$\boxed{\text{Mathematica Homework \#4}}$

Email notebook to corbin@physics.ucla.edu
with a subject line: [Physics 105B]
Due: Friday, 3 June

Please note: The last day to submit Mathematica projects for a grade is Friday, 3 June. There will be no exceptions.

Relativity is all about transforming the description of events observed in one frame of reference into valid descriptions in some other frame of reference.

Events happen some-where and some-when. The mathematical representation of an event looks like:

$$r = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

where t, x, y and z are the coordinates of the event in the relevant frame of reference, and c is the (frame invariant!) speed of light in vacuum.

We'll follow the usual convention where the *rocket frame* (S') moves through the *lab frame* (S) with a relative velocity $\vec{v} = \vec{\beta}c$. Coordinates measured in the the rocket frame will usually be primed, their counterparts in the lab frame will usually be unprimed.

Unless otherwise noted, we'll assume that the axes in S and S' coincide at $t = t' = 0$ and the rocket is moving in the $+x$ direction in the lab. The transformation operator (often called a *Lorentz boost*), will then look like

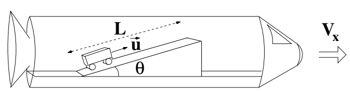
$$\begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{where} \qquad \gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$

and transformations of event descriptions from the rocket frame to the lab frame will look like

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix}$$

Ok, we should all be on the same page, let's proceed to the fun stuff ...

- In the first cell, enter all the usual stuff: your **name**, **student ID**, **email address** and the **assignment identifier** (eg. "HW 5").
- 1) Let's start by checking for consistency. This is an easy problem, but the result is important. Find the inverse operator that takes descriptions of events in the lab and maps them into the rocket frame. Compare your result to the boost operator from the rocket to the lab and discuss.
- 2) Still on the consistency theme. According to Einstein, anything that travels with a speed c in one inertial frame of reference travels with a speed c in *all* inertial frames of reference. To test this, let's launch a photon from the origin in the rocket frame, at $t' = 0$, in the direction (polar angle $= \theta'$, azimuthal angle $= \phi'$).
  - i) Write an event description that gives the space-time coordinates of the photon at any time t' in the rocket frame.
  - ii) Boost that description into the lab frame (that is, transform it into a valid description of how things would look in the lab).
  - iii) Use the description of the photon's motion through the lab to verify that it is indeed traveling with a speed c through the lab. Be careful with the temporal component of your event description!
- 3) This is fun to do by hand, more so by computer (since you have the luxury of stepping away from the math and thinking about what it means) - derive the following special relativity 'magic tricks'...
  - i) Time Dilation. Don't forget, the events must be co-located in the rocket frame!
  - ii) The Relativity of Simultaneity. Show that events that simultaneous events in the rocket frame are not simultaneous in the lab.
  - iii) Length Contraction. Hint: How does one measure length? (*ii*) is more important than it looks.
  - iv) For fun: Take a stick of length 5 units, inclined at an angle of 30 degrees from the $+x'$ axis in the rocket frame. Use LineListPlot and Manipulate to draw the stick as it would appear in the lab frame as a function of β.
- 4)



Astronauts have set up an inclined plane (length L, angle θ, as measured in the rocket) to test a robotic rover that has been programmed to roll over terrain with a constant speed u. The plane has been oriented along the length of the rocket, which is traveling relative to the nearest space outpost with a speed V_x, as shown.

- i) How long will it take the rover to climb the full length of the plane in the rest-frame of the outpost?
- ii) How far will the rover have traveled in the outpost's rest-frame?
- iii) How fast is the rover moving in the outpost's rest-frame?
- iv) How does your answer to part b compare to the length of the plane as measured in the outpost's rest-frame? Explain.

- 5) To understand how "velocity addition" works in special relativity, let's take a massive particle and send it from the origin in the rocket frame with a velocity (u'_x, u'_y, u'_z) at time $t' = 0$.
  - i) Write an event description that gives the space-time coordinates of the particle at any time t' in the rocket frame.
  - ii) Boost that description into the lab frame.
  - iii) Use the description in the lab frame to find the velocity components of the particle as seen in the lab.

- 6) In three-space, the inner product $\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$. In Lorentz four-space, the inner product is evaluated $A \cdot B = A_t B_t - A_x B_x - A_y B_y - A_z B_z$.
  - i) Show that the inner-product in four-space is frame-invariant.
  - ii) The four-momentum of a particle is given by

$$P = \begin{pmatrix} E/c \\ P_x \\ P_y \\ P_z \end{pmatrix} = \gamma_p \begin{pmatrix} mc \\ mv_x \\ mv_y \\ mv_z \end{pmatrix}$$

    where γ_p is the Lorentz gamma factor evaluated using the speed of the particle in the lab frame. What is the value of P^2 (that is, the inner product of P with itself?). What does this say about the energy and momentum of 'massless' particles like photons, neutrinos, relativistic electrons and so on?
  - iii) Tangential, but fun. Taylor-expand the temporal component of P in β^2 and compare the result to classical physics.

- 7) Create a space-time diagram showing the light cone, a set of at least five points inside the light cone and a set of at least five points outside the light cone (different colors, inside and outside). Lorentz-boost the cone and points and plot them in a new frame. Do this for a range of relative speeds (Manipulate might be cool).