Multi-Level and Multi-Index Monte Carlo Discontinuous Galerkin Methods for Uncertainty Quantification of Nonlinear Hyperbolic Problems

Stanislav Polishchuk (Monash University), Hans De Sterck (University of Waterloo) and Tiangang Cui (Monash University)

Introduction

Many physical phenomena are modelled by partial differential equations (PDEs). As a result of measurement noise and uncertainties in model-driven factors such as initial conditions, boundary conditions, domain geometry and other model inputs, it is required to quantify uncertainties in the solutions.

- Input uncertainty: quantified by some probability distribution of the model parameters, $\pi(\omega)$;
- Quantity of interest: $P(\omega)$, reguires solving PDEs;
- Uncertainty quantification: $\mathbb{E}_\pi[P(\omega)]$ or some other statistics of QoI.

Model problem

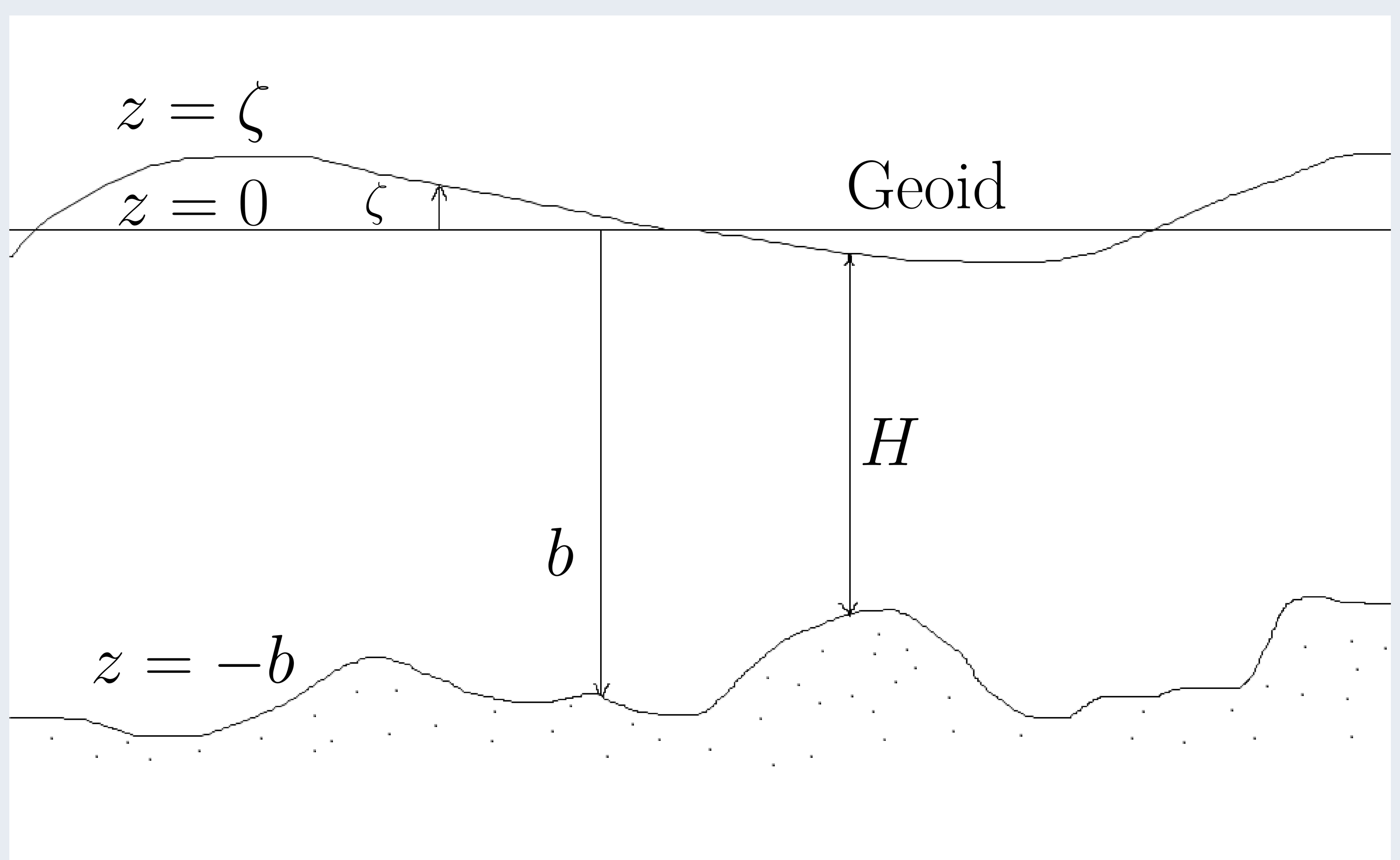

Figure – Shallow water model

Nonlinear shallow water equations in conservative form are

$$\frac{\partial H}{\partial t}+\frac{\partial}{\partial x}(Hu)+\frac{\partial}{\partial y}(Hv)=0$$
$$\frac{\partial Hu}{\partial t}+\frac{\partial}{\partial x}(Hu^2+\frac{1}{2}g(H^2-h^2))+\frac{\partial}{\partial y}(Huv)=-gH\frac{\partial\zeta}{\partial x}$$
$$\frac{\partial Hv}{\partial t}+\frac{\partial}{\partial x}(Huv)+\frac{\partial}{\partial y}(Hv^2+\frac{1}{2}g(H^2-h^2))=-gH\frac{\partial\zeta}{\partial y}$$

- $D=[-1000;1000]\times[-1000;1000]$ and $T=[0;10]$;
- $\zeta=\zeta(t,x,y)$: the elevation of the free surface relative to the geoid;
- $b=b(x,y;\omega)$: the bathymetry modelled as a lognormal random field;
- $H(t,x,y;\omega)=b+\zeta$ is the total depth of the water column.
- QoI: the average total depth over the region $[-500;500]\times[-500;500]$:

$$P(\omega)=D^{-1}\int_D H(t=10,x,y;\omega)\,dD.$$

Discontinuous Galerkin method

$$(U_t,v)_{D_k}=(A(U),\nabla v)_{D_k}+(A^*\cdot n,v)_{\partial D}$$

- $(\cdot,\cdot)_{D_k}$ is the scalar product in L_2-space;
- $A(U)$ is the flux matrix;
- the function v is the test function in some test space.

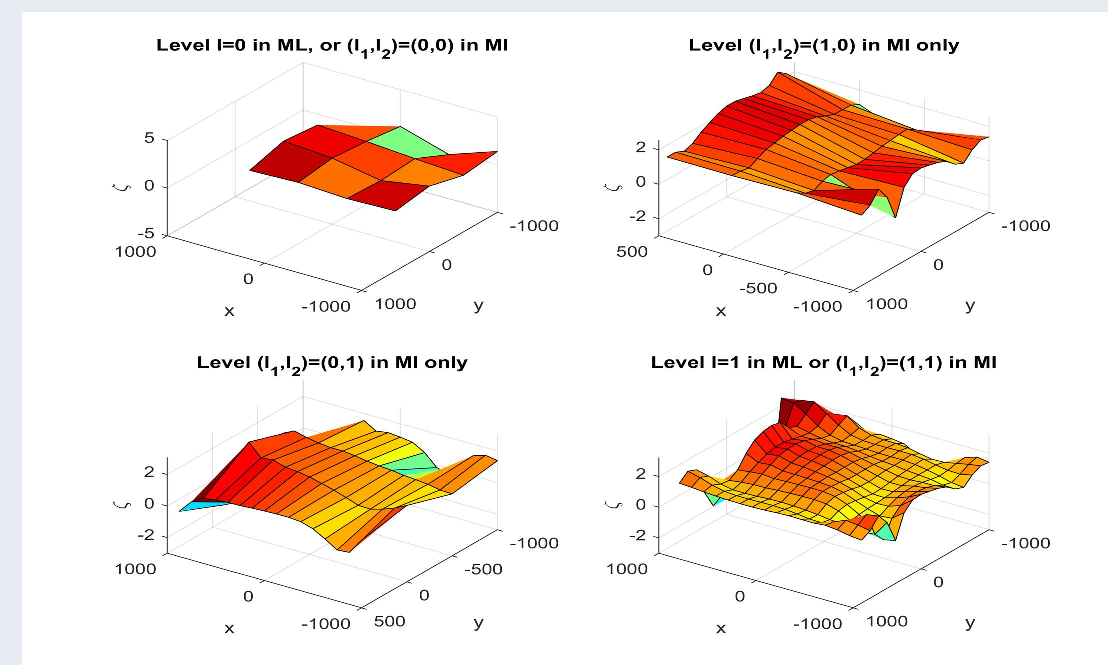

Figure – Free surface elevation, $\zeta(x,y,t=1.14)$

Multi-level Monte Carlo

Given a sequence of QoIs $P_0,\dots,P_L$

$$\mathbb{E}[P]\approx\mathbb{E}[P_L]=\mathbb{E}[P_0]+\sum_{l=1}^{L}\mathbb{E}[P_\ell-P_{\ell-1}]$$

The ML estimator

$$N_0^{-1}\sum_{n=1}^{N_0}P_0^{(0,n)}+\sum_{\ell=1}^{L}\left(N_\ell^{(-1)}\sum_{n=1}^{N_\ell}(P_\ell^{(\ell,n)}-P_{\ell-1}^{(\ell,n)})\right).$$

$$N_\ell=2\varepsilon^{-2}\sqrt{V_\ell/C_\ell}\sum_{i=0}^{L}\sqrt{V_iC_i}$$

Multi-index Monte Carlo

Level ℓ is now a vector of indices $\boldsymbol{\ell}=(\ell_1,\ell_2,\dots,\ell_M)$ and we define a backward difference operator in one particular dimension,

$$\Delta_m P_{\boldsymbol{\ell}}\equiv P_{\boldsymbol{\ell}}-P_{\boldsymbol{\ell}-\boldsymbol{e}_m}$$

Then defining the cross-difference

$$\Delta P_{\boldsymbol{\ell}}\equiv\left(\prod_{m=1}^{M}\Delta_m\right)P_{\boldsymbol{\ell}},$$

where $\mathfrak{L}$ is some index set.
The MI telescopic sum

$$\mathbb{E}[P]=\sum_{\boldsymbol{\ell}\in\mathfrak{L}}\mathbb{E}[\Delta P_{\boldsymbol{\ell}}].$$

The estimator Y for $\mathbb{E}[P]$

$$Y=\sum_{\boldsymbol{\ell}\in\mathfrak{L}}Y_{\boldsymbol{\ell}},\quad Y_{\boldsymbol{\ell}}=\frac{1}{N_{\boldsymbol{\ell}}}\sum_{n=1}^{N_{\boldsymbol{\ell}}}\Delta P_{\boldsymbol{\ell}}^{(\boldsymbol{\ell},n)}.$$

Results

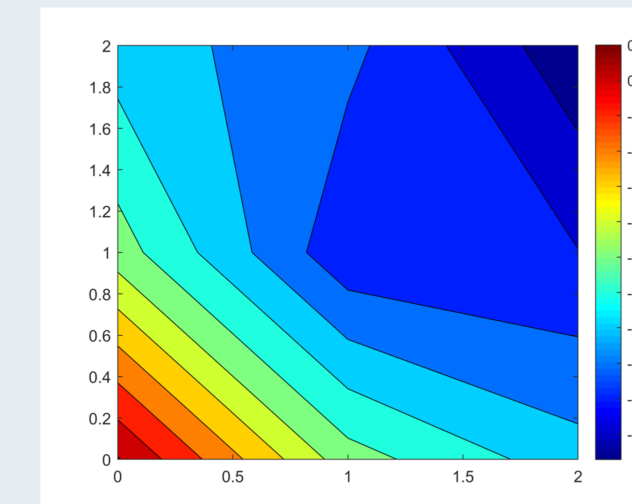

Figure – MI mean, $\log(\mathbb{E}_{\boldsymbol{\ell}})$

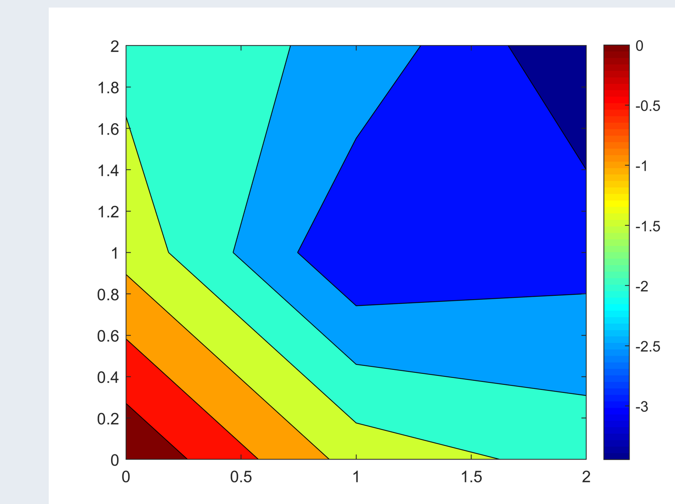

Figure – Profits, $\log\mathbb{P}_{\boldsymbol{\ell}}=\log\frac{\mathbb{E}_{\boldsymbol{\ell}}}{\sqrt{\mathbb{V}_{\boldsymbol{\ell}}C_{\boldsymbol{\ell}}}}$

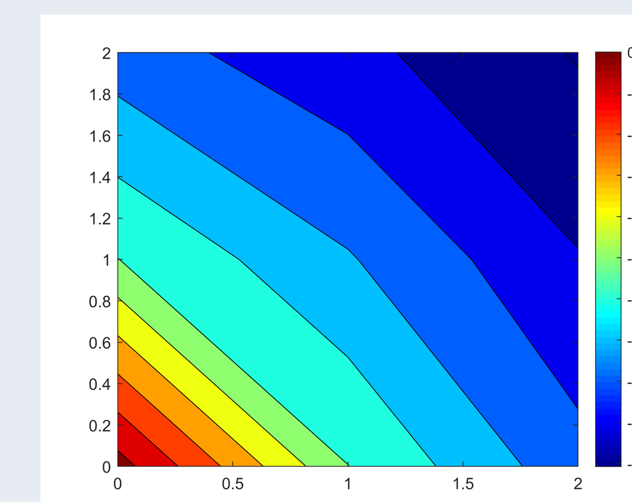

Figure – MI variance, $\log\mathbb{V}_{\boldsymbol{\ell}}$

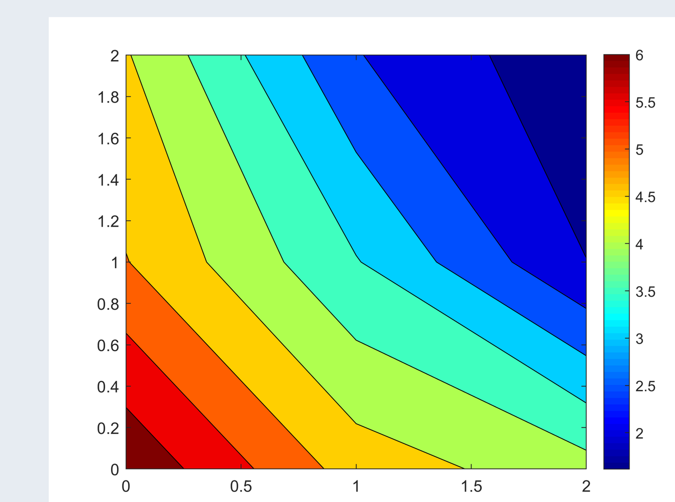

Figure – MI number of samples, $\log N_{\boldsymbol{\ell}}$

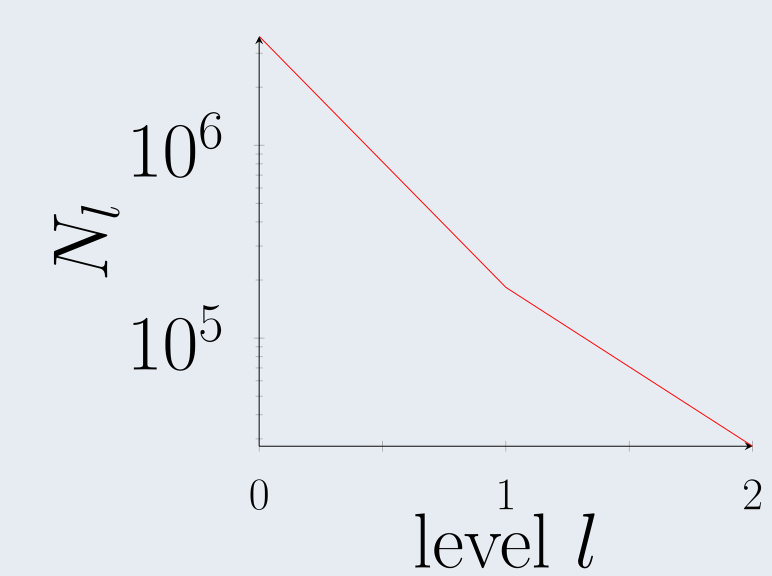

Figure – ML number of samples

	MC	MLMC	MIMC
Complexity	$O(\epsilon^{-5})$	$O(\epsilon^{-2})$	$O(\epsilon^{-2})$
Computational time	17 hours (estimated)	1.03 h	1.05 h

Table – Computational cost, ε =1e-3

References

[1] M.B. Giles. Multilevel Monte Carlo methods. *Acta Numerica*, 24:259–328, 2015.

[2] J.S. Hesthaven and T. Warburton. *Nodal Discontinuous Galerkin Methods Algorithms, Analysis, and Applications*. Springer, 2007.

[3] Abdul-Lateef Haji-Ali, Fabio Nobile, and Raúl Tempone. Multi-index Monte Carlo: when sparsity meets sampling. *Numerische Mathematik*, 132:767–806, 2016.