

HERMITE INTERPOLATION

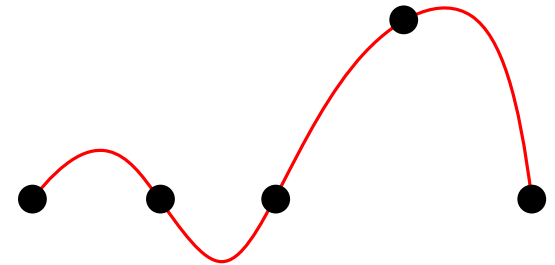
Rodrigo Silveira

Curve and Surface Design
Facultat d'Informàtica de Barcelona
Universitat Politècnica de Catalunya

POLYNOMIAL INTERPOLATION

Recall: Issues with polynomial interpolation

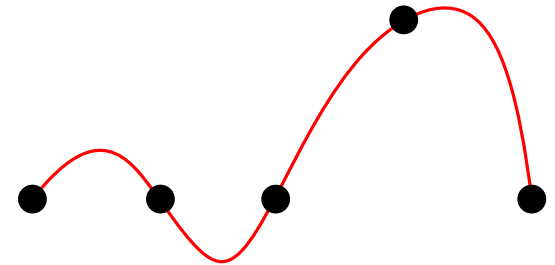
- The high degree of the polynomial produces a curve with higher roughness (i.e., it can wiggle a lot) than probably desired
→ The variation diminishing property is not satisfied!
- Intuitively: adding data points should improve the resulting polynomial curve. But that is not always the case! This is known as *Runge's phenomenon*
- Lagrange's formula requires $\Theta(n^2)$ additions and products, which is quite a lot (although more efficient versions exist)
- If one has computed $\gamma(t)$ for n points and needs to add one extra point, everything needs to be recomputed
- Lagrange's formula is not numerically stable: small variations in the input points can produce large variations in the final curve
- The method is not easy to make interactive: if the curve is not what one wants, (and you cannot modify the data points) all you can do is to add more points



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A more interactive interpolation method

Practical curve design methods need to be interactive:

- Based on user-controlled parameters that modify the shape of the curve in an intuitive (and thus predictable) way

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Idea: design a curve that interpolates **two** points, and whose shape is controlled by the **tangent vectors** at those points

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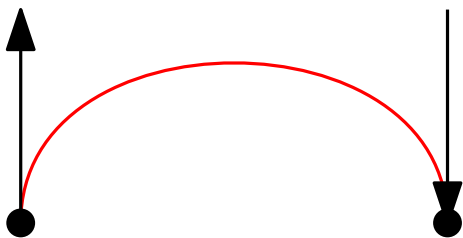
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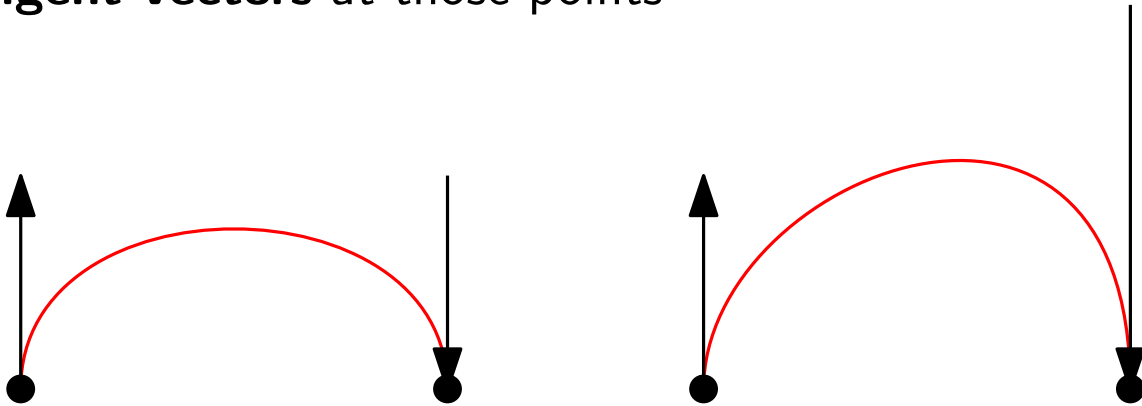
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Note the effect of modifying one of the tangent vectors

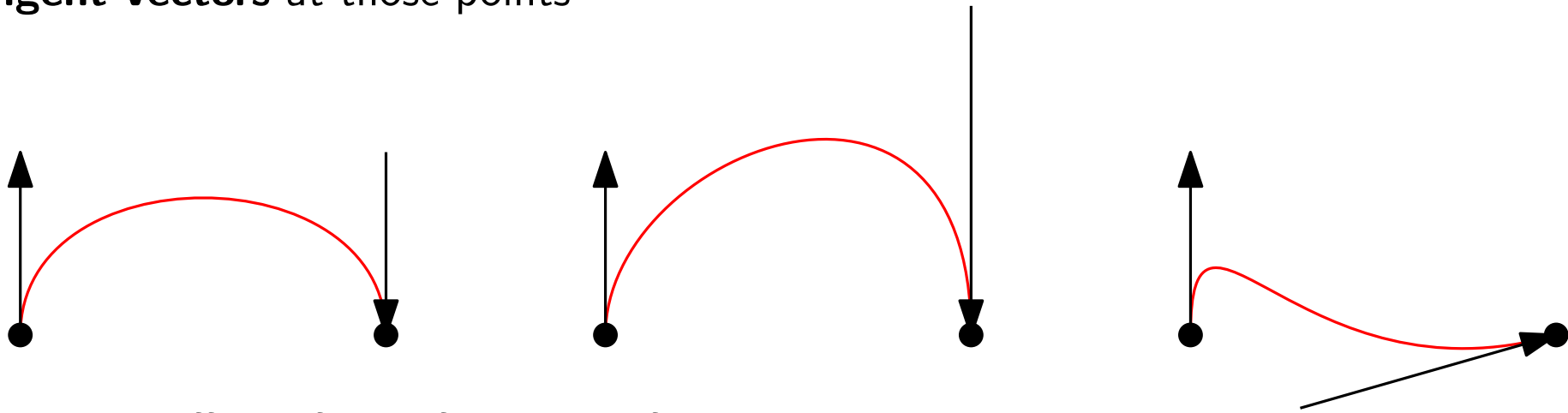
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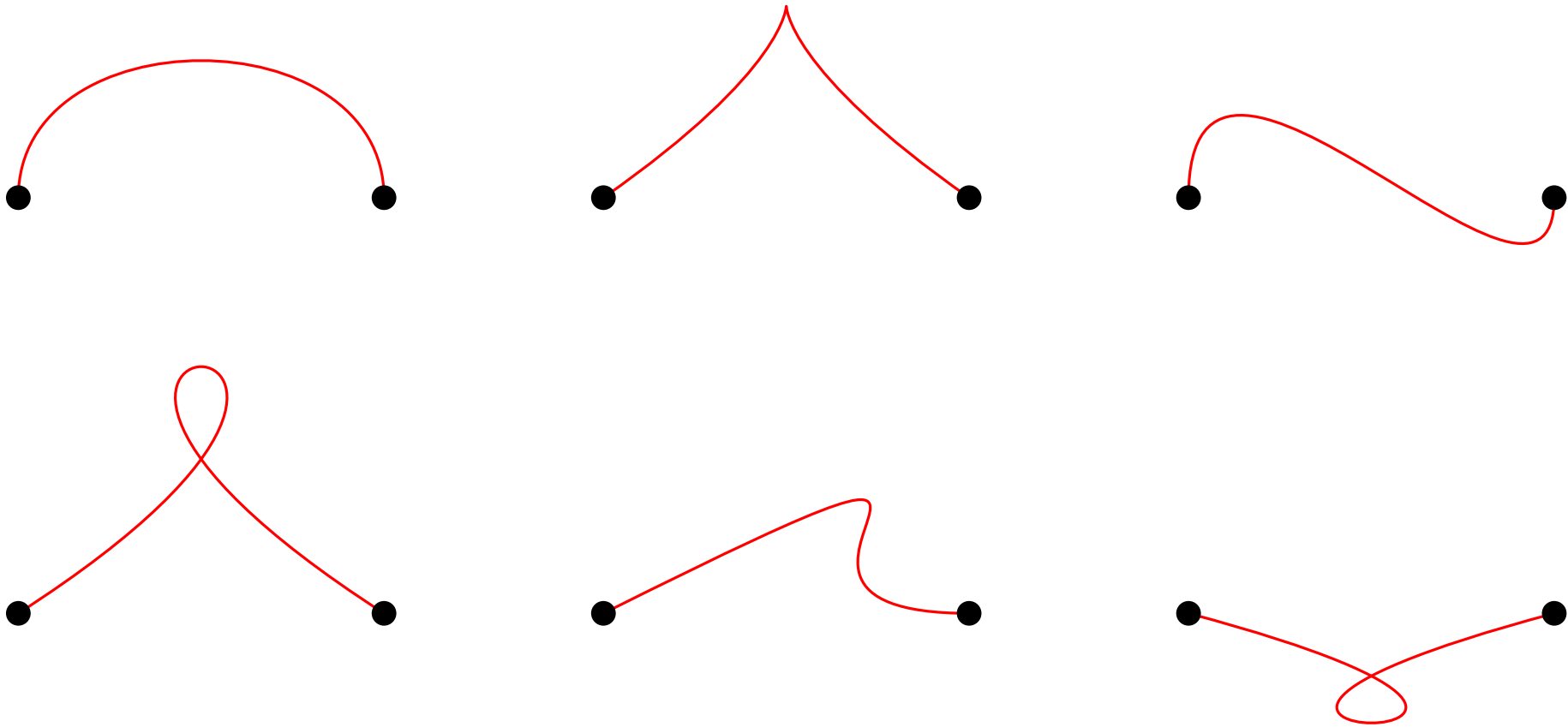
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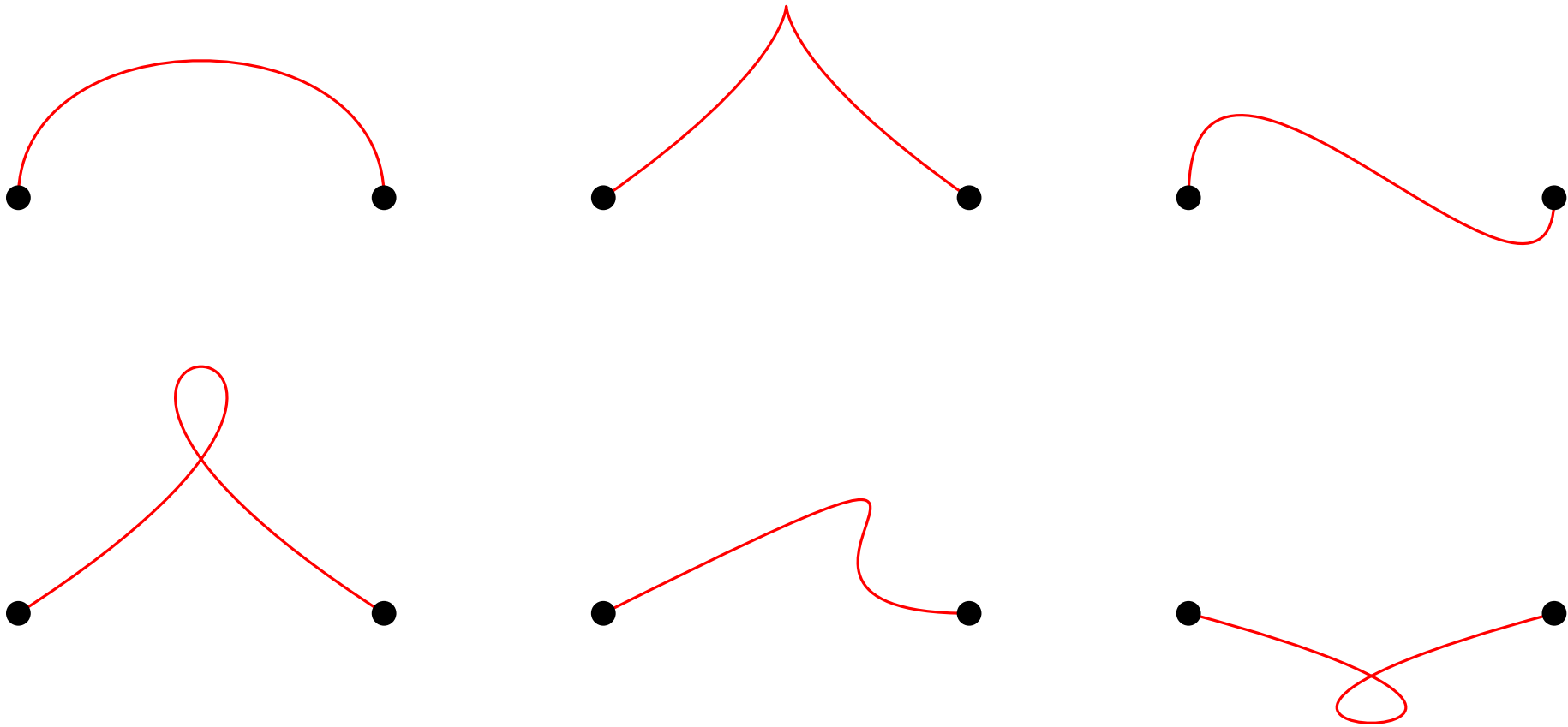
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A single Hermite curve can take many different shapes
(between the same two points)



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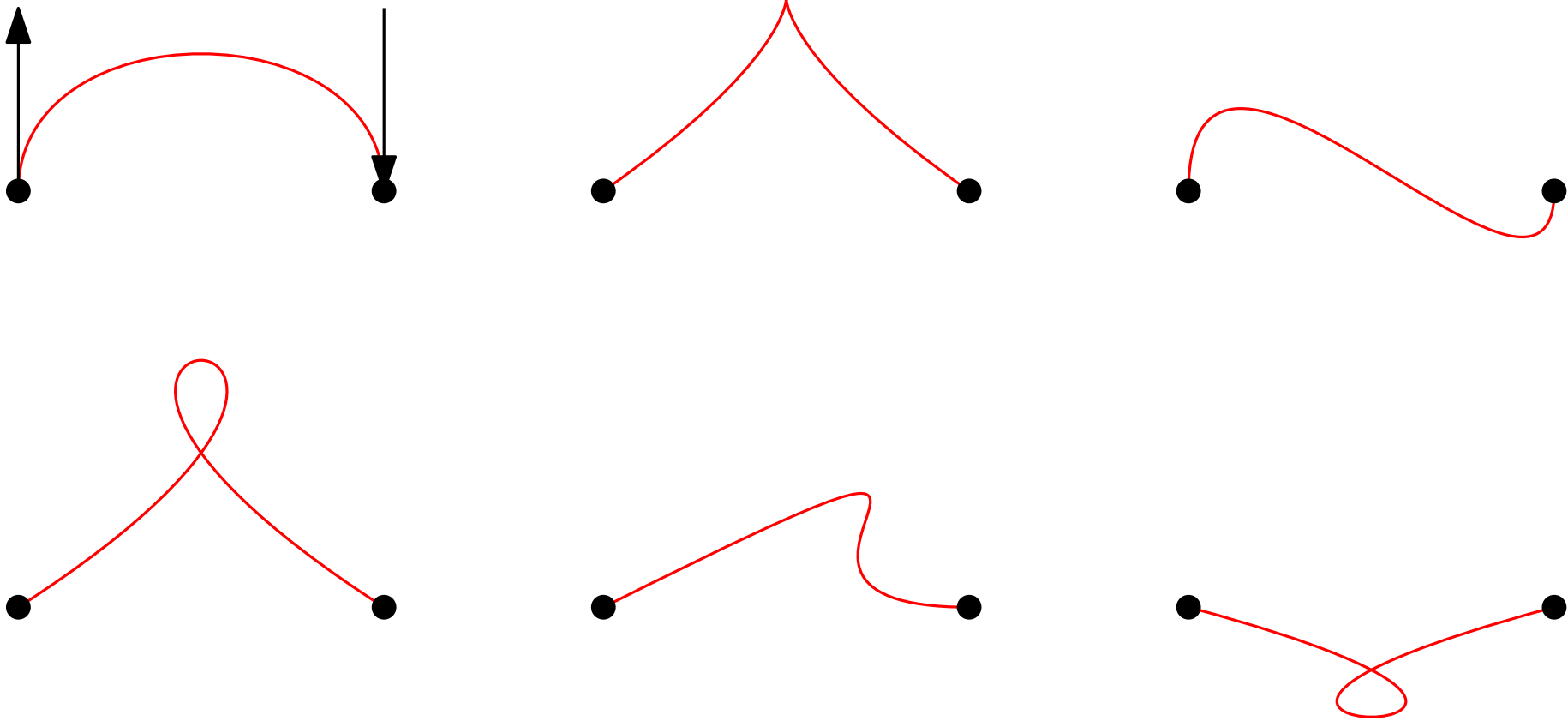
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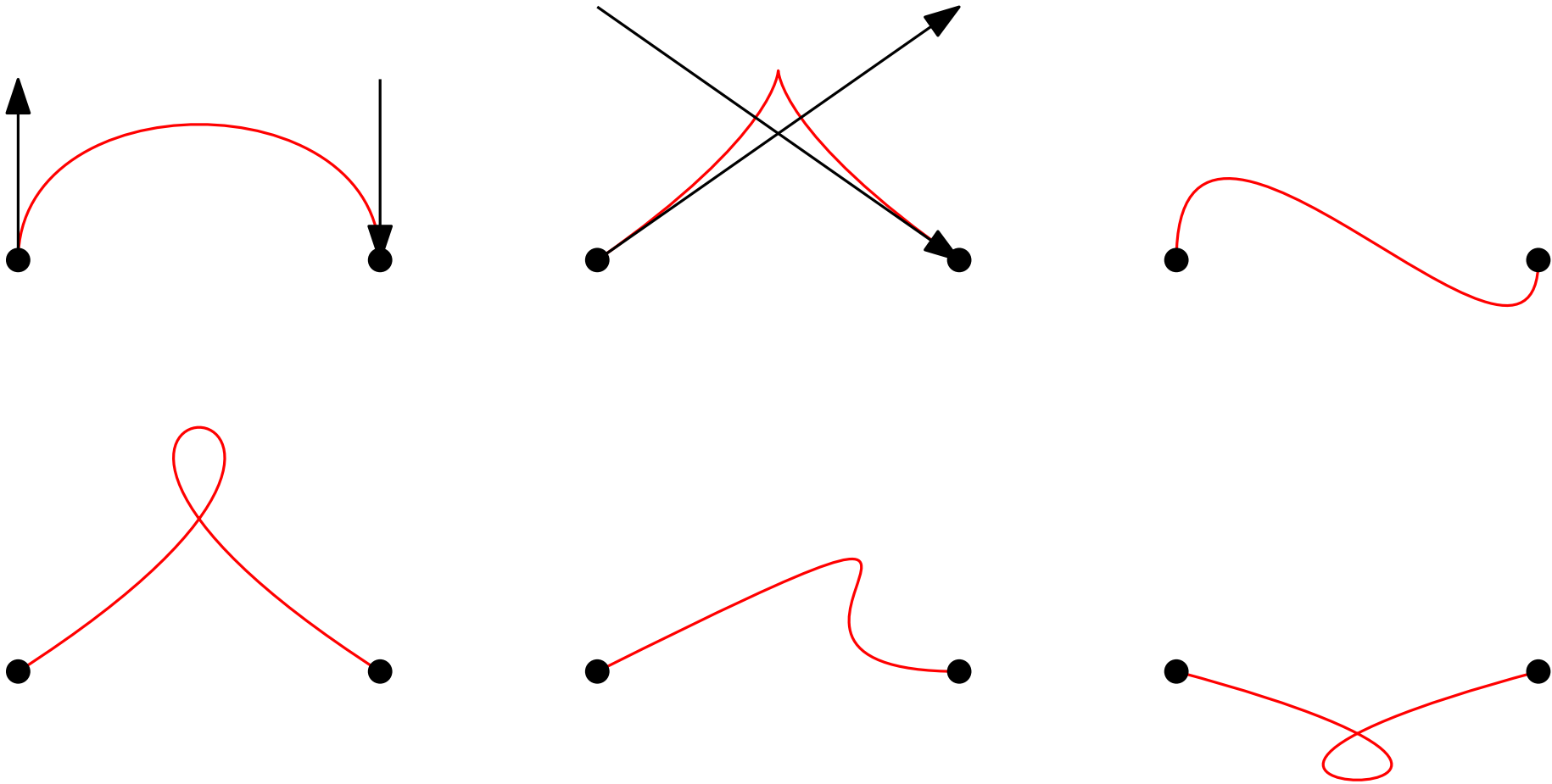
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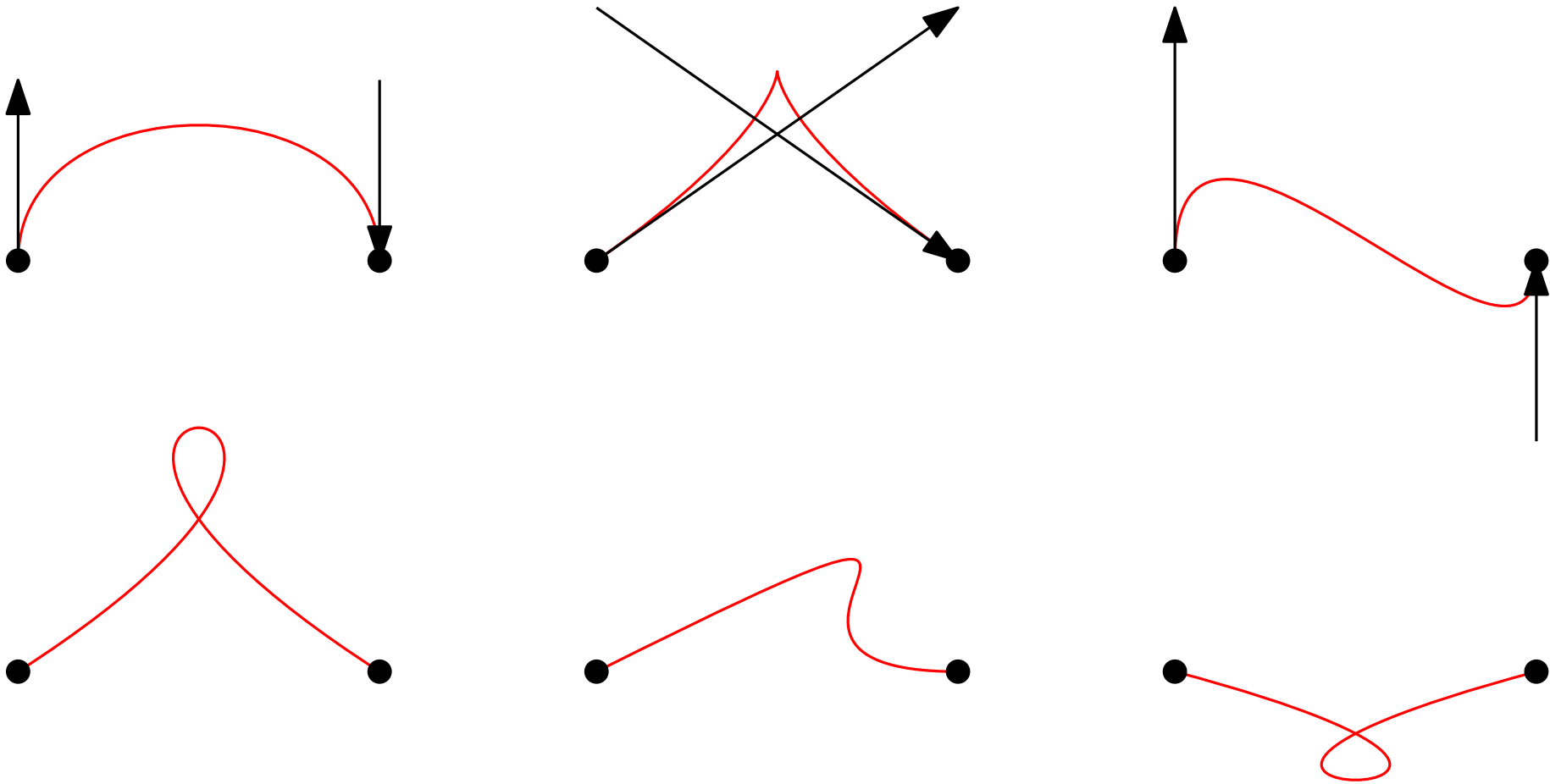
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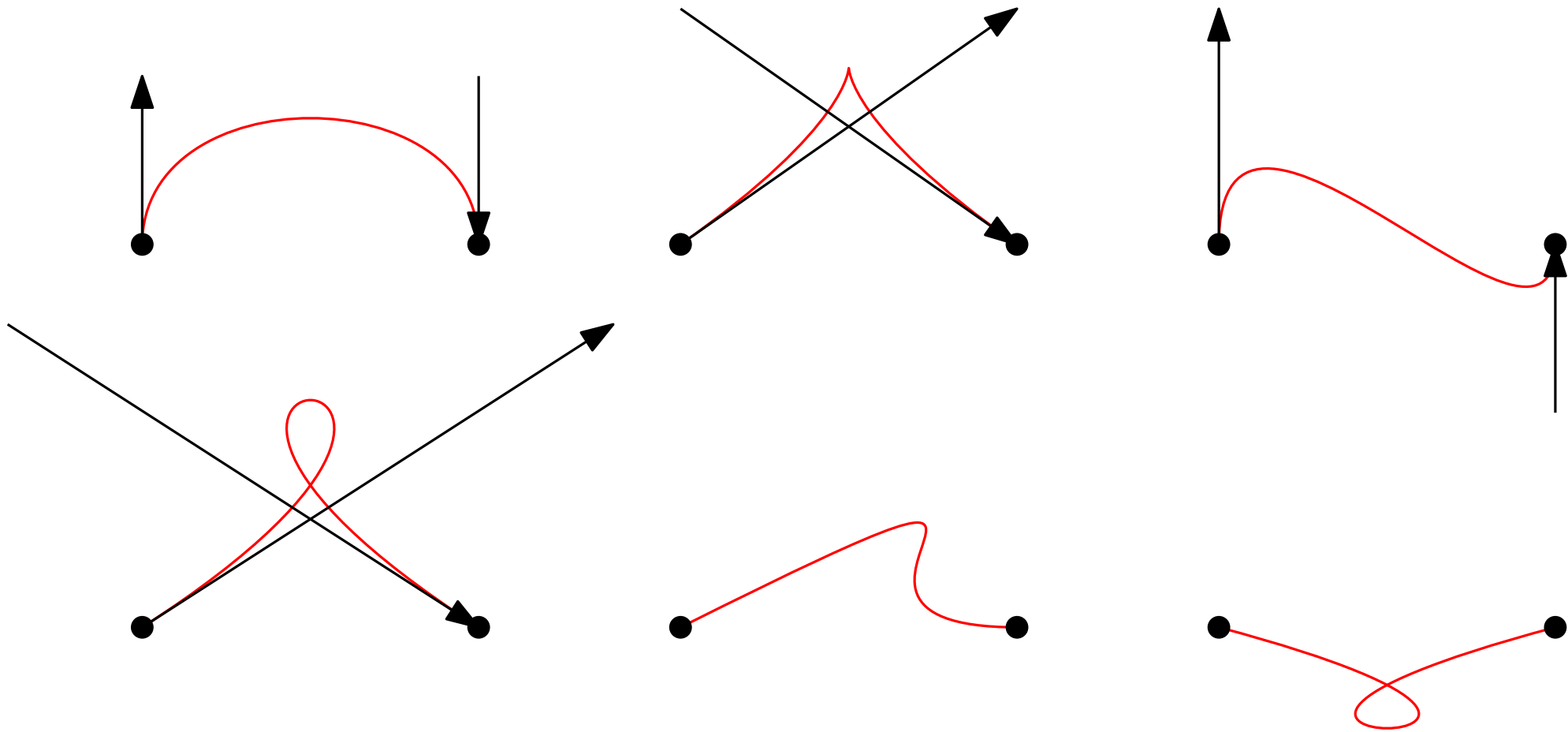
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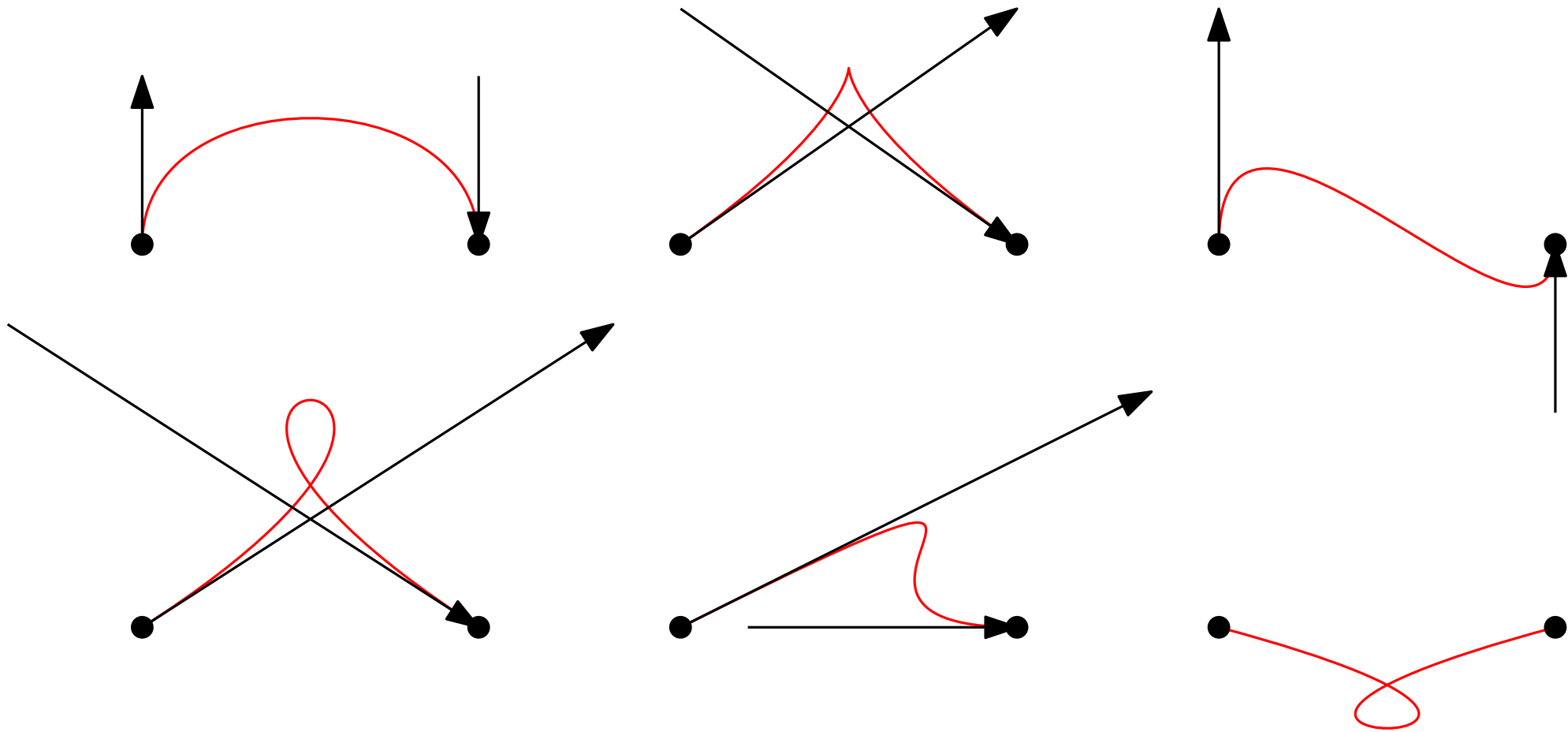
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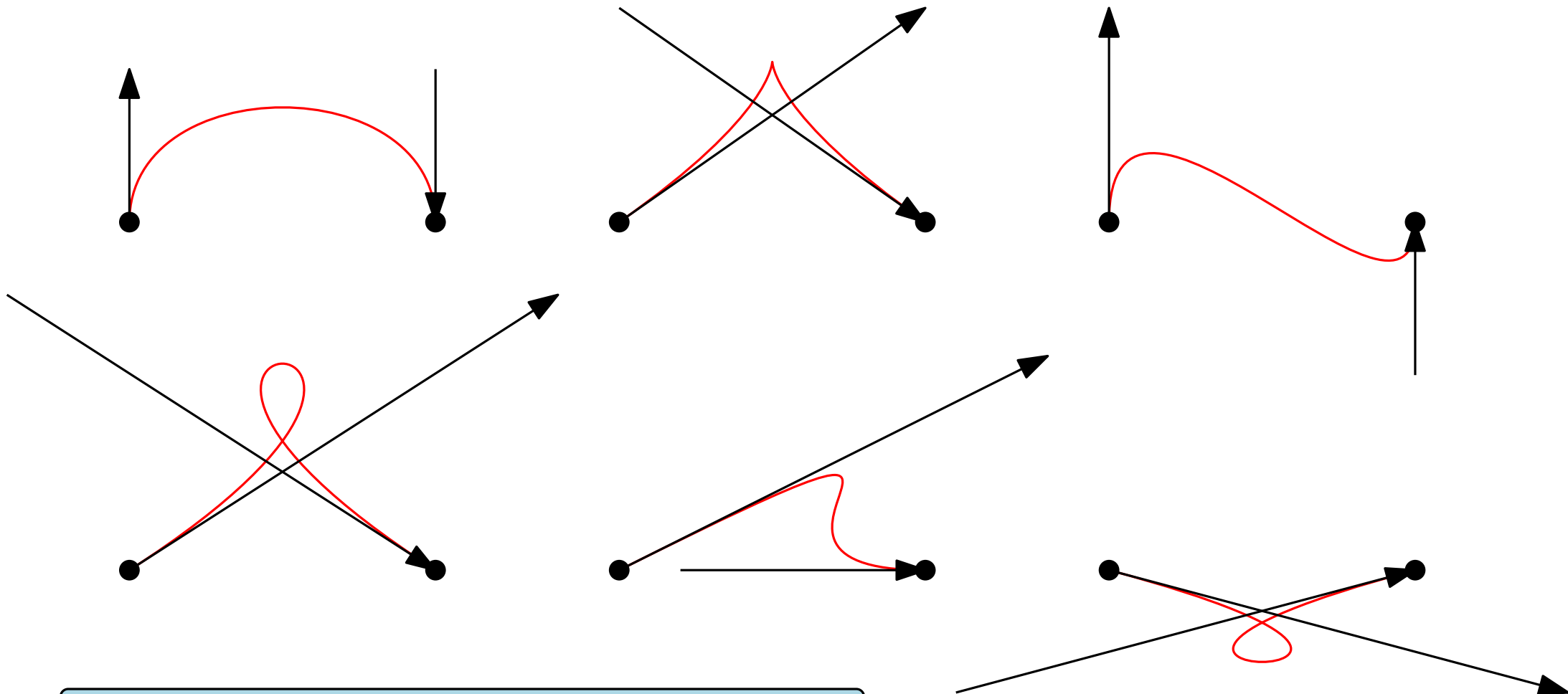
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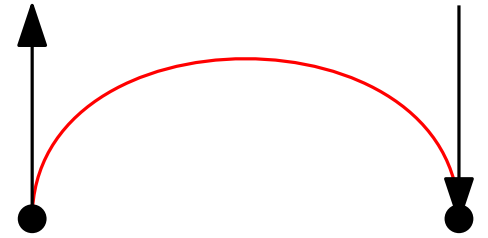
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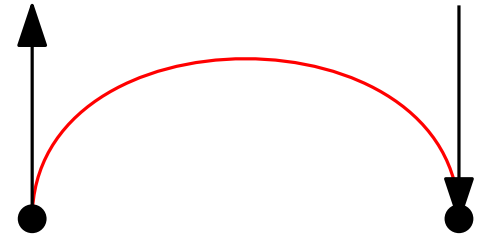
Cubic Hermite interpolation



HERMITE INTERPOLATION

Cubic Hermite interpolation

Each curve is a (parametric) cubic polynomial

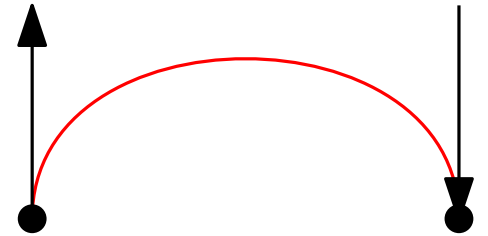


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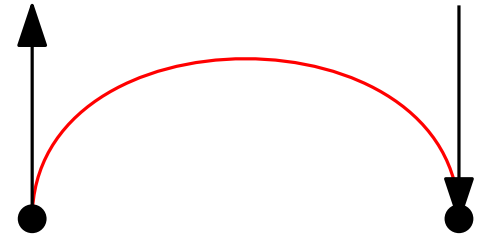


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Theorem: Given two points P_0, P_1 and two vectors \vec{v}_0, \vec{v}_1 , there exists a unique curve $\gamma(t)$ parametrized as a cubic polynomial in t such that:

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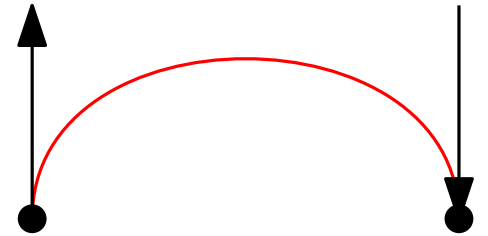
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Proof. First we prove uniqueness, as we did with Lagrange interpolation.

Secondly, we prove that it exists, by deducing an expression for it.

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Proof. First we prove uniqueness, as we did with Lagrange interpolation.

Let γ and δ be two curves that satisfy the constraints above, and consider a third curve r defined as $r(t) = \gamma(t) - \delta(t)$. Clearly, $r(t)$ is a polynomial of degree at most three. Since $r(0) = 0$ and $r(1) = 0$, we can write it as $r(t) = at(t-1)(t-t_0)$, for two unknown values a and t_0 .

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Now consider $r'(t)$. We know that $r'(t) = \gamma'(t) - \delta'(t)$, so we have that $r'(0) = 0$ and $r'(1) = 0$. We can also write $r'(t)$ as follows

$$r'(t) = at(t-1) + at(t-t_0) + a(t-1)(t-t_0)$$

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Similarly, since $r'(1) = 0$, we have $0 = a(1)(1-t_0) = a(1-t_0)$

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iff a is 0, thus $r(t) = 0$
for all t , and therefore
 $\gamma(t) = \delta(t)$

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Hence $\gamma(t)$ can be written as $\gamma(t) = At^3 + Bt^2 + Ct + D$, for $A, B, C, D \in \mathbb{R}^2$

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- $P_0 = \gamma(0) = D$, thus $D = P_0$
- $\vec{v}_0 = \gamma'(0) = C$, thus $C = \vec{v}_0$
- $P_1 = \gamma(1) = A + B + C + D$, thus $B = P_1 - P_0 - \vec{v}_0 - A$
- $\vec{v}_1 = \gamma'(1) = 3A + 2B + C = 3A + 2(P_1 - P_0 - \vec{v}_0 - A) + \vec{v}_0 = A + 2P_1 - 2P_0 - \vec{v}_0$

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After simplifying and grouping by the input points and vectors, this is:

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Hermite blending functions

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Hermite blending functions

Exercise: verify that $\gamma(t)$ satisfies the four constraints of the theorem

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Cubic Hermite blending functions

The concept of blending functions is fundamental for many curve design methods

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These are the four blending functions in Hermite interpolation:

$$F_1(t) = 2t^3 - 3t^2 + 1$$

$$F_3(t) = t^3 - 2t^2 + t$$

$$F_2(t) = -2t^3 + 3t^2$$

$$F_4(t) = t^3 - t^2$$

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$$\gamma(t) = \underbrace{(2t^3 - 3t^2 + 1)}_{F_1} P_0 + \underbrace{(-2t^3 + 3t^2)}_{F_2} P_1 + \underbrace{(t^3 - 2t^2 + t)}_{F_3} \vec{v}_0 + \underbrace{(t^3 - t^2)}_{F_4} \vec{v}_1, \text{ for } t \in [0, 1]$$

These are the four blending functions in Hermite interpolation:

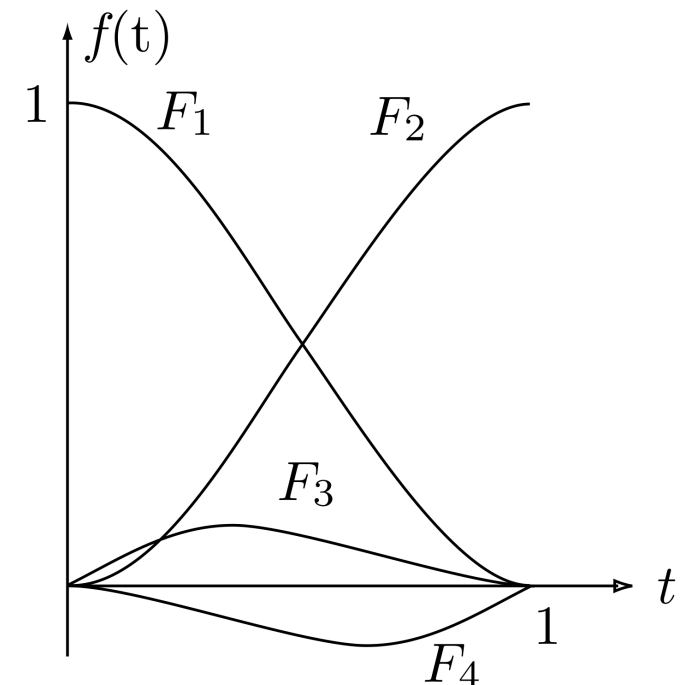
$$F_1(t) = 2t^3 - 3t^2 + 1$$

$$F_3(t) = t^3 - 2t^2 + t$$

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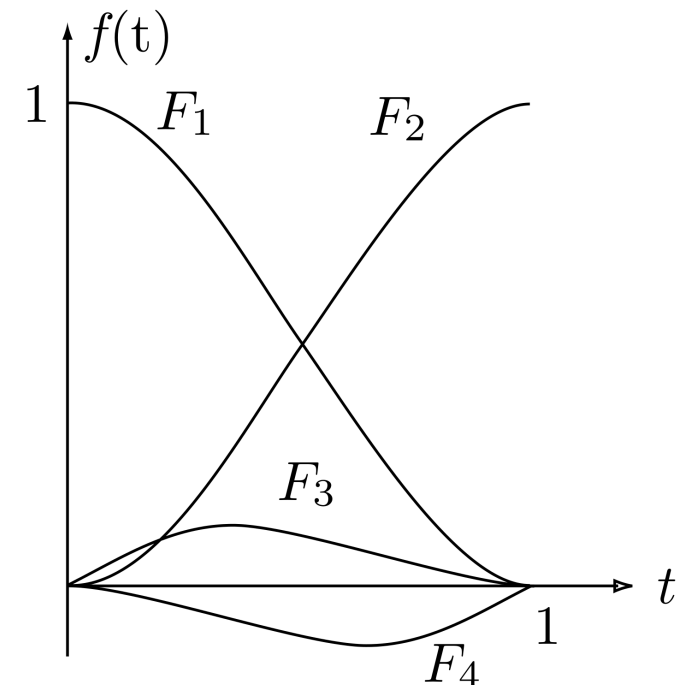
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Let's see how these functions look like:

The functions control the weights of $P_0, P_1, \vec{v}_0, \vec{v}_1$:

- For $t = 0$, $F_1(t) = 1$, and all others are 0: this makes the curve start at P_0 . Similarly, at $t = 1$, $F_2(t) = 1$, and all others are 0, so the curve ends at P_1 .
- $F_3(t)$ has a less clear behavior: for small values of t , it has little effect (the curve stays close to P_0). For t around $1/3$, $F_3(t)$ has its maximum influence, pulling the curve in direction \vec{v}_0 . For larger t , $F_3(t)$ again has almost no effect.
- $F_4(t)$ behaves in a symmetric way to $F_3(t)$.



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Affine invariance

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- The weights of the points add up to 1:

$$(2t^3 - 3t^2 + 1) + (-2t^3 + 3t^2) = 1$$

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Consider $f(x) = Ax + W$. Recall only linear part of f applies to a vector, i.e., $f(\vec{u}) = A\vec{u}$

$$\gamma(t) = a(t)P_0 + (1 - a(t))P_1 + \alpha(t)\vec{v}_0 + \beta(t)\vec{v}_1, \text{ for some functions } a(t), \alpha(t), \beta(t)$$

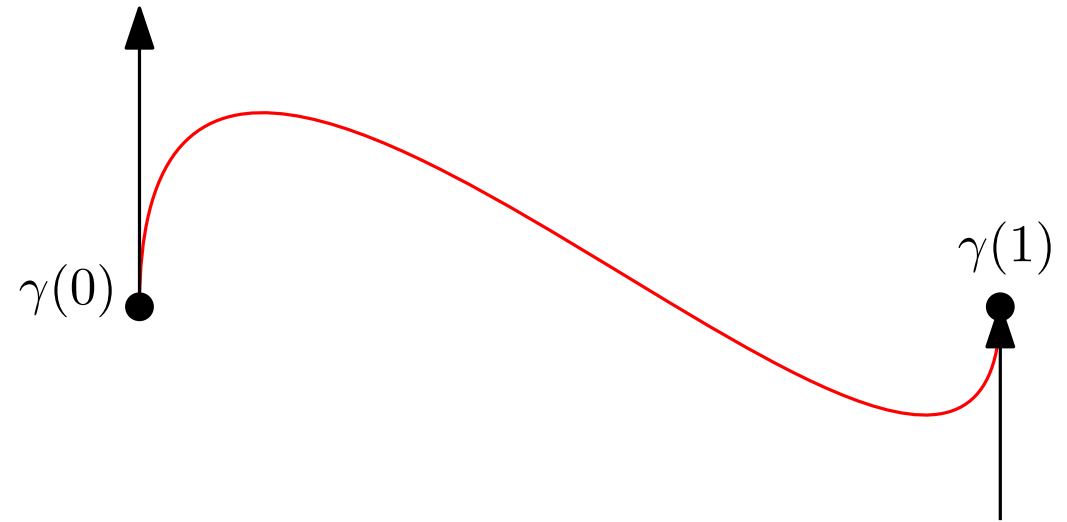
$$\begin{aligned} f(\gamma(t)) &= A(a(t)P_0 + (1 - a(t))P_1 + \alpha(t)\vec{v}_0 + \beta(t)\vec{v}_1) + W \\ &= a(t)AP_0 + (1 - a(t))AP_1 + \alpha(t)A\vec{v}_0 + \beta(t)A\vec{v}_1 + W \quad (\text{using } W = (a(t) + (1 - a(t)))W) \\ &= a(t)AP_0 + (1 - a(t))AP_1 + \alpha(t)A\vec{v}_0 + (a(t) + (1 - a(t)))W + \beta(t)A\vec{v}_1 \\ &= a(t)(AP_0 + W) + (1 - a(t))(AP_1 + W) + \alpha(t)A\vec{v}_0 + \beta(t)A\vec{v}_1 \\ &= a(t)f(P_0) + (1 - a(t))f(P_1) + \alpha(t)f(\vec{v}_0) + \beta(t)f(\vec{v}_1) \quad \therefore \text{ it is affine invariant!} \end{aligned}$$

HERMITE INTERPOLATION

Clipping Hermite curves

Clipping is a basic operation with curves:
extract a continuous part of an Hermite curve
 $\gamma(t)$ into a new Hermite curve

- $\gamma(t)$ is parametrized in $[0, 1]$

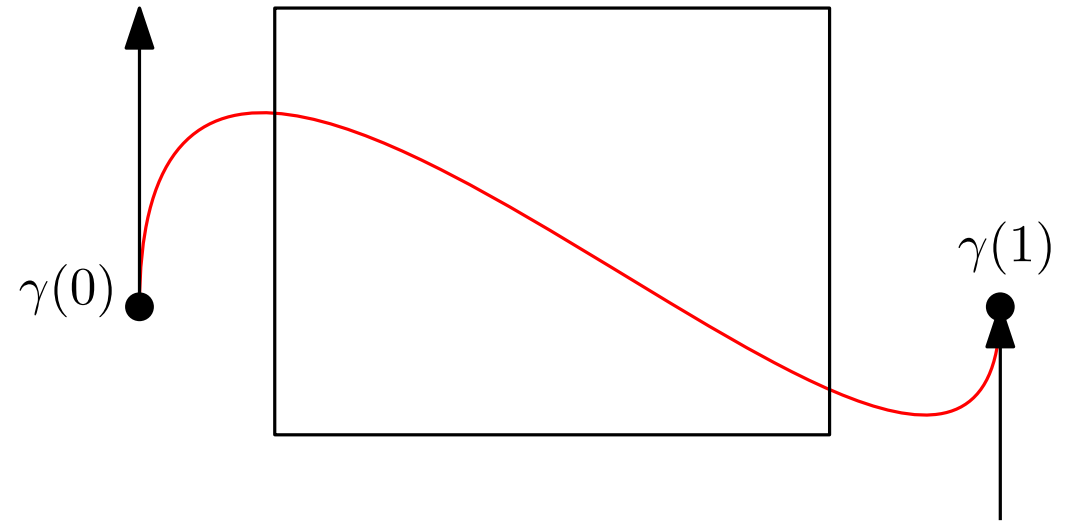


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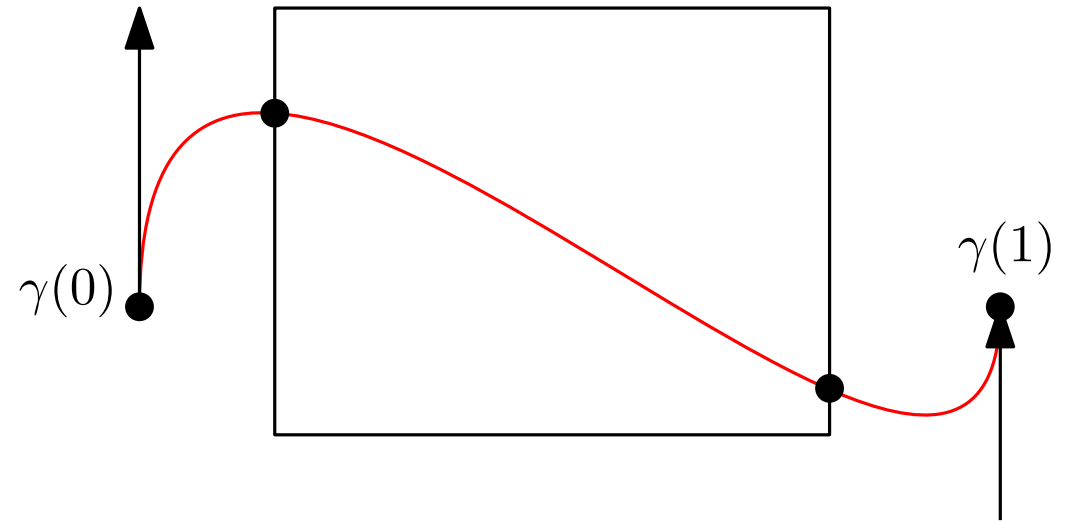


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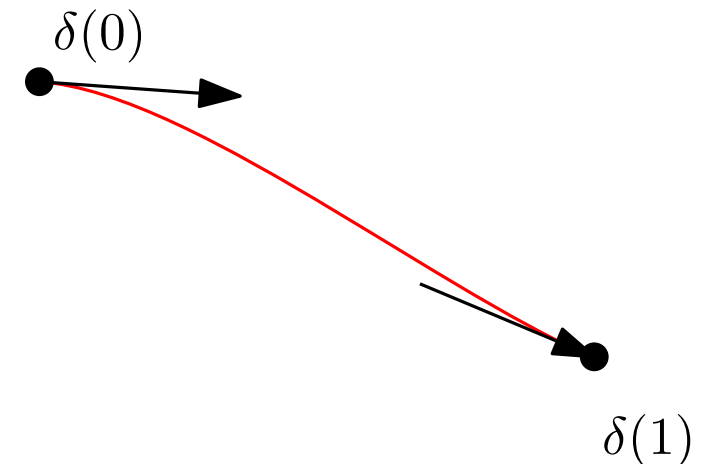
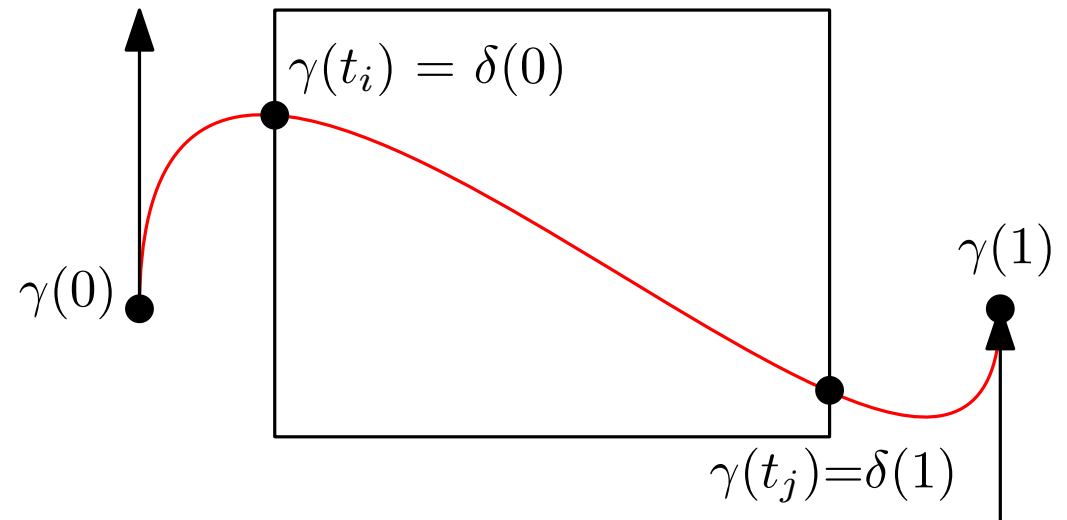


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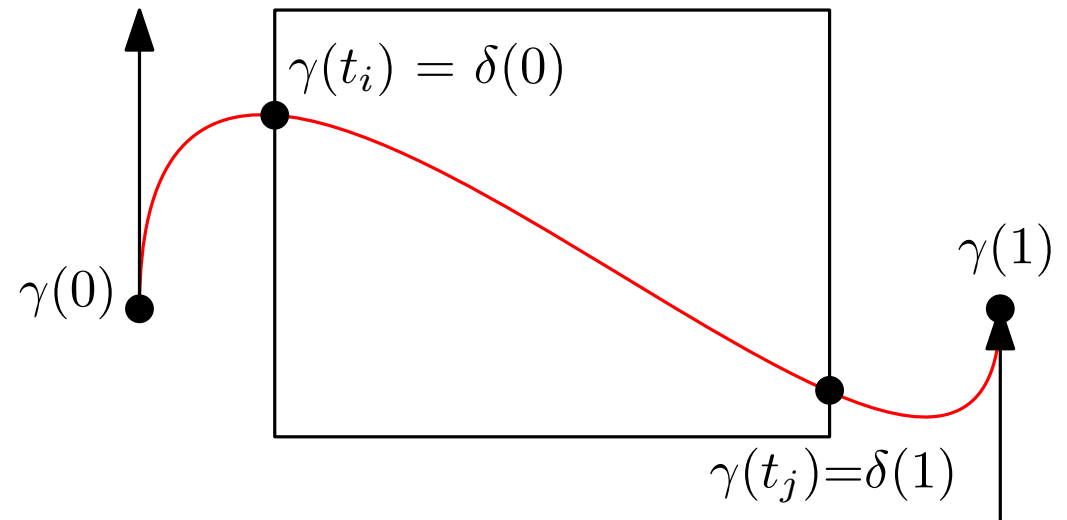


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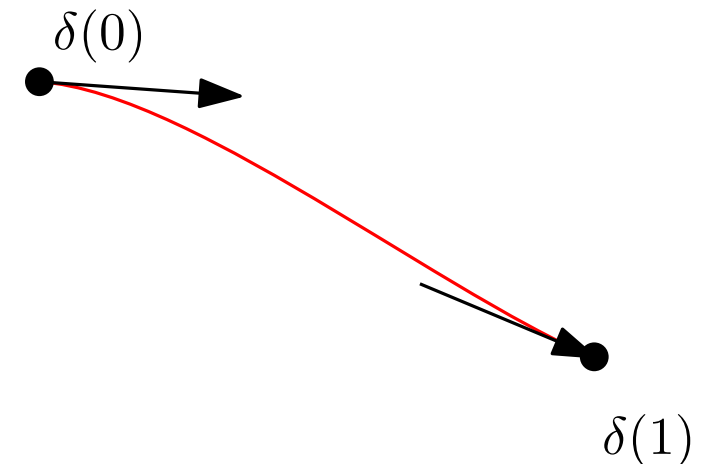
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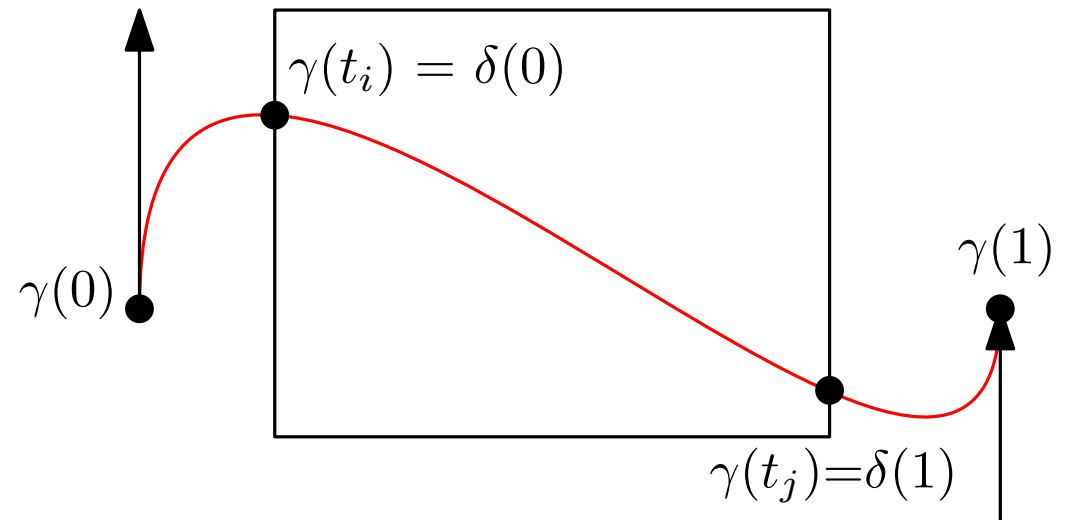


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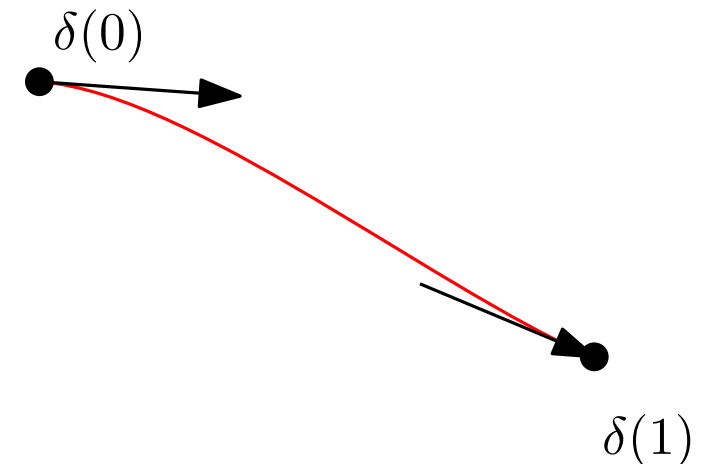
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$$s \longleftrightarrow t(s) = t_i + s(t_j - t_i)$$

and make sure that:

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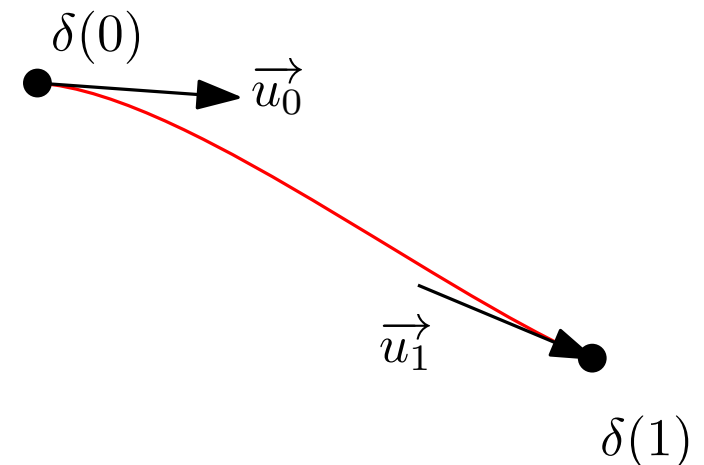
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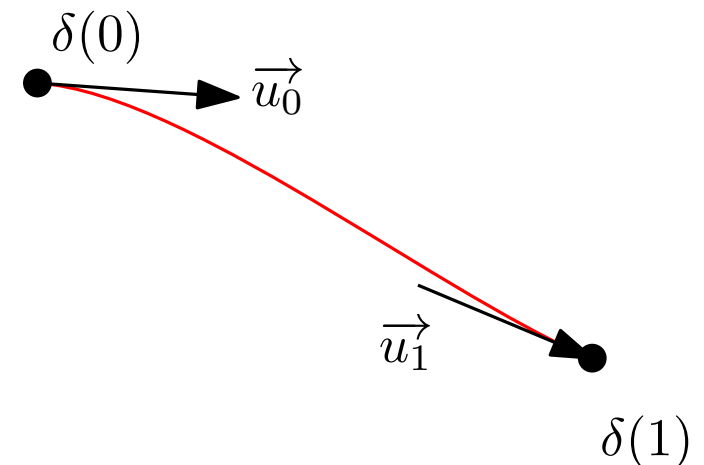
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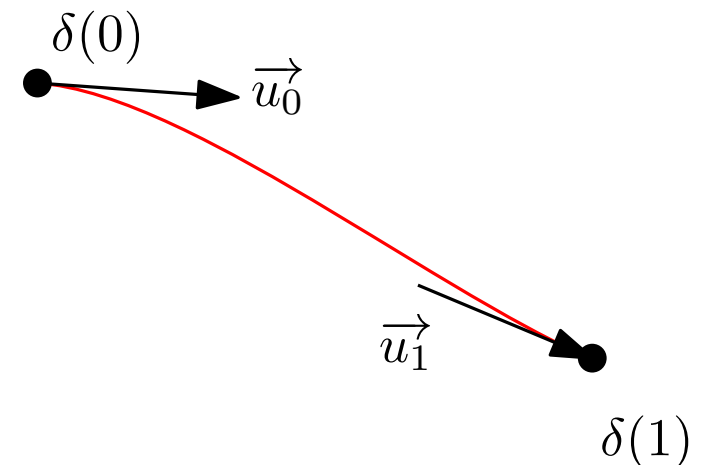
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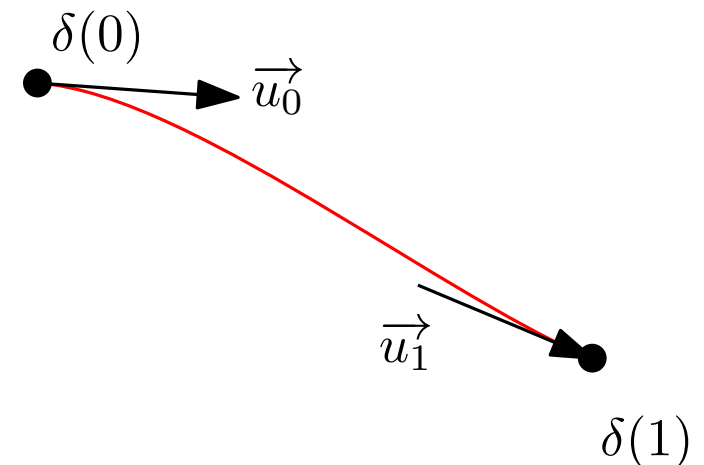
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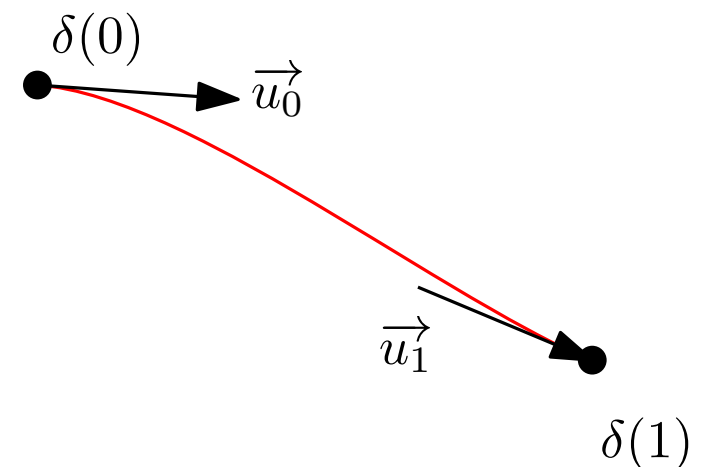
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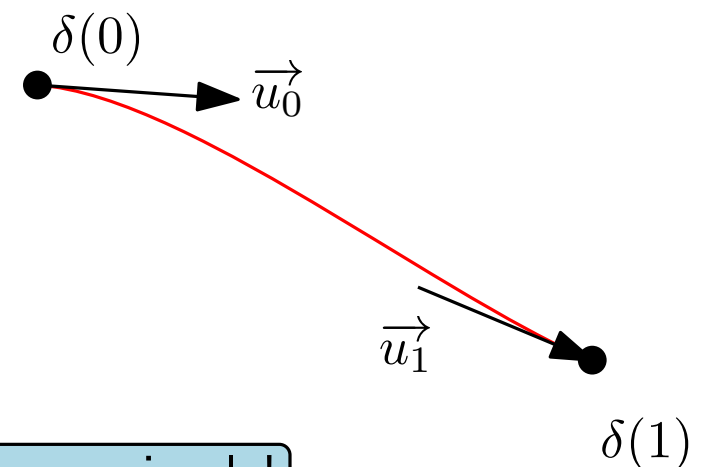
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Notice that clipping in a Lagrange polynomial would not be as simple!

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Matrix formulation

Similarly to most curve design methods, Hermite interpolation can be expressed in terms of matrices. This is sometimes convenient.

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is equivalent to:

$$\gamma(t) = (t^3, t^2, t, 1) \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} P_0 \\ P_1 \\ \vec{v}_0 \\ \vec{v}_1 \end{pmatrix}$$

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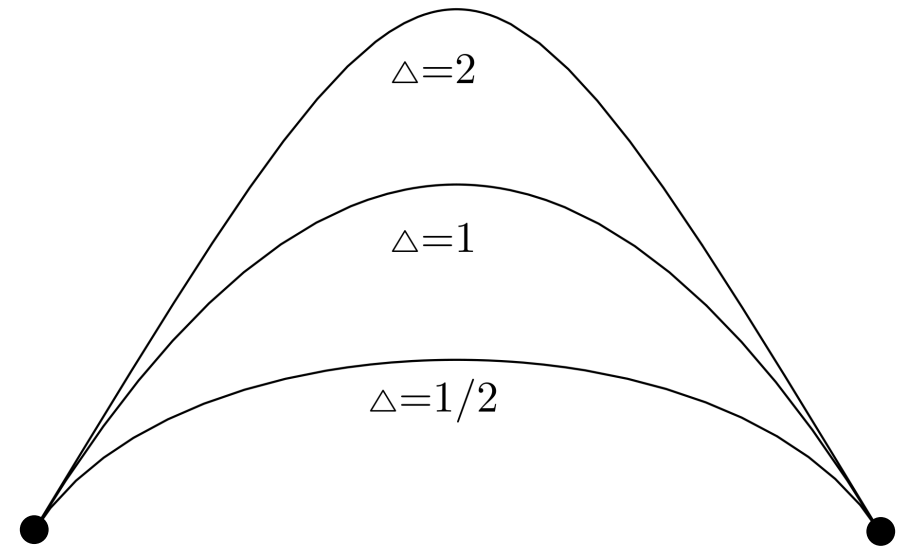
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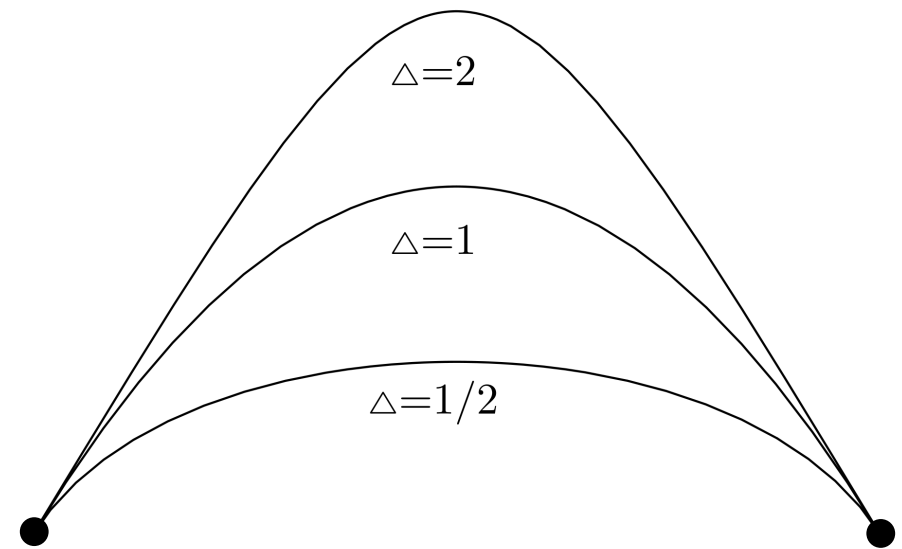
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The smaller the Δ , the higher the tension in the curve

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- For degree-5 we can use two endpoints, two tangent vectors, and two second derivative vectors (i.e., principal normal vectors) at the endpoints
- In general, for degree $2k + 1$ we can use two endpoints and the first k derivatives at each of them ($2k + 2$ items)

The formulas for such polynomials can be derived as we did for degree 3

However, higher degree Hermite polynomials are not of much use in practice!