Truncated Normal Distribution

Meesum Qazalbash

December 8, 2023

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$$f_X(x|\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma\mathcal{Z}}\exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right),\ x\in[L,H] \tag{1}$$

First we will find the normalization constant.

$$1 = \int_L^H \frac{1}{\sqrt{2\pi}\sigma\mathcal{Z}}\exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)dx \tag{2}$$

$$\mathcal{Z} = \int_L^H \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)dx \tag{3}$$

$$\mathcal{Z} = \int_{-\infty}^H \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)dx - \int_{-\infty}^L \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)dx \tag{4}$$

$$\mathcal{Z} = \Phi\left(\frac{H-\mu}{\sigma}\right) - \Phi\left(\frac{L-\mu}{\sigma}\right) \tag{5}$$

$$\mathcal{Z} = \Phi(\beta) - \Phi(\alpha) \tag{6}$$

Where, $\alpha = \dfrac{L-\mu}{\sigma}$ and $\beta = \dfrac{H-\mu}{\sigma}$.

2 CDF

$$F_X(x) = \int_L^x \frac{1}{\sqrt{2\pi}\sigma\mathcal{Z}}\exp\left(-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2\right)dt \tag{7}$$

$$= \frac{1}{\mathcal{Z}}\int_{-\infty}^x \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2\right)dt - \frac{1}{\mathcal{Z}}\int_{-\infty}^L \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2\right)dt \tag{8}$$

$$= \frac{1}{\mathcal{Z}}\Phi\left(\frac{x-\mu}{\sigma}\right) - \frac{1}{\mathcal{Z}}\Phi\left(\frac{L-\mu}{\sigma}\right) \tag{9}$$

$$= \frac{\Phi(\zeta) - \Phi(\alpha)}{\mathcal{Z}} \tag{10}$$

Where, $\zeta = \dfrac{x-\mu}{\sigma}$.

3 Inverse CDF

$$x = \frac{\Phi\left(\frac{F_X^{-1}(x)-\mu}{\sigma}\right) - \Phi\left(\alpha\right)}{\mathcal{Z}} \tag{11}$$

$$F_X^{-1}(x) = \mu + \sigma\Phi^{-1}\left(\Phi\left(\alpha\right) + \mathcal{Z}x\right) \tag{12}$$

$f_X(x)$	$\frac{1}{\sqrt{2\pi}\sigma\mathcal{Z}}\exp\left(-\frac{1}{2}\zeta^2\right)$	$\log\left(f_X(x)\right)$	$-\frac{1}{2}\zeta^2 - \frac{1}{2}\log\left(2\pi\right) - \log\left(\sigma\right) - \log\left(\mathcal{Z}\right)$
$\mathcal{Z}(\beta)$	$\Phi\left(\beta\right) - \Phi\left(\alpha\right)$	$\log\left(\mathcal{Z}(\beta)\right)$	$\log\left(\Phi\left(\beta\right) - \Phi\left(\alpha\right)\right)$
$F_X(x)$	$\frac{\Phi\left(\zeta\right) - \Phi\left(\alpha\right)}{\mathcal{Z}}$	$\log\left(F_X(x)\right)$	$\log\left(\Phi\left(\zeta\right) - \Phi\left(\alpha\right)\right) - \log\left(\mathcal{Z}\right)$
$F_X^{-1}(x)$	$\mu + \sigma\Phi^{-1}\left(\Phi\left(\alpha\right) + \mathcal{Z}x\right)$	$\log\left(F_X^{-1}(x)\right)$	$\log\left(\mu + \sigma\Phi^{-1}\left(\Phi\left(\alpha\right) + \mathcal{Z}x\right)\right)$