The Kerr theorem and multi-particle Kerr-Schild solutions

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We discuss and prove an extended version of the Kerr theorem which allows one to construct the exact solutions of the Einstein-Maxwell field equations from a holomorphic generating function F of twistor variables. The exact multiparticle Kerr-Schild solutions are obtained from generating function of the form $F = \prod_i^k F_i$, where F_i are partial generating functions for the spinning particles $i = 1...k$. Solutions have an unusual multi-sheeted structure. Twistorial structures of the i-th and j-th particles do not feel each other, forming a type of its internal space. Gravitational and electromagnetic interaction of the particles occurs via the light-like singular twistor lines. As a result, each particle turns out to be 'dressed' by singular pp-strings connecting it to other particles. We argue that this solution may have a relation to quantum theory and hints a geometrical (twistorial) way to quantum gravity.

I. INTRODUCTION

In the fundamental work by Debney, Kerr and Schild [1], the Einstein-Maxwell field equations were integrated out for the Kerr-Schild form of metric

$$g_{\mu\nu} = \eta_{\mu\nu} + 2h e_\mu^3 e_\nu^3, \tag{1}$$

where $\eta_{\mu\nu}$ is metric of an auxiliary Minkowski spacetime M^4, and vector field $e_\mu^3(x)$ is null ($e_\mu^3 e^{3\mu} = 0$) and tangent to a principal null congruence (PNC) which is geodesic and shear-free (GSF) [1]. PNC is determined by a complex function $Y(x)$ via the one-form

$$e^3 = du + \bar{Y} d\zeta + Y d\bar{\zeta} - Y\bar{Y} dv \tag{2}$$

written in the null Cartesian coordinates[1]

$$2^{\frac{1}{2}}\zeta = x + iy, \qquad 2^{\frac{1}{2}}\bar{\zeta} = x - iy,$$
$$2^{\frac{1}{2}}u = z + t, \qquad 2^{\frac{1}{2}}v = z - t. \tag{3}$$

One of the most important solution of this class is the Kerr-Newman solution for the rotating and charged black hole. It has been mentioned long ago that the Kerr-Newman solution displays some relationships to the quantum world. It has the anomalous gyromagnetic ratio $g = 2$, as that of the Dirac electron [2], stringy structures [3–6] and other features allowing one to construct a semiclassical model of the extended electron [6–11] which has the Compton size and possesses the wave properties [6,8,12]. So, we will be speaking on *the Kerr spinning particle*, assuming a particle-like source of the Einstein-Maxwell field equations generating the external Kerr-Newman field and having a definite position of centrum of mass, and the definite momentum and orientation of angular momentum.

Principal null congruence (PNC) of the Kerr-Newman solution represents a vortex of the light-like rays (see Fig.1) which are *twistors* indeed. So, the Kerr geometry is supplied by a *twistorial structure* which is described in twistor terms by the Kerr theorem. In addition to the very important meaning for twistor theory [14–16], the Kerr theorem represents in the Kerr-Schild approach [1] *a very useful technical instrument* allowing one to obtain the Kerr-Newman solution and its generalizations.

In accordance with the Kerr theorem, the general geodesic and shear-free congruence on M^4 is generated by the simple algebraic equation

$$F = 0, \tag{4}$$

where $F(Y, \lambda_1, \lambda_2)$ is any holomorphic function of the projective twistor coordinates

$$Y, \quad \lambda_1 = \zeta - Yv, \quad \lambda_2 = u + Y\bar{\zeta}. \tag{5}$$

Since the twistor coordinates λ_1, λ_2 are itself the functions of Y, one can consider F as a function of Y and $x \in M^4$, so the solution of (4) is a function $Y(x)$ which allows one to restore PNC by using the relation (2).

We shall call the function F as *generating function of the Kerr theorem*.

It should be noted that the Kerr theorem has never been published by R. Kerr as a theorem. First, it has been published without a proof in the Penrose work "Twistor Algebra" [14]. However, in a restricted form[2] it has practically been used in [1] by derivation of the Kerr-Newman solution. The text of paper [1] contains some technical details which allows one to reconstruct the proof of the Kerr theorem in a general form which is valid for the Kerr-Schild class of metrics [17,18]. We reproduce this proof in the Appendix B.

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[1]Coordinates $x = x^\mu$ are Cartesian coordinates of Minkowski space $x = \{x, y, z, t\} \in M^4$. It is assumed that they may be analytically extended to a complexified Minkowski space CM^4. The function Y is a projective angular coordinate, i.e. projection of sphere S^2 on a complex plane.

[2]For a special type of generating function F.

The basic results of the fundamental work [1] were obtained for the quadratic in Y generating function F, which corresponds to the Kerr PNC. In particular, for the Kerr-Newman solution the equation $F(Y) = 0$ has two roots $Y^\pm(x)$ [3,19,17], and the space-time is double sheeted, which is one of the mysteries of the Kerr geometry, since the (+) and (−) sheets are imbedded in the same Minkowski background having dissimilar gravitation (and electromagnetic) fields, and the fields living on the (+)-sheet do not feel the existence of different fields on the (−)-sheet.

It has been mentioned in [21], that for quadratic in Y functions F the Kerr theorem determines not only congruence, but also allows one to determine the metric and electromagnetic field (up to an arbitrary function $\psi(Y)$).

In this paper we give the proof of an *extended version of the Kerr theorem* which allows one to determine the corresponding geodesic and shear-free PNC for a very broad class of holomorphic generating functions F, and also to reconstruct the metric and electromagnetic field, i.e. to describe fully corresponding class of the exact solutions of the Einstein-Maxwell field equations.

In particular, we consider polynomial generating functions F of higher degrees in Y which lead to the multiparticle Kerr-Schild solutions. These solutions have a new peculiarity: the space-time and corresponding twistorial structures turn out to be multi-sheeted.

The wonderful twosheetedness of the usual Kerr spacetime is generalized in these solutions to multi-sheeted space-times which are determined by multi-sheeted Riemann holomorphic surfaces and induce the corresponding multi-sheeted twistorial structures.

Twistorial structures of the i-th and j-th particles do not feel each other, forming a type of its internal space. However, the corresponding exact solutions of the Einstein-Maxwell field equations show that particles interact via the common singular twistor lines – the light-like pp-strings. The appearance of singular strings is typical for the known multiparticle solutions [20]. However, contrary to the usual cases, these strings are light-like and do not have conical singularities.

We find out that the mystery of the known twosheetedness of the Kerr geometry is generalized to some more mystical multi-sheetedness of the multiparticle solutions.

We find out that besides the usual Kerr-Newman solution for an isolated spinning particle, there is a series of the exact solutions, in which the selected Kerr-Newman particle is surrounded by external particles and interacts with them by singular pp-strings. It is reminiscent of the known from quantum theory difference between the "naked" one-particle electron of the Dirac theory and a multi-particle structure of a "dressed" electron which is surrounded by virtual photons in accordance with QED. The multiparticle space-time turns out to be penetrated by a multi-sheeted web of twistors. Since they are not observable at the classical level, we conjecture that these multi-sheeted twistor congruences are related to a twisto-

rial structure of vacuum and may be a substitute for the usual spin-foam of quantum gravity, in the spirit of the spin net-work proposals by Penrose. Taking also into account the other known relations of the Kerr geometry to quantum theory, one can conjecture that multisheeted Kerr geometry gives a hint for a new way to quantum gravity.

Basically, we use the notations of the fundamental paper by Debney, Kerr and Schild [1], however, the the Kerr-Schild form of metric (1) is very convenient for derivation, but for applications an another equivalent form is more convenient

$$g_{\mu\nu} = \eta_{\mu\nu} + 2H k_\mu k_\nu. \qquad (6)$$

It is obtained by a simple replacement $h = HP^{-2}$, $e_\mu^3 = Pk_\mu$, where $P = P(Y, \bar{Y})$ is a function which normalizes the vector field e_μ^3, removing singularity by $Y \to \infty$. Function P is also determined by the Kerr Theorem.

For the reader convenience we give in the Appendix A a brief description of the basic relations of the Kerr-Schild formalism. In the Appendix B we reproduce the derivation of the Kerr theorem for the Kerr-Schild backgrounds, following to the papers [17,18]. In the Appendix C we describe briefly the results of [1], the basic Kerr-Schild equations for the general geodesic and shearfree congruences. Appendix D is written for the mathematically oriented readers and contains a description of the double twistor bundle forming the geodesic and shear-free congruences on the Minkowski and the Kerr-Schild space-times.

II. THE KERR THEOREM AND ONE-PARTICLE KERR-SCHILD SOLUTIONS

The Kerr-Schild form of metric (1) has the remarkable properties allowing one to apply *rigorously* the Kerr Theorem to the curved spaces. It is related to the fact that the PNC field e_μ^3, being null and GSF with respect to the Kerr-Schild metric g, $e^{3\mu} e_\mu^3|_g = 0$, will also be null and GSF with respect to the auxiliary Minkowski metric, and this relation remains valid by an analytic extension to the complex region. [3] In the Appendix A we show that the geodesic and shear-free conditions on PNC coincide in the Minkowski space and in the Kerr-Schild background. Therefore, obtaining a geodesic and shear-free PNC in Minkowski space in accordance with the Kerr theorem, and using the corresponding null vector field $e_\mu^3(x)$ in the

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[3]Indeed $e^{3\mu} e_\mu^3|_g = e^{3\mu} e_\mu^3|_\eta + 2h(e^{3\mu} e_\mu^3|_g)^2$, which yields $e^{3\mu} e_\mu^3|_\eta = 0$. And vice versa, it follows from $e^{3\mu} e_\mu^3|_\eta = 0$ two solutions for $e^{3\mu} e_\mu^3|_g$, one of which corresponds to definition of the Kerr-Schild metric, i.e. $e^{3\mu} e_\mu^3|_g = 0$.

Kerr-Schild form of metric, one obtains a curved Kerr-Schild space-time where PNC will also be null, geodesic and shearfree.
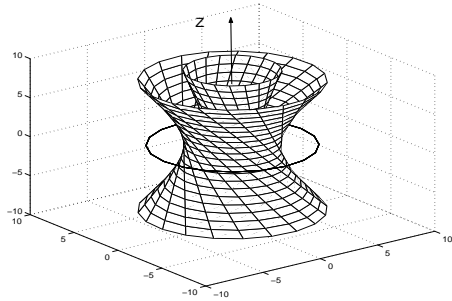

FIG. 1. The Kerr singular ring and 3-D section of the Kerr principal null congruence. Singular ring is a branch line of space, and PNC propagates from "negative" sheet of the Kerr space to "positive " one, covering the space-time twice.

It was shown in [17,21] that the quadratic in Y generating function of the Kerr theorem can be expressed via a set of parameters q which determine the position, motion and orientation of the Kerr spinning particle.

For some selected particle i, function $F_i(Y)$, may be represented in general form

$$F_i(Y|q_i) = A(x|q_i)Y^2 + B(x|q_i)Y + C(x|q_i). \qquad (7)$$

The equation $F_i(Y|q_i) = 0$ can be resolved explicitly, leading to two roots $Y(x) = Y^{\pm}(x|q_i)$ which correspond to two sheets of the Kerr space-time. The root $Y^+(x)$ determines via (2) the out-going congruence on the $(+)$-sheet, while the root $Y^-(x)$ gives the in-going congruence on the $(-)$-sheet. By using these root solutions, one can represent function $F_i(Y)$ in the form

$$F_i(Y) = A_i(x)(Y - Y_i^+(x))(Y - Y_i^-(x)). \qquad (8)$$

The relation (2) determines the vector field $e_\mu^{3(i)}$ of the Kerr-Schild ansatz (1), and metric acquires the form

$$g_{\mu\nu}^{(i)} = \eta_{\mu\nu} + 2h^{(i)}e_\mu^{3(i)}e_\nu^{3(i)}. \qquad (9)$$

Based on this ansatz, after rather long calculations and integration of the Einstein-Maxwell field equations *performed in the work [1]* under the conditions that PNC is geodesic and shearfree (which means $Y_{,2} = 0$ and $Y_{,4} = 0$, see Appendix A), one can represent the function $h^{(i)}$ in the form

$$h^{(i)} = \frac{1}{2}M^{(i)}(Z^{(i)} + \bar{Z}^{(i)}) - \frac{1}{2}A^{(i)}\bar{A}^{(i)}Z^{(i)}\bar{Z}^{(i)}, \quad (10)$$

where

$$M^{(i)} = m^{(i)}(P^{(i)})^{-3} \qquad (11)$$

and

$$A^{(i)} = \psi^{(i)}(Y)(P^{(i)})^{-2}. \qquad (12)$$

Here $m^{(i)}$ is mass and $\psi^{(i)}(Y)$ is arbitrary holomorphic function.

Electromagnetic field is determined by two complex self-dual components of the Kerr-Schild tetrad form $\mathcal{F} = \mathcal{F}_{ab}e^a \wedge e^b$,

$$\mathcal{F}_{12}^{(i)} = A^{(i)}(Z^{(i)})^2 \qquad (13)$$

and

$$\mathcal{F}_{31}^{(i)} = -(A^{(i)}Z^{(i)})_{,1}, \qquad (14)$$

see Appendix C.

We added here the indices i to underline that these functions depend on the parameters q_i of i-th particle.

Setting $\psi^{(i)}(Y) = e = const.$ we have the charged Kerr-Newman solution for i-th particle, vector potential of which may be represented in the form

$$\mathcal{A}_\mu^{(i)} = \Re e(eZ^{(i)})e_\mu^{3(i)}(P^{(i)})^{-2}. \qquad (15)$$

It should be emphasized (!), that integration of the field equations has been performed in [1] in a general form, *before concretization of the form of congruence,* only under the general conditions that PNC is geodesic and shear free.

On the other hand, it was shown in [17,21] (see also Appendix B), that the unknown so far functions $P^{(i)}$ and $Z^{(i)}$ can also be determined from the generating function of the Kerr theorem $F^{(i)}$. Namely,

$$P^{(i)} = \partial_{\lambda_1}F_i - \bar{Y}\partial_{\lambda_2}F_i, \quad P^{(i)}/Z^{(i)} = - \quad dF_i/dY. \qquad (16)$$

Therefore, for the quadratic in Y functions (7), we arrive at the first **extended version of the Kerr theorem**

1/ For a given quadratic in Y generating function F_i, solution of the equation $F_i = 0$ determines the geodesic and shear free PNC in the Minkowski space M^4 and in the associated Kerr-Schild background (9).

2/ The given function F_i determines the exact stationary solution of the Einstein-Maxwell field equations with metric given by (9), (10), (11) and electromagnetic field given by (15), where functions $P^{(i)}$ and $Z^{(i)}$ are given by (16).

As we have mentioned in introduction, for practical calculations the Kerr-Schild form (6) is more useful, where function $H = H^{(i)} = h^{(i)}(P^{(i)})^2$, and the normalized null vector field is $k_\mu^{(i)} = e_\mu^{3(i)}/P^{(i)}$.

In this form function $H^{(i)}$ will be

$$H^{(i)} = \frac{m}{2}(\frac{1}{\tilde{r}_i} + \frac{1}{\tilde{r}_i^*}) + \frac{e^2}{2|\tilde{r}_i|^2}, \qquad (17)$$

and the Kerr-Newman electromagnetic field is determined by the vector potential

$$A_\mu^{(i)} = \Re e(e/\tilde{r}_i)k_\mu^{(i)}, \qquad (18)$$

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where

$$\tilde{r}_i = P^{(i)}/Z^{(i)} = - \quad dF_i/dY \qquad (19)$$

is the so called *complex radial distance* which is related to a complex representation of the Kerr geometry [17,18,21,23].

For a standard oriented Kerr solution in the rest, $\tilde{r} = \sqrt{x^2 + y^2 + (z - ia)^2} = r + ia\cos\theta$, which corresponds to the distance from a *complex point* source positioned at the complex point $\vec{x} = (0, 0, ia)$. One sees, that the Kerr singular ring is determined by $\tilde{r} = 0 \Rightarrow r = \cos\theta = 0$. For the Kerr geometry this representation was initiated by Newman, however it works rigorously in the Kerr-Schild approach, where the complex source represents a complex world line $x(\tau)$ in the complexified auxiliary Minkowski space-time CM^4. We will use this approach in sec.5.

In accordance with Corollary 4 (Appendix B), position of the Kerr singular ring is determined by the system of equations

$$F_i = 0, \quad \tilde{r}_i = - \quad dF_i/dY = 0 \ . \qquad (20)$$

Extended version of the Kerr theorem allows one to get exact solution for an arbitrary oriented and boosted charged spinning particle [21].

III. MULTI-SHEETED TWISTOR SPACE

Let us now consider the case of a system of k spinning particles having the arbitrary displacement, orientations and boosts. One can form the function F as a product of the corresponding blocks $F_i(Y)$,

$$F(Y) \equiv \prod_{i=1}^{k} F_i(Y). \qquad (21)$$

The solution of the equation $F = 0$ acquires $2k$ roots $Y_i^{\pm}(x)$, forming a multisheeted covering space over the Riemann sphere $S^2 = CP^1 \ni Y$.

Indeed, $Y = e^{i\phi}\tan\frac{\theta}{2}$ is a complex projective angular coordinate on the Minkowski space-time and on the corresponding Kerr-Schild space-time.[4]

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[4]Two other coordinates in the Kerr-Schild space-time may be chosen as $\tilde{r} = PZ^{-1}$ and $\rho = x^\mu e_\mu^3$, where x^μ are the four Cartesian coordinates in M^4.
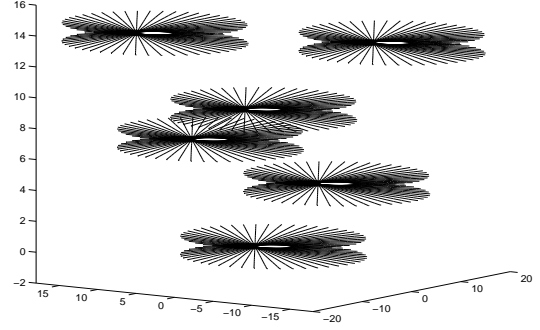

FIG. 2. Multi-sheeted twistor space over the auxiliary Minkowski space-time of the multiparticle Kerr-Schild solution. Each particle has two-sheeted structure.

The twistorial structure on the i-th $(+)$ or $(-)$ sheet is determined by the equation $F_i = 0$ and does not depend on the other functions F_j, $j \neq i$. Therefore, the particle i does not feel the twistorial structures of other particles.

The equations for singular lines

$$F = 0, \ dF/dY = 0 \qquad (22)$$

acquires the form

$$\prod_{l=1}^{k} F_l = 0, \qquad \sum_{i=1}^{k}\prod_{l \neq i}^{k} F_l dF_i/dY = 0 \qquad (23)$$

which splits into k independent relations

$$F_i = 0, \quad \prod_{l \neq i}^{k} F_l dF_i/dY = 0. \qquad (24)$$

One sees, that the Kerr singular ring on the sheet i is determined by the usual relations $F_i = 0$, $dF_i/dY = 0$, and i-th particle does not feel also the singular rings of the other particles. The space-time splits on the independent twistorial sheets, and the twistorial structure related to the i-th particle plays the role of its "internal space". One should mention that it is a direct generalization of the well known two-sheetedness of the usual Kerr space-time.

Since the twistorial structures of different particles are independent, it seems that the k-particle solutions $\{Y_i^{\pm}(x)\}$, $i = 1, 2..k$ form a trivial covering space K over the sphere S^2, i.e. K is a trivial sum of k disconnected two-sheeted subspaces $K = \bigcup_i^k S_i^2$.

However, there is one more source of singularities on K which corresponds to the multiple roots: the cases when some of twistor lines of one particle i coincides with a twistor line of another particle j, forming a common (ij)-twistor line. Indeed, for each pair of particles i and j, there are two such common twistor lines: one of them $\overrightarrow{(ij)}$ is going from the positive sheet of particle i, $Y_i^+(x)$ to negative sheet of particle j, $Y_j^-(x)$ and corresponds to the solution of the equation $Y_i^+(x) = Y_j^-(x)$, another

one $(\vec{ji})$ is going from the positive sheet of particle j, $Y_j^+(x)$ to negative sheet of particle i and corresponds to the equation $Y_j^+(x) = Y_i^-(x)$. We will consider the corresponding simple example in sec.5.

The common twistor lines are also described by the solutions of the equations (22) and correspond to the multiple roots which give a set of "points" A_j, where the complex analyticity of the map $Y_i^\pm(x) \to S^2$ is broken.[5]

The solutions $Y_i(x)$, which determine PNC on the i-th sheet of the covering space, induce multisheeted twistor fields over the corresponding Kerr-Schild manifold $\mathcal{K}^4$.

IV. MULTIPARTICLE KERR-SCHILD SOLUTIONS.

As we have seen, the quadratic in Y functions F generate exact solutions of the Einstein-Maxwell field equations. In the same time, the considered above generating functions $\prod_{i=1}^k F_i(Y) = 0$, leads to a multisheeted covering space over S^2 and to the induced multisheeted twistor structures over the Kerr-Schild background which look like independent ones. Following to the initiate naive assumption that twistorial sheets are fully independent, one could expect that the corresponding multisheeted solutions of the Einstein-Maxwell field equations will be independent on the different sheets, and the solution on i-th sheet will reproduce the result for an isolated i-th particle. However, It is obtained that the result is different.

Formally, we have to replace F_i by

$$F = \prod_{i=1}^k F_i(Y) = \mu_i F_i(Y), \qquad (25)$$

where

$$\mu_i = \prod_{j \neq i}^k F_j(Y) \qquad (26)$$

is a normalizing factor which takes into account the external particles. In accordance with (16) this factor will also appear in the new expression for P/Z which we mark now by capital letter $\tilde{R}_i$

$$\tilde{R}_i = P/Z = -d_Y F = \mu_i P^{(i)}/Z^{(i)}, \qquad (27)$$

and in the new function P_i which we will mark by hat

$$\hat{P}_i = \mu_i P_i. \qquad (28)$$

Functions Z and $\bar{Z}$ will not be changed.

By substitution of the new functions P_i in the relations (10), (11) and (12), we obtain the new relations

$$M^{(i)} = m^{(i)}(\mu_i(Y)P^{(i)})^{-3}, \qquad (29)$$

$$A^{(i)} = \psi^{(i)}(Y)(\mu_i(Y)P^{(i)})^{-2} \qquad (30)$$

and

$$h_i = \frac{m}{2(\mu_i(Y)P_i)^3}(Z^{(i)} + \bar{Z}^{(i)}) - \frac{|\psi|^2}{2|\mu_i(Y)P_i|^4}Z^{(i)}\bar{Z}^{(i)}. \qquad (31)$$

For new components of electromagnetic field we obtain

$$\mathcal{F}_{12}^{(i)} = \psi^{(i)}(Y)(\mu_i P^{(i)})^{-2}(Z^{(i)})^2 \qquad (32)$$

and

$$\mathcal{F}_{31}^{(i)} = -\left(\frac{\psi^{(i)}(Y)}{(\mu_i(Y)P^{(i)})^2}Z^{(i)}\right)_{,1}. \qquad (33)$$

In the terms of $\tilde{r}_i$ and H_i the Kerr-Newman metric takes the form:

$$H_i = \frac{m_0}{2}\left(\frac{1}{\mu_i \tilde{r}_i} + \frac{1}{\mu_i^* \tilde{r}_i^*}\right) + \frac{e^2}{2|\mu_i \tilde{r}_i|^2}, \qquad (34)$$

where we have set $\psi^{(i)}(Y) = e$, for the Kerr-Newman solution.

The simple expression for vector potential (15) is not valid more.[6]

One sees, that in general case metric turns out to be complex for the complex mass factor $m(\mu_i(Y))^{-3}$, and one has to try to reduce it to the real one.

This problem of reality M was also considered in [1]. Function M satisfies the equation

$$(\ln M + 3\ln P)_{,\bar{Y}} = 0 \qquad (35)$$

which has the general solution

$$M = m(Y)/P^3(Y, \bar{Y}), \qquad (36)$$

where $m(Y)$ is an arbitrary holomorphic function. The simplest real solution is given by the real constant m and the real function $P(Y, \bar{Y})$. As it was shown in [1], it resulted in the one-particle solutions.

In our case functions P_i have also to be real, since they relate the real one-forms e^3 and k,

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[5]The given in [25] analysis of the equations (22) shows that for the holomorphic functions $F(Y)$ the covering space K turns out to be connected and forms a multisheeted Riemann surface over the sphere with the removed branch points $S^2 \setminus \cup_j A_j$.

[6]Besides the related with $\mu_i(Y)$ singular string factor, it acquires an extra vortex term.

$$e^{3(i)} = P_i k_\mu^{(i)} dx. \qquad (37)$$

Functions $\mu_i(Y)$ are the holomorphic functions given by (26), and functions $m_i = m_i(Y)$ are arbitrary holomorphic functions which may be taken in the form

$$m_i(Y) = m_0(\mu_i(Y))^3 \qquad (38)$$

to provide reality of the mass terms $M_i = m_i(Y)/\hat{P}_i^3$ on the each i-th sheet of the solution.

Therefore, we have achieved the reality of the multi-sheeted Kerr-Schild solutions, and *the extended version of the Kerr theorem is now applicable for the general multiplicative form of the functions F, given by (21).*

One can specify the form of functions μ_i by using the known structure of blocks F_i

$$\mu_i(Y_i) = \prod_{j \neq i} A_j(x)(Y_i - Y_j^+)(Y_i - Y_j^-). \qquad (39)$$

If the roots $Y_i^\pm$ and $Y_j^\pm$ coincide for some values of $Y_i^\pm$, it selects a common twistor for the sheets i and j. Assuming that we are on the i-th $(+)$-sheet, where congruence is out-going, this twistor line will also belong to the in-going $(-)$-sheet of the particle j. The metric and electromagnetic field will be singular along this twistor line, because of the pole $\mu_i \sim A(x)(Y_i^+ - Y_j^-)$. This singular line is extended to the semi-infinite line which is common for the $i-th$, and $j-th$ particle. However, the considered in the following section simple example shows that there exists also a second singular line related to interaction of two particles. It is out-going on the Y_j^+-sheet and belongs to the in-going $(-)$-sheet of the particle i, Y_i^-.

Therefore, each pair of the particles (ij) creates two opposite oriented in the space (future directed) singular twistor lines, pp-strings. The field structure of this string is described by singular pp-wave solutions (the Schild strings) [6,12].

If we have k particles, then in general, for the each Kerr's particle $2k$ twistor lines belonging to its PNC will turn into singular null strings.

As a result, one sees, that in addition to the well known Kerr-Newman solution for an isolated particle, there are series of the corresponding solutions which take into account presence of the surrounding particles, being singular along the twistor lines which are common with them.

By analogue with QED, we call these solutions as 'dressed' ones to differ them from the original 'naked' Kerr-Newman solution. The 'dressed' solutions have the same position and orientation as the 'naked' ones, and differ only by the appearance of singular string along some of the twistor lines of the Kerr PNC.
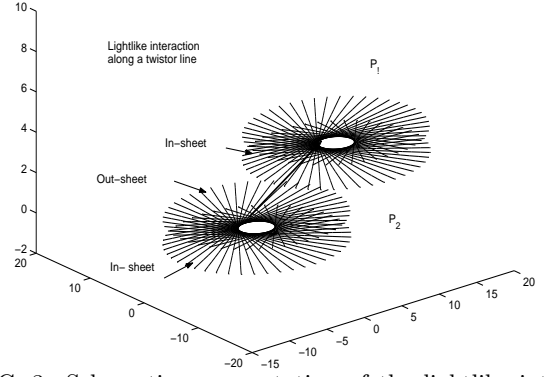

FIG. 3. Schematic representation of the lightlike interaction via a common twistor line connecting out-sheet of one particle to in-sheet of another.

V. EXAMPLE. TWO-PARTICLE SOLUTION.

Let us concretize solution for two particles $P1$ and $P2$. Taking $F(Y) = F_1(Y)F_2(Y)$, where

$$F_1 = A_1(x)(Y - Y_1^+)(Y - Y_1^-), \qquad (40)$$
$$F_2 = A_2(x)(Y - Y_2^+)(Y - Y_2^-), \qquad (41)$$

one obtains that the 'dressed' complex distance is

$$\tilde{R} = -d_Y F = -A_1 A_2[(Y - Y_1^-)(Y - Y_2^+)(Y - Y_2^-)$$
$$+ (Y - Y_1^+)(Y - Y_2^+)(Y - Y_2^-)$$
$$+ (Y - Y_1^-)(Y - Y_1^+)(Y - Y_2^-)$$
$$+ (Y - Y_1^-)(Y - Y_1^+)(Y - Y_2^+)]. \quad (42)$$

If we are positioned on the sheet 2^+, we have $Y = Y_2^+$, and $\tilde{R}$ contains only one term,

$$\tilde{R} = \mu_2 \tilde{r}_2^+, \qquad (43)$$

where

$$\mu_2 = A_1(Y_2^+ - Y_1^-)(Y_2^+ - Y_1^+), \qquad (44)$$
$$\tilde{r}_2^+ = -A_2(Y_2^+ - Y_2^-). \qquad (45)$$

One sees that μ_2 has the null at the set of space-time points where $Y_2^+ = Y_1^-$ or $Y_2^+ = Y_1^+$. This set is a semi-infinite twistor line on the sheet Y_2^+ consisting of the segment connecting particles P2 and P1, as it is shown on FIG.3, and of the semi-infinite extension of this segment which is the common twistor line of the sheets Y_2^+ and Y_1^+.

Let us consider two simple cases:

a/ the particles are in the rest at the points $P1 = (d, 0, 0)$ and $P2 = (0, 0, 0)$, and the spins are orientated along the z-axis,

b/the same, but the points $P1 = (0, 0, d)$, i.e. the both particles lie on the z-axis.

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Function $F_1(Y)$ and $F_2(Y)$ may be used in the form (7) where coefficients A, B, C are determined by complex world lines $X_0^{(1)}(\tau)$ and $X_0^{(2)}(\tau)$ of these particles [17,21].

In the null Cartesian coordinates $(u,\ v,\ \bar\zeta,\ \zeta)$ they are given by the relations [21]

$$A = (\bar\zeta - \bar\zeta_0)\dot v_0 - (v - v_0)\dot{\bar\zeta}_0, \tag{46}$$
$$B = (u - u_0)\dot v_0 + (\zeta - \zeta_0)\dot{\bar\zeta}_0 - (\bar\zeta - \bar\zeta_0)\dot\zeta_0 - (v - v_0)\dot u_0,$$
$$C = (\zeta - \zeta_0)\dot u_0 - (u - u_0)\dot\zeta_0.$$

Particles $P1$ and $P2$ are described by complex world lines

Case a/ :

$$X_0^{(1)\mu} = \{\tau, d, 0, ia\}, \quad X_0^{(2)\mu} = \{\tau, 0, 0, ia\}, \tag{47}$$

or in the null coordinates

$$X_0^{(1)} = 2^{-1/2}\{\tau + ia,\ ia - \tau,\ d,\ d\}, \tag{48}$$
$$X_0^{(2)} = 2^{-1/2}\{\tau + ia,\ ia - \tau,\ 0,\ 0\}, \tag{49}$$

Case b/ :

$$X_0^{(1)\mu} = \{\tau, 0, 0, d + ia\}, \quad X_0^{(2)\mu} = \{\tau, 0, 0, ia\}, \tag{50}$$

and in the null coordinates

$$X_0^{(1)} = 2^{-1/2}\{\tau + d + ia,\ d + ia - \tau,\ 0,\ 0\}, \tag{51}$$
$$X_0^{(2)} = 2^{-1/2}\{\tau + ia,\ ia - \tau,\ 0,\ 0\}, \tag{52}$$

Since particles are in the rest, we have for the both particles in the both cases $\dot u_0 = -\dot v_0 = 2^{-1/2}$, and $\dot\zeta_0 = \dot{\bar\zeta}_0 = 0$.

Coefficients for the functions F_1 and F_2 take the form:

Cases a/ and b/ :

$$A_2 = -\frac{1}{2}(x - iy), \tag{53}$$
$$B_2 = -z + ia,$$
$$C_2 = \frac{1}{2}(x + iy).$$

Case a/ :

$$A_1 = -\frac{1}{2}(x - iy - d), \tag{54}$$
$$B_1 = -z + ia,$$
$$C_1 = \frac{1}{2}(x + iy - d),$$

Case b/ :

$$A_1 = -\frac{1}{2}(x - iy), \tag{55}$$
$$B_1 = -z + d + ia,$$
$$C_1 = \frac{1}{2}(x + iy),$$

The roots of the equations $F_1 = 0$ and $F_2 = 0$ will be:
Case a/ :

$$Y_1^\pm = \frac{ia - z \pm \tilde r_1}{x - iy - d}, \qquad Y_2^\pm = \frac{ia - z \pm \tilde r_2}{x - iy}, \tag{56}$$

where

$$\tilde r_1 = \sqrt{(x - d)^2 + y^2 + (z - ia)^2},$$
$$\tilde r_2 = \sqrt{x^2 + y^2 + (z - ia)^2}. \tag{57}$$

One can find two singular twistor lines which are determined by the equation $Y_2^+ = Y_1^-$, or

$$\frac{ia - z + \tilde r_2}{x - iy} = \frac{ia - z - \tilde r_1}{x - iy - d}. \tag{58}$$

It is enough to find at least one point for each of these lines. Looking for such a point at $\tilde r_2 = 0$, we obtain

$$r_1 = d, \quad y = -a, \quad x = z = 0, \tag{59}$$

which gives

$$Y_1^- = \frac{ia - d}{ia - d} = 1, \quad Y_2^+ = \frac{ia}{ia} = 1, \tag{60}$$

and the null direction corresponding $Y = Y_1^- = Y_2^+ = 1$.

This singular line is going via the point (59) with null direction $Y = 1$. Another line is going via the point

$$\tilde r_2 = 0, r_1 = -d, \quad y = a, \quad x = z = 0, \tag{61}$$

and has the opposite space direction $Y = Y_1^- = Y_2^+ = -1$. These two lines lie in the common equatorial plane of the both particles and are tangent to the both singular rings.

Case b/ :
Similar,

$$Y_1^\pm = \frac{ia + d - z \pm \tilde r_1}{x - iy}, \qquad Y_2^\pm = \frac{ia - z \pm \tilde r_2}{x - iy}, \tag{62}$$

where

$$\tilde r_1 = \sqrt{x^2 + y^2 + (z - d - ia)^2},$$
$$\tilde r_2 = \sqrt{x^2 + y^2 + (z - ia)^2}. \tag{63}$$

The equation $Y_1^- = Y_2^+$ takes the form

$$\frac{d + ia - z - \tilde r_1}{x - iy} = \frac{ia - z + \tilde r_2}{x - iy}. \tag{64}$$

Solutions are given by the equation $\frac{d - \tilde r_1 - \tilde r_2}{x - iy} = 0$. Although, there appears indefinite limit by $x - iy \to 0$, one can show that this expression tends to zero by $d - r_1 - r2 = 0$ and one obtains two singular lines which

lie on the axis z and have the opposite directions $Y = 0$ and $Y = \infty$.[7]

Let us summarize the expressions for the 'dressed' solution on the sheet $i = 2+$, where we will omit indices 2^+ for coordinate Y

In the both cases it has the Kerr-Schild form

$$g^{(2)}_{\mu\nu} = \eta_{\mu\nu} + 2H^{(2)}P_2^{-2}e^3_\mu e^3_\nu, \tag{65}$$

where

$$H^{(2)} = \frac{m_0}{2}\left(\frac{1}{\mu_2 \tilde{r}_2} + \frac{1}{\mu_2^* \tilde{r}_2^*}\right) + \frac{e}{2|\mu_2 \tilde{r}_2|^2}, \tag{66}$$

Electromagnetic field is determined by the two complex tetrad components

$$\mathcal{F}^{(2)}_{12} = A^{(2)}(Z^{(2)})^2 \tag{67}$$

and

$$\mathcal{F}^{(2)}_{31} = -(A^{(2)}Z^{(2)})_{,1} \ , \tag{68}$$

where for $\psi = e$ we have

$$A^{(2)} = \frac{e}{(\mu_2 P_2)^2}. \tag{69}$$

Congruence is $e^3 = P_2 k^{(2)}_\mu dx^\mu$ given by the form

$$e^3 = du + \bar{Y}d\zeta + Yd\bar{\zeta} - Y\bar{Y}dv, \tag{70}$$

where function P_2 is determined by the relation $P_2 = \dot{X}^{(2)\mu}_0 e^3_\mu$ (see [21]) which gives for the particle $P2$ in the rest the known expression [1]

$$P_2 = 2^{-1/2}(1 + Y\bar{Y}). \tag{71}$$

Function Y has the form $Y = e^{i\phi}\tan\frac{\theta}{2}$, where ϕ and θ are asymptotically the usual angular coordinates. However, their relations to Cartesian coordinates is strongly deformed near singular ring, so the dependencies $Y^\pm_i(x)$ turn out to be different for the different sheets and rather complicate. They are nontrivial even for the Kerr's particle $P2$ which has the standard position and orientation. The corresponding function

$$Y \equiv Y^+_2 = \frac{ia - z + \tilde{r}_2}{x - iy}, \tag{72}$$

where

$$\tilde{r}_2 = \sqrt{x^2 + y^2 + (z - ia)^2}, \tag{73}$$

and one can get for particle $P2$ the known (Kerr-Schild) coordinate relations

$$\tilde{r}_2 = r + ia\cos\theta, \tag{74}$$

$$x + iy = (r + ia)e^{i\phi}\sin\theta, \quad z = r\cos\theta. \tag{75}$$

Finally, the function μ_2 has the complicate form:

in the case a/ :

$$\mu_2 = -\frac{1}{2}(x - iy - d)(Y - Y^-_1)(Y - Y^+_1), \tag{76}$$

where

$$Y^\pm_1 = \frac{ia - z \pm \tilde{r}_1}{x - iy - d}, \tag{77}$$

and

$$\tilde{r}_1 = \sqrt{(x - d)^2 + y^2 + (z - ia)^2}, \tag{78}$$

and **in the case b/ :**

$$\mu_2 = -\frac{1}{2}(x - iy)(Y - Y^-_1)(Y - Y^+_1), \tag{79}$$

where

$$Y^\pm_1 = \frac{ia + d - z \pm \tilde{r}_1}{x - iy}, \tag{80}$$

and

$$\tilde{r}_1 = \sqrt{x^2 + y^2 + (z - d - ia)^2}. \tag{81}$$

VI. CONCLUSION

We considered the extended version of the Kerr theorem which, being incorporated in the Kerr-Schild formalism, allows one to get exact multiparticle Kerr-Schild solutions.

One of the principal new properties of these solutions is that they are multi-sheeted. This property is a natural generalization of the known two-sheetedness of the Kerr geometry and is related with multi-sheetedness of the corresponding twistorial spaces of the geodesic and shear-free principal null congruences.

Another new feature is the existence of the 'naked' and corresponding 'dressed' Kerr-Newman solutions, which resembles the structure of dressed particles in QED[8] The 'naked' solution is the usual Kerr-Newman solution for an isolated particle, while the 'dressed' solutions are the

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[7]Since particle P2 has the standard Kerr position and orientation, one can use the known relations for the Kerr angular coordinates, $x - iy = (r_2 - ia)e^{-i\phi}\sin\theta, \quad z = r_2\cos\theta$ and $\tilde{r}_2 = r_2 + ia\cos\theta$, which allows one to find the limit by $x - iy \to 0$.

[8]Similar structure appears also in DN sector of D-strings in superstring theory [27,5].

series of corresponding solutions, in which the selected Kerr-Newman particle is surrounded by other particles.

Finally, the following from these solutions gravitational and electromagnetic interaction of the particles via singular pp-strings is also surprising and gives a hint that the photons and gravitons may have the structure of pp-strings.

Moreover, the obtained multiparticle solutions are related to multi-sheeted twistorial space-times, showing that a twistorial web net of pp-string covers the space-time.

It suggests that the obtained multiparticle solutions may have a relation to twistorial structure of vacuum which was conjectured in many Penrose publications. It looks not too wonder, since the multiplicative generating function of the Kerr theorem (21) has been taken in analogue with the structure of higher spin gauge theory [26] and is reminiscent of a twistorial version of the Fock space.

On the other hand, the Kerr-Newman solution has the double gyromagnetic ratio $g = 2$ as that of the Dirac electron, which raises the question on the relation of the Kerr spinning particle to the Dirac electron [23], as well as to the other elementary particles. In these cases the relation $a >> m$ is valid, and the black hole horizons disappear providing for the one-particle Kerr-Newman solution a simple two-sheeted topology, where the analytic extension of Kerr-Newman space-time may be represented by splitting of the known conformal diagram, as it is shown in Fig.4.
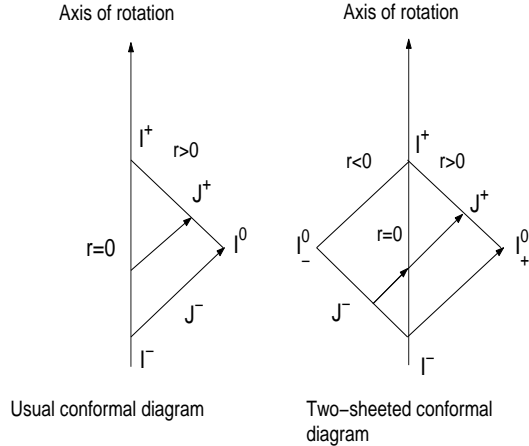

FIG. 4. The usual conformal diagram of the flat Minkowski space-time (left figure) and the twosheeted conformal diagram of the almost flat Kerr-Newman space-time with $a >> m$. The space-like infinity I^0 splits into I^0_-, corresponding to $r = -\infty$, and I^0_+, corresponding to $r = +\infty$,.

It allows one to conjecture that the presented multiparticle solutions has a relation to the structure of spinning particle and may also shed some light on some problems of quantum gravity, which is also supported by the old and modern expectations of the important role of twistors in quantum theory [28,29].

The opposite case, $a < m$, is important for astrophysical applications. Although, the appearance of the multisheeted structures in this case seems very problematic, the solutions with axial singular lines are actual, since the semi-infinite singular strings may be related to jet formation. Treatment of the causal structure of these solutions will be given in a separate paper [30]. However, one can preliminarily note, that these axial singularities change drastically the topological structure of horizon, leading to formation of a hole in the event horizon. So, the resulting 'black hole' turns out to be not black.

APPENDIX A. BASIC RELATIONS OF THE KERR-SCHILD FORMALISM

Following the notations of the work [1], the Kerr-Schild null tetrad $e^a = e^a_\mu dx^\mu$ is determined by relations:

$$e^1 = d\zeta - Y dv, \qquad e^2 = d\bar\zeta - \bar Y dv,$$
$$e^3 = du + \bar Y d\zeta + Y d\bar\zeta - Y \bar Y dv,$$
$$e^4 = dv + h e^3, \tag{82}$$

and

$$g_{ab} = e^\mu_a e_{b\mu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{83}$$

Vectors e^3, e^4 are real, and e^1, e^2 are complex conjugate.

The inverse (dual) tetrad has the form

$$\partial_1 = \partial_\zeta - \bar Y \partial_u;$$
$$\partial_2 = \partial_{\bar\zeta} - Y \partial_u;$$
$$\partial_3 = \partial_u - h \partial_4;$$
$$\partial_4 = \partial_v + Y \partial_\zeta + \bar Y \partial_{\bar\zeta} - Y \bar Y \partial_u, \tag{84}$$

where $\partial_a \equiv ,_a \equiv e^\mu_a \partial,_\mu$.

The Ricci rotation coefficients are given by

$$\Gamma^a_{bc} = - \quad e^a_{\mu;\nu} e^\mu_b e^\nu_c. \tag{85}$$

The PNC have the e^3 direction as tangent. PNC is geodesic if and only if $\Gamma_{424} = 0$ and shear free if and only if $\Gamma_{422} = 0$. corresponding complex conjugate terms are $\Gamma_{414} = 0$ and $\Gamma_{411} = 0$.

The connection forms in this basis are

$$\Gamma_{42} = \Gamma_{42a} e^a = -dY - hY,_4 e^4. \tag{86}$$

The congruence e^3 is geodesic if $\Gamma_{424} = -Y,_4\,(1-h) = 0$, and is shear free if $\Gamma_{422} = -Y,_2 = 0$.

Therefore, the conditions

$$Y,_2 = Y,_4 = 0, \tag{87}$$

define a shear-free and geodesic congruence.

APPENDIX B. PROOF OF THE KERR THEOREM FOR THE KERR-SCHILD BACKGROUND.

Starting from the geodesic and shear free conditions $Y,_2 = Y,_4 = 0$, one obtains the differential of function Y in the form

$$dY = Y,_a\,e^a = Y,_1\,e^1 + Y,_3\,e^3. \tag{88}$$

$Y,_1 = Z$ is an important parameter: complex expansion of congruence, $Z = \rho + i\omega$, where $\rho = expansion$ and $\omega = rotation$.

One needs to work out the form $Y,_3$. By using relations (84) and their commutators one finds

$$Z,_2 = (Z - \bar{Z})Y,_3\,. \tag{89}$$

Straightforward differentiation of $Y,_3$ gives the equation

$$Y,_{32} = (Y,_3\,)^2. \tag{90}$$

Then, by using (89) and (90) one obtains the equation

$$(Z^{-1}Y,_3\,),_2 = \bar{Z}(Z^{-1}Y,_3\,)^2. \tag{91}$$

This is a first-order differential equation for the function $Z^{-1}Y,_3$. Its general solution can be obtained by substitution $x = Z(Y,_3\,)^{-1}$ and has the form

$$Y,_3 = Z(\phi - \bar{Y})^{-1}, \tag{92}$$

where ϕ is an arbitrary solution of the equation $\phi,_2 = 0$. Analogously, by using the relation $Y,_{34} = -ZY,_3$ one obtains $\phi,_4 = 0$, and therefore ϕ may be an arbitrary function satisfying

$$\phi,_2 = \phi,_4 = 0. \tag{93}$$

One can easily check that the three projective twistor coordinates $\lambda_1 = \zeta - Yv, \qquad \lambda_2 = u + Y\bar{\zeta}$, and Y satisfy the similar relations $(.),_2 = (.),_4 = 0$.

Since the surface $\phi = const$ forms a sub-manifolds of CM^4 which has the complex dimension three, an arbitrary function ϕ satisfying (2.6) may be presented as function of three projective twistor coordinates $\phi = \phi(Y, \lambda_1, \lambda_2)$. Now we can substitute $Y,_3$ in (88) that implies

$$Z^{-1}(\bar{Y} - \phi)dY = \phi(d\zeta - Ydv) + (du + Yd\bar{\zeta}). \tag{94}$$

If an arbitrary holomorphic function $F(Y, \lambda_1, \lambda_2)$ is given, then differentiating the equation $F(Y, \lambda_1, \lambda_2) = 0$ and comparing the result with (94) one finds that

$$PZ^{-1} = -\quad dF/dY, \tag{95}$$

where

$$P = \partial_{\lambda_1} F - \bar{Y}\partial_{\lambda_2} F, \tag{96}$$

where the function P can also be defined as

$$P = (\phi - \bar{Y})\partial_{\lambda_2} F. \tag{97}$$

Therefore, we have:

Corollary 1:
For arbitrary holomorphic function of the projective twistor variables $F(Y, \lambda_1, \lambda_2)$, the equation $F = 0$ determines function $Y(x)$ which gives the congruence of null directions e^3, (82) satisfying the geodesic and shearfree conditions $Y,_2 = Y,4 = 0$.

Corollary 2:
Using (84), one sees that the explicit form of the geodesic and shearfree conditions $Y,_2 = Y,4 = 0$ is $(\partial_{\bar{\zeta}} - Y\partial_u)Y = 0$ and $(\partial_v + Y\partial_{\zeta})\partial_u)Y = 0$. It does not depend on function h and coincides with these conditions in Minkowski space. Therefore, the resulting PNC is geodesic and shearfree with respect to the Kerr-Schild background as well as with respect to the auxiliary Minkowski metric $\eta_{\mu\nu}$.

Corollary 3:
Function F (which we call generating function) determines two important functions

$$P = \partial_{\lambda_1} F - \bar{Y}\partial_{\lambda_2} F, \tag{98}$$

and

$$PZ^{-1} = -\quad dF/dY, \tag{99}$$

Corollary 4:
Singular points of the congruence, where the complex divergence Z blows up, is defined by the system of equations

$$F = 0, \quad dF/dY = 0 \tag{100}$$

Note, that

$$\tilde{r} = PZ^{-1}. \tag{101}$$

is complex radial distance which is related to complex representation of the Kerr geometry [17].

Corollary 5:
The following useful relations are valid

$$\bar{Z}Z^{-1}Y,_3 = -(\log P),_2\ ,\quad P,_4 = 0. \tag{102}$$

Proof. So far as $\partial_2\partial_{\lambda_2} F = 0$, one sees that

$$(\log P),_2 = -\bar{Z}(\phi - \bar{Y})^{-1}, \tag{103}$$

then (92) leads to first equality of (102). The relation $P,_4 = 0$ follows from (98) and properties of the twistor components $Y,_4 = (\lambda_1),_4 = (\lambda_2),_4 = 0$.

APPENDIX C. INTEGRATION OF THE EINSTEIN-MAXWELL FIELD EQUATIONS FOR ARBITRARY GEODESIC AND SHEARFREE PNC

Following [1], the Einstein-Maxwell field equations written in the above null tetrad form become:

$$R_{ab} = -2(F_{ac}F_b{}^c - \frac{1}{4}F_{cd}F^{cd}), \tag{104}$$

$$\mathcal{F}^{ab}{}_{;b} = \mathcal{F}^{ab}{}_{,b} + \Gamma^a{}_{cb}\mathcal{F}^{cb} + \Gamma^b{}_{cb}\mathcal{F}^{ac} = 0, \tag{105}$$

where electromagnetic field is represented by complex tensor

$$\mathcal{F}_{ab} = -\mathcal{F}_{ba} = F_{ab} + \frac{i}{2}\eta_{abcd}F^{cd}, \tag{106}$$

and η_{abcd} is completely skew-symmetric and

$$\eta_{1234} = i. \tag{107}$$

The geodesic and shearfree conditions $Y_{,2} = Y_{,4} = 0$ reduce strongly the list of gravitational and Maxwell equations. As a result, one obtains for the tetrad components

$$R_{24} = R_{22} = R_{44} = R_{14} = R_{11} = R_{41} = R_{42} = 0. \tag{108}$$

The equation $R_{44} = 0$ gives $F_{42} = F_{41} = 0$, which simplifies strongly $\mathcal{F}_{ab}$, leaving only two nonzero complex components

$$\mathcal{F}_{12} = \mathcal{F}_{34} = F_{12} + F_{34}, \tag{109}$$

and

$$\mathcal{F}_{31} = 2F_{31}. \tag{110}$$

The survived gravitational equations are

$$(Z + \bar{Z})[h_{,4} + (\bar{Z} - Z)h] + 2Z^2 h = \mathcal{F}_{12}\bar{\mathcal{F}}_{12}, \tag{111}$$

and

$$[h_{,4} + (\bar{Z} - Z)h]_{,4} + 2Z[h_{,4} + (\bar{Z} - Z)h] = -\mathcal{F}_{12}\bar{\mathcal{F}}_{12}. \tag{112}$$

Adding these equations, one obtains the equation for h

$$h_{,44} + 2(Z + \bar{Z})h_{,4} + 2Z\bar{Z}h = 0 \tag{113}$$

which has the general solution

$$h = \frac{1}{2}M(Z + \bar{Z}) + BZ\bar{Z}, \quad M_{,4} = B_{,4} = 0. \tag{114}$$

By obtaining this solution, the very useful relation

$$Z_{,4} = -Z^2 \tag{115}$$

was used.[9] In the same time, one of the Maxwell equations

$$\mathcal{F}_{12,4} + 2Z F_{12} = 0 \tag{116}$$

may easily be integrated, leading to

$$\mathcal{F}_{12} = AZ^2, \quad A_{,4} = 0. \tag{117}$$

This equation, being inserted in (112) yields $B = -\frac{1}{2}A\bar{A}$, which gives the final general form for h for any geodesic and shearfree PNC

$$h = \frac{1}{2}M(Z + \bar{Z}) + A\bar{A}Z\bar{Z}. \tag{118}$$

Other Maxwell equations take the form

$$\mathcal{F}_{31} = \gamma Z - (AZ)_{,1}, \quad \gamma_{,4} = 0, \tag{119}$$

$$A_{,2} - 2Z^{-1}\bar{Z}Y_{,3}A = 0, \tag{120}$$

$$A_{,3} - Z^{-1}Y_{,3}A_{,1} - \bar{Z}^{-1}\bar{Y}_{,3}A_{,2} + \bar{Z}^{-1}\gamma_{,2} - Z^{-1}Y_{,3}\gamma = 0. \tag{121}$$

And finally, after teddious calculations, two last gravitational equations may be obtained

$$M_{,2} - 3Z^{-1}\bar{Z}Y_{,3}M - A\bar{\gamma}\bar{Z} = 0, \tag{122}$$

and

$$M_{,3} - Z^{-1}Y_{,3}M_{,1} - \bar{Z}^{-1}\bar{Y}_{,3}M_{,2} - \frac{1}{2}\gamma\bar{\gamma} = 0. \tag{123}$$

These equations may further be reduced for $\gamma = 0$, which means the restriction by a stationary electromagnetic field, without wave excitations. It yields

$$\mathcal{F}_{31} = -(AZ)_{,1}, \tag{124}$$

$$A_{,2} - 2Z^{-1}\bar{Z}Y_{,3}A = 0, \tag{125}$$

$$A_{,3} - Z^{-1}Y_{,3}A_{,1} - \bar{Z}^{-1}\bar{Y}_{,3}A_{,2} = 0. \tag{126}$$

$$M_{,2} - 3Z^{-1}\bar{Z}Y_{,3}M = 0, \tag{127}$$

and

$$M_{,3} - Z^{-1}Y_{,3}M_{,1} - \bar{Z}^{-1}\bar{Y}_{,3}M_{,2} = 0, \tag{128}$$

$$M_{,4} = A_{,4} = 0. \tag{129}$$

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[9]It was obtained from commutation relation for $Y_{,[14]}$.

These equations contain the function $Z^{-1}Y_{,3}$ which is determined by the Consequence 5 of the Kerr theorem, (102), as follows

$$\bar{Z}Z^{-1}Y_{,3} = -(\log P)_{,2} \ , \quad P_{,4} = 0. \tag{130}$$

So, by using this expression one obtains

$$(\log AP^2)_{,2} = 0, \quad A_{,4} = P_{,4} = 0 \tag{131}$$

$$(\log MP^3)_{,2} = 0, \quad M_{,4} = P_{,4} = 0. \tag{132}$$

Since $Y_{,2} = Y_{,4} = 0$, the general solutions of these equation have the form

$$A = \psi(Y)/P^2 \tag{133}$$

and

$$M = m(Y)/P^3. \tag{134}$$

APPENDIX D. MULTISHEETED PNC AND DOUBLE TWISTOR BUNDLES

In general case, the twistorial structure for an i-th particle is determined by solution $Y_i^{\pm}(x)$ and forms a double twistor bundle $E = CM^4 \times CP^1$, where $CP^1 = S^2$ is the Riemann sphere, parametrized by projective angular coordinate $Y \in CP^1$, and CM^4 is a complexified Minkowski space-time:

$$\begin{array}{ccc} & E^5 & \quad (135) \\ \pi_2 \swarrow & & \searrow \pi_1 \\ CP^3 & & CM^4. \end{array}$$

The bundle with the base CM^4 has the fiber $F_x = \pi_1^{-1}(x) = CP^1 = S^2$, in which the complex coordinate $Y \in CP^1$ parametrizes the subspace of complex null α-planes incident to the point x.

The base of dual bundle, CP^3, is the space of projective twistor coordinates $Z = \{Y, \quad \lambda_1 = \zeta - Yv, \quad \lambda_2 = u + Y\bar{\zeta} \in CP^3\}$. The fiber of this bundle $F_Z = \pi_2^{-1}(Z)$ represents a fixed complex null plane (twistor) in CM^4.

The subset PN of "null" twistors $Z_N = \{Y, \quad \zeta - Yv, \quad u + Y\bar{\zeta}\} \in PN \subset CP^3$, is determined by the coordinates $\{Y, \quad \lambda_1 = \zeta - Yv, \quad \lambda_2 = u + Y\bar{\zeta}\}$, where $(u, v, \zeta, \bar{\zeta})$ are the null Cartesian coordinates of the real Minkowski space-time M^4, i.e. u and v are real, and ζ, and $\bar{\zeta}$ are complex conjugate. The subset PN may also be selected by quadric

$$PN = \{Z : Z_\alpha \bar{Z}^\alpha = 0\}, \tag{136}$$

$PN \in CP^3 \times \bar{CP}^3$.

Therefore, the above double twistor bundle may be restricted to the real twistor bundle: $E = M^4 \times CP^1$

$$\begin{array}{ccc} & E^5 & \quad (137) \\ \pi_2 \swarrow & & \searrow \pi_1 \\ PN & & M^4, \end{array}$$

where the base M^4 is the real Minkowski space, and the fiber is again the Riemann sphere CP^1, parametrized by complex projective angular coordinate Y. The fiber of this bundle $F_x = \pi_1^{-1}(x) = CP^1$ is the subspace of the real null rays incident to the point $x \in M^4$.

The dual bundle has the base PN which is parametrized by the null projective twistors which lie on the quadric (136), $Z \in PN$, and the fiber of this bundle $F_Z = \pi_2^{-1}(Z)$ represents a fixed null ray (real twistor) in M^4. One can also note that $Y_i(x) = Y_j(x)$ on the common (ij)-twistor line. Since the tetrad vectors are determined by the values of Y, $\bar{Y}$, it has a consequence that all the tetrad vectors e^a match on this line for the different sheets over M^4, and also over $\mathcal{K}^4$ for exception the singular lines. It confirms that the sheets are connected, forming the covering space over CP^1 with the set of branch points which are determined by the equations (22). Since these null rays are determined by the tetrad null vector $e^3(Y, \bar{Y})$ at the point x, this twistor bundle may be extended to the cotangent bundle of the Kerr-Schild null tetrad e^a, $a = 1, 2, 3, 4$ which is also fixed by the fields of $Y(x) \in CP^1$ on the base space M^4 (See Appendix A). On the other hand, the twistor bundle $E = CP^1 \times M^4$ may be extended to the complexified Minkowski space, $E = CP^1 \times CM^4$, with base $CM^4 \ni x$ and fibers F_x which are incident to x complex null planes (α-planes), spanned by the tetrad vectors $e^1 \wedge e^3$ and controlled by $Y \in CP^1 = S^2$.

PNC, or twistorial structure, related to i-th particle is a section of fibre bundle $s_i(E) = s(E, Y_i(x)) = s\{x, e^3(Y_i(x))\}$. Multisheeted twistor space is formed by the set of sections $s_i(E) = s(E, Y_i(x))$, which are controlled by multisheeted function $Y(x)$.

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