Description: The exam has two types of problems. Problems 1–3 should be answered on paper and delivered to the professor when you are done. Problems 4–5 are practical and must be delivered at `https://examens.fib.upc.edu` before the exam ends. Note that problems 4 and 5 include writing some explanation in the HTML file.

Duration: 3:30 hours.

Publication of final grades: Wednesday, January 24, through the Racó.

Revision: Anyone wishing to revise the grades obtained should: 1) send the professor an e-mail by Friday, January 26; 2) show up in the following room on Tuesday, January 30 at 16:00: Omega Building, 4th floor, office 422.

Problem 1. [1 point] Deliver on paper. Suppose that you are asked to add a new functionality to a drawing software based on Bézier curves. The new functionality is to allow the user to split curves into two. That is, the user will select a curve, then click a point on the curve, and then the program should break the existing curve into two at the selected point.

What method for Bézier curves would you use to implement this? Why? Explain how the method works.

Solution sketch. Since the goal is to split a curve into two curves, while preserving the shape of the original curve, the most appropriate tool seen in class for this is using De Casteljau's algorithm to produce curve subdivision.

Other options like breaking the curve manually and creating two curves using the breaking point as endpoint have the serious problem that cannot guarantee that the resulting two curves will preserve the shape of the original curve.

Problem 2. [1 point] Deliver on paper. Suppose that you need to construct a software to design 3D objects using piece-wise curves, to be used to produce high-quality images. The software will also take care of producing the necessary projections from 3D to 2D.

Which of the curves studied in class would you use? Why?

Solution sketch. If projections from 3D to 2D are required, the type of curve must be invariant under (perspective) projections. That limits the options, among the ones seen in the course, to rational Bézier or B-Splines. Another reason to choose rational curves is that they can represent conics exactly, and that can be important if the images produced must be of high quality.

Problem 3. [1.5 points] Deliver on paper. Let $C(t)$ and $Q(t)$ be two B-spline curves defined using the same control points, but with two different knot vectors. The knot vector of $C(t)$ is given by $[0, 0, 0, 0.1, 0.2, \ldots, 0.8, 0.9, 1, 1, 1]$, while the knot vector of $Q(t)$ is given by $[0, 0, 0, 1, 2, \ldots, 8, 9, 10, 10, 10]$.

Prove that $C(t)$ and $Q(t)$ are the same curves.

Solution sketch. The proof relies on showing that the basis functions of the two curves are essentially the same, up to a change of variable (i.e., $v = 10u$). Below we include a complete, formal proof. For the correction of the exam, less formal answers were partially accepted as long as they were justified based on the formula of the basis functions of the two curves, and as long as the arguments used were correct.

Complete proof

Since the control points are the same, we have that C and Q only differ on the knot vector. Let $C(u) = \sum_{i=0}^{n} P_i N_{i,k}^u(u)$, $Q(v) = \sum_{i=0}^{n} P_i N_{i,k}^v(v)$ (the use of two different variables, u and v, is just to avoid confusion between the two).

We have $C(u)$ defined on $[0, 1]$, with knot vector $[u_i] = [0, 0, 0, 0.1, \ldots, 0.8, 0.9, 1, 1, 1]$, and $Q(v)$ defined on $[0, 10]$ with knot vector $[v_i] = [0, 0, 0, 1, \ldots, 8, 9, 10, 10, 10]$.

To show that C and Q define the same curve, we need to show that $C[0, 1]$ and $Q[0, 10]$ define the same points. That is easiest seen by comparing the basis functions of each. Observe that the knots are related as follows: $v_i = 10u_i$ (for $i = 0..14$). It is then intuitive that the two sets of basis functions are related by a change of variables of the type $v = 10u$.

We formalize that by proving that, indeed, $N_{i,k}^u(u) = N_{i,k}^v(10u)$ for $u \in [0, 1]$.

We prove it by using induction on k.

We start from the base case, $k = 0$:

$$N_{i,1}^u(u) = \begin{cases} 1, & \text{if } u \in [u_i, u_{i+1}) \\ 0, & \text{otherwise} \end{cases} \qquad N_{i,1}^v(v) = \begin{cases} 1, & \text{if } v \in [v_i, v_{i+1}) \\ 0, & \text{otherwise} \end{cases}$$

$N_{i,1}^u$ will have value 1 if and only if $u \in [0, 1]$, and 0 otherwise, and $N_{i,1}^v$ will have value 1 if and only if $v \in [0, 10]$, and 0 otherwise. Therefore $N_{i,1}^u(u) = N_{i,1}^v(10u)$.

Now we consider the general case. The inductive hypothesis is that $N_{i,k-1}^u(u) = N_{i,k-1}^v(10u)$. We want to prove that is also the case for k.

The general expressions for $N_{i,k}^u(u)$ and $N_{i,k}^v(v)$ are:

$$N_{i,k}^u(u) = \frac{u - u_i}{u_{i+k-1} - u_i} N_{i,k-1}^u(u) + \frac{u_{i+k} - u}{u_{i+k} - u_{i+1}} N_{i+1,k-1}^u(u)$$

$$N^v_{i,k}(v) = \frac{v - v_i}{v_{i+k-1} - v_i} N^v_{i,k-1}(v) + \frac{v_{i+k} - v}{v_{i+k} - v_{i+1}} N^v_{i+1,k-1}(v)$$

Now, since $v_i = 10u_i$, we can replace in $N^v_{i,k}(v)$ the occurrences of v_i by $10u_i$:

$$N^v_{i,k}(v) = \frac{v - 10u_i}{10u_{i+k-1} - 10u_i} N^v_{i,k-1}(v) + \frac{10u_{i+k} - v}{10u_{i+k} - 10u_{i+1}} N^v_{i+1,k-1}(v)$$

Next, we evaluate the previous expression for $v = 10u$, yielding:

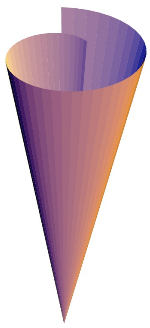
$$\begin{aligned} N^v_{i,k}(10u) &= \frac{10u - 10u_i}{10u_{i+k-1} - 10u_i} N^v_{i,k-1}(10u) + \frac{10u_{i+k} - 10u}{10u_{i+k} - 10u_{i+1}} N^v_{i+1,k-1}(10u) \\ &= \frac{10(u - u_i)}{10(u_{i+k-1} - u_i)} N^v_{i,k-1}(10u) + \frac{10(u_{i+k} - u)}{10(u_{i+k} - u_{i+1})} N^v_{i+1,k-1}(10u) \\ &= \frac{u - u_i}{u_{i+k-1} - u_i} N^v_{i,k-1}(10u) + \frac{u_{i+k} - u}{u_{i+k} - u_{i+1}} N^v_{i+1,k-1}(10u) \end{aligned}$$

It only remains to apply the inductive hypothesis, which tells us that $N^v_{i,k-1}(10u) = N^u_{i,k-1}(u)$, to obtain:

$$= \frac{u - u_i}{u_{i+k-1} - u_i} N^u_{i,k-1}(u) + \frac{u_{i+k} - u}{u_{i+k} - u_{i+1}} N^u_{i+1,k-1}(u) = N^u_{i,k}(u).$$

We conclude that $N^u_{i,k}(u) = N^v_{i,k}(10u)$ for $u \in [0, 1]$, proving that the curves defined by C and Q are the same. $\square$

Problem 4. Deliver through the Racó. For a 3D scene depicting a chestnut stand, we want to model a paper cone like the one shown below (left).



1. **[2 points]** Give a parametrization of the surface of the cone, formed by the segments that connect the apex (i.e., bottom vertex) of the cone to the (partial)

spiral curve defining the top edge of the cone. Assume that the apex is at point $(0, 0, -h)$ (for some $h > 0$), and that the spiral lies on the plane $z = 0$.

Justify in detail how you obtained the parametrization in the HTML file.

2. [**1 point**] Write a program to draw such a cone on the screen.

Solution sketch.

1. A point in a 2D Archimedean spiral, like the one defining the top edge of the cone, that lives on the $z = 0$ plane has parametric equation $(b\theta \cos\theta, b\theta \sin\theta, 0)$, for some b that controls the distance between turnings, and $\theta \in [0, +\infty)$. To imitate the cone in the figure, the spiral should do a bit more than one turn (2π), so we can take, for instance, $\theta \in [0, 2.1\pi]$.

The parametrization of the surface is obtained by parametrizing the line segments connecting the apex at $(0, 0, -h)$ with the points on the spiral. That is given by:

$$(1 - \lambda)(0, 0, -h) + \lambda(b\theta \cos\theta, b\theta \sin\theta, 0), \text{ for } \lambda \in [0, 1].$$

The final parametrization is obtained by rewriting the previous expression:

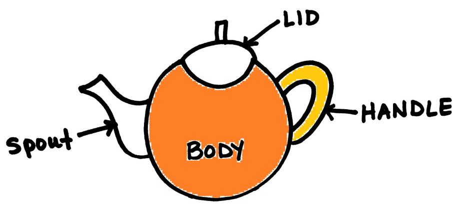
$$(\lambda b\theta \cos\theta, \lambda b\theta \sin\theta, h(\lambda - 1)), \text{ for } \theta \in [0, 2.1\pi] \text{ and } \lambda \in [0, 1].$$

2. Having the parametrization, the drawing is straightforward, only requiring to set appropriate values of h, b, and to rescale accordingly.

Problem 5. [3.5-4.5 points] Deliver through the Racó. Write a program to draw part of the outer surface of a *Utah teapot*, similar to the one below (left). **You must draw at least the body and the handle** [3.5 points] (see figure, right). You can get up to one extra point by drawing also the spout and the lid.

You don't need to worry about the connection of the different parts to the body.

3D view of Utah teapot



Main parts of the teapot

Important: In the HTML file that you will deliver, include a brief description of the strategy that you have followed and the design decisions made.

Solution sketch. This exercise has many possible solutions. The body and the lid were easiest to model as surfaces of revolution, producing the profile with a Bézier or B-spline curve.

For the handle two approaches were most common: (i) Drawing the central axis of the handle as a curve, and placing copies of a circle or ellipse along the axis, each of them placed orthogonally with respect to the axis curve. (ii) Two Bézier surfaces, one for each side of the handle.

The spout is very similar to the handle, with the additional complication that the radius of the circle or ellipse varies along the central axis. Using Bézier surfaces is also possible in this case.

For the exam grading, we took into account:

- A body and lid with shape as similar as possible to the original one (i.e., not excessively simplified).
- A handle and spout with a shape similar to the ones in the figure. Simpler handles and spouts gave some (lower) score as well.
- Separating the body from the lid (these are two separate pieces).
- Placing the different parts in their corresponding places with respect to the body.
- The description included in the HTML file.