

Kerr-Schild Tetrads and the Nijenhuis Tensor

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Abstract

We write the Kerr-Schild tetrads in terms of the flat space-time tetrads and of a (1,1) tensor S_μ^λ . This tensor can be considered as a projection operator, since it transforms (i) flat space-time tetrads into non-flat tetrads, and vice-versa, and (ii) the Minkowski space-time metric tensor into a non-flat metric tensor, and vice-versa. The S_μ^λ tensor and its inverse are constructed in terms of the standard null vector field l_μ that defines the Kerr-Schild form of the metric tensor in general relativity, and that yields black holes and non-linear gravitational waves as solutions of the vacuum Einstein's field equations. We show that the condition for the vanishing of the Ricci tensor obtained by Kerr and Schild, in empty space-time, is also a condition for the vanishing of the Nijenhuis tensor constructed out of S_μ^λ . Thus, a theory based on the Nijenhuis tensor yields an important class of solutions of the Einstein's field equations, namely, black holes and non-linear gravitational waves.

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1 The Kerr-Schild form of the metric tensor

The Kerr-Schild form of the metric tensor [1] is a quite interesting construction that allows obtaining certain solutions of the vacuum Einstein's field equations. According to the book by Stephani *et. al.* [2], the Kerr-Schild ansatz was previously studied by Trautman [3]. It is given by

$$g_{\mu\nu} = \eta_{\mu\nu} + l_\mu l_\nu, \quad (1)$$

where $\eta_{\mu\nu}$ is the metric for the Minkowski space-time in any coordinate system, and l_μ is a null vector field. In general, the Minkowski metric tensor is taken be $\eta_{\mu\nu} = (-1, +1, +1, +1)$ in a Cartesian coordinate system (which, however, is not necessarily rectangular). The form of the metric tensor (1) allows one to easily transform covariant into contravariant components, and vice-versa, by means of the flat Minkowski metric tensor. The contravariant components of the metric tensor are given by [1]

$$g^{\lambda\mu} = \eta^{\lambda\mu} - l^\lambda l^\mu. \quad (2)$$

The vector l_μ is null with respect to both the metric tensors $g_{\mu\nu}$ and $\eta_{\mu\nu}$ [1], i.e.,

$$g^{\mu\nu} l_\mu l_\nu = \eta^{\mu\nu} l_\mu l_\nu = l^\mu l_\mu = 0, \quad (3)$$

where $l^\lambda = g^{\lambda\mu} l_\mu = \eta^{\lambda\mu} l_\mu$. In addition, the determinant of the metric tensor satisfies [1] $(-g) = -\det(g_{\mu\nu}) = 1$.

Two gravitational field configurations that are of great relevance in general relativity, black holes (Schwarzschild and Kerr) and non-linear gravitational waves (pp-waves), may be described by means of the Kerr-Schild ansatz [4, 5]. These are vacuum solutions of Einstein's field equations. The conditions under which matter fields may be included in the framework of the Kerr-Schild ansatz (as for instance, the electromagnetic field) are discussed in Ref. [2] (see also Ref. [6]). In particular, the metric tensor in the form of Eqs. (1), (2) may be treated as *exact linear perturbations* of the Minkowski space-time [7, 8].

As noted by Kerr and Schild, the vacuum field equations $R_{\mu\nu} l^\mu l^\nu = 0$ yield [1]

$$(l^\nu \partial_\nu l_\mu)(l^\lambda \partial_\lambda l^\mu) = 0. \quad (4)$$

However, from the derivative of the null condition $l^\mu l_\mu = 0$ we obtain $l^\mu \partial_\lambda l_\mu = 0$, and consequently

$$l^\mu (l^\lambda \partial_\lambda l_\mu) = 0. \quad (5)$$

Thus, the vector $l^\lambda \partial_\lambda l_\mu$ is null, from Eq. (4), and is itself orthogonal to the null vector l^μ . Therefore we conclude that the vector $l^\lambda \partial_\lambda l_\mu$ besides being null, is collinear to the vector l_μ , i.e.,

$$l^\lambda \partial_\lambda l_\mu = \sigma l_\mu. \quad (6)$$

where σ is a multiplicative factor. We are using mostly the notation of Ref. [1], with partial derivatives and adopting Cartesian coordinates, but the passage to arbitrary coordinates (and the corresponding covariant derivatives in the flat space-time) is straightforward [9]. At this point, all indices may be raised and lowered by means of the flat metric tensor $\eta_{\mu\nu}$.

The equations above express the basics of the Kerr-Schild ansatz. In the following section we will construct the Kerr-Schild tetrads, and arrive at the (1,1) tensor S_μ^λ . Then in Section 3 we address the Nijenhuis tensor, and conclude that the latter vanishes provided Eq. (6) is satisfied, i.e., as long as the Ricci tensor vanishes, according to Eq. (4).

2 Kerr-Schild tetrads

The Kerr-Schild tetrads $e^a{}_\mu(x)$ that yield the metric tensors (1) and (2) are constructed by observing all issues of consistency. We first denote by $E^a{}_\mu$ the flat space-time tetrads (i.e., tetrads for the flat Minkowski space-time) in an arbitrary inertial state. The latin index a is a $\text{SO}(3,1)$ tangent space index. Given some matrix representation $\Lambda^a{}_b$ of the Lorentz Group, the tetrad fields $E^a{}_\mu$ transform as $\tilde{E}^a{}_\mu = \Lambda^a{}_b E^b{}_\mu$. We also denote the flat tangent space-time metric tensor as $\eta_{ab} = (-1, +1, +1, +1)$. Therefore, the starting point for the construction of the Kerr-Schild tetrads is the expression

$$e^a{}_\mu = E^a{}_\mu + \frac{1}{2} l^a l_\mu, \quad (7)$$

where

$$l^a = E^a{}_\mu l^\mu. \quad (8)$$

Tetrad fields of this type were recently considered in Ref. [10], in the context of teleparallel gravity. The flat space-time tetrad fields are required to satisfy

$$\eta_{ab} E^a{}_\mu E^b{}_\nu = \eta_{\mu\nu}. \quad (9)$$

Thus, it follows from Eqs. (8) and (9) that

$$\eta_{ab} l^a l^b = \eta_{\mu\nu} l^\mu l^\nu. \quad (10)$$

As a consequence of the expressions above, we have

$$\eta_{ab} e^a{}_\mu e^b{}_\nu = \eta_{\mu\nu} + l_\mu l_\nu = g_{\mu\nu}. \quad (11)$$

The inverse tetrads are defined by

$$e_b{}^\lambda = E_b{}^\lambda - \frac{1}{2} l_b l^\lambda. \quad (12)$$

where $l_b = E_b{}^\mu l_\mu$. It is important to note that all indices of both $E^a{}_\mu$ and $E_b{}^\lambda$ are raised and lowered by $\eta_{\mu\nu}$, η_{ab} and their inverses. The null vectors l^λ and l^a are related by $l^\lambda = E_a{}^\lambda l^a$. The vector l^a is also a null vector, $l^a l_a = 0$. This condition can be obtained directly from Eq. (10). The expressions so far obtained allow to verify the following orthogonality properties,

$$E_b{}^\lambda E^b{}_\mu = \delta_\mu^\lambda, \quad (13)$$

$$e_b{}^\lambda e^b{}_\mu = \delta_\mu^\lambda. \quad (14)$$

The relations below can be verified by simple, direct calculations and ensure the consistency and validity of the whole formulation presented here:

$$\eta^{ab} E_a{}^\mu E_b{}^\nu = \eta^{\mu\nu}, \quad (15)$$

$$\eta^{ab} e_a{}^\mu e_b{}^\nu = \eta^{\mu\nu} - l^\mu l^\nu = g^{\mu\nu}, \quad (16)$$

$$e^a{}_\mu e^b{}_\nu g^{\mu\nu} = \eta^{ab}. \quad (17)$$

Now we return to Eqs. (7) and (8) and observe that after simple rearrangements, the tetrad fields (7) may be first rewritten as

$$e^a{}_\mu = \frac{1}{2} E^a{}_\lambda (\delta_\mu^\lambda + \eta^{\lambda\sigma} g_{\sigma\mu}). \quad (18)$$

It is interesting to note, already at this point, (i) that the transformations properties of $e^a{}_\mu$ under Lorentz transformations are determined by the flat space-time tetrads $E^a{}_\mu$, and that (ii) the tetrad fields are written in terms of the metric tensors $\eta^{\lambda\mu}$ and $g_{\lambda\mu}$. The inverse tetrads may be rewritten in a similar form,

$$e_a{}^\mu = \frac{1}{2} E_a{}^\lambda (\delta_\lambda^\mu + \eta_{\lambda\rho} g^{\rho\mu}). \quad (19)$$

By means of straightforward manipulations, we again rewrite the tetrad field $e^a{}_\mu$ given by Eq. (18) in the form

$$e^a{}_\mu = E^a{}_\lambda S_\mu^\lambda, \quad (20)$$

where the (1,1) tensor S_μ^λ is defined by

$$S_\mu^\lambda = \delta_\mu^\lambda + \frac{1}{2} l^\lambda l_\mu. \quad (21)$$

It can be easily verified that the inverse of the tensor above is

$$(S^{-1})_\beta^\mu = \delta_\beta^\mu - \frac{1}{2} l^\mu l_\beta, \quad (22)$$

i.e., $S_\mu^\lambda (S^{-1})_\beta^\mu = \delta_\beta^\lambda$. In terms of the inverse tensor $(S^{-1})_\beta^\mu$, we may rewrite the inverse tetrads (19) in the form

$$e_a{}^\mu = E_a{}^\lambda (S^{-1})_\lambda^\mu. \quad (23)$$

The (1,1) tensors S_μ^λ and $(S^{-1})_\beta^\mu$ exhibit very interesting properties for both the metric tensor and for the tetrad fields. First, for the metric tensor, it can be verified by simple calculations the following relations,

$$S_\rho^\mu S_\sigma^\nu \eta_{\mu\nu} = g_{\rho\sigma}, \quad (24)$$

$$S_\rho^\mu S_\sigma^\nu g^{\rho\sigma} = \eta^{\mu\nu}, \quad (25)$$

$$(S^{-1})_\rho^\mu (S^{-1})_\sigma^\nu \eta^{\rho\sigma} = g^{\mu\nu}, \quad (26)$$

$$(S^{-1})_\rho^\mu (S^{-1})_\sigma^\nu g_{\mu\nu} = \eta_{\rho\sigma}. \quad (27)$$

As for the tetrad fields, besides equations (20) and (23) above, we have

$$E^a{}_\lambda = e^a{}_\mu (S^{-1})^\mu_\lambda, \quad (28)$$

$$E_a{}^\rho = e_a{}^\mu S_\mu^\rho. \quad (29)$$

In conclusion, we have the following rules, at least for the metric tensor and for the tetrad fields. For a given tensor S_λ^μ ,

- the contravariant index μ converts flat space-time quantities into non-flat quantities,
- the covariant index λ converts non-flat space-time quantities into flat quantities.

As for the inverse tensor $(S^{-1})_\beta^\alpha$, the opposite takes place,

- the contravariant index α converts non-flat quantities into flat quantities,
- the covariant index β converts flat space-time quantities into non-flat quantities.

The general conclusion of the present analysis is that a (1,1) tensor of the type S_λ^μ may play some role in gravity theories. On the other hand, such a (1,1) tensor plays a relevant role in the construction of the Nijenhuis tensor. In the following section, we will investigate the implications of the S_λ^μ tensor given by Eq. (21) in the context of the Nijenhuis tensor.

3 The Nijenhuis tensor

The Nijenhuis tensor is a very interesting geometrical quantity since it is entirely independent of any affine connection. It is defined by [11, 12]

$$N_{\mu\nu}^\lambda = S_\mu^\alpha \partial_\alpha S_\nu^\lambda - S_\nu^\alpha \partial_\alpha S_\mu^\lambda - S_\alpha^\lambda (\partial_\mu S_\nu^\alpha - \partial_\nu S_\mu^\alpha). \quad (30)$$

This tensor may be established in any differentiable manifold M of arbitrary dimension D. To our knowledge, the Nijenhuis tensor has been considered in physics in two different contexts. First, in the study of dynamical integral models [13, 14]. In this context, the vanishing of the Nijenhuis tensor yields interesting properties in manifolds with dual symplectic structures. One

of these properties is the emergence of a number of conserved quantities in involution, that are necessary for the complete integrability of certain dynamical systems [13, 14]. The Nijenhuis tensor is constructed out of a (1,1) tensor S_μ^λ not related to Eq. (21), but to a dual symplectic structure of the manifold.

In the context of complex manifolds, however, the vanishing of the Nijenhuis tensor is related to the existence of integrable almost complex structures [12]. Some formulations of string theory demand that the internal six extra dimensions correspond to a complex manifold, endowed with a complex structure for which the Nijenhuis tensor vanishes [15], while the ordinary four dimensional space-time establish the Minkowski space. Here we show that the Nijenhuis tensor may play some role in gravity.

Let us analyse the expression of the Nijenhuis tensor constructed out of the tensor S_μ^λ given by Eq. (21). By just using the null conditions $l^\mu l_\mu = 0$ and $l^\mu \partial_\alpha l_\mu = 0$, we obtain

$$N_{\mu\nu}^\lambda = \frac{1}{2}[l^\lambda l_\mu(l^\alpha \partial_\alpha l_\nu) - l^\lambda l_\nu(l^\alpha \partial_\alpha l_\mu)]. \quad (31)$$

Considering now the validity of Eq. (6), which follows from the vanishing of the Ricci tensor in empty space-times, we see that the Nijenhuis tensor vanishes,

$$N_{\mu\nu}^\lambda = 0. \quad (32)$$

Therefore some solutions of Einstein's field equations in vacuum may be obtained from a theory based on the Nijenhuis tensor, provided the tensor S_μ^λ is given by Eq. (21). The metric tensor is determined by Eq. (24), in which case the geometrization of the space-time is achieved after solving some field equations in flat space-time.

It is not the purpose of the present article to establish a complete theory for gravity entirely based on the Nijenhuis tensor, because of the limitation to vacuum solutions, or to solutions for which the energy-momentum tensor for the matter fields satisfies $T_{\mu\nu} l^\mu l^\nu = 0$ [2] (of course, this limitation takes place in the context of Einstein's general relativity). But we may easily address two theories constructed out of $N_{\mu\nu}^\lambda$ in flat space-time. The first one is determined by the Lagrangian density

$$L_1 = k \sqrt{-g} N^\mu N_\mu, \quad (33)$$

where k is a constant, $\sqrt{-g}$ refers to the flat space-time in arbitrary coordinates, $N_\mu = N_{\lambda\mu}^\lambda = S_\mu^\rho \partial_\rho S_\lambda^\lambda - S_\rho^\lambda \partial_\mu S_\lambda^\rho$ and $N^\mu = \eta^{\mu\lambda} N_\lambda$. By neglecting the surface terms that arise in the variation of L_1 , we find that this Lagrangian density yields the field equations

$$\sqrt{-g} S_\rho^\mu \partial_\lambda N^\lambda + \sqrt{-g} N^\mu \partial_\rho S_\lambda^\lambda - \delta_\rho^\mu \partial_\lambda (\sqrt{-g} N^\nu S_\nu^\lambda) = 0. \quad (34)$$

The second theory is the Yang-Mill type theory, also in flat space-time, defined by the Lagrangian density

$$L_2 = k \sqrt{-g} N_{\beta\gamma}^\alpha N_{\mu\nu}^\lambda \eta_{\alpha\lambda} \eta^{\beta\mu} \eta^{\gamma\mu} \equiv k \sqrt{-g} N_\lambda^{\mu\nu} N_{\mu\nu}^\lambda, \quad (35)$$

where we have defined $N_\lambda^{\mu\nu} = N_{\beta\gamma}^\alpha \eta_{\alpha\lambda} \eta^{\beta\mu} \eta^{\gamma\mu}$; k and $\sqrt{-g}$ are the same as in L_1 . Again, we neglect the surface terms that arise from the variation of the Lagrangian density above. The field equations that follow from L_2 are

$$\begin{aligned} & \sqrt{-g} N_\lambda^{\mu\nu} \partial_\alpha S_\nu^\lambda - \sqrt{-g} N_\alpha^{\lambda\nu} \partial_\lambda S_\nu^\mu \\ & - \partial_\lambda (\sqrt{-g} N_\alpha^{\nu\mu} S_\nu^\lambda) + \partial_\lambda (\sqrt{-g} N_\nu^{\lambda\mu} S_\nu^\lambda) = 0. \end{aligned} \quad (36)$$

Obviously, $N_{\mu\nu}^\lambda = 0$ is a solution of both Eqs. (34) and (36).

One interesting feature of a theory based on the Nijenhuis tensor is that the theory cannot be linearised, although, of course, the solutions may be linearised in the sense of weak field limits, for instance. Let us consider an expression for S_μ^λ in the form

$$S_\mu^\lambda = \delta_\mu^\lambda + \varepsilon h_\mu^\lambda \quad (37)$$

where ε is an infinitesimal parameter. It is easy to verify that

$$N_{\mu\nu}^\lambda(S) = \varepsilon^2 N_{\mu\nu}^\lambda(h). \quad (38)$$

Therefore, a theory based on the Nijenhuis tensor is definitely a non-linear theory.

One straightforward consequence of expression (21) for the tensor S_μ^λ is the following. The null vector l^μ may be considered an eigenvalue of the tensor S_μ^λ in the sense that $S_\mu^\lambda l^\mu = l^\lambda$. In fact, this relation may be generalised to several products of tensor S_μ^λ , as for instance $S_\sigma^\lambda S_\rho^\sigma S_\mu^\rho l^\mu = l^\lambda$. An arbitrary product of the tensor S_μ^λ yields

$$S_{\alpha_1}^\lambda S_{\alpha_2}^{\alpha_1} \cdots S_\mu^{\alpha_{n-1}} = \delta_\mu^\lambda + \frac{n}{2} l^\lambda l_\mu. \quad (39)$$

Before closing this section, we present an alternative expression for the Nijenhuis tensor. We consider an arbitrary (1,1) tensor S_μ^λ and define the quantity $\mathcal{T}_{\mu\nu}^\lambda$ according to

$$\mathcal{T}_{\mu\nu}^\lambda = \partial_\mu S_\nu^\lambda - \partial_\nu S_\mu^\lambda. \quad (40)$$

This quantity was considered in Ref. [16], in the analysis of a consistent theory for massless spin 2 fields. In terms of the expression above, it is possible to rewrite the Nijenhuis tensor as

$$N_{\mu\nu}^\lambda = -S_\sigma^\lambda [\mathcal{T}_{\mu\nu}^\sigma(S) + S_\mu^\alpha S_\nu^\beta \mathcal{T}_{\alpha\beta}^\sigma(S^{-1})]. \quad (41)$$

We see that $\mathcal{T}_{\mu\nu}^\lambda$ may be understood as a building block for the Nijenhuis tensor. Indeed, in Ref. [17] a very clear exposition of complex manifolds is presented, which supports this interpretation. In summary, a complex structure is determined by a real mixed tensor \mathbf{J} . In a $2n$ dimensional manifold, this mixed tensor satisfies (in the notation of Ref. [17]) $J_N^P J_M^N = -\delta_M^P$ and, in this context, the Nijenhuis tensor is defined by

$$N_{MN}^P = J_M^Q (\partial_Q J_N^P - \partial_N J_Q^P) - J_M^Q (\partial_Q J_M^P - \partial_M J_Q^P). \quad (42)$$

The almost complex structure \mathbf{J} defines a complex structure if and only if the associated Nijenhuis tensor vanishes [17].

4 Comments

In this article we have established tetrads for the Kerr-Schild form of the metric tensor in general relativity. The Kerr-Schild ansatz has been considered in the literature as a manifestation of the gravitational field on a flat Minkowski background. In spite of the limitations of this formulation, the Kerr-Schild ansatz and its consequences are very interesting. The tetrad fields obtained in the present analysis, Eqs. (20) and (23), are given by multiplication of the tetrad fields for the flat space-time with a (1,1) tensor S_μ^λ . The emergence of this tensor immediately leads us to address the Nijenhuis tensor. We have seen that the condition for the vanishing of the Ricci tensor in the context of general relativity, as obtained by Kerr and Schild, and which leads to important gravitational field configurations, is also a condition for the vanishing of the Nijenhuis tensor. Thus, the connection between gravitational field configurations and the Nijenhuis tensor should be explored. In principle,

the latter tensor would lead to theories that yield field equations on a flat space-time. By means of Eq. (24), we cast the results obtained from S_μ^λ in geometrical form. The analysis of field equations in flat space-time, and the *a posteriori* geometrization by means of Eq. (24), would greatly simplify the analysis of gravitational field configurations. The Nijenhuis tensor dispenses the consideration of any space-time affine connection, and this feature also simplifies the whole formulation.

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