**Duration:** 4:00 hours.

**Description:** The exam has two types of problems. The answers to Problems 1–3 should be delivered in a PDF file each. Solutions to Problems 4–5 must be delivered as HTML files or in a zip file with all required files. **Upload one separate file (PDF/ZIP) for each problem**.
All files should be uploaded to the Racó before the exam ends. Note that problems 4 and 5 include writing some explanation in the HTML file. All explanations should be visible by simply opening the HTML file with a web browser (code comments do not count as explanations).

**Publication of grades:** June 26, through the Racó.

**Revision:** Students wishing to revise their exam should send an email to the professor stating so by June 27. The revision will be on June 29 (17:00) using Meet.

**Problem 1.** [**2 points**] Suppose you need to design a piece like the ones shown below (these are *support washers*, used in assemblies to ensure tightness). The piece is roughly a flat and thick ring. The inner hole is bounded by a circle, while the outer boundary is bounded by a concentric larger circle, but with a dent (with the shape of a circular arc). In addition, the ring has a small circular hole.

Your design of the piece must be as accurate as possible.

Review the different types of curves seen in the course, and argue which type would be the most suitable one to represent the shape of this piece accurately.

**Important:** provide a detailed justification of your choice, comparing against other types of curves, and stating clearly which properties of the chosen type of curve make it the best choice.

*Sketch of solution* The shape of the piece that needs to be designed has two key properties that influence the choice of the type of curve: (i) it is composed of several circles (four, if we count the circular arc of the dent), and (ii) it has a dent in the outer circle that creates a $C^1$-discontinuity.

The most important constraint is (i), since the fact that we need to represent circles, and we need to do it as accurately as possible, restricts the types of curves to rational Bézier or rational B-splines. This is because, if we review the different types of curves seen in the course, these are the only types of curves that can represent circles exactly (and conics, in general). It should be noted that one can also represent circles exactly by using a direct parametrization of the circle, such as $(\cos t, \sin t)$. However, while this is possible, it is somewhat less practical to model the outer boundary with the dent.

As for (ii), the $C^1$-discontinuity in the outer boundary could be a reason to choose between rational Bézier or rational B-splines. Both are possible, but non-uniform rational B-splines have the advantage of allowing to represent the outer boundary, composed of two circular arcs, with a single curve (by configuring the knot vector appropriately). In contrast, when using rational Bézier curves we will be forced to use two different curves.

Any of these solutions were considered correct for the exam, as long as they were properly justified, and it was explained in some detail how the different parts of the pieces would be modeled.

**Problem 2.** [**2 points**] A very important feature of curves is having *local control* (also referred to as *local support*), as opposed to *global control*. Discuss this aspect for the following three types of curves: (i) Lagrange interpolation curves, (ii) Bézier curves, (iii) B-spline curves. State clearly, for each of them, whether it offers local or global control. Justify your answers in a mathematically sound way, using the formulas of each of the curves to support your argument.

*Sketch of solution* The idea here was to present the formulas of each of the three type of curves, and to argue based on them what kind of control they provide.

For Lagrange this is global control, as the formula clearly shows that the position of each point on the interpolating curve is influenced by each of the control points.

For Bézier curves the formula also shows this behavior: all control points have a positive weight in the position of each point on the curve. However, the weight of each point fades away as control points are further from the curve point. More precisely, the Bernstein polynomial that gives the weight to the $i$th control point has its maximum at $t = i/n$, and quickly fades away to zero as $t$ is far from $i/n$. For this reason, Bézier curves, although technically offer global control, are considered to offer pseudo-local control.

Finally, B-splines offer true local control, as seen by analyzing the basis functions: the basis function $N_{i,k}(t)$ that determines the weight of control point $P_i$ is non-zero only in the interval $[t_i, t_{i+k}]$. This immediately implies that each control point only has an influence in a few knot intervals around the point, and has absolutely no influence in the rest of the curve.

**Problem 3. [1 point]** Consider the cubic uniform B-Spline defined by the following control points $P_0, \ldots, P_4$: $\{P_0 = (1,0), (0,0), (0,1), (0,2), P_4 = (1,2)\}$ .

1. Give the parametric formula for each segment of the curve, together with the first and last point of each segment.

2. What would be needed in order to make the curve start at $P_0$?

*Sketch of solution* The first part simply consisted in applying the definition of cubic uniform B-splines. Since the b-spline is cubic and there are five control points, we will have two segments.

The formula of the $i$th segment in a uniform cubic B-spline (following the notation in the course slides) is:

$$P_i(t) = \frac{1}{6}(-t^3+3t^2-3t+1)P_{i-1}+\frac{1}{6}(3t^3-6t^2+4)P_i+\frac{1}{6}(-3t^3+3t^2+3t+1)P_{i+1}+\frac{t^3}{6}P_{i+2}$$

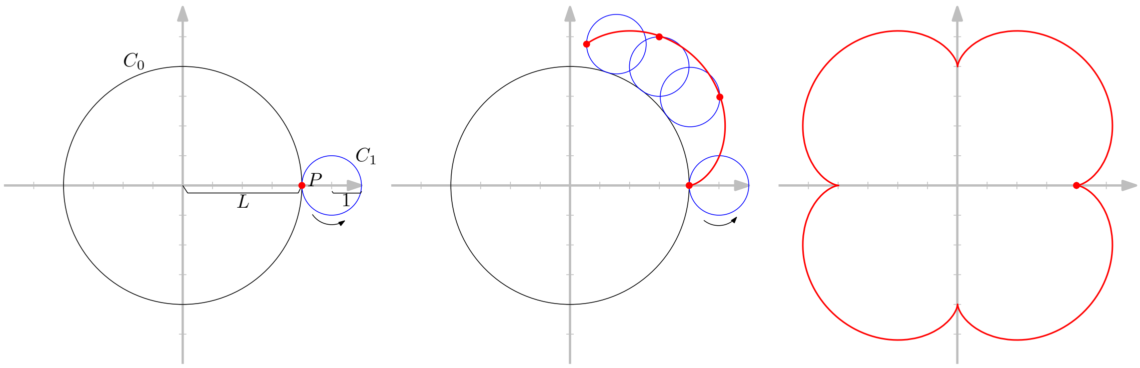
Replacing $\{P_{i-1}, P_i, P_{i+1}, P_{i+2}\}$ by $\{(1,0), (0,0), (0,1), (0,2)\}$ gives the parametric formula for the first segment. Replacing by $\{(0,0), (0,1), (0,2), (1,2)\}$ results in the parametric formula for the second segment.

To obtain the endpoints of the segments it is enough to evaluate each segment at $t = 0$ and $t = 1$. For instance, the first segment at $t = 0$ gives point $(1/6, 1/6)$, at $t = 1$ it gives point $(0, 1)$, and the second segment at $t = 1$ results in point $(1/6, 11/6)$.

For the second part of the problem it is enough to add one dummy control point $P_{-1}$, whose position should be equal to $2P_0 - P_1 = (2, 0)$.

**Problem 4. [2.5 points]** Let $C_0$ be a circle of radius $L$ in the plane, centered at the origin $O = (0,0)$. Let $C_1$ be a circle of radius one outside of $C_0$, which is tangent to $C_0$ at a point $P$ belonging to the positive semi-axis $Ox^+$, as illustrated in the left figure.

Consider the curve traced by the point $P$ as the circle $C_1$ rolls around $C_0$. The figure in the middle shows a few intermediate positions, while the figure on the right shows the resulting curve, for the case $L = 4$.




1. Give a parametrization of the curve described by point $P$ along the rolling movement, as a function of $L$ and a parameter $t \in \mathbb{R}$. **Describe in detail, in the same HTML file or in a PDF, the steps followed to obtain the parametrization.**

2. Write a program that draws the curve for a given positive integer value of $L$. The user should be able to enter the value of $L$.

*Sketch of solution* The curve in this exercise is very similar to the cardioid seen in class. In fact, this curve, called *epicycloid*, is a generalization that includes the cardiod as a particular case when $L = 1$.

There are several ways to deduce a parametric equation of the curve.

One option is to first trace the center of $C_1$, which has center with coordinates $((L+1)\cos t, (L+1)\sin t)$, for $t \in [0, 2\pi)$.

Then one has to combine the rotation of $C_1$ around its own center in order to find out where $P$ is exactly for each value of $t$,

For this, observe that $C_1$ makes several full turns for each turn of the circle of radius $L + 1$ where its centered.

To see this for $L = 4$, consider the middle figure. There one can see the position where $C_1$ makes its first full turn. Further, observe that at the point where $P$ touches $C_0$ again (not shown in the figure), $C_1$ will have rotated around its center by $2\pi + (1/2)\pi$, thus 1.25 full turns, while the center of $C_1$ will have rotated with respect to the origin by $\pi/4$. Therefore, after a full rotation of the center of $C_1$ around the origin, which occurs when $t$ ranges from 0 to $2\pi$, $P$ will have made $4 \times 1.25 = 5$ full turns. It follows that $C_1$ rotates around its own center 5 times faster than its center rotates around the origin.

In general, when $t$ varies from 0 to $2\pi$, $C_1$ will makes $L + 1$ full turns around its center.

Assuming for now that $C_1$ is centered at the origin, this suggest a parametrization for $P$ of the type

$$(\cos((L+1)t), \sin((L+1)t)).$$

However, for $t = 0$ this would place $P$ at $(L+1, 0)$, instead of at $(-(L+1), 0)$. To force this to happen, instead we use

$$(-\cos((L+1)t), -\sin((L+1)t)).$$

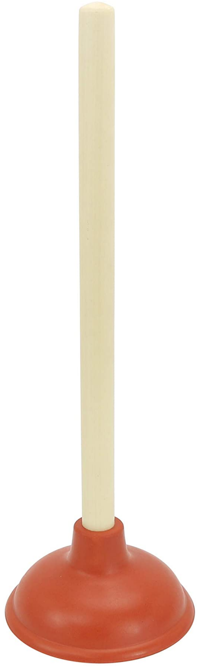
Finally we need to translate this to the right position, by adding the vector to the center $C_1$, obtaining the following formula for $t \in [0, 2\pi)$:

$$((L+1)\cos t, (L+1)\sin t) + (-\cos((L+1)t), -\sin((L+1)t))$$

$$= ((L+1)\cos t - \cos((L+1)t), (L+1)\sin t - \sin((L+1)t)).$$

For the correction of the exam, only answers that were justified with enough detail in the HTML file (not in code comments) or in a separate PDF document obtained the full score.

**Problem 5. [2.5 points]** Write a program to draw a plunger similar to the one below. Design the profile of the object using one or more Bézier curves.



**Important:** In the HTML file that you will deliver, include a brief description of the strategy that you have followed and the design decisions made.

*Sketch of solution* The best way to model this object is by using a surface of revolution. This is in fact already hinted in the statement, where it refers to the "profile" of the object.

Some people implemented this by using a single surface of revolution, some used two. To that end, one may want to use two or more Bézier curves to represent the profile, since there is at least one clear $C^1$-discontinuity in the profile. However, for visual purposes this can be avoided by using extra control points to simulate a $C^1$-discontinuity (note, however, that this may not be good enough if the model was to be used for purposes).

Some other people used a Bézier curve only for the bottom part of the plunger, and a cylinder for the handle. This is also fine, although the top of the handle ends up hollow, which is not as it is originally.

In general, the full score was given to solutions that used Bézier curves to produce a surface of revolution, resulted in an accurate model of the object, and that explained the strategy chosen in the HTML file (not in code comments) or in a separate PDF document.