Truncated Power Distribution

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1 PDF

$$f_X(x \mid \alpha) = \frac{x^\alpha}{\mathcal{Z}},\ x \in [L, H] \tag{1}$$

Before we continure futher lets assume that $\beta = 1 + \alpha$. To find $\mathcal{Z}$ we will normalize the PDF.

$$\begin{align}
\int_L^H f_X(x \mid \alpha) dx = 1 \tag{2} \\
\int_L^H \frac{x^\alpha}{\mathcal{Z}} dx = 1 \tag{3} \\
\mathcal{Z} = \int_L^H x^\alpha dx \tag{4} \\
\mathcal{Z}(\beta) = \begin{cases} \log(H) - \log(L) & \beta = 0 \\ \dfrac{H^\beta - L^\beta}{\beta} & \beta \neq 0 \end{cases} \tag{5}
\end{align}$$

2 CDF

$$\begin{align}
F_X(x \mid \alpha) &= \int_L^x f_X(x \mid \alpha) dx \tag{6} \\
&= \frac{1}{\mathcal{Z}} \int_L^x x^\alpha dx \tag{7} \\
&= \frac{1}{\mathcal{Z}(\beta)} \begin{cases} \log(x) - \log(L) & \beta = 0 \\ \dfrac{x^\beta - L^\beta}{\beta} & \beta \neq 0 \end{cases} \tag{8}
\end{align}$$

3 Inverse CDF

For $\beta = 0$,

$$x = \frac{1}{\mathcal{Z}(-1)}\left(\log\left(F_X^{-1}(x \mid \alpha)\right) - \log(L)\right) \tag{9}$$

$$\log\left(F_X^{-1}(x \mid \alpha)\right) = (\log(H) - \log(L))x + \log(L) \tag{10}$$

$$\log\left(F_X^{-1}(x \mid \alpha)\right) = x\log(H) + (1-x)\log(L) \tag{11}$$

$$F_X^{-1}(x \mid \alpha) = \exp(x\log(H) + (1-x)\log(L)) \tag{12}$$

$$F_X^{-1}(x \mid \alpha) = e^{x\log(H)}e^{(1-x)\log(L)} \tag{13}$$

$$F_X^{-1}(x \mid \alpha) = \left(e^{\log(H)}\right)^x \left(e^{\log(L)}\right)^{1-x} \tag{14}$$

$$F_X^{-1}(x \mid \alpha) = H^x L^{1-x} \tag{15}$$

For $\beta \neq 0$,

$$x = \frac{1}{\mathcal{Z}(\beta)}\frac{\left(F_X^{-1}(x \mid \alpha)\right)^\beta - L^\beta}{\beta} \tag{16}$$

$$x = \frac{\left(F_X^{-1}(x \mid \alpha)\right)^\beta - L^\beta}{H^\beta - L^\beta} \tag{17}$$

$$\left(F_X^{-1}(x \mid \alpha)\right)^\beta = xH^\beta + (1-x)L^\beta \tag{18}$$

$$F_X^{-1}(x \mid \alpha) = \left(xH^\beta + (1-x)L^\beta\right)^{1/\beta} \tag{19}$$

$f_X(x \mid \alpha)$	$\frac{x^\alpha}{\mathcal{Z}}$	$\log(f_X(x \mid \alpha))$	$\alpha\log(x) - \log(\mathcal{Z})$
$\mathcal{Z}(\beta)$	$\begin{cases} \log(H) - \log(L) & \beta = 0 \\ \frac{H^\beta - L^\beta}{\beta} & \beta \neq 0 \end{cases}$	$\log(\mathcal{Z}(\beta))$	$\begin{cases} \log(\log(H) - \log(L)) & \beta = 0 \\ \log\left(H^\beta - L^\beta\right) - \log(\beta) & \beta \neq 0 \end{cases}$
$F_X(x \mid \alpha)$	$\frac{1}{\mathcal{Z}(\beta)}\begin{cases} \log(x) - \log(L) & \beta = 0 \\ \frac{x^\beta - L^\beta}{\beta} & \beta \neq 0 \end{cases}$	$\log(F_X(x \mid \alpha))$	$\begin{cases} \log(\log(x) - \log(L)) & \beta = 0 \\ \log\left(x^\beta - L^\beta\right) - \log(\beta) & \beta \neq 0 \end{cases}$
$F_X^{-1}(x \mid \alpha)$	$\begin{cases} H^x L^{1-x} & \beta = 0 \\ \left(xH^\beta + (1-x)L^\beta\right)^{1/\beta} & \beta \neq 0 \end{cases}$	$\log\left(F_X^{-1}(x \mid \alpha)\right)$	$\begin{cases} x\log(H) + (1-x)\log(L) & \beta = 0 \\ \frac{1}{\beta}\log\left(xH^\beta + (1-x)L^\beta\right) & \beta \neq 0 \end{cases}$