DCS 2020-21

2D/3D

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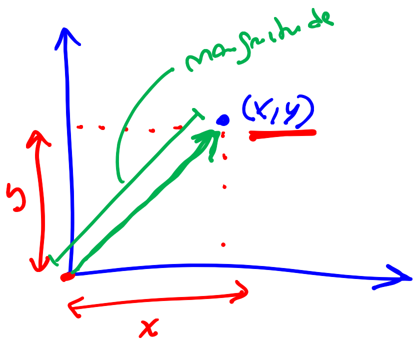
VECTORS AND POINTS



POINT: location

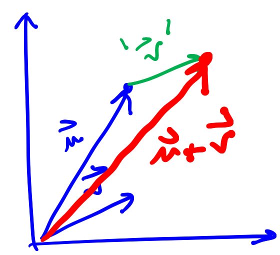
VECTOR: direction, orientation, magnitude

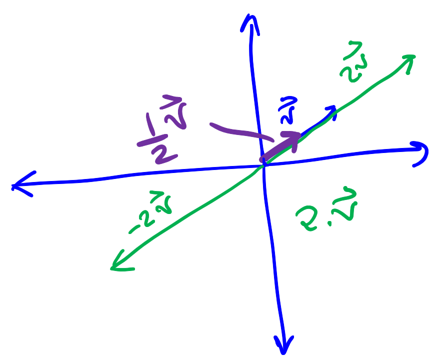
VECTORS

A vector space over $\mathbb{R}$ (→ scalars) is a set V (→ vectors) together with two binary operations:

- vector addition: $V \times V \to V$
  $(\vec{u}, \vec{v}) \to \vec{u} + \vec{v}$

- scalar multiplication $\mathbb{R} \times V \to V$
  $\lambda, \vec{v} \to \lambda . \vec{v}$





$\frac{1}{2}\vec{v}$?

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1) Example of vector space: Coordinate space

$$\mathbb{R}^n = \{ (x_1, x_2, \dots, x_n),\ x_i \in \mathbb{R} \}$$

Example: (points) in Euclidean plane, $\mathbb{R}^2$

$$V = \mathbb{R}^2 = \{ (x,y) : x,y \in \mathbb{R} \}$$

- Addition: $\vec{v}_1 = (x_1, y_1)$ $\quad \vec{v}_1 + \vec{v}_2 = (\underbrace{x_1 + x_2}_{\in \mathbb{R}}, \underbrace{y_1 + y_2}_{\in \mathbb{R}}) \in V$
  $\vec{v}_2 = (x_2, y_2)$

• Scalar multiplication: $\lambda \in \mathbb{R}$ (scalar), $\vec{v}_1 = (x_1, y_1)$

$$\underbrace{\lambda}_{\in \mathbb{R}} . \vec{v}_1 = \underbrace{(\underbrace{\lambda . x_1}_{\in \mathbb{R}}, \lambda . y_1)}_{\in \mathbb{R}^2}$$

$\mathbb{R}^2$, $\mathbb{R}^3$

• Example of a different vector space: quadratic polynomials

$$\underbrace{ax^2 + bx + c}_{\text{a quadratic polynomial}} \quad , a, b, c \in \mathbb{R}$$

$$V = \{ P[x] : P \text{ is a polynomial on } x \text{ of degree } 2 \}$$

$$P(x) = ax^2 + bx + c$$
$$Q(x) = a'x^2 + b'x + c'$$

• $$P(x) + Q(x) = ax^2 + bx + c + a'x^2 + b'x + c'$$
$$= \underbrace{(a+a')x^2 + (b+b')x + (c+c')}_{\text{A quadratic polynomial}}$$

• $\lambda \in \mathbb{R}$

$$\lambda . P(x) = \lambda(ax^2 + bx + c) = \underbrace{\overbrace{(\lambda . a)}^{\in \mathbb{R}} x^2 + \overbrace{(\lambda . b)}^{\in \mathbb{R}} x + \overbrace{(\lambda c)}^{\in \mathbb{R}}}_{\in V}$$

$\therefore$ They form a vector space

Linear combinations

$$\vec{v} = \lambda_1 . \vec{v}_1 + \lambda_2 . \vec{v}_2 + \cdots + \lambda_n \vec{v}_n = \sum_{i=1}^{n} \overbrace{\lambda_i . \vec{v}_i}^{\text{a vector}}$$

coefficients

$\vec{v}$ is a linear combination of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$

A basis of V is set of vectors of V, $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$

such that:

i) they span the whole space V:

i.e. $\forall \vec{v} \in V \Rightarrow \exists\, \lambda_1, \lambda_2, \dots, \lambda_n$ s.t.

$$\vec{v} = \lambda_1 \cdot \vec{e}_1 + \lambda_2 \vec{e}_2 + \dots + \lambda_n \cdot \vec{e}_n = \sum_{i=1}^{n} \lambda_i \cdot \vec{e}_i$$

ii) $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$ are linearly independent:

i.e. $\vec{0} = \sum_{i=1}^{n} \lambda_i \vec{e}_i \Rightarrow \lambda_i = 0 \;\; \forall i$

$\vec{0} = (0, 0, \dots, 0)$

$$\underbrace{\vec{e}_1}_{} = \lambda_2 \vec{e}_2 + \lambda_3 \vec{e}_3 + \dots + \vec{\lambda}_n e_n \quad \left(\begin{matrix}\exists i \\ \lambda_i \neq 0\end{matrix}\right)$$

$$\vec{0} = -\vec{e}_1 + \lambda_2 \vec{e}_2 + \dots + \vec{\lambda}_n e_n$$

The coefficients of the unique combinations are called **coordinates** of the vector

Example basis of $\mathbb{R}^2$: $\{(1,0), (0,1)\}$

$$(5,3) = \lambda_1 (1,0) + \lambda_2 (0,1) \Rightarrow \begin{cases} \lambda_1 = 5 \\ \lambda_2 = 3 \end{cases}$$

In general: $(x,y) = \lambda_1 (1,0) + \lambda_2 (0,1) \Rightarrow \begin{cases} \lambda_1 = x \\ \lambda_2 = y \end{cases}$

Should show: $\lambda_1 (1,0) + \lambda_2 (0,1) = (0,0)$

$$\Rightarrow \lambda_1 = \lambda_2 = 0$$

Once a basis is fixed, we have

$$V \longleftrightarrow \mathbb{R}^d$$

vector $\vec{v}$ coordinates $(x_1, x_2, \dots, x_d)$

$\{(1,0)\,(0,1)\}$

$(5,3)$

$(5,3) = 5(1,0) + 3(0,1)$

coordinates of $(5,3)$ in basis $\{(1,0)\,(0,1)\}$

coords: $(6,2)$

$6(1,0) + 2(0,1)$
$= (6,2)$

$\boxed{\text{This allows us to treat all vector space as } \mathbb{R}^d, \text{ in other words: any vector space has a bijection with } \mathbb{R}^d}$

$\vec{v} = (1,2) \longleftrightarrow$ coordinate $(1,2) \to \mathbb{R}^2$ point

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Another example: quadratic polynomials

$P_2 = \{ax^2 + bx + c : a, b, c \in \mathbb{R}\}$

Basis: $\{ \overset{\vec{e}_1}{1}, \overset{\vec{e}_2}{x}, \overset{\vec{e}_3}{x^2} \}$

quadratic polynomials

An $\mathbb{R}^3$ basis $\{(1\,0\,0)\,(0\,1\,0)\,(0\,0\,1)\}$

$\boxed{5x^2 + 3x - 2} = (-2)\vec{e}_1 + 3\vec{e}_2 + 5\vec{e}_3$
$= (-2) \cdot 1 + 3 \cdot x + 5 \cdot x^2$
$= 5x^2 + 3x - 2$

or $\{(2,3,5)\,(0,1,8)\,(0,0,-3)\}$

coordinates in basis are: $\boxed{(-2, 3, 5)}$

Measuring vectors

We call inner product the following operation on vectors:

$$V \times V \to \mathbb{R}$$
$$(\vec{u}, \vec{v}) \to \vec{u}.\vec{v} = \sum_{i=1}^{n} u_i . v_i$$

E.g. $\left.\begin{array}{l}\vec{u} = (1,2)\\ \vec{v} = (3,4)\end{array}\right\} \vec{u}.\vec{v} = 1.3 + 2.4 = 3 + 8 = 11$

Also written as $\langle \vec{u}, \vec{v} \rangle$

The dot product allows to measure $\begin{cases}\text{length} \\ \text{angle}\end{cases}$ $\quad \vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\|\vec{v}\| = \sqrt{\vec{v}.\vec{v}} = \sqrt{x.x + y.y} = \sqrt{x^2 + y^2}$

The norm of a vector is $V \to \mathbb{R}$

$$\boxed{\vec{v} \to \underbrace{\|\vec{v}\|}_{\text{norm of } \vec{v}} = \sqrt{\underbrace{\vec{v} \bullet \vec{v}}_{\text{dot product}}}}$$

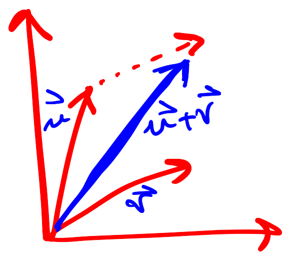
Properties

i) $\|\vec{v}\| = 0 \iff \vec{v} = 0$

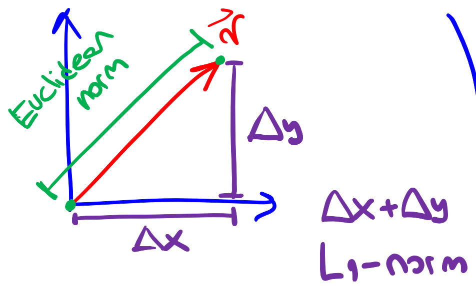
$\|\vec{v}\| \geq 0 \quad \forall \vec{v} \in V$

ii) $\|\lambda . \vec{v}\| = |\lambda| . \|\vec{v}\| \quad \forall \vec{v} \in V, \forall \lambda \in \mathbb{R}$

iii) $\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\| \quad \forall \vec{u}, \vec{v} \in V$

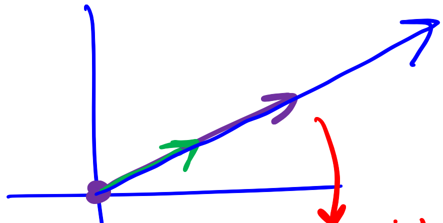


Note there are many different norms!



- We call $\vec{v}$ a unit vector iff $\|\vec{v}\| = 1$.
- We normalize a vector $\vec{v}$ by: $\boxed{\dfrac{\vec{v}}{\|\vec{v}\|}}$

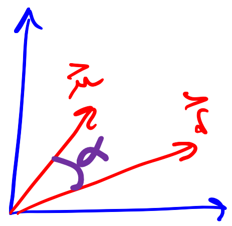
Is this a unit vector? $\left\| \dfrac{\vec{v}}{\|\vec{v}\|} \right\| = \left\| \underbrace{\dfrac{1}{\|\vec{v}\|}}_{\in \mathbb{R}} . \vec{v} \right\| = \left| \dfrac{1}{\|\vec{v}\|} \right| . \|\vec{v}\| = \dfrac{1}{\|\vec{v}\|} . \|\vec{v}\| = 1.$

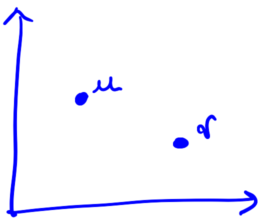


3 vectors with same direction

The <u>angle</u> between two vectors $\vec{u}, \vec{v}$ $(\neq \vec{0})$ is defined as the unique $\alpha \in [0, \pi)$ such that: $\boxed{\cos\alpha} = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}$

$$\boxed{= \frac{\vec{u}}{\|\vec{u}\|} \cdot \frac{\vec{v}}{\|\vec{v}\|}}$$



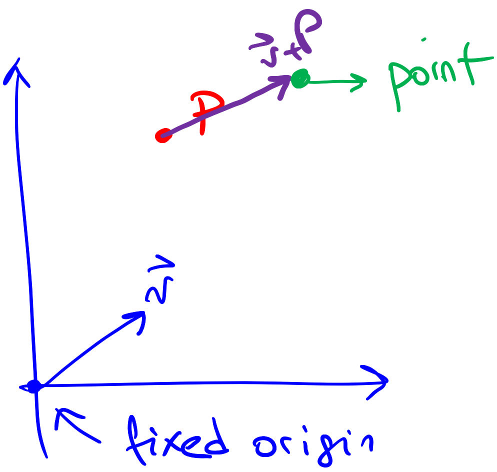


- Two vectors are <u>orthogonal</u> when $\vec{u}\cdot\vec{v} = 0$

POINTS

An <u>affine space</u> (over $\mathbb{R}$) is a set of points A together with a vector space V (over $\mathbb{R}$) and a map (addition operation)

$$A \times V \longrightarrow A$$
$$(P, \vec{v}) \longrightarrow P + \vec{v}$$



Properties:

addition in affine space   addition in vector space

i) $(P+\vec{v}) + \vec{w} = P + (\vec{v} + \vec{w})$

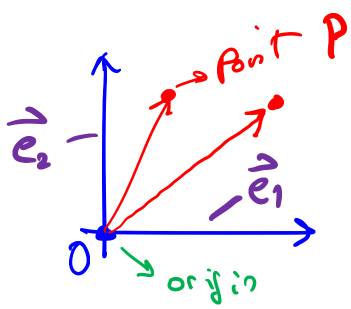
ii) If <u>P is fixed</u>, the map is a bijection

$\forall\, P, Q \in A,\ \exists!\, \vec{v} \in V : P + \vec{v} = Q$

written as $\vec{v} = \overrightarrow{PQ} = Q - P$

A <u>coordinate system</u> in A is:

- a point $O \in A$ called <u>origin</u>.
- a basis $\{\vec{e}_1, \dots, \vec{e}_n\}$ of V.



<u>We have</u>:

$$\begin{array}{ccccc} A & \xleftrightarrow{O \text{ fixed}} & V & \xleftrightarrow{\text{fixed } \{\vec{e}_1, \dots, \vec{e}_n\}} & \mathbb{R}^d \\ P & \longleftrightarrow & \overrightarrow{OP} & \longleftrightarrow & (x_1, \dots, x_n) \end{array}$$

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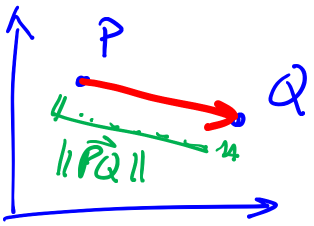
Distances

The distance between two points P and Q is:

$$d : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}$$

$$(P, Q) \longmapsto d(P,Q) = \|\vec{PQ}\| = \|Q-P\| = \boxed{\sqrt{\sum_{i=1}^{d} (q_i - p_i)^2}}$$

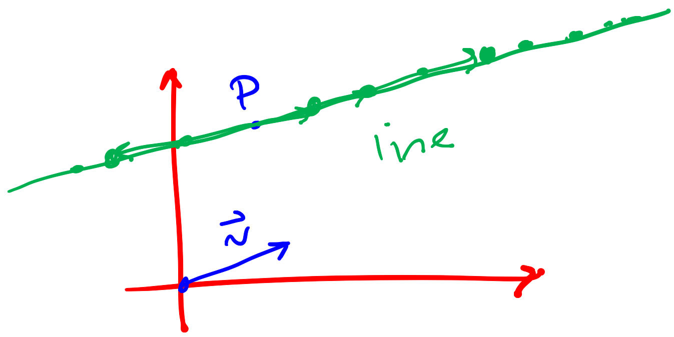
$$= \sqrt{(Q-P)\cdot(Q-P)}$$



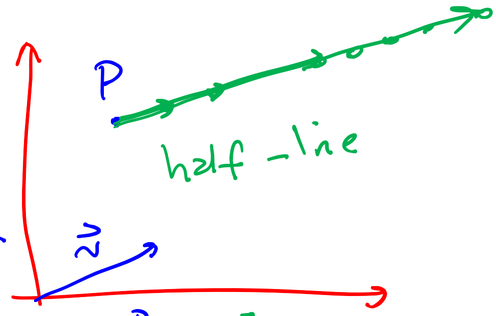
Lines, half-lines, line segments

A line in $\mathbb{R}^d$ is defined by a point P (support point) and a vector $\vec{v} \neq \vec{0}$ as follows:

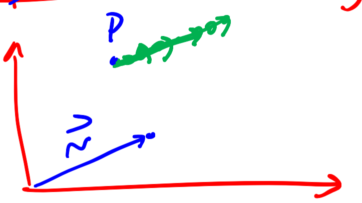
$$\ell = \{X \in \mathbb{R}^d \,/\, X = P + \lambda.\vec{v},\ \lambda \in \mathbb{R}\}$$



Notice:
- if $\lambda \in \mathbb{R}$, we get a line
- if $\lambda \in \mathbb{R}_{\geq 0}$, we get a half-line (or ray)



- if $\lambda \in [0,1]$, we get a line segment (of length $\|\vec{v}\|$)
- if $\lambda \in [0,2]$, we get a segment of length $2\|\vec{v}\|$.



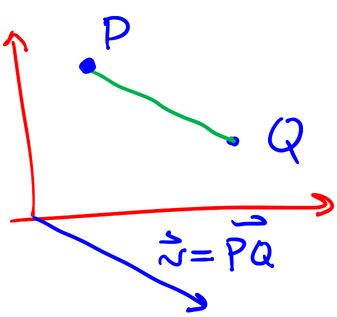
Consider two points P, Q

$$X = P + \lambda.\vec{PQ} \quad \lambda \in [0,1]$$

or $X = P + \lambda(\vec{Q} - \vec{P})$

$$\boxed{= (1-\lambda)P + \lambda Q} \quad \lambda \in [0,1]$$

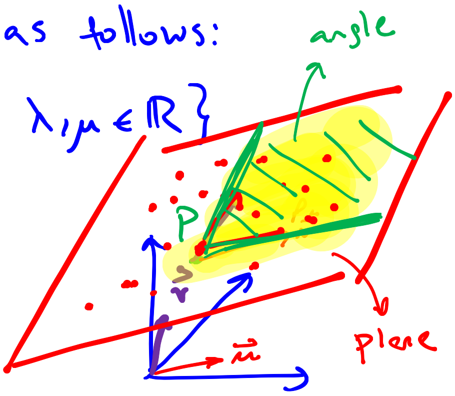
$$= s.P + t.Q,\quad t,s \in \mathbb{R} \,/\, \boxed{t+s=1} \rightarrow \text{affine combination}$$



Planes, angles, and triangles

A plane is defined by a supporting point P and two linearly independent vectors $\vec{u}, \vec{v}$ (spanning vectors) as follows:

$$\Pi = \{ X \in \mathbb{R}^d \,/\, X = P + \lambda.\vec{u} + \mu.\vec{v} \,,\ \lambda,\mu \in \mathbb{R} \}$$



Notice :
- $\lambda, \mu \in \mathbb{R} \longrightarrow$ plane
- $\lambda, \mu \in \mathbb{R}_{\geq 0} \longrightarrow$ angle (a.k.a. wedge)
- $\lambda, \mu \in [0,1] \longrightarrow$ quadrilateral

Given three points P, Q, R (not aligned) they define a plane: $X = P + \lambda.\vec{PQ} + \mu.\vec{PR} \quad \lambda,\mu \in \mathbb{R}$

We can also write

$$X = (1-\lambda-\mu)\, P + \lambda\, Q + \mu . R \qquad \lambda, \mu \in \mathbb{R}$$

$(1-\lambda-\mu) + \lambda + \mu = 1$ (affine combination)

- What happens if $X = s.P + t.Q + \mu.R \qquad s+t+\mu = 1$

  AND $\begin{cases} s \in [0,1] \\ t \in [0,1] \\ \mu \in [0,1] \end{cases}$

  $\Longrightarrow$ we get a triangle !

Affine combinations

- Expressions $X = \sum_{i=1}^{n} \lambda_i . P_i$ with $\sum_{i=1}^{n} \lambda_i = 1$

  are called affine combinations.

- Expressions $X = \sum_{i=1}^{n} \lambda_i . P_i$ with $\sum_{i=1}^{n} \lambda_i = 1$ AND $0 \leq \lambda_i \leq 1$

  are called convex combinations

Alternative coordinate systems

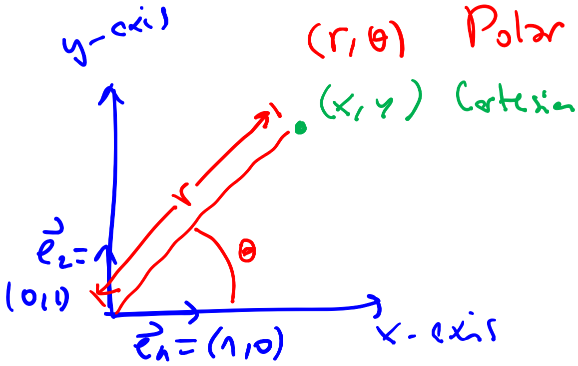
Cartesian coordinates (the usual ones)

$S = \{0, \vec{e}_1, \vec{e}_2\}$ (2D) $\quad \vec{e}_1 = (1,0) \quad \vec{e}_2 = (0,1)$

$S = \{0, \vec{e}_1, \vec{e}_2, \vec{e}_3\}$ (3D) $\quad \vec{e}_1 = (1,0,0) \quad \vec{e}_2 = (0,1,0). \ \vec{e}_3 = (0,0,1)$

Other well-known coord. systems

2D: polar coordinates



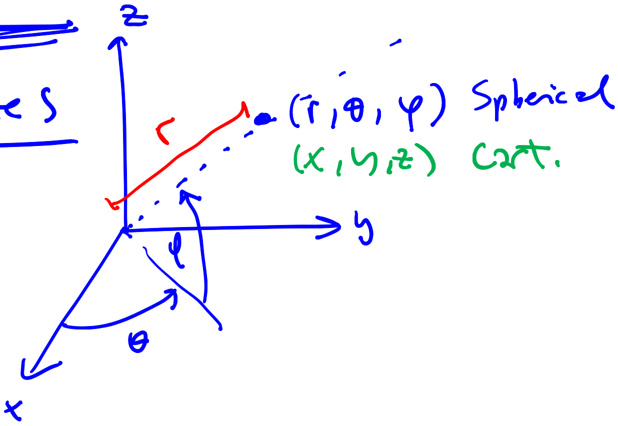
Cartesian $\qquad$ Polar

$x = r \cdot \cos\theta$
$y = r \cdot \sin\theta$ $\Leftrightarrow$ $r = \sqrt{x^2 + y^2}$
$\theta = \operatorname{atan}\left(\frac{y}{x}\right)$ (if $x \geq 0$)

3D: Spherical coordinates

(r, θ, φ) → 2 angles

r → "length"

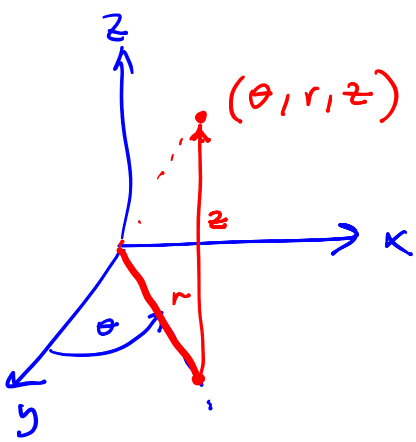


Cylindrical coordinates

(θ, r, z)

θ → angle

r, z → 2 'distances'



AFFINE TRANSFORMATIONS

An affine transformation (map) is a map $f: \mathbb{R}^n \to \mathbb{R}^n$ such that, once a coordinate system is fixed, we have

$$f(X) = \underbrace{A.X}_{\text{matrix in } \mathbb{R}^{n \times n}} + \overset{\text{a point}}{w}$$

A linear transformation

vector or point

LINEAR TRANSFORMATION

Let U, V be two vector spaces over $\mathbb{R}$.

A linear trasformation is a map from U to V such that:

1) $T(x+y) = T(x) + T(y) \quad \forall\, x,y \in U$

2) $T(\alpha . x) = \alpha . T(x) \quad x \in U,\ \alpha \in \mathbb{R}$

In practice

linear trasf in $\mathbb{R}^d$ $\longleftrightarrow$ Matrix multiplication

f

$$f(\vec{v}) = \underbrace{A}_{\text{matrix in } \mathbb{R}^{d \times d}}.\vec{v}$$

Examples

1) Scaling (with respect to the origin)

$$(x_1, x_2 \dots x_n) \longrightarrow (\lambda . x_1, \lambda . x_2, \dots, \lambda . x_n) \quad \lambda \in \mathbb{R}$$

(isotetic)

$$(x_1, x_2, \dots, x_n \longrightarrow (\lambda_1 . x_1, \lambda_2 x_2, \dots, \lambda_n x_n) \quad \lambda_1, \dots, \lambda_n \in \mathbb{R}$$

(non-isotetic)

$$f(X) = \overset{\text{matrix}}{A}\, X \qquad Ex.\ d=3 \quad f\overset{X}{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \lambda_1 x_1 \\ \lambda_2 x_2 \\ \lambda_3 x_3 \end{pmatrix}$$

$\Longrightarrow$ linear transformation

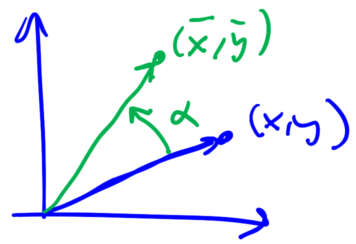
2) Rotation (2D)

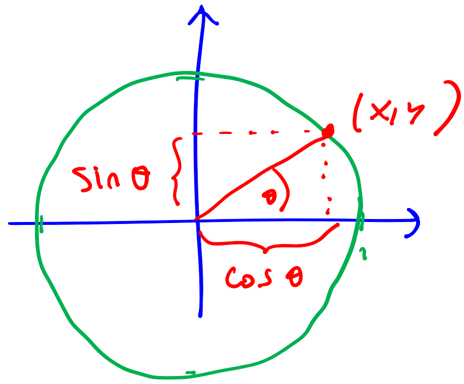
matrix in $\mathbb{R}^{2\times 2}$

$$R^{o}_{\alpha}(X) = \boxed{\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} \cos\alpha\, x - \sin\alpha\, y \\ \sin\alpha \cdot x + \cos\alpha \cdot y \end{pmatrix}$$

$\Rightarrow$ linear transformation





3) Translation

$$T_w(X) = X + W$$

Exemple in $\mathbb{R}^3$, $w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$

$$T_w\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + w_1 \\ y + w_2 \\ z + w_3 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} + W$$

$\rightarrow$ not a linear transformation.

$\rightarrow$ $T_w(X) = \underbrace{I}_{\text{identity matrix}} . X + W$ $\rightarrow$ affine transformation

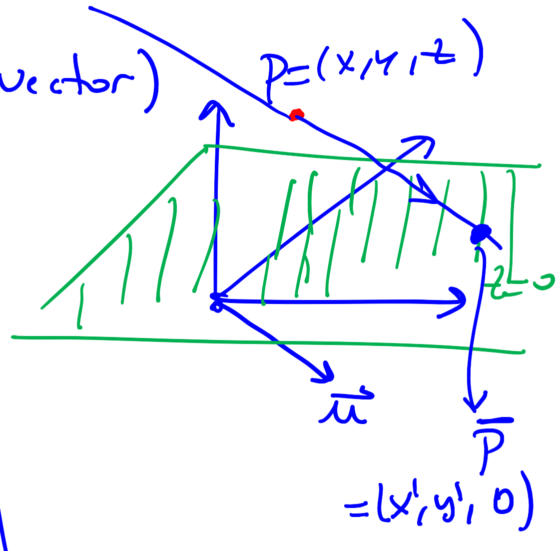
2) Parallel Projection (3D)

Given $\vec{u} = (u_1, u_2, u_3)$ (projection vector)

A projection plane, here $\boxed{z=0}$

$$f(x,y,z) = (x', y', z') \text{ s.t. } \begin{cases} x' = x + \lambda u_1 \\ y' = y + \lambda . u_2 \\ \boxed{z' = z + \lambda . u_3} \end{cases}$$

Also, $z' = 0$ $\Rightarrow$ $z + \lambda u_3 = 0$ $\Rightarrow$ $\boxed{\lambda = \dfrac{-z}{u_3 \neq 0}}$



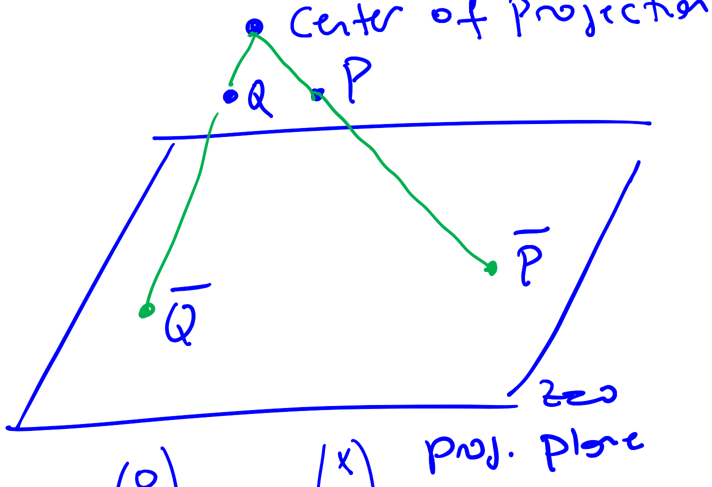
$$\begin{cases} x' = x - \frac{m_1}{m_3} \cdot z \\ y' = y - \frac{m_2}{m_3} \cdot z \\ z' = 0 \end{cases}$$

In matrix form:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & -\frac{m_1}{m_3} \\ 0 & 1 & -\frac{m_2}{m_3} \\ 0 & 0 & 0 \end{pmatrix}}_{\text{matrix in } \mathbb{R}^{3\times 3}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$\Rightarrow$ liner tranformation!

• Central projection



- Assume center of proj: $(0,0,1)$
- proj. plane $z=0$

Combine:

- $f(P)$ belongs to line through $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and P $\rightarrow$ $\begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = t \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + (1-t) \begin{pmatrix} x \\ y \\ z \end{pmatrix}$
- $f(P)$ is on plane $z=0$

$$\bar{x} = (1-t)x$$

$$\bar{y} = (1-t)y$$

$$0 = \bar{z} = (1-t) \cdot z + t \longrightarrow t = \frac{z}{z-1}$$

$$\begin{cases} \bar{x} = \frac{x}{1-z} \\ \bar{y} = \frac{y}{1-z} \\ \bar{z} = 0 \end{cases}$$

- Not an affine transformation

Some properties of affine transform. 2/3/21

- Scaling
- rotation
- translation
- parallel projection
- central projection

linear transf

affine transf

fixed (constant)

$$\bar{x} = x - \frac{\mu_1}{\mu_2} \cdot z$$

variables

$$= x - \underbrace{c}_{\text{fixed number}} \cdot z$$

central projection:

Not affine

(this is a projective transf.)

$$\bar{X} = \frac{X}{1-Z}$$

$$= \underbrace{\frac{1}{1-Z}}_{\text{variable}} \cdot \underbrace{X}_{\text{variable}}$$

$A \in \mathbb{R}^{n \times m}$, $B \in \mathbb{R}^{m \times d}$

$A.B \in \mathbb{R}^{n \times d}$

Some properties

- Composition of two affine transformations is an affine transformation.

Let T_1, T_2 affine maps.

$$T_1(X) = A_1 X + W_1$$
$$T_2(X) = A_2 X + W_2$$

- variable
- constant

$A_1, A_2 \in \mathbb{R}^{3\times 3}$

$W_1, W_2 \in \mathbb{R}^3$

The composition looks like this

$$\underbrace{T_2 \circ T_1}_{\circ\ \text{composition}}(X) = T_2(T_1(X))$$

$$= A_2.(T_1(X)) + W_2$$

$$= A_2.(A_1.X + W_1) + W_2$$

$$= A_2.A_1.X + A_2.W_1 + W_2$$

$$= \underbrace{(A_2.A_1)}_{\substack{\in \mathbb{R}^{3\times 3} \\ \text{(constant)}}} X + \underbrace{(A_2.W_1 + W_2)}_{\substack{\in \mathbb{R}^3 = \in \mathbb{R}^{3\times 1} \\ \text{(constant)}}}$$

$\therefore$ composition is also affine

- **Inversion**: affine transformations that are bijective are called regular.

i.e. has an inverse

- rotation
- translation

$$T_w(x) = \begin{pmatrix} x_1 + w_1 \\ x_2 + w_2 \\ x_3 + w_3 \end{pmatrix}$$

inverse

$$T_w^{-1}(x) = \begin{pmatrix} x_1 - w_1 \\ x_2 - w_2 \\ x_3 - w_3 \end{pmatrix}$$

Proof

$$T_w^{-1}(T_w(x)) = \begin{pmatrix} T_w(x)_x - w_1 \\ {}_y - w_2 \\ {}_z - w_3 \end{pmatrix}$$

$$= \begin{pmatrix} x_1 + w_1 - w_1 \\ x_2 + w_2 - w_2 \\ x_3 + w_3 - w_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

- Scaling:

$$T_{\lambda_1 \lambda_2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 x_1 \\ \lambda_2 x_2 \end{pmatrix}$$

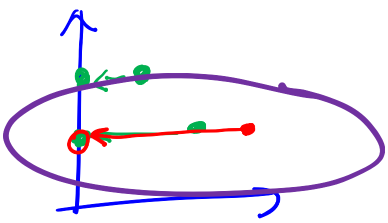
$\lambda_1, \lambda_2 \in \mathbb{R}$ (constant)

$$T^{-1}_{\lambda_1 \lambda_2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\lambda_1} \cdot x_1 \\ \frac{1}{\lambda_2} \cdot x_2 \end{pmatrix}$$

is also regular as long as $\lambda_1 \neq 0$, $\lambda_2 \neq 0$

Example

$\lambda_1 = 0$, $\lambda_2 = 1$



- Parallel projection: no inverse!
  Central/

- Transforming vectors: In an affine map, vectors are only affected by the $\underbrace{\text{linear part}}_{\text{matrix product}}$ of the transformation.

Proof

Let $\vec{u} = \overrightarrow{PQ} = Q - P$

$$\left.\begin{aligned} f(P) &= A.P + W \\ f(Q) &= A.Q + W \end{aligned}\right\} \quad \overrightarrow{f(P)f(Q)} = f(Q) - f(P)$$

$$= (AQ + W) - (A.P + W)$$

$$= AQ + W - AP - W$$

$$= A(Q - P)$$

$$= A\vec{u} \quad //$$

- Affine transformations preserve affine combinations

Consider

$$X = \sum_{i=1}^{n} \underbrace{\lambda_i}_{\text{coordinates of } X \text{ with respect to } P_1, P_2, \dots, P_n} P_i \quad \text{with } \sum_{i=1}^{n} \lambda_i = 1$$

(X is an affine combination of $P_1 \dots P_n$)

Let f be an affine combination, $f(X) = AX + W$.

Then

$$f(X) = \sum_{i=1}^{n} \underbrace{\lambda_i}_{\text{coordinates of } f(X) \text{ w.r.t. } f(P_1), f(P_2), \dots, f(P_n)} . f(P_i)$$

(i.e. the Coords of X are preserved)

Proof

$$f(X) \underset{\text{def of } X}{=} f\left(\sum_{i=1}^{n} \lambda_i P_i\right) \underset{\text{def of } f}{=} A\left(\sum_{i=1}^{n} \lambda_i P_i\right) + W$$

$$\sum_{i=1}^{n} \lambda_i$$

$$= \sum_{i=1}^{n} \lambda_i (AP_i) + W = \sum_{i=1}^{n} \lambda_i (AP_i) + 1.W$$

$$= \sum_{i=1}^{n} \lambda_i (AP_i) + \left(\sum_{i=1}^{n} \lambda_i\right).W = \sum_{i=1}^{n} \lambda_i (AP_i + W)$$

using that $\sum_{i=1}^{n} \lambda_i = 1$ (i.e., combination is affine)

$$= \sum_{i=1}^{n} \lambda_i . f(P_i) \quad //$$

Idea for L1 - Problem 4

P ---- • ---- R ---- • ---- Q

$$R = \underbrace{\tfrac{1}{2}}P + \underbrace{\tfrac{1}{2}}Q$$ (R is an affine coord. of P, Q)

coords.

f : central proj.

$$f(R) \overset{?}{=} \frac{1}{2} f(P) + \frac{1}{2} f(Q)$$

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