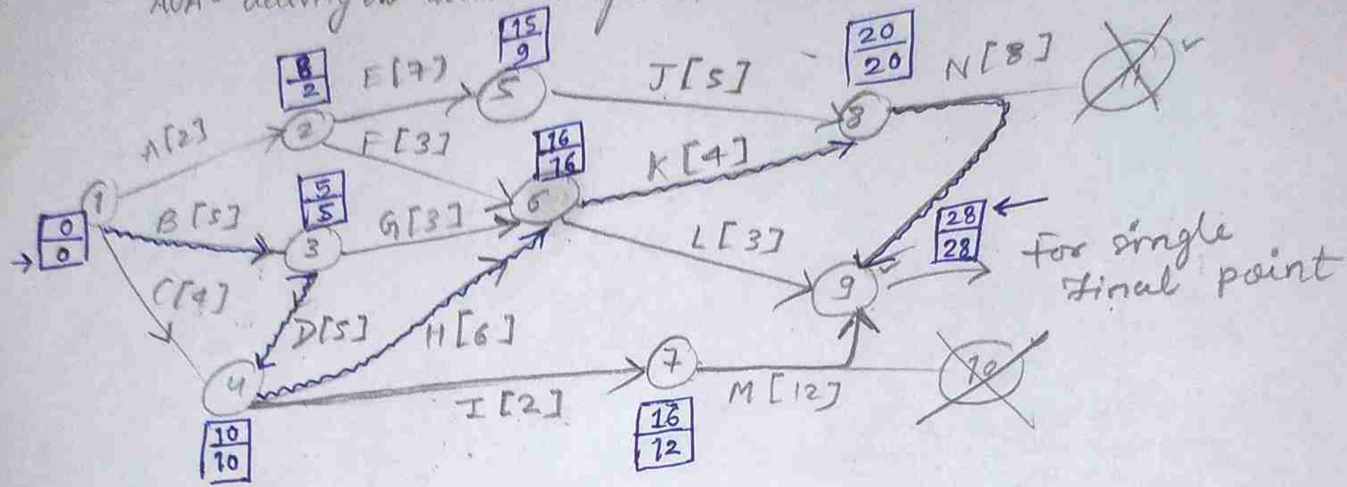
Operational Research

★ Critical path ~~management~~ method

- completion time of a project
- AOA - activity ~~on~~ arrow diagram



★ Determine the critical path and project completion time.

In order to figure out $\boxed{\text{critical path}}$, we have two phases —

Phase 1 : Determine the earliest start time (ES) of all the nodes. This is called forward phase.

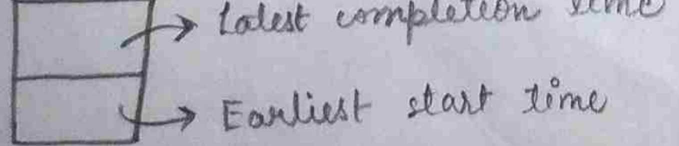
$$\boxed{ES_j = \max_i (ES_i + D_{ij})}$$

where D_{ij} be the duration of the activity (i, j)

j: ending node
i: starting node

Phase 2 : Determine the latest completion time (LC) of all nodes. This is called backward phase.

$$\boxed{LC_i = \min_j (LC_j - D_{ij})}$$



critical path :—

1 3 4 6 8 9

critical activities :—

B, D, H, K, N

(activities which cannot be removed)

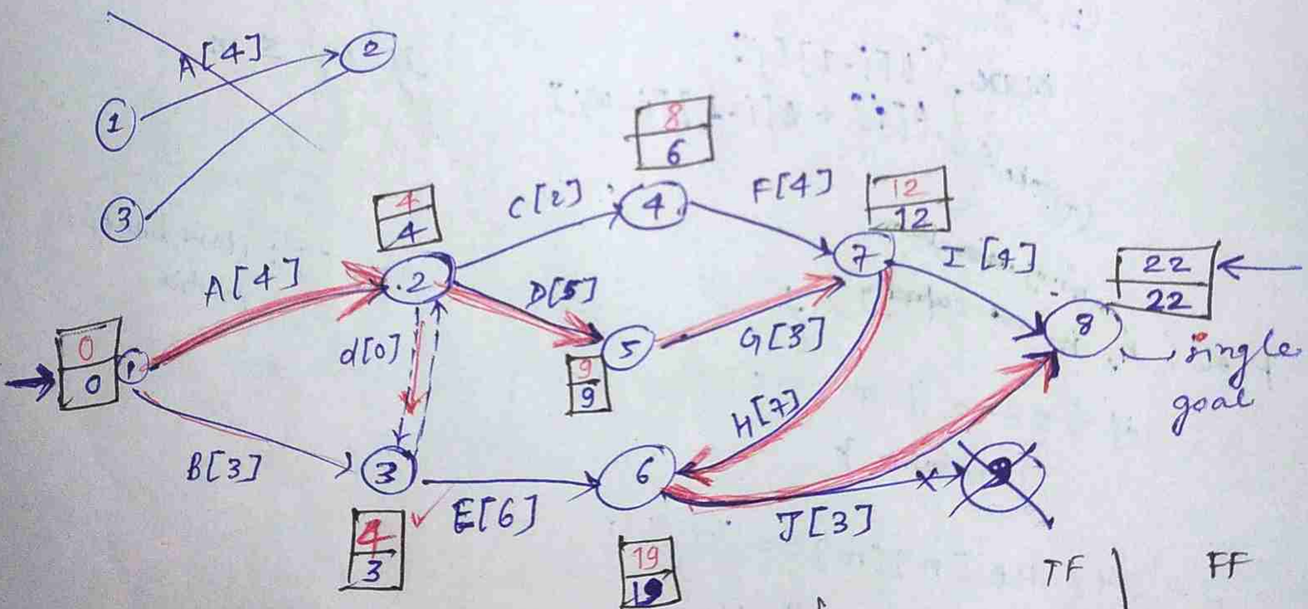
Condition for critical path :—

(1) $ES_i = LC_i$

(2) $ES_j = LC_j$

(3) $ES_j - ES_i = LC_j - LC_i = D_{ij}$

Activity	immediate predecessor	duration
A	-	4
B	-	3
C	A, B	2
D	A, B	5
E	B	6
F	C	4
G	D	3
H	F, G	7
I	F, G	4
J	E, H	3



critical path : 1 2 5 7 6 8

critical activities : A, D, G, H, J

★ Total float: $TF_{ij} = LC_j - ES_i - D_{ij}$

Free-float: $FF_{ij} = ES_j - ES_i - D_{ij}$

	TF	FF
B	1	0
C	2	0
E	10	10
F	2	2
I	6	6

Gantt chart (Time chart) :-

Gantt chart is a commonly used graphical depiction of a project or a schedule where start and completion time ~~it~~ will be represented on this chart. This chart gives ~~you~~ clear calendar schedule for whole project when there are limitations on available resources like manpower, equipments money etc, using this chart one can depending upon their total-float to maintain the pre-requirement of resources.

Operational Research

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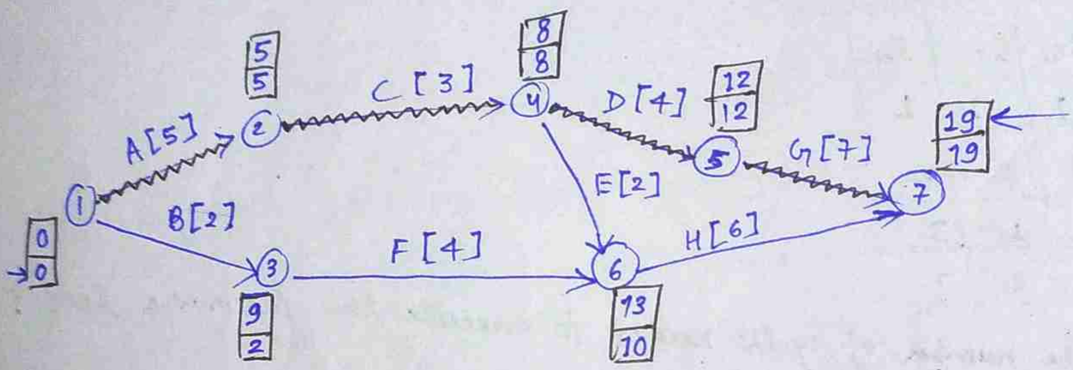
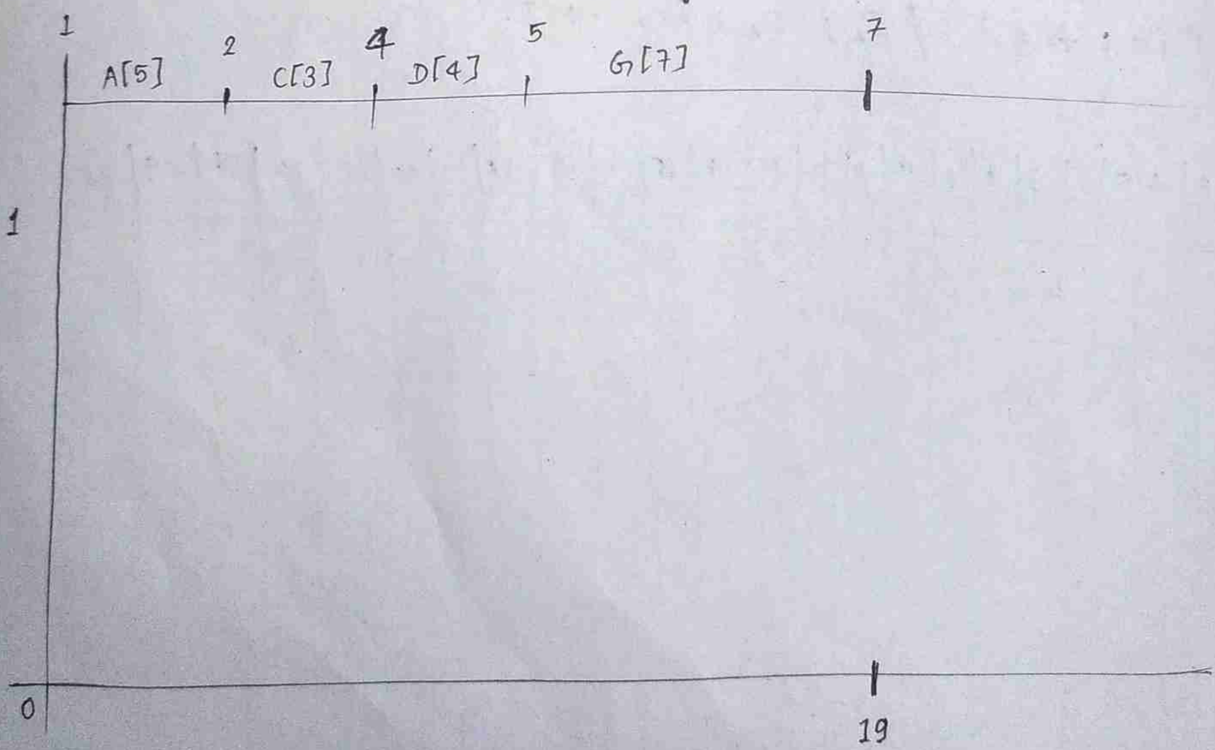
Activity	Immediate predecessor	Duration (Months)
A	–	5
B	–	2
C	A	3
D	C	4
E	C	2
F	B	4
G	D	7
H	E, F	6



critical path :-
1 2 4 5 7

critical activities :-
A, C, D, G



Project Evaluation and Review Technique (PERT) :-

Activity	mean	Immediate predecessor	Variance	Duration ~~(months)~~ (weeks) optimistic (O)	most likely (M)	pessimistic (P)
A	6	–	0.11	5	6	7
B	3	–	0.44	1	3	5
C	4	–	1.00	1	4	7
D	2	A	0.11	1	2	3
E	3	B	1.78	1	2	9
F	5	C	1.78	1	5	9
G	3	C	1.00	2	2	8
H	5	E, F	1.00	4	4	10
I	5	D	1.00	2	5	8
J	3	H, G	1.00	2	2	8

0 - Construct the project network.

Q 1- Find the ~~exp~~ critical path and expected project completion time.

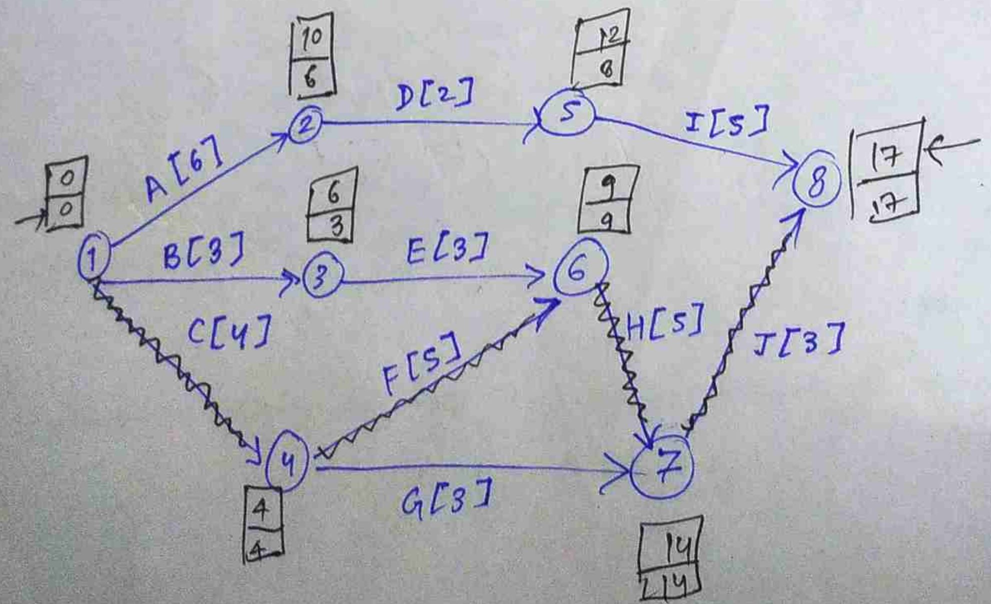
2 - What is the probability of completing the project on or before 22 weeks.

3 - find the expected duration and variance.

\# Assignment Question ✓

★ Mean (t_e expected duration) = $\dfrac{t_o + 4t_m + t_p}{6}$

★ variance (σ^2) = $\left(\dfrac{t_p - t_o}{6}\right)^2$



critical path :-

14678

Critical activities :-

C, F, H, J

critical activity	mean	variance
C	4	1.00
F	5	1.78
H	5	1.00
J	3	1.00
	17	4.78

$$P(x \le 22) = P\left(\frac{x-\mu}{\sigma}\right)$$

$$= P\left\{\frac{x-\mu}{\sigma} \le \frac{22-17}{\sqrt{4.78}}\right\}$$

$$= P(z \le 2.28)$$

OR

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Game Theory

deterministic situations
Probabilistic situations
Uncertainty situation

Terminologies of Game theory

Players : 2 players in a game
strategy : course of action taken by player ; $\begin{cases} \boxed{\text{Pure strategy}} \\ \boxed{\text{Mixed strategy}} \end{cases}$

Pure strategy : single strategy is followed with a probability 1

Mixed strategy : more than 1 strategy followed, sum will be 1

Payoff matrix : outcome associated with a_{ij}
if a_{ij} is +ve : gain of Player A, loss for player B
" " -ve : loss " " A, gain " " B

Maximin principle : maximizes the min guaranteed gains of player A.
maximum of these minimum gains is called maximin value.

Minimax principle : minimizes the max losses.

Saddle point : if maximin == minimax {game has a saddle point}

Value of the game : if saddle point then value at saddle point will be value of the game.
else
calculated using expected values.

Game with pure Strategies :—

Ex : Find the optimal strategies of the players in the following games —

$$\begin{array}{cc|ccc|c} & & & B & & \\ & & 1 & 2 & 3 & \text{Row min} \\ \hline & 1 & 25 & 20 & 35 & 20 \\ A & 2 & 50 & 45 & 55 & \boxed{45} \text{ — maximin} \\ & 3 & 58 & 40 & 42 & 40 \\ \hline \text{Col max} & & 58 & \boxed{45} & 55 & \\ & & & \text{minimax} & & \end{array}$$

∵ maximin = minimax ∴ game has a saddle point
hence,
value of the game will be 45. hence, in order to maximize the gain and

minimize the loss,

A will follow strategy 2

B " " " 2

optimal probabilities —

$A[P_1, P_2, P_3] = [0, 1, 0]$

$B[Q_1, Q_2, Q_3] = [0, 1, 0]$

Game with mixed strategies :-

Example :-

		B 1	B 2
A	1	a	b
A	2	c	d

Algorithm :-

step 1 : find absolute value of a-b, write it against R2

step 2 : |c-d|, write against R_1

step 3 : |a-c|, C_2

step 4 : |b-d|, C_1

} oddment calculation

step 5 : Calculate, (Probabilities of selection of alternatives of A and B)

$$P_1 = \frac{|c-d|}{|a-b|+|c-d|} \qquad P_2 = \frac{|a-b|}{|a-b|+|c-d|}$$

$$Q_1 = \frac{|b-d|}{|a-c|+|b-d|} \qquad Q_2 = \frac{|a-c|}{|a-c|+|b-d|}$$

step 6 : Calculate, value of the game

$$V = \frac{a|c-d|+c|a-b|}{|a-b|+|c-d|} = \frac{b|c-d|+d|a-b|}{|a-b|+|c-d|} = \frac{a|b-d|+b|a-c|}{|b-d|+|a-c|}$$

$$= \frac{c|b-d|+d|a-c|}{|a-c|+|b-d|}$$

OR
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Dominance property

Example:-

Player B

Player A	1	2	3	4	5	Row min	Total sum
1	4	6	5	10	6	4	31
2	7	8	5	9	10	5	39
3	8	9	11	10	9	$\boxed{8}$ maximin	47
4	6	4	10	6	4	4	$\boxed{30}$ → smallest sum value
Col max	$\boxed{8}$ minimax	9	11	10	10		

Check if any row greater than this row, If Y, then delete the smallest value selected row.

since, minimax = maximin, = 8
game has a saddle point, V = 8

Player A = [0, 0, 1, 0, 0]
Player B = [1, 0, 0, 0, 0]

B

A	1	2	3	4	5	Total sum
~~1~~	~~4~~	~~6~~	~~5~~	~~10~~	~~6~~	~~31~~
2	7	8	5	9	10	39
3	8	9	11	10	9	47
Total sum	15	17	16	$\boxed{19}$	19	

dominance can't be figured out moving to column dominance

max sum value
check if any col has smaller value than this,
If Y, then delete that column

Dominance property with mixed strategy

A \ B	1	2	3	4	Row min	Total Sum
1	5	-3	3	4	-3	9
2	-4	5	4	5	-4	10
3	4	-4	-3	3	-4	$\boxed{0}$
Col max	5	5	4	5		
Total sum	1	2	7	$\boxed{9}$		

↳ dominated by Row 1

dominated by col 2

	1	2	3
1	5	-3	3
2	-4	5	4
Total sum	1	2	7

Total sum	oddments
2	9
1	8

Neither of row or column dominance can be applied.

oddments 8 9

$p_1 = \frac{9}{17}$, $p_2 = \frac{8}{17}$, $q_1 = \frac{8}{17}$, $q_2 = \frac{9}{17}$

$V = \frac{13}{17}$ = value of game.

Matrix oddment for nxn games for arithmetic mean

let $A = (a_{ij})$ be nxn payoff matrix, obtain a new matrix C, whose first column is obtained from A by subtracting 2nd column from 1st; second col = 3rd from 2nd and so on.

$$A = \begin{bmatrix} 3 & -1 & -3 \\ -3 & 3 & -1 \\ -4 & -3 & 3 \end{bmatrix}_{3\times 3} \qquad C = \begin{bmatrix} 4 & 2 \\ -6 & 4 \\ -1 & -6 \end{bmatrix}_{3\times 2}$$

$$R = \begin{bmatrix} 6 & -4 & -2 \\ 1 & 6 & -4 \end{bmatrix}_{2\times 3}$$

$$|C_1| = \begin{vmatrix} -6 & 4 \\ -1 & -6 \end{vmatrix} = 40 \qquad |R_1| = \begin{vmatrix} -4 & -2 \\ 6 & -4 \end{vmatrix} = 28 \qquad |R_3| = \begin{vmatrix} 6 & -4 \\ 1 & 6 \end{vmatrix} = 40$$

$$|C_2| = \begin{vmatrix} 4 & 2 \\ -1 & -6 \end{vmatrix} = 22 \qquad |R_2| = \begin{vmatrix} 6 & -2 \\ 1 & -4 \end{vmatrix} = 22$$

$$|C_3| = \begin{vmatrix} 4 & 2 \\ -6 & 4 \end{vmatrix} = 28$$

Row oddments

$$A = \begin{array}{ccc|c} & & & \text{Row oddments} \\ 3 & -1 & -3 & 40 \\ 3 & 3 & -1 & 22 \\ -4 & -3 & 3 & 28 \\ \hline 28 & 22 & 40 & 90 \end{array}$$

column oddments

strategy of row player

$$\left(\frac{40}{90}, \frac{22}{90}, \frac{28}{90}\right) = \left(\frac{4}{9}, \frac{11}{45}, \frac{14}{45}\right)$$

strategy for column player

$$\left(\frac{28}{90}, \frac{22}{90}, \frac{40}{90}\right) = \left(\frac{14}{45}, \frac{11}{45}, \frac{4}{9}\right)$$

value of the game :- $\frac{4}{9}\times 3 + \frac{11}{45}\times(-3) + \frac{14}{45}\times(-4)$

$$= \frac{-29}{45}$$

Q. $A = \begin{bmatrix} -1 & 2 & 1 \\ 1 & -2 & 2 \\ 3 & 4 & -3 \end{bmatrix}$

$C = \begin{bmatrix} -3 & 1 \\ 3 & -4 \\ -1 & 7 \end{bmatrix}$ $\quad$ $R = \begin{bmatrix} -2 & 4 & -1 \\ -2 & -6 & 5 \end{bmatrix}$

$C_1 = \begin{vmatrix} 3 & -4 \\ -1 & 7 \end{vmatrix} = 17$ $\quad$ $R_1 = \begin{vmatrix} 4 & -1 \\ -6 & 5 \end{vmatrix} = 14$

$C_2 = \begin{vmatrix} -3 & 1 \\ -1 & 7 \end{vmatrix} = 20$ $\quad$ $R_2 = \begin{vmatrix} -2 & -1 \\ -2 & 5 \end{vmatrix} = 12$

$C_3 = \begin{vmatrix} -3 & 1 \\ 3 & -4 \end{vmatrix} = 9$ $\quad$ $R_3 = \begin{vmatrix} -2 & 4 \\ -2 & -6 \end{vmatrix} = 20$

$$A = \begin{array}{c|c} \begin{bmatrix} -1 & 2 & 1 \\ 1 & -2 & 2 \\ 3 & 4 & -3 \end{bmatrix} & \begin{array}{c} \text{row} \\ 17 \\ 20 \\ 9 \end{array} \\ \hline \text{col} \quad 14 \quad 12 \quad 20 & 46 \end{array}$$

strategy of Row player

strategy of col. player

val of game

Graphical Soln

Example of case 2 : 2×4 rectangular games

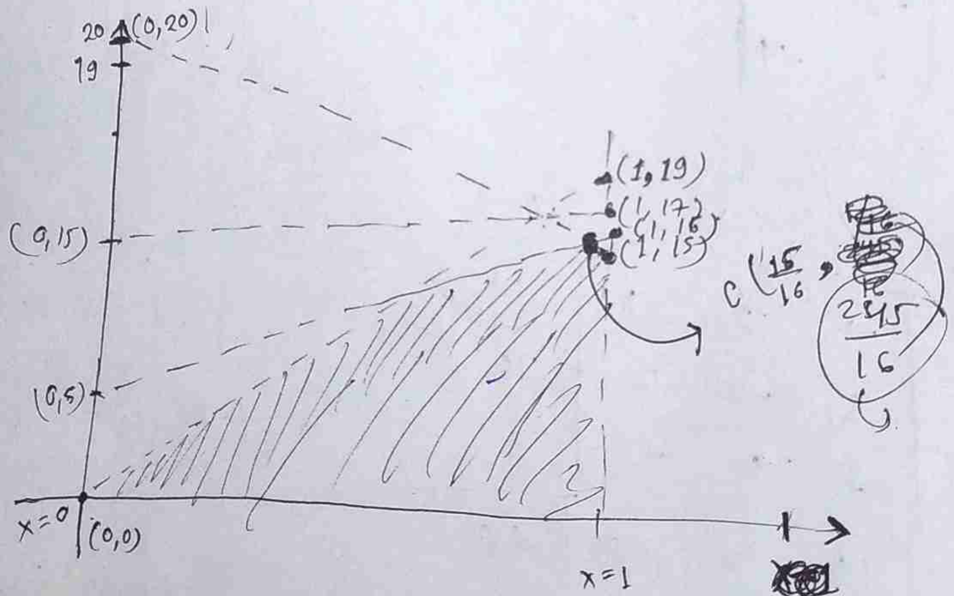
$2 \times n$ → नीचे वाला area

$n \times 2$ → ऊपर वाला area

		P_2: 1	2	3	4	
P_1	1	19	15	17	16	x
	2	0	20	15	5	$1-x$

Expected payoff function and gain of player A

strategy of B	Expected payoff function of A	Expected gain of A: $x=0$	$x=1$
1	$19x$	0	19
2	$20-5x$	20	15
~~2~~ 3	$\cancel{19x}$ $15+2x$	15	17
4	$\cancel{16x}$ $5+11x$	5	16



$20-5x = 5+11x$

$16x = 15$

$x = \frac{15}{16}$

$$P = \left(\frac{15}{16}, \frac{1}{16}\right)$$

Value of game $= 20-5x$ or $5+11x$

$= \frac{245}{16}$ or $\frac{5+11\times15}{16} \Rightarrow \frac{245}{16}$