Chapter 5: Statistical Models in Simulation

The world the model builder sees is probabilistic rather than deterministic.

Deterministic: model does not include elements of randomness. Every time you run the model with the same initial conditions you will get the same results.

Probabilistic: model includes elements of randomness. Every time you run the model, you are likely to get different results, even with the same initial conditions. A probabilistic model is one which incorporates some aspect of random variation.

An appropriate model can be developed:

→ by sampling the phenomenon of interest

→ selecting a known distribution form through educated guesses

→ make an estimate of the parameters of this distribution

→ then test to see how good a fit has been obtained.

Through continued efforts in the selection of an appropriate distribution form, a postulated model could be accepted.

Review of terminology and concepts.

1. Discrete random variables.

Let X be a random variable. If the number of possible values of X is finite or countably infinite, X is called a discrete random variable.

The possible values of X may be listed as $x_1, x_2 \ldots$. In the finite case, the list terminates. In the countably infinite case, the list continues indefinitely.

Example 5.1

The number of jobs arriving each week at a job shop is observed. The random variable of interest is X, where

X = number of jobs arriving each week.

The possible values of X are given by the range space of X, which is denoted by R_x.

Here, $R_x = \{0, 1, 2, \ldots\}$

Let X be a discrete random variable. With each possible outcome x_i in R_x, a number $p(x_i) = P(X = x_i)$ gives the probability that random variable equals the value of x_i.

The numbers $p(x_i)$, $i = 1, 2, \ldots$ must specify the following two conditions :

1. $p(x_i) \geq 0$ for all i

2. $\sum_{i=1}^{\infty} p(x_i) = 1$

The collection of pairs $(x_i, p(x_i))$, $i = 1, 2 \ldots$ is called the probability distribution of X and $p(x_i)$ is called the probability mass function (pmf) of X.

2. Continuous Random Variables . If the range space R_x of the random variable X is an interval or a collection of intervals, X is called a continuous random variable. For a continuos random variable X, the probability that X lies in the interval $[a, b]$ is given by ,

$$P(a \leq x \leq b) = \int_a^b f(x)\,dx \quad \text{------} \quad 5.1$$

The function $f(x)$ is called the probability density function (pdf) of the random variable X.

The pdf satisfies the following conditions:

a. $f(x) \geq 0$ for all x in R_x

b. $\int_{R_x} f(x)\,dx = 1$

c. $f(x) = 0$ if x is not in R_x.

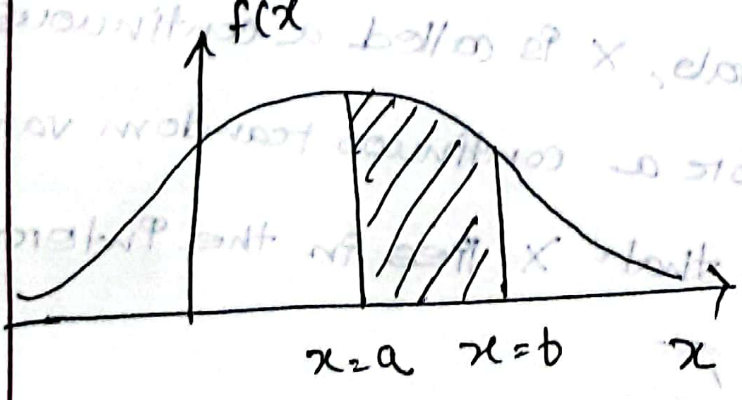
Properties →

1. $P(X = x_0) = 0$ because $\int_{x_0}^{x_0} f(x)\,dx = 0$

2. $P(a \leq x \leq b) = P(a < x \leq b) = P(a \leq x < b)$

$= P(a < X < b)$

The graphical representation of eq 5.1 →



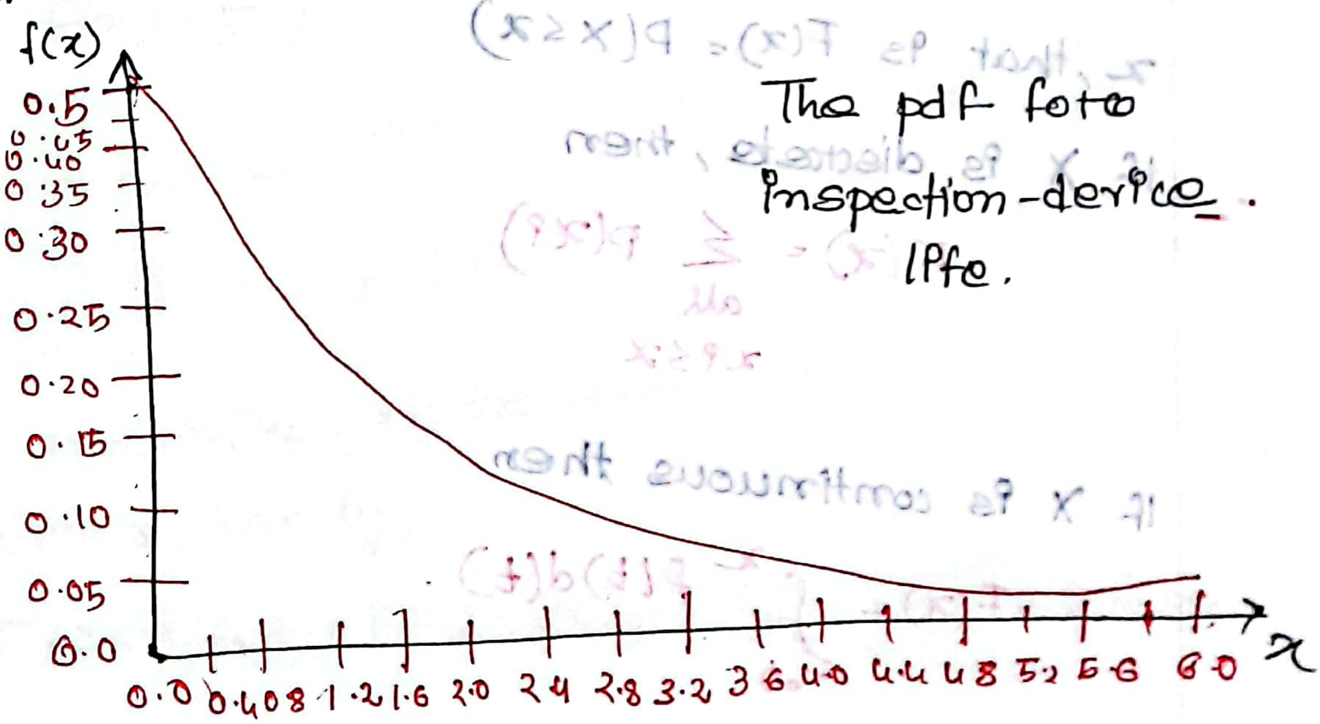
The shaded area represent the probability that X lies in the interval [a,b]

Example 5.3

The life of a device used to inspect cracks in aircraft wings is given by X, a continuous random variable assuming all values in the range $x \geq 0$. The pdf of the lifetime, in years, is as follows:

$$f(x) = \begin{cases} \frac{1}{2} e^{-x/2}, & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

the random variable X is said to have an exponential distribution with mean 2 years.



The pdf for inspection-device life.

The probability that the life of the device is between 2 and 3 years is calculated as.

$$P(2 \leq X \leq 3) = \frac{1}{2} \int_{2}^{3} e^{-x/2} dx$$

$$\left[\cancel{\int e^{x} dx} = \int e^{ax} dx = \frac{1}{a} e^{ax} + c \right]$$

$$= e^{-\frac{3}{2}} + e^{-1}$$

$$= -0.223 + 0.368$$

$$= 0.145$$

3. Cumulative Distribution Function.

The cumulative distribution function (cdf), denoted by $F(x)$, measures the probability that the random variable X assumes a value less than or equal to x, that is $F(x) = P(X \leq x)$

If X is discrete, then

$$F(x) = \sum_{\substack{\text{all} \\ x_i \leq x}} p(x_i)$$

If X is continuous then

$$F(x) = \int_{-\infty}^{x} f(t)\, d(t).$$

Some properties of cdf:

a. F is a nondecreasing function.

If $a < b$ then, $F(a) \leq F(b)$

b. $\lim_{x \to \infty} F(x) = 1$

c. $\lim_{x \to -\infty} F(x) = 0$

All probability questions about X can be answered in terms of the cdf. For example,

$$P(a < X \leq b) = F(b) - F(a) \text{ for all } a < b .$$

Example 5.5

The cdf for the device in Example 5.3 is given by.

$$F(x) = \frac{1}{2} \int_0^x e^{-t/2} dt$$

$$= -e^{-x/2} + e^{-0/2}$$

$$= 1 - e^{-x/2}$$

The probability that the device will last for less than 2 years is given by,

$$P(0 \leq x \leq 2) = F(2) - F(0) = F(2) = 1 - e^{-1} = 0.632$$

The probability that the life of the device is between 2 and 3 years is calculated as,

$$P(2 \le x \le 3) = F(3) - F(2)$$

$$= (1 - e^{-3/2}) - (1 - e^{-1})$$

$$= e^{-1} - e^{-3/2}$$

$$= 0.368 - 0.223$$

$$= 0.145$$

4. Expectation.

If X is a random variable, the expected value of X, denoted by $E(x)$, for discrete and continuous variables is defined as follows:

$$E(x) = \sum_{\text{all } i} x_i \, P(x_i) \quad \text{if } X \text{ is discrete.}$$

and $$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx \quad \text{if } X \text{ is continuous.}$$

The expected value $E(X)$ of a random variable X is also referred to as the mean μ, or the first moment of X.

→ The mean $E(x)$ is a measure of central tendency of a random variable.

四 The variance of a random variable X, denoted by $V(x)$ or $var(X)$ or σ^2, is defined by,

$$V(x) = E[(X - E[X])^2]$$

Also, $V(x) = E(X^2) - [E(x)]^2$

→ The variance of X measures the expected value of the squared difference between the random variable and its mean.

→ Thus, the variance $V(x)$ is a measure of the spread or variation of the possible values of X around the mean $E(X)$.

→ The standard deviation, σ ~~= √V(X), are~~, is defined to the square root of the variance σ^2.

so $\sigma = \sqrt{V(x)}$

The standard deviation σ, and the mean $E(x)$ are expressed in the same units.

Example 5.7.

The mean and variance of the life of the device are computed as follows:

$$E(X) = \frac{1}{2}\int_0^\infty x e^{-x/2} dx$$

$$= -xe^{-x/2}\Big|_0^\infty + \int_0^\infty e^{-x/2} dx$$

$$= 0 + \frac{1}{1/2} e^{-x/2}\Big|_0^\infty$$

$$= 2 \text{ years.}$$

Formula: $\int x e^{ax} dx$

Integration by parts

$u = x \rightarrow du = dx$

$dv = e^{ax}dx \rightarrow v = \frac{1}{a}e^{ax}$

$$\int x e^{ax} dx = \int u\,dv$$

$$= uv - \int v\,du$$

$$= \frac{x}{a}e^{ax} - \frac{1}{a}\int e^{ax} dx$$

$$= \frac{x}{a} e^{ax} - \frac{1}{a}\left(\frac{1}{a} e^{ax}\right) + c$$

$$\boxed{= \frac{x}{a} e^{ax} - \frac{1}{a^2} e^{ax} + c}$$

To compute $V(x)$

$$E(X^2) = \frac{1}{2}\int_0^{\infty} x^2 e^{-x/2}\, dx$$

$$\boxed{\left[\because\ E(X^n) = \int_{-\infty}^{\infty} x^n f(x)\, dx \text{ if } X \text{ is continuos}\right]}$$

Thus,

$$E(X^2) = \frac{1}{2}\cdot\frac{x^2}{-1/2} e^{-x/2}\Big|_0^{\infty} - \frac{2x}{2}\frac{1}{-1/2}\int_0^{\infty} x e^{-x/2}\, dx$$

$$\cancel{= -x^2 e^{-x/2}\Big|_0^{\infty} + 2\left(-\frac{1}{2} e^{-x/2}\right)\Big|_0^{\infty}}$$

$$\cancel{= 0 -} \left[\right.$$

$$= -x^2 e^{-x/2}\Big|_0^{\infty} + 2\int_0^{\infty} x e^{-x/2}\, dx$$

$$= 0 + 2 \times 4 = 8$$

so $V(x) = 8 - 2^2 = 4$ years.

so $\sigma = \sqrt{V(x)} = 2$ years

With a mean life of 2 years and a standard deviation of 2 years, most analysts would conclude the actual lifetimes have a fairly large variability.

Useful Statistical Model.

Here, statistical models appropriate to some application areas are presented. The areas include:

1. Queueing systems
2. Inventory and supply-chain system.
3. Reliability and maintainability
4. Limited data.

1. Queueing systems.

In a queueing system, interarrival and service-time patterns can be probabilistic

Ex sample statisticals models for interarrival or service time distribution:

☐ Exponential distribution;

If service times are completely random

☐ Normal distribution:

service times are constant but some random variability causes fluctuations in either a positive or negative way.

☐ Truncated normal distribution:

Similar to normal distribution, but the random variable is restricted to be greater than or less than a certain value.

☐ Gamma and Weibull distributions:

more general than exponential distribution.

The differences between the exponential, gamma and Weibull distribution involve the location of the modes of the pdf's and the shapes of their tails for large and small times.

In practice, if there are more large service times than an exponential distribution can account for, a Weibull distribution might provide a better model of these service times.

Discrete Distribution

Discrete random variables are used to describe random phenomena in which only integer values can occur.

→ Bernoulli trials and Bernoulli distribution
→ Binomial distribution
→ Geometric and negative binomial distribution
→ Poisson distribution

Bernoulli Trials and Bernoulli Distribution.

Bernoulli Trials

Consider an experiment consisting of n trials, each can be a success or a failure.

- Let $x_i = 1$, if the ith experiment is a success p
- and $x_i = 0$ if the ith experiment is a failure q.

The n Bernoulli trials are called a Bernoulli process if the trials are independent, each trial has only two possible outcomes (success or failure), and the probability of a success remains constant from trial to trial. Thus,

$$P(x_1, x_2 \ldots x_n) = P_1(x_1) \cdot P_2(x_2) \cdots P_n(x_n)$$

and $P_j(x_j) = p(x_j) = \begin{cases} p & x_j = 1, j = 1,2 \cdots n \\ 1-p = q & x_j = 0, j = 1,2, \cdots n \\ 0 & \text{otherwise.} \end{cases}$

For one trial, the ~~dist~~ distribution in the above equation is called Bernoulli distribution. The mean and variance of x_j are calculated as follows:

$$E(x_j) = 0 \cdot q + 1 \cdot p = p$$

and $V(x_j) = [(0^2 \cdot q) + (1^2 \cdot p)] - p^2 = p(1-p)$

Binomial Distribution

The random variable x that denotes the number of successes in n Bernoulli trials has a binomial distribution given by $p(x)$, where

$$p(x) = \begin{cases} \binom{n}{x} p^x q^{n-x}, & x = 0,1,2,\cdots n \\ 0 & \text{otherwise.} \end{cases}$$

The above equation is motivated by computing the probability of a particular outcome with all the successes, each denoted by S, occurring in the first x trials, followed by the $n-x$ failures, each denoted by an F.

that is,

$$P(\underbrace{SSS\ldots SS}_{x \text{ of these}}\underbrace{FF\ldots\ldots FF}_{n-x \text{ of these}}) = p^x q^{n-x}$$

where $q = 1-p$. There are

$$\binom{n}{x} = \frac{n!}{x!(n-x)!}$$

and the mean, $E(x)$, is given by,

$$E(x) = p+p+\cdots p = np$$

and the variance $V(x)$ is given by

$$V(x) = pq+pq+\cdots+pq = npq$$

Geometric Distribution.

The geometric distribution is related to a sequence of Bernoulli trials: the random variable of interest, X, is defined to be the number of trials to achieve the first success. The distribution of X is given by.

$$p(x) = \begin{cases} q^{x-1}p & x = 1,2\ldots \\ 0 & \text{otherwise} \end{cases}$$

The event $\{X = x\}$ occurs when there are $x-1$ failures followed by a success. Each of the failures has an associated probability of $q = 1-p$ and each success has probability p. Thus,

$$P(FFF\ldots FS) = q^{x-1}p$$

The mean and variance are given by $E(x) = \frac{1}{p}$

and $V(x) = \frac{q}{p^2}$

Negative Binomial Distribution

More generally, the negative binomial distribution is the distribution of the number of trials until the kth success, for $k = 1, 2, \ldots$ If Y has a negative binomial distribution with parameters p and k, then the distribution of Y is given by,

$$P(y) = \begin{cases} \binom{y-1}{k-1} q^{y-k} p^{k}, & y = k, k+1, k+2 \ldots \\ 0 & \text{otherwise} \end{cases}$$

Because we can think of the negative binomial random variable Y as the sum of k independent geometric random variables, it is easy to see that

$$E(Y) = \frac{k}{p} \quad \text{and} \quad V(X) = \frac{kq}{p^2}$$

Poisson distribution (for random occurrence)

The Poisson probability mass function is given by

$$P(x) = \begin{cases} \frac{e^{-\alpha}\alpha^{x}}{x!}, & x = 0, 1, \dots \\ 0 & \text{otherwise} \end{cases}$$

where $\alpha > 0$. One of the important properties of the Poisson distribution is that the mean and variance are both equal to α, that is

$$E(X) = \alpha = V(X).$$

The cumulative distribution function is given by.

$$F(x) = \sum_{i=0}^{x} \frac{e^{-\alpha}\alpha^{i}}{i!}$$

~~The pmf and cdf for a Poisson distribution with α=~~

Ex A computer repair person is "beeped" each time there is call for service. The number of beeps per hour is known to occur in accordance with a Poisson distribution with a mean of $\alpha = 2$ per h

1. Calculate the probability for 3 beeps.

$$P(3) = \frac{e^{-2} 2^{3}}{3!} = 0.18$$

The same result can be found by

$$F(3) - F(2) = \sum_{i=0}^{3} \frac{e^{-2} 2^i}{i!} - \sum_{i=0}^{2} \frac{e^{-2} 2^i}{i!}$$

$$= \left\{ e^{-2} + e^{-2} \times 2 + \frac{e^{-2} \times 2^2}{2!} + \frac{e^{-2} \times 2^3}{3!} \right\}$$

$$- \left\{ e^{-2} + e^{-2} \times 2 + \frac{e^{-2} \times 2^2}{2!} \right\}$$

$$= 0.857 - 0.677 = 0.18$$

2. Calculate the probability of two or more beeps in one hour.

$$P(2 \text{ or more}) = 1 - P(0) - P(1)$$

$$= 1 - \frac{e^{-2} 2^0}{0!} - \frac{e^{-2} 2^1}{1!}$$

$$= 0.594$$

or $P(2 \text{ or more}) = 1 - F(1) = 0.594$

Continuous Distribution

Continuous random variables can be used to describe random phenomena in which the variable can take on any value in some interval.

The distributions in this case are:

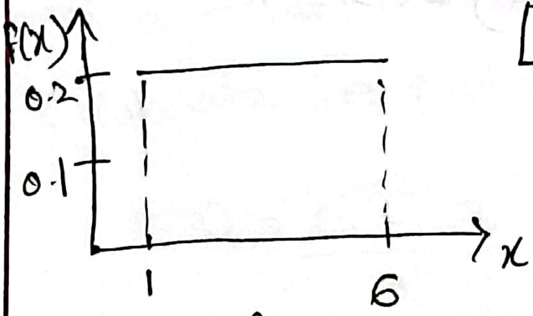
- Uniform
- Exponential
- Normal
- Weibull
- Lognormal
- Erlang
- Gamma Distribution
- Beta Distribution
- Poisson Distribution

Uniform Distribution

A random variable X is uniformly distributed on the interval [a,b] if its pdf is given by

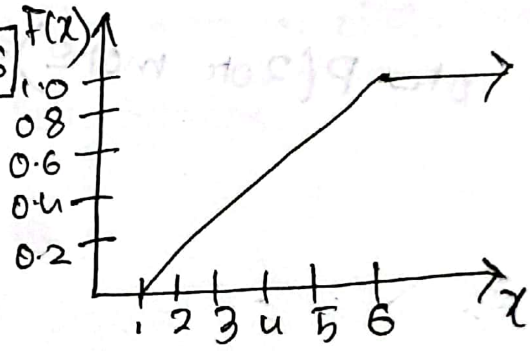
$$f(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

The cdf is given by, $F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \le x < b \\ 1 & x \ge b \end{cases}$



pdf

$\boxed{1 \le x \le 6}$



cdf

Note that, $P(x_1 < x < x_2) = F(x_2) - F(x_1) = \dfrac{x_2 - x_1}{b-a}$

is proportional to the length of the interval, for all x_1 and x_2 satisfying $a \leq x_1 < x_2 \leq b$. The mean and variance of the distribution are given by

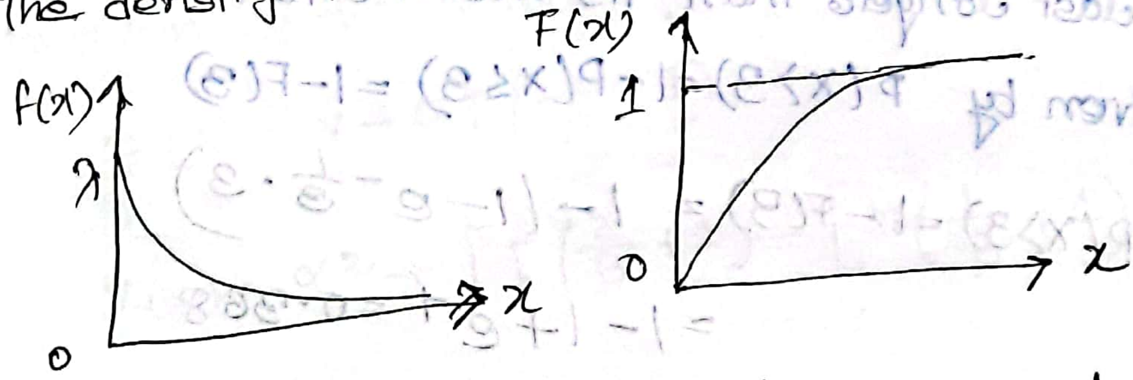
$$E(x) = \frac{a+b}{2} \quad \text{and} \quad V(x) = \frac{(b-a)^2}{12}$$

<u>Exponential Distribution</u>

A random variable X is said to be exponentially distributed with parameter $\lambda > 0$ if its pdf is given by,

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0 & \text{elsewhere} \end{cases} \quad \text{--- (1)}$$

The density function and cdf is shown below:



The exponential distribution has mean and variance given by, $E(x) = \dfrac{1}{\lambda}$ and $V(x) = \dfrac{1}{\lambda^2}$

Thus the mean and standard deviation are equal.

The cdf can be exhibited by integrating equation ①

to obtain,

$$F(x) = \begin{cases} 0 & x < 0 \\ \int_0^x \lambda e^{-\lambda t} dt = 1 - e^{-\lambda x} & x \geq 0 \end{cases}$$

$$\lambda \int_0^x e^{-\lambda t} dt = \lambda \cdot \frac{-1}{\lambda}\left[e^{-\lambda t}\right]_0^x = -\{e^{-\lambda x} - e^0\}$$

$$= -(e^{-\lambda x} - 1) = 1 - e^{-\lambda x}$$

Ex: Suppose that the life of an industrial lamp, in thousand of hours, is exponentially distributed with failure rate $\lambda = 1/3$ (one failure every 3000 hours on the average). The probability that the lamp will last longer than its mean life, 3000 hours is given by $P(X>3) = 1 - P(X \leq 3) = 1 - F(3)$

$$P(X>3) = 1 - F(3) = 1 - (1 - e^{-\frac{1}{3} \cdot 3})$$

$$= 1 - 1 + e^{-1} = 0.368$$

Regardless of the value of λ, this result will always be the same. That is, the probability that an exponential random variable is ~~greatere the~~ greatere that its mean is 0.368, for any value of λ.

The probability that the industrial lamp will last between 2000 and 3000 hours is computed as

$$P(2 \le X \le 3) = F(3) - F(2)$$

$$= \left(1 - e^{-\frac{1}{3} \cdot 3}\right) - \left(1 - e^{-\frac{1}{3} 2}\right)$$

$$= \cancel{0.632 - 0.490}$$

$$= e^{-\frac{2}{3}} - e^{-1}$$

$$= 0.145$$

Weibull Distribution.

The mean and variance of the Weibull distribution are given by the following expressions:

$$E(X) = v + \alpha \Gamma\left(\frac{1}{\beta} + 1\right)$$

$$V(X) = \alpha^2 + \left[\Gamma\left(\frac{2}{\beta} + 1\right) - \left[\Gamma\left(\frac{1}{\beta} + 1\right)\right]^2\right]$$

where $\Gamma(\beta) = (\beta - 1)!$

Thus, the location parameter ν has no effect on the variance; however, the mean is increased or decreased by ν. The cdf of the Weibull distribution is given by.

$$F(x) = \begin{cases} 0 & x < \nu \\ 1 - \exp\left[-\left(\frac{x-\nu}{\alpha}\right)^{\beta}\right] & x \geq \nu \end{cases}$$

Q. What is memoryless property of the exponential distribution?

One of the most important properties of exponential distribution is that it is memoryless, which means that, for all $s \geq 0$ and $t \geq 0$.

$$P(X > s+t \mid X > s) = P(X > t) \quad \text{--- (1)}$$

Let X represent the life of a component and assume that X is exponentially distributed. Equation (1) states that the probability that the component lives for at least $s+t$ hours, given that it has survived s hours, is the same as the ~~ptc~~ initial probability that it ~~leo~~ lives for at least t hours. If the component is alive at time s (if $x > s$), then the distribution of remaining amount of time that ~~is~~ it survives namely $x-s$, is the same as the original distribution of a new component. That is, the component does not remember that it has already been in use for ~~t~~ a time s. A used component is as good as new.

$$P(X > s+t \mid X > s) = \frac{P(X > s+t)}{P(X > s)}$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t}$$

$$= P(X > t).$$

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Chapter 4: Simulation Software.

Classification of Simulation Software:

1. General-purpose programming language
   - = Flexible and familiar
   - = Well suited for learning DES principles and techniques.
   - = E.g. C, C++, Java
2. Simulation Programming language
   - = GPSS, SIMAN
   - = GPSS/H is a highly structured process - interaction simulation language.
3. Simulation languages (simulation environment)
   - = Good for building models quickly
   - = Provide built-in features (e.g. queue structure)
   - = Graphics and animation provided
   - = E.g. Arena, Automod

Selection of simulation software

1. Model building feature.
2. Runtime environment
3. Animation of layout features
4. Output features
5. Vendor support and product documentation.

Model building features

1. Modeling world-view

Process interaction, event perspectives, and continuous modeling, depending on needs.

2. Input-data analysis capability

Estimate empirical or statistical distributions from trace data.

3. Graphical model-building

Process-flow, block-diagram or network approach

4. Conditional routing.

Route entities based on prescribed conditions or attributes.

5. Simulation programming

Capability to add procedural logic through a high-level powerful simulation language.

6. Syntax:

Easily understood, consistent, unambiguous, English

7. Input flexibility

Accepts data from external files, databases, spreadsheets or interactively.

8. Modeling conciseness.

Powerful actions, blocks or nodes.

9. Randomness

Random-variate generators for all common distributions, e.g., -exponential -uniform -normal -triangular

10. Specialized components and templates.

Material handling vehicles, conveyors, bridge cranes etc. Communication system, computer systems, call centers ..

11. User-built custom objects

Reusable objects, templates and submodels.

12. Continuous flow

Tanks and pipes or bulk conveyors.

13. Interface with general-programming language.

Link code in C, C++, Java or other general programming language.

Runtime Environment.

1. Execution speed.

Many runs needed for scenarios and replication

Impacts development as well as experimentation

2. Model size; number of variables and attributes.

should be no built in limits

3. Interactive debuggers.

Monitor the simulation in detail as it progresses. Ability to break, trap, run, until, step's to display status, attributes and variables, etc.

4. Model status and statistics

Display at any time during simulation

3. Runtime license - Ability to change parameters and run a model (but not to change logic or build a new model)

Animation and Layout Features:

1. Type of animation

True to scale or iconic (such as process flow diagram)

2. Import drawing and object files

From CAD (vector formats) drawings or icons (bit-mapped or raster graphics)

3. Dimension: 2-D, 2-D with perspective, 3-D

4. Movement: Motion of entities or status indicators.

5. Quality of motion: smooth or jerky

6. Libraries of common objects: Extensive predrawn graphics.

7. Navigation: Panning, zooming, rotation.

8. Views: User defined, named.

9. Display step: control of animation speed.

10. Selectable objects:

Dynamic status and statistics displayed upon selection.

11. Hardware requirements

Standard or special video card, RAM requirements

Output Features.

1. Scenario Managers.
   Create user-defined scenarios to simulate.

2. Run manager.
   Make all runs (scenarios and replications) and save results for future analysis.

3. Warmup capability:
   For steady state analysis.

4. Independent replication.
   Using a different set of random numbers.

5. Optimization
   Genetic algorithms, tabu search etc.

6. Standardized reports.
   Summary reports including averages, counts minimum and maximum etc.

7. Customized reports.
   Tailored presentations for managers.

8. Statistical analysis.
   Confidence intervals, designed experiments etc.

9. Business graphics
   Bar charts, pie charts, timelines etc.

10. Costing module Activity based costing included

11. File export

Input to spreadsheet or database for custom processing and analysis.

12. Database maintenance

Store output in an organized manner.

Vendor support and Product Documentation.

1. Training

Regularly scheduled classes of high quality.

2. Documentation. Quality, completeness, online.

3. Help system General or context sensitive

4. Tutorials For learning the package or specific features.

5. Support Telephone, e-mail, web

6. Upgrades, maintenance

Regularity of new versions and maintenance releases that address customer needs.

7. Track record. Stability, history, customer relations

Example 4.1 : The checkout Counter : A typical single-server Queue.

→ The simulation will run until 1000 customers have been served.

→ The inter arrival times of customers are exponentially distributed with a mean of 4.5 minutes.

→ The service times are normally distributed with a mean of 3.2 minutes and a standard deviation of 0.6 minute.

→ When the cashier is busy, a queue forms with no customers turned away.

☐ Definitions of Variables, Functions and subroutines in the Java model of the single-Server Queue below:

Variables	Description.
System state	
QueueLength	Number of customers enqueued (but not in service) at current simulated time.
NumberInService	Number being served at current simulated time.
Entity attributes and sets	
Customers	FCFS Queue of customers in system.
Future event list	
FutureEventList	Priority-ordered list of pending events.
Activity durations	
MeanInterArrivalTime	The interarrival time between the previous customer's arrival and the next arrival.
MeanServiceTime	The service time of the most recent customer to begin service.
Input parameters	
MeanInterArrivalTime	mean Interarrival time (4.5 mnts)
MeanServiceTime	mean service time (~~3.6~~ 3.2 mnts)
SIGMA	standard deviation of service time (0.6 minutes)
Total Customers	The stopping criterion - number of customers to be served (1000).

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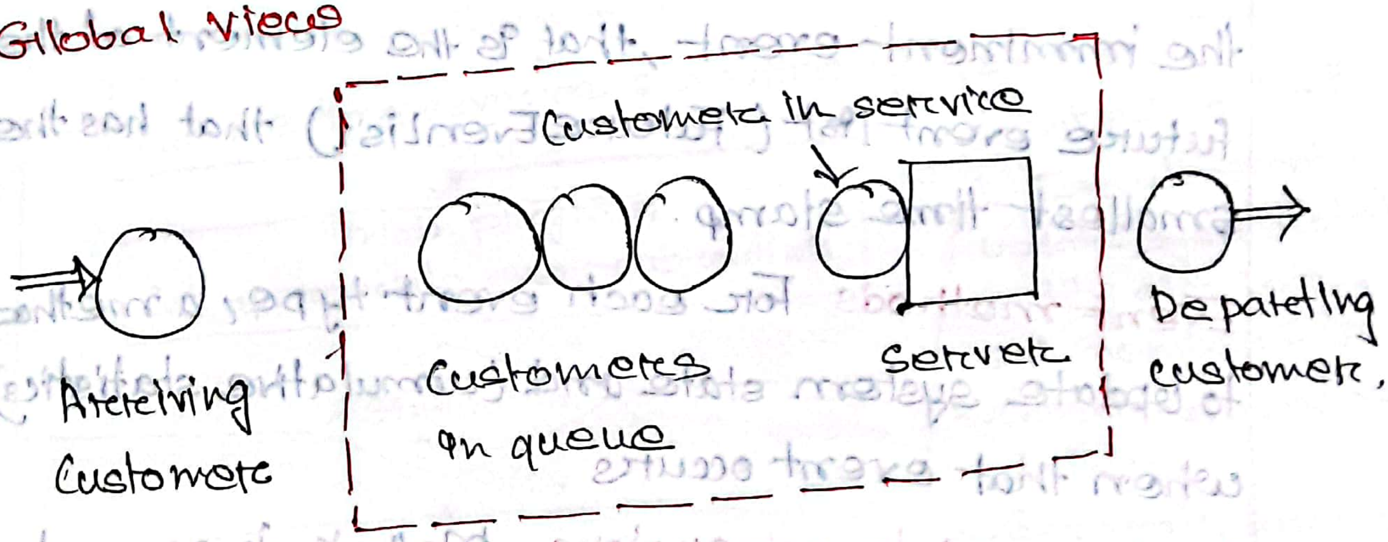
Variables	Description
Simulation variables	
Clock	The current value of simulated time
Statistical Accumulators	
Last Event Time	Time of occurrence of the last event.
Total Busy	Total busy time of server so far
Max Queue Length	Maximum length of waiting line (so far)
Sum Response Time	Sum of customer response times for all customers who have departed (so far)
Number of Departures	Number of Departures (so far).
Long Service	Number of customers who spent 4 or more minutes at the checkout counter (so far)
Summary statistics	
RHO = BusyTime / Clock	Proportion of time server is busy (here the value of Clock is the final value of simulated time)
AVGR	Average response time (SumResponseTime / Total Customers)
PC4	Proportion of customers who spent 4 or more minutes at the checkout counter.

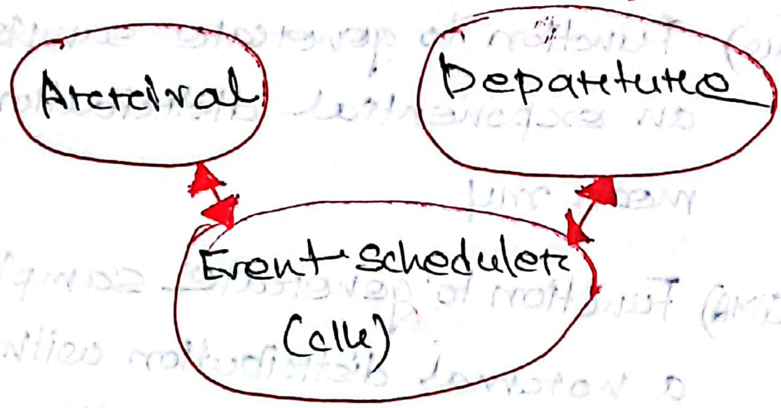
Functions	Description
exponential (mu)	Function to generate samples from an exponential distribution with mean mu
normal (xmu, SIGMA)	Function to generate samples from a normal distribution with mean xmu and standard deviation SIGMA.

Methods	Description
Initialization	Initialization method
Process Arrival	Event method that executes the arrival event
Process Departure	Event method that executes the departure event
Report Generation	Report generator.

Global View



Modules (Functions, Objects)



Simulation in Java

The following components are common to almost all models written in Java:

Clock: A variable defining simulated time

Initialization method: A method to define the system state at time 0

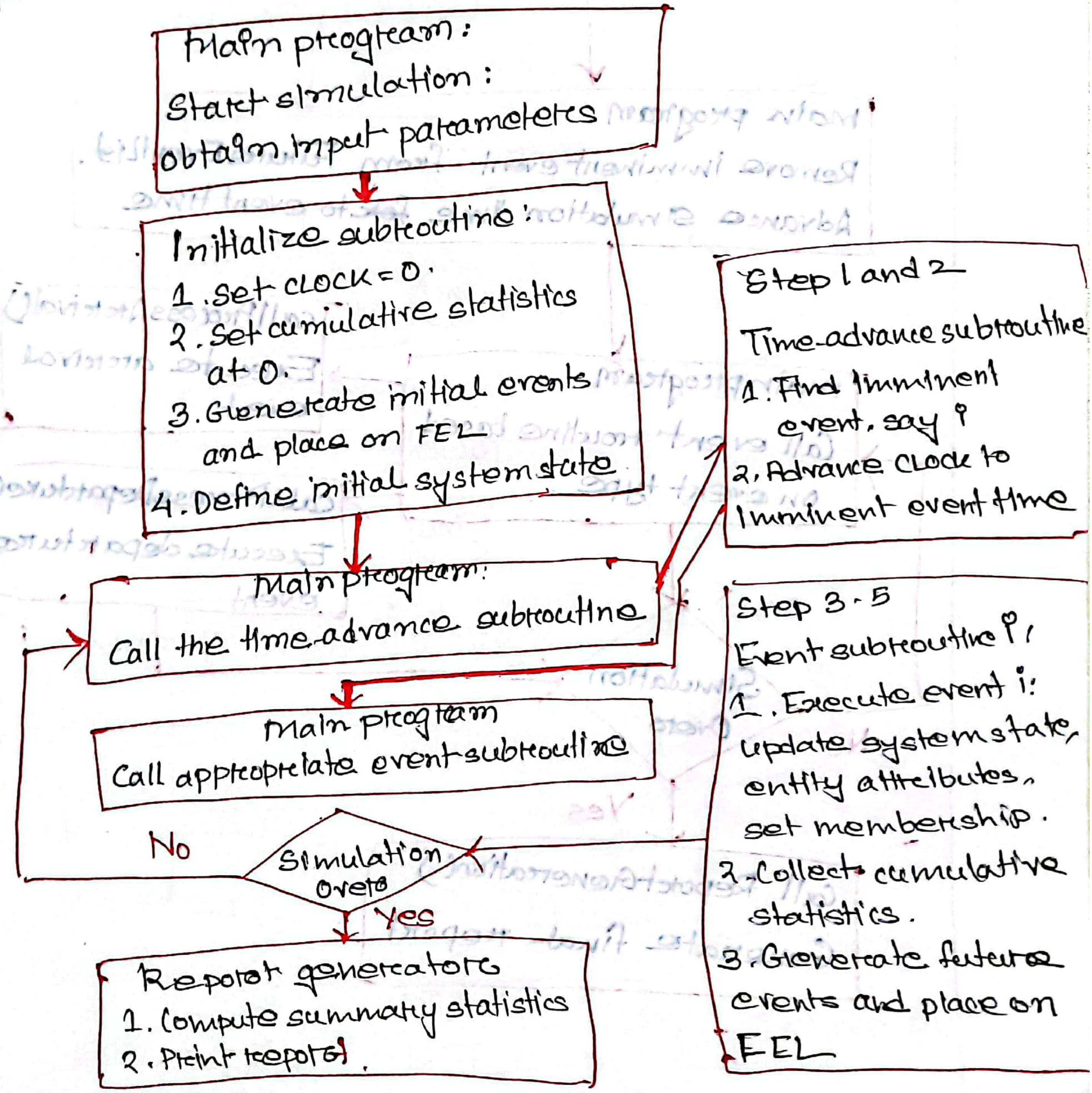
Min-time event method: A method that identifies the imminent event, that is the element of the future event list (FutureEvenList) that has the smallest time stamp.

Event methods: For each event type, a method to update system state and (cumulative, statistics) when that event occurs.

Random-variate generators: Methods to generate samples from desired probability distributions

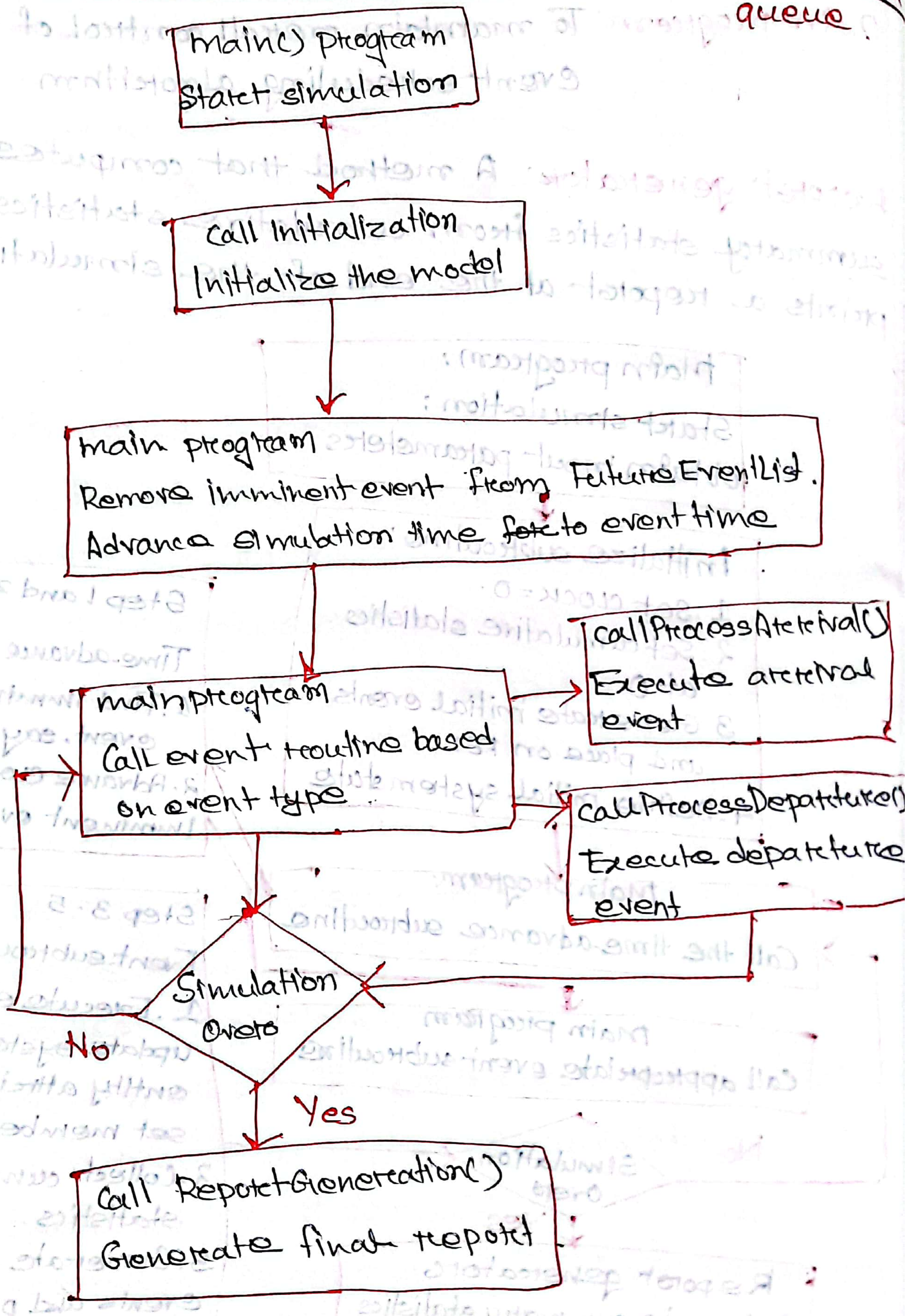
Main program: To maintain overall control of the event scheduling algorithm.

Report generator: A method that computes cumulative statistics and summary statistics from cumulative statistics and prints a report at the end of the simulation.



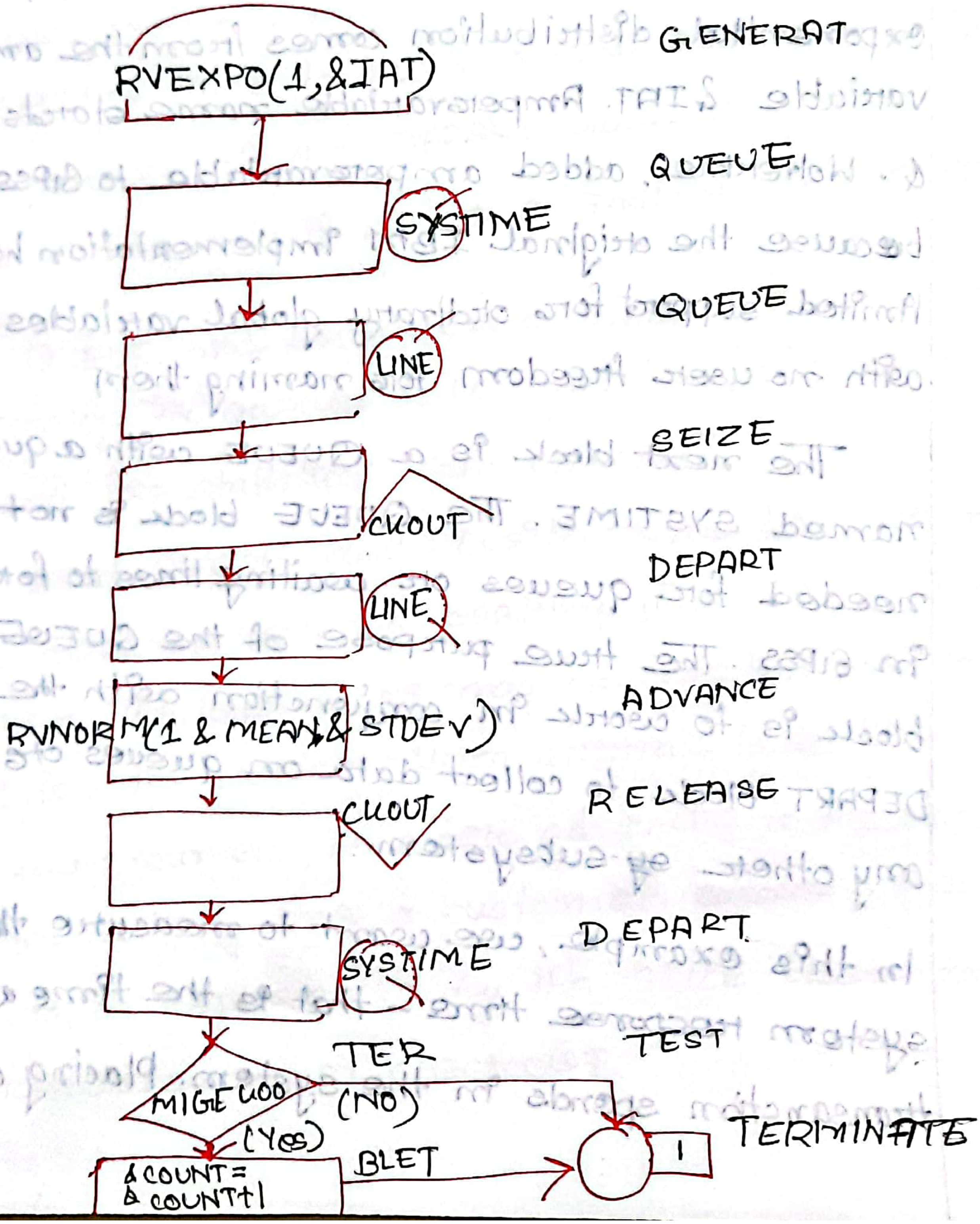
Overall structure of an event scheduling simulation program

Overall structure of Java simulation of a single-server queue



Simulation in GIPSS

GIPSS is a highly structured, special purpose simulation programming language based on the process interaction approach and oriented towards queueing system.



GIPSS block diagram for the single-server queue simulation

The GENERATE block represents the arrival event with the interarrival time specified by RVEXPO(1,&IAT) RVEXPO stands for "random variable, exponentially distribute", 1 indicates the random-number stream to use. and &IAT indicates that mean time for exponential distribution comes from the ampers-variable &IAT. Ampervariable name starts with & . Wolverine added ampervariable to GPSS because the original IBM implementation had limited support for ordinary global variables with no user freedom for naming them

The next block is a QUEUE with a queue named SYSTIME. The QUEUE block is not needed for queues or waiting lines to form in GPSS. The true purpose of the QUEUE block is to work in conjunction with the DEPART block to collect data on queues or any other ~~of~~ subsystem.

In this example, we want to measure the system response time - that is the time a transaction spends in the system. Placing a

QUEUE block, ~~the DEPART block, at the~~ at the point that transactions enter the system and placing the counterpart of the QUEUE block, the DEPART block, at the point that the transactions complete their processing causes the response times to be collected automatically.

The purpose of the DEPART block is to signal the end of data collection for an individual transaction

The QUEUE and DEPART block combination is not necessary for queues to be modeled, but rather is used for statistical data collection.

The next QUEUE block (with name LINE) begins the data collection for the waiting line before the cashier. The customers may or may not have to wait for the cashier. Upon arrival to an idle checkout counter, or after advancing to the head of the waiting line, a customer captures the cashier, as represented by the SEIZE block with the resource named CHECKOUT.

Once the transaction representing captures the cashier represented by the resource CHECKOUT, the data collection for the waiting time statistics ends, as represented by the DEPART block for the queue named LINE.

The transaction's service time at the cashier is represented by an ADVANCE block. RVNORM indicates "random variable, normally distributed." Again, random number stream 1 is being used, the mean time for the normal distribution is given by ampervariable &MEAN and its standard deviation is given by ampervariable &STDEV.

Next, the customer gives up the use of the facility CHECKOUT with a RELEASE block. The end of the data collection for response times is indicated by the DEPART block for the queue SYSTIME.

Next, there is a TEST block that checks to see whether the time in the system, M1, is greater than or equal to 4 minutes. M1 is a reserved word in GPSS/H, it automatically tracks transaction total time in system.
In GPSS/H, the maxim is "if true, pass through".
Thus, if the customer has been in the system 4 minutes or longer, the next BLET block (for block LET), adds one to the counter &COUNT.

If not true, the escape route is the block labelled TER. That label appears before the TERMINATE block whose purpose is the removal of the transaction from the system. The TERMINATE block has a value "1" indicating that one more transaction is added toward the limiting value (for "transactions to go")

Other simulation tools

- Arena
- Automod
  - Autostat
- Extend
- Flexlm
- Micro Saint
- Promodel
- SIMUL8
- SMPL

Chapter 3: General Principles in Simulation.

Concepts in Discrete-Event Simulation

System: A collection of entities (e.g. people and machines) that interact together over time to accomplish one or more goals.

Model: An abstract representation of a system, usually containing structural, logical or mathematical relationships that describe a system in terms of state, entities and their attributes, sets, processes, events, activities and delays.

System state: A collection of variables that contain all the information necessary to describe the system at any time.

Entity: Any object or component in the system that requires explicit representation in the model (e.g. a server, a customer, a machine).

Attributes: The properties of a given entity (e.g. the priority of a waiting custome, the routing of a job through a job shop)

List: A collection of (permanently or temporarily) associated entities, ordered in some logical fashion (such as all customers currently in a waiting line, ordered by "first come, first served", or by priority).

Event: An instantaneous occurrence that changes the state of a system (such as an arrival of a new customer)

Event notice: A record of an event to occur at the current or some future time, along with any associated data necessary to execute the event; at a minimum, the record includes the event type and event time.

Event list: A list of event notices for future events, ordered by time of occurrence, also known as the future event list (FEL).

Activity: A duration of time of specified length (e.g. a service time or interarrival time) which is known when it begins although it may be defined in terms of a statistical distribution.

Delay: A duration of time of unspecified length.

which is not known until it ends
(e.g. a customer's delay in a last-in-first-out waiting line which, when it begins, depends on future arrivals)

Clock A variable representing simulated time.

Example 3.1: Able-Baker Call Center Problem

Consider a computer technical support center where personnel take calls and provide service. There are two technical support people - Able and Baker.

A discrete-event model has the following components:

System state

$LQ(t)$, the number of callers waiting to be served at time t;

$L_A(t)$, 0 or 1 to indicate Able as being idle or busy at time t;

$L_B(t)$, 0 or 1 to indicate Baker as being idle or busy at time t.

Entities: Neither the caller nor the servers need to be explicitly represented, except in terms of the state variables, unless certain caller averages are desired.

Events:

Arrival event;

Service completion by Able;

Service completion by Baker.

Activities

Interarrival time

Service time by Able

Service time by Baker

Delay A caller's wait in queue until Able or Baker becomes free.

The definition of the model components provide a static description of the model. In addition, a description of the dynamic relationships and interactions between the components is also needed. Some questions that needs answers include:

1. How does each event affect system state, entity attributes and set contents?

2. How are activities defined (i.e., deterministic, probabilistic or some other mathematical equation)?

- What event marks the beginning or end of each activity?
- Can the activity begin regardless of system state or is its beginning conditioned on the system being in a certain state?

  * For example, a machine "activity" cannot begin unless the machine is idle, not broken, not in maintenance).

3. Which events trigger the beginning and end of each type of delay?

- Under what conditions does a delay begin or end?

4. What is the system state at time 0?

- What events should be generated at time 0 to "prime" the model – that is, to get the simulation started?

A discrete-event simulation is the modeling over time of a system all of whose state changes occur at discrete points in time – those points when an event occurs.

A simulation proceeds by producing a sequence of system snapshots that represent the evolution of the system through time. A given snapshot at a given time (CLOCK = t) includes, ~~not only the~~

- system state at time t.
- a list (FEL) of all activities currently in progress when each such activities will end
- the status of all entities and current membership of all sets
- the current values of cumulative statistics and counters that will be used to calculate summary statistics at the end of the simulation.

Clock	System state	Entities and attributes	Set 1	Set 2	...	Future event list, FEL	Cummulative statistics and counters
t	$(x, y, z, \ldots)$					$(3, t_1)$ - Type 3 event to occur at time t_1 $(1, t_2)$ - Type 1 event to occur at time t_2	

Prototype system snapshot at simulation time t

The Event Scheduling / Time Advance Algorithm

→ The mechanism for advancing simulation time and guaranteeing that all events occur in correct chronological order is based on the future event list (FEL). This list contains all event notices for events that have been scheduled to occur at a future time.

→ Scheduling a future event means that, at the instant an activity begins, its duration is computed or drawn as a sample from a statistical distribution and the end-activity event, together with its event time is placed on the future event list.

→ At any given time t, the FEL contains all previously scheduled future events and their associated event times called $t_1, t_2, \ldots$. The FEL is ordered by event time, meaning that the events are arranged chronologically — that is, the event times satisfy

$$t < t_1 \leq t_2 \leq t_3 \leq \cdots \leq t_n$$

Time t is the value of clock, the current value of simulated time. The event associated with time t_1 is called the imminent event; that is, it is the next event that will occur.

After the system snapshot at simulation time clock = t has been updated, the clock is advanced to simulation time clock = t_1, the imminent event notice is removed from the FEL, and the event is executed. Execution of the imminent event means that a new system snapshot for time t_1 is created, based on the old snapshot at time t and the nature of imminent event.

At time t_1, new future events may or might not be generated, but if any are, they are scheduled by creating event notices and putting them into their proper position on the FEL.

After the new system snapshot for time t_1 has been updated, the clock is advanced to the time of the new imminent event and that event is executed. This process repeats until the simulation is over.

The sequence of actions that a simulator must perform to advance the clock and build a new system snapshot is called the event-scheduling/time-advance algorithm.

Old system snapshot at time t

Clock	System state	...	Future event list	...
t	(5,1,6)		$(3, t_1)$ – Type 3 event to occur at time t_1	
			$(1, t_2)$ – Type 1 event to occur at time t_2	
			$(1, t_3)$ – Type 1 event to occur at time t_3	
			⋮	
			$(2, t_n)$ – Type 2 event to occur at time t_n	

Event scheduling/time-advance algorithm

Step 1: Remove the event notice for the imminent event (event 3, time t_1) from FEL.

Step 2: Advance clock to imminent event time (i.e. advance clock from t to t_1).

Step 3: Execute imminent event: update system state change entity attributes, and set memberships as needed.

Step 4: Generate future events (if necessary) and place their event notices on FEL, ranked by event time.

[Example: Event 4 to occur at time t^*, where $t_2 < t^* < t_3$]

New system snapshot at time t_1

Clock	System state	...	Future event list	...
t_1	(5,1,5)		$(1, t_2)$ - Type 1 event to occur at time t_2 $(4, t^*)$ - Type 4 event to occur at time t^* $(1, t_3)$ - Type 1 event to occur at time t_3 . . . $(2, t_n)$ - Type 2 event to occur at time t_n	

→ The length and contents of the FEL are constantly changing as the simulation progresses and thus its efficiently management in a computerized simulation will have a major impact on the efficiency of the computer program representing the model.

→ The management of a list is called list processing. The major list processing performed on a FEL are

- removal of the imminent event.
- addition of a new event to the list
- occassionally removal of some event (called cancellation of an event)

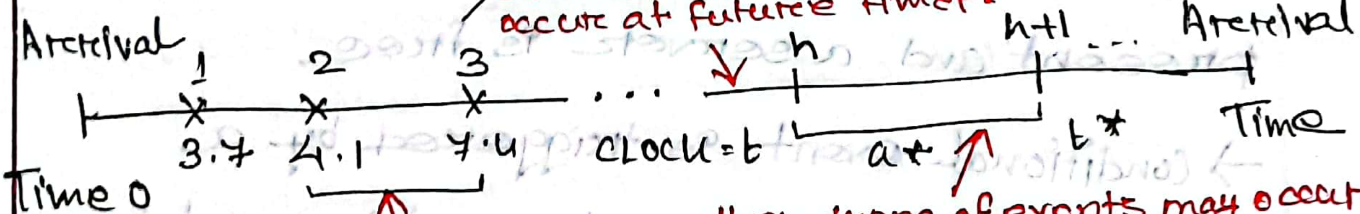
→ As the imminent event is usually at the top of the list, its removal is as efficient as possible.

→ Addition of a new event and cancellation of an old event requires a search of the list. The efficiency of this search depends on the logical organization of the list and on how the search is conducted.

→ The system snapshot at time 0 is defined by the initial conditions and the generation of the so-called exogenous events. An exogenous event is a happening "outside the system" that impinges on the system.

→ At time 0, the first arrival event is generated and is scheduled on the FEL meaning that its event notice is placed on the FEL. When the clock eventually is advanced to the time of this first arrival, a second arrival even is generated. First an interarrival time is generated, a^*; it is added to the current time, CLOCK = t, the resulting event time, $t + a^* = t^*$ is used to position the new arrival event notice on FEL. This method of generating an external arrival stream is called bootstrapping.

At simulated time t, assumed to be the instant of the nth arrival, generate interarrival time a^*, compute $t^* = t + a^*$ and schedule future arrival on FEL to occur at future time t^*



Between successive arrival events, other types of events may occur, causing system state to change

→ When one customer completes service, at current time CLOCK = t, if the next customer is present, then a new service time, s* will be generated for the next customer.

→ The next service completion event will be scheduled to occur at future time $t^* = t + s^*$, by placing onto the FEL, a new event notice, of type service completion with event time t^*.

→ In addition, a service completion event will be generated and scheduled at the time of an arrival event provided that, upon arrival, there is at least one idle server in the server group.

→ Beginning service is a conditional event because its occurrence is triggered only on the condition that a customer is present and a server is free.

→ Conditional event is triggered by a primary event's occurring and by certain conditions prevailing in the system.

→ only Primary events appear on the FEL.
The end of an interarrival ~~event~~ interval and service completion are primary event.

☐ Every simulation must have a stopping event called E, which defines how long the simulation will run. There are generally two ways to stop a simulation:

1. At time 0, schedule a stop simulation event at a specified future time T_E. Thus before simulating, it is known that the simulation will run over the time interval $(0, T_E)$.
Ex: Simulate a job shop for $T_E = 40$ hours.

2. Run length T_E is determined by the simulation itself. Generally, T_E is the time of occurrence of some specified event E.

Ex. - T_E is the time of breakdown of a complex system
- " " " " " disengagement or total kill
- " " (whichever occurs first) in a combat simulation
- T_E is the time at which a distribution center ships the last carton of in a day's order.

In case 2, T_E is not known ahead of time. Indeed, it could be one of the statistics of primary interest to be produced by the simulation.

World view

When using a simulation package, or even when doing a manual simulation, a modeler adopts a world view or orientation for developing a model. The most prevalent world views are:

1. Event-scheduling world view
2. Process-interaction world view
3. Activity-scanning world-view

1. Event-scheduling world view

- A simulation analyst concentrates on events and their effect on system state.

2. Process-interaction world view

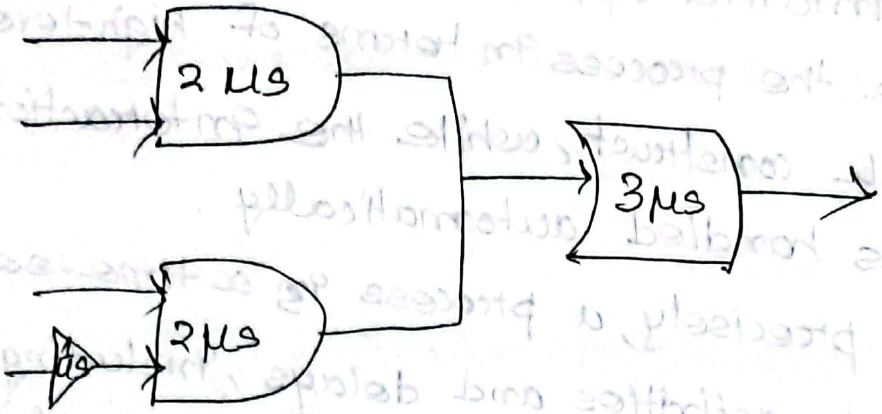
- The analyst defines the simulation model in terms of entities or objects and their life cycle as they flow through the system, demanding resources and queueing to wait for resources.

- A process is the life cycle of one entity.
- It has intuitive appeal and it allows an analyst to describe the process in terms of high-level block or network construct, while the interaction among processes is handled automatically.
- More precisely, a process is a time-sequenced list of events, activities and delays, including demands for resources, that define the life cycle of one entity as it moves through a system.
- Event scheduling is hidden.
- Both event scheduling and the process-interaction approach use a variable time advance.

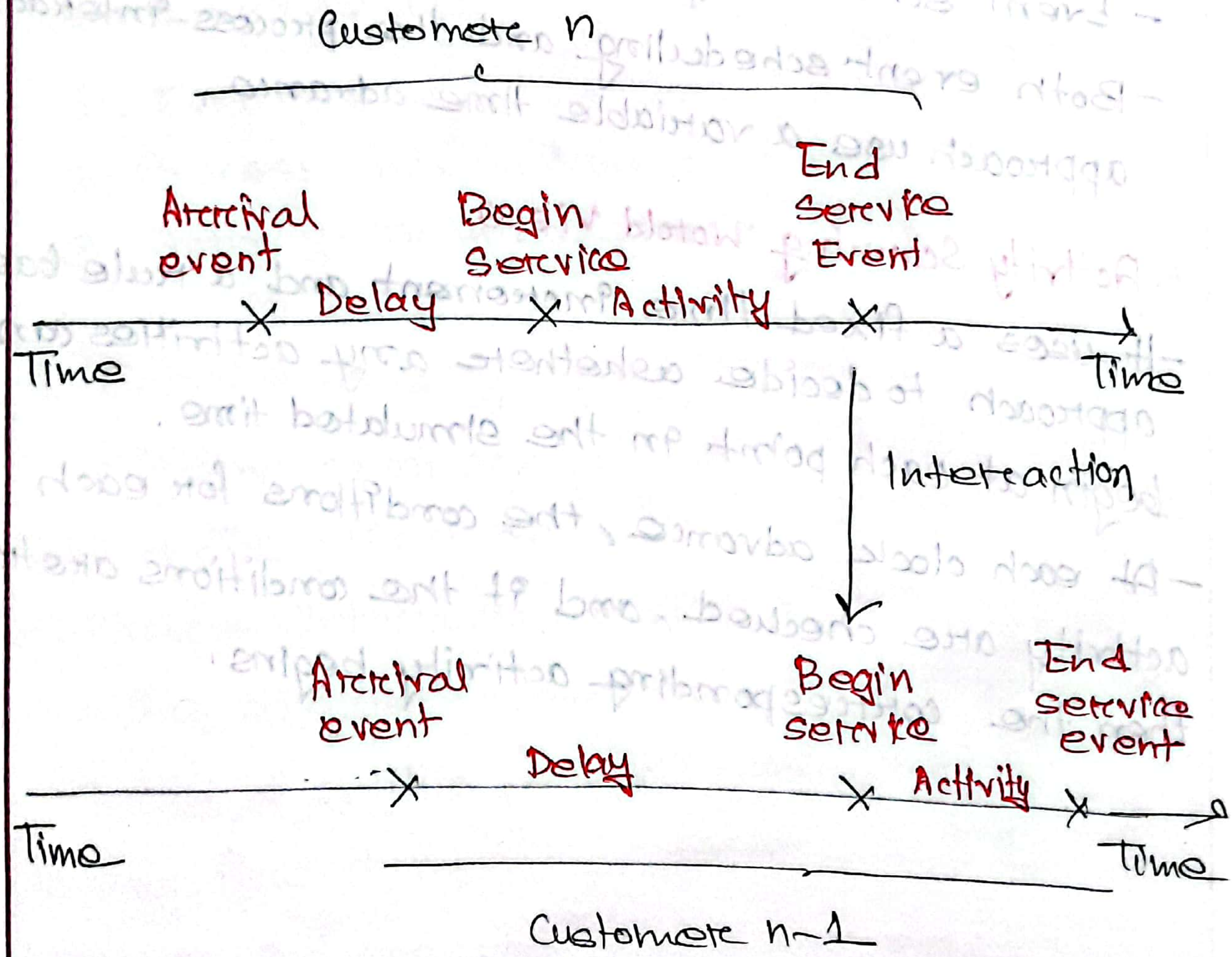
3. Activity Scanning World View.

- It uses a fixed time increment and a rule based approach to decide whether any activities can begin at each point in the simulated time.
- At each clock advance, the conditions for each activity are checked, and if the conditions are true, then the corresponding activity begins.

Activity scanning example (Gate simulation)



Two customer processes interaction for a single server queue.



Example 3.2: Single Channel Queue

The system consists of ~~60 minutes~~ those customers in the waiting line plus the one (if any) checking out. A stopping time of 60 minutes is set for this example.

The model has the following components:

System state (LQ(t), LS(t))

LQ(t) → the number of customers in the waiting line.

LS(t) → the number being served (0 or 1) at time t

Entities The server and customers are not explicitly modeled, except in terms of the state variables.

Events

Arrival (A)

Departure (D)

Stopping event (E), scheduled to occur at ~~future~~ time 60.

Event notices

(A, t), representing an arrival event to occur at future time t;

(D, t), representing a customer departure at future time t;

(E, 60), representing the simulation stop event at future time 60

Activities

Interarrival time

Service time

Delay Customer time spent in waiting line.

Only two statistics will be collected which are:
1. server utilization
2. maximum queue length.

Server utilization is defined by total server busy time (B) divided by total time (T_E).

Total busy time B and maximum queue length, MQ, will be accumulated as the simulation progresses

Execution of the arrival event

Arrival event
occurs at clock = t

Set LS(t) = 1

Is
LS(t) = 1
?

Increase LQ(t)
by 1

Generate service time s*;
schedule next arrival
event at time t + s*

Generate interarrival time a*
schedule next arrival
event at time t + a*

Collect statistics

Return control to time-
advance routine to
continue simulation.

2017

(1)(a) What is simulation? List some advanta

(2)(a) i average waiting time =

$$\frac{\text{total time customers wait in queue}}{\text{total number of customer}}$$

$= \frac{3+2}{5} = 1$ minute.

ii) Probability that a customer has to wait

$= \frac{\text{number of customers who wait}}{\text{total number of customers}}$

$= \frac{2}{5} = 0.4$

iii) average waiting of those who wait

$= \frac{\text{total time customers wait in queue}}{\text{total number of customers that wait}}$

$= \frac{3+2}{2} = 2.5$ mnt

(iv) The average time a customer spend in the system $= \frac{\text{total time customers spend in the system}}{\text{total number of customers}}$

$$= \frac{4+1+4+6+4}{5} = \frac{19}{5} = 3.8 \text{ mnt}$$

9p) The average service time =

$$\frac{\text{total service time}}{\text{total number of customers}}$$

$$= \frac{4+1+4+3+2}{5} = \frac{14}{5} = 2.8 \text{ mnt}$$

The average interarrival time =

$$= \frac{\text{sum of all times between arrivals}}{\text{number of arrivals} - 1}$$

$$= \frac{8+6+1+4}{5-1} = \frac{19}{4} = 4.75 \text{ mnt}$$

The expected service time = $\sum_{x=}^{0} s\,p(s)$

$= 1(.1) + 2(.2) + 3(.3) + 4(.25) + 5(.1) + 6(.05)$

$= 3.2$ mnt

The expected interarrival time = $\frac{a+b}{2} = \frac{1+8}{2}$

$= 4.5$ mnt.

2(c)

Clock	Future event list		
0	(A,8) (D,4)	0	(D,4) (A,8)
4	(A,8)	4	(A,8)
8	(A,14), (D,9)	8	(D,9) (A,14)
9	(A,14)	9	(A,14)
14	(A,15) (D,18)	14	~~(D,18)~~ (A,15) (D,18)
15	(A,19) (D,18)	15	(D,18) (A,19)
18	(A,19) (D,21)	18	(A,19) (D,21)
19	(D,21)	19	D(21)
21	(D,23)	21	(D,23)
		23	

$P(X>3) = 1 - P(X \le 3)$

$= 1 - F(3)$

$= 1 - (1 - e^{-\lambda x})$

$= 1 - 1 + e^{-\frac{1}{3}3}$

$= e^{-1}$

$= 0.368$

$P(2 \le X \le 3)$

$= F(3) - F(2)$

$= \cancel{0.368} - (1 - e^{-\frac{1}{3}2})$

$(1 - e^{-\frac{1}{3}3})$

$= e^{-\frac{2}{3}} - e^{-1}$

$= 0.513 - 0.368$

$= 0.145$

4(a) $\rho = 80\% = 0.8, \quad \lambda = 100$

$$L_Q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{\rho^2}{1-\rho} = \frac{0.8^2}{1-\rho(0.8)} = 3.2$$

$$\rho = \frac{\lambda}{\mu} \text{ or } \mu = \frac{\lambda}{\rho} = \frac{10}{0.8} = 12.5$$

$$\mu = 12.5 \quad \omega = \frac{1}{\mu-\lambda} = \frac{1}{12.5-10} = 0.4$$

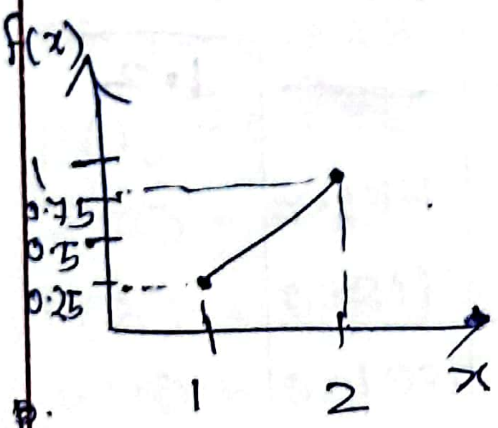
$$\frac{1}{\mu-\lambda} = \frac{1}{\mu-\mu\rho} = \frac{1}{\mu(1-\rho)}$$

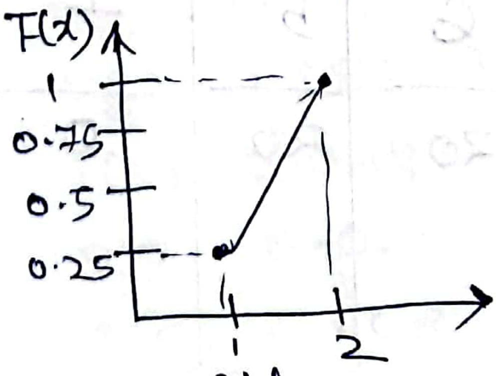
4.c $E(x) = (1)0.25 + (2)0.75 = 1.75$

$$V(x) = E(x^2) - [E(x)]^2$$

$$= \{1^2 \times 0.25 + 2^2 \times 0.75\} - \{1.75\}^2$$

$$= 0.1875$$





cdf

卌
卌 I   III

0.11   0.97
0.10   1.00   0.

4(d) Step 1: Define the hypothesis

H_0 : $R_i \sim U[0,1]$

H_1 : $R_i \nsim U[0,1]$

Step 2: Divide the total no of classes into mutually exclusive equal no of classes $n = 4$

$$E_i = \frac{N}{n} = \frac{20}{4} = 5 \geq 5$$

Step 3: Test statistics $\chi_0^2 = \sum (O_i - E_i)^2 / E_i$

Interval	O_i	E_i	$O_i - E_i$	$(O_i - E_i)^2$	$\frac{(O_i - E_i)^2}{E_i}$
1 (0.10-0.25)	5	5	0	0	0
2 (0.26-0.50)	6	5	1	1	0.2
3 (0.51-0.75)	3	5	-2	4	0.8
4 (0.76-1.00)	6	5	1	1	0.2
	20	20	0		1.2

so $\chi_0^2 = 1.21$

Step 4 Determine critical value using $n-1$ and

~~$n-1$~~ $\alpha = 0.05$

$$\chi^2_{0\,\alpha, n-1} = \chi^2_{0\,0.05, 3} = 7.81$$

$\chi_0^2 < \chi^2_{0\,\alpha, n-1}$ = infact it is smaller ~~that~~ compared to all α value.

5(b) $n = 100$

$$\bar{X} = \alpha = \frac{\sum_{j=1}^{k} f_j x_j}{n} = \frac{364}{100} = 3.64$$

The pmf for the Poisson distribution

$$P(x) = \begin{cases} \dfrac{e^{-\alpha}\alpha^x}{x!}, & x = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

$P(0) = 0.026$

$P(1) = 0.096$

$P(2) = 0.174$

$P(3) = 0.211$

$P(4) = 0.192$

$P(5) = 0.140$

$P(6) = 0.085$

$P(7) = 0.044$

$P(8) = 0.020$

$P(9) = 0.008$

$P(10) = 0.003$

$P(11) = 0.001$

Expected Frequency $E = np$

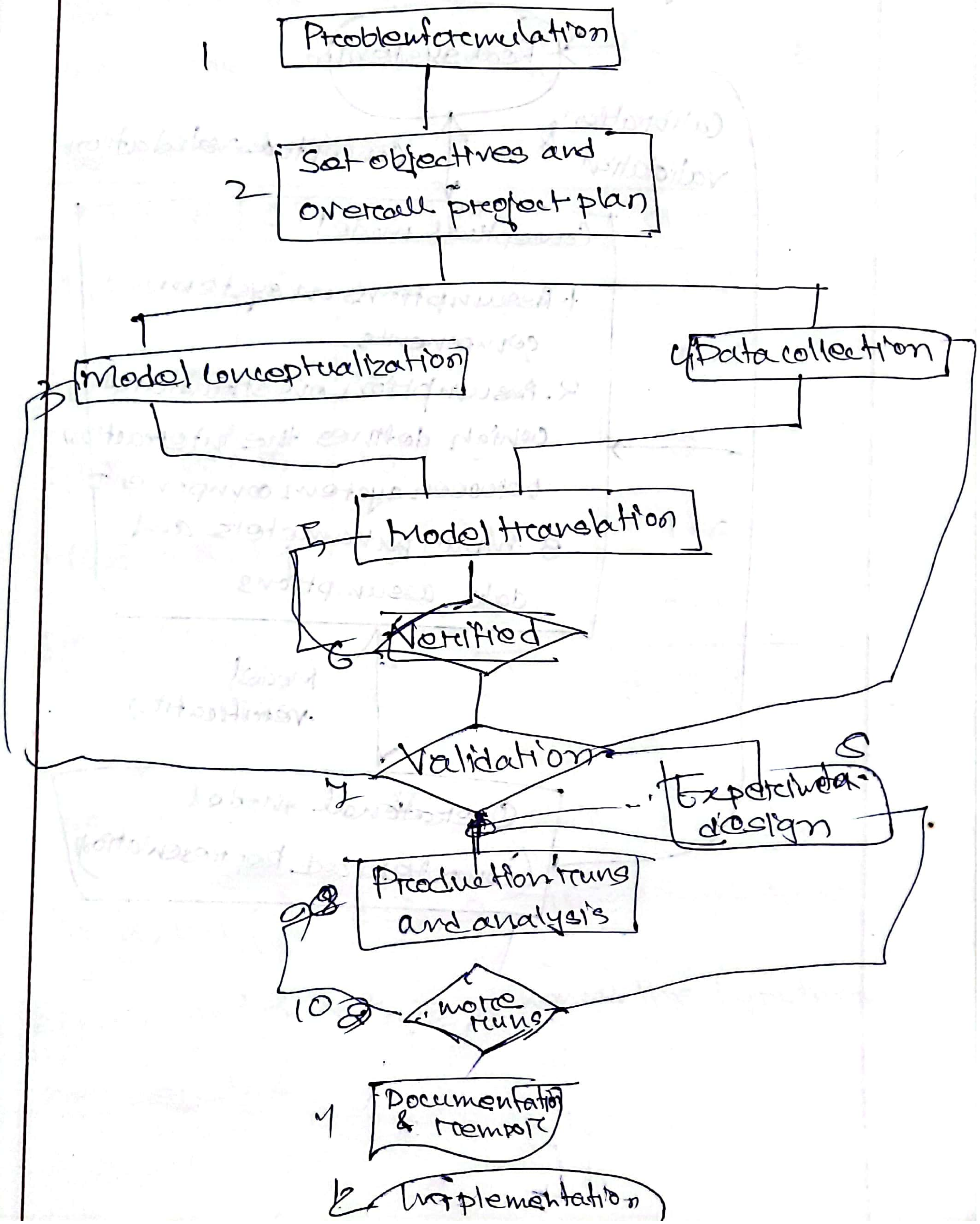
x_i	Observed frequency O_i		Expected frequency E_i		$\frac{(O_i-E_i)^2}{E_i}$
0	12	22	2.6	12.2	7.87
1	10		9.6		
2	19		17.4		0.15
3	17		21.1		0.80
4	10		19.2		4.41
5	8		14.0		2.57
6	7		8.5		0.26
7	5	17	4.4	7.6	11.62
8	5		2.0		
9	3		0.8		
10	3		0.3		
11	1		0.1		
					27.68

$\chi_0^2 = 27.68$

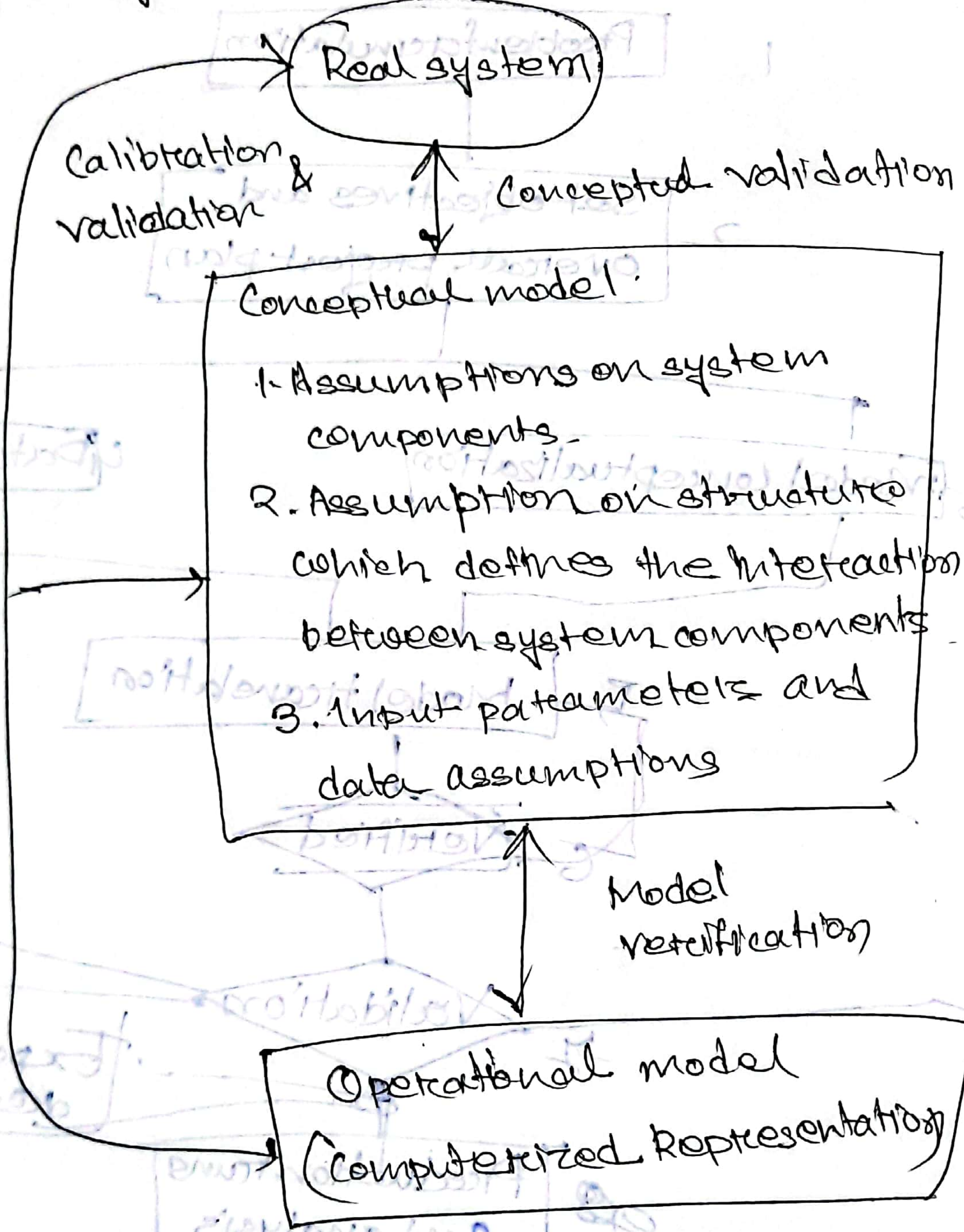
$n = 7$, $s = 1$ since one parameter α was estimated $n-s-1 = 7-1-1 = 5$

$\chi^2_{0.05,5}$ H_0 is rejected

The basic flowchart in discrete event simulation.



Model building verification and validation



0.02 - 0.99

2013 (b) Here $N = 50$

We assume $n = 5$ so $E = \frac{N}{n} = \frac{50}{5} = 10 \geq 5$

~~The critical value~~

Intervals	O_i	E_i	$(O_i - E_i)$	$(O_i - E_i)^2 / E_i$
1 (0.05-0.20)	6	10	-4	1.6
2 (0.21-0.40)	9	10	-1	0.1
3 (0.41-0.60)	9	10	-1	0.1
4 (0.61-0.80)	14	10	4	1.6
5 (0.81-1.00)	12	10	2	0.4
	50	50	0	$\Sigma = 3.8$

so $\chi_0^2 = 3.8$

$\chi_0^2{}_{\alpha, n-1} = \chi_0^2{}_{0.05, 9} = 16.9$

Since $\chi_0^2 < \chi_0^2{}_{0.05, 9}$ hence the hypothesis is not rejected

2013 3(c)

$\lambda = 2$ $\rho = \frac{\lambda}{\mu} = \frac{2}{3}$

$\mu = 3$

$P_0 = (1-\rho)\rho^n = (1-\frac{2}{3})(\frac{2}{3})^0 = \frac{1}{3}$

$P_1 = (1-\rho)\rho^n = \frac{1}{3} \times (\frac{2}{3})^1 = \frac{2}{9}$

$P_2 = (1-\rho)\rho^n = \frac{1}{3} \times (\frac{2}{3})^2 = \frac{4}{27}$

$P_3 = \frac{1}{3} \times (\frac{2}{3})^3 = \frac{8}{81}$

$P_{\geq 4} = 1 - \sum_{n=0}^{3} P_n = 1 - \left(\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \frac{8}{81}\right)$

$= \frac{16}{81}$

The time-average number of customers in the system $L = \frac{\lambda}{\mu-\lambda} = \frac{\rho}{1-\rho} = \frac{\frac{2}{3}}{1-\frac{2}{3}}$

$= 2$ customers

average time a customer spends in the system,

$w = \frac{L}{\lambda} = \frac{1}{\mu-\lambda} = \frac{1}{\mu(1-\rho)} = \frac{2}{2} = 1$ hours average time the customers wait in the queue

$$W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{\rho}{\mu(1-\rho)} = \frac{L}{\mu} = \frac{2}{3}$$

$$= W - \frac{1}{\mu} = 1 - \frac{1}{3} = \frac{2}{3}$$

Time average number of customers in queue

$$L_q = L - \frac{\lambda}{\mu} = 2 - \rho = 2 - \frac{2}{3} = \frac{4}{3}$$

$$L_q = \frac{\rho^2}{1-\rho} = \rho \cdot \frac{\rho}{1-\rho} = \frac{2}{3} \cdot L = \frac{2}{3} \cdot 2 = \frac{4}{3}$$

6(a) $X_0 = 117$, $a = 43$, $m = 1000$

$X_1 = aX_0 \% m = (43 \times 117) \% 1000 = 31$

$X_2 = (43 \times 31) \% 1000 = 333$

$X_3 = (43 \times 333) \% 1000 = 319$

$X_4 = (43 \times 319) \% 1000 = 717$

$X_5 = (43 \times 717) \% 1000 = 831$

ans.

Here $R = 4$, $T_E = 2$ hours

5(b) point estimator.

$$\bar{Y}_{..} = \hat{p} = \frac{1}{R}\sum_{i=1}^{R} Y_i$$

$$= \frac{0.808 + 0.875 + 0.708 + 0.842}{4}$$

$$= 0.808$$

variance, $S^2 = \frac{1}{R-1}\sum_{i=1}^{R}(Y_i - \bar{Y}_{..})^2$

$$= \frac{(0.808-0.808)^2 + (0.875-0.808)^2 + (0.708-0.808)^2 + (0.842-0.808)^2}{3}$$

$$= \frac{0.015645}{3} = 5.215 \times 10^{-3} = (0.072)^2$$

standard error

$$s.e(\hat{p}) = \frac{S}{\sqrt{R}} = \frac{0.072}{\sqrt{4}} = 0.036$$

~~5.215×10^{-3}~~, ~~0.0026075~~

95% confidence interval halfwidth,

$(1-\alpha)100\% = 95\%$ $\alpha = 0.05$

$$H = t_{\alpha/2, R-1}\frac{S}{\sqrt{R}} = t_{0.025,3}\frac{0.072}{\sqrt{4}}$$

~~5.215×10^{-3}~~

$$= 3.18 \times 0.036 = 0.114$$

$0.694 \leq p \leq 0.922$

Runlength T	Batch j	Average Batch Mean $\bar{Y}_{.j}$	Cumu Avg $\bar{Y}_{..}(j,0)$	Cum Avg $\bar{Y}_{..}(j,1)$	Cum Avg $\bar{Y}_{..}(j,2)$
4	1	3.32	3.32	—	
	2	5.86	4.59	5.86	—
	3	8.61	5.93	7.24	8.61

4(b) Here, $R = 120$ $\quad S = 1.60$

$\bar{Y} = 5.80$ hr

95% confidence Interval for the long run expected daily average cycle:

$\bar{Y} \pm t_{\alpha/2, R-1} \frac{S}{\sqrt{R}}$

$5.80 \pm t_{0.025, 119} \frac{1.60}{\sqrt{120}}$

$5.80 \pm 1.98 \times 0.146$

5.80 ± 0.29 hours

prediction Interval

$\bar{Y} \pm t_{\alpha/2, R-1} S\sqrt{1+\frac{1}{R}}$

$\bar{Y} \pm 1.98 \times 1.60 \sqrt{1+\frac{1}{120}}$

5.8 ± 3.18

0.025

4(c)

The point estimatote $\hat{\theta} = 1000 \times 0.8 = 800$th smallest value of sorted data $= 212.03$

$$p_l = p - z_{\alpha/2}\sqrt{\frac{p(1-p)}{R-1}} = 0.8 - 1.96\sqrt{\frac{0.8 \times 0.2}{999}}$$

$$= 0.775$$

$$p_u = p + z_{\alpha/2}\sqrt{\frac{p(1-p)}{R-1}} = 0.8 + 1.96\sqrt{\frac{0.8 \times 0.2}{999}}$$

$$= \cancel{8.02}\ 0.825$$

The lowere bound of confidence interval:

$\hat{\theta}_l = 1000 \times 0.775 = 775$th smallest value

The upper bound of the confidence interval

$\hat{\theta}_u = 1000 \times 0.825 = 825$

$$v = \frac{\left(\frac{118.9}{10} + \frac{244.3}{10}\right)^2}{\left\{\left(\frac{118.9}{10}\right)^2 \div 9\right\} + \left\{\left(\frac{244.3}{10}\right)^2 \div 9\right\}} = \frac{1319.1424}{15.71 + 66.31}$$

$$= 16.08 \simeq 17$$

~~$x_1 = 3.25$~~ $x = 0.0083$

$x_1 = 0.005$ $x_2 = 0.01$

$y_1 = 3.25$ $y_2 = 2.82$

$x_1 = 0.01$ $x_2 = 0.025$

$x = 0.0167$ $y_1 = 2.82$ $y_2 = 2.26$

$-\frac{469}{1875}$

12.4 $k = 4$, $1 - \alpha = 0.95$, $\epsilon = 2$ minutes

$\theta_1 - \min_{j \neq 1} \theta_j$, $R_0 = 10$

$t = t_{\alpha/(k-1), R_0 - 1} = t_{0.0167, 9}$

Customer	Time since last Arrival (Minutes)	Arrival Clock time	Service time	Time Service Begins	Time Customer waits in queue	Time Service ends.	Time Customer spends in the System	Idle time of server (minutes)
1								
2								
3								
4								
5								
6								

3(c)

Travel time	Probability	Cumulative Property	Random Numbers
110	0.35	0.35	01-35
130	0.40	0.75	36-75
140	0.15	0.90	76-90
180	0.10	1.00	91-00

1(a)

Interarrivals	Probability	Cumulative Prob	Random digit assignment
1	0.25	0.25	01-25
2	0.40	0.65	26-65
3	0.20	0.85	66-85
4	0.15	1.00	86-00

Able Service Time	Prob	Cum Prob	Random digit assignment
2	0.30	0.30	~~04~~ 01-30
3	0.28	0.58	31-58
4	0.25	0.83	59-83
5	0.17	1.00	84-00

Barber Service Time	Probability	Cum Prob	Random digit assignment
3	0.35	0.35	01-35
4	0.25	0.60	36-60
5	0.20	0.80	61-80
6	0.20	1.00	81-00

Average caller delay $= \dfrac{\text{total caller delay}}{\text{total \# of customers}}$

$$= \frac{0}{10} = 0$$

Percentage of customers who waits =

$$\frac{\text{\# of cust who wait}}{\text{total \# of cust}} = \frac{0}{10} = 0$$

Utilization of able $= \dfrac{\text{total service time of able}}{\text{total system time}} = \dfrac{20}{38}$

Utilization of baker $= \dfrac{18}{38}$

Caller number	Inter Arrival time	Clock Arrival time	Clock when Able available	Clock when Baker available	Server chosen	Service time	Clock Time service begin	Clock Able service completion time	Clock Bakers service completion time	Caller delay	Time in system
1	–	0	0	0	Able	2	0	2	–	0	2
2	1	1	2	0	Baker	3	1	–	4	0	3
3	2	3	2	4	Able	2	3	5	–	0	2
4	2	5	5	4	Able	3	5	8	–	0	3
5	2	7	8	4	Baker	4	7	–	11	0	4
6	2	9	8	11	Able	4	9	13	–	0	4
7	3	12	13	11	Baker	5	12	–	17	0	5
8	3	15	13	17	Able	4	15	19	–	0	4
9	4	19	19	17	Able	5	19	24	–	0	5
10	4	23	24	17	Baker	6	23	~~29~~ –	29	0	6
										0	38

Random numbers	Service time	Random numbers	Inter Arrival times	Arrival times
20	2	100	1	1
40	3	300	3	4
60	4	500	4	8
80	4	400	6	14
96	6	900	8	22

2014 2(b)

Time between Arrival	Prob	Cum Prob	Random digit Assign.
1	0.125	0.125	001-125
2	0.125	0.250	126-250
3	0.125	0.375	251-375
4	0.125	0.500	376-500
5	0.125	0.625	501-625
6	0.125	0.750	626-750
7	0.125	0.875	751-875
8	0.125	1.000	876-000

Service time	Prob	Cum Prob	Random digit Assign
1	0.10	0.10	01-10
2	0.20	0.30	11-30
3	0.30	0.60	31-60
4	0.25	0.85	61-85
5	0.10	0.95	86-95
6	0.05	1.00	96-00

Travel time	Prob	Cum Prob	Random digit Assignment
40	0.4	0.4	1-4
60	0.3	0.7	5-7
80	0.2	0.9	8-9
100	0.1	1.0	0

Clock	System state $L_Q(t)$	$L_S(t)$	Checkout line	Future Event list	B	MQ	S	F	N_D
0	0	1	(C1, 0)	(D, 2) ~~(A, 2)~~ (A, 2)	0	0	0	0	0
2	0	1	(C2, 2)	(D, 3) (A, 6)	2	0	2	0	1
3	0	0		(A, 6)	3	0	3	0	2
6	0	1	(C3, 6)	(A, 7) (D, 9)	3	0	3	0	2
7	1	1	(C3, 6) (C4, 7)	(D, 9) (A, 9)	4	1	3	0	2
9	1	1	(C4, 7) (C5, 9)	(D, 12) (A, 15)	6	1	6	0	3
10	0	1	(C5, 9)	(D, 12) (A, 15)	8	1	10	0	4
12	0	0		(A, 15)	9	1	13	0	5
15	0	1	(C6, 15)	(D, 19)	9	1	13	0	5
19	0	0	—	—	12	1	17	1	6

i) percentage of customers waited in queue

$$2+1+3 = \frac{4}{6} = \frac{2}{3}$$

ii) ~~Average~~ server utilization

$11-7$

4 $12-9$

$$= \frac{22}{23}$$

$\frac{3}{2}^{2}$

$2+2$

average time spent in the system

Chapter 5

4(b) Example 5.3 Exponential distribution

The life of a device is given by X, a continuos random variable assuming all values in the range $x \geq 0$

The pdf $f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$

Now, its mean $E(X) = 2 = \frac{1}{\lambda}$ hence $\lambda = \frac{1}{2}$

so $f(x) = \begin{cases} \frac{1}{2} e^{-x/2} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$

The probability that the life of device is between 2 and 3 years, is calculated as,

$$P(2 \leq x \leq 3) = \frac{1}{2} \int_2^3 e^{-x/2} dx$$

$$= \frac{1}{2} \times \frac{-2}{1} \left[e^{-x/2} \right]_2^3$$

$$= \cancel{\frac{1}{2} \left[\quad \frac{1}{2} \right.}$$

$$= -\left(e^{-3/2} - e^{-1} \right)$$

$$= e^{-1} - e^{-3/2} = 0.145.$$

The cdf $F(x) = \int_0^x p(x)\,dx$

$$= \frac{1}{2}\int_0^x e^{-x/2}\,dx$$

$$= \frac{1}{2} \times -\frac{2}{1}\left[e^{-\frac{x}{2}}\right]_0^x$$

$$= -\left(e^{-\frac{x}{2}} - e^{0}\right)$$

$$= 1 - e^{-\frac{x}{2}}$$

$P(a \le x \le b) = F(b) - F(a)$

Now

$P(2 \le x \le 3) = F(3) - F(2)$

$$= \left(1 - e^{-\frac{3}{2}}\right) - \left(1 - e^{-\frac{2}{2}}\right)$$

$$= e^{-1} - e^{-\frac{3}{2}} = 0.145$$

$\boxed{\text{Ex}}$ Ampere meter with failure rate

$\lambda = \frac{1}{4}$

$P(x \geq 4) = 1 - P(x \le 4)$

$$= 1 - F(4) = 1 - \left(1 - e^{-8/2}\right) = e^{-2} = 0.135$$

$$E(x) = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_{0}^{\infty} x \frac{1}{2} e^{-x/2} dx$$

$$= \frac{1}{2} \int_{0}^{\infty} x e^{-x/2} dx$$

Now $u = x \rightarrow du = dx$

$$dv = e^{-x/2} dx \rightarrow v = -2e^{-x/2}$$

$$= \frac{1}{2} \int_{0}^{\infty} u\, dv$$

$$= \frac{1}{2} \left\{ uv - \int_{0}^{\infty} v\, du \right\}$$

$$= \frac{1}{2} \left\{ \left[x \times -2e^{-x/2} \right]_{0}^{\infty} - \int_{0}^{\infty} -2e^{-x/2} dx \right\}$$

$$= - \left\{ \left[x e^{-x/2} \right]_{0}^{\infty} - \int_{0}^{\infty} e^{-x/2} dx \right\}$$

$$= - \left\{ \left[x e^{-x/2} \right]_{0}^{\infty} + \left[2e^{-x/2} \right]_{0}^{\infty} \right\}$$

$= \cancel{e^{-x/2}} 0 + 2 = 2$

$E(x) = 2$

$V(x) = E(x^2) - \{E(x)\}^2$

$$= \int_0^\infty x^2 e^{-x/2} dx - 2^2$$

$$= \frac{1}{2} \frac{x^2}{-1/2} e^{-x/2} \Big|_0^\infty - \int_0^\infty x e^{-x/2} dx - 4$$

$$= 0 + 2 \cancel{e^{-x/2}} \Big|_0^\infty \times 4 - 4$$

$= 8 - 4 = 4$

$\sigma = \sqrt{V(x)} = \sqrt{4} = 2$ years.

Binomial

Example 5.20

Given $p = 0.2$, $n = 50$

so $P(x) = \binom{50}{x} (0.2)^x (0.98)^{50-x}$

Now $P(X \geq 2) = 1 - P(x \leq 2)$

$$P(x \le 2) = \sum_{x=0}^{2} \binom{50}{x} (0.02)^x (0.98)^{50-x}$$

$$= 1 \times 0.36 + 50 \times 0.02 \times 0.37 + 1225 \times \overset{0.0004}{\cancel{0.04}} \times 0.38$$

$$= 0.36 + 0.37 + 0.1862 = 0.92$$

$$P(x \ge 2) = 1 - 0.92 = 0.08$$

$$E(x) = np = 50 \times 0.02 = 1$$

$$V(x) = npq = 50 \times 0.02 \times 0.98 = 0.98$$

Poisson Distribution

Example 5.12

$\alpha = 2$, $\cancel{P(x) = e^{-\alpha} x}$

$$P(x) = \frac{e^{-\alpha} \alpha^x}{x!} \qquad P(3) = \frac{e^{-2} 2^3}{3!} = 0.18$$

$$P(x \ge 2) = 1 - P(x \le 2) + P(2)$$

$$= 1 - F(2) + P(2)$$

$$= 1 - \sum_{x=0}^{2} \frac{e^{-\alpha} \alpha^x}{x!} + \frac{e^{-2} 2^3}{3!}$$

$$= 1 - \left(\frac{e^{-2} 2^0}{0!} + \frac{e^{-2} 2^1}{1!} + \frac{e^{-2} 2^2}{2!} \right) + \frac{e^{-2} 2^3}{3!}$$

$$= 0.323$$

$P(x \geq 2) = 1 - P(0) - P(1)$

$= 1 - (P(0) + P(1))$

$= 1 - F(1)$

$= 1 - \sum_{x=0}^{1} \frac{e^{-\alpha} \alpha^{x}}{x!}$

$= 1 - \left(\frac{e^{-2} 2^{0}}{0!} + \frac{e^{-2} 2^{1}}{1!} \right)$

$= 0.594$

Exponential.

5.14 $\lambda = \frac{1}{3}$

$P(x > 3) = 1 - P(x \leq 3) = 1 - F(3) = 1 - (1 - e^{-\lambda 3})$

$= e^{-3/3} = e^{-1}$

$= 0.368$

$P(2 \leq x \leq 3) = F(3) - F(2)$

$(1 - e^{-3/3}) - (1 - e^{-\frac{2}{3}})$

$= 0.632 - 0.486$

$= 0.145$

$P(x > 3.5 \mid x > 2.5) = P(x > 1) = 1 - P(x \leq 1)$

$= 1 - F(1)$

$= 1 - (1 - e^{-\frac{1}{3}})$

$= e^{-\frac{1}{3}} =$

Normal Distribution μ, σ^2

5.22 Given, $x=10$, $N(12,4)$ hence $\mu=12$

$\sigma^2 = 4 \rightarrow \sigma = 2$

$$F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right) \qquad \phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

$$F(10) = \Phi\left(\frac{10-12}{2}\right) = \Phi(-1) = \frac{1}{\sqrt{2\pi}} e^{-1/2}$$

$$=$$

NSPP

Example 5.34

$$\lambda(t) = \begin{cases} 2 & 0 \le t < 4 \\ \frac{1}{2} & 4 \le x \le 8 \end{cases} \qquad \Lambda(t) = \begin{cases} 2t & 0 \le t < 4 \\ \frac{t}{2} + 6 & 4 \le t \le 8 \end{cases}$$

$$\Lambda(t) = \int_0^t 2\,dt = 2[t]_0^t = 2(t-0) = 2t$$

$$\Lambda(t) = \int_0^t \lambda(s)\,d(s) = \int_0^4 2\,ds + \int_4^t \frac{1}{2}\,ds$$

$$= 2s\Big|_0^4 + \frac{1}{2} s\Big|_4^t$$

$$= 8 + \frac{1}{2}(t-4)$$

$$= 8 + \frac{t}{2} - 2 = \frac{t}{2} + 6$$

$$P(N(6)-N(3)=k) = P(N(\Lambda(6)) - N(\Lambda(3)) = k)$$

$$= P(N(9) - N(6) = k)$$

$$= \frac{e^{-(9-6)}(9-6)^k}{k!} = \frac{e^{-3}3^k}{k!}$$

7.5 Given, $k=2$.

$m_1 = 2147483563 \quad a_1 = 40014$

$m_2 = 2147483399 \quad a_2 = 40692$

$\alpha = 0.05$

$n = 5$

$$x_i^0 = \left(\sum_{j=1}^{k} (-1)^{j-1} x_{i,j}\right) \bmod m_1 - 1$$

$$R_i = \begin{cases} \frac{x_i}{m_1} \\ \frac{m_1-1}{m_1} \end{cases} \qquad P = \frac{(m_1-1)(m_1-2)\cdots(m_k-1)}{2^{k-1}}$$

$n=5$

~~Ri 0 44 0.~~

R_i	0.05	0.14	0.44	0.81	0.93
i/n	0.2	0.4	0.6	0.8	1.0
$i/n - R_i$	0.15	0.26	0.16	–	0.07
$\frac{i-1}{n}$	0	0.2	0.4	0.6	0.8
$R_i - \frac{i-1}{n}$	0.05	–	0.04	0.21	0.13

$D^+ = 0.26$

$D^- = 0.21$

$D = \max\{D^+, D^-\}$

$= 0.26$

0.565

$-1.96 \le -1.519 \le 1.96$

$\alpha = 0.05$

3rd, 8th, 13th $\quad i + (M+1)m \le N \qquad Z_{\alpha/2} = 1.96$

$i = 3 \quad m = 5 \qquad$ or $3 + (M+1)5 \le 30$

$N = 30$

$M \to 3 + (0+1)5 = 8 \le 30 \qquad Z_0 = \dfrac{\hat{\rho}_m}{\sigma_{\rho m}}$

$M \to 3 + (1+1)5 = 13 \le 30$

so $M = 4 \qquad M \to 3 + (2+1)5 = 18 \le 30 \qquad = \dfrac{-0.1945}{0.128}$

$Z_0 = \dfrac{\hat{\rho}_m}{\sigma_{\rho m}} \qquad M \to 3 + (4+1)5 = 28 \le 30 \qquad = -1.519$

$M \to 3 + (5+1)6 = 33 \not\le 30$

$$\hat{\rho} = \frac{1}{m+1} \sum_{k=0}^{M} [R_{i+km} R_{i+(k+1)m}] - 0.25$$

$$= \frac{1}{4+1} \sum_{k=0}^{4} [R_{3+5k} R_{3+5(k+1)}] - 0.25$$

$$= \frac{1}{5} \{ R_3 R_8 + R_8 R_{13} + R_{13} R_{18} + R_{18} R_{23} + R_{23} R_{28}] - 0.25$$

$$= \frac{1}{5} \{ 0.23 \times 0.28 + 0.28 \times 0.33 + 0.33 \times 0.27 + 0.27 \times 0.05$$

$$+ 0.05 \times 0.36 \} - 0.25 = -0.1945$$

$$\sigma_{\rho m} = \frac{\sqrt{13M+7}}{12(M+1)} = \frac{\sqrt{13 \times 4 + 7}}{12(4+1)} = 0.128$$

$$F(x) = \frac{1}{\sqrt{2\pi}} e^{-\left(\frac{x-u}{6}\right)^2/2} = y$$

$$e^{-\left(\frac{x-u}{\sigma}\right)^2/2} = \sqrt{2\pi}.y$$

$$L = \rho + \frac{\rho^2(1+\mu^2\sigma^2)}{2(1-\rho)}$$

$$\omega = \frac{1}{\mu} + \frac{\lambda\left(\frac{1}{\mu^2}+\sigma^2\right)}{2(1-\rho)}$$

$$\frac{-\left(\frac{x-u}{6}\right)^2}{2} = \ln(\sqrt{2\pi}y)$$

$y = 0025$

$$\left(\frac{x-u}{\sigma}\right)^2 = -2\ln(\sqrt{2\pi}\,y)$$

$$\frac{x-u}{\sigma} = \sqrt{-2\ln(\sqrt{2\pi}y)} \qquad y = \frac{i-\frac{1}{2}}{n}$$

$$x = 6\sqrt{-2\ln(\sqrt{2\pi}\,y)} + u$$

$$\frac{\lambda\left(\frac{1}{\mu^2}+\sigma^2\right)}{2(1-\rho)} = \frac{\lambda(1+\sigma^2\mu^2)}{\mu^2\, 2(1-\rho)}$$

$$= \frac{\rho(1+\sigma^2\mu^2)}{2\mu(1-\rho)} + \frac{1}{\mu}$$

$p_0 = 1-\rho$

$F(x) = 1 - e^{-\lambda x} = y$

$e^{-\lambda x} = 1-y$

$-\lambda x = \ln(1-y)$

$x = -\frac{1}{\lambda}\ln(1-y)$ where $y = \frac{i-\frac{1}{2}}{n}$

$\rho_B = \frac{\lambda}{\mu_B} = \frac{25}{30} = 0.96$

$L_{Q_B} = \frac{(0.96)^2\left(1+\left(\frac{1}{25}\right)^2 \cdot 2^2\right)}{2(1-0.96)}$

$= \frac{0.604}{0.08}$ 1.0064 1.7004 0.34

M/G/1

$L = \rho + \frac{\rho^2(1+\mu^2\sigma^2)}{2(1-\rho)}$

$\omega = \frac{1}{\mu} + \frac{\lambda\left(\frac{1}{\mu^2}+\sigma^2\right)}{2(1-\rho)}$

$\frac{\lambda}{\mu_A} = \frac{24}{30} = 0.8$

0.333

$L_{Q_A} = \frac{\rho^2(1+\mu^2\sigma^2)}{2(1-\rho)}$

$\lambda = \frac{1}{30}$

$\frac{1}{\mu_A} = 24$ $\quad \sigma_A = 20$

$\frac{1}{\mu_B} = 25$ $\quad \sigma_B = 2$

$= \frac{(0.8)^2\left(1+\left(\frac{1}{24}\right)^2 \cdot 20^2\right)}{2(1-0.8)}$

$= \frac{0.64 \times 1.7}{0.4}$

$= 2.72$

$$L = \rho + \frac{\rho^2 (1+\mu^2\sigma^2)}{2(1-\rho)}$$

$$= \rho + \rho^2\left(1+\mu^2 \cdot \frac{1}{\mu^2}\right)$$

$$= \rho + \frac{\cancel{2}\rho^2}{\cancel{2}(1-\rho)} = \frac{\rho(1-\rho)+\rho^2}{(1-\rho)} = \frac{\rho-\rho^2+\rho^2}{1-\rho}$$

$$\boxed{= \frac{\rho}{1-\rho}}$$

$$L_Q = \boxed{\frac{\rho^2}{(1-\rho)}}$$

$$W = \frac{1}{\mu} + \frac{\lambda\left(\frac{1}{\mu^2}+\sigma^2\right)}{2(1-\rho)}$$

$$= \frac{1}{\mu} + \frac{\lambda\left(\frac{1}{\mu^2}+\frac{1}{\mu^2}\right)}{2(1-\rho)} = \frac{1}{\mu} + \frac{\frac{2}{\mu}\cdot\frac{\lambda}{\mu}}{2(1-\rho)}$$

$$= \frac{1}{\mu} + \frac{\rho}{\mu(1-\rho)}$$

$$= \frac{1-\rho+\rho}{\mu(1-\rho)}$$

$$\boxed{= \frac{1}{\mu(1-\rho)}}$$

$$W_q = \frac{\rho}{\mu(1-\rho)}$$

$$P_n = \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n$$

$$= (1-\rho)\rho^n$$

$\lambda = 2$ $\rho = \frac{\lambda}{\mu} = \frac{2}{3}$. $L = \frac{\rho}{1-\rho} = \frac{\frac{2}{3}}{1-\frac{2}{3}}$

$\mu = 3$

$P_0 = 1 - \rho = 1 - \frac{2}{3} = \frac{1}{3}$ $= 2$

$P_1 = (1-\rho)\rho = \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}$

$P_2 = (1-\rho)\rho^2 = \frac{1}{3}\left(\frac{2}{3}\right)^2 = \frac{4}{27}$ $W = \frac{1}{\mu(1-\rho)}$

$P_3 = (1-\rho)\rho^3 = \frac{1}{3}\left(\frac{2}{3}\right)^3 = \frac{8}{81}$ $= \frac{1}{}$

$P_{\geq k} = 1 - \sum_{n=0}^{3} P_n$ $= 1 - \left(\frac{8}{81} + \frac{4}{27} + \frac{2}{9} + \frac{1}{3}\right)$

$= 1 - \frac{8+12+18+27}{81}$

$W_Q = \frac{2}{3}$ $= \frac{16}{81}$

$L_Q = \frac{4}{3}$ Steady State Parameters for M/M/c/k/k

$$P_0 = \left[\sum_{n=0}^{c-1} \binom{k}{n}\left(\frac{\lambda}{\mu}\right)^n + \sum_{n=c}^{k} \frac{k!}{(k-n)!\,c!\,c^{n-c}}\left(\frac{\lambda}{\mu}\right)^n\right]^{-1}$$

$$P_n = \begin{cases} \binom{k}{n}\left(\frac{\lambda}{\mu}\right)^n P_0 & n = 0, 1 \ldots c-1 \\ \frac{k!}{(k-n)!\,c!\,c^{n-c}}\left(\frac{\lambda}{\mu}\right)^n P_0 & n = c, c+1 \ldots k \end{cases}$$

C.I → Confidence interval

Statistically significant

practically significant.

$$L = \sum_{n=0}^{K} n P_n \qquad \lambda_e = \sum_{n=0}^{K} (K-n)\lambda P_n$$

$$L_Q = \sum_{n=c+1}^{K} (n-c) P_n \qquad W = \frac{L}{\lambda_e} \qquad W_Q = \frac{L_Q}{\lambda_e}$$

$$\rho = \frac{L - L_Q}{c} = \frac{\lambda_e}{c\mu}$$

$$\bar{y}_{.1} - \bar{y}_{.2} \pm t_{\alpha/2,\nu}\, se(\bar{y}_{.1} - \bar{y}_{.2})$$

$$\bar{y}_{.i} = \frac{1}{R_p} \sum_{r=1}^{R_p} Y_{ri}$$

$K = 10$ $\quad \lambda = \frac{1}{20}$

$c = 2$ $\quad \mu = \frac{1}{5}$

$$P_0 = \left[\sum_{n=0}^{1} \binom{10}{n} \left(\frac{5}{20}\right)^n + \sum_{n=2}^{10} \frac{10!}{(10-n)!\,2!\,2^{n-2}} \left(\frac{5}{20}\right)^n\right]^{-1}$$

$$= \left[\left(1 + 10 \times \frac{5}{20}\right) + \left(180 \times \left(\frac{5}{20}\right)^3 + 630 \times \left(\frac{5}{20}\right)^4 + 1890 \times \left(\frac{5}{20}\right)^5 + 4725 \times \left(\frac{5}{20}\right)^6 + 9450 \times \left(\frac{5}{20}\right)^7 + 14175 \times \left(\frac{5}{20}\right)^8 + 14175 \times \left(\frac{5}{20}\right)^9 + 14175 \times \left(\frac{5}{20}\right)^{10}\right)\right]$$

$$45 \times \left(\frac{5}{20}\right)^2 = \frac{45}{16}$$

$$= \left[3.5 + \left(\frac{45}{16} + \frac{315}{128} + \frac{945}{512} + \frac{4725}{4096} + \frac{4725}{8192}\right.\right.$$

$$\left.\left. + 0.2163 + 0.0541 + 0.0135\right)\right]^{-1}$$

$$\left[3.5 + 9.133 + \frac{45}{16}\right]^{-1} = 0.065$$

$$L_Q = \sum_{n=c+1}^{K} (n-c) P_n = \sum_{n=3}^{10} (n-2) \cancel{0.065}\, P_n$$

$$= \cancel{K} \sum_{n=3}^{10} (n-2) \frac{K!}{(K-n)!\, c!\, c^{n-c}} \left(\frac{\lambda}{\mu}\right)^n$$

$$= \sum_{n=3}^{10} \frac{10!}{(10-n)!\, 2!\, 2^{n-2}} \left(\frac{5}{20}\right)^n$$

$$= \sum_{n=3}^{10} \frac{10! \times 2}{(10-n)!\, 2^n} \left(\frac{1}{4}\right)^n$$