ISTANBUL TECHNICAL UNIVERSITY ⋆ INSTITUTE OF SCIENCE AND TECHNOLOGY

KERR-SCHILD SPACETIMES

MSc Thesis by

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SEPTEMBER 2007

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CONTENTS

ABBREVIATIONS

RN	:	Reissner-Nordström
EF	:	Eddington-Finkelstein
BL	:	Boyer-Lindquist
NJ	:	Newman-Janis

LIST OF SYMBOLS

$\mu, \nu, \rho...$	:	Spacetime indices
$T_{\mu\nu}$	:	Energy-Momentum Tensor
η	:	Minkowski Metric Tensor
w^a_b	:	Connection one form
R^a_b	:	Curvature two form
$R_{\mu\nu}$	:	Ricci Tensor
$\hat{\epsilon}_{\mu\nu\alpha\beta}$	:	Levi-Civita Tensor for curved space-time
$\epsilon_{\mu\nu\alpha\beta}$	:	Levi-Civita Tensor for Minkovski space-time
$\xi^{(\mu)}$	:	Killing Vector
$\Gamma^\alpha_{\beta\gamma}$	:	Christoffel Symbols
$d\Omega^2$	:	Line element of unit two-sphere
$d\Omega^2_{N-3}$	:	Line element of unit $(N-3)$-sphere

KERR-SCHILD SPACETIMES

SUMMARY

Using differential forms for finding solutions to Einstein equations is a powerful tool since the calculations by the basis vectors are not related to any coordinate system. Also decomposition of the space-time by complex null tetrad formalism is a powerful tool for finding solutions to Einstein equations. By the use of complex transformation, rotating black hole solutions can be obtained from static black hole solutions. More generally if a metric can be written in Kerr-Schild space-time form, which allows one to separate metric into two parts as one part is flat and the other is having a mass times a null vector field structure, it is allowed in general relativity that a complex transformation can be performed on these type of metrics. Besides known ones, this method leads some new solutions for Einstein equations, as Kerr-Newman (rotating-charged) black hole solutions first obtained by this complexification method. In this study differential forms are used for black hole solutions and after introducing complex null tetrad formalism complex coordinate transformation method is applied first onto the null-tetrad one forms and then onto the contravariant metric components for obtaining Kerr metric from Schwarzschild metric first in 4 dimensions and then in $N+1$ dimensions with a single rotation parameter.

KERR-SCHILD UZAY-ZAMANLARI

ÖZET

Diferansiyel formlar kullanılarak Einstein denklemlerinin çözülmesi, hesapların koordinat seçiminden bağımsız olması sebebiyle etkin ve daha kolay bir yöntemdir. Ayrıca uzay-zamanın kompleks null-tetrat formalizmi ile ifade edilmesi de Einstein denklemlerine çözüm bulmada etkin bir yöntemdir. Kompleks öteleme yöntemi ile durgun karadelik çözümlerinden dönen karadelik çözümleri elde etmek mümkündür. En genel olarak eğer metrik Kerr-Schild metrik formunda yazılabiliyorsa, genel relativitede bu kompleks öteleme yöntemi kullanılabilinir. Kerr-Schild türü metriklerde metrik düz Minkowski metriği ve karadeliğin kütlesinin çarpan olarak bulunduğu iki kısımdan oluşur. Bu yöntemle ilk olarak bilinen bir çözüm olan Kerr metriği elde edilmiştir ayrıca daha önce bilinmeyen Kerr-Newman (dönen yüklü karadelik) çözümü ilk kez bu yöntemle elde edilmiştir. Bu çalışmada karadelik çözümleri için diferansiyel formlar kullanılarak kompleks null-tetrat formalizmi ile Schwarzschild metriğinin null 1-formları üzerine kompleks koordinat öteleme yöntemi denenmiştir ve ayrıca kompleks koordinat öteleme yöntemi metriğin tersine uygulanarak 4 boyutta ve $N+1$ boyutta tek dönme parametresine sahip Kerr metriği verilmiştir.

1. INTRODUCTION

The properties of black holes in four dimensions are well established starting from the Schwarzschild solution for a static spherically symmetric black hole where black hole is characterized only by its mass and the Kerr solution for a stationary black hole where black hole is characterized by two parameters, its mass and angular momentum.

Finding solutions of Einstein equations using differential forms is a powerful tool since the calculations do not depend on a chosen coordinate system where the metric can be written in an orthonormal basis as

$$ds^2 = \eta_{a\,b}\, e^a \otimes e^b\,. \tag{1.1}$$

The use of complex null-tetrad formalism, where tetrad is written as a combination of basis 1-forms, is also a powerful method for finding black hole solutions. For example, Kerr found the rotating black hole solution using null-tetrad methods in his paper [1]. Later, Newman and Janis [2] showed that Kerr metric can be obtained from the Schwarzschild metric by a complex coordinate transformation writing the Schwarzschild metric in outgoing Eddington-Finkelstein coordinates, and using complex null tetrad formalism by writing the inverse metric in the following form

$$g^{-1} = \tilde{l} \otimes \tilde{n} + \tilde{n} \otimes \tilde{l} + \tilde{m} \otimes \tilde{m}^* + \tilde{m}^* \otimes \tilde{m}\,. \tag{1.2}$$

This complexification method is reminiscent of Trautman's method [3], where Trautman generalized a procedure to construct new solutions to linear, real partial differential equations starting from known solutions using complex transformations.

In NJ method the metric is used in contravariant form and the resultant metric after complex transformation can be written in Kerr-Schild form [4], which is the following

$$g^{-1} = \eta^{-1} + H\, \tilde{l} \otimes \tilde{l}\,, \tag{1.3}$$

where the metric is decomposed into two parts: linear Minkowski metric plus a scalar function times square of a null-geodesic vector $\tilde{l}$. Also using this complexification procedure a new exact solution of Einstein-Maxwell equations

$$R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R = 8\pi T_{\mu\nu}\,, \qquad (1.4)$$
$$\partial_\mu\, F^{\,\mu}_{\nu} = 0$$

is found, called the Kerr-Newman black hole where the black hole is characterized by three parameters, its mass, angular momentum and charge [5]. Later, it has been shown that if a metric can be written in Kerr-Schild form then a complex transformation is allowed in general relativity [6] .

A different approach to complexification is also done [7] by getting use of a generating potential, which gives Schwarzschild solution as the simplest case and it is shown that Kerr solution can be obtained from this generating function if the origin is translated by a complex transformation, $z \mapsto z - ia$. Furthermore by getting use of the octonion algebra, Kerr metric in eight dimensions is obtained from Schwarzschild metric by a complex transformation in Cartesian coordinates where the metric was defined in Kerr-Schild form in eight dimensions. Here the key point is making use of the cross product, since cross product is defined only in eight dimensions after 4 dimensions [8].

The generalization of exact solutions of static black holes in higher dimensions is first found by Tangherlini [9], where the generalization of spherically symmetric Schwarzschild and Reissner-Nordström metrics are given. Rotating black hole solutions in higher dimensions are obtained by getting use of Kerr-Schild form of the metric by Myers and Perry, where the metric is characterized by $\lfloor N/2 \rfloor + 1$ parameters: its mass and $\lfloor N/2 \rfloor$ rotation parameters in $N+1$ dimensional spacetime [10] . Generalization of Kerr-de Sitter metrics to higher dimensions is recently achieved by Gibbons *at all* [11], where Kerr-Schild type of the metric is again used, however, in this case flat Minkowski metric is replaced by de Sitter metric in the Kerr-Schild form

$$g^{-1} = \eta'^{-1} + H\,\tilde{l} \otimes \tilde{l}\,,$$

where again black hole is characterized by $\lfloor N/2 \rfloor + 1$ parameters: its mass and maximal number $\lfloor N/2 \rfloor$ of independent rotation parameters in $N+1$ dimensional space-time and η'^{-1} is the contravariant metric of the de-Sitter space-time which is free of black hole mass.

In the Chapter 2 of this thesis differential forms are introduced and spherically symmetric black hole solution is obtained using differential forms. Then in the second section we introduce complex null-tetrad formalism

$$g = l \otimes n + n \otimes l + m \otimes m^* + m^* \otimes m \,, \tag{1.5}$$

and using these null vector formalism Schwarzschild metric is obtained in Kerr-Schild form

$$g = \eta + 2H \, l \otimes l \,. \tag{1.6}$$

In Chapter 3, starting from the ingoing Eddington-Finkelstein coordinates

$$ds^2 = -f(r)\, du^2 + 2du\, dr + r^2 \left(d\theta^2 + sin^2\theta \, d\varphi^2\right), \tag{1.7}$$

we apply complex coordinate transformation method to the null basis 1-forms. Then the same method is applied to the inverse metric components of the Schwarzschild metric to obtain Kerr metric from static black hole solution. The result is given first in Boyer-Lindquist coordinates where metric has only $g_{t\phi}$ off diagonal terms in it and also in Kerr-Schild form using Cartesian coordinates.

In the last Chapter, we give the $N + 1$ dimensional Kerr black hole solutions obtained from the $N + 1$ dimensional static black hole solution, where the Kerr black hole has only one rotation parameter which is a special case of known solutions [10, 11].

2. SOLUTIONS BY DIFFERENTIAL FORMS

2.1. Differential forms in General Relativity

A metric can be written in an orthonormal basis as

$$ds^2 = \eta_{a\,b}\, e^a \otimes\, e^b \,= g_{\mu\nu}\, dx^\mu \otimes\, dx^\nu . \tag{2.1}$$

Given vector fields X and $Y \in TM$, $g(X,Y) = g_{\mu\nu}\, \xi^\mu\, \xi^\nu$. Suppose $\{X_a\}$ is an orthonormal frame, then

$$g(X_a, X_b) = \eta_{ab}$$

where metric signature $(-,+,+,+)$ is used.

Since X_a's belong to tangent manifold, orthonormal coframes e^a's belong to dual of this tangent manifold, called the cotangent manifold

$$\begin{gathered} X_a \in TM, \\ e^a \in T^*M. \end{gathered}$$

So

$$e^a\,(x_a) = \delta^a_b\,.$$

In a coordinate chart (x^μ) we have

$$\begin{aligned} dx^\mu &\in T^*M, \\ \partial_\mu = \frac{\partial}{\partial x^\mu} &\in TM. \end{aligned}$$

The definition of a tensor field is given as a multilinear map

$$\mathbf{T} = \underbrace{TM \otimes TM \otimes \ldots}_{\text{p times}} \underbrace{\otimes T^*M \otimes \ldots T^*M}_{\text{q times}} \rightarrow R,$$

$$(X_{a_1},\, X_{a_2}, \ldots,\, X_{a_p},\, e^{b_1},\, e^{b_2},\, \ldots\, , e^{b_q}) \rightarrow (p\,, q)$$

This is the definition of a type $(p\,, q)$ tensor field.

The space-time, (M, g, ∇) is defined by a differentiable manifold M, metric g

and connection ∇ which carries one point to another point. Space-time is called Riemanian if ∇ is the Levi-Civita connection:

$$\nabla : \underbrace{\mathbf{T}}_{(\mathrm{p}\,,\,\mathrm{q})} \rightarrow \underbrace{\nabla\mathbf{T}}_{(\mathrm{p+1}\,,\,\mathrm{q})}$$

g and ∇ are compatible if and only if

$$\nabla\, g = 0\,.$$

Then the connection 1-forms are antisymmetric

$$w_{a\,b} = -w_{b\,a}\,.$$

The first Cartan equations are

$$de^a + w^a_b \wedge e^b = 0$$

and

$$w^a_b = w^a_{c,\,b}\, e^c\,.$$

After finding connection 1-forms from the 1^{st} Cartan equations, curvature two forms can be found from the 2^{nd} Cartan equations

$$dw^a_b + w^a_c \wedge w^c_b = R^a_b\,. \tag{2.2}$$

To derive Bianchi identities one can take the d derivatives of 1^{st} and 2^{nd} Cartan equations. From the 1^{st} Cartan equations we find

$$\underbrace{d^2e^a}_{0} + dw^a_b \wedge e^b - w^a_b \wedge de^b = 0$$

$$(R^a_b - w^a_c \wedge w^c_b)\;\wedge\; e^b - w^a_b \wedge (-w^b_c \wedge e^c) = R^a_b \wedge e^b = 0,$$

so we get the first Bianchi identity

$$R^a_b \wedge e^b = 0\,. \tag{2.3}$$

From the d derivative of 2^{nd} Cartan equations we get

$$\begin{aligned} &\underbrace{d^2w^a_b}_{0} + dw^a_c \wedge w^c_b - w^a_c \wedge dw^c_b = dR^a_b \\ &= (\, R^a_c - w^a_\alpha \wedge w^\alpha_c\,)\;\wedge w^c_b - w^a_c \wedge (\, R^c_b - w^c_d \wedge w^d_b)\,. \end{aligned} \tag{2.4}$$

So we get the 2^{nd} Bianchi identity as

$$dR^a_b = R^a_c \wedge w^c_b - w^a_c \wedge R^c_b \,. \tag{2.5}$$

The relation between the curvature 2 forms and the Riemann tensor is the following:

$$R^a_b = \frac{1}{2} R^a_{\;c\,d\,,b}\, e^c \wedge e^d \,. \tag{2.6}$$

To find Ricci one forms, we define an interior product which satisfies the Leibnitz rule,

$$l_X : \underbrace{\alpha}_{p\ form} \rightarrow \underbrace{l_X\alpha}_{(p-1)\ form} \tag{2.7}$$

such that

$$l_{X_a}(e^b) = \delta^b_a \,. \tag{2.8}$$

For example, if we define $X = \xi^a\, X_a$ and $\alpha = \alpha_b\, e^b$ then

$$\begin{aligned} l_X\alpha &= l_{\xi^a\,X_a}(\alpha_b\, e^b) = \xi^a\, l_{X_a}(\alpha_b\, e^b) \\ &= \xi^a\, \alpha_b\, \underbrace{l_{X_a}(e^b)}_{\delta^b_a} = \xi^a\, \alpha_a = \alpha(X)\,. \end{aligned} \tag{2.9}$$

As for the Leibnitz rule we have

$$\begin{aligned} l_{X_a}(e^b \wedge e^c) &= l_{X_a}(e^b)\, e^c - e^b\, l_{X_a}\,(e^c) \\ &= \delta^b_a\, e^c - \delta^c_a\, e^b \,. \end{aligned} \tag{2.10}$$

To find the Ricci 1-forms we apply l_{X_a} on R^a_b:

$$\begin{aligned} l_{X_a}\left(\frac{1}{2} R^a_{\;c\,d\,,b}\, e^c \wedge e^d\right) &= \frac{1}{2} R^a_{\;c\,d\,,b}(\,\delta^c_a\, e^d - \delta^d_a\, e^c\,) \\ &= R^a_{\;c\,d\,,b}\, \delta^c_a\, e^d = (\,R^a_{\;a\,d\,,b}\,)\, e^d \,, \end{aligned} \tag{2.11}$$

which results in

$$l_{X_a} R^a_b = Ric_b = (Ric)_{ab}\, e^a \,. \tag{2.12}$$

The curvature scalar is obtained by one more contraction:

$$l_{X_b}\, l_{X_a} R^{a\,b} = l_{X_b}(Ric^b_a\, e^a) = Ric^a_a = R\,. \tag{2.13}$$

As an example for solving Einstein equations by differential forms we can look at the 4 dimensional spherically symmetric solution.

The most general metric for a spherically symmetric system is given by

$$ds^2 = -e^{2\nu(r,t)}dt^2 + e^{2\lambda(r,t)}dr^2 + r^2\,(d\theta^2 + sin^2\theta\, d\varphi^2)\,. \tag{2.14}$$

In order to find the arbitrary functions $e^{2\nu(r,t)}$ and $e^{2\lambda(r,t)}$, we will use differential forms starting by defining basis one forms as

$$
\begin{aligned}
e^0 &= e^{\nu(r,t)} dt\,, \\
e^1 &= e^{\lambda(r,t)} dr\,, \\
e^2 &= r\, d\theta\,, \\
e^3 &= r\, sin\theta\, d\varphi\,.
\end{aligned} \tag{2.15}
$$

The d derivatives of these basis one forms are

$$
\begin{aligned}
de^0 &= \nu^{'} e^{\nu}\, dr \wedge dt = -\nu^{'} e^{-\lambda}\, e^0 \wedge e^1\,, \\
de^1 &= \dot{\lambda} e^{\lambda} dt \wedge dr = \dot{\lambda} e^{-\nu} e^0 \wedge e^1\,, \\
de^2 &= dr \wedge d\theta = \frac{e^{-\lambda}}{r} e^1 \wedge e^2\,, \\
de^3 &= sin\theta\, dr \wedge d\varphi + r\, cos\theta\, d\theta \wedge d\varphi = \frac{e^{-\lambda}}{r} e^1 \wedge e^3 + \frac{cot\theta}{r} e^2 \wedge e^3\,,
\end{aligned} \tag{2.16}
$$

where $'$ denotes the derivative with respect to r and $\dot{}$ denotes the derivative with respect to t.

The first Cartan equations, written out explicitly are

$$
\begin{aligned}
de^0 + w^0_1 \wedge e^1 + w^0_2 \wedge e^2 + w^0_3 \wedge e^3 &= 0\,, \\
de^1 + w^1_0 \wedge e^0 + w^1_2 \wedge e^2 + w^1_3 \wedge e^3 &= 0\,, \\
de^2 + w^2_0 \wedge e^0 + w^2_1 \wedge e^1 + w^2_3 \wedge e^3 &= 0\,, \\
de^3 + w^3_0 \wedge e^0 + w^3_1 \wedge e^1 + w^3_2 \wedge e^2 &= 0\,.
\end{aligned} \tag{2.17}
$$

By comparing these structure equations with d derivatives of basis one forms, we can determine the following connection one forms:

$$
\begin{aligned}
w^0_1 &= \nu^{'} e^{-\lambda}\, e^0 + \dot{\lambda}\, e^{-\nu}\, e^1\,, \\
w^0_2 &= w^0_3 = 0\,, \\
w^1_2 &= -\frac{e^{-\lambda}}{r}\, e^2\,, \\
w^1_3 &= -\frac{e^{-\lambda}}{r}\, e^3\,, \\
w^2_3 &= -\frac{cot\theta}{r}\, e^3\,.
\end{aligned} \tag{2.18}
$$

Then using the second Cartan equations, the curvature two forms are found as

$$
\begin{aligned}
R^0_1 &= \Big[(\dot{\lambda}\, e^{(\lambda-\nu)})^{\cdot} - (\nu'\, e^{(\nu-\lambda)})' \Big]\, e^{-(\nu+\lambda)}\, e^0 \wedge e^1 \,, \\
R^2_3 &= \frac{1}{r^2}\,(\,1 - e^{-2\lambda}\,)\, e^2 \wedge e^3 \,, \\
R^0_2 &= -\frac{\nu'}{r}\, e^{-2\lambda}\, e^0 \wedge e^2 - \frac{\dot{\lambda}}{r}\, e^{-(\nu+\lambda)}\, e^1 \wedge e^2 \,, \\
R^1_3 &= \frac{\lambda'}{r}\, e^{-2\lambda}\, e^1 \wedge e^3 + \frac{\dot{\lambda}\, e^{-(\lambda+\nu)}}{r}\, e^0 \wedge e^3 \,, \\
R^0_3 &= -\frac{\nu'\, e^{-2\lambda}}{r}\, e^0 \wedge e^3 - \frac{\dot{\lambda}\, e^{-(\nu+\lambda)}}{r}\, e^1 \wedge e^3 \,, \\
R^1_2 &= \frac{\lambda'\, e^{-2\lambda}}{r}\, e^1 \wedge e^2 + \frac{\dot{\lambda}\, e^{-(\nu+\lambda)}}{r}\, e^0 \wedge e^2 \,.
\end{aligned}
\tag{2.19}
$$

The corresponding Ricci one forms are

$$
\begin{aligned}
(Ric)_0 &= l_1\, R^1_0 + l_2 R^2_0 + l_3 R^3_0 \\
&= \Big[(\,\nu'\, e^{(\nu-\lambda)})' - (\,\dot{\lambda} e^{(\lambda-\nu)}\,)^{\cdot} \Big]\, e^{-(\nu+\lambda)}\, e^0 \\
&+ \frac{2\nu'}{r}\, e^{-2\lambda}\, e^0 + \frac{2\dot{\lambda}}{r}\, e^{-(\lambda+\nu)}\, e^1 = 0 \,, \qquad (2.20) \\
(Ric)_1 &= l_0\, R^0_1 + l_2 R^2_1 + l_3 R^3_1 \\
&= \Big[(\,\dot{\lambda} e^{(\lambda-\nu)}\,)^{\cdot} - (\,\nu'\, e^{(\nu-\lambda)})' \Big]\, e^{-(\nu+\lambda)}\, e^1 \\
&+ \frac{2\lambda'}{r}\, e^{-2\lambda}\, e^1 = 0 \,, \qquad (2.21) \\
\cdots\cdot &\quad \cdots\cdot \\
\cdots\cdot &\quad \cdots\cdot
\end{aligned}
$$

From the $(Ric)_0$ component we see that

$$
\frac{2\dot{\lambda}}{r}\, e^{-(\lambda+\nu)} = 0 \,.
$$

Therefore $\lambda = \lambda(r)$ only. Taking this into account we have

$$
\begin{aligned}
(Ric)_0 &= (\,\nu'\, e^{(\nu-\lambda)})'\, e^{-(\nu+\lambda)} + \frac{2\nu'}{r}\, e^{-2\lambda} = 0 \,, \qquad (2.22) \\
(Ric)_1 &= (\,\nu'\, e^{(\nu-\lambda)})'\, e^{-(\nu+\lambda)} - \frac{2\lambda'}{r}\, e^{-2\lambda} = 0 \,, \qquad (2.23) \\
\cdots\cdot &\quad \cdots\cdot \\
\cdots\cdot &\quad \cdots\cdot
\end{aligned}
$$

Comparing $(Ric)_0$ and $(Ric)_1$ we have

$$
\nu(r,t) = -\lambda(r) + f(t)
$$

So the metric becomes

$$ds^2 = -e^{-2\lambda(r)}\, e^{2f(t)}\, dt^2 + e^{2\lambda(r)}\, dr^2 + r^2\, d\Omega^2\,.$$

We can make the coordinate transformation

$$d\widetilde{t} = e^{1/2\, f(t)}\, dt\,,$$

and after dropping the $\;\widetilde{}\;$ on t the metric becomes

$$ds^2 = -e^{-2\lambda(r)}\, dt^2 + e^{2\lambda(r)}\, dr^2 + r^2\, d\Omega^2\,.$$

We have two different equations remaining from Ricci one forms

$$(Ric)_0\,,\,(Ric)_1 \;\;\rightarrow\;\; (\,\lambda'\, e^{-2\lambda}\,)' + \frac{2\lambda'}{r}\, e^{-2\lambda} = 0\,, \tag{2.24}$$

$$(Ric)_2\,,\,(Ric)_3 \;\;\rightarrow\;\; \frac{2\lambda'}{r}\, e^{-2\lambda} + \frac{1}{r^2}\,(1 - e^{-2\lambda}) = 0\,. \tag{2.25}$$

So we must satisfy

$$(\,\lambda'\, e^{-2\lambda}\,)' = \frac{1}{r^2}\,(1 - e^{-2\lambda})\,. \tag{2.26}$$

This is achieved if we choose

$$e^{-2\lambda} = \left(1 + \frac{A}{r}\right).$$

Taking the logarithm of both sides and derivation with respect to r it is shown to satisfy (2.26) . Lastly to determine the arbitrary constant A in the solution, we can make use of Komar mass integral [12]

$$M = \frac{1}{8\pi}\int_{\mathbf{S}^2} \xi^{\mu;\,\nu}_{\,(\mathbf{t})}\, d^2\mathbf{\Sigma}_{\mu\,\nu} \tag{2.27}$$

where $\xi^{\mu}_{(t)}$ is a timelike Killing vector field

$$\xi^{\mu}_{(t)} = (\,1, 0, 0, 0)$$

and

$$d^2\Sigma_{\mu\,\nu} = \frac{1}{2}\,\hat{\epsilon}_{\,\mu\,\nu\alpha\beta}\, dx^\alpha \wedge dx^\beta$$

where $\hat{\epsilon}_{\mu\nu\alpha\beta}$ is the Levi-Civita tensor in curved space-time and it is defined as

$$\hat{\epsilon}_{\mu\nu\alpha\beta} = \sqrt{-g}\,\epsilon_{\mu\nu\alpha\beta} \tag{2.28}$$

where $\epsilon_{\mu\nu\alpha\beta}$ is the Levi-Civita tensor in Minkowski space-time. We can take $t = constant$ and $r = constant$ so we are dealing with a spherical surface, then

$$d^2\Sigma_{01} = \sqrt{-g}\, d\theta \wedge d\varphi\,,$$

where g is the determinant of the metric and covariant derivative of Killing vector is

$$\xi^{\mu\,;\,\nu} = g^{\nu\,\alpha}\,\xi^{\mu}_{;\alpha} = g^{\nu\,\alpha}\,\Gamma^{\mu}_{\alpha\,\lambda}\xi^{\lambda}\,.$$

Since we have chosen r and t constant,

$$\xi^{\mu\,;\,\nu} \rightarrow \xi^{0\,;\,1} = g^{11}\,\Gamma^{\,0}_{\,10}$$

for this metric and the Christoffel symbols $\Gamma^{\,\alpha}_{\,\beta\,\gamma}$ are defined as

$$w^{\alpha}_{\beta} = \Gamma^{\,\alpha}_{\,\beta\,\gamma}\, dx^{\gamma}\,.$$

The Komar integral becomes

$$M = \frac{1}{4\pi}\int \sqrt{-g}\, g^{11}\,\Gamma^{0}_{10}\, d\theta \wedge d\varphi\,,$$

where $-g = r^4\, sin^2\theta$ and

$$M = \frac{1}{4\pi}\int r^2\, sin\theta \left(1 + \frac{A}{r}\right) \frac{(-A)}{2r\,(r+A)}\, d\theta \wedge d\varphi\,.$$

As a result it is found that

$$A = -2\,M\,,$$

so the exact solution becomes

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\,(d\theta^2 + sin^2\theta\, d\varphi^2)\,,$$

where $G = \hbar = c = 1$ units are used and despite we started by a metric which is depended on time, the result is free of time outside of the black hole (Birkhoff's theorem).

2.2. Complex Null-Tetrad Formalism

Starting from the metric written in an orthonormal basis

$$g = -e^0 \otimes e^0 + e^1 \otimes e^1 + e^2 \otimes e^2 + e^3 \otimes e^3 = \eta_{a\,b}\, e^a \otimes e^b, \tag{2.29}$$

the complex null tetrad components can be defined as

$$l = \frac{e^3 + e^0}{\sqrt{2}}, \quad n = \frac{e^3 - e^0}{\sqrt{2}}, \quad m = \frac{e^1 + ie^2}{\sqrt{2}}\,. \tag{2.30}$$

m^* being the complex conjugate of m, we can write the metric in the following form:

$$g = l \otimes n + n \otimes l + m \otimes m^* + m^* \otimes m\,. \tag{2.31}$$

Then we get the following structure equations:

$$\begin{aligned}
dl + w^0_3 \wedge l - \frac{1}{2}\,(w^1_3 - w^0_1 - iw^2_3 + iw^0_2) \wedge m - \frac{1}{2}\,(w^1_3 - w^0_1 + iw^2_3 - iw^0_2) \wedge m^* = 0\,,\\
dn - w^0_3 \wedge n - \frac{1}{2}\,(w^1_3 + w^0_1 - iw^2_3 - iw^0_2) \wedge m - \frac{1}{2}\,(w^1_3 + w^0_1 + iw^2_3 - iw^0_2) \wedge m^* = 0\,,\\
dm - iw^1_2 \wedge m + \frac{1}{2}\,(w^0_1 + iw^0_2 + w^1_3 + iw^2_3) \wedge l - \frac{1}{2}\,(w^0_1 + iw^0_2 - w^1_3 - iw^2_3) \wedge n = 0\,.
\end{aligned}$$

We define complex valued connection one-forms

$$w_j = -\frac{1}{2}\,(\,iw^0_j + w^k_l\,)\,, \tag{2.32}$$

where $j = 1, 2, 3$ and jkl are cyclic. We introduce the complex one-forms

$$\begin{aligned}
w_1 &= -\frac{1}{2}\,(\,iw^0_1 + w^2_3\,)\,,\\
w_2 &= -\frac{1}{2}\,(\,iw^0_2 - w^1_3\,)\,,\\
w_3 &= -\frac{1}{2}\,(\,iw^0_3 + w^1_2\,)\,,
\end{aligned} \tag{2.33}$$

and w_+ and w_- defined as

$$\begin{aligned}
w_+ = w_1 + iw_2 &= -\frac{1}{2}\,(\,iw^0_1 + w^2_3 - w^0_2 - iw^1_3\,)\,,\\
w_- = w_1 - iw_2 &= -\frac{1}{2}\,(\,iw^0_1 + w^2_3 + w^0_2 + iw^1_3\,)\,.
\end{aligned} \tag{2.34}$$

We may write $w^0_3 = (w_3 - w^*_3)i$ and $w^1_2 = -(w_3 + w^*_3)$ where w^*_3 is complex conjugate of w_3.

Finally we get the structure equations in newly defined complex one-forms

$$\begin{aligned}
dl + i(w_3 - w^*_3) \wedge l - iw^*_+ \wedge m + iw_+ \wedge m^* &= 0\,,\\
dn - i(w_3 - w^*_3) \wedge n - iw_- \wedge m + iw^*_- \wedge m^* &= 0\,,\\
dm + i(w_3 + w^*_3) \wedge m - iw^*_- \wedge l - iw_+ \wedge n &= 0\,.
\end{aligned} \tag{2.35}$$

Now we try to find connection one-forms of the following metric written in Kerr-Schild form by comparing its structure equations with (2.35):

$$g = du \otimes dv + dv \otimes du + r^2(d\theta^2 + sin^2\theta\, d\varphi^2) + 2H(r)du^2 \,, \tag{2.36}$$

where u and v are given by

$$u = \frac{r+t}{\sqrt{2}} \,, \quad v = \frac{r-t}{\sqrt{2}}$$

and H is a function of r only. For this metric the null tetrad can be chosen as

$$\begin{aligned} l &= du \,, \\ n &= dv + H du \,, \\ m &= \frac{r}{\sqrt{2}}(d\theta + i sin\theta\, d\varphi) \end{aligned} \tag{2.37}$$

and their d derivatives are

$$\begin{aligned} dl &= 0 \,, \\ dn &= \frac{H'}{\sqrt{2}} n \wedge l \,, \\ dm &= \frac{1}{\sqrt{2}\, r} n \wedge m + \frac{(1-H)}{\sqrt{2}\, r} l \wedge m - \frac{cot\theta}{\sqrt{2}\, r} m \wedge m^* \,. \end{aligned} \tag{2.38}$$

By comparing (2.35) and (2.38) the following complex connection one-forms are found:

$$\begin{aligned} w_3 - w_3^* &= i \frac{H'}{\sqrt{2}} l \,, \\ w_3 + w_3^* &= i \frac{cot\theta}{\sqrt{2}\, r} (m^* - m) \,, \\ w_+ &= i \frac{m}{\sqrt{2}\, r} \,, \\ w_- &= i \frac{m^*}{\sqrt{2}\, r} (H-1) \,. \end{aligned} \tag{2.39}$$

So we get the orthonormal connection one-forms as

$$\begin{aligned} w^0_1 = -\frac{H}{2} d\theta \quad &, \quad w^2_3 = \frac{(2-H)}{2} sin\theta\, d\varphi \,, \\ w^0_2 = -\frac{H}{2} sin\theta\, d\varphi \quad &, \quad w^1_3 = \frac{(2-H)}{2} d\theta \,, \\ w^1_2 = -cos\theta\, d\varphi \quad &, \quad w^0_3 = -\frac{H'}{\sqrt{2}} du \,. \end{aligned} \tag{2.40}$$

From the second Cartan structure equations we get the curvature two-forms

$$\begin{aligned} R^0_1 &= -\frac{H'}{2\,r}\, e^0 \wedge e^1 \,, \\ R^0_2 &= -\frac{H'}{2\,r}\, e^0 \wedge e^2 \,, \\ R^0_3 &= -\frac{H''}{2}\, e^0 \wedge e^3 \,, \\ R^1_2 &= \frac{H}{r^2}\, e^1 \wedge e^2 \,, \\ R^1_3 &= \frac{H'}{2\,r}\, e^1 \wedge e^3 \,, \\ R^2_3 &= \frac{H'}{2\,r}\, e^2 \wedge e^3 \,. \end{aligned} \tag{2.41}$$

Substituting these into the Einstein equations we get the following ordinary differential equation for $H(r)$:

$$\frac{H''}{2} = \frac{H}{r^2}. \tag{2.42}$$

So $H(r) = \mathrm{A}/\,\mathrm{r}$. To find the constant A, we first write the standard Schwarzschild metric

$$ds^2 = -(1 - \frac{2M}{r})\, dt^2 + (1 - \frac{2M}{r})^{-1}\, dr^2 + r^2\, (d\theta^2 + sin^2\theta\, d\varphi^2)\,, \tag{2.43}$$

and apply the transformation

$$\bar{t} = t + 2M\, log\, (\frac{r}{2M} - 1)\,, \tag{2.44}$$

to we get the Kerr-Schild form of the Schwarzschild metric as

$$ds^2 = -d\bar{t}^2 + dr^2 + r^2\, (d\theta^2 + sin^2\theta\, d\varphi^2) + \frac{2M}{r}\, (d\bar{t} + dr)^2\,. \tag{2.45}$$

Comparing (2.36) and (2.45) and remembering

$$du = \frac{(dr + dt)}{\sqrt{2}}\,, \tag{2.46}$$

we find $A = 2M$ and the metric (2.36) becomes

$$g = du \otimes dv + dv \otimes du + r^2(d\theta^2 + sin^2\theta\, d\varphi^2) + \frac{4M}{r} du^2\,. \tag{2.47}$$

3. COMPLEX COORDINATE TRANSFORMATION

The Schwarzschild metric written in standard coordinates

$$ds^2 = -(1 - \frac{2M}{r})\, dt^2 + (1 - \frac{2M}{r})^{-1}\, dr^2 + r^2\, (d\theta^2 + sin^2\theta\, d\varphi^2) \tag{3.1}$$

can be written in advanced Eddington-Finkelstein coordinates by the following transformation

$$dt = du - \frac{r}{r - 2M}\, dr \tag{3.2}$$

and in the new coordinates the metric becomes

$$ds^2 = -(1 - \frac{2M}{r})\, du^2 + 2du\, dr + r^2\, (d\theta^2 + sin^2\theta\, d\varphi^2)\,. \tag{3.3}$$

This metric is the advanced Eddington-Finkelstein form of the Schwarzschild solution and the surface $u = constant$ is a spherically symmetric null surface.
We can write this metric in complex null tetrad formalism

$$g = l \otimes n + n \otimes l + m \otimes m^* + m^* \otimes m \tag{3.4}$$

choosing the corresponding null tetrad as

$$\begin{aligned} l &= du\,, \\ n &= dr - \frac{1}{2}\left(1 - \frac{2M}{r}\right) du\,, \\ m &= \frac{r}{\sqrt{2}}\,(\, d\theta + isin\theta\, d\varphi)\,. \end{aligned} \tag{3.5}$$

The inverse metric can be defined using dual null tetrad

$$g^{-1} = \tilde{l} \otimes \tilde{n} + \tilde{n} \otimes \tilde{l} + \tilde{m} \otimes \tilde{m}^* + \tilde{m}^* \otimes \tilde{m}\,, \tag{3.6}$$

where $\tilde{l}$ $\tilde{n}$ and $\tilde{m}$ are duals of l, n, m vectors and given by

$$\begin{aligned} \tilde{l} &= \frac{\partial}{\partial r}\,, \\ \tilde{n} &= \frac{\partial}{\partial u} + \frac{1}{2}\left(1 - \frac{2M}{r}\right) \frac{\partial}{\partial r}\,, \\ \tilde{m} &= \frac{1}{\sqrt{2}r}\left(\frac{\partial}{\partial \theta} + \frac{i}{sin\theta}\,\frac{\partial}{\partial \varphi}\right). \end{aligned} \tag{3.7}$$

These vectors satisfy the duality relations

$$
\begin{aligned}
&l\,(\tilde{n}) = n\,(\tilde{l}) = 1\,,\\
&l\,(\tilde{l}) = n\,(\tilde{n}) = 0\,,\\
&m\,(\tilde{m}^*) = m^*\,(\tilde{m}) = 1\,,\\
&m\,(\tilde{m}) = m^*\,(\tilde{m}^*) = 0\,.
\end{aligned}
\tag{3.8}
$$

One can apply a complex coordinate transformation on these frame vectors and derive Kerr metric from Schwarzschild metric [2]. This method is used by Newman and Janis on contravariant components of the advanced Eddington-Finkelstein form of the Schwarzschild metric and the Kerr metric is obtained. To try this method on coframe vectors l, n and m, we will use the same complex transformation

$$
\begin{aligned}
&u = u' + ia\,cos\theta'\,,\\
&r = r' + ia\,cos\theta'\,,\\
&\varphi = \varphi'\,, \quad \theta = \theta'\,.
\end{aligned}
\tag{3.9}
$$

Since we let r take complex values, the new null tetrad becomes

$$
\begin{aligned}
l &= du\,,\\
n &= dr - \frac{1}{2}\left[1 - M\Big(\frac{1}{r} + \frac{1}{r^*}\Big)\right] du\,,\\
m &= \frac{r}{\sqrt{2}}\,(\,d\theta + isin\theta\,d\varphi)\,.
\end{aligned}
\tag{3.10}
$$

After applying (3.9) and dropping the primes they become

$$
\begin{aligned}
l &= du - ia\,sin\theta\,d\theta\,,\\
n &= dr - ia\,sin\theta\,d\theta - \frac{1}{2}\Big(1 - \frac{2M\,r}{r^2 + cos^2\theta}\Big)\,(\,du - ia\,sin\theta\,d\theta\,)\,,\\
m &= \frac{r + ia\,cos\theta}{\sqrt{2}}\,(\,d\theta + isin\theta\,d\varphi)\,.
\end{aligned}
\tag{3.11}
$$

To make the metric stay real after this complex transformation we can use

$$
g = l^* \otimes n + n^* \otimes l + m \otimes m^* + m^* \otimes m\,, \tag{3.12}
$$

which results in

$$
\begin{aligned}
ds^2 &= -\Big(\,1 - \frac{2Mr}{\Sigma}\,\Big)\,du^2 + 2du\,dr - \Big(\,1 + \frac{2Mr}{\Sigma}\,\Big)\,a^2\,sin^2\theta\,d\theta^2\\
&+ \;\Sigma\,(\,d\theta^2 + sin^2\theta\,d\varphi^2)\,,
\end{aligned}
\tag{3.13}
$$

where

$$\Sigma = r^2 + a^2\, cos^2\theta\,.$$

But this metric does not satisfy the Einstein equations. On the contrary, the metric obtained by Newman-Janis method which applies the complex coordinate transformation onto the contravariant components of the metric, satisfies the Einstein equations. We show it by the following procedure.
Starting from the metric (3.3), the non-zero inverse metric components are

$$\begin{aligned} g^{01} &= 1\,, \quad g^{11} = \left(1 - \frac{2M}{r}\right), \\ g^{22} &= \frac{1}{r^2}\,, \quad g^{33} = \frac{1}{r^2\, sin^2\theta}\,. \end{aligned} \tag{3.14}$$

They can be written in the complex null tetrad system (3.6) where $\tilde{l}$, $\tilde{m}$ and $\tilde{n}$ are the following vectors

$$\begin{aligned} \tilde{l} &= \partial_r\,, \\ \tilde{n} &= \partial_u + \frac{1}{2}\left(1 - \frac{2M}{r}\right)\partial_r\,, \\ \tilde{m} &= \frac{1}{\sqrt{2}r}\,(\partial_\theta + \frac{i}{sin\theta}\,\partial_\varphi)\,, \end{aligned} \tag{3.15}$$

where $\partial_{x^\mu} = \partial/\partial x^\mu$.
If we let r take complex values, the new vectors become

$$\begin{aligned} \tilde{l} &= \partial_r\,, \\ \tilde{n} &= \partial_u + \frac{1}{2}\left[1 - M\left(\frac{1}{r} + \frac{1}{r^*}\right)\right]\partial_r\,, \\ \tilde{m} &= \frac{1}{\sqrt{2}r^*}\,(\partial_\theta + \frac{i}{sin\theta}\,\partial_\varphi)\,. \\ \tilde{m}^* &= \frac{1}{\sqrt{2}r}\,(\partial_\theta - \frac{i}{sin\theta}\,\partial_\varphi)\,, \end{aligned} \tag{3.16}$$

where the important thing is to keep $\tilde{l}$ and $\tilde{n}$ real while $\tilde{m}$ and $\tilde{m}^*$ are complex conjugate of each other.
Then the following transformation is applied

$$\begin{aligned} u' &= u - ia\, cos\theta\,, \\ r' &= r - ia\, cos\theta\,, \\ \phi' &= \phi\,, \quad \theta' = \theta\,. \end{aligned} \tag{3.17}$$

After dropping the primes, the new null tetrad becomes

$$\begin{aligned}
\tilde{l} &= \partial_r \,, \\
\tilde{n} &= \partial_u + \frac{1}{2}\left(1 - \frac{2M\,r}{r^2 + a^2\,cos^2\theta}\right)\partial_r \,, \\
\tilde{m} &= \frac{1}{\sqrt{2}(r - ia\,cos\theta)}\left[ia\,sin\theta\,(\partial_u + \partial_r) + \partial_\theta + \frac{i}{sin\theta}\,\partial_\varphi\right] .
\end{aligned} \tag{3.18}$$

Then using the definition

$$g^{-1} = \tilde{l} \otimes \tilde{n} + \tilde{n} \otimes \tilde{l} + \tilde{m} \otimes \tilde{m}^* + \tilde{m}^* \otimes \tilde{m} \,, \tag{3.19}$$

the new non-zero contravariant metric components are read as

$$\begin{aligned}
g^{00} &= \frac{a^2\,sin^2\theta}{r^2 + a^2\,cos^2\theta}\,, \quad g^{01} = 1 + \frac{a^2\,sin^2\theta}{r^2 + a^2\,cos^2\theta}\,, \\
g^{03} &= \frac{a}{r^2 + a^2\,cos^2\theta}\,, \quad g^{11} = 1 - \frac{2M\,r}{r^2 + a^2\,cos^2\theta} + \frac{a^2\,sin^2\theta}{r^2 + a^2\,cos^2\theta}\,, \\
g^{13} &= \frac{a}{r^2 + a^2\,cos^2\theta}\,, \quad g^{22} = \frac{1}{r^2 + a^2\,cos^2\theta}\,, \quad g^{33} = \frac{1}{(r^2 + a^2\,cos^2\theta)\,sin^2\theta} .
\end{aligned} \tag{3.20}$$

Therefore the metric becomes

$$\begin{aligned}
ds^2 &= -(1 - \frac{2M\,r}{\Sigma})\,du^2 + 2du\,dr - \frac{4a\,M\,r\,sin^2\theta}{\Sigma}\,du\,d\varphi - 2a\sin^2\theta\,dr\,d\varphi \\
&\quad + \Sigma\,d\theta^2 + sin^2\theta\,(r^2 + a^2 + \frac{2M\,ra^2\,sin^2\theta}{\Sigma})\,d\varphi^2 \,,
\end{aligned} \tag{3.21}$$

where

$$\Sigma = r^2 + a^2\,cos^2\theta \,.$$

The corresponding null basis one forms are found to be

$$\begin{aligned}
l &= du - a\,sin^2\theta\,d\varphi \,, \\
n &= dr - \frac{1}{2}\left(1 - \frac{2M\,r}{r^2 + cos^2\theta}\right)du - \frac{a\,sin^2\theta}{2}\left(1 + \frac{2M\,r}{r^2 + cos^2\theta}\right)d\varphi \,, \\
m &= \frac{(r + ia\,cos\theta)}{\sqrt{2}}\,(\,d\theta + isin\theta\,d\varphi) .
\end{aligned} \tag{3.22}$$

These are dual of frame vectors (3.18) and they satisfy the duality relations (3.8) , but can not be obtained by applying complex transformations (3.9) on null basis one forms (3.10).

The metric (3.21) is called advanced Eddington-Finkelstein form of the Kerr solution. To write this metric in Boyer-Lindquist form, where only the $dt\, d\varphi$ off-diagonal terms remain, one should do the following coordinate transformation

$$\begin{aligned} du &= dt + \frac{(r^2+a^2)}{\Delta}\, dr\,, \\ d\varphi &= d\phi + \frac{a}{\Delta}\, dr\,, \end{aligned} \tag{3.23}$$

where

$$\Delta = r^2 + a^2 - 2M\, r\,.$$

So we get the Kerr metric in Boyer-Lindquist form

$$\begin{aligned} ds^2 &= -(\,1 - \frac{2M\,r}{\Sigma})\, dt^2 - \frac{4M\,r\,a\,sin^2\theta}{\Sigma}\, dt\, d\phi\, + \Sigma\, d\theta^2 + \frac{\Sigma}{\Delta}\, dr^2 \\ &+ \; sin^2\theta\, (\, r^2 + a^2 + \frac{2M\,r\,a^2\,sin^2\theta}{\Sigma})\, d\phi^2\,. \end{aligned} \tag{3.24}$$

We can further write the metric (3.21) which is driven by a complex transformation, in the Kerr-Schild form by writing it in the following way

$$\begin{aligned} ds^2 &= -du^2 + 2du\, dr - 2a\sin^2\theta\, dr\, d\varphi + (\, r^2 + a^2\,)\, sin^2\theta\, d\varphi^2 \\ &+ \; \Sigma\, d\theta^2 + \frac{2M\,r}{\Sigma}\, (\, du - a\, sin^2\theta\, d\varphi\,)^2\,. \end{aligned} \tag{3.25}$$

This form is similar to Kerr-Schild form

$$g_{\alpha\,\beta} = \eta_{\alpha\,\beta} + H\, l_\alpha\, l_\beta\,, \tag{3.26}$$

where H is a scalar function and l_α is a null vector both with respect to $g_{\alpha\,\beta}$ and $\eta_{\alpha\,\beta}$

$$g_{\alpha\,\beta}\, l^\alpha\, l^\beta = \eta_{\alpha\,\beta}\, l^\alpha\, l^\beta = 0\,,$$

and it is also geodesic both with respect to $g_{\alpha\,\beta}$ and $\eta_{\alpha\,\beta}$

$$l^\alpha\, \partial_\alpha\, l_\beta = l^\alpha\, \nabla_\alpha\, l_\beta = 0\,.$$

The advantage of writing the metric in Kerr-Schild form is that we can divide it in two parts and can get advantage of null geodesics properties. To obtain the standard Minkowski flat metric in the first part of (3.25) the following transformations should be applied

$$
\begin{aligned}
x &= r\,cos\varphi\,sin\theta - a\,sin\theta\,sin\varphi\,, \\
y &= r\,sin\varphi\,sin\theta + a\,sin\theta\,sin\varphi\,, \qquad (3.27)\\
z &= r\,cos\theta\,, \\
t &= u - r\,.
\end{aligned}
$$

Then the following equation is obtained

$$-du^2+2du\,dr-2a\sin^2\theta\,dr\,d\varphi+(\,r^2+a^2\,)\,sin^2\theta\,d\varphi^2+\Sigma\,d\theta^2 = -dt^2+dx^2+dy^2+dz^2. \tag{3.28}$$

For the remaining part of (3.25),

$$\frac{2\,M\,r}{\Sigma}\,(\,du - a\,sin^2\theta\,d\varphi\,)^2\,, \tag{3.29}$$

we can make use of null geodesics while transforming it to the new coordinates since it is similar to

$$\frac{2M\,r}{\Sigma}\,l_\alpha\,l_\beta\,. \tag{3.30}$$

From the contravariant metric components stated in (3.20) we see that the only nonvanishing component of l^α is $l^r = -1$, so we find

$$-l_\nu = \{1, 0, 0, -a\,sin\theta^2\}\,. \tag{3.31}$$

Since the only nonvanishing component of the tangent vector is $l^r = -1$ so the null geodesics move with constant values of u, θ and φ, then we have

$$
\begin{aligned}
dx &= cos\varphi\,sin\theta\,dr\,, \\
dy &= sin\varphi\,sin\theta\,dr\,, \qquad (3.32)\\
dz &= cos\theta\,dr, \\
dt &= -dr\,.
\end{aligned}
$$

Using vector transformation for l^ν

$$l'^\nu = \frac{\partial x'^\nu}{\partial x^\mu}\,l^\mu\,, \tag{3.33}$$

and lowering the indices, we find that

$$-l_\nu = \left\{ 1, \, \frac{r\,x + a\,y}{r^2 + a^2}, \, \frac{r\,y - a\,x}{r^2 + a^2}, \, \frac{z}{r} \right\}. \tag{3.34}$$

Finally we end up with the Kerr metric in Kerr-Schild form

$$ds^2 = \eta_{\mu\nu} + \frac{2M\,r^3}{r^4 + a^2\,z^2} \left(dt + \frac{r\,x + a\,y}{r^2 + a^2}\,dx + \frac{r\,y - a\,x}{r^2 + a^2}\,dy + \frac{z}{r}\,dz \right)^2, \tag{3.35}$$

where $\eta_{\mu\nu}$ is the standard Minkowski metric signed as (-, +, +, +) . Also it should be noted that r is defined by

$$r^4 - (\,x^2 + y^2 + z^2 - a^2\,)\,r^2 - a^2\,z^2 = 0\,.$$

The metric (3.35) is in the form which Kerr originally discovered it in his paper using null tetrad formalism [1] .

As a result we can say that this method works only on contravariant components of the advanced Eddington-Finkelstein form of the Schwarzschild metric and applying the complex coordinate transformation on the null basis one forms does not give a solution to Einstein equations.

4. KERR SOLUTION DERIVED FROM SCHWARZSCHILD SOLUTION IN N+1 DIMENSIONS

To find the Kerr solution in $N+1$ dimensional space-time, which has a single rotation parameter, by a complex transformation from Schwarzschild metric we first derive the $N+1$ dimensional static spherically symmetric solution.

In $N+1$ dimensional spacetime, general metric for a spherically symmetric and static spacetime is given by

$$\begin{aligned} ds^2 &= -e^{2\mu(r)}\, dt^2 + e^{-2\mu(r)} dr^2 + r^2 (d\theta_1^2 + sin^2\theta_1\, d\theta_2^2 \\ &+ sin^2\theta_1\, sin^2\theta_2\, d\theta_3^2 + \ldots + sin^2\theta_1 \,\ldots\, sin^2\theta_{N-2}\, d\theta_{N-1}^2)\,. \end{aligned} \tag{4.1}$$

To determine the arbitrary function $e^{2\mu(r)}$ we use differential forms starting by defining basis one forms as

$$\begin{aligned} &e^0 = e^{\mu(r)} dt\,, \\ &e^1 = e^{-\mu(r)} dr\,, \\ &e^2 = r\, d\theta_1\,, \\ &e^3 = r\, sin\theta_1\, d\theta_2\,, \\ &e^4 = r \sin\theta_1\, sin\theta_2\, d\theta_3\,, \\ &e^5 = r \sin\theta_1\, sin\theta_2\, sin\theta_3\, d\theta_4\,, \\ &\ldots \quad \ldots \\ &\ldots \quad \ldots\ldots \\ &e^N = r \sin\theta_1\, sin\theta_2 \,\ldots\, sin\theta_{N-2}\, d\theta_{N-1}\,. \end{aligned} \tag{4.2}$$

d derivatives of these basis one forms are

$$
\begin{aligned}
de^0 &= -\mu' e^\mu \, e^0 \wedge e^1 \,, \\
de^1 &= 0 \,, \\
de^2 &= \frac{e^\mu}{r} \, e^1 \wedge e^2 \,, \\
de^3 &= \frac{e^\mu}{r} \, e^1 \wedge e^3 + \frac{cot\theta_1}{r} \, e^2 \wedge e^3 \,, \\
de^4 &= \frac{e^\mu}{r} \, e^1 \wedge e^4 + \frac{cot\theta_1}{r} \, e^2 \wedge e^4 + \frac{cot\theta_2}{r \, sin\theta_1} \, e^3 \wedge e^4 \,, \\
de^5 &= \frac{e^\mu}{r} \, e^1 \wedge e^5 + \frac{cot\theta_1}{r} \, e^2 \wedge e^5 \ + \frac{cot\theta_2}{r \, sin\theta_1} \, e^3 \wedge e^5 + \frac{cot\theta_3}{r \, sin\theta_1 \, sin\theta_2} \, e^4 \wedge e^5 \,, \\
\ldots & \quad \ldots \\
\ldots & \quad \ldots\ldots \\
de^N &= \frac{e^\mu}{r} \, e^1 \wedge e^N + \frac{cot\theta_1}{r} \, e^2 \wedge e^N + \ldots + \frac{cot\theta_{N-2}}{r \, sin\theta_1 \, sin\theta_2 \ldots \, sin\theta_{N-3}} \, e^{N-1} \wedge e^N \,,
\end{aligned}
\tag{4.3}
$$

where $'$ denotes the derivative with respect to r .

The first Cartan equations are

$$
\begin{aligned}
de^0 + w^0_1 \wedge e^1 + w^0_2 \wedge e^2 + \ldots + w^0_N \wedge e^N &= 0 \,, \\
de^1 + w^1_0 \wedge e^0 + w^1_2 \wedge e^2 + \ldots + w^1_N \wedge e^N &= 0 \,, \\
de^2 + w^2_0 \wedge e^0 + w^2_1 \wedge e^1 + \ldots + w^2_N \wedge e^N &= 0 \,, \\
\ldots \quad \ldots & \\
\ldots\ldots \quad \ldots\ldots & \\
de^N + w^N_0 \wedge e^0 + w^N_1 \wedge e^1 + \ldots + w^N_{N-1} \wedge e^{N-1} &= 0 \,.
\end{aligned}
\tag{4.4}
$$

By comparing the structure equations with d derivatives of basis one forms one can find the following connection one forms

$$
\begin{aligned}
w^0_1 &= \mu' e^\mu \, e^0 \,, \\
w^0_2 &= w^0_3 = \ldots\ldots = w^0_N = 0 \,, \\
w^1_2 &= -\frac{e^\mu}{r} \, e^2 \,, \ w^1_3 = -\frac{e^\mu}{r} \, e^3 \,, \ldots \,, \ w^1_N = -\frac{e^\mu}{r} \, e^N \,, \\
w^2_3 &= -\frac{cot\theta_1}{r} \, e^3 \,, \ w^2_4 = -\frac{cot\theta_1}{r} \, e^4 \,, \ldots \,, \ w^2_N = -\frac{cot\theta_1}{r} \, e^N \,, \\
w^3_4 &= -\frac{cot\theta_2}{r \, sin\theta_1} \, e^4 \,, \ w^3_5 = -\frac{cot\theta_2}{r \, sin\theta_1} \, e^5 \,, \ldots \,, \ w^3_N = -\frac{cot\theta_2}{r \sin\theta_1} \, e^N \,, \\
\ldots & \quad \ldots \\
\ldots & \quad \ldots\ldots \\
w^{N-1}_N &= -\frac{cot\theta_{N-2}}{r \, sin\theta_1 \, sin\theta_2 \, \ldots \, sin\theta_{N-3}} \, e^N \,.
\end{aligned}
\tag{4.5}
$$

Then using the second Cartan equations, the curvature two forms are found to be

$$\begin{aligned}
R^0_1 &= -(\,\mu'\,e^{2\mu})'\,e^0\wedge e^1\,,\\
R^2_3 &= \frac{1}{r^2}\,(\,1-e^{2\mu}\,)\,e^2\wedge e^3\,,\ldots\,,R^2_N=\frac{1}{r^2}\,(\,1-e^{2\mu}\,)\,e^2\wedge e^N\,,\ \ for\ N>2\\
R^0_2 &= -\frac{\mu'}{r}\,e^{2\mu}\,e^0\wedge e^2\,,\ldots\,,R^0_N=-\frac{\mu'}{r}\,e^{2\mu}\,e^0\wedge e^N\,,\ \ for\ N>1\\
R^1_2 &= -\frac{\mu'}{r}\,e^{2\mu}\,e^1\wedge e^3\,,\ldots\,,R^1_N=-\frac{\mu'}{r}\,e^{2\mu}\,e^1\wedge e^N\,,\ \ for\ N>2\,.
\end{aligned}\tag{4.6}$$

The corresponding Ricci one forms are

$$\begin{aligned}
(Ric)_0 &= l_1\,R^1_0+\underbrace{l_2R^2_0+l_3R^3_0+\ldots+l_NR^N_0}_{(n-1)\,times\,R^0_2}\\
&= (\,\mu'\,e^{2\mu})'\,e^0+(n-1)\,\frac{\mu'}{r}\,e^{2\mu}\,e^0=0\,, && (4.7)\\
(Ric)_1 &= l_0\,R^0_1+\underbrace{l_2R^2_1+l_3R^3_1+\ldots+l_NR^N_1}_{(n-1)\,times\,R^1_2}\\
&= -(\,\mu'\,e^{2\mu})'\,e^1-(n-1)\,\frac{\mu'}{r}\,e^{2\mu}\,e^1=0\,, && (4.8)\\
(Ric)_2 &= l_0\,R^0_2+l_1R^1_2+\underbrace{l_3R^3_2+\ldots+l_NR^N_2}_{(n-2)\,times\,R^2_3}\\
&= -\frac{2\mu'}{r}\,e^{2\mu}\,e^2+\frac{(n-2)}{r^2}\,(\,1-e^{2\,\mu})\,e^2=0\,, && (4.9)\\
(Ric)_3 &= l_0\,R^0_3+l_1R^1_3+\underbrace{l_2R^2_3+\ldots+l_NR^N_3}_{(n-2)\,times\,R^2_3}\\
&= -\frac{2\mu'}{r}\,e^{2\mu}\,e^3+\frac{(n-2)}{r^2}\,(\,1-e^{2\,\mu})\,e^3=0\,. && (4.10)
\end{aligned}$$

From $(Ric)_0$ and $(Ric)_2$ components we have

$$(\,\mu'\,e^{2\mu})'+(n-1)\,\frac{\mu'}{r}\,e^{2\mu}=0\,,$$

$$-\frac{2\mu'}{r}\,e^{2\mu}+\frac{(n-2)}{r^2}\,(\,1-e^{2\,\mu})=0\,.$$

These imply

$$(\,\mu'\,e^{2\mu})'=\frac{(n-2)\,(n-1)}{2r^2}\,(e^{2\mu}-1)\,.\tag{4.11}$$

If we now choose

$$e^{2\mu}=\left(1+\frac{A}{r^{(N-2)}}\right)\,,$$

taking the logarithm of both sides and derivation with respect to r it satisfies the related equations.

The metric becomes

$$\begin{aligned} ds^2 &= -\left(1+\frac{A}{r^{N-2}}\right) dt^2 + \left(1+\frac{A}{r^{N-2}}\right)^{-1} dr^2 + r^2 (d\theta_1^2 + sin^2\theta_1\, d\theta_2^2 \\ &+ sin^2\theta_1\, sin^2\theta_2\, d\theta_3^2 + ... + sin^2\theta_1 .... sin^2\theta_{N-2}\, d\theta_{N-1}^2)\,, \end{aligned} \tag{4.12}$$

by the transformation $sin^2\theta_1\, sin^2\theta_2 = cos^2\theta_1$ it can be written as

$$\begin{aligned} ds^2 &= -\left(1+\frac{A}{r^{N-2}}\right) dt^2 + \left(1+\frac{A}{r^{N-2}}\right)^{-1} dr^2 + r^2 (d\theta_1^2 + sin^2\theta_1\, d\theta_2^2) \\ &+ r^2\, cos^2\theta_1\, d\Omega_{N-3}^2\,, \end{aligned} \tag{4.13}$$

where

$$d\Omega_{N-3}^2 = d\theta_3^2 + sin^2\theta_3\, d\theta_4^2 + ... + sin^2\theta_3 \,...sin^2\theta_{N-2}\, d\theta_{N-1}^2\,. \tag{4.14}$$

The arbitrary constant A can be determined by using Komar integral in $N+1$ dimensions [10, 12],

$$m = \frac{1}{(N-2)\, A_{N-1}} \int\limits_{\mathbf{S}_{N-1}} \xi_{(\mathbf{t})}^{\mu;\,\nu}\, d^{N-1}\mathbf{\Sigma}_{\mu\,\nu}\,, \tag{4.15}$$

and the relation between m and the total mass of black hole M is given by

$$m = \frac{16\pi\, M}{(N-1)\, A_{N-1}}\,. \tag{4.16}$$

Here A_{N-1} is the area of unit $(N-1)$-sphere and

$$d^{N-1}\Sigma_{\mu\,\nu} = \frac{1}{(N-1)!}\,\hat{\epsilon}_{\mu\,\nu\alpha_1\,\alpha_2...\alpha_{N-1}}\, dx^{\alpha_1}\wedge dx^{\alpha_2}\wedge\ldots\wedge dx^{\alpha_{N-1}}$$

again

$$\hat{\epsilon}_{\mu\,\nu\alpha_1\,\alpha_2...\alpha_{N-1}} = \sqrt{-g}\,\epsilon_{\mu\,\nu\alpha_1\,\alpha_2...\alpha_{N-1}}\,.$$

To find g for this metric

$$\begin{aligned} N=3 \rightarrow &\qquad \sqrt{-g} = r^2\, sin\theta_1\,, \\ N=4 \rightarrow &\qquad \sqrt{-g} = r^3\, sin\theta_1\, cos\theta_1\,, \\ N=5 \rightarrow &\qquad \sqrt{-g} = r^4\, sin\theta_1\, cos\theta_1^2\, sin\theta_3\,, \\ N=6 \rightarrow &\qquad \sqrt{-g} = r^5\, sin\theta_1\, cos\theta_1^3\, sin\theta_3^2\, sin\theta_4\,, \\ \cdots &\qquad \cdots \\ N=N+1 \rightarrow &\qquad \sqrt{-g} = r^{N-1}\, sin\theta_1\, cos\theta_1^{N-3}\,\sqrt{\chi}\,, \end{aligned} \tag{4.17}$$

where χ is the determinant of the metric $d\Omega^2_{N-3}$.
Again here $\xi^{\mu}_{(t)}$ is the timelike Killing vector field

$$\xi^{\mu}_{(t)} = (\,1, 0, 0, 0, \dots\,, 0)$$

and

$$\xi^{0\,;\,1} = g^{11}\,\Gamma^{0}_{10} = -\frac{A(N-2)}{2\,r^{N-1}}\,.$$

Inserting these into (4.15)

$$m = \frac{2}{(N-2)\,A_{N-1}} \int \xi^{\mathbf{0;1}}_{\mathbf{(t)}}\, d^{N-1}\mathbf{\Sigma}_{0\,1}\,, \tag{4.18}$$

which becomes

$$m = \frac{-A}{A_{N-1}} \underbrace{\int\; sin\theta_1\, cos\theta_1^{N-3}\,\sqrt{\chi}\; d\theta_1\, d\theta_2\, \dots\, d\theta_{N-1}}_{A_{N-1}}\,. \tag{4.19}$$

So the unknown constant is found as

$$A = -m\,.$$

Therefore the $N+1$ dimensional Schwarzschild metric reads

$$\begin{aligned} ds^2 \;&=\; -\left(1-\frac{m}{r^{N-2}}\right)\,dt^2 + \left(1-\frac{m}{r^{N-2}}\right)^{-1} dr^2 + r^2(d\theta_1^2 + sin^2\theta_1\, d\theta_2^2 \\ &+\; sin^2\theta_1\, sin^2\theta_2\, d\theta_3^2 + ... + sin^2\theta_1 ....sin^2\theta_{N-2}\, d\theta^2_{N-1})\,. \end{aligned} \tag{4.20}$$

For $N = 3$ using (4.16) it can be seen that $m = 2M$, so it gives the four dimensional Schwarzschild metric (2.43).
We can start to apply NJ method on this metric. To write Schwarzschild metric in advanced EF coordinates the following transformation is applied

$$dt = du - \frac{r^{N-2}}{r^{N-2} - m}\, dr \tag{4.21}$$

and in the new coordinates the metric becomes

$$\begin{aligned} ds^2 \;&=\; -(1-\frac{m}{r^{N-2}})\, du^2 + 2du\, dr + r^2\,(d\theta_1^2 + sin^2\theta_1\, d\theta_2^2 \\ &+\; sin^2\theta_1\, sin^2\theta_2\, d\theta_3^2 + ... + sin^2\theta_1 ....sin^2\theta_{N-2}\, d\theta^2_{N-1})\,. \end{aligned} \tag{4.22}$$

The non-zero inverse metric components are

$$
\begin{aligned}
g^{01} &= 1\,, \quad g^{11} = (1 - \frac{m}{r^{N-2}}), \\
g^{22} &= \frac{1}{r^2}\,, \quad g^{33} = \frac{1}{r^2\, sin^2\theta_1}\,, \\
g^{44} &= \frac{1}{r^2\, sin^2\theta_1\, sin^2\theta_2}\,, \quad g^{55} = \frac{1}{r^2\, sin^2\theta_1\, sin^2\theta_2\, sin^2\theta_3}\,, \\
.... &\quad .... \\
.... &\quad .... \\
g^{N-1\,N-1} &= \frac{1}{r^2\, sin^2\theta_1 ...\, sin^2\theta_{N-3}}\,, \quad g^{N\,N} = \frac{1}{r^2\, sin^2\theta_1 ...\, sin^2\theta_{N-2}}\,,
\end{aligned}
\tag{4.23}
$$

which can be written in the following complex null form

$$
\begin{aligned}
g^{-1} &= \tilde{l} \otimes \tilde{n} + \tilde{n} \otimes \tilde{l} + \tilde{m}_1 \otimes \tilde{m}_1{}^* + \tilde{m}_1{}^* \otimes \tilde{m}_1 + \ldots\ldots \\
&+ \tilde{m}_{(N-1)/2} \otimes \tilde{m}^*_{(N-1)/2} + \tilde{m}^*_{(N-1)/2} \otimes \tilde{m}_{(N-1)/2}\,,
\end{aligned}
\tag{4.24}
$$

where we have assumed $N+1$ is even.

Corresponding null vectors are

$$
\begin{aligned}
\tilde{l} &= \partial_r\,, \\
\tilde{n} &= \partial_u + \frac{1}{2}\left(1 - \frac{m}{r^{N-2}}\right)\partial_r\,, \\
\tilde{m}_1 &= \frac{1}{\sqrt{2}r}\,(\partial_{\theta_1} + \frac{i}{sin\theta_1}\,\partial_{\theta_2})\,, \\
\tilde{m}_2 &= \frac{1}{\sqrt{2}r\, sin\theta_1\, sin\theta_2}\,(\partial_{\theta_3} + \frac{i}{sin\theta_3}\,\partial_{\theta_4})\,, \\
.... &\quad .... \\
.... &\quad .... \\
\tilde{m}_{(N-1)/2} &= \frac{1}{\sqrt{2}r\, sin\theta_1 ...\, sin\theta_{N-3}}\left(\partial_{\theta_{N-2}} + \frac{i}{sin\theta_{N-2}}\,\partial_{\theta_{N-1}}\right).
\end{aligned}
\tag{4.25}
$$

If we let r take complex values, the new vectors become

$$\begin{aligned}
\tilde{l} &= \partial_r \,, \\
\tilde{n} &= \partial_u + \frac{1}{2}\left(1 - \frac{m}{r^{N-4}\, r\, r^*}\right)\partial_r \,, \\
\tilde{m}_1 &= \frac{1}{\sqrt{2}r^*}\,(\partial_{\theta_1} + \frac{i}{sin\theta_2}\,\partial_{\theta_2})\,, \\
\tilde{m}_2 &= \frac{1}{\sqrt{2}r\; sin\theta_1\; sin\theta_2}\,(\partial_{\theta_3} + \frac{i}{sin\theta_3}\,\partial_{\theta_4})\,, \\
.... &\quad .... \\
.... &\quad .... \\
\tilde{m}_{(N-1)/2} &= \frac{1}{\sqrt{2}r\; sin\theta_1 ...\; sin\theta_{N-3}}\,\big(\partial_{\theta_{N-2}} + \frac{i}{sin\theta_{N-2}}\,\partial_{\theta_{N-1}}\big)\,.
\end{aligned} \tag{4.26}$$

The important thing here is to obtain a solution satisfying Einstein equations with a complexification that applies only on l, n and m_1 while other null vector pairs remain unchanged.

If the following transformation is applied

$$\begin{aligned}
&u' = u - ia\, cos\theta_1 \,, \\
&r' = r - ia\, cos\theta_1 \,, \\
&\theta_1' = \theta_1 \,, \quad \theta_2' = \theta_2 \,, \;\; .... \,, \;\; \theta_{N-1}' = \theta_{N-1}
\end{aligned} \tag{4.27}$$

and the primes are dropped, the new null vectors became

$$\begin{aligned}
\tilde{l} &= \partial_r \,, \\
\tilde{n} &= \partial_u + \frac{1}{2}\left(1 - \frac{m}{r^{N-4}\,(r^2 + a^2\, cos^2\theta_1)}\right)\partial_r \,, \\
\tilde{m}_1 &= \frac{1}{\sqrt{2}(r - ia\, cos\theta_1)}\left(ia\, sin\theta_1\,(\partial_u + \partial_r) + \partial_{\theta_1} + \frac{i}{sin\theta_1}\,\partial_{\theta_2}\right), \\
\tilde{m}_2 &= \frac{1}{\sqrt{2}r\; sin\theta_1\; sin\theta_2}\,(\partial_{\theta_3} + \frac{i}{sin\theta_3}\,\partial_{\theta_4})\,, \\
.... &\quad .... \\
.... &\quad .... \\
\tilde{m}_{(N-1)/2} &= \frac{1}{\sqrt{2}r\; sin\theta_1 ...\; sin\theta_{N-3}}\,\big(\partial_{\theta_{N-2}} + \frac{i}{sin\theta_{N-2}}\,\partial_{\theta_{N-1}}\big)\,.
\end{aligned} \tag{4.28}$$

Using (4.24) for the inverse components of the metric, the new non-zero contravariant metric components are found as

$$\begin{aligned}
g^{00} &= \frac{a^2\, sin^2\theta_1}{\Sigma}\,, \quad g^{01} = 1 + \frac{a^2\, sin^2\theta_1}{\Sigma}\,, \\
g^{03} &= \frac{a}{\Sigma}\,, \quad g^{11} = 1 - \frac{m}{r^{N-4}\,\Sigma} + \frac{a^2\, sin^2\theta_1}{\Sigma}\,, \\
g^{13} &= \frac{a}{\Sigma}\,, \quad g^{22} = \frac{1}{\Sigma}\,, \quad g^{33} = \frac{1}{\Sigma\, sin^2\theta_1}\,, \\
g^{44} &= \frac{1}{r^2\, cos^2\theta_1}\,, \quad .....\,, g^{NN} = \frac{1}{r^2\, cos^2\theta_1\, sin^2\theta_3 \ldots sin^2\theta_{N-2}}\,,
\end{aligned} \tag{4.29}$$

where again the transformation $sin^2\theta_1\, sin^2\theta_2 = cos^2\theta_1$ is done and

$$\Sigma = r^2 + a^2\, cos^2\theta_1\,.$$

Finally the metric reads

$$\begin{aligned}
ds^2 &= -(1 - \frac{m}{r^{N-4}\Sigma})\, du^2 + 2du\, dr - \frac{2a\, m\, sin^2\theta_1}{r^{N-4}\Sigma}\, du\, d\theta_2 - 2a \sin^2\theta_1\, dr\, d\theta_2 \\
&+ \Sigma\, d\theta_1^2 + sin^2\theta_1\, (r^2 + a^2 + \frac{m\, a^2\, sin^2\theta_1}{r^{N-4}\Sigma})\, d\theta_2^2 + r^2\, cos^2\theta_1\, d\Omega_{N-3}^2\,.
\end{aligned} \tag{4.30}$$

To write this metric in Boyer-Lindquist form, one should do the following coordinate transformation

$$\begin{aligned}
du &= dt + \frac{(r^2 + a^2)}{\Delta}\, dr\,, \\
d\theta_2 &= d\theta_2' + \frac{a}{\Delta}\, dr\,,
\end{aligned} \tag{4.31}$$

where

$$\Delta = r^2 + a^2 - m\, r^{4-N}.$$

So dropping the primes we get the Kerr metric in BL form

$$\begin{aligned}
ds^2 &= -\left(1 - \frac{m}{r^{N-4}\Sigma}\right) dt^2 - \frac{2m\, a\, sin^2\theta_1}{r^{N-4}\Sigma}\, dt\, d\theta_2 + \Sigma\, d\theta_1^2 + \frac{\Sigma}{\Delta}\, dr^2 \\
&+ sin^2\theta_1 \left(r^2 + a^2 + \frac{m\, a^2\, sin^2\theta}{r^{N-4}\Sigma}\right) d\theta_2^2 + r^2\, cos^2\theta_1\, d\Omega_{N-3}^2\,.
\end{aligned} \tag{4.32}$$

To summarize, we started from $N+1$ dimensional Schwarzschild metric and by performing a single complex transformation on it, we obtained the $N+1$ dimensional Kerr metric with only one rotation parameter.

5. DISCUSSION

In this thesis, the concept of differential forms in general relativity is studied. Since the metric is written in an orthonormal basis which are not related to any chosen coordinate system, this method is very powerful. Furthermore we introduced the complex null-tetrad formalism that is a very efficient form of the metric to find solutions to Einstein equations.

If a metric can be written in Kerr-Schild form, then one can apply the complex coordinate transformation method which is applicable to these kind of metrics. In 1962, Trautman gave a generalized procedure to get new solutions starting from given ones by a complex transformation. Later, Newman and Janis derived the Kerr and Kerr- Newman solutions by applying complex coordinate transformations on the contravariant components of the Schwarzschild and Reissner-Nordström metrics written in null tetrad formalism.

A completely different approach with the same results as Newman-Janis was also studied [7] where a complex transformation is performed on the solution so there is no arbitrariness in this method.

After introducing differential forms in general relativity, spherically symmetric black hole solutions are studied in the Chapter 2 of this thesis. Then after introducing complex null-tetrad formalism, complex coordinate transformation on the null basis 1-forms has been tried, which gives no solution to Einstein equations. On the other hand the same complex coordinate transformation applied on null tetrad vectors as was first done by NJ gives a new solution to Einstein equations. To understand this better, complex gravity studies can be done as a future work.

In the Chapter 4 a very specific Kerr metric with a single rotation parameter in $N+1$ dimensional spacetime is obtained by applying NJ complex coordinate transformation method. This solution is a special case of known solutions [10, 11] .

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BIOGRAPHY

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