Description: The exam has two parts. On the front of this page you have three problems to be answered by hand on paper and delivered to the professor when you are done. On the back of this page you have two problems to be delivered at `https://examens.fib.upc.edu` before the exam ends. Both consist of thinking and programming something, and both include writing some explanation in the HTML file.

Duration: 4 hours.

Problem 1. [2 points] Deliver on paper. Let C_1 and C_2 be two Bézier curves of degree 9 in the plane, with the last control point of C_1 equal to the first control point of C_2. We want to connect them so that the resulting curve is C^2-continuous at the connecting point.

1. Describe what we need to do.
2. Prove why it works.

Solution: If $B(t)$, with $t \in [0,1]$, is a Bézier curve of degree n, defined by the control points $P_0, \ldots, P_n$, it is easy to see that

$$\begin{array}{ll} B(0) = P_0, & B(1) = P_n, \\ B'(0) = n(P_1 - P_0), & B'(1) = n(P_n - P_{n-1}), \\ B''(0) = n(n-1)(P_2 - 2P_1 + P_0), & B''(1) = n(n-1)(P_n - 2P_{n-1} + P_{n-2}). \end{array}$$

Suppose that C_1 is defined by the control points $P_0, \ldots, P_9$ and C_2 is defined by the control points $Q_0, \ldots, Q_9$. If we want to connect them so that the resulting curve is C^2-continuous at the connecting point, we need the following conditions to hold:

$$\begin{aligned} C_1(1) &= C_2(0), \\ C_1'(1) &= C_2'(0), \\ C_1''(1) &= C_2''(0). \end{aligned}$$

This translates into the following conditions on the control points:

$$\begin{aligned} P_9 &= Q_0, \\ P_9 - P_8 &= Q_1 - Q_0, \\ P_9 - 2P_8 + P_7 &= Q_2 - 2Q_1 + Q_0. \end{aligned}$$

Problem 2. [1 point] Deliver on paper. We need to construct a piece-wise curve in the plane, where one of the pieces of the curve must be a half-circle.

1. Which of the many types of curves studied in class would you use?
2. Why?
3. Describe all the necessary details of how you would do it.

Solution: The right type of curve to use is a rational Bézier or rational B-Spline, because these are the only types of curves seen in the course that can represent a half-circle exactly.

In the details, we expected at least some reference to how to place the control points and how to choose their weights to represent the half-circle, and some mention of how to connect the different pieces to guarantee some continuity.

Still, people who chose a different type of curve in the first question were given some score for the second and third parts if the answers there were correct and consistent with the choice of curve.

Problem 3. [1 point] Deliver on paper. Describe the strategy you would use to design in 3D a pipe like the one in the image.

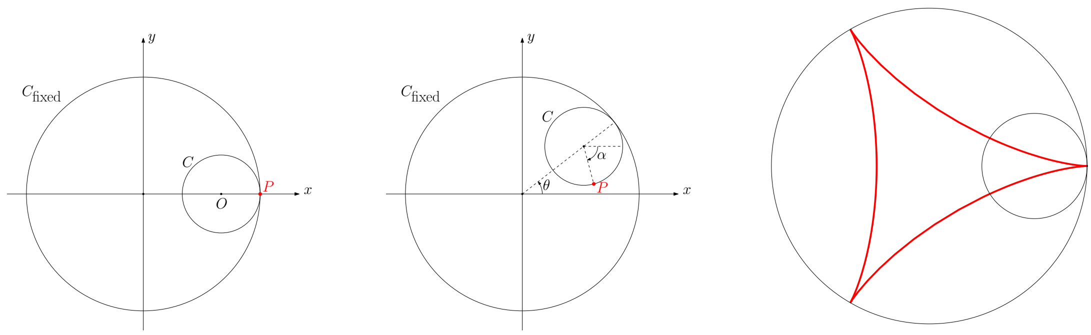
Solution: There exist different strategies to solve this problem. The simplest one is probably as follows:

1. Using a Bézier curve or a spline, design the curve Γ, with parametrization $\gamma(t)$, $t \in (0, 1)$, to describe the central axis if the pipe.
2. For each value of the parameter $t \in (0, 1)$, compute $\overrightarrow{v}(t) = \gamma'(t)$, the vector tangent to the curve Γ at point $\gamma(t)$, and parametrize a circle $C_t(s)$, with $s \in (0, 2\pi)$, located in the plane orthogonal to $\overrightarrow{v}(t)$ at $\gamma(t)$, with center in $\gamma(t)$, and whose radius is a linear function of t. The result is the interior surface of the pipe.

3. In order to produce the exterior, modify the previous surface:
   (a) first enlarge the radius of the circles $C_t(s)$,
   (b) then modify it for low values of t, in order to squeeze the mouthpiece of the pipe in the direction of one of the coordinate axes,
   (c) finally modify it for large values of t, in order to produce the torus-like annulus at the other end of the pipe.

Any alternative solution should take into account, at least, that the pipe surface has distinct portions, and is almost completely made out of circles.

Problem 4. [3 points] Deliver through the Racó. In the plane, let C_{fixed} be a circle centered at the origin $(0, 0)$. The radius of C_{fixed} is three times the radius of another circle C, which is interior to C_{fixed} and tangent to it at a point P belonging to the positive semi-axis Ox^+, as illustrated in the left figure. Assume that the interior circle C rolls inside the fix circle C_{fixed}. Your goal is to parametrize the curve described by point P along this rolling movement, and to write a program to show the result on your screen. The result should be similar to the red curve in the rightmost figure.



In order to obtain a parametrization of the curve, please use the parameter θ, which is the polar angle of the center O of the rolling circle C, as illustrated in the middle figure. Hint: It may become useful for you to compute the value of the angle α that you can see in the middle figure.

Note: In the HTML file that you will deliver, please include a brief description of the strategy you have followed and a justification of the correctness of your steps.

Solution: As the interior circle C rolls over C_{fixed}, the length of the perimeter of C that has been in touch with C_{fixed} coincides with the length of the perimeter of C_{fixed} that has been in touch with C, i.e., $3r\theta = r(\theta+\alpha)$. From this equation we obtain that

$\alpha = 2\theta$. Therefore, the coordinates of P are $P = 2r(\cos\theta, \sin\theta) + r(\cos\alpha, -\sin\alpha)$. In other words, a parametrization of the curve described by P is:

$$\left.\begin{array}{l} x(\theta) = 2r\cos\theta + r\cos\alpha = 2r\cos\theta + r\cos 2\theta \\ x(\theta) = 2r\sin\theta - r\sin\alpha = 2r\sin\theta - r\sin 2\theta \end{array}\right\} \text{ for } \theta \in [0, 2\pi].$$

As for the implementation part, we expected to see at least this parametrization implemented and graphically shown in the screen. Animation was a plus.

Problem 5. [3 points] Deliver through the Racó. Write a program to draw the outer surface (ignoring all little details) of the following canteen:

front

back

Note: In the HTML file that you will deliver, please include a brief description of the strategy you have followed and the design decisions you have made.

Solution: This exercise has many possible solutions. In our grading, we took into account:

- Producing the three main pieces that form the canteen.
- Producing a non-cylindrical shape.
- Including the bottom and the top of the cap.
- Making the connection between the bottom and middle piece smooth.
- Making the connection between the middle and top piece not smooth.
- Including the concavity on the back side.
- Including the rim that connects the bottom and middle pieces.
- The description included in the HTML file.