y)}{-\sin(xy)(x) - \cos y} = -\frac{y\sin(xy)}{\cos y + x\sin(xy)}.$$

29. $\tan^{-1}(x^2y) = x + xy^2$, so let $F(x,y) = \tan^{-1}(x^2y) - x - xy^2 = 0$. Then

$$F_x(x,y) = \frac{1}{1+(x^2y)^2}\,(2xy) - 1 - y^2 = \frac{2xy}{1+x^4y^2} - 1 - y^2 = \frac{2xy - (1+y^2)(1+x^4y^2)}{1+x^4y^2},$$

$$F_y(x,y) = \frac{1}{1+(x^2y)^2}\,(x^2) - 2xy = \frac{x^2}{1+x^4y^2} - 2xy = \frac{x^2 - 2xy(1+x^4y^2)}{1+x^4y^2}$$

and

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{[2xy - (1+y^2)(1+x^4y^2)]/(1+x^4y^2)}{[x^2 - 2xy(1+x^4y^2)]/(1+x^4y^2)} = \frac{(1+y^2)(1+x^4y^2) - 2xy}{x^2 - 2xy(1+x^4y^2)}$$

$$= \frac{1 + x^4y^2 + y^2 + x^4y^4 - 2xy}{x^2 - 2xy - 2x^5y^3}$$

30. $e^y \sin x = x + xy$, so let $F(x,y) = e^y\sin x - x - xy = 0$. Then $\dfrac{dy}{dx} = -\dfrac{F_x}{F_y} = -\dfrac{e^y\cos x - 1 - y}{e^y\sin x - x} = \dfrac{1 + y - e^y\cos x}{e^y\sin x - x}$.

31. $x^2 + 2y^2 + 3z^2 = 1$, so let $F(x,y,z) = x^2 + 2y^2 + 3z^2 - 1 = 0$. Then by Equations 7

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2x}{6z} = -\frac{x}{3z} \quad\text{and}\quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{4y}{6z} = -\frac{2y}{3z}.$$

32. $x^2 - y^2 + z^2 - 2z = 4$, so let $F(x,y,z) = x^2 - y^2 + z^2 - 2z - 4 = 0$. Then by Equations 7

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2x}{2z-2} = \frac{x}{1-z} \quad\text{and}\quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{-2y}{2z-2} = \frac{y}{z-1}.$$

33. $e^z = xyz$, so let $F(x,y,z) = e^z - xyz = 0$. Then $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = -\dfrac{-yz}{e^z - xy} = \dfrac{yz}{e^z - xy}$ and

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{-xz}{e^z - xy} = \frac{xz}{e^z - xy}.$$

34. $yz + x\ln y = z^2$, so let $F(x,y,z) = yz + x\ln y - z^2 = 0$. Then $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = -\dfrac{\ln y}{y - 2z} = \dfrac{\ln y}{2z - y}$ and

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{z + (x/y)}{y - 2z} = \frac{x + yz}{2yz - y^2}.$$

35. Since x and y are each functions of t, $T(x, y)$ is a function of t, so by the Chain Rule, $\dfrac{dT}{dt} = \dfrac{\partial T}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial T}{\partial y}\dfrac{dy}{dt}$. After 3 seconds, $x = \sqrt{1+t} = \sqrt{1+3} = 2$, $y = 2 + \frac{1}{3}t = 2 + \frac{1}{3}(3) = 3$, $\dfrac{dx}{dt} = \dfrac{1}{2\sqrt{1+t}} = \dfrac{1}{2\sqrt{1+3}} = \dfrac{1}{4}$, and $\dfrac{dy}{dt} = \dfrac{1}{3}$.

Then $\dfrac{dT}{dt} = T_x(2,3)\dfrac{dx}{dt} + T_y(2,3)\dfrac{dy}{dt} = 4\left(\frac{1}{4}\right) + 3\left(\frac{1}{3}\right) = 2$. Thus the temperature is rising at a rate of 2°C/s.

36. (a) Since $\partial W/\partial T$ is negative, a rise in average temperature (while annual rainfall remains constant) causes a decrease in wheat production at the current production levels. Since $\partial W/\partial R$ is positive, an increase in annual rainfall (while the average temperature remains constant) causes an increase in wheat production.

(b) Since the average temperature is rising at a rate of 0.15°C/year, we know that $dT/dt = 0.15$. Since rainfall is decreasing at a rate of 0.1 cm/year, we know $dR/dt = -0.1$. Then, by the Chain Rule,

$\dfrac{dW}{dt} = \dfrac{\partial W}{\partial T}\dfrac{dT}{dt} + \dfrac{\partial W}{\partial R}\dfrac{dR}{dt} = (-2)(0.15) + (8)(-0.1) = -1.1$. Thus we estimate that wheat production will decrease at a rate of 1.1 units/year.

37. $C = 1449.2 + 4.6T - 0.055T^2 + 0.00029T^3 + 0.016D$, so $\dfrac{\partial C}{\partial T} = 4.6 - 0.11T + 0.00087T^2$ and $\dfrac{\partial C}{\partial D} = 0.016$.

According to the graph, the diver is experiencing a temperature of approximately 12.5°C at $t = 20$ minutes, so

$\dfrac{\partial C}{\partial T} = 4.6 - 0.11(12.5) + 0.00087(12.5)^2 \approx 3.36$. By sketching tangent lines at $t = 20$ to the graphs given, we estimate

$\dfrac{dD}{dt} \approx \dfrac{1}{2}$ and $\dfrac{dT}{dt} \approx -\dfrac{1}{10}$. Then, by the Chain Rule, $\dfrac{dC}{dt} = \dfrac{\partial C}{\partial T}\dfrac{dT}{dt} + \dfrac{\partial C}{\partial D}\dfrac{dD}{dt} \approx (3.36)\left(-\frac{1}{10}\right) + (0.016)\left(\frac{1}{2}\right) \approx -0.33$.

Thus the speed of sound experienced by the diver is decreasing at a rate of approximately 0.33 m/s per minute.

38. $V = \dfrac{\pi r^2 h}{3}$, so
$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} = \frac{2\pi rh}{3}(4.6) + \frac{\pi r^2}{3}(-6.5) = \frac{2\pi(300)(350)}{3}(4.6) + \frac{\pi(300)^2}{3}(-6.5)$$
$$= 322{,}000\pi - 195{,}000\pi = 127{,}000\pi \text{ cm}^3\text{/s}$$

39. (a) $V = \ell wh$, so by the Chain Rule,
$$\frac{dV}{dt} = \frac{\partial V}{\partial \ell}\frac{d\ell}{dt} + \frac{\partial V}{\partial w}\frac{dw}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} = wh\frac{d\ell}{dt} + \ell h\frac{dw}{dt} + \ell w\frac{dh}{dt} = 2\cdot 2\cdot 2 + 1\cdot 2\cdot 2 + 1\cdot 2\cdot(-3) = 6 \text{ m}^3\text{/s}.$$

(b) $S = 2(\ell w + \ell h + wh)$, so by the Chain Rule,
$$\begin{aligned}\frac{dS}{dt} &= \frac{\partial S}{\partial \ell}\frac{d\ell}{dt} + \frac{\partial S}{\partial w}\frac{dw}{dt} + \frac{\partial S}{\partial h}\frac{dh}{dt} = 2(w+h)\frac{d\ell}{dt} + 2(\ell+h)\frac{dw}{dt} + 2(\ell+w)\frac{dh}{dt}\\ &= 2(2+2)2 + 2(1+2)2 + 2(1+2)(-3) = 10 \text{ m}^2\text{/s}\end{aligned}$$

(c) $L^2 = \ell^2 + w^2 + h^2 \quad\Rightarrow\quad 2L\dfrac{dL}{dt} = 2\ell\dfrac{d\ell}{dt} + 2w\dfrac{dw}{dt} + 2h\dfrac{dh}{dt} = 2(1)(2) + 2(2)(2) + 2(2)(-3) = 0 \quad\Rightarrow$ $dL/dt = 0$ m/s.

40. $I = \dfrac{V}{R} \quad \Rightarrow$

$$\frac{dI}{dt} = \frac{\partial I}{\partial V}\frac{dV}{dt} + \frac{\partial I}{\partial R}\frac{dR}{dt} = \frac{1}{R}\frac{dV}{dt} - \frac{V}{R^2}\frac{dR}{dt} = \frac{1}{R}\frac{dV}{dt} - \frac{I}{R}\frac{dR}{dt} = \frac{1}{400}(-0.01) - \frac{0.08}{400}(0.03) = -0.000031 \text{ A/s}$$

41. $\dfrac{dP}{dt} = 0.05$, $\dfrac{dT}{dt} = 0.15$, $V = 8.31\dfrac{T}{P}$ and $\dfrac{dV}{dt} = \dfrac{8.31}{P}\dfrac{dT}{dt} - 8.31\dfrac{T}{P^2}\dfrac{dP}{dt}$. Thus when $P = 20$ and $T = 320$,

$\dfrac{dV}{dt} = 8.31\left[\dfrac{0.15}{20} - \dfrac{(0.05)(320)}{400}\right] \approx -0.27 \text{ L/s}.$

42. $P = 1.47L^{0.65}K^{0.35}$ and considering P, L, and K as functions of time t we have

$\dfrac{dP}{dt} = \dfrac{\partial P}{\partial L}\dfrac{dL}{dt} + \dfrac{\partial P}{\partial K}\dfrac{dK}{dt} = 1.47(0.65)L^{-0.35}K^{0.35}\dfrac{dL}{dt} + 1.47(0.35)L^{0.65}K^{-0.65}\dfrac{dK}{dt}$. We are given that $\dfrac{dL}{dt} = -2$ and $\dfrac{dK}{dt} = 0.5$, so when $L = 30$ and $K = 8$, the rate of change of production $\dfrac{dP}{dt}$ is $1.47(0.65)(30)^{-0.35}(8)^{0.35}(-2) + 1.47(0.35)(30)^{0.65}(8)^{-0.65}(0.5) \approx -0.596$. Thus production at that time is decreasing at a rate of about \$596,000 per year.

43. Let x be the length of the first side of the triangle and y the length of the second side. The area A of the triangle is given by $A = \frac{1}{2}xy\sin\theta$ where θ is the angle between the two sides. Thus A is a function of x, y, and θ, and x, y, and θ are each in turn functions of time t. We are given that $\dfrac{dx}{dt} = 3$, $\dfrac{dy}{dt} = -2$, and because A is constant, $\dfrac{dA}{dt} = 0$. By the Chain Rule,

$\dfrac{dA}{dt} = \dfrac{\partial A}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial A}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial A}{\partial \theta}\dfrac{d\theta}{dt} \quad\Rightarrow\quad \dfrac{dA}{dt} = \frac{1}{2}y\sin\theta\cdot\dfrac{dx}{dt} + \frac{1}{2}x\sin\theta\cdot\dfrac{dy}{dt} + \frac{1}{2}xy\cos\theta\cdot\dfrac{d\theta}{dt}$. When $x = 20$, $y = 30$, and $\theta = \pi/6$ we have

$$\begin{aligned} 0 &= \tfrac{1}{2}(30)\left(\sin\tfrac{\pi}{6}\right)(3) + \tfrac{1}{2}(20)\left(\sin\tfrac{\pi}{6}\right)(-2) + \tfrac{1}{2}(20)(30)\left(\cos\tfrac{\pi}{6}\right)\frac{d\theta}{dt} \\ &= 45\cdot\tfrac{1}{2} - 20\cdot\tfrac{1}{2} + 300\cdot\frac{\sqrt{3}}{2}\cdot\frac{d\theta}{dt} = \tfrac{25}{2} + 150\sqrt{3}\,\frac{d\theta}{dt} \end{aligned}$$

Solving for $\dfrac{d\theta}{dt}$ gives $\dfrac{d\theta}{dt} = \dfrac{-25/2}{150\sqrt{3}} = -\dfrac{1}{12\sqrt{3}}$, so the angle between the sides is decreasing at a rate of $1/\left(12\sqrt{3}\right) \approx 0.048$ rad/s.

44. $f_o = \left(\dfrac{c + v_o}{c - v_s}\right) f_s = \left(\frac{332+34}{332-40}\right) 460 \approx 576.6$ Hz. v_o and v_s are functions of time t, so

$$\begin{aligned} \frac{df_o}{dt} &= \frac{\partial f_o}{\partial v_o}\frac{dv_o}{dt} + \frac{\partial f_o}{\partial v_s}\frac{dv_s}{dt} = \left(\frac{1}{c - v_s}\right) f_s\cdot\frac{dv_o}{dt} + \frac{c + v_o}{(c - v_s)^2} f_s\cdot\frac{dv_s}{dt} \\ &= \left(\tfrac{1}{332-40}\right)(460)(1.2) + \tfrac{332+34}{(332-40)^2}(460)(1.4) \approx 4.65 \text{ Hz/s} \end{aligned}$$

45. (a) By the Chain Rule, $\dfrac{\partial z}{\partial r} = \dfrac{\partial z}{\partial x}\cos\theta + \dfrac{\partial z}{\partial y}\sin\theta$, $\dfrac{\partial z}{\partial \theta} = \dfrac{\partial z}{\partial x}(-r\sin\theta) + \dfrac{\partial z}{\partial y}r\cos\theta$.

(b) $\left(\dfrac{\partial z}{\partial r}\right)^2 = \left(\dfrac{\partial z}{\partial x}\right)^2\cos^2\theta + 2\,\dfrac{\partial z}{\partial x}\dfrac{\partial z}{\partial y}\cos\theta\,\sin\theta + \left(\dfrac{\partial z}{\partial y}\right)^2\sin^2\theta$,

$\left(\dfrac{\partial z}{\partial \theta}\right)^2 = \left(\dfrac{\partial z}{\partial x}\right)^2 r^2\sin^2\theta - 2\,\dfrac{\partial z}{\partial x}\dfrac{\partial z}{\partial y}r^2\cos\theta\,\sin\theta + \left(\dfrac{\partial z}{\partial y}\right)^2 r^2\cos^2\theta$. Thus

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right](\cos^2\theta + \sin^2\theta) = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2.$$

46. $\dfrac{\partial z}{\partial s} = \dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y}$ and $\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}$. Thus $\dfrac{\partial z}{\partial s}\dfrac{\partial z}{\partial t} = \left(\dfrac{\partial z}{\partial x}\right)^2 - \left(\dfrac{\partial z}{\partial y}\right)^2$.

47. Let $u = x - y$ and $v = x + y$. Then $z = \dfrac{1}{x}\left[f(u) + g(v)\right]$ and

$$\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{1}{x}\left[\frac{df}{du}\frac{\partial u}{\partial x} + \frac{dg}{dv}\frac{\partial v}{\partial x}\right] + \left[f(u) + g(v)\right]\left(-\frac{1}{x^2}\right) \\
&= \frac{1}{x}\left[f'(u)(1) + g'(v)(1)\right] - \frac{1}{x^2}\left[f(u) + g(v)\right] = \frac{1}{x}\left[f'(u) + g'(v)\right] - \frac{1}{x^2}\left[f(u) + g(v)\right]
\end{aligned}$$

$$\frac{\partial z}{\partial y} = \frac{1}{x}\left[\frac{df}{du}\frac{\partial u}{\partial y} + \frac{dg}{dv}\frac{\partial v}{\partial y}\right] = \frac{1}{x}\left[f'(u)(-1) + g'(v)(1)\right] = \frac{1}{x}\left[-f'(u) + g'(v)\right]$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{1}{x}\left[\frac{d}{du}\left[-f'(u)\right]\frac{\partial u}{\partial y} + \frac{d}{dv}\left[g'(v)\right]\frac{\partial v}{\partial y}\right] = \frac{1}{x}\left[-f''(u)(-1) + g''(v)(1)\right] = \frac{1}{x}\left[f''(u) + g''(v)\right]$$

Thus

$$\begin{aligned}
\frac{\partial}{\partial x}\left(x^2\frac{\partial z}{\partial x}\right) &= \frac{\partial}{\partial x}\left(x\left[f'(u) + g'(v)\right] - \left[f(u) + g(v)\right]\right) \\
&= x\left[f''(u)(1) + g''(v)(1)\right] + \left[f'(u) + g'(v)\right](1) - \left[f'(u)(1) + g'(v)(1)\right] \\
&= x\left[f''(u) + g''(v)\right] + f'(u) + g'(v) - f'(u) - g'(v) = x\left[f''(u) + g''(v)\right] \\
&= x^2\cdot\frac{1}{x}\left[f''(u) + g''(v)\right] = x^2\frac{\partial^2 z}{\partial y^2}
\end{aligned}$$

48. Let $u = ax + y$ and $v = ax - y$. Then $z = \dfrac{1}{y}\left[f(u) + g(v)\right]$ and

$$\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{1}{y}\left[\frac{df}{du}\frac{\partial u}{\partial x} + \frac{dg}{dv}\frac{\partial v}{\partial x}\right] = \frac{1}{y}\left[f'(u)(a) + g'(v)(a)\right] = \frac{a}{y}\left[f'(u) + g'(v)\right] \\
\frac{\partial^2 z}{\partial x^2} &= \frac{a}{y}\left[\frac{d}{du}\left[f'(u)\right]\frac{\partial u}{\partial x} + \frac{d}{dv}\left[g'(v)\right]\frac{\partial v}{\partial x}\right] = \frac{a}{y}\left[f''(u)(a) + g''(v)(a)\right] = \frac{a^2}{y}\left[f''(u) + g''(v)\right] \\
\frac{\partial z}{\partial y} &= \frac{1}{y}\left[\frac{df}{du}\frac{\partial u}{\partial y} + \frac{dg}{dv}\frac{\partial v}{\partial y}\right] + \left[f(u) + g(v)\right]\left(-\frac{1}{y^2}\right) \\
&= \frac{1}{y}\left[f'(u)(1) + g'(v)(-1)\right] - \frac{1}{y^2}\left[f(u) + g(v)\right] = \frac{1}{y}\left[f'(u) - g'(v)\right] - \frac{1}{y^2}\left[f(u) + g(v)\right]
\end{aligned}$$

[continued]

$$\frac{\partial}{\partial y}\left(y^2\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial y}\left(y\left[f'(u)-g'(v)\right]-\left[f(u)+g(v)\right]\right)$$
$$=y\left[f''(u)(1)-g''(v)(-1)\right]+\left[f'(u)-g'(v)\right](1)-\left[f'(u)(1)+g'(v)(-1)\right]$$
$$=y\left[f''(u)+g''(v)\right]+f'(u)-g'(v)-f'(u)+g'(v)=y\left[f''(u)+g''(v)\right]$$

Thus $\dfrac{\partial^2 z}{\partial x^2}=\dfrac{a^2}{y}\left[f''(u)+g''(v)\right]=\dfrac{a^2}{y^2}\cdot y\left[f''(u)+g''(v)\right]=\dfrac{a^2}{y^2}\dfrac{\partial}{\partial y}\left(y^2\dfrac{\partial z}{\partial y}\right)$.

49. Let $u=x+at,\ v=x-at$. Then $z=f(u)+g(v)$, so $\partial z/\partial u=f'(u)$ and $\partial z/\partial v=g'(v)$.

Thus $\dfrac{\partial z}{\partial t}=\dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial t}+\dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial t}=af'(u)-ag'(v)$ and

$$\frac{\partial^2 z}{\partial t^2}=a\frac{\partial}{\partial t}\left[f'(u)-g'(v)\right]=a\left(\frac{df'(u)}{du}\frac{\partial u}{\partial t}-\frac{dg'(v)}{dv}\frac{\partial v}{\partial t}\right)=a^2f''(u)+a^2g''(v).$$

Similarly, $\dfrac{\partial z}{\partial x}=f'(u)+g'(v)$ and $\dfrac{\partial^2 z}{\partial x^2}=f''(u)+g''(v)$. Thus $\dfrac{\partial^2 z}{\partial t^2}=a^2\dfrac{\partial^2 z}{\partial x^2}$.

50. By the Chain Rule, $\dfrac{\partial u}{\partial s}=e^s\cos t\,\dfrac{\partial u}{\partial x}+e^s\sin t\,\dfrac{\partial u}{\partial y}$ and $\dfrac{\partial u}{\partial t}=-e^s\sin t\,\dfrac{\partial u}{\partial x}+e^s\cos t\,\dfrac{\partial u}{\partial y}$.

Then $\dfrac{\partial^2 u}{\partial s^2}=e^s\cos t\,\dfrac{\partial u}{\partial x}+e^s\cos t\,\dfrac{\partial}{\partial s}\left(\dfrac{\partial u}{\partial x}\right)+e^s\sin t\,\dfrac{\partial u}{\partial y}+e^s\sin t\,\dfrac{\partial}{\partial s}\left(\dfrac{\partial u}{\partial y}\right)$. But

$$\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial s}+\frac{\partial^2 u}{\partial y\,\partial x}\frac{\partial y}{\partial s}=e^s\cos t\,\frac{\partial^2 u}{\partial x^2}+e^s\sin t\,\frac{\partial^2 u}{\partial y\,\partial x}\ \text{ and}$$

$$\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial y}\right)=\frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial s}+\frac{\partial^2 u}{\partial x\,\partial y}\frac{\partial x}{\partial s}=e^s\sin t\,\frac{\partial^2 u}{\partial y^2}+e^s\cos t\,\frac{\partial^2 u}{\partial x\,\partial y}.$$

Also, by continuity of the partials, $\dfrac{\partial^2 u}{\partial x\,\partial y}=\dfrac{\partial^2 u}{\partial y\,\partial x}$. Thus

$$\frac{\partial^2 u}{\partial s^2}=e^s\cos t\,\frac{\partial u}{\partial x}+e^s\cos t\left(e^s\cos t\,\frac{\partial^2 u}{\partial x^2}+e^s\sin t\,\frac{\partial^2 u}{\partial x\,\partial y}\right)+e^s\sin t\,\frac{\partial u}{\partial y}+\ e^s\sin t\left(e^s\sin t\,\frac{\partial^2 u}{\partial y^2}+e^s\cos t\,\frac{\partial^2 u}{\partial x\,\partial y}\right)$$
$$=e^s\cos t\frac{\partial u}{\partial x}+e^s\sin t\,\frac{\partial u}{\partial y}+e^{2s}\cos^2 t\,\frac{\partial^2 u}{\partial x^2}+2e^{2s}\cos\,t\sin t\,\frac{\partial^2 u}{\partial x\,\partial y}+e^{2s}\sin^2 t\,\frac{\partial^2 u}{\partial y^2}$$

Similarly

$$\frac{\partial^2 u}{\partial t^2}=-e^s\cos t\,\frac{\partial u}{\partial x}-e^s\sin t\,\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)-e^s\sin t\,\frac{\partial u}{\partial y}+e^s\cos t\,\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\right)$$
$$=-e^s\cos t\,\frac{\partial u}{\partial x}-e^s\sin t\left(-e^s\sin t\,\frac{\partial^2 u}{\partial x^2}+e^s\cos t\,\frac{\partial^2 u}{\partial x\,\partial y}\right)$$
$$-e^s\sin t\,\frac{\partial u}{\partial y}+e^s\cos t\left(e^s\cos t\,\frac{\partial^2 u}{\partial y^2}-e^s\sin t\,\frac{\partial^2 u}{\partial x\,\partial y}\right)$$
$$=-e^s\cos t\,\frac{\partial u}{\partial x}-e^s\sin t\,\frac{\partial u}{\partial y}+e^{2s}\sin^2 t\,\frac{\partial^2 u}{\partial x^2}-2e^{2s}\cos\,t\sin t\,\frac{\partial^2 u}{\partial x\,\partial y}+e^{2s}\cos^2 t\,\frac{\partial^2 u}{\partial y^2}$$

Thus $e^{-2s}\left(\dfrac{\partial^2 u}{\partial s^2}+\dfrac{\partial^2 u}{\partial t^2}\right)=(\cos^2 t+\sin^2 t)\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}\right)=\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}$, as desired.

51. $\dfrac{\partial z}{\partial s} = \dfrac{\partial z}{\partial x}\,2s + \dfrac{\partial z}{\partial y}\,2r$. Then

$$\begin{aligned}
\frac{\partial^2 z}{\partial r\,\partial s} &= \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}2s\right) + \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}2r\right) \\
&= \frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial r}\,2s + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\frac{\partial y}{\partial r}\,2s + \frac{\partial z}{\partial x}\frac{\partial}{\partial r}\,2s + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial r}\,2r + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\frac{\partial x}{\partial r}\,2r + \frac{\partial z}{\partial y}\,2 \\
&= 4rs\,\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y\,\partial x}\,4s^2 + 0 + 4rs\,\frac{\partial^2 z}{\partial y^2} + \frac{\partial^2 z}{\partial x\,\partial y}\,4r^2 + 2\,\frac{\partial z}{\partial y}
\end{aligned}$$

By the continuity of the partials, $\dfrac{\partial^2 z}{\partial r\partial s} = 4rs\,\dfrac{\partial^2 z}{\partial x^2} + 4rs\,\dfrac{\partial^2 z}{\partial y^2} + (4r^2 + 4s^2)\,\dfrac{\partial^2 z}{\partial x\,\partial y} + 2\dfrac{\partial z}{\partial y}$.

52. By the Chain Rule,

(a) $\dfrac{\partial z}{\partial r} = \dfrac{\partial z}{\partial x}\cos\theta + \dfrac{\partial z}{\partial y}\sin\theta$ (b) $\dfrac{\partial z}{\partial \theta} = -\dfrac{\partial z}{\partial x}\,r\sin\theta + \dfrac{\partial z}{\partial y}\,r\cos\theta$

(c)
$$\begin{aligned}
\frac{\partial^2 z}{\partial r\,\partial\theta} &= \frac{\partial^2 z}{\partial\theta\,\partial r} = \frac{\partial}{\partial\theta}\left(\frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta\right) = -\sin\theta\,\frac{\partial z}{\partial x} + \cos\theta\,\frac{\partial}{\partial\theta}\left(\frac{\partial z}{\partial x}\right) + \cos\theta\,\frac{\partial z}{\partial y} + \sin\theta\,\frac{\partial}{\partial\theta}\left(\frac{\partial z}{\partial y}\right) \\
&= -\sin\theta\,\frac{\partial z}{\partial x} + \cos\theta\left(\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial\theta} + \frac{\partial^2 z}{\partial y\,\partial x}\frac{\partial y}{\partial\theta}\right) + \cos\theta\,\frac{\partial z}{\partial y} + \sin\theta\,\frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial\theta} + \frac{\partial^2 z}{\partial x\,\partial y}\frac{\partial x}{\partial\theta} \\
&= -\sin\theta\,\frac{\partial z}{\partial x} + \cos\theta\left(-r\sin\theta\,\frac{\partial^2 z}{\partial x^2} + r\cos\theta\,\frac{\partial^2 z}{\partial y\,\partial x}\right) + \cos\theta\,\frac{\partial z}{\partial y} + \sin\theta\left(r\cos\theta\,\frac{\partial^2 z}{\partial y^2} - r\sin\theta\,\frac{\partial^2 z}{\partial x\,\partial y}\right) \\
&= -\sin\theta\,\frac{\partial z}{\partial x} - r\cos\theta\,\sin\theta\,\frac{\partial^2 z}{\partial x^2} + r\cos^2\theta\,\frac{\partial^2 z}{\partial y\,\partial x} + \cos\theta\,\frac{\partial z}{\partial y} + r\cos\,\theta\sin\theta\,\frac{\partial^2 z}{\partial y^2} - r\sin^2\theta\,\frac{\partial^2 z}{\partial y\,\partial x} \\
&= \cos\theta\,\frac{\partial z}{\partial y} - \sin\theta\,\frac{\partial z}{\partial x} + r\cos\,\theta\sin\theta\left(\frac{\partial^2 z}{\partial y^2} - \frac{\partial^2 z}{\partial x^2}\right) + r(\cos^2\theta - \sin^2\theta)\,\frac{\partial^2 z}{\partial y\,\partial x}
\end{aligned}$$

53. $\dfrac{\partial z}{\partial r} = \dfrac{\partial z}{\partial x}\cos\theta + \dfrac{\partial z}{\partial y}\sin\theta$ and $\dfrac{\partial z}{\partial\theta} = -\dfrac{\partial z}{\partial x}\,r\sin\theta + \dfrac{\partial z}{\partial y}\,r\cos\theta$. Then

$$\begin{aligned}
\frac{\partial^2 z}{\partial r^2} &= \cos\theta\left(\frac{\partial^2 z}{\partial x^2}\cos\theta + \frac{\partial^2 z}{\partial y\,\partial x}\sin\theta\right) + \sin\theta\left(\frac{\partial^2 z}{\partial y^2}\sin\theta + \frac{\partial^2 z}{\partial x\,\partial y}\cos\theta\right) \\
&= \cos^2\theta\,\frac{\partial^2 z}{\partial x^2} + 2\cos\theta\,\sin\theta\,\frac{\partial^2 z}{\partial x\,\partial y} + \sin^2\theta\,\frac{\partial^2 z}{\partial y^2}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2 z}{\partial\theta^2} &= -r\cos\theta\,\frac{\partial z}{\partial x} + (-r\sin\theta)\left(\frac{\partial^2 z}{\partial x^2}(-r\sin\theta) + \frac{\partial^2 z}{\partial y\,\partial x}\,r\cos\theta\right) \\
&\qquad -r\sin\theta\,\frac{\partial z}{\partial y} + r\cos\theta\left(\frac{\partial^2 z}{\partial y^2}\,r\cos\theta + \frac{\partial^2 z}{\partial x\,\partial y}(-r\sin\theta)\right) \\
&= -r\cos\theta\,\frac{\partial z}{\partial x} - r\sin\theta\,\frac{\partial z}{\partial y} + r^2\sin^2\theta\,\frac{\partial^2 z}{\partial x^2} - 2r^2\cos\theta\,\sin\theta\,\frac{\partial^2 z}{\partial x\,\partial y} + r^2\cos^2\theta\,\frac{\partial^2 z}{\partial y^2}
\end{aligned}$$

[continued]

Thus

$$\frac{\partial^2 z}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 z}{\partial\theta^2}+\frac{1}{r}\frac{\partial z}{\partial r}=(\cos^2\theta+\sin^2\theta)\frac{\partial^2 z}{\partial x^2}+\left(\sin^2\theta+\cos^2\theta\right)\frac{\partial^2 z}{\partial y^2}$$

$$-\frac{1}{r}\cos\theta\frac{\partial z}{\partial x}-\frac{1}{r}\sin\theta\frac{\partial z}{\partial y}+\frac{1}{r}\left(\cos\theta\frac{\partial z}{\partial x}+\sin\theta\frac{\partial z}{\partial y}\right)$$

$$=\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\text{ as desired.}$$

54. (a) $\dfrac{\partial z}{\partial t}=\dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t}+\dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial t}$. Then

$$\frac{\partial^2 z}{\partial t^2}=\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}\right)+\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}\right)=\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial x}\right)\frac{\partial x}{\partial t}+\frac{\partial^2 x}{\partial t^2}\frac{\partial z}{\partial x}+\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial y}\right)\frac{\partial y}{\partial t}+\frac{\partial^2 y}{\partial t^2}\frac{\partial z}{\partial y}$$

$$=\frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial t}\right)^2+\frac{\partial^2 z}{\partial y\,\partial x}\frac{\partial x}{\partial t}\frac{\partial y}{\partial t}+\frac{\partial^2 x}{\partial t^2}\frac{\partial z}{\partial x}+\frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial t}\right)^2+\frac{\partial^2 z}{\partial x\,\partial y}\frac{\partial y}{\partial t}\frac{\partial x}{\partial t}+\frac{\partial^2 y}{\partial t^2}\frac{\partial z}{\partial y}$$

$$=\frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial t}\right)^2+2\frac{\partial^2 z}{\partial x\,\partial y}\frac{\partial x}{\partial t}\frac{\partial y}{\partial t}+\frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial t}\right)^2+\frac{\partial^2 x}{\partial t^2}\frac{\partial z}{\partial x}+\frac{\partial^2 y}{\partial t^2}\frac{\partial z}{\partial y}$$

(b) $\dfrac{\partial^2 z}{\partial s\,\partial t}=\dfrac{\partial}{\partial s}\left(\dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t}+\dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial t}\right)$

$$=\left(\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial s}+\frac{\partial^2 z}{\partial y\,\partial x}\frac{\partial y}{\partial s}\right)\frac{\partial x}{\partial t}+\frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial s\,\partial t}+\left(\frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial s}+\frac{\partial^2 z}{\partial x\,\partial y}\frac{\partial x}{\partial s}\right)\frac{\partial y}{\partial t}+\frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial s\,\partial t}$$

$$=\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial s}\frac{\partial x}{\partial t}+\frac{\partial^2 z}{\partial x\,\partial y}\left(\frac{\partial y}{\partial s}\frac{\partial x}{\partial t}+\frac{\partial y}{\partial t}\frac{\partial x}{\partial s}\right)+\frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial s\,\partial t}+\frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial s\,\partial t}+\frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial s}\frac{\partial y}{\partial t}$$

55. (a) Since f is a polynomial, it has continuous second-order partial derivatives, and

$$f(tx,ty)=(tx)^2(ty)+2(tx)(ty)^2+5(ty)^3=t^3x^2y+2t^3xy^2+5t^3y^3=t^3(x^2y+2xy^2+5y^3)=t^3f(x,y).$$

Thus, f is homogeneous of degree 3.

(b) Differentiating both sides of $f(tx,ty)=t^nf(x,y)$ with respect to t using the Chain Rule, we get

$$\frac{\partial}{\partial t}f(tx,ty)=\frac{\partial}{\partial t}\left[t^nf(x,y)\right]\quad\Leftrightarrow$$

$$\frac{\partial}{\partial(tx)}f(tx,ty)\cdot\frac{\partial(tx)}{\partial t}+\frac{\partial}{\partial(ty)}f(tx,ty)\cdot\frac{\partial(ty)}{\partial t}=x\frac{\partial}{\partial(tx)}f(tx,ty)+y\frac{\partial}{\partial(ty)}f(tx,ty)=nt^{n-1}f(x,y).$$

Setting $t=1$: $x\dfrac{\partial}{\partial x}f(x,y)+y\dfrac{\partial}{\partial y}f(x,y)=nf(x,y)$.

56. Differentiating both sides of $f(tx,ty)=t^nf(x,y)$ with respect to t using the Chain Rule, we get

$$\frac{\partial}{\partial(tx)}f(tx,ty)\cdot\frac{\partial(tx)}{\partial t}+\frac{\partial}{\partial(ty)}f(tx,ty)\cdot\frac{\partial(ty)}{\partial t}=x\frac{\partial}{\partial(tx)}f(tx,ty)+y\frac{\partial}{\partial(ty)}f(tx,ty)=nt^{n-1}f(x,y)\text{ and}$$

differentiating again with respect to t gives

$$x\left[\frac{\partial^2}{\partial (tx)^2} f(tx,ty)\cdot\frac{\partial (tx)}{\partial t}+\frac{\partial^2}{\partial (ty)\,\partial (tx)} f(tx,ty)\cdot\frac{\partial (ty)}{\partial t}\right]$$

$$+\,y\left[\frac{\partial^2}{\partial (tx)\,\partial (ty)} f(tx,ty)\cdot\frac{\partial (tx)}{\partial t}+\frac{\partial^2}{\partial (ty)^2} f(tx,ty)\cdot\frac{\partial (ty)}{\partial t}\right]=n(n-1)t^{n-1}f(x,y).$$

Setting $t=1$ and using the fact that $f_{yx}=f_{xy}$, we have $x^2f_{xx}+2xyf_{xy}+y^2f_{yy}=n(n-1)f(x,y)$.

57. Differentiating both sides of $f(tx,ty)=t^nf(x,y)$ with respect to x using the Chain Rule, we get

$$\frac{\partial}{\partial x} f(tx,ty)=\frac{\partial}{\partial x}\left[t^nf(x,y)\right] \quad\Leftrightarrow$$

$$\frac{\partial}{\partial (tx)} f(tx,ty)\cdot\frac{\partial (tx)}{\partial x}+\frac{\partial}{\partial (ty)} f(tx,ty)\cdot\frac{\partial (ty)}{\partial x}=t^n\frac{\partial}{\partial x}f(x,y) \quad\Leftrightarrow\quad tf_x(tx,ty)=t^nf_x(x,y).$$

Thus $f_x(tx,ty)=t^{n-1}f_x(x,y)$.

58. $F(x,y,z)=0$ is assumed to define z as a function of x and y, that is, $z=f(x,y)$. So by (7), $\dfrac{\partial z}{\partial x}=-\dfrac{F_x}{F_z}$ since $F_z\neq 0$.

Similarly, it is assumed that $F(x,y,z)=0$ defines x as a function of y and z, that is $x=h(x,z)$. Then $F(h(y,z),y,z)=0$ and by the Chain Rule, $F_x\dfrac{\partial x}{\partial y}+F_y\dfrac{\partial y}{\partial y}+F_z\dfrac{\partial z}{\partial y}=0$. But $\dfrac{\partial z}{\partial y}=0$ and $\dfrac{\partial y}{\partial y}=1$, so $F_x\dfrac{\partial x}{\partial y}+F_y=0 \quad\Rightarrow\quad \dfrac{\partial x}{\partial y}=-\dfrac{F_y}{F_x}$.

A similar calculation shows that $\dfrac{\partial y}{\partial z}=-\dfrac{F_z}{F_y}$. Thus $\dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial y}\dfrac{\partial y}{\partial z}=\left(-\dfrac{F_x}{F_z}\right)\left(-\dfrac{F_y}{F_x}\right)\left(-\dfrac{F_z}{F_y}\right)=-1$.

59. Given a function defined implicitly by $F(x,y)=0$, where F is differentiable and $F_y\neq 0$, we know that $\dfrac{dy}{dx}=-\dfrac{F_x}{F_y}$. Let $G(x,y)=-\dfrac{F_x}{F_y}$ so $\dfrac{dy}{dx}=G(x,y)$. Differentiating both sides with respect to x and using the Chain Rule gives

$$\frac{d^2y}{dx^2}=\frac{\partial G}{\partial x}\frac{dx}{dx}+\frac{\partial G}{\partial y}\frac{dy}{dx} \text{ where } \frac{\partial G}{\partial x}=\frac{\partial}{\partial x}\left(-\frac{F_x}{F_y}\right)=-\frac{F_yF_{xx}-F_xF_{yx}}{F_y^2},\ \frac{\partial G}{\partial y}=\frac{\partial}{\partial y}\left(-\frac{F_x}{F_y}\right)=-\frac{F_yF_{xy}-F_xF_{yy}}{F_y^2}.$$

Thus

$$\begin{aligned}\frac{d^2y}{dx^2}&=\left(-\frac{F_yF_{xx}-F_xF_{yx}}{F_y^2}\right)(1)+\left(-\frac{F_yF_{xy}-F_xF_{yy}}{F_y^2}\right)\left(-\frac{F_x}{F_y}\right)\\&=-\frac{F_{xx}F_y^2-F_{yx}F_xF_y-F_{xy}F_yF_x+F_{yy}F_x^2}{F_y^3}\end{aligned}$$

But F has continuous second derivatives, so by Clauraut's Theorem, $F_{yx}=F_{xy}$ and we have

$\dfrac{d^2y}{dx^2}=-\dfrac{F_{xx}F_y^2-2F_{xy}F_xF_y+F_{yy}F_x^2}{F_y^3}$ as desired.