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The Schrödinger Wave Functional and Vacuum States in Curved Spacetime

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Abstract

The Schrödinger picture description of vacuum states in curved spacetime is further developed. General solutions for the vacuum wave functional are given for both static and dynamic (Bianchi type I) spacetimes and for conformally static spacetimes of Robertson-Walker type. The formalism is illustrated for simple cosmological models with time-dependent metrics and the phenomenon of particle creation is related to a special form of the kernel in the vacuum wave functional.

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1 Introduction

Many of the unusual and potentially paradoxical phenomena which occur in quantum field theories in curved spacetimes reflect the nature of the vacuum state, for a review see for example [1, 2, 3, 4]. However, whereas in Minkowski spacetime we have a simple and unambiguous prescription for determining the vacuum, the specification of the vacuum state in a general curved spacetime involves many subtleties. Indeed, it is not assured that a state satisfying the defining attributes of the Minkowski vacuum even exists for a general spacetime.

In this and a companion paper [5], we discuss the nature of the vacuum state for a broad class of flat and curved spacetimes and develop a formalism of sufficient generality to allow a future application to the vacuum structure of black hole spacetimes and the associated Hawking radiation [6, 4].

This formalism is based on the Schrödinger wave functional. The essential reason for this choice is to avoid the conventional description of the vacuum as the ‘no-particle’ state in a Fock space of states generated by the creation operators of particles defined with respect to a particular mode decomposition of the quantum field. This approach is inherently problematic in curved spacetime, where there is no uniquely favoured mode decomposition and no guarantee that the usual concept of a particle is a good description of the spectrum of the theory. The Schrödinger wave functional circumvents this second problem by giving an intrinsic description of the vacuum without reference to the spectrum of excited states.

The wave functional that defines the vacuum state satisfies a functional Schrödinger equation, which presupposes a foliation of spacetime into a series of space-like hypersurfaces, the Schrödinger equation describing the evolution of the wave functional between successive hypersurfaces. The wave functional therefore depends on the choice of foliation.³ Our expectation is that physical quantities should be independent of this foliation. However, intermediate quantities such as Green functions may have a foliation dependence. This dependence should be controlled by a functional Ward identity (somewhat analogous to the dependence of Green functions on the choice of gauge in ordinary quantum field theory). Physical quantities should depend only on the spacetime manifold and boundary conditions.

³Notice that the choice of foliation is quite distinct from a choice of coordinates. The wave functional is independent of any particular coordinates used to describe either the spacetime or the foliation (see section 3). In the Schrödinger formalism, the choice of foliation replaces the mode decomposition in the conventional approach. One of the advantages of this formalism is the clear separation it provides between the choice of vacuum state and the issue of coordinate independence.

In this paper, which is intended as the first step in a general programme, we consider a class of globally hyperbolic spacetimes, viz. manifolds with the topology $\mathcal{M} = \mathcal{R} \times \Sigma$ where Σ are spacelike hypersurfaces. Within this category, we distinguish manifolds which are ‘static’ and ‘dynamic’, essentially implying that the metric depends only on the spacelike and timelike coordinates respectively. The class referred to as ‘dynamic’ is in fact the Bianchi type I spacetimes. We also consider more general conformally static spacetimes where the conformal factor is purely time dependent. This includes the cosmologically important class of Robertson-Walker spacetimes.

The ultimate goal of our work is a full analysis of the physics of the vacuum state in the Schwarzschild and Kruskal spacetimes describing collapsed and eternal black holes. In this and the companion paper, we develop all the necessary theoretical formalism and, as a warm-up, illustrate the techniques on a number of simple illustrative examples. The solutions for static and dynamic spacetimes are sufficient to encompass the full Kruskal metric and we check explicitly that subtleties involving foliation dependence and the interpretation of vacuum states are under control. The extension to black hole spacetimes is in principle a straightforward application of these techniques, although the presence of real event horizons separating the static and dynamic regions is an important new feature.

A number of the results and techniques described here can be found already elsewhere in the literature. In these cases, we have referenced the original works as fully as possible, but have included the material as we have been concerned to present a coherent and systematic study of the Schrödinger wave functional.

The paper is organised as follows. In section 2 we review the Schrödinger picture formulation of quantum field theory in Minkowski spacetime and derive the Schrödinger equation and its solutions for the vacuum and excited states for a free scalar field. We also discuss some properties of a general Gaussian state which will be needed in sections 4 and 5.

Then in section 3 we construct the Schrödinger wave functional equation for an arbitrary foliation of a globally hyperbolic spacetime with topology $\mathcal{M} = \mathcal{R} \times \Sigma$. This makes explicit the dependence of the equation, and therefore the vacuum wave functional, on the ‘lapse’ and ‘shift’ functions specifying the foliation. The simplification for special coordinate choices which mirror the foliation is shown. We find explicit solutions for the vacuum wave functional for the special cases of static and dynamic spacetimes and a particular class of conformally static metrics including the Robertson-Walker spacetimes. This analysis shows clearly that while there is frequently an essentially unique vacuum state for a static spacetime, in the dynamic case there is necessarily a one-parameter family of vacuum solutions of the Schrödinger equation.

In sections 4 and 5, these techniques are illustrated using two simple two-dimensional ‘cosmological models’ which have been much discussed in the literature. The first represents an asymptotically static universe that undergoes a period of expansion while the second describes a universe which expands from an initial singularity before recollapsing. These are of particular interest as models illustrating cosmological particle creation. We solve the Schrödinger wave functional equation exactly to determine the candidate vacuum states and discuss carefully the interpretation that particles are created by the expansion.

A number of technical results on solutions and Green functions for various differential equations occurring in the text are given in the appendices.

2 Schrödinger Picture in Minkowski Spacetime

To introduce the Schrödinger formalism, we begin by reviewing the wave functional equation and its solutions in Minkowski spacetime. (For a review see, e.g. [7].) Consider a massive scalar field $\varphi(x)$ with action⁴

$$S = \int d^{d+1}x \mathcal{L} \quad (1)$$

where the Lagrangian density is

$$\mathcal{L} = \frac{1}{2} \left\{ \eta^{\mu\nu} (\partial_\mu \varphi)(\partial_\nu \varphi) - m^2 \varphi^2 \right\} \quad (2)$$

The canonical momentum conjugate to the field is defined by

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \varphi)} = \partial_0 \varphi(x) \quad (3)$$

The Hamiltonian is constructed by a Legendre transform of the Lagrangian

$$H(\pi, \varphi) = \int d^d \underline{x} [\pi \dot{\varphi} - \mathcal{L}] \quad (4)$$

$$= \frac{1}{2} \int d^d \underline{x} \left\{ \pi^2 - \eta^{ij} (\partial_i \varphi)(\partial_j \varphi) + m^2 \varphi^2 \right\} \quad (5)$$

The system can now be canonically quantised by treating the fields as operators and imposing appropriate commutation relations. This involves the choice of a foliation of the spacetime into a succession of spacelike hypersurfaces. In Minkowski spacetime, it is natural to choose these to be the hypersurfaces of fixed t and impose the equal-time commutation relations

$$[\varphi(\underline{x}, t), \pi(\underline{y}, t)] = i\delta^d(\underline{x} - \underline{y})$$

⁴Our conventions are as follows. For generality we consider a spacetime of $d+1$ dimensions. The metric signature is $(+, -, \dots, -)$. Coordinates for the spacetime point x are denoted either by (x^0, x^i) , where $i = 1, \dots, d$, or by (t, \underline{x}) .

$$[\varphi(\underline{x}, t), \varphi(\underline{y}, t)] = [\pi(\underline{x}, t), \pi(\underline{y}, t)] = 0 \quad (6)$$

In the Schrödinger picture, we take the basis vectors of the state vector space to be the eigenstates of the field operator $\varphi(x)$ on a fixed t hypersurface, with eigenvalues $\phi(\underline{x})$,

$$\varphi(x)|\phi(\underline{x}), t\rangle = \phi(\underline{x})|\phi(\underline{x}), t\rangle \quad (7)$$

Notice that the set of field eigenvalues $\phi(\underline{x})$ is independent of the value of t labelling the hypersurfaces. This will be especially important in the curved space-time context.

In this picture, the quantum states are explicit functions of time and are represented by wave functionals $\Psi[\phi(\underline{x}), t]$. Operators $O(\pi, \varphi)$ acting on these states may be represented by

$$O(\pi(x), \varphi(x)) \sim O\left(-i\frac{\delta}{\delta\phi(\underline{x})}, \phi(\underline{x})\right) \quad (8)$$

consistent with the commutation relations (6). The Schrödinger equation, which governs the evolution of the wave functional between the spacelike hypersurfaces, is

$$\begin{aligned} i\frac{\partial\Psi[\phi, t]}{\partial t} &= H\left(-i\frac{\delta}{\delta\phi(\underline{x})}, \phi(\underline{x})\right)\Psi[\phi, t] \\ &= \frac{1}{2}\int d^d\underline{x} \left\{ -\frac{\delta^2}{\delta\phi^2} - \eta^{ij}(\partial_i\phi)(\partial_j\phi) + m^2\phi^2 \right\} \Psi[\phi, t] \end{aligned} \quad (9)$$

To solve this equation, we make the ansatz that, up to a time-dependent phase, the vacuum wave functional is a simple Gaussian. This is in direct analogy with the standard solution of the harmonic oscillator in quantum mechanics. We therefore write

$$\Psi_0[\phi, t] = N_0(t) \exp\left\{-\frac{1}{2}\int d^d\underline{x} \int d^d\underline{y} \phi(\underline{x}) G(\underline{x}, \underline{y}) \phi(\underline{y})\right\} = N_0(t)\psi_0[\phi] \quad (10)$$

Substituting into (9) and comparing coefficients of $[\phi^0]$ determines $N_0(t)$ from

$$\frac{d\ln N_0(t)}{dt} = -\frac{i}{2}\int d^d\underline{x} G(\underline{x}, \underline{x}) \quad (11)$$

while comparing coefficients of $[\phi^2]$ gives an equation for the kernel $G(\underline{x}, \underline{y})$:

$$\int d^d\underline{z} G(\underline{x}, \underline{z}) G(\underline{z}, \underline{y}) = (-\nabla^2 + m^2)\delta^d(\underline{x} - \underline{y}) \quad (12)$$

The kernel equation can be solved by using several methods, either by working in momentum space, or coordinate space, or using the Schwinger-DeWitt representation of the propagator [8, 9].

We show here the momentum space approach, which is the most convenient in Minkowski spacetime. Introducing the Fourier transform of the kernel,

$$G(\underline{x}, \underline{y}) = \int \frac{d^d k}{(2\pi)^d} e^{i\underline{k} \cdot (\underline{x} - \underline{y})} \tilde{G}(\underline{k}) \quad (13)$$

the kernel eq.(12) reduces to

$$\tilde{G}(\underline{k})^2 = \underline{k}^2 + m^2 \quad (14)$$

The inverse of the kernel, $\Delta(\underline{x}, \underline{y})$, is defined as

$$\int d^d z G(\underline{x}, \underline{z}) \Delta(\underline{z}, \underline{y}) = \delta^d(\underline{x} - \underline{y}) \quad (15)$$

and is therefore given explicitly by

$$\Delta(\underline{x}, \underline{y}) = \int \frac{d^d k}{(2\pi)^d} \frac{e^{i\underline{k} \cdot (\underline{x} - \underline{y})}}{\sqrt{\underline{k}^2 + m^2}} \quad (16)$$

The kernel is simply related to the Green functions of the theory (see appendix B). A general Green function can be written as

$$\mathcal{G}(x, y) = \int \frac{d^{d+1} k}{(2\pi)^{d+1}} \frac{e^{i\underline{k} \cdot (\underline{x} - \underline{x}') - ik_0(x_0 - y_0)}}{k_0^2 - \omega^2} \quad (17)$$

where $\omega^2 = \underline{k}^2 + m^2$. The k_0 integral has poles at $k_0 = \pm \omega$ and can be evaluated with different boundary conditions, determining how the contour goes round the poles. In particular, the Wightman function $\mathcal{G}_+(x, y)$ is defined by choosing the contour integral to close around the $k_0 = \omega$ pole as in Fig(3) of appendix B. For equal times, this Green function reduces to

$$\mathcal{G}_+(x, y)|_{ET} = \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \frac{e^{i\underline{k} \cdot (\underline{x} - \underline{y})}}{\sqrt{\underline{k}^2 + m^2}} \quad (18)$$

$$= \frac{1}{2\pi} \left(\frac{m}{2\pi|\underline{x} - \underline{y}|} \right)^{\frac{d}{2} - \frac{1}{2}} K_{\frac{d}{2} - \frac{1}{2}}(m|\underline{x} - \underline{y}|) \quad (19)$$

The inverse kernel is therefore simply twice the Wightman function evaluated at equal times, i.e.

$$\Delta(\underline{x}, \underline{y}) = 2\mathcal{G}_+(x, y)|_{ET} \quad (20)$$

The reason for this identification is made clear in appendix C where we discuss the representation of vacuum expectation values and two-point functions in the Schrödinger picture.

The kernel itself is

$$G(\underline{x}, \underline{y}) = \frac{1}{2} \mathcal{G}_+(x, y)|_{ET}^{-1} \quad (21)$$

$$= \int \frac{d^d \underline{k}}{(2\pi)^d} \sqrt{\underline{k}^2 + m^2} e^{i\underline{k} \cdot (\underline{x} - \underline{y})} \quad (22)$$

$$= -2 \left(\frac{m}{2\pi |\underline{x} - \underline{y}|} \right)^{\frac{d}{2} + \frac{1}{2}} K_{\frac{d}{2} + \frac{1}{2}}(m|\underline{x} - \underline{y}|) \quad (23)$$

The time dependent phase factor in the vacuum wave functional is

$$N_0(t) = e^{-iE_0 t} \quad (24)$$

where

$$E_0 = \frac{1}{2} \int d^d \underline{x} G(\underline{x}, \underline{x}) \quad (25)$$

$$= \frac{1}{2} \int d^d \underline{k} \sqrt{\underline{k}^2 + m^2} \delta^d(0) \quad (26)$$

is the (divergent) ground state energy.

There are several ways to derive the excited state solutions to the Schrödinger wave functional equation, but with a view to the curved spacetime generalisation, we start with a method that does not presuppose a particle interpretation of the excited states.

Making the following ansatz for the first excited state

$$\Psi_1[\phi, t] = 2 N_1(t) \int d^d \underline{x} \int d^d \underline{y} \phi(\underline{x}) G(\underline{x}, \underline{y}) f(\underline{y}) \psi_0[\phi] = N_1(t) \psi_1[\phi] \quad (27)$$

the Schrödinger equation is solved with the same kernel (12) as for the ground state, provided

$$N_1(t) = e^{-iE_1 t} \quad (28)$$

where the energy gap $\Delta E_1 = E_1 - E_0$ satisfies

$$\Delta E_1 \int d^d \underline{y} G(\underline{x}, \underline{y}) f(\underline{y}) = (-\nabla^2 + m^2) f(\underline{x}) \quad (29)$$

This is readily solved if $f(\underline{x})$ is an eigenfunction of the operator $(-\nabla^2 + m^2)$. We therefore choose $f_k(\underline{x}) = e^{i\underline{k} \cdot \underline{x}}$, which has an eigenvalue of $(\underline{k}^2 + m^2)$. We then find

$$\Delta E_1 \int d^d \underline{y} G(\underline{x}, \underline{y}) f_k(\underline{y}) = (\underline{k}^2 + m^2) f_k(\underline{x})$$

which reduces to

$$\Delta E_1 \tilde{G}(\underline{k}) = \underline{k}^2 + m^2$$

so the energy gap is

$$\Delta E_1 = \sqrt{\underline{k}^2 + m^2} = \omega \quad (30)$$

Clearly this admits a particle interpretation where the first excited state Ψ_1 is a one-particle state and E_1 is its energy. This is apparent from the energy gap, $\Delta E_1 = \sqrt{\underline{k}^2 + m^2}$, which is the energy of one particle with momentum \underline{k} . The wave functional carries a label \underline{k} , introduced through the eigenfunction $f_{\underline{k}}$, which specifies the momentum of the particle.

The Schrödinger equation can also be solved directly (as in the quantum mechanical harmonic oscillator) for an n -particle state, giving

$$\Psi_n[\phi, t] = e^{-iE_n t} H_n[\phi] \psi_0[\phi] \quad (31)$$

where the H_n are generalised Hermite polynomials defined using the kernel and eigenfunctions $f_{\underline{k}_j}$ contracted with the field ϕ as in (27). The energy of the n th excited state is simply

$$E_n = E_0 + \sum_{i=1}^n \omega(\underline{k}_i) \quad (32)$$

where \underline{k}_i is the momentum of the i th particle.

The excited states can also be described using creation and annihilation operators. We define

$$\hat{a}(\underline{k}) = \int d^d \underline{x} \left[\int d^d \underline{y} G(\underline{x}, \underline{y}) \phi(\underline{y}) + \frac{\delta}{\delta \phi(\underline{x})} \right] f_{\underline{k}}^*(\underline{x}) \quad (33)$$

$$= \int d^d \underline{x} \left[\omega \phi(\underline{x}) + \frac{\delta}{\delta \phi(\underline{x})} \right] f_{\underline{k}}^*(\underline{x}) \quad (34)$$

and

$$\hat{a}^\dagger(\underline{k}) = \int d^d \underline{x} \left[\int d^d \underline{y} G(\underline{x}, \underline{y}) \phi(\underline{y}) - \frac{\delta}{\delta \phi(\underline{x})} \right] f_{\underline{k}}(\underline{x}) \quad (35)$$

$$= \int d^d \underline{x} \left[\omega \phi(\underline{x}) - \frac{\delta}{\delta \phi(\underline{x})} \right] f_{\underline{k}}(\underline{x}) \quad (36)$$

consistent with the commutation relations

$$[\hat{a}(\underline{k}), \hat{a}^\dagger(\underline{p})] = 2(2\pi)^d \omega \delta^d(\underline{k} - \underline{p}) \quad (37)$$

These can be used iteratively to construct all the excited states, starting from

$$\hat{a}(\underline{k}) \psi_0 = 0 \quad \hat{a}^\dagger(\underline{k}) \psi_0 = \psi_1 \quad (38)$$

The n th excited state is labelled by the n particle momenta, viz.

$$\psi_n(\underline{k}_1, \dots, \underline{k}_n) = a^\dagger(\underline{k}_1) \dots a^\dagger(\underline{k}_n) \psi_0 \quad (39)$$

Orthogonality relations between these states follow directly from the commutation relations of the creation and annihilation operators. We have

$$\begin{aligned} \langle \psi_n(\underline{k}_1, \dots, \underline{k}_n) | \psi_m(\underline{p}_1, \dots, \underline{p}_m) \rangle &= \delta_{mn} \left\{ \prod_{i=1}^n 2\omega_i (2\pi)^d \delta^d(\underline{k}_i - \underline{p}_i) \right. \\ &\quad \left. + \dots n! \text{ perms} \right\} \langle \psi_0 | \psi_0 \rangle \end{aligned} \quad (40)$$

where the inner product is defined by

$$\langle \Psi_1 | \Psi_2 \rangle = \int \mathcal{D}\phi \Psi_1^*(\phi) \Psi_2(\phi) \quad (41)$$

and $\omega_i = \sqrt{\underline{k}_i^2 + m^2}$. The Hamiltonian and particle number operators are written in terms of a and a^\dagger as usual as

$$H = \frac{1}{4} \int \frac{d^d \underline{k}}{(2\pi)^d} \left[\hat{a}^\dagger(\underline{k}) \hat{a}(\underline{k}) + \hat{a}(\underline{k}) \hat{a}^\dagger(\underline{k}) \right] \quad (42)$$

and

$$N = \int \frac{d^d \underline{k}}{(2\pi)^d} \frac{1}{2\omega} \hat{a}^\dagger(\underline{k}) \hat{a}(\underline{k}) \quad (43)$$

It will often be convenient to express the wave functional for the vacuum and excited states in terms of the Fourier transform $\tilde{\phi}(\underline{k})$ of the field eigenvalues defined in eq.(7). We define

$$\phi(\underline{x}) = \int \frac{d^d \underline{k}}{(2\pi)^d} e^{i\underline{k} \cdot \underline{x}} \tilde{\phi}(\underline{k}) \quad (44)$$

and since $\phi(\underline{x})$ is real, $\tilde{\phi}^*(\underline{k}) = \tilde{\phi}(-\underline{k})$. The vacuum wave functional is therefore

$$\psi_0[\tilde{\phi}] = \exp \left\{ -\frac{1}{2} \int \frac{d^d \underline{k}}{(2\pi)^d} \omega |\tilde{\phi}(\underline{k})|^2 \right\} \quad (45)$$

In terms of $\tilde{\phi}(\underline{k})$, the creation and annihilation operators simplify:

$$\hat{a}(\underline{k}) = \left[\omega \tilde{\phi}(\underline{k}) + \frac{\delta}{\delta \tilde{\phi}^*(\underline{k})} \right] \quad (46)$$

$$\hat{a}^\dagger(\underline{k}) = \left[\omega \tilde{\phi}^*(\underline{k}) - \frac{\delta}{\delta \tilde{\phi}(\underline{k})} \right] \quad (47)$$

where $\delta \tilde{\phi}(\underline{p}) / \delta \tilde{\phi}(\underline{k}) = (2\pi)^d \delta^d(\underline{k} - \underline{p})$ and $\delta \tilde{\phi}^*(\underline{p}) / \delta \tilde{\phi}(\underline{k}) = (2\pi)^d \delta^d(\underline{k} + \underline{p})$. The excited states for particles with momenta \underline{k}_i are therefore given by

$$\psi_1 = 2[\omega_1 \tilde{\phi}^*(\underline{k}_1)] \psi_0 \quad (48)$$

$$\psi_2 = \left\{ 4[\omega_1 \tilde{\phi}^*(\underline{k}_1)][\omega_2 \tilde{\phi}^*(\underline{k}_2)] - 2\omega_1 (2\pi)^d \delta^d(\underline{k}_1 + \underline{k}_2) \right\} \psi_0 \quad (49)$$

etc. The general n -particle wave functional is expressed in terms of generalised Hermite polynomials with n momentum labels, viz.

$$\begin{aligned}\Psi_n[\tilde{\phi}, t; \underline{k}_1, \dots, \underline{k}_n] &= e^{-iE_n t} H_n[\tilde{\phi}; \underline{k}_1, \dots, \underline{k}_n] \psi_0[\tilde{\phi}] \\ &= e^{-iE_n t} \sum_{m=0}^{\left[\frac{n}{2}\right]} \left\{ (-1)^m 2^{n-m} \left[\prod_{i=1}^m \omega_{2i-1} (2\pi)^d \delta(\underline{k}_{2i-1} + \underline{k}_{2i}) \right] \right. \\ &\quad \left. \left[\prod_{j=2m+1}^n \omega_j \tilde{\phi}^*(\underline{k}_j) \right] + \dots (2m-1)!! \binom{n}{2m} \text{perms} \right\} \psi_0 \quad (50)\end{aligned}$$

where there are $(2m-1)!! \binom{n}{2m}$ permutations of the pairing of momenta at each stage of the sum. The sum is from zero to $\left[\frac{n}{2}\right]$ - which is the largest integer $\leq \frac{n}{2}$. Products $\prod_{i=a}^b$ with $b < a$ are defined to be 1.

All this analysis is based on the specific foliation of Minkowski spacetime in which the family of spacelike hypersurfaces Σ are equal-time hypersurfaces and the Hamiltonian evolution is along the timelike Killing vector field $\frac{\partial}{\partial t}$. While this is convenient, it is by no means a unique choice. In section 3, we set up the Schrödinger equation for an arbitrary foliation in curved spacetime. This formalism may of course be specialised to Minkowski spacetime by simply replacing the metric $g_{\mu\nu}$ by $\eta_{\mu\nu}$ throughout.

2.1 General Gaussian state

In the discussion of the cosmological models in sections 4 and 5 we encounter Gaussian states where the kernel is time dependent and not simply equal to ω , the eigenvalue of the wave operator. In anticipation of this, we show here how such a state in Minkowski spacetime may be expressed as a superposition of excited many-particle states.

Consider a solution of the Schrödinger equation of the form

$$\Psi[\tilde{\phi}, t] = N(t) \exp \left\{ -\frac{1}{2} \int \frac{d^d \underline{k}}{(2\pi)^d} \Omega(\underline{k}; t) |\tilde{\phi}(\underline{k})|^2 \right\} \quad (51)$$

where

$$N(t) = \exp \left\{ -\frac{i}{2} \int^t dt \int \frac{d^d \underline{k}}{(2\pi)^d} \Omega(\underline{k}; t) \right\} \quad (52)$$

and $\Omega(\underline{k}; t)$ is a solution of

$$i \frac{\partial \Omega(\underline{k}; t)}{\partial t} = \omega^2 - \Omega^2 \quad (53)$$

This can be written as a linear superposition of n -particle states as follows:

$$\Psi = \sum_{n=0}^{\infty} \left[\prod_{i=1}^n \int \frac{d^d \underline{k}_i}{(2\pi)^d} \right] c_n(\underline{k}_1, \dots, \underline{k}_n) \psi_n(\underline{k}_1, \dots, \underline{k}_n) \quad (54)$$

where the (time-dependent) expansion coefficients are

$$c_n(\underline{k}_1, \dots, \underline{k}_n) = \frac{\int \mathcal{D}\tilde{\phi} \psi_n^*(\underline{k}_1, \dots, \underline{k}_n) \Psi}{n! [\prod_{i=1}^n 2\omega_i]} \quad (55)$$

and we are assuming constant normalisation factors are included such that $\langle \psi_0 | \psi_0 \rangle$ and $\langle \Psi | \Psi \rangle$ are both 1. (For the justification that this is possible for the general Gaussian state, see section 3.) The n -particle states have been given earlier. We therefore have

$$\int \mathcal{D}\tilde{\phi} \psi_n^*(\underline{k}_1, \dots, \underline{k}_n) \Psi = \int \mathcal{D}\tilde{\phi} H_n[\tilde{\phi}; \underline{k}_1, \dots, \underline{k}_n] e^{-\frac{1}{2} \int \frac{d^d k}{(2\pi)^d} (\Omega + \omega) |\tilde{\phi}(\underline{k})|^2} \quad (56)$$

The odd moments of the integral vanish and therefore the coefficients c_n vanish for n odd. For even n , we find

$$c_n(\underline{k}_1, \dots, \underline{k}_n) = \left\{ \frac{1}{n!} \left[\prod_{i=1}^{n/2} \frac{(2\pi)^d \delta^d(\underline{k}_{2i-1} + \underline{k}_{2i})}{2\omega_{2i-1}} \left(\frac{\omega_{2i-1} - \Omega_{2i-1}}{\omega_{2i-1} + \Omega_{2i-1}} \right) \right] + \dots (n-1)!! \text{ perms} \dots \right\} \langle \psi_0 | \Psi \rangle \quad (57)$$

where $\Omega_i = \Omega(\underline{k}_i; t)$ and the $(n-1)!!$ permutations specify different combinations of the paired momenta.

It is particularly interesting to look at the expectation value in the general Gaussian state of the number operator $N(\underline{k})$, which counts the number of particles with prescribed momentum in the expansion (54). This is

$$\begin{aligned} \langle \Psi | N(\underline{k}) | \Psi \rangle &= \frac{1}{2\omega} \langle \Psi | \hat{a}^\dagger(\underline{k}) \hat{a}(\underline{k}) | \Psi \rangle \\ &= \frac{1}{2} \left\{ \omega \langle \Psi | \tilde{\phi}(\underline{k}) \tilde{\phi}(-\underline{k}) | \Psi \rangle - \frac{1}{\omega} \langle \Psi | \frac{\delta^2}{\delta \tilde{\phi}(\underline{k}) \delta \tilde{\phi}(-\underline{k})} | \Psi \rangle - (2\pi)^d \delta^d(0) \right\} \end{aligned} \quad (58)$$

The expectation values can be easily evaluated using techniques described in appendix C. For complex Ω , we find the expectation value of the number density to be

$$\langle \mathcal{N}(\underline{k}) \rangle = \frac{(\omega - \Omega)(\omega - \Omega^*)}{2\omega(\Omega + \Omega^*)} \quad (59)$$

This result can also be recovered using the explicit expressions for the expansion coefficients c_n given above.

3 Schrödinger Picture in Curved Spacetime

We consider globally hyperbolic spacetimes of topology $\mathcal{M} = \mathcal{R} \times \Sigma$ with a global timelike vector field. To provide a general foliation of the spacetime, we use the formalism of embedding variables [10, 11, 12]. We therefore choose a family of spacelike hypersurfaces Σ , with intrinsic coordinates ξ^i , labelled by a ‘time’ parameter s and consider evolution along the integral curves of $\frac{\partial}{\partial s}$. The embeddings characterising the foliation are maps $x : \Sigma \rightarrow \mathcal{M}$ which take a point on the hypersurface Σ to a spacetime point $x^\mu = x^\mu(s, \xi^i)$.

First we need to construct the projections of spacetime quantities normal and tangential to the hypersurface Σ . The tangential projections are defined using

$$p_i^\mu = \frac{\partial x^\mu}{\partial \xi^i} \quad (60)$$

The normal to the surface, n_μ , is timelike and is uniquely defined by

$$n_\mu p_i^\mu = 0 \quad (61)$$

with $g^{\mu\nu} n_\mu n_\nu = 1$. Any spacetime tensor can be expanded in terms of its projections normal and tangential to Σ . We can also introduce an induced metric on Σ ,

$$h_{ij} = g_{\mu\nu} p_i^\mu p_j^\nu \quad (62)$$

and raise and lower hypersurface indices, e.g. define $p_\mu^i = h^{ij} p_j^\nu g_{\mu\nu}$.

Now consider the deformation vector of the foliation, defined as

$$\begin{aligned} N^\mu &= \dot{x}^\mu \equiv \frac{\partial x^\mu}{\partial s} \\ &= N n^\mu + N^i p_i^\mu \end{aligned} \quad (63)$$

where N and N^i are the ‘lapse’ and ‘shift’ functions of the embeddings given by

$$N = n_\mu N^\mu \quad (64)$$

$$N^i = p_\mu^i N^\mu \quad (65)$$

In terms of these functions, the metric interval can be written as

$$g_{\mu\nu} dx^\mu dx^\nu = (N^2 + N^i N_i) ds^2 + 2N_i ds d\xi^i + h_{ij} d\xi^i d\xi^j \quad (66)$$

We also need the inverse relations

$$\frac{\partial s}{\partial x^\mu} = \frac{n_\mu}{N} \quad (67)$$

$$\frac{\partial \xi^i}{\partial x^\mu} = \frac{-n_\mu N^i}{N} + p_\mu^i \quad (68)$$

and the Jacobian

$$\det\left[\frac{\partial x^\mu}{\partial(s, \xi^i)}\right] = \frac{N\sqrt{-h}}{\sqrt{-g}} \quad (69)$$

With these preliminaries complete, we can now construct the Hamiltonian describing the evolution appropriate to this foliation, i.e. along the integral curves of the timelike vector field $\frac{\partial}{\partial s}$. We consider a massive scalar field theory coupled to gravity. The action is

$$S = \frac{1}{2} \int_{\mathcal{M}} d^{d+1}x \sqrt{-g} \left[g^{\mu\nu} \frac{\partial\varphi}{\partial x^\mu} \frac{\partial\varphi}{\partial x^\nu} - (m^2 + \xi R)\varphi^2 \right] \quad (70)$$

and so in terms of the embedding variables,

$$\begin{aligned} S = & \frac{1}{2} \int_R ds \int_{\Sigma} d^d\xi \det\left[\frac{\partial x^\mu}{\partial(s, \xi^i)}\right] \sqrt{-g} \left\{ g^{\mu\nu} \left[\frac{\partial s}{\partial x^\mu} \frac{\partial s}{\partial x^\nu} \dot{\varphi}^2 \right. \right. \\ & \left. \left. + 2 \frac{\partial s}{\partial x^\mu} \frac{\partial \xi^i}{\partial x^\nu} \dot{\varphi} (\partial_i \varphi) + \frac{\partial \xi^i}{\partial x^\mu} \frac{\partial \xi^j}{\partial x^\nu} (\partial_i \varphi)(\partial_j \varphi) \right] - (m^2 + \xi R)\varphi^2 \right\} \end{aligned} \quad (71)$$

where $\dot{\varphi} \equiv \frac{\partial\varphi}{\partial s}$ and $\partial_i \varphi \equiv \frac{\partial\varphi}{\partial \xi^i}$. Using eqs.(67), (68) and (69), the action can be written in the form

$$S = \int_R ds \int_{\Sigma} d^d\xi \mathcal{L} \quad (72)$$

where the Lagrangian density is,

$$\mathcal{L} = \frac{1}{2} N \sqrt{-h} \left\{ \frac{\dot{\varphi}^2}{N^2} - \frac{2N^i}{N^2} \dot{\varphi} (\partial_i \varphi) + \left[\frac{N^i N^j}{N^2} + h^{ij} \right] (\partial_i \varphi)(\partial_j \varphi) - (m^2 + \xi R)\varphi^2 \right\} \quad (73)$$

The conjugate momentum is defined as usual to be

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \frac{\sqrt{-h}}{N} \left\{ \dot{\varphi} - N^i (\partial_i \varphi) \right\} \quad (74)$$

The Hamiltonian is given by the Legendre transform

$$H = \int_{\Sigma} d^d\xi [\pi \dot{\varphi} - \mathcal{L}] \quad (75)$$

$$= \int_{\Sigma} d^d\xi [N \mathcal{H} + N^i \mathcal{H}_i] \quad (76)$$

where

$$\mathcal{H} = \frac{1}{2} \sqrt{-h} \left\{ -\frac{\pi^2}{h} - h^{ij} (\partial_i \varphi)(\partial_j \varphi) + (m^2 + \xi R)\varphi^2 \right\} \quad (77)$$

$$\mathcal{H}_i = \pi (\partial_i \varphi) \quad (78)$$

Canonical quantisation is implemented by imposing equal- s commutation relations for the field operators on the hypersurface Σ , viz.

$$[\varphi(\underline{\xi}, s), \pi(\underline{\zeta}, s)] = i\delta^d(\underline{\xi} - \underline{\zeta})$$

$$[\varphi(\underline{\xi}, s), \varphi(\underline{\zeta}, s)] = [\pi(\underline{\xi}, s), \pi(\underline{\zeta}, s)] = 0 \quad (79)$$

In the Schrödinger picture, the quantum states are represented by wave functionals $\Psi[\phi(\underline{\xi}), s; N, N^i, h_{ij}]$ where the variables $\phi(\underline{\xi})$ are the eigenvalues of the field operator on the equal- s hypersurfaces, i.e.

$$\varphi(x)|\phi(\underline{\xi}), s\rangle = \phi(\underline{\xi})|\phi(\underline{\xi}), s\rangle \quad (80)$$

As before, the commutation relations are realised by representing operators in terms of the field eigenvalues $\phi(\underline{\xi})$ and functional derivatives $\frac{\delta}{\delta\phi}$, viz.⁵

$$O(\pi(x), \varphi(x)) \sim O\left(-i\frac{\delta}{\delta\phi(\underline{x})}, \phi(\underline{x})\right) \quad (81)$$

The Schrödinger wave functional equation gives the evolution of $\Psi[\phi(\underline{\xi}), s]$:

$$i\frac{\partial\Psi}{\partial s} = \int_{\Sigma} d^d\underline{\xi} \left\{ \frac{1}{2}N\sqrt{-h}\left(\frac{1}{h}\frac{\delta^2}{\delta\phi^2} - h^{ij}\partial_i\phi\partial_j\phi + (m^2 + \xi R)\phi^2\right) - iN^i\partial_i\phi\frac{\delta}{\delta\phi} \right\} \Psi \quad (82)$$

This equation holds for an arbitrary foliation (specified by N , N^i and h_{ij}) and makes explicit the foliation dependence of the wave functional. In practical applications, however, it is often convenient to choose spacetime coordinates which reflect the desired foliation. In this case, we simply identify the embedding variables $(s, \underline{\xi})$ with the spacetime coordinates (t, \underline{x}) . The lapse and shift functions are then $N = \sqrt{g_{00}}$ and $N^i = 0$ (so that $g_{0i} = 0$) while the induced metric is just $h_{ij} = g_{ij}$. The Schrödinger equation reduces to

$$i\frac{\partial\Psi}{\partial t} = \frac{1}{2}\int d^d\underline{x} \sqrt{-g} \left\{ \frac{g_{00}}{g}\frac{\delta^2}{\delta\phi^2} - g^{ij}(\partial_i\phi)(\partial_j\phi) + (m^2 + \xi R)\phi^2 \right\} \Psi \quad (83)$$

This is the form we shall usually use in the remainder of this paper.

We can search for solutions to this Schrödinger equation in analogy to those found in Minkowski space. For the vacuum state, we make the ansatz

$$\Psi_0[\phi(\underline{x}), t] = N_0(t)\psi_0[\phi, t] \quad (84)$$

where

$$\psi_0[\phi, t] = \exp\left\{-\frac{1}{2}\int d^d\underline{x}\sqrt{-h_x}\int d^d\underline{y}\sqrt{-h_y}\phi(\underline{x})G(\underline{x}, \underline{y}; t)\phi(\underline{y})\right\} \quad (85)$$

This has the same form as before, except that the kernel may now depend on the ‘time’ t since the metric is in general time dependent. The spatial integrals include the correct measure to ensure that they are invariant d dimensional volume elements.

⁵The functional derivative is defined so that $\delta\phi(\underline{y})/\delta\phi(\underline{x}) = \delta^d(\underline{x} - \underline{y})$. The delta function density is given by $\delta^d(\underline{x}, \underline{y}) = (\sqrt{-h_x})^{-1}\delta^d(\underline{x} - \underline{y})$ and satisfies $\int d^d\underline{x}\sqrt{-h_x}\delta^d(\underline{x}, \underline{y})f(\underline{x}) = f(\underline{y})$.

Substituting this ansatz into the functional Schrödinger equation (83) results in the time dependence equation

$$\frac{d \ln N_0(t)}{dt} = -\frac{i}{2} \int d^d \underline{x} \sqrt{-h_x} \sqrt{g_{00}^x} G(\underline{x}, \underline{x}; t) \quad (86)$$

and the kernel equation

$$\begin{aligned} i \frac{\partial}{\partial t} \left(\sqrt{h_x h_y} G(\underline{x}, \underline{y}; t) \right) &= \int d^d \underline{z} \sqrt{-h_z} \sqrt{g_{00}^z} \sqrt{h_x h_y} G(\underline{x}, \underline{z}; t) G(\underline{z}, \underline{y}; t) \\ &\quad - \sqrt{h_x h_y} \sqrt{g_{00}^x} (\square_i + m^2 + \xi R)_x \delta^d(\underline{x}, \underline{y}) \end{aligned} \quad (87)$$

where $\square_i = \frac{1}{\sqrt{-g}} \partial_i (g^{ij} \sqrt{-g} \partial_j)$ is the spatial part of the Laplacian (see appendix A). Notice that this assumes that on the boundaries of the hypersurface Σ the fields or their derivatives vanish, since the surface terms in an integration by parts have been omitted.

We can also mimic the Minkowski solutions for excited states, although in the general case with explicit t dependence in all the quantities the physical interpretation is not immediately evident. Nevertheless, we can look for solutions of the form

$$\Psi_1[\phi, t] = N_1(t) \psi_1[\phi, t] \quad (88)$$

$$= 2N_1(t) \int d^d \underline{x} \sqrt{-h_x} \int d^d \underline{y} \sqrt{-h_y} \phi(\underline{x}) G(\underline{x}, \underline{y}; t) f_{(\lambda)}(\underline{y}) \psi_0[\phi, t] \quad (89)$$

This solves the Schrödinger equation with the same kernel as for the vacuum state. The analogue of the energy gap equation is

$$\Delta E_1(t) \int d^d \underline{y} \sqrt{-h_y} G(\underline{x}, \underline{y}; t) f_{(\lambda)}(\underline{y}) = \sqrt{g_{00}^x} (\square_i + m^2 + \xi R)_x f_{(\lambda)}(\underline{x}) \quad (90)$$

where

$$\Delta E_1(t) = i \frac{d}{dt} \left[\ln \frac{N_1}{N_0} \right] \quad (91)$$

The nature of these states depends on the functions $f_{(\lambda)}(\underline{x})$. These may be chosen to be eigenfunctions of the operator $(\square_i + m^2 + \xi R)$, the suffix λ denoting the set of quantum numbers specifying the degenerate eigenfunction (the analogues of the momenta \underline{k} in Minkowski spacetime).

We can also write down analogues of the creation and annihilation operators of section 2. If we write

$$\hat{a}(\lambda; t) = \int d^d \underline{x} \sqrt{-h_x} \left[\int d^d \underline{y} \sqrt{-h_y} G(\underline{x}, \underline{y}; t) \phi(\underline{y}) + \frac{1}{\sqrt{-h_x}} \frac{\delta}{\delta \phi(\underline{x})} \right] f_{(\lambda)}^*(\underline{x}) \quad (92)$$

and

$$\hat{a}^\dagger(\lambda; t) = \int d^d \underline{x} \sqrt{-h_x} \left[\int d^d \underline{y} \sqrt{-h_y} G(\underline{x}, \underline{y}; t) \phi(\underline{y}) - \frac{1}{\sqrt{-h_x}} \frac{\delta}{\delta \phi(\underline{x})} \right] f_{(\lambda)}(\underline{x}) \quad (93)$$

then

$$\hat{a}(\lambda; t) \psi_0[\phi, t] = 0 \quad \hat{a}^\dagger(\lambda; t) \psi_0[\phi, t] = \psi_1[\phi, t] \quad (94)$$

and

$$[\hat{a}(\lambda; t), \hat{a}^\dagger(\rho; t)] = 2 \int d^d \underline{x} \sqrt{-h_x} \int d^d \underline{y} \sqrt{-h_y} f_{(\lambda)}^*(\underline{x}) G(\underline{x}, \underline{y}; t) f_{(\rho)}(\underline{y}) \quad (95)$$

However, in contrast to Minkowski spacetime, there is no simple particle interpretation associated with these operators for a general spacetime.

The kernel equation (87) is difficult to solve in general, so we shall specialise to three cases of particular interest – ‘static’ spacetimes, where the metric is a function only of the space coordinates, ‘dynamic’ or Bianchi type I spacetimes, where the metric depends only on the time coordinate, and a special class of conformally static metrics of sufficient generality to include all Robertson-Walker spacetimes. Before restricting to these special cases, however, it is of some interest to give a partial solution valid in general.

This partial solution is given in terms of a solution $\psi(x)$ to the wave equation

$$\sqrt{-g} [\square + m^2 + \xi R] \psi(x) = 0 \quad (96)$$

Any $G(\underline{x}, \underline{y}; t)$ satisfying the equation

$$\int d^d \underline{y} \sqrt{-h_y} G(\underline{x}, \underline{y}; t) \psi(t, \underline{y}) = -i \sqrt{g_x^{00}} \frac{\partial}{\partial t} \psi(t, \underline{x}) \quad (97)$$

is then a solution of the full kernel eq.(87).

To see this, multiply the kernel equation by $\psi(t, \underline{y})$ and integrate with respect to \underline{y} and then substitute (97). This gives

$$\begin{aligned} i \int d^d \underline{y} \psi(t, \underline{y}) \frac{\partial}{\partial t} [\sqrt{h_x h_y} G(\underline{x}, \underline{y}; t)] &+ i \int d^d \underline{y} \sqrt{h_x h_y} G(\underline{x}, \underline{y}; t) \frac{\partial}{\partial t} [\psi(t, \underline{y})] \\ &= -\sqrt{-g_x} (\square_i + m^2 + \xi R)_x \psi(t, \underline{x}) \end{aligned}$$

and combining the two differential terms we find

$$i \frac{\partial}{\partial t} \left[\int d^d \underline{y} \sqrt{-h_x} \sqrt{-h_y} G(\underline{x}, \underline{y}; t) \psi(t, \underline{y}) \right] + \sqrt{-g_x} (\square_i + m^2 + \xi R)_x \psi(t, \underline{x}) = 0$$

Using eq.(97) for the kernel again now gives

$$\frac{\partial}{\partial t} \left(g_x^{00} \sqrt{-g_x} \frac{\partial}{\partial t} \psi(t, \underline{x}) \right) + \sqrt{-g_x} (\square_i + m^2 + \xi R)_x \psi(t, \underline{x}) = 0$$

which is just the wave equation satisfied by $\psi(t, \underline{x})$.

Later, when we have found explicit solutions for static or dynamic spacetimes, we see that these may be found from eq.(97) by writing the function $\psi(x)$ as an appropriate Fourier transform, using eqs.(216) or (220) respectively, and using orthonormality and completeness relations where applicable.

3.1 Static spacetimes

For static spacetimes, the kernel equation is solved with a time-independent kernel satisfying

$$\int d^d \underline{z} \sqrt{-g_z} G(\underline{x}, \underline{z}) G(\underline{z}, \underline{y}) = \sqrt{g_{00}^x} (\square_i + m^2 + \xi R)_x \delta^d(\underline{x}, \underline{y}) \quad (98)$$

The solutions can be expressed in terms of the eigenfunctions of $(\square_i + m^2 + \xi R)$, which are the Fourier transformed solutions $\tilde{\psi}_{(\lambda)}(\omega, \underline{x})$ of the wave equation for a static metric, viz.

$$(\square_i + m^2 + \xi R) \tilde{\psi}_{(\lambda)}(\omega, \underline{x}) = g^{00} \omega^2(\lambda) \tilde{\psi}_{(\lambda)}(\omega, \underline{x}) \quad (99)$$

The eigenfunctions are specified by a set of discrete or continuous quantum numbers λ which generalise the momentum label \underline{k} in flat space. The eigenvalue $\omega(\lambda)$ is a function of the λ . These eigenfunctions satisfy orthonormality and completeness relations described in appendix A.

It is then straightforward to check that the kernel equation is solved by

$$G(\underline{x}, \underline{y}) = \sqrt{g_x^{00} g_y^{00}} \int \frac{d\mu(\lambda)}{(2\pi)^d} \omega(\lambda) \tilde{\psi}_{(\lambda)}(\omega, \underline{x}) \tilde{\psi}_{(\lambda)}^*(\omega, \underline{y}) \quad (100)$$

In passing, we also note that this expression follows by expressing $\psi(x)$ in eq.(97) as a Fourier transform and using the completeness relation (219). It is also clear that the kernel is a real function.

In Minkowski spacetime, we showed that the kernel was related to the inverse Wightman function evaluated at equal time. We can derive a similar result for a general static metric. Introducing the inverse of the kernel, $\Delta(\underline{x}, \underline{y})$, defined by

$$\int d^d \underline{z} \sqrt{-h_z} \Delta(\underline{x}, \underline{z}) G(\underline{z}, \underline{y}) = \delta^d(\underline{x}, \underline{y}) \quad (101)$$

we find using the orthonormality and completeness relations that

$$\Delta(\underline{x}, \underline{y}) = \int \frac{d\mu(\lambda)}{(2\pi)^d} \frac{1}{\omega(\lambda)} \tilde{\psi}_{(\lambda)}(\omega, \underline{x}) \tilde{\psi}_{(\lambda)}^*(\omega, \underline{y}) \quad (102)$$

The Wightman function is evaluated in appendix B, eq.(246) and as expected we confirm

$$\Delta(\underline{x}, \underline{y}) = 2 \mathcal{G}_+(x, y)|_{ET} \quad (103)$$

Again the reason for this identification is given in appendix C.

The complete Schrödinger vacuum wave functional can therefore be written as

$$\Psi_0[\phi, t] = N_0(t) \psi_0[\phi] \quad (104)$$

where

$$\psi_0 = \exp \left\{ - \int \frac{d\mu(\lambda)}{(2\pi)^d} \frac{\omega(\lambda)}{2} \int d^d \underline{x} \sqrt{-h_x} \sqrt{g_x^{00}} \phi(\underline{x}) \tilde{\psi}_{(\lambda)}(\omega, \underline{x}) \int d^d \underline{y} \sqrt{-h_y} \sqrt{g_y^{00}} \phi(\underline{y}) \tilde{\psi}_{(\lambda)}^*(\omega, \underline{y}) \right\} \quad (105)$$

Notice how the fields $\phi(\underline{x})$ effectively project out the kernel functions $\tilde{\psi}_{(\lambda)}(\omega, \underline{x})$ appropriate to the boundary conditions on ϕ on the hypersurface Σ . We shall discuss this point in more detail in the examples which follow. The time dependent phase factor is

$$N_0(t) = e^{-iE_0 t} \quad (106)$$

where E_0 is the divergent vacuum energy,

$$E_0 = \frac{1}{2} \int d^d \underline{x} \sqrt{-h_x} \sqrt{g_x^{00}} G(\underline{x}, \underline{x}) \quad (107)$$

$$= \frac{1}{2} \int d\mu(\lambda) \omega(\lambda) \delta^d(0) \quad (108)$$

The analysis given above of the excited states for a general spacetime may now be applied. For a static spacetime, all the quantities are time independent, so the states generated by the creation and annihilation operators are genuine stationary states with respect to the chosen Hamiltonian evolution.

The wave functional for the first excited state is

$$\Psi_1[\phi, t] = 2N_1(t) \int d^d \underline{x} \sqrt{-h_x} \int d^d \underline{y} \sqrt{-h_y} \phi(\underline{x}) G(\underline{x}, \underline{y}) f_{(\lambda)}(\underline{y}) \psi_0[\phi] \quad (109)$$

where $G(\underline{x}, \underline{y})$ is the static kernel, $N_1(t) = e^{-i\omega(\lambda)t} N_0(t)$ and $f_{(\lambda)}(\underline{y})$ is a particular eigenfunction of $(\square_i + m^2 + \xi R)$ specifying the state. The corresponding creation and annihilation operators simplify to

$$\hat{a}(\lambda) = \int d^d \underline{x} \left[\sqrt{-h_x} \sqrt{g_x^{00}} \omega(\lambda) \phi(\underline{x}) + \frac{\delta}{\delta \phi(\underline{x})} \right] f_{(\lambda)}^*(\underline{x}) \quad (110)$$

$$\hat{a}^\dagger(\lambda) = \int d^d \underline{x} \left[\sqrt{-h_x} \sqrt{g_x^{00}} \omega(\lambda) \phi(\underline{x}) - \frac{\delta}{\delta \phi(\underline{x})} \right] f_{(\lambda)}(\underline{x}) \quad (111)$$

and satisfy the commutation relations

$$[\hat{a}(\lambda), \hat{a}^\dagger(\rho)] = 2\omega(\lambda) (2\pi)^d \delta^d(\lambda, \rho) \quad (112)$$

As an elementary example, it is straightforward to check that in Minkowski spacetime choosing $\tilde{\psi}_{(\lambda)}(\omega; \underline{x}) = \exp i\underline{k} \cdot \underline{x}$, where $\omega(\underline{k}) = \sqrt{\underline{k}^2 + m^2}$ and $d\mu(\lambda) = d^d \underline{k}$, reproduces precisely the results of section 2.

3.2 Dynamic spacetimes

By a ‘dynamic’ spacetime, we mean one where the metric depends only on the ‘time’ coordinate t . The general metric, after rescaling the t coordinate, can therefore be written as

$$ds^2 = dt^2 - a_{ij}^2(t)dx^i dx^j \quad (113)$$

and defines a Bianchi type I spacetime [15].

To simplify the kernel equation for the special case of a dynamic spacetime, we introduce the Fourier transform $\tilde{G}(\underline{k}, t)$ (see appendix A for definitions),

$$G(\underline{x}, \underline{y}; t) = \int \frac{d^d k}{(2\pi)^d} e^{i\underline{k} \cdot (\underline{x} - \underline{y})} \tilde{G}(\underline{k}; t) \quad (114)$$

The kernel equation reduces to

$$i \frac{\partial}{\partial t} [h \tilde{G}(\underline{k}; t)] = h \sqrt{-g} \tilde{G}^2(\underline{k}; t) + \sqrt{-g} \omega^2(\underline{k}; t) \quad (115)$$

where $\omega^2(\underline{k}; t) = -g^{ij} k_i k_j + m^2 + \xi R$.

The solution⁶ may be expressed in terms of the functions $\tilde{\psi}(t, \underline{k})$ introduced in appendix A which satisfy the Fourier transform of the wave equation, viz.

$$(\square_0 - g^{ij} k_i k_j + m^2 + \xi R) \tilde{\psi}(t, \underline{k}) = 0 \quad (116)$$

We find

$$\tilde{G}(\underline{k}; t) = \frac{-i}{\sqrt{-g}} \frac{\partial}{\partial t} \ln \tilde{\psi}(t, \underline{k}) \quad (117)$$

and thus for the kernel itself,

$$G(\underline{x}, \underline{y}; t) = -i \sqrt{g^{00}} \frac{1}{\sqrt{-h}} \int \frac{d^d k}{(2\pi)^d} e^{i\underline{k} \cdot (\underline{x} - \underline{y})} \frac{\partial}{\partial t} \ln \tilde{\psi}(t, \underline{k}) \quad (118)$$

Notice that this is again consistent with the general expression (97) above.

To show this [13], we convert the non-linear first-order differential equation for $\tilde{G}(\underline{k}; t)$ into a linear second-order equation. Define

$$F(\underline{k}; t) = \exp \left\{ -i \int dt a h \tilde{G}(\underline{k}; t) \right\} \quad (119)$$

⁶The Schrödinger formalism in flat Robertson-Walker spacetime has been extensively studied in refs.[13, 14].

where $a = \frac{\sqrt{-g}}{-h}$ and let $\dot{a} = \frac{\partial a}{\partial t}$. We then have

$$\frac{\partial^2}{\partial t^2} F = -ia \frac{\partial}{\partial t} (h\tilde{G}) - a^2 h^2 \tilde{G}^2 F - i\dot{a} h \tilde{G} F \quad (120)$$

and using eq.(115),

$$\frac{\partial^2}{\partial t^2} F = \frac{\dot{a}}{a} \frac{\partial}{\partial t} F - a\sqrt{-g}\omega^2 F \quad (121)$$

However, this is just the equation satisfied by the functions $\tilde{\psi}(t, \underline{k})$, since by expanding \square_0 in eq.(116) we see

$$\left(\partial_0^2 - \frac{\dot{a}}{a} \partial_0 + a\sqrt{-g}\omega^2 \right) \tilde{\psi}(t, \underline{k}) = 0 \quad (122)$$

We therefore identify $F(\underline{k}; t)$ with $\tilde{\psi}(t, \underline{k})$ and eq.(117) follows.

The function $\tilde{\psi}(t, \underline{k})$ is the solution of a second-order ordinary differential equation and is therefore a linear combination of two independent solutions with arbitrary coefficients. Since the kernel itself is the derivative of the logarithm of $\tilde{\psi}(t, \underline{k})$, the overall normalisation is unimportant. The kernel therefore depends on one arbitrary parameter.

This is a physically important difference from the static spacetime, where the kernel is essentially unique. For a dynamic spacetime, the ‘vacuum’ states described by the Gaussian solution of the Schrödinger wave functional equation form a one-parameter family [13]. The selection of one of these as the vacuum must then be made on further physical grounds. We shall return to this point in the context of particular examples later.

The Schrödinger vacuum wave functional can therefore be written as

$$\Psi_0[\phi, t] = N_0(t)\psi_0[\phi, t] \quad (123)$$

where

$$\psi_0 = \exp \left\{ \frac{i}{2} \sqrt{g^{00}} \sqrt{-h} \int \frac{d^d k}{(2\pi)^d} \frac{\partial}{\partial t} [\ln \tilde{\psi}(t, \underline{k})] \int d^d \underline{x} \int d^d \underline{y} \phi(\underline{x}) e^{i\underline{k} \cdot (\underline{x} - \underline{y})} \phi(\underline{y}) \right\} \quad (124)$$

and

$$N_0(t) = e^{-i \int^t dt E_0(t)} \quad (125)$$

with

$$E_0(t) = \frac{1}{2} \sqrt{g_{00}} \sqrt{-h} \int d^d \underline{k} \tilde{G}(\underline{k}; t) \delta^d(0) \quad (126)$$

In this case, however, E_0 does not have a useful interpretation as an energy.

The analogue of the higher excited states can be found by substituting for the kernel in the general formulae following eq.(89), where $f_{\underline{k}}(\underline{x}) = e^{i\underline{k} \cdot \underline{x}}$ is an

eigenstate of $(\square_i + m^2 + \xi R)$ with eigenvalue $\omega^2(\underline{k}; t)$. Again, however, we emphasise that the physical interpretation of these states is not as straightforward as for a static spacetime where they are genuine stationary states of the energy associated with the chosen Hamiltonian evolution.

It is important to check the stability of the vacuum state defined by the time-dependent wave functional (124). Vacuum stability requires that

$$\int \mathcal{D}\phi \Psi_0^*(\phi, t) \Psi_0(\phi, t) = |N_0(t)|^2 \int \mathcal{D}\phi \psi_0^*(\phi, t) \psi_0(\phi, t) \quad (127)$$

should be time independent. From eqs.(125) and (126) we have

$$|N_0(t)|^2 = \exp\left\{-\frac{1}{2} \int d^d \underline{x} \int \frac{d^d \underline{k}}{(2\pi)^d} \ln |\tilde{\psi}|^2\right\} \quad (128)$$

where we have used eq.(117) for the kernel. Obviously this will in general be time dependent.

The second contribution to eq.(127) is

$$\begin{aligned} \int \mathcal{D}\phi |\psi_0|^2 &\sim [\det(h G_{\Re}(\underline{x}, \underline{y}; t))]^{-\frac{1}{2}} \\ &= \exp\left\{-\frac{1}{2} \int d^d \underline{x} \int \frac{d^d \underline{k}}{(2\pi)^d} \ln[h \tilde{G}_{\Re}(\underline{k}; t)]\right\} \end{aligned} \quad (129)$$

where \tilde{G}_{\Re} is the real part of the (generally complex) kernel. From eq.(117) we see

$$\tilde{G}_{\Re}(\underline{k}; t) = \frac{i W[\tilde{\psi}, \tilde{\psi}^*]}{2\sqrt{-g} |\tilde{\psi}|^2} \quad (130)$$

while for the imaginary part,

$$\tilde{G}_{\Im}(\underline{k}; t) = \frac{-1}{2\sqrt{-g} |\tilde{\psi}|^2} \frac{\partial}{\partial t} |\tilde{\psi}|^2 \quad (131)$$

where $W[\tilde{\psi}, \tilde{\psi}^*] = \tilde{\psi}(\frac{\partial}{\partial t} \tilde{\psi}^*) - \tilde{\psi}^*(\frac{\partial}{\partial t} \tilde{\psi})$ is the Wronskian of the solution and its complex conjugate. Using eqs.(114) and (130) therefore gives

$$\int \mathcal{D}\phi |\psi_0|^2 \sim \exp\left\{+\frac{1}{2} \int d^d \underline{x} \int \frac{d^d \underline{k}}{(2\pi)^d} \ln |\tilde{\psi}|^2\right\} \quad (132)$$

up to time-independent terms. These remaining terms are seen to be time independent using Abel's theorem, viz. that $g^{00} \sqrt{-g} W$ is a constant, where W is the Wronskian defined above.

The overall time dependence therefore cancels as required, confirming the stability of the vacuum state in a dynamic spacetime.

3.3 Conformally static and Robertson-Walker spacetimes

A straightforward generalisation of the results of sections 3.1 and 3.2 allows us to find the kernel for the class of $(d+1)$ -dimensional spacetimes with metric

$$ds^2 = g_{00}(t)dt^2 - a^2(t)d\underline{\sigma}^2 \quad (133)$$

where $d\underline{\sigma}^2$ is a d -dimensional static space with metric $\sigma_{ij}(\underline{x})$. Let $\sigma = \det \sigma_{ij}$. Introducing a rescaled ‘conformal time’ η , the metric can be written as

$$ds^2 = C(\eta) [d\eta^2 - \sigma_{ij}(\underline{x})dx^i dx^j] \quad (134)$$

The spacetime is therefore conformally static, but with the conformal scale factor restricted to depend only on time. This class includes the important Robertson-Walker spacetimes, which are special cases of eq.(133) where $g_{00} = 1$ and $d\underline{\sigma}^2$ is a 3-dimensional static space of constant curvature $\kappa = -1, 0, +1$, viz.

$$d\underline{\sigma}^2 = d\chi^2 + f^2(\chi)[d\theta^2 + \sin^2 \theta d\phi^2] \quad (135)$$

where

$$f(\chi) = \begin{cases} \sin \chi & 0 \leq \chi < 2\pi & \kappa = +1 \\ \chi & 0 \leq \chi < \infty & \kappa = 0 \\ \sinh \chi & 0 \leq \chi < \infty & \kappa = -1 \end{cases} \quad (136)$$

For $\kappa = -1, 0, +1$ this metric describes hyperbolic (open), flat, and spherical (closed) spaces respectively.

Since the spatial metric is static, general theorems guarantee the existence of orthonormal spatial modes $\mathcal{Y}_{(\lambda)}(\underline{x})$ which play the rôle of the Fourier functions $\exp ik.x$ in the previous section. A full discussion of the solutions of the wave equation for these spacetimes is given in appendix A.2.

We therefore write the kernel as

$$G(\underline{x}, \underline{y}; t) = \int \frac{d\mu(\lambda)}{(2\pi)^d} \tilde{G}(\lambda; t) \mathcal{Y}_{(\lambda)}(\underline{x}) \mathcal{Y}_{(\lambda)}^*(\underline{y}) \quad (137)$$

Then, using the solutions $\mathcal{T}(t, \lambda)$ of the transformed wave equation (see eq.(225)), we readily find (c.f. eqs.(117) or (97))

$$\tilde{G}(\lambda; t) = -i \frac{\sqrt{\sigma}}{\sqrt{-g}} \frac{\partial}{\partial t} \ln \mathcal{T}(t, \lambda) \quad (138)$$

The kernel for a conformally static spacetime with a conformal factor which is a function of time only is therefore

$$G(\underline{x}, \underline{y}; t) = \frac{-i}{\sqrt{g_{00}} a^d} \int \frac{d\mu(\lambda)}{(2\pi)^d} \left[\frac{\partial}{\partial t} \ln \mathcal{T}(t, \lambda) \right] \mathcal{Y}_{(\lambda)}(\underline{x}) \mathcal{Y}_{(\lambda)}^*(\underline{y}) \quad (139)$$

The interpretation is identical to that for the dynamic metric case. Again, there is a one-parameter family of vacuum states corresponding to the freedom in selecting the solution $\mathcal{T}(t, \lambda)$ of the wave equation.

4 Cosmological Model I

To illustrate this formalism, we now consider two simple two-dimensional ‘cosmological models’ of Robertson-Walker type. We give an exact description of the vacuum states, emphasising the rôle of boundary conditions, and discuss carefully the phenomenon of particle creation due to the cosmological expansion.

The spacetime metric is chosen to be of Robertson-Walker form,

$$ds^2 = dt^2 - a(t)^2 dx^2 \quad (140)$$

Rescaling the time coordinate such that $dt = a(t)d\eta$, the metric may be rewritten in terms of the ‘conformal time’ η as

$$ds^2 = C(\eta)(d\eta^2 - dx^2) \quad (141)$$

The metric is conformal to Minkowski spacetime.

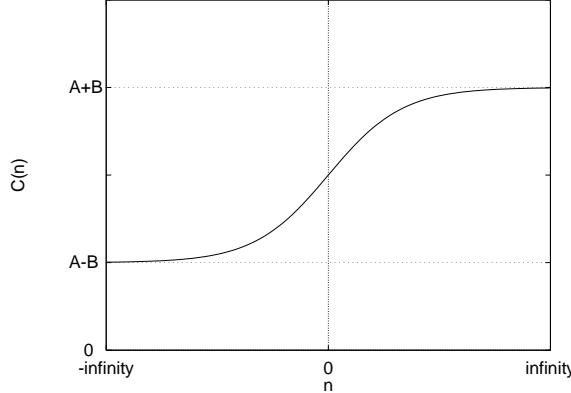


Figure 1: Conformal scale factor for the cosmological model with expansion.

The first model [1] is chosen to have asymptotically static regions separated by a period of expansion (see Fig(1)). The conformal scale factor is chosen to be

$$C(\eta) = A + B \tanh(\rho\eta) \quad (142)$$

where A, B, ρ are constants with $A > B > 0$. In the asymptotic IN ($\eta \rightarrow -\infty$) and OUT ($\eta \rightarrow \infty$) regions,

$$C(\eta) \rightarrow A \pm B \quad \text{as } \eta \rightarrow \pm\infty \quad (143)$$

This choice allows explicit analytic solutions for the wave functionals in terms of hypergeometric functions.

The metric is in the class we have called ‘dynamic’, so the construction of the vacuum wave functional is a straightforward application of the techniques described in section 3.2. The first step is to find the solutions of the wave equation in the metric specified by eq.(142).

4.1 Wave equation

Taking the Fourier transform,

$$\psi(\eta, x) = \int_{-\infty}^{\infty} \frac{dk}{(2\pi)} e^{ikx} \tilde{\psi}(\eta; k) \quad (144)$$

the wave equation becomes

$$[\partial_{\eta}^2 + (k^2 + Am^2) + Bm^2 \tanh(\rho\eta)] \tilde{\psi}(\eta, k) = 0 \quad (145)$$

Notice that the only occurrence of the conformal scale factor is with the mass m . This is a trivial consequence of the conformal invariance of the scalar field equation for $m = 0$ (we have taken $\xi = 0$ for simplicity). Obviously any non-Minkowskian behaviour of the vacuum must involve the mass.

It is convenient at this point to make the definitions:

$$\omega^2(k; \eta) = k^2 + m^2 C(\eta) \quad (146)$$

$$\omega_{in}^2 = k^2 + m^2(A - B) \quad (147)$$

$$\omega_{out}^2 = k^2 + m^2(A + B) \quad (148)$$

$$\omega_{\pm} = \frac{1}{2}(\omega_{out} \pm \omega_{in}) \quad (149)$$

With the substitution

$$\tilde{\psi}(\eta, k) = e^{-iw_+\eta} [2 \cosh(\rho\eta)]^{-\frac{iw_-}{\rho}} \tilde{\chi}(\eta, k) \quad (150)$$

and a change of variable $z = \frac{1}{2}[1 + \tanh(\rho\eta)]$, the wave equation reduces to the hypergeometric equation

$$\left\{ z(1-z)\partial_z^2 + \left[1 - \frac{iw_{in}}{\rho} - z(2 + 2\frac{iw_-}{\rho})\right] \partial_z - \frac{iw_-}{\rho} + \frac{w_-^2}{\rho^2} \right\} \tilde{\chi}(z, k) = 0 \quad (151)$$

The properties of this equation and its solutions are discussed in [16, 17]. Here, since $-\infty < \eta < \infty$, we are only concerned with solutions in the range $0 \leq z \leq 1$.

Because of the singular points at $z = 0$ and $z = 1$ in the hypergeometric equation we have to consider two sets of solutions to cover this range completely. The first set are valid in the range $0 \leq z < 1$, while the second set are valid for $0 < z \leq 1$. The radius of convergence of the two sets of solutions is strictly less than 1.

Within the radius of convergence of the $z = 0$ singular point there are two linearly independent solutions. We can choose these so that as $z \rightarrow 0$, one, $\tilde{\psi}_+^{in}$,

behaves as a positive frequency solution while the other, $\tilde{\psi}_+^{in}$, becomes a negative frequency solution. Explicitly,

$$\begin{aligned}\tilde{\psi}_+^{in}(\eta, k) &= e^{-iw_+\eta} [2 \cosh(\rho\eta)]^{-\frac{iw_-}{\rho}} \times \\ &\quad {}_2F_1\left(\frac{iw_-}{\rho}, 1 + \frac{iw_-}{\rho}; 1 - \frac{iw_{in}}{\rho}; z\right)\end{aligned}\quad (152)$$

$$\stackrel{\eta \rightarrow -\infty}{\sim} e^{-iw_{in}\eta} \quad (153)$$

$$\begin{aligned}\tilde{\psi}_-^{in}(\eta, k) = (\tilde{\psi}_+^{in})^* &= e^{-iw_+\eta} [2 \cosh(\rho\eta)]^{-\frac{iw_-}{\rho}} \left[\frac{1}{2}(1 + \tanh(\rho\eta))\right]^{\frac{iw_{in}}{\rho}} \times \\ &\quad {}_2F_1\left(\frac{iw_+}{\rho}, 1 + \frac{iw_+}{\rho}; 1 + \frac{iw_{in}}{\rho}; z\right)\end{aligned}\quad (154)$$

$$\stackrel{\eta \rightarrow -\infty}{\sim} e^{+iw_{in}\eta} \quad (155)$$

The solutions valid around the $z = 1$ singular point can similarly be chosen so that they correspond to positive and negative frequency solutions in the limit $z \rightarrow 1$. These are

$$\begin{aligned}\tilde{\psi}_+^{out}(\eta, k) &= e^{-iw_+\eta} [2 \cosh(\rho\eta)]^{-\frac{iw_-}{\rho}} \times \\ &\quad {}_2F_1\left(\frac{iw_-}{\rho}, 1 + \frac{iw_-}{\rho}; 1 + \frac{iw_{out}}{\rho}; 1 - z\right)\end{aligned}\quad (156)$$

$$\stackrel{\eta \rightarrow \infty}{\sim} e^{-iw_{out}\eta} \quad (157)$$

$$\begin{aligned}\tilde{\psi}_-^{out}(\eta, k) = (\tilde{\psi}_+^{out})^* &= e^{-iw_+\eta} [2 \cosh(\rho\eta)]^{-\frac{iw_-}{\rho}} \left[\frac{1}{2}(1 - \tanh(\rho\eta))\right]^{-\frac{iw_{out}}{\rho}} \times \\ &\quad {}_2F_1\left(-\frac{iw_+}{\rho}, 1 - \frac{iw_+}{\rho}; 1 - \frac{iw_{out}}{\rho}; 1 - z\right)\end{aligned}\quad (158)$$

$$\stackrel{\eta \rightarrow \infty}{\sim} e^{+iw_{out}\eta} \quad (159)$$

In the overlap region $0 < z < 1$, both sets of solutions are valid and we may express one in terms of the other. Specifically, we have

$$\tilde{\psi}_-^{in}(\eta, k) = \alpha(k) \tilde{\psi}_-^{out} + \beta(k) \tilde{\psi}_+^{out} \quad (160)$$

where

$$\alpha(k) = \frac{\Gamma(1 + \frac{iw_{in}}{\rho}) \Gamma(\frac{iw_{out}}{\rho})}{\Gamma(\frac{iw_+}{\rho}) \Gamma(1 + \frac{iw_+}{\rho})} \quad (161)$$

$$\beta(k) = \frac{\Gamma(1 + \frac{iw_{in}}{\rho}) \Gamma(\frac{-iw_{out}}{\rho})}{\Gamma(\frac{-iw_-}{\rho}) \Gamma(1 - \frac{iw_-}{\rho})} \quad (162)$$

The magnitudes of these two coefficients are

$$|\alpha(k)|^2 = \left(\frac{w_{in}}{w_{out}}\right) \frac{\sinh^2\left(\frac{\pi w_+}{\rho}\right)}{\sinh\left(\frac{\pi w_{out}}{\rho}\right) \sinh\left(\frac{\pi w_{in}}{\rho}\right)} \quad (163)$$

$$|\beta(k)|^2 = \left(\frac{w_{in}}{w_{out}}\right) \frac{\sinh^2\left(\frac{\pi w_-}{\rho}\right)}{\sinh\left(\frac{\pi w_{out}}{\rho}\right) \sinh\left(\frac{\pi w_{in}}{\rho}\right)} \quad (164)$$

and satisfy

$$|\alpha(k)|^2 - |\beta(k)|^2 = \left(\frac{w_{in}}{w_{out}}\right) \quad (165)$$

Notice that the solution which is positive frequency in the $z \rightarrow 0$ IN region is a mixture of the solutions of both positive and negative frequencies in the $z \rightarrow 1$ OUT region. As we shall see, this mixing is at the heart of the ‘particle creation’ interpretation. The α and β coefficients above are just the Bogoliubov coefficients in conventional treatments [18, 1].

4.2 Vacuum wave functional

The vacuum wave functional is given by the standard formula (118) for a dynamic spacetime. That is,

$$G(x, y; \eta) = \int_{-\infty}^{\infty} \frac{dk}{(2\pi)} e^{ik(x-y)} \tilde{G}(k; \eta) \quad (166)$$

where

$$\tilde{G}(k; \eta) = -\frac{i}{C(\eta)} \frac{\partial}{\partial \eta} \ln[a\tilde{\psi}_+^{in}(\eta, k) + b\tilde{\psi}_-^{in}(\eta, k)] \quad (167)$$

To specify the vacuum beyond this one-parameter ambiguity, we need to impose a boundary condition. We choose this so that in the IN region, the wave functional becomes the usual positive frequency Minkowski vacuum

$$\psi_0^{IN}[\tilde{\phi}] = \exp\left\{-\frac{1}{2} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \omega_{in} |\tilde{\phi}(k)|^2\right\} \quad (168)$$

This implies we choose $a = 0, b = 1$ above, i.e. use the negative frequency solution of the wave equation in the dynamic kernel.

Explicitly,

$$\tilde{G}(k; \eta) = -\frac{i}{C(\eta)} \frac{\partial}{\partial \eta} [\ln \tilde{\psi}_-^{in}(\eta, k)] \quad (169)$$

$$= \frac{-1}{C(\eta)} \left\{ \omega_+ + \omega_- \tanh(\rho\eta) - \omega_{in} [1 - \tanh(\rho\eta)] \times \left[\frac{{}_2F_1\left(\frac{i}{\rho}\omega_+, 1 + \frac{i}{\rho}\omega_+; \frac{i\omega_{in}}{\rho}; z\right)}{{}_2F_1\left(\frac{i}{\rho}\omega_+, 1 + \frac{i}{\rho}\omega_+; 1 + \frac{i\omega_{in}}{\rho}; z\right)} \right] \right\} \quad (170)$$

which in the asymptotic limit is simply

$$\tilde{G}(k; \eta \rightarrow -\infty) = \frac{\omega_{in}}{(A - B)} \quad (171)$$

as required to reproduce eq.(168). The complete vacuum wave functional is therefore

$$\Psi_0[\tilde{\phi}, \eta] = N_0(\eta) \psi_0[\tilde{\phi}, \eta] \quad (172)$$

where

$$\psi_0[\tilde{\phi}, \eta] = \exp \left\{ -\frac{1}{2} C(\eta) \int_{-\infty}^{\infty} \frac{dk}{2\pi} \tilde{G}(k; \eta) |\tilde{\phi}(k)|^2 \right\} \quad (173)$$

and

$$N_0(\eta) = \exp \left\{ -\frac{i}{2} \int^{\eta} d\eta' C(\eta') \int_{-\infty}^{\infty} dk \tilde{G}(k; \eta) \delta(0) \right\} \quad (174)$$

This is the full analytic expression, valid for all values of the time η , for the vacuum state chosen to satisfy a positive frequency boundary condition in the IN region. It is stable for all times, the proof being precisely that given in section 3 for a general dynamic metric.

The next step is to examine the behaviour of this state in the asymptotic future OUT region. For this, we use eq.(160) to reexpress the kernel in terms of the wave equation solutions $\tilde{\psi}_+^{out}$ and $\tilde{\psi}_-^{out}$ which have simple asymptotic behaviour as $z \rightarrow 1$. This gives

$$\tilde{G}(k; \eta) = \frac{-i}{C(\eta)} \left[\frac{\alpha \frac{\partial}{\partial \eta} (\tilde{\psi}_-^{out}) + \beta \frac{\partial}{\partial \eta} (\tilde{\psi}_+^{out})}{\alpha(\tilde{\psi}_-^{out}) + \beta(\tilde{\psi}_+^{out})} \right] \quad (175)$$

As $\eta \rightarrow \infty$ this function does not tend to a fixed value but rather to a limit circle in the complex plane. We find

$$\tilde{G}(k; \eta \rightarrow \infty) = \frac{\omega_{out}}{(A + B)} \left[\frac{1 - \gamma e^{-2i\omega_{out}\eta}}{1 + \gamma e^{-2i\omega_{out}\eta}} \right] \quad (176)$$

where $\gamma = \frac{\beta}{\alpha}$. The field-dependent part of the wave functional is therefore

$$\psi_0[\tilde{\phi}, \eta \rightarrow \infty] = \exp \left\{ -\frac{1}{2} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \omega_{out} \left[\frac{1 - \gamma e^{-2i\omega_{out}\eta}}{1 + \gamma e^{-2i\omega_{out}\eta}} \right] |\tilde{\phi}(k)|^2 \right\} \quad (177)$$

4.3 Vacuum states and particle creation

We now discuss the interpretation of this vacuum state and in particular the phenomenon of particle creation. First, notice that eq.(177) is *not* the standard

Minkowski vacuum corresponding to the asymptotic limit of the metric, viz.

$$\psi_0^{OUT}[\tilde{\phi}] = \exp \left\{ -\frac{1}{2} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \omega_{out} |\tilde{\phi}(k)|^2 \right\} \quad (178)$$

In fact, this is the asymptotic limit of a different state $\Psi'_0[\tilde{\phi}, \eta]$ chosen from the one-parameter family of vacuum solutions to the Schrödinger equation, specified by the kernel (c.f. eq.(169))

$$\tilde{G}'(k; \eta) = -\frac{i}{C(\eta)} \frac{\partial}{\partial \eta} [\ln \tilde{\psi}_-^{out}(\eta, k)] \quad (179)$$

Completeness of the solutions to the Schrödinger equation means that any state can be expressed as a superposition of one of the one-parameter family of vacuum solutions and the excited states built from it. In particular, we can express the chosen vacuum state $\Psi'_0[\tilde{\phi}, \eta]$ as a linear combination of the state $\Psi'_0[\tilde{\phi}, \eta]$ and its excited states. In the OUT region, this amounts to expressing the Gaussian state $\psi_0[\tilde{\phi}, \eta \rightarrow \infty]$ of eq.(177) in terms of the excited states of the Minkowski state $\psi_0^{OUT}[\tilde{\phi}]$.

In the limit $\eta \rightarrow \infty$, these excitations are standard Minkowski particle states and we can characterise them by their particle number and momenta. The problem therefore reduces to the one solved in section 2.1, i.e. expressing a general, time-dependent Gaussian state in Minkowski spacetime in terms of a superposition of many-particle states. The time-dependent kernel Ω is read off from eq.(177), viz.

$$\Omega(k; \eta) = \omega_{out} \left[\frac{1 - \gamma e^{-2i\omega_{out}\eta}}{1 + \gamma e^{-2i\omega_{out}\eta}} \right] \quad (180)$$

Transcribing results directly from section 2.1, we find that the expectation value of the number density of particles with momentum k in the state $\Psi_0[\tilde{\phi}, \eta \rightarrow \infty]$ is

$$\langle \Psi_0[\tilde{\phi}, \eta \rightarrow \infty] | \mathcal{N}(k) | \Psi_0[\tilde{\phi}, \eta \rightarrow \infty] \rangle = \frac{(\omega_{out} - \Omega)(\omega_{out} - \Omega^*)}{2\omega_{out}(\Omega + \Omega^*)} \quad (181)$$

Remarkably, despite the residual time-dependence in Ω , this expectation value is independent of η . Explicitly,

$$\langle \mathcal{N}(k) \rangle = \frac{|\gamma|^2}{1 - |\gamma|^2} = \frac{\omega_{out}}{\omega_{in}} |\beta|^2 \quad (182)$$

$$= \frac{\sinh^2\left(\frac{\pi\omega_-}{\rho}\right)}{\sinh\left(\frac{\pi\omega_{out}}{\rho}\right) \sinh\left(\frac{\pi\omega_{in}}{\rho}\right)} \quad (183)$$

This is the Schrödinger picture derivation of the traditional result on particle creation in this expanding universe model. Notice that the particle number density is proportional to $|\beta|^2$, the Bogoliubov coefficient in the conventional treatment, which is only non-zero if there is a mixing between the positive and negative frequency solutions of the wave equation in the respective asymptotic regions. As expected, $\langle \mathcal{N}(k) \rangle$ also vanishes in the conformal limit $m \rightarrow 0$.

It is also illuminating to evaluate the expectation value of the energy momentum tensor in these states. In particular, $\langle \Psi_0 | T_{00} | \Psi_0 \rangle$ evaluated in the asymptotic IN and OUT regions measures the time-independent energy eigenvalue of the state Ψ_0 . In the intermediate region, of course, Ψ_0 , like all the other ‘vacuum’ solutions, is not an energy eigenstate. With the definition of T_{00} implicit in eq.(83) (recall that $H = \int d^d \underline{x} \sqrt{-g} g^{00} T_{00}$), we find

$$\langle \Psi_0[\tilde{\phi}, \eta] | T_{00} | \Psi_0[\tilde{\phi}, \eta] \rangle = \frac{1}{2} C(\eta) \int_{-\infty}^{\infty} \frac{dk}{2\pi} \tilde{G}(k; \eta) \left[1 - \frac{1}{2\tilde{G}\tilde{G}_{\Re}} \left(\tilde{G}^2 - \frac{\omega^2}{C(\eta)^2} \right) \right] \quad (184)$$

where we have normalised Ψ_0 such that $\langle \Psi_0 | \Psi_0 \rangle = 1$.

In the IN region, we simply have

$$\langle \Psi_0 | T_{00} | \Psi_0 \rangle_{IN} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \omega_{in} \quad (185)$$

This is just the usual Minkowski zero-point energy density. On the other hand, substituting (176) into eq.(184) we find, after some calculation,

$$\langle \Psi_0 | T_{00} | \Psi_0 \rangle_{OUT} = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \omega_{out} \left(\langle \mathcal{N}(k) \rangle + \frac{1}{2} \right) \quad (186)$$

Notice that the η dependence from the kernel $\tilde{G}(k; \eta \rightarrow \infty)$ has again cancelled out. $\Psi_0[\tilde{\phi}, \eta \rightarrow \infty]$ is therefore not the minimum energy state in the Minkowski OUT region. Rather, eq.(186) describes the energy of a many-particle state with $\langle \mathcal{N}(k) \rangle$ particles of energy $\omega_{out}(k)$ as described above.

The Schrödinger picture therefore brings a particular clarity to the issue of particle creation in this model. It is evident that there is no state which satisfies all the properties of a standard Minkowski vacuum state. As we have repeatedly emphasised, there is a one-parameter family of Gaussian ‘vacuum’ solutions to the wave functional equation for this class of dynamic spacetimes. If we choose to consider the state $\Psi_0[\tilde{\phi}, \eta]$ which in the asymptotic past is the Minkowski vacuum state $\psi_0^{IN}[\tilde{\phi}]$ and follow its evolution, we find that in the asymptotic future OUT region it is described by the time-dependent kernel (176). At this point, there is really no more to be said – that is a complete description of the state. It is stable and any physical quantity may be evaluated by taking appropriate operator matrix elements. The remaining question is simply to describe what the state $\Psi_0[\tilde{\phi}, \eta]$ looks like. As we have shown above, in the OUT region where the spacetime is again asymptotically Minkowskian, it is precisely a superposition of many-particle states built from the minimum energy Minkowski vacuum state $\psi_0^{OUT}[\tilde{\phi}]$. In the asymptotic OUT region, therefore, it is true to say that particles have been created as a result of the expansion. For any intermediate value of the time, $\Psi_0[\tilde{\phi}, \eta]$ continues to provide a complete description of the state, although in this case there is no simple interpretation in terms of Minkowski particles.

5 Cosmological Model II

In this section, we consider a second cosmological model of the same type [3], this time with a conformal scale factor

$$C(\eta) = \frac{A^2}{\cosh^2(\rho\eta)} \quad (187)$$

This is shown in Fig(2). This describes a universe which expands exponentially

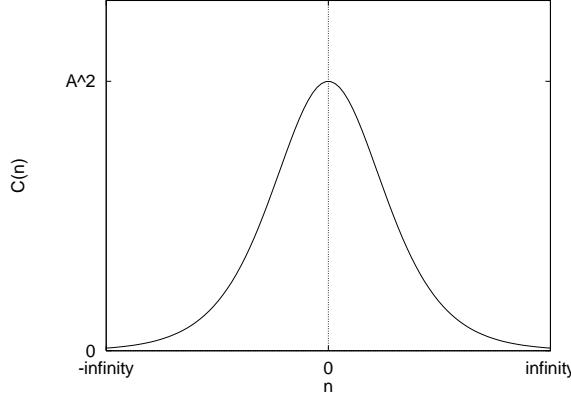


Figure 2: Conformal scale factor for the recollapse model.

from an initial singularity before eventually recollapsing. (Since $C(\eta) \rightarrow 0$ as $\eta \rightarrow \pm\infty$, the size of the universe tends to zero in the past and future asymptotic limits.) Notice that although the universe is infinitely long-lived in terms of the conformal time η , the range of the Robertson-Walker time coordinate t in eq.(140) is only from $t = 0$ to $t = T = A\pi/\rho$. In fact, solving the relation $dt^2 = C(\eta)d\eta^2$, we find $a(t) = A \sin \rho t / A$, representing an oscillating universe of which we are considering only the first cycle.

The analysis of this model follows closely that of section 4. Again we have two asymptotically flat regions in which we can compare the vacuum solutions of the dynamic spacetime Schrödinger equation with the equivalent Minkowski vacua. With some additional subtleties, the results are similar and we again find particle creation occurring as a result of the gravitational expansion and recontraction.

5.1 Wave equation

In this case, the Fourier transformed wave equation is

$$[\partial_\eta^2 + k^2 + A^2 m^2 \cosh^{-2}(\rho\eta)]\tilde{\psi}(\eta, k) = 0 \quad (188)$$

Changing variables from η to z as before and making the substitution

$$\tilde{\psi}(\eta, k) = [2 \cosh(\rho\eta)]^{-ik\rho} \tilde{\chi}(\eta, k) \quad (189)$$

the wave equation reduces to the hypergeometric equation [16, 17]

$$\left\{ z(1-z)\partial_z^2 + \left[1 + \frac{ik}{\rho} - z(2 + 2\frac{ik}{\rho})\right]\partial_z + \left[\frac{k^2}{\rho^2} - \frac{ik}{\rho} + \frac{A^2 m^2}{\rho^2}\right]\right\} \tilde{\chi}(z, k) = 0 \quad (190)$$

Again, there are two sets of solutions of interest, valid in the neighbourhood of the singular points $z = 0$ and $z = 1$ with a radius of convergence strictly less than 1. Choosing a basis of asymptotically positive or negative frequency solutions, we have

$$\begin{aligned} \tilde{\psi}_-^{in}(\eta, k) &= [2 \cosh(\rho\eta)]^{-\frac{ik}{\rho}} {}_2F_1\left(\frac{ik}{\rho} - \sigma, 1 + \frac{ik}{\rho} + \sigma; 1 + \frac{ik}{\rho}; z\right) \\ &\stackrel{\eta \rightarrow -\infty}{\sim} e^{+ik\eta} \end{aligned} \quad (191)$$

$$\begin{aligned} \tilde{\psi}_+^{in}(\eta, k) = (\tilde{\psi}_-^{in})^* &= [2 \cosh(\rho\eta)]^{-\frac{ik}{\rho}} [z]^{-\frac{ik}{\rho}} {}_2F_1\left(-\sigma, 1 + \sigma; 1 - \frac{ik}{\rho}; z\right) \\ &\stackrel{\eta \rightarrow -\infty}{\sim} e^{-ik\eta} \end{aligned} \quad (192)$$

and

$$\begin{aligned} \tilde{\psi}_-^{out}(\eta, k) &= [2 \cosh(\rho\eta)]^{-\frac{ik}{\rho}} [1-z]^{-\frac{ik}{\rho}} {}_2F_1\left(-\sigma, 1 + \sigma; 1 - \frac{ik}{\rho}; 1-z\right) \\ &\stackrel{\eta \rightarrow \infty}{\sim} e^{+ik\eta} \end{aligned} \quad (193)$$

$$\begin{aligned} \tilde{\psi}_+^{out}(\eta, k) = (\tilde{\psi}_-^{out})^* &= [2 \cosh(\rho\eta)]^{-\frac{ik}{\rho}} {}_2F_1\left(\frac{ik}{\rho} - \sigma, 1 + \frac{ik}{\rho} + \sigma; 1 + \frac{ik}{\rho}; 1-z\right) \\ &\stackrel{\eta \rightarrow \infty}{\sim} e^{-ik\eta} \end{aligned} \quad (194)$$

where

$$\sigma = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1 + \frac{4A^2 m^2}{\rho^2}} \quad (195)$$

Notice that these solutions are independent of the mass m in the asymptotic limit. The mass decouples when the size of the universe shrinks beyond the scale of m^{-1} .

In the intermediate region $0 < z < 1$, we can express these solutions in terms of each other. In particular, we will need the relation

$$\tilde{\psi}_-^{in}(\eta, k) = \alpha(k)\tilde{\psi}_-^{out} + \beta(k)\tilde{\psi}_+^{out} \quad (196)$$

where

$$\alpha(k) = \frac{\Gamma(1 + \frac{ik}{\rho})\Gamma(\frac{ik}{\rho})}{\Gamma(\frac{ik}{\rho} - \sigma)\Gamma(1 + \frac{ik}{\rho} + \sigma)} \quad (197)$$

$$\beta(k) = \frac{\Gamma(1 + \frac{ik}{\rho})\Gamma(-\frac{ik}{\rho})}{\Gamma(1 + \sigma)\Gamma(-\sigma)} \quad (198)$$

The magnitudes of these coefficients are

$$|\alpha(k)|^2 = \frac{\sin^2(\pi\sigma)\cosh^2(\pi k) + \cos^2(\pi\sigma)\sinh^2(\pi k)}{\sinh^2(\pi k)} \quad (199)$$

$$|\beta(k)|^2 = \frac{\sin^2(\pi\sigma)}{\sinh^2(\pi k)} \quad (200)$$

and satisfy

$$|\alpha(k)|^2 - |\beta(k)|^2 = 1 \quad (201)$$

5.2 Vacuum Wave Functional and Particle Creation

The construction of the vacuum wave functional is identical to section 4. Choosing boundary conditions such that the vacuum wave functional $\Psi_0[\tilde{\phi}, \eta]$ becomes the standard Minkowski vacuum in the asymptotic IN region, viz.

$$\psi_0^{IN}[\tilde{\phi}] = \exp \left\{ -\frac{1}{2} \int_{-\infty}^{\infty} \frac{dk}{(2\pi)} k |\tilde{\phi}(k)|^2 \right\} \quad (202)$$

fixes the one-parameter freedom of vacuum states. This choice determines the kernel,

$$\tilde{G}(k; \eta) = -\frac{i}{C(\eta)} \frac{\partial}{\partial \eta} [\ln \tilde{\psi}_-^{in}(\eta, k)] \quad (203)$$

Explicitly,

$$\begin{aligned} \tilde{G}(k; \eta) &= -\frac{i}{C(\eta)} \left\{ -ik \tanh(\rho\eta) + \frac{1}{2}[1 - \tanh^2(\rho\eta)] \times \right. \\ &\quad \left. \frac{(\frac{ik}{\rho} - \sigma)(1 + \frac{ik}{\rho} + \sigma)}{(1 + \frac{ik}{\rho})} \frac{{}_2F_1\left(1 + \frac{ik}{\rho} - \sigma, 2 + \frac{ik}{\rho} + \sigma; 2 + \frac{ik}{\rho}; z\right)}{{}_2F_1\left(\frac{ik}{\rho} - \sigma, 1 + \frac{ik}{\rho} + \sigma; 1 + \frac{ik}{\rho}; z\right)} \right\} \end{aligned} \quad (204)$$

Rewriting eq.(203) in terms of the wave equation solutions $\tilde{\psi}_{\pm}^{out}$ valid in the OUT region, we have

$$\tilde{G}(k; \eta) = \frac{-i}{C(\eta)} \left[\frac{\alpha \frac{\partial}{\partial \eta}(\tilde{\psi}_-^{out}) + \beta \frac{\partial}{\partial \eta}(\tilde{\psi}_+^{out})}{\alpha(\tilde{\psi}_-^{out}) + \beta(\tilde{\psi}_+^{out})} \right] \quad (205)$$

Then, taking the asymptotic future limit $\eta \rightarrow \infty$, we find the following expression for the $\tilde{\phi}$ dependent part of the vacuum wave functional:

$$\psi_0[\tilde{\phi}, \eta \rightarrow \infty] = \exp \left\{ -\frac{1}{2} \int_{-\infty}^{\infty} \frac{dk}{(2\pi)} k \left[\frac{1 - \gamma(1-z)^{\frac{ik}{\rho}}}{1 + \gamma(1-z)^{\frac{ik}{\rho}}} \right] |\tilde{\phi}(k)|^2 \right\} \quad (206)$$

where $\gamma = \frac{\beta}{\alpha}$. This is to be compared with the equivalent Minkowski vacuum in the OUT region,

$$\psi_0^{OUT}[\tilde{\phi}] = \exp \left\{ -\frac{1}{2} \int_{-\infty}^{\infty} \frac{dk}{(2\pi)} k |\tilde{\phi}(k)|^2 \right\} \quad (207)$$

which is of course identical to ψ_0^{IN} in this case. The same argument as in section 4.3 now shows that the asymptotic vacuum state (206) is a linear superposition of many-particle states built from the Minkowski vacuum ψ_0^{OUT} . The expectation value of the particle number density is

$$\begin{aligned} \langle \Psi_0[\tilde{\phi}, \eta \rightarrow \infty] | \mathcal{N}(k) | \Psi_0[\tilde{\phi}, \eta \rightarrow \infty] \rangle &= \frac{|\gamma|^2}{1 - |\gamma|^2} \\ &= \frac{\sin^2(\pi\sigma)}{\sinh^2(\pi k)} \end{aligned} \quad (208)$$

in agreement with the canonical approach [3]. Once again, therefore, we see that the initial Minkowski vacuum has evolved in the asymptotic future to a state described by a Gaussian wave functional with a time-dependent kernel, which may be interpreted as a many-particle state. We conclude that despite the recontraction following the initial exponential expansion, particles have been created in this cosmological model as a result of the time-dependence of the metric.

A remarkable feature of this model, however, is that for certain values of the parameters, there is *no* particle creation. From eq.(208) we see that $\langle \mathcal{N}(k) \rangle$ vanishes when σ is an integer n , i.e.

$$\frac{Am}{\rho} = \sqrt{n(n+1)} \quad (209)$$

Equivalently, there is no particle creation when the cycle period of the Robertson-Walker metric and the mass are related by

$$T = \frac{1}{m} \sqrt{n(n+1)} \pi \quad (210)$$

6 Outlook

These simple models already illustrate some of the power of the Schrödinger picture techniques to describe vacuum states in spacetimes with time-dependent metrics. As we have seen in the discussion of cosmological particle creation, there is a clear advantage in such spacetimes of representing the vacuum state by a Gaussian wave functional, entirely characterised by a simple kernel function, rather than the usual Fock space description. This approach also makes clear that it is only in the very special case of certain static spacetimes that ‘vacuum’ states exist which share all the properties associated with the usual Minkowski vacuum.

In the companion paper, we extend this investigation of the Schrödinger picture to simple model spacetimes with boundaries. In this case, even when the spacetime is static, the Unruh effect sharpens the question of defining a vacuum state by requiring us to identify which, if any, class of observers (modelled in the Schrödinger picture by the choice of foliation⁷) perceives the chosen state to resemble a Minkowski vacuum.

A complete understanding of all these issues is a necessary precursor to describing vacuum states and particle creation in Schwarzschild-Kruskal black hole spacetimes, which comprise both static and dynamic regions separated by boundaries which are event horizons.

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⁷This statement requires careful qualification, given the fact that for specified boundary conditions, physical quantities are independent of the choice of foliation. See ref.([5]) for an analysis of the subtleties associated with observer and foliation dependence.

A Wave Equation

The wave equation for a massive scalar field is

$$(\square + m^2 + \xi R)\psi(x) = 0 \quad (211)$$

where the Laplacian operator is

$$\square = \frac{1}{\sqrt{-g}} \partial_\mu (g^{\mu\nu} \sqrt{-g} \partial_\nu) \quad (212)$$

and $g = \det(g_{\mu\nu})$. The wave equation can be rewritten as

$$(\square_0 + \square_i + m^2 + \xi R)\psi(x) = 0 \quad (213)$$

where

$$\square_i = \frac{1}{\sqrt{-g}} \partial_i (g^{ij} \sqrt{-g} \partial_j) \quad (214)$$

$$\square_0 = \frac{1}{\sqrt{-g}} \partial_0 (g^{00} \sqrt{-g} \partial_0) \quad (215)$$

in coordinates where the metric components g_{0i} vanish.

A.1 Static and dynamic spacetimes

If the metric is independent of a coordinate, then there exists a Killing vector associated with the corresponding translation symmetry. We use this to introduce convenient Fourier transforms for the special cases of static and dynamic spacetimes.

In the static case, because of the translational invariance of the wave equation with respect to t , the solution may be written as

$$\psi(x) = \int_{-\infty}^{\infty} \frac{dw}{(2\pi)} e^{-i\omega t} \tilde{\psi}(\omega, \underline{x}) \quad (216)$$

where

$$(\square_i - g^{00}\omega^2 + m^2 + \xi R) \tilde{\psi}(\omega, \underline{x}) = 0 \quad (217)$$

This equation has degenerate solutions labelled by a discrete or continuous set of quantum numbers (λ) , which generalise the momentum \underline{k} in flat space.

An extensive discussion of the general properties of such solutions in static spacetimes has been given by Fulling [2, 19]. They satisfy the orthonormality and completeness relations

$$\int \frac{d^d \underline{x}}{(2\pi)^d} \sqrt{-h_x} \sqrt{g_x^{00}} \tilde{\psi}_{(\lambda)}^*(\omega, \underline{x}) \tilde{\psi}_{(\rho)}(\omega, \underline{x}) = \delta(\lambda, \rho) \quad (218)$$

and

$$\int \frac{d\mu(\lambda)}{(2\pi)^d} \tilde{\psi}_{(\lambda)}^*(\omega, \underline{x}) \tilde{\psi}_{(\lambda)}(\omega, \underline{y}) = \sqrt{g_{00}^x} \delta^d(\underline{x}, \underline{y}) \quad (219)$$

where the measure $\mu(\lambda)$ and delta function $\delta(\lambda, \rho)$ are such that

$$\int d\mu(\lambda) \delta(\lambda, \rho) \tilde{f}(\lambda) = \tilde{f}(\rho)$$

In the dynamic case, because of the translational invariance of the wave equation with respect to x^i , the solution may be written as

$$\psi(x) = \int \frac{d^d k}{(2\pi)^d} e^{i\underline{k} \cdot \underline{x}} \tilde{\psi}(t, \underline{k}) \quad (220)$$

where $\underline{k} \cdot \underline{x} \equiv g^{ij} k_i x^j$, the measure is defined in terms of the covariant vector components as $d^d \underline{k} = dk_1 \dots dk_d$ and $\tilde{\psi}(t, \underline{k})$ satisfies

$$(\square_0 - g^{ij} k_i k_j + m^2 + \xi R) \tilde{\psi}(t, \underline{k}) = 0 \quad (221)$$

A.2 Conformally static and Robertson-Walker spacetimes

Now consider spacetimes with a conformally static metric of the form (133). The Laplacian \square_0 becomes

$$\square_0 = \sqrt{g^{00}} a^{-d} \partial_0 (\sqrt{g^{00}} a^d \partial_0) \quad (222)$$

and all dependence on the spatial coordinates drops out. The Laplacian \square_i is

$$\begin{aligned} \square_i &= \frac{-1}{a^2 \sqrt{\sigma}} \partial_i (\sigma^{ij} \sqrt{\sigma} \partial_j) \\ &= -a^{-2} \square_d \end{aligned} \quad (223)$$

where \square_d is the Laplacian on the d -dimensional static space with metric σ_{ij} . The wave equation takes the form

$$[a^2 \square_0 - \square_d + a^2(m^2 + \xi R)] \psi(x) = 0 \quad (224)$$

The solutions factorise and we find

$$\psi(t, \underline{x}) = \mathcal{Y}_{(\lambda)}(\underline{x}) \mathcal{T}(t, \lambda) \quad (225)$$

where $\mathcal{Y}_{(\lambda)}(\underline{x})$ are eigenfunctions of \square_d with eigenvalues $K^2(\lambda)$, i.e.

$$\square_d \mathcal{Y}_{(\lambda)}(\underline{x}) = -K^2(\lambda) \mathcal{Y}_{(\lambda)}(\underline{x}) \quad (226)$$

and $\mathcal{T}(t, \lambda)$ is a solution of the transformed wave equation

$$[a^2 \square_0 + K^2(\lambda) + a^2(m^2 + \xi R)]\mathcal{T}(t, \lambda) = 0 \quad (227)$$

All the analysis above for static spacetimes applies to eq.(226). General theorems assure the existence of a complete, orthonormal set of solutions $\mathcal{Y}_{(\lambda)}(\underline{x})$ satisfying

$$\int \frac{d^d \underline{x}}{(2\pi)^d} \sqrt{\sigma} \mathcal{Y}_{(\lambda)}(\underline{x}) \mathcal{Y}_{(\rho)}^*(\underline{x}) = \delta(\lambda, \rho) \quad (228)$$

$$\int \frac{d\mu(\lambda)}{(2\pi)^d} \mathcal{Y}_{(\lambda)}(\underline{x}) \mathcal{Y}_{(\lambda)}^*(\underline{z}) = \frac{\delta^d(\underline{x} - \underline{z})}{\sqrt{\sigma}} \quad (229)$$

Robertson-Walker spacetimes are a special case of this class of conformally static metrics. The solutions of the wave equation for $\kappa = -1, 0, +1$ are therefore

$$\psi(t, \underline{x}) = \mathcal{T}(t, \lambda) \mathcal{Y}_\lambda(\chi, \theta, \phi) \quad (230)$$

where \mathcal{Y}_λ are eigenfunctions of the 3-dimensional Laplacian \square_3 ,

$$\square_3 \mathcal{Y}_\lambda(\chi, \theta, \phi) = -K^2(\lambda) \mathcal{Y}_\lambda(\chi, \theta, \phi) \quad (231)$$

and $\mathcal{T}(t, \lambda)$ satisfies

$$[a^{-1} \partial_t(a^3 \partial_t) + K^2(\lambda) + a^2(m^2 + \xi R)]\mathcal{T}(t, \lambda) = 0 \quad (232)$$

Here, $K^2(\lambda) = k^2 - \kappa$, where $(\lambda) = (k, j, m)$ and the eigenfunctions $\mathcal{Y}_{(\lambda)}(\underline{x})$, [20, 21] are explicitly

$$\mathcal{Y}_{(\lambda)}(\underline{x}) = \begin{cases} (2\pi)^{\frac{3}{2}} \Pi_{kj}^{(-)}(\chi) Y_j^m(\theta, \phi) & \kappa = -1 \\ (2\pi)^{\frac{3}{2}} \sqrt{\frac{k}{\chi}} J_{j+\frac{1}{2}}(k\chi) Y_j^m(\theta, \phi) & \kappa = 0 \\ (2\pi)^{\frac{3}{2}} \Pi_{kj}^{(+)}(\chi) Y_j^m(\theta, \phi) & \kappa = +1 \end{cases} \quad (233)$$

The Y_j^m are spherical harmonics and the function $\Pi^{(-)}$ is defined by

$$\Pi_{kj}^{(-)}(\chi) = [\frac{1}{2}\pi k^2(k^2 + 1) \dots (k^2 + j^2)]^{-\frac{1}{2}} \sinh^j \chi \left(\frac{d^{j+1} \cos(k\chi)}{d(\cosh \chi)^{j+1}} \right) \quad (234)$$

The function $\Pi_{kj}^{(+)}$ is obtained from $\Pi_{kj}^{(-)}$ by replacing k with $-ik$ and χ with $-i\chi$. Alternative forms of this decomposition are given in [3, 22].

The orthonormality and completeness relations are given by eqs.(228) and (229), viz.

$$\int \frac{d^3 \underline{x}}{(2\pi)^3} \sqrt{\sigma} \mathcal{Y}_{(\lambda)}(\underline{x}) \mathcal{Y}_{(\rho)}^*(\underline{x}) = \delta(\lambda, \rho) \quad (235)$$

$$\int \frac{d\mu(\lambda)}{(2\pi)^3} \mathcal{Y}_{(\lambda)}(\underline{x}) \mathcal{Y}_{(\lambda)}^*(\underline{y}) = \frac{\delta^3(\underline{x} - \underline{y})}{\sqrt{\sigma}} \quad (236)$$

where $\sqrt{\sigma} = f^2(\chi) \sin \theta$. The integration measures for the different types of Robertson-Walker metric are

$$\int d\mu(\lambda) = \begin{cases} \int_0^\infty dk \sum_{j=0}^\infty \sum_{m=-j}^j & \kappa = -1, 0 \\ \sum_{k=1}^\infty \sum_{j=0}^{k-1} \sum_{m=-j}^j & \kappa = 1 \end{cases} \quad (237)$$

which also specifies the form and range of the quantum numbers k, j , and m .

For the spatially flat $\kappa = 0$ metrics, it is of course frequently more convenient to choose the eigenfunctions in Cartesian form

$$\mathcal{Y}_{(\lambda)}(\underline{x}) = \exp(ik_j x^j) \quad (238)$$

where $(\lambda) = (k_1, k_2, k_3)$ and $K^2(\lambda) = k^2$. The measure is just $\int d\mu(\lambda) = \int d^3 \underline{k}$.

B Green Functions

The Green function is given by the equation

$$\sqrt{-g}(\square + m^2 + \xi R)\mathcal{G}(x, y) = -\delta^{(d+1)}(x - y) \quad (239)$$

As in the previous section, if the equation is translationally invariant with respect to a particular coordinate then the Green function may be expressed as a Fourier transform.

In the static case,

$$\mathcal{G}(x, y) = \int_{-\infty}^{\infty} \frac{d\omega}{(2\pi)} e^{-i\omega(x^0 - y^0)} \tilde{\mathcal{G}}(\omega; \underline{x}, \underline{y}) \quad (240)$$

where

$$\sqrt{-g}(\square_i - g^{00}\omega^2 + m^2 + \xi R)\tilde{\mathcal{G}}(\omega; \underline{x}, \underline{y}) = -\delta^d(\underline{x} - \underline{y}) \quad (241)$$

For dynamic spacetimes,

$$\mathcal{G}(x, y) = \int \frac{d^d \underline{k}}{(2\pi)^d} e^{i\underline{k} \cdot (\underline{x} - \underline{y})} \tilde{\mathcal{G}}(\underline{k}; x^0, y^0) \quad (242)$$

where

$$\sqrt{-g}(\square_0 - g^{ij}k_i k_j + m^2 + \xi R)\tilde{\mathcal{G}}(\underline{k}; x^0, y^0) = -\delta(x^0 - y^0) \quad (243)$$

We cannot say more about the dynamic Green function in general. However, in the static case, we can give a general solution. Using the orthonormality relations of solutions to the wave equation, we find

$$\mathcal{G}(x, y) = \int_{-\infty}^{\infty} \frac{d\omega}{(2\pi)} \int \frac{d\mu(\lambda)}{(2\pi)^d} \frac{e^{-i\omega(x^0 - y^0)} \tilde{\psi}_{(\lambda)}(\nu, \underline{x}) \tilde{\psi}_{(\lambda)}^*(\nu, \underline{y})}{(\omega^2 - \nu(\lambda)^2)} \quad (244)$$

plus solutions of the homogeneous equation. The ω integral is along the real axis, but there are poles at $w = \pm\nu$. The choice of contour ($i\epsilon$ prescription) selects one of a variety of different Green functions [1, 2].

For example, the contours giving the Wightman function are shown in Fig(3). Evaluating the integral gives

$$\mathcal{G}_+(x, y) = \langle 0 | \varphi(x) \varphi(y) | 0 \rangle \quad (245)$$

$$= \frac{1}{2} \int \frac{d\mu(\lambda)}{(2\pi)^d} \frac{1}{\omega} e^{-i\omega(x_0-y_0)} \tilde{\psi}_{(\lambda)}(\omega, \underline{x}) \tilde{\psi}_{(\lambda)}^*(\omega, \underline{y}) \quad (246)$$

As the Wightman function is just the difference between two Green functions, it satisfies the homogeneous wave equation.

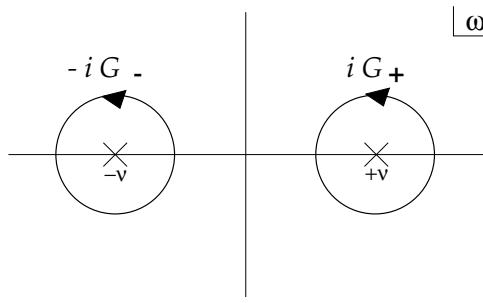


Figure 3: The omega contour integral for the Wightman functions.

C Expectation values and two-point functions

The vacuum expectation value of an operator $O(\varphi, \pi)$ (a function of the field φ and conjugate momentum π) in the Schrödinger picture is

$$\langle 0 | O(\varphi, \pi) | 0 \rangle = \int \mathcal{D}\phi \Psi_0^* O(\phi, -i\frac{\delta}{\delta\phi}) \Psi_0 \quad (247)$$

The action of the momentum is simply

$$\begin{aligned} \pi(\underline{x})\Psi_0 &= -i\frac{\delta}{\delta\phi(\underline{x})}\Psi_0 \\ &= i \int d^d\underline{y} \sqrt{h_y h_x} \phi(\underline{y}) G(\underline{y}, \underline{x}; t) \Psi_0 \end{aligned} \quad (248)$$

To evaluate the action of the field, it is convenient to modify the wave functional by formally introducing a ‘source’, viz.

$$\Psi_J = \Psi_0 \exp \left\{ \frac{1}{2} \int d^d\underline{x} \sqrt{-h_x} J(\underline{x}) \phi(\underline{x}) \right\} \quad (249)$$

Omitting field independent terms for clarity, the vacuum amplitude is then

$$\langle 0|0\rangle_J \sim \int \mathcal{D}\phi e^{-\int d^d\underline{x} \int d^d\underline{y} \sqrt{h_x h_y} \phi(\underline{x}) G_{\Re}(\underline{x}, \underline{y}; t) \phi(\underline{y}) + \int d^d\underline{x} \sqrt{-h_x} J(\underline{x}) \phi(\underline{x})} \quad (250)$$

$$\sim \exp\left\{\frac{1}{4} \int d^d\underline{x} \int d^d\underline{y} \sqrt{h_x h_y} J(\underline{x}) \Delta_{\Re}(\underline{x}, \underline{y}; t) J(\underline{y})\right\} \quad (251)$$

Notice that the functional integral involves just the real part G_{\Re} of the kernel. Also note that Δ_{\Re} is defined here as the inverse of G_{\Re} (*not* the real part of the inverse kernel). Expectation values of products of fields can then be found by repeated differentiation with respect to the source, e.g.

$$\begin{aligned} \langle 0|\varphi(\underline{x})\varphi(\underline{y})|0\rangle &= \int \mathcal{D}\phi \Psi_J^* \phi(\underline{x})\phi(\underline{y}) \Psi_J \Big|_{J=0} \\ &= \frac{1}{\sqrt{-h_x}} \frac{\delta}{\delta J(\underline{x})} \frac{1}{\sqrt{-h_y}} \frac{\delta}{\delta J(\underline{y})} \langle 0|0\rangle_J \Big|_{J=0} \\ &= \frac{1}{2} \Delta_{\Re}(\underline{x}, \underline{y}; t) \end{aligned} \quad (252)$$

This is the equal-time Wightman function, confirming the identifications (20) and (103) in the text.

Other useful two-point functions are

$$\begin{aligned} \langle 0| \pi(\underline{x}) \pi(\underline{y}) |0\rangle &= \frac{1}{2} \sqrt{h_x h_y} \left\{ G_{\Re}(\underline{x}, \underline{y}; t) \right. \\ &\quad \left. + \int d^d\underline{u} \int d^d\underline{v} \sqrt{h_u h_v} G_{\Im}(\underline{x}, \underline{u}; t) \Delta_{\Re}(\underline{u}, \underline{v}; t) G_{\Im}(\underline{v}, \underline{y}; t) \right\} \end{aligned} \quad (253)$$

and

$$\langle 0| [\varphi(\underline{x}), \pi(\underline{y})] |0\rangle = i\delta^d(\underline{x} - \underline{y}) \quad (254)$$

$$\langle 0| \{\varphi(\underline{x}), \pi(\underline{y})\} |0\rangle = - \int d^d\underline{u} \sqrt{h_x h_u} G_{\Im}(\underline{x}, \underline{u}; t) \Delta_{\Re}(\underline{u}, \underline{y}; t) \quad (255)$$

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