B-SPLINE CURVES

Rodrigo Silveira

Curve and Surface Design
Facultat d'Informàtica de Barcelona
Universitat Politècnica de Catalunya

Improving over Bézier curves

Improving over Bézier curves

Bézier curves have some drawbacks:

- Degree is proportional to number of control points

- Does not offer true global control (at most "pseudo-local")

- C^2 continuity is not so easy to obtain for composite curves

Improving over Bézier curves

Bézier curves have some drawbacks:

- Degree is proportional to number of control points

- Does not offer true global control (at most "pseudo-local")

- C^2 continuity is not so easy to obtain for composite curves

To overcome this: **B-splines**

Improving over Bézier curves

Bézier curves have some drawbacks:

- Degree is proportional to number of control points

- Does not offer true global control (at most "pseudo-local")

- C^2 continuity is not so easy to obtain for composite curves

To overcome this: **B-splines**

- "Discovered" by Schoenberg (1940s) (actually, even earlier), developed as we will see them by Riesenfeld and others in 1970s

- B-splines = Basis splines

- Several flavors: uniform, non-uniform, rational non-uniform (NURBs)...

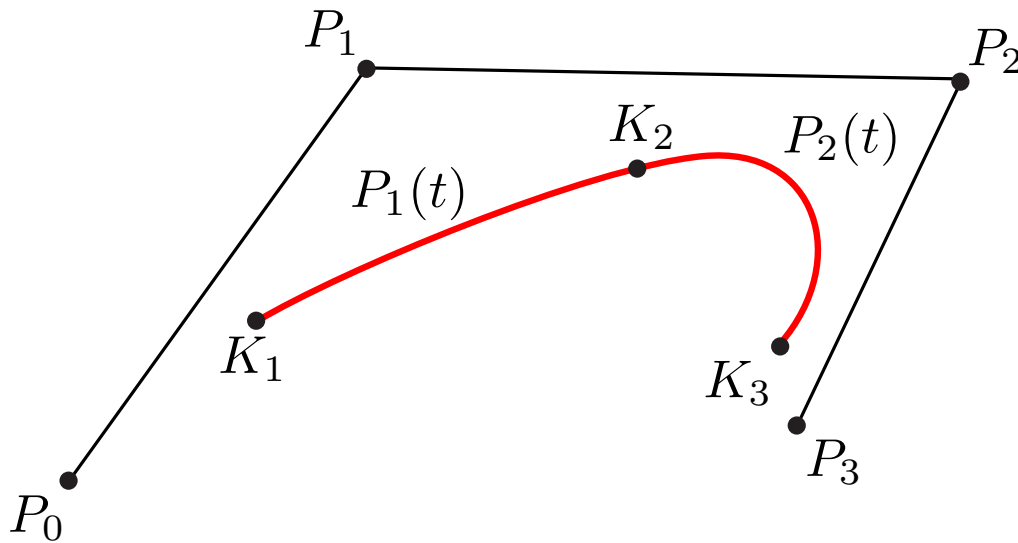## Deriving the formula for the quadratic B-splines

Setting:

- Input: $n + 1$ control points $P_0, \ldots, P_n$

- Output: **spline curve** where each segment $P_i(t)$ is a **quadratic** parametric polynomial based on P_{i-1}, P_i and P_{i+1}. Segments start and end at points K_i, called *knots*.

Deriving the formula for the quadratic B-splines

Setting:

- Input: $n + 1$ control points $P_0, \ldots, P_n$

- Output: **spline curve** where each segment $P_i(t)$ is a **quadratic** parametric polynomial based on P_{i-1}, P_i and P_{i+1}. Segments start and end at points K_i, called *knots*.
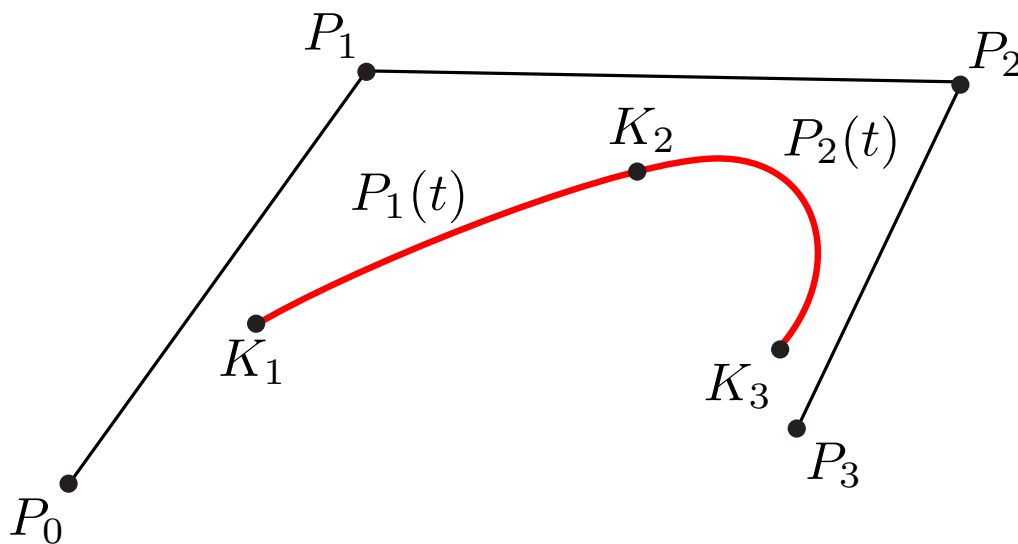

sketch of setting for $n = 3$ (not accurate!)

Deriving the formula for the quadratic B-splines

Setting:

- Input: $n+1$ control points $P_0, \ldots, P_n$

- Output: **spline curve** where each segment $P_i(t)$ is a **quadratic** parametric polynomial based on P_{i-1}, P_i and P_{i+1}. Segments start and end at points K_i, called *knots*.

$$P_i(t) = (t^2, t, 1) \begin{pmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{pmatrix} \begin{pmatrix} P_{i-1} \\ P_i \\ P_{i+1} \end{pmatrix}$$

sketch of setting for $n = 3$ (not accurate!)

Deriving the formula for the quadratic B-splines

Setting:

- Input: $n+1$ control points $P_0, \ldots, P_n$

- Output: **spline curve** where each segment $P_i(t)$ is a **quadratic** parametric polynomial based on P_{i-1}, P_i and P_{i+1}. Segments start and end at points K_i, called *knots*.

$$P_i(t) = (t^2, t, 1) \begin{pmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{pmatrix} \begin{pmatrix} P_{i-1} \\ P_i \\ P_{i+1} \end{pmatrix}$$
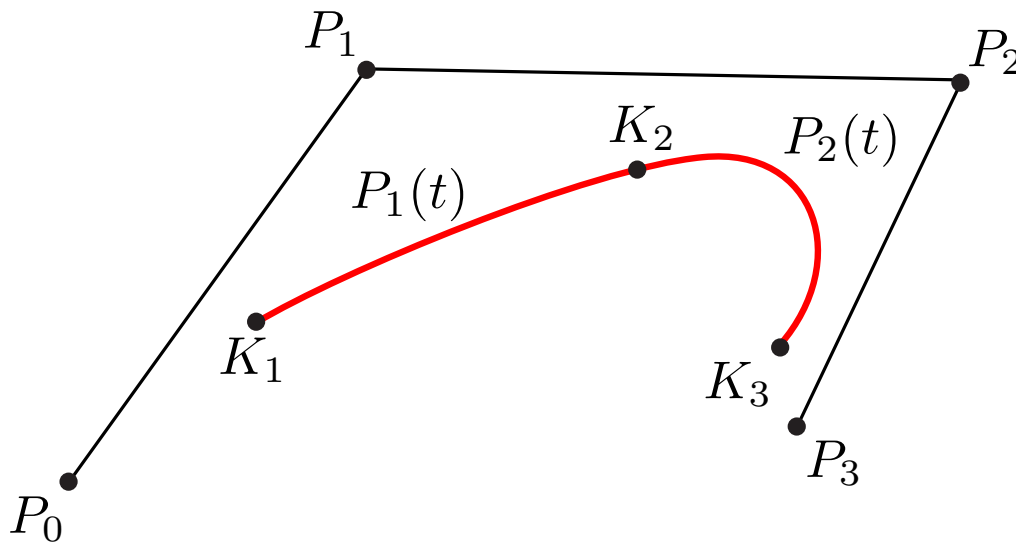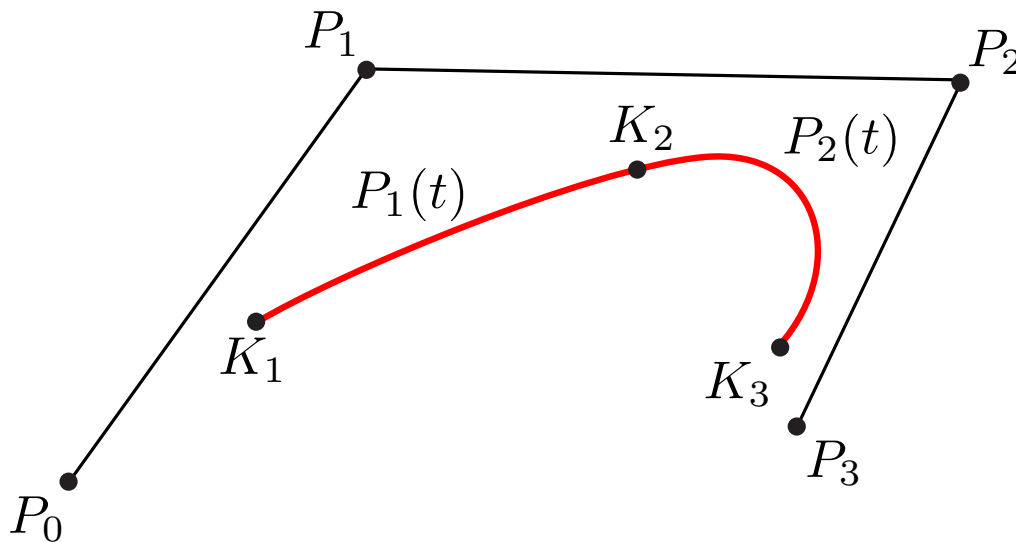
$$= (t^2, t, 1) \begin{pmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_0 & b_0 & c_0 \end{pmatrix} \begin{pmatrix} P_{i-1} \\ P_i \\ P_{i+1} \end{pmatrix}$$

sketch of setting for $n = 3$ (not accurate!)

Deriving the formula for the quadratic B-splines

Setting:

- Input: $n+1$ control points $P_0, \ldots, P_n$

- Output: **spline curve** where each segment $P_i(t)$ is a **quadratic** parametric polynomial based on P_{i-1}, P_i and P_{i+1}. Segments start and end at points K_i, called *knots*.

$$P_i(t) = (t^2, t, 1) \begin{pmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{pmatrix} \begin{pmatrix} P_{i-1} \\ P_i \\ P_{i+1} \end{pmatrix}$$

$$= (t^2, t, 1) \begin{pmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_0 & b_0 & c_0 \end{pmatrix} \begin{pmatrix} P_{i-1} \\ P_i \\ P_{i+1} \end{pmatrix}$$

sketch of setting for $n=3$ (not accurate!)

Requirements:

1. $P_1(t)$ and $P_2(t)$ meet smoothly at common point
2. Affine combination of control points

Question: what is the matrix?

Deriving the formula for the quadratic B-splines

$$P_i(t) = \frac{1}{2}(t^2, t, 1) \begin{pmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} P_{i-1} \\ P_i \\ P_{i+1} \end{pmatrix}, i = 1, 2$$

$$= \frac{1}{2}(t^2 - 2t + 1)P_{i-1} + \frac{1}{2}(-2t^2 + 2t + 1)P_i + \frac{t^2}{2}P_{i+1}$$

Deriving the formula for the quadratic B-splines

$$P_i(t) = \frac{1}{2}(t^2, t, 1) \begin{pmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} P_{i-1} \\ P_i \\ P_{i+1} \end{pmatrix}, i = 1, 2$$

$$= \frac{1}{2}(t^2 - 2t + 1)P_{i-1} + \frac{1}{2}(-2t^2 + 2t + 1)P_i + \frac{t^2}{2}P_{i+1}$$

- Start and endpoints: K_i and K_{i+1}

Deriving the formula for the quadratic B-splines

$$P_i(t) = \frac{1}{2}(t^2, t, 1) \begin{pmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} P_{i-1} \\ P_i \\ P_{i+1} \end{pmatrix}, i = 1, 2$$

$$= \frac{1}{2}(t^2 - 2t + 1)P_{i-1} + \frac{1}{2}(-2t^2 + 2t + 1)P_i + \frac{t^2}{2}P_{i+1}$$

- Start and endpoints: K_i and K_{i+1}

  Since $K_i = P_i(0)$ and $K_{i+1} = P_i(1)$, we have:

  $K_i = P_i(0) = \frac{1}{2}(P_{i-1} + P_i)$
  $K_{i+1} = P_i(1) = \frac{1}{2}(P_i + P_{i+1})$

Deriving the formula for the quadratic B-splines
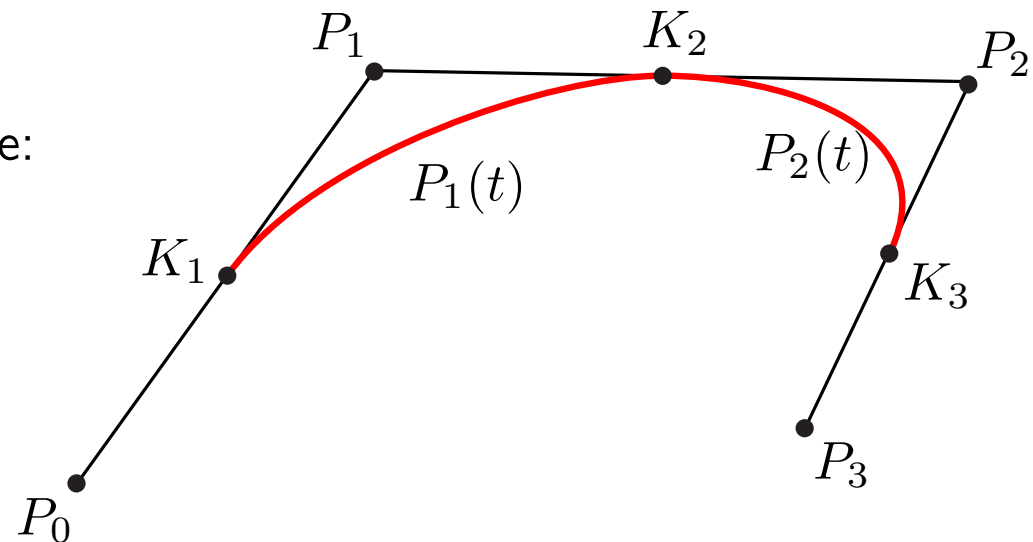
$$P_i(t) = \frac{1}{2}(t^2, t, 1) \begin{pmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} P_{i-1} \\ P_i \\ P_{i+1} \end{pmatrix}, i = 1, 2$$

$$= \frac{1}{2}(t^2 - 2t + 1)P_{i-1} + \frac{1}{2}(-2t^2 + 2t + 1)P_i + \frac{t^2}{2}P_{i+1}$$

- Start and endpoints: K_i and K_{i+1}

  Since $K_i = P_i(0)$ and $K_{i+1} = P_i(1)$, we have:

  $K_i = P_i(0) = \frac{1}{2}(P_{i-1} + P_i)$
  $K_{i+1} = P_i(1) = \frac{1}{2}(P_i + P_{i+1})$

More accurate picture

Deriving the formula for the quadratic B-splines

$$P_i(t) = \frac{1}{2}(t^2, t, 1) \begin{pmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} P_{i-1} \\ P_i \\ P_{i+1} \end{pmatrix}, i = 1, 2$$

$$= \frac{1}{2}(t^2 - 2t + 1)P_{i-1} + \frac{1}{2}(-2t^2 + 2t + 1)P_i + \frac{t^2}{2}P_{i+1}$$

- Start and endpoints: K_i and K_{i+1}

  Since $K_i = P_i(0)$ and $K_{i+1} = P_i(1)$, we have:

  $K_i = P_i(0) = \frac{1}{2}(P_{i-1} + P_i)$
  $K_{i+1} = P_i(1) = \frac{1}{2}(P_i + P_{i+1})$

Question: what are the tangent vectors at the ends?

More accurate picture

Deriving the formula for the quadratic B-splines

$$P_i(t) = \frac{1}{2}(t^2, t, 1) \begin{pmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} P_{i-1} \\ P_i \\ P_{i+1} \end{pmatrix}, i = 1, 2$$

$$= \frac{1}{2}(t^2 - 2t + 1)P_{i-1} + \frac{1}{2}(-2t^2 + 2t + 1)P_i + \frac{t^2}{2}P_{i+1}$$
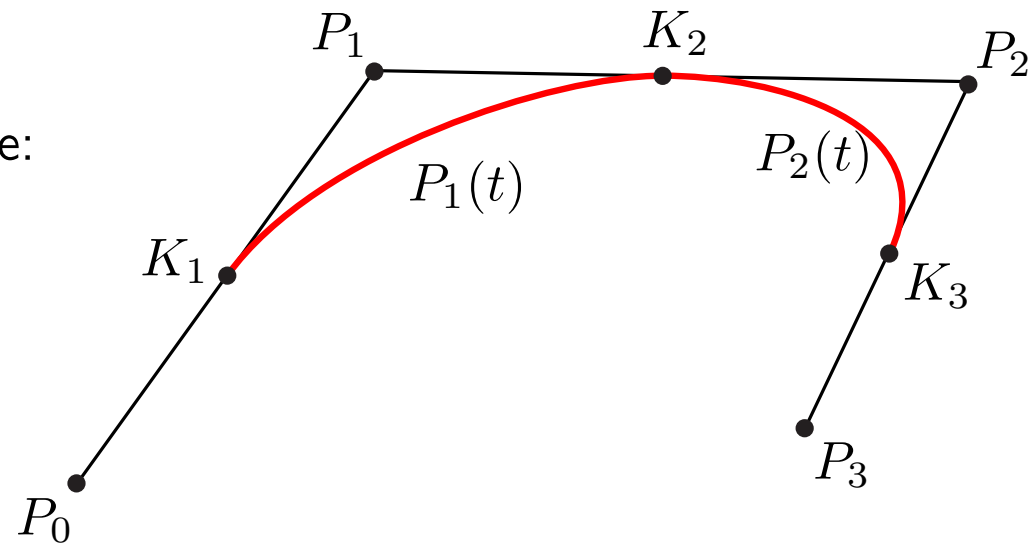
- Start and endpoints: K_i and K_{i+1}
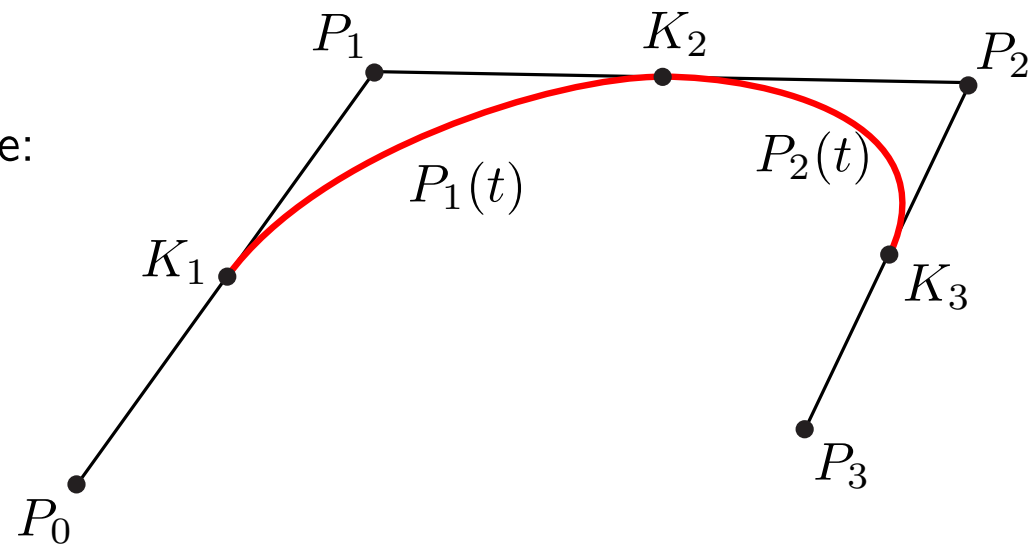 Since $K_i = P_i(0)$ and $K_{i+1} = P_i(1)$, we have:

  $K_i = P_i(0) = \frac{1}{2}(P_{i-1} + P_i)$
  $K_{i+1} = P_i(1) = \frac{1}{2}(P_i + P_{i+1})$

Question: what are the tangent vectors at the ends?

$P_i'(0) = P_i - P_{i-1}$
$P_i'(1) = P_{i+1} - P_i$

More accurate picture

Deriving the formula for the quadratic B-splines

$$P_i(t) = \frac{1}{2}(t^2, t, 1) \begin{pmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} P_{i-1} \\ P_i \\ P_{i+1} \end{pmatrix}, i = 1, 2$$

$$= \frac{1}{2}(t^2 - 2t + 1)P_{i-1} + \frac{1}{2}(-2t^2 + 2t + 1)P_i + \frac{t^2}{2}P_{i+1}$$

- Start and endpoints: K_i and K_{i+1}

  Since $K_i = P_i(0)$ and $K_{i+1} = P_i(1)$, we have:

  $K_i = P_i(0) = \frac{1}{2}(P_{i-1} + P_i)$
  $K_{i+1} = P_i(1) = \frac{1}{2}(P_i + P_{i+1})$

More accurate picture

Question: what are the tangent vectors at the ends?

$P_i'(0) = P_i - P_{i-1}$
$P_i'(1) = P_{i+1} - P_i$

Example: use control points
$\{(1, 0), (1, 1), (2, 1), (2, 0)\}$
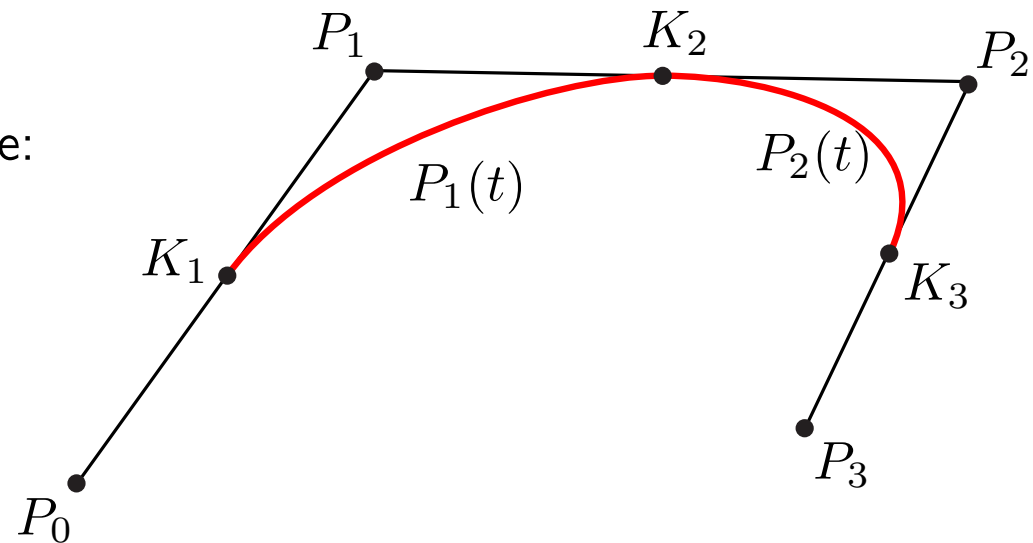
Deriving the formula for the cubic B-splines

Deriving the formula for the cubic B-splines

Setting:

- Input: $n + 1$ control points $P_0, \ldots, P_n$

- Output: spline curve where each segment $P_i(t)$ is a **cubic** parametric polynomial based on $P_{i-1}, P_i, P_{i+1}, P_{i+2}$

Deriving the formula for the cubic B-splines

Setting:

- Input: $n + 1$ control points $P_0, \ldots, P_n$

- Output: spline curve where each segment $P_i(t)$ is a **cubic** parametric polynomial based on $P_{i-1}, P_i, P_{i+1}, P_{i+2}$

sketch of setting for $n = 4$ (not accurate!)

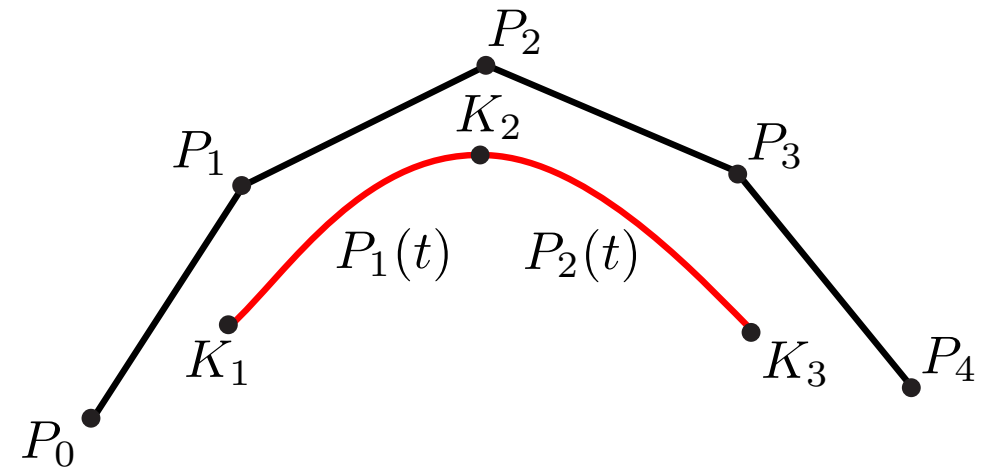## Deriving the formula for the cubic B-splines

Setting:

- Input: $n + 1$ control points $P_0, \ldots, P_n$

- Output: spline curve where each segment $P_i(t)$ is a **cubic** parametric polynomial based on $P_{i-1}, P_i, P_{i+1}, P_{i+2}$

$$P_i(t) = (t^3, t^2, t, 1)M \begin{pmatrix} P_{i-1} \\ P_i \\ P_{i+1} \\ P_{i+2} \end{pmatrix}$$

Requirements:

1. Consecutive segments meet with C^2-continuity
2. Entire curve is affine combination of control points

sketch of setting for $n = 4$ (not accurate!)

Deriving the formula for the cubic B-splines

$$P_i(t) = (t^3, t^2, t, 1) \begin{pmatrix} a_3 & b_3 & c_3 & d_3 \\ a_2 & b_2 & c_2 & d_2 \\ a_1 & b_1 & c_1 & d_1 \\ a_0 & b_0 & c_0 & d_0 \end{pmatrix} \begin{pmatrix} P_{i-1} \\ P_i \\ P_{i+1} \\ P_{i+2} \end{pmatrix}$$

Requirements:

1. Consecutive segments meet with C^2 continuity

2. Entire curve is affine combination of control points

Deriving the formula for the cubic B-splines

$$P_i(t) = (t^3, t^2, t, 1) \begin{pmatrix} a_3 & b_3 & c_3 & d_3 \\ a_2 & b_2 & c_2 & d_2 \\ a_1 & b_1 & c_1 & d_1 \\ a_0 & b_0 & c_0 & d_0 \end{pmatrix} \begin{pmatrix} P_{i-1} \\ P_i \\ P_{i+1} \\ P_{i+2} \end{pmatrix}$$

Requirements:

1. Consecutive segments meet with C^2 continuity

2. Entire curve is affine combination of control points

Matrix derived in similar way:

Deriving the formula for the cubic B-splines

$$P_i(t) = (t^3, t^2, t, 1) \begin{pmatrix} a_3 & b_3 & c_3 & d_3 \\ a_2 & b_2 & c_2 & d_2 \\ a_1 & b_1 & c_1 & d_1 \\ a_0 & b_0 & c_0 & d_0 \end{pmatrix} \begin{pmatrix} P_{i-1} \\ P_i \\ P_{i+1} \\ P_{i+2} \end{pmatrix}$$

Requirements:

1. Consecutive segments meet with C^2 continuity

2. Entire curve is affine combination of control points

Matrix derived in similar way:

- Equations for equal endpoints, equal first and second derivatives at $t = 0$ and $t = 1$?

Deriving the formula for the cubic B-splines

$$P_i(t) = (t^3, t^2, t, 1) \begin{pmatrix} a_3 & b_3 & c_3 & d_3 \\ a_2 & b_2 & c_2 & d_2 \\ a_1 & b_1 & c_1 & d_1 \\ a_0 & b_0 & c_0 & d_0 \end{pmatrix} \begin{pmatrix} P_{i-1} \\ P_i \\ P_{i+1} \\ P_{i+2} \end{pmatrix}$$

Requirements:

1. Consecutive segments meet with C^2 continuity

2. Entire curve is affine combination of control points

Matrix derived in similar way:

- Equations for equal endpoints, equal first and second derivatives at $t = 0$ and $t = 1$?

15 equations

Deriving the formula for the cubic B-splines

$$P_i(t) = (t^3, t^2, t, 1) \begin{pmatrix} a_3 & b_3 & c_3 & d_3 \\ a_2 & b_2 & c_2 & d_2 \\ a_1 & b_1 & c_1 & d_1 \\ a_0 & b_0 & c_0 & d_0 \end{pmatrix} \begin{pmatrix} P_{i-1} \\ P_i \\ P_{i+1} \\ P_{i+2} \end{pmatrix}$$

Requirements:

1. Consecutive segments meet with C^2 continuity

2. Entire curve is affine combination of control points

Matrix derived in similar way:

- Equations for equal endpoints, equal first and second derivatives at $t = 0$ and $t = 1$?

  15 equations

- Equations for affine combination

Deriving the formula for the cubic B-splines

$$P_i(t) = (t^3, t^2, t, 1) \begin{pmatrix} a_3 & b_3 & c_3 & d_3 \\ a_2 & b_2 & c_2 & d_2 \\ a_1 & b_1 & c_1 & d_1 \\ a_0 & b_0 & c_0 & d_0 \end{pmatrix} \begin{pmatrix} P_{i-1} \\ P_i \\ P_{i+1} \\ P_{i+2} \end{pmatrix}$$

Requirements:

1. Consecutive segments meet with C^2 continuity

2. Entire curve is affine combination of control points

Matrix derived in similar way:

- Equations for equal endpoints, equal first and second derivatives at $t = 0$ and $t = 1$?

  15 equations

- Equations for affine combination

  4 equations

Deriving the formula for the cubic B-splines

$$P_i(t) = (t^3, t^2, t, 1) \begin{pmatrix} a_3 & b_3 & c_3 & d_3 \\ a_2 & b_2 & c_2 & d_2 \\ a_1 & b_1 & c_1 & d_1 \\ a_0 & b_0 & c_0 & d_0 \end{pmatrix} \begin{pmatrix} P_{i-1} \\ P_i \\ P_{i+1} \\ P_{i+2} \end{pmatrix}$$

Requirements:

1. Consecutive segments meet with C^2 continuity

2. Entire curve is affine combination of control points

Matrix derived in similar way:

- Equations for equal endpoints, equal first and second derivatives at $t = 0$ and $t = 1$?

  15 equations

- Equations for affine combination

  4 equations

16 of them are independent $\implies$ unique solution

Formula for the cubic B-splines

Formula for the cubic B-splines

$$P_i(t) = \frac{1}{6}(t^3, t^2, t, 1) \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix} \begin{pmatrix} P_{i-1} \\ P_i \\ P_{i+1} \\ P_{i+2} \end{pmatrix}$$

Formula for the cubic B-splines

$$P_i(t) = \frac{1}{6}(t^3, t^2, t, 1)\begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix}\begin{pmatrix} P_{i-1} \\ P_i \\ P_{i+1} \\ P_{i+2} \end{pmatrix}$$

$$= \frac{1}{6}(-t^3 + 3t^2 - 3t + 1)P_{i-1} + \frac{1}{6}(3t^3 - 6t^2 + 4)P_i + \frac{1}{6}(-3t^3 + 3t^2 + 3t + 1)P_{i+1} + \frac{t^3}{6}P_{i+2}$$
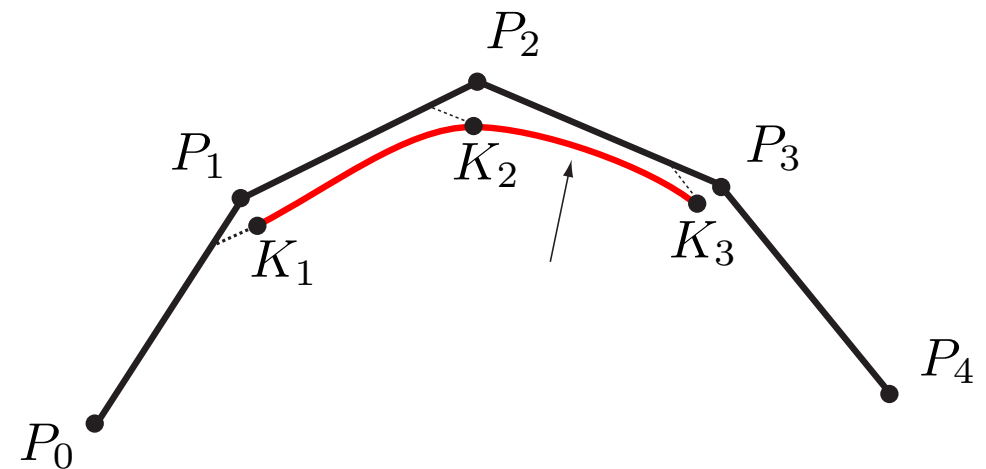
Formula for the cubic B-splines

$$P_i(t) = \frac{1}{6}(t^3, t^2, t, 1) \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix} \begin{pmatrix} P_{i-1} \\ P_i \\ P_{i+1} \\ P_{i+2} \end{pmatrix}$$

$$= \frac{1}{6}(-t^3 + 3t^2 - 3t + 1)P_{i-1} + \frac{1}{6}(3t^3 - 6t^2 + 4)P_i + \frac{1}{6}(-3t^3 + 3t^2 + 3t + 1)P_{i+1} + \frac{t^3}{6}P_{i+2}$$

Formula for the cubic B-splines

$$P_i(t) = \frac{1}{6}(t^3, t^2, t, 1) \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix} \begin{pmatrix} P_{i-1} \\ P_i \\ P_{i+1} \\ P_{i+2} \end{pmatrix}$$

$$= \frac{1}{6}(-t^3 + 3t^2 - 3t + 1)P_{i-1} + \frac{1}{6}(3t^3 - 6t^2 + 4)P_i + \frac{1}{6}(-3t^3 + 3t^2 + 3t + 1)P_{i+1} + \frac{t^3}{6}P_{i+2}$$
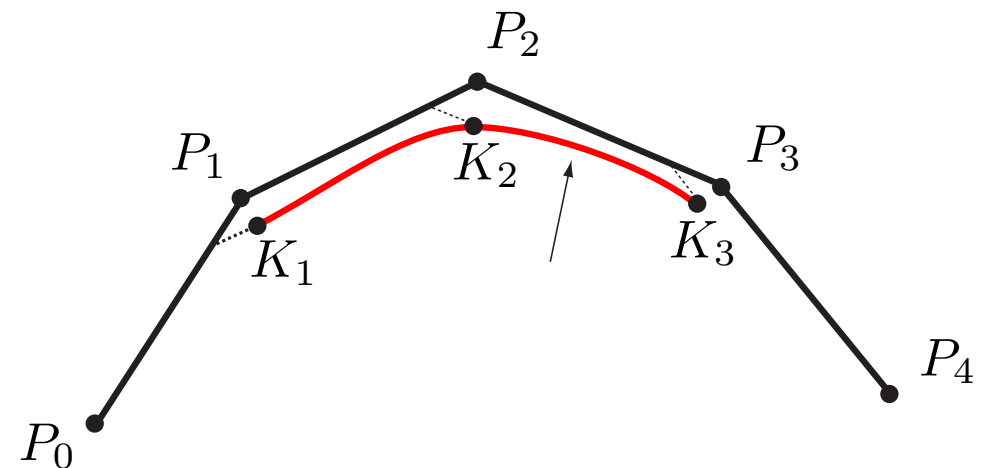
The two endpoints of each curve segment are

Formula for the cubic B-splines

$$P_i(t) = \frac{1}{6}(t^3, t^2, t, 1) \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix} \begin{pmatrix} P_{i-1} \\ P_i \\ P_{i+1} \\ P_{i+2} \end{pmatrix}$$

$$= \frac{1}{6}(-t^3 + 3t^2 - 3t + 1)P_{i-1} + \frac{1}{6}(3t^3 - 6t^2 + 4)P_i + \frac{1}{6}(-3t^3 + 3t^2 + 3t + 1)P_{i+1} + \frac{t^3}{6}P_{i+2}$$

The two endpoints of each curve segment are

$$K_i = P_i(0) = \tfrac{1}{6}(P_{i-1} + 4P_i + P_{i+1})$$
$$K_{i+1} = P_i(1) = \tfrac{1}{6}(P_i + 4P_{i+1} + P_{i+2})$$
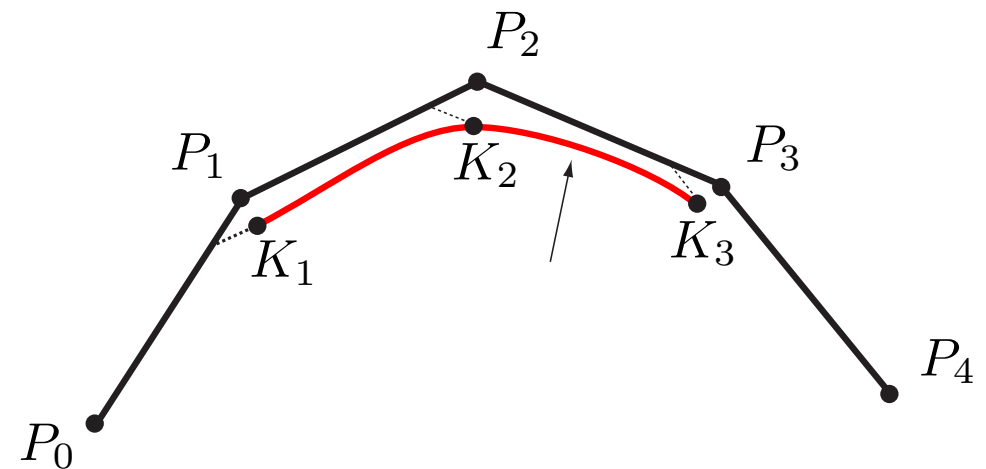
Formula for the cubic B-splines

$$P_i(t) = \frac{1}{6}(t^3, t^2, t, 1) \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix} \begin{pmatrix} P_{i-1} \\ P_i \\ P_{i+1} \\ P_{i+2} \end{pmatrix}$$

$$= \frac{1}{6}(-t^3 + 3t^2 - 3t + 1)P_{i-1} + \frac{1}{6}(3t^3 - 6t^2 + 4)P_i + \frac{1}{6}(-3t^3 + 3t^2 + 3t + 1)P_{i+1} + \frac{t^3}{6}P_{i+2}$$

The two endpoints of each curve segment are

$$K_i = P_i(0) = \tfrac{1}{6}(P_{i-1} + 4P_i + P_{i+1})$$
$$K_{i+1} = P_i(1) = \tfrac{1}{6}(P_i + 4P_{i+1} + P_{i+2})$$

Geometrically, it makes more sense to rewrite as

$$K_i = \left(\tfrac{1}{6}P_{i-1} + \tfrac{5}{6}P_i\right) + \tfrac{1}{6}\left(P_{i+1} - P_i\right)$$
$$K_{i+1} = \left(\tfrac{1}{6}P_i + \tfrac{5}{6}P_{i+1}\right) + \tfrac{1}{6}\left(P_{i+2} - P_{i+1}\right)$$

Formula for the cubic B-splines

$$P_i(t) = \frac{1}{6}(t^3, t^2, t, 1) \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix} \begin{pmatrix} P_{i-1} \\ P_i \\ P_{i+1} \\ P_{i+2} \end{pmatrix}$$

$$= \frac{1}{6}(-t^3 + 3t^2 - 3t + 1)P_{i-1} + \frac{1}{6}(3t^3 - 6t^2 + 4)P_i + \frac{1}{6}(-3t^3 + 3t^2 + 3t + 1)P_{i+1} + \frac{t^3}{6}P_{i+2}$$
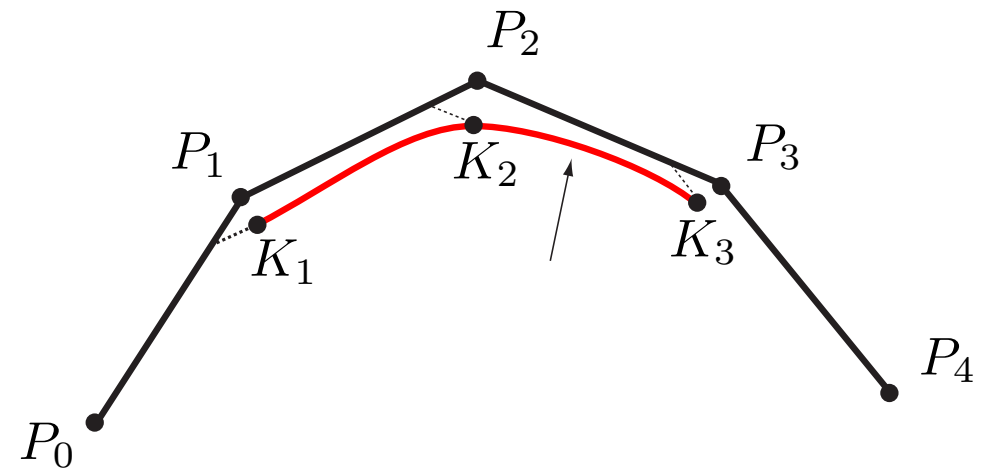
The two endpoints of each curve segment are

$$K_i = P_i(0) = \tfrac{1}{6}(P_{i-1} + 4P_i + P_{i+1})$$
$$K_{i+1} = P_i(1) = \tfrac{1}{6}(P_i + 4P_{i+1} + P_{i+2})$$

Geometrically, it makes more sense to rewrite as

$$K_i = \left(\tfrac{1}{6}P_{i-1} + \tfrac{5}{6}P_i\right) + \tfrac{1}{6}\left(P_{i+1} - P_i\right)$$
$$K_{i+1} = \left(\tfrac{1}{6}P_i + \tfrac{5}{6}P_{i+1}\right) + \tfrac{1}{6}\left(P_{i+2} - P_{i+1}\right)$$

Other geometric interpretations exist (e.g., $\frac{2}{3}$ rule)

Making the curve go from P_0 to P_n

How can we force the curve go through P_0 and P_n?

Making the curve go from P_0 to P_n

How can we force the curve go through P_0 and P_n?

Answer: add *dummy* control points P_{-1} and P_{n+1}

Making the curve go from P_0 to P_n

How can we force the curve go through P_0 and P_n?

Answer: add *dummy* control points P_{-1} and P_{n+1}

Question: what are P_{-1} and P_{n+1} points exactly?

Making the curve go from P_0 to P_n

How can we force the curve go through P_0 and P_n?

Answer: add *dummy* control points P_{-1} and P_{n+1}

Question: what are P_{-1} and P_{n+1} points exactly?

$P_{-1} = 2P_0 - P_1$ and $P_{n+1} = 2P_n - P_{n-1}$

Making the curve go from P_0 to P_n

How can we force the curve go through P_0 and P_n?

Answer: add *dummy* control points P_{-1} and P_{n+1}

Question: what are P_{-1} and P_{n+1} points exactly?

$P_{-1} = 2P_0 - P_1$ and $P_{n+1} = 2P_n - P_{n-1}$

Example: use control points
$\{(0,0), (0,1), (1,1), (2,1), (2,0)\}$.
(i) Draw the first segment.
(ii) Add P_{-1} to fix beginning of curve at P_0.

P_1 P_2 P_3

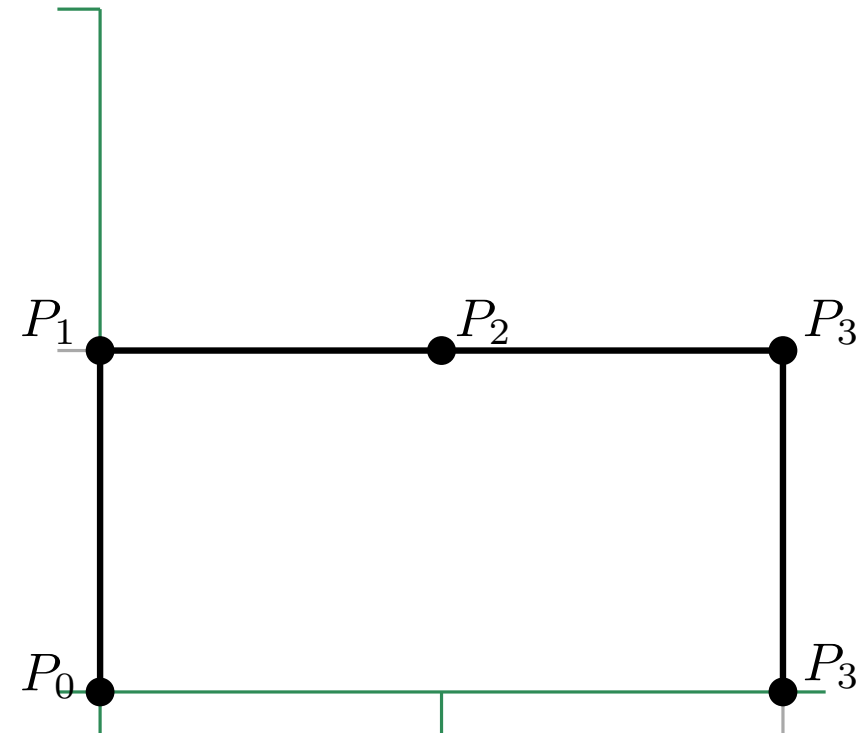P_0 P_3

Making the curve go from P_0 to P_n

How can we force the curve go through P_0 and P_n?

Answer: add *dummy* control points P_{-1} and P_{n+1}

Question: what are P_{-1} and P_{n+1} points exactly?

$P_{-1} = 2P_0 - P_1$ and $P_{n+1} = 2P_n - P_{n-1}$

Example: use control points
$\{(0,0), (0,1), (1,1), (2,1), (2,0)\}$.
(i) Draw the first segment.
(ii) Add P_{-1} to fix beginning of curve at P_0.
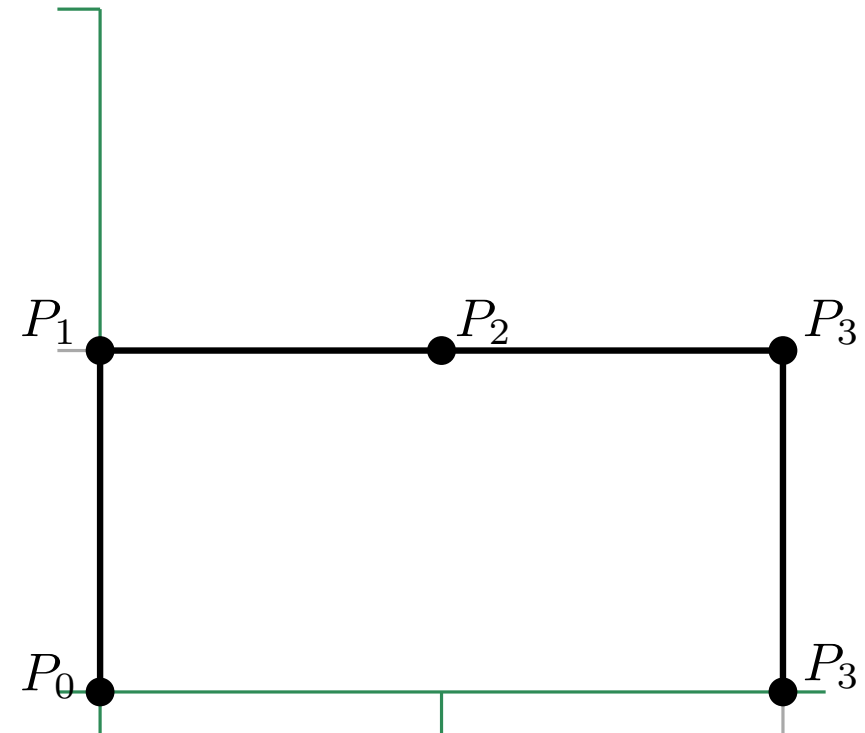
Making the curve go from P_0 to P_n

How can we force the curve go through P_0 and P_n?

Answer: add *dummy* control points P_{-1} and P_{n+1}

Question: what are P_{-1} and P_{n+1} points exactly?

$P_{-1} = 2P_0 - P_1$ and $P_{n+1} = 2P_n - P_{n-1}$

Example: use control points
$\{(0,0),(0,1),(1,1),(2,1),(2,0)\}$.
(i) Draw the first segment.
(ii) Add P_{-1} to fix beginning of curve at P_0.

You can choose the order (=degree+1)

You can choose the order (=degree+1)

As opposed to Bézier curves, in B-splines the same $n + 1$ points can be used to construct curve of order 2 (linear), 3 (quadratic), 4 (cubic) ...

You can choose the order (=degree+1)

As opposed to Bézier curves, in B-splines the same $n+1$ points can be used to construct curve of order 2 (linear), 3 (quadratic), 4 (cubic) ...

Example: use control points $\{(0,0), (0,1), (1,1), (2,1), (2,0)\}$.

$\longrightarrow$

How would a quadratic B-spline curve look like?

You can choose the order (=degree+1)

As opposed to Bézier curves, in B-splines the same $n + 1$ points can be used to construct curve of order 2 (linear), 3 (quadratic), 4 (cubic) …

Example: use control points $\{(0,0), (0,1), (1,1), (2,1), (2,0)\}$.

How would a quadratic B-spline curve look like?

You can choose the order (=degree+1)

As opposed to Bézier curves, in B-splines the same $n + 1$ points can be used to construct curve of order 2 (linear), 3 (quadratic), 4 (cubic) ...

Example: use control points $\{(0, 0), (0, 1), (1, 1), (2, 1), (2, 0)\}$.

How would a quadratic B-spline curve look like?

You can choose the order (=degree+1)

As opposed to Bézier curves, in B-splines the same $n+1$ points can be used to construct curve of order 2 (linear), 3 (quadratic), 4 (cubic) ...

Example: use control points
$\{(0,0),(0,1),(1,1),(2,1),(2,0)\}$.
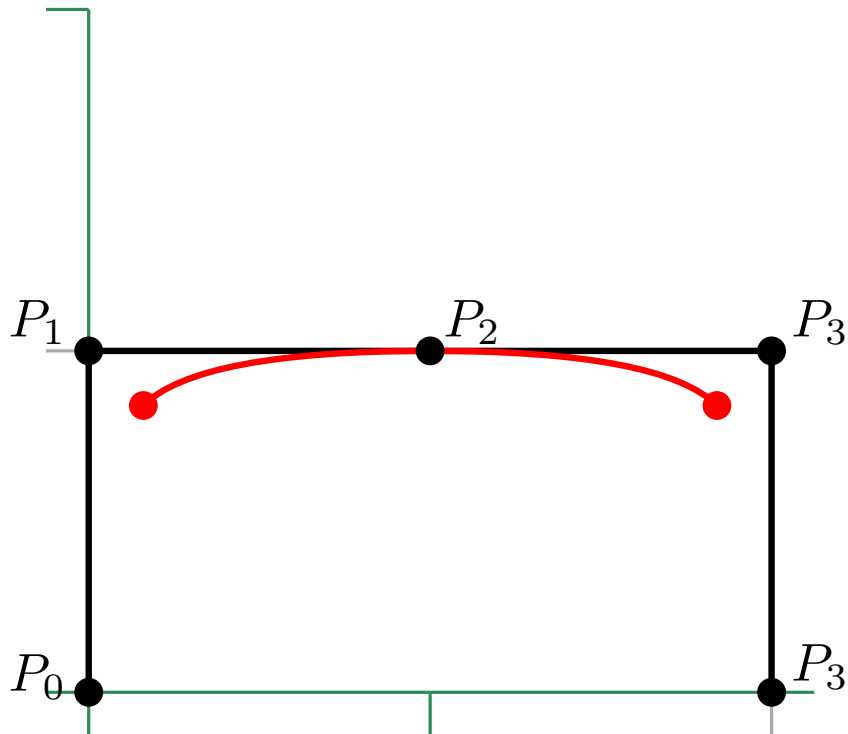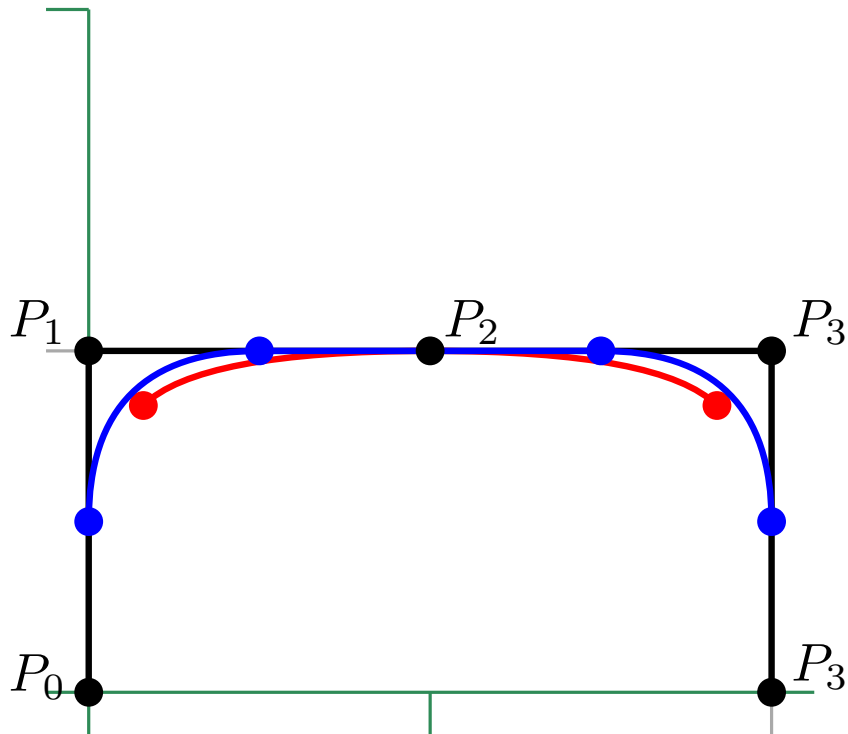
How would a quadratic B-spline curve look like?
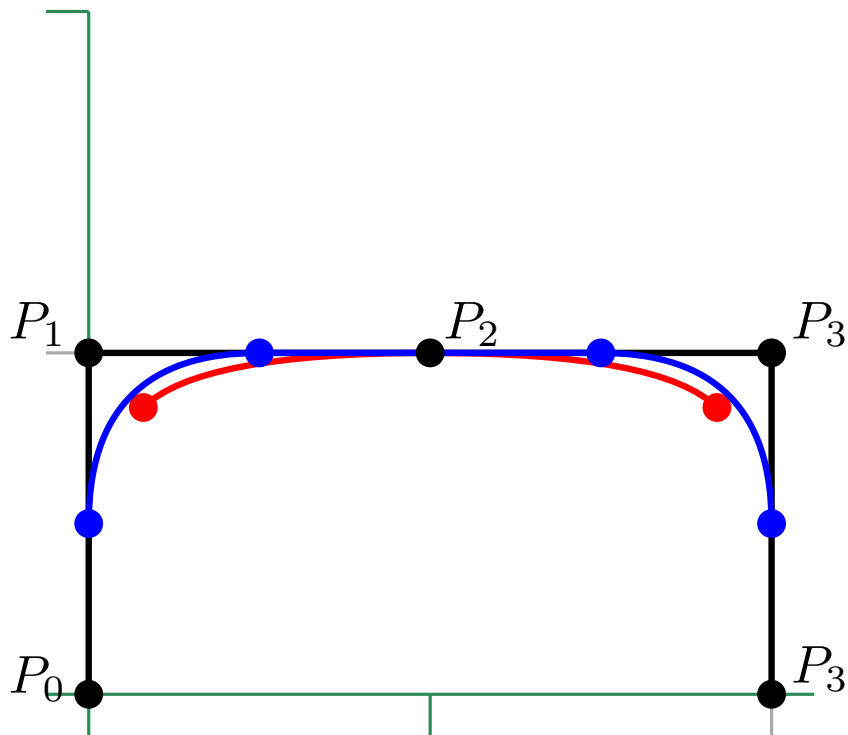
Quadratic

3 segments

Cubic

2 segments

Higher order, better fit
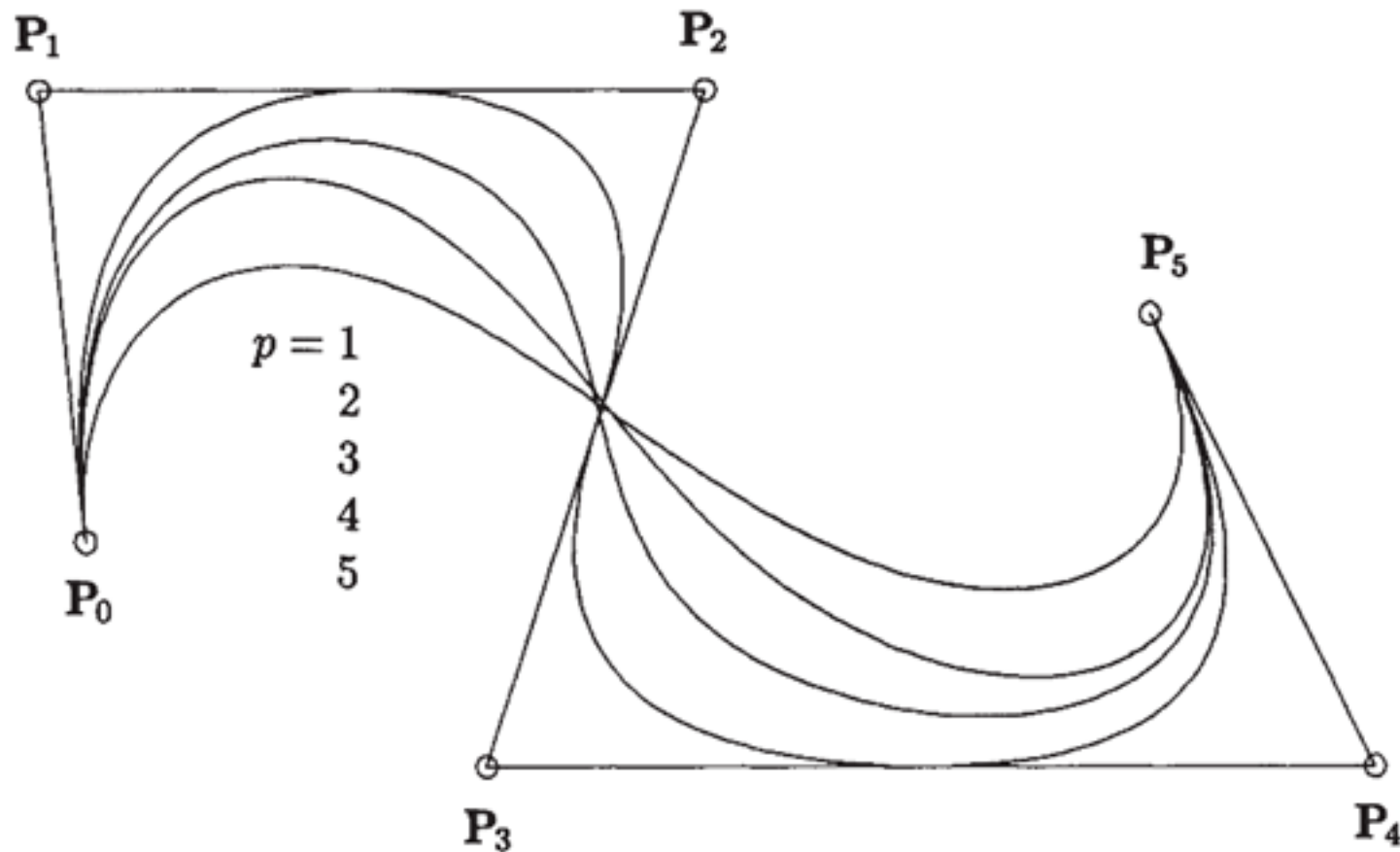
Figure 3.9. B-spline curves of different degree, using the same control polygon.

Where p is the degree of curve (i.e., $p = k - 1$)   Figure from [Piegl and Tiller]

From a cubic B-Spline to a cubic Bézier curve

From a cubic B-Spline to a cubic Bézier curve

Given a cubic B-spline segment $P(t)$ based on control points $P_0, \cdots, P_3$, how can we find a cubic Bézier curve $C(t)$ with the same shape?

From a cubic B-Spline to a cubic Bézier curve

Given a cubic B-spline segment $P(t)$ based on control points $P_0, \cdots, P_3$, how can we find a cubic Bézier curve $C(t)$ with the same shape?

$$P(t) = \frac{1}{6}(t^3, t^2, t, 1) \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix} \begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix}$$

Matrix formulation of cubic B-spline

From a cubic B-Spline to a cubic Bézier curve

Given a cubic B-spline segment $P(t)$ based on control points $P_0, \cdots, P_3$, how can we find a cubic Bézier curve $C(t)$ with the same shape?

$$P(t) = \frac{1}{6}(t^3, t^2, t, 1) \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix} \begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix}$$

Matrix formulation of cubic B-spline

$$C(t) = (t^3, t^2, t, 1) \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} Q_0 \\ Q_1 \\ Q_2 \\ Q_3 \end{pmatrix}$$

Matrix formulation of cubic Bézier

From a cubic B-Spline to a cubic Bézier curve

Given a cubic B-spline segment $P(t)$ based on control points $P_0, \cdots, P_3$, how can we find a cubic Bézier curve $C(t)$ with the same shape?

$$P(t) = \frac{1}{6}(t^3, t^2, t, 1) \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix} \begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix} \qquad C(t) = (t^3, t^2, t, 1) \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} Q_0 \\ Q_1 \\ Q_2 \\ Q_3 \end{pmatrix}$$

Matrix formulation of cubic B-spline   Matrix formulation of cubic Bézier

Unknowns

From a cubic B-Spline to a cubic Bézier curve

Given a cubic B-spline segment $P(t)$ based on control points $P_0, \cdots, P_3$, how can we find a cubic Bézier curve $C(t)$ with the same shape?

$$P(t) = \frac{1}{6}(t^3, t^2, t, 1) \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix} \begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix}$$

Matrix formulation of cubic B-spline

$$C(t) = (t^3, t^2, t, 1) \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} Q_0 \\ Q_1 \\ Q_2 \\ Q_3 \end{pmatrix}$$

Matrix formulation of cubic Bézier

Unknowns

Question: what is the solution to the problem?

From a cubic B-Spline to a cubic Bézier curve

Given a cubic B-spline segment $P(t)$ based on control points $P_0, \cdots, P_3$, how can we find a cubic Bézier curve $C(t)$ with the same shape?

$$P(t) = \frac{1}{6}(t^3, t^2, t, 1) \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix} \begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix}$$

Matrix formulation of cubic B-spline

$$C(t) = (t^3, t^2, t, 1) \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} Q_0 \\ Q_1 \\ Q_2 \\ Q_3 \end{pmatrix}$$

Matrix formulation of cubic Bézier

Unknowns

Question: what is the solution to the problem?

Solution: Solve $P(t) = C(t)$ for the unknowns $Q_0, \ldots, Q_3$

From a cubic B-Spline to a cubic Bézier curve

Given a cubic B-spline segment $P(t)$ based on control points $P_0, \cdots, P_3$, how can we find a cubic Bézier curve $C(t)$ with the same shape?

$$P(t) = \frac{1}{6}(t^3, t^2, t, 1) \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix} \begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix}$$

Matrix formulation of cubic B-spline

$$C(t) = (t^3, t^2, t, 1) \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} Q_0 \\ Q_1 \\ Q_2 \\ Q_3 \end{pmatrix}$$

Matrix formulation of cubic Bézier

Unknowns

Question: what is the solution to the problem?

Solution: Solve $P(t) = C(t)$ for the unknowns $Q_0, \ldots, Q_3$

Result:
$Q_0 = \frac{1}{6}(P_0 + 4P_1 + P_2)$,
$Q_1 = \frac{1}{6}(4P_1 + 2P_2)$,
$Q_2 = \frac{1}{6}(2P_1 + 4P_2)$,
$Q_3 = \frac{1}{6}(P_1 + 4P_2 + P_3)$

B-splines of order higher than four

B-splines of order higher than four

The formulas for quadratic and cubic uniform B-splines generalize to arbitrary order

B-splines of order higher than four

The formulas for quadratic and cubic uniform B-splines generalize to arbitrary order

$$P_i(t) = \frac{1}{6}(t^3, t^2, t, 1) \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix} \begin{pmatrix} P_{i-1} \\ P_i \\ P_{i+1} \\ P_{i+2} \end{pmatrix}$$

Matrix formulation of cubic
B-spline segment $P_i(t)$

B-splines of order higher than four

The formulas for quadratic and cubic uniform B-splines generalize to arbitrary order

$$P_i(t) = \frac{1}{6}(t^3, t^2, t, 1) \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix} \begin{pmatrix} P_{i-1} \\ P_i \\ P_{i+1} \\ P_{i+2} \end{pmatrix}$$

Matrix formulation of cubic B-spline segment $P_i(t)$

$$P(t) = (t^n, \ldots, t^2, t, 1) M_n \begin{pmatrix} P_{i-1} \\ P_i \\ P_{i+1} \\ \vdots \\ P_{i+n-1} \end{pmatrix}$$

Matrix formulation of degree-n B-spline segment $P_i(t)$

B-splines of order higher than four

The formulas for quadratic and cubic uniform B-splines generalize to arbitrary order

$$P_i(t) = \frac{1}{6}(t^3, t^2, t, 1) \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix} \begin{pmatrix} P_{i-1} \\ P_i \\ P_{i+1} \\ P_{i+2} \end{pmatrix}$$

Matrix formulation of cubic
B-spline segment $P_i(t)$

$$P(t) = (t^n, \ldots, t^2, t, 1) M_n \begin{pmatrix} P_{i-1} \\ P_i \\ P_{i+1} \\ \vdots \\ P_{i+n-1} \end{pmatrix}$$

Matrix formulation of degree-n
B-spline segment $P_i(t)$

$$m_{ij} = \frac{1}{n!}\binom{n}{i}\sum_{k=j}^{n}(n-k)^i(-1)^{k-j}\binom{n+1}{k-j}$$

B-splines of order higher than four

The formulas for quadratic and cubic uniform B-splines generalize to arbitrary order

$$P_i(t) = \frac{1}{6}(t^3, t^2, t, 1) \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix} \begin{pmatrix} P_{i-1} \\ P_i \\ P_{i+1} \\ P_{i+2} \end{pmatrix}$$

$$P(t) = (t^n, \ldots, t^2, t, 1) M_n \begin{pmatrix} P_{i-1} \\ P_i \\ P_{i+1} \\ \vdots \\ P_{i+n-1} \end{pmatrix}$$

Matrix for
B-spline s

$$m_{ij} = \frac{1}{n!}\binom{n}{i}\sum_{k=j}^{n}(n-k)^i(-1)^{k-j}\binom{n+1}{k-j}$$

$$M_3 = \frac{1}{6}\begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix}$$

$$M_4 = \frac{1}{24}\begin{pmatrix} -1 & -4 & 6 & -4 & 1 \\ -4 & 12 & -12 & 4 & 0 \\ 6 & -6 & -6 & 6 & 0 \\ -4 & -12 & 12 & 4 & 0 \\ 1 & 11 & 11 & 1 & 0 \end{pmatrix}$$

$$M_5 = \frac{1}{120}\begin{pmatrix} -1 & 5 & -10 & 10 & -5 & 1 \\ 5 & -20 & 30 & -20 & 5 & 0 \\ -10 & 20 & 0 & -20 & 10 & 0 \\ 10 & 20 & -60 & 20 & 10 & 0 \\ -5 & -50 & 0 & 50 & 5 & 0 \\ 1 & 26 & 66 & 26 & 1 & 0 \end{pmatrix}$$

How to build an interpolating cubic B-spline curve

How to build an interpolating cubic B-spline curve

B-spline curve approximates control points. How can we make it interpolate?

How to build an interpolating cubic B-spline curve

B-spline curve approximates control points. How can we make it interpolate?

B-spline curve: given $n + 1$ given control points, produces curve through $n - 1$ joints K_j

How to build an interpolating cubic B-spline curve

B-spline curve approximates control points. How can we make it interpolate?

B-spline curve: given $n + 1$ given control points, produces curve through $n - 1$ joints K_j

We want: given $n + 1$ data points $K_0, \ldots, K_n$, produce n-segment curve **through them**

How to build an interpolating cubic B-spline curve

B-spline curve approximates control points. How can we make it interpolate?

B-spline curve: given $n+1$ given control points, produces curve through $n-1$ joints K_j

We want: given $n+1$ data points $K_0, \ldots, K_n$, produce $\boxed{n\text{-segment curve}}$ **through them**

To produce n segments, a <u>cubic</u> B-spline requires $n+3$ control points, $P_{-1}, \ldots, P_{n+1}$

How to build an interpolating cubic B-spline curve

Unknowns

B-spline curve approximates control points. How can we make it interpolate?

B-spline curve: given $n+1$ given control points, produces curve through $n-1$ joints K_j

We want: given $n+1$ data points $K_0, \ldots, K_n$, produce n-segment curve **through them**

To produce n segments, a cubic B-spline requires $n+3$ control points, $P_{-1}, \ldots, P_{n+1}$

How to build an interpolating cubic B-spline curve Unknowns

B-spline curve approximates control points. How can we make it interpolate?

B-spline curve: given $n + 1$ given control points, produces curve through $n - 1$ joints K_j

We want: given $n + 1$ data points $K_0, \ldots, K_n$, produce $\boxed{n\text{-segment curve}}$ **through them**

To produce n segments, a <u>cubic</u> B-spline requires $\boxed{n + 3 \text{ control points, } P_{-1}, \ldots, P_{n+1}}$

Recall formula of cubic B-spline:

$$P_i(t) = \frac{1}{6}(-t^3+3t^2-3t+1)P_{i-1}+\frac{1}{6}(3t^3-6t^2+4)P_i+\frac{1}{6}(-3t^3+3t^2+3t+1)P_{i+1}+\frac{t^3}{6}P_{i+2}$$

How to build an interpolating cubic B-spline curve

Unknowns

B-spline curve approximates control points. How can we make it interpolate?

B-spline curve: given $n+1$ given control points, produces curve through $n-1$ joints K_j

We want: given $n+1$ data points $K_0, \ldots, K_n$, produce $\boxed{n\text{-segment curve}}$ **through them**

To produce n segments, a <u>cubic</u> B-spline requires $\boxed{n+3 \text{ control points, } P_{-1}, \ldots, P_{n+1}}$

Recall formula of cubic B-spline:

$$P_i(t) = \frac{1}{6}(-t^3+3t^2-3t+1)P_{i-1}+\frac{1}{6}(3t^3-6t^2+4)P_i+\frac{1}{6}(-3t^3+3t^2+3t+1)P_{i+1}+\frac{t^3}{6}P_{i+2}$$

$$P_i(1) = \frac{1}{6}(P_i + 4P_{i+1} + P_{i+2})$$

How to build an interpolating cubic B-spline curve

Unknowns

B-spline curve approximates control points. How can we make it interpolate?

B-spline curve: given $n+1$ given control points, produces curve through $n-1$ joints K_j

We want: given $n+1$ data points $K_0, \ldots, K_n$, produce $\boxed{n\text{-segment curve}}$ **through them**

To produce n segments, a <u>cubic</u> B-spline requires $\boxed{n+3 \text{ control points, } P_{-1}, \ldots, P_{n+1}}$

Recall formula of cubic B-spline:

$$P_i(t) = \frac{1}{6}(-t^3+3t^2-3t+1)P_{i-1}+\frac{1}{6}(3t^3-6t^2+4)P_i+\frac{1}{6}(-3t^3+3t^2+3t+1)P_{i+1}+\frac{t^3}{6}P_{i+2}$$

$$P_i(1) = \frac{1}{6}(P_i + 4P_{i+1} + P_{i+2}) = K_{i+1}$$

we require this

How to build an interpolating cubic B-spline curve

Unknowns

B-spline curve approximates control points. How can we make it interpolate?

B-spline curve: given $n+1$ given control points, produces curve through $n-1$ joints K_j

We want: given $n+1$ data points $K_0, \ldots, K_n$, produce $\boxed{n\text{-segment curve}}$ **through them**

To produce n segments, a <u>cubic</u> B-spline requires $\boxed{n+3 \text{ control points, } P_{-1}, \ldots, P_{n+1}}$

Recall formula of cubic B-spline:

$$P_i(t) = \frac{1}{6}(-t^3 + 3t^2 - 3t + 1)P_{i-1} + \frac{1}{6}(3t^3 - 6t^2 + 4)P_i + \frac{1}{6}(-3t^3 + 3t^2 + 3t + 1)P_{i+1} + \frac{t^3}{6}P_{i+2}$$

$$P_i(1) = \frac{1}{6}(P_i + 4P_{i+1} + P_{i+2}) = K_{i+1}$$

we require this → this gives one equation for each K_j ($n+1$ in total)

How to build an interpolating cubic B-spline curve Unknowns

B-spline curve approximates control points. How can we make it interpolate?

B-spline curve: given $n+1$ given control points, produces curve through $n-1$ joints K_j

We want: given $n+1$ data points $K_0, \ldots, K_n$, produce $\boxed{n\text{-segment curve}}$ **through them**

To produce n segments, a <u>cubic</u> B-spline requires $\boxed{n+3 \text{ control points, } P_{-1}, \ldots, P_{n+1}}$

Recall formula of cubic B-spline:

$$P_i(t) = \frac{1}{6}(-t^3+3t^2-3t+1)P_{i-1}+\frac{1}{6}(3t^3-6t^2+4)P_i+\frac{1}{6}(-3t^3+3t^2+3t+1)P_{i+1}+\frac{t^3}{6}P_{i+2}$$

$$P_i(1) = \frac{1}{6}(P_i + 4P_{i+1} + P_{i+2}) = K_{i+1}$$

we require this

this gives one equation for each K_j ($n+1$ in total)

We need $n+3$ equations, so we need two more: we fix extreme tangent vectors T_1 and T_n

How to build an interpolating cubic B-spline curve

Unknowns

B-spline curve approximates control points. How can we make it interpolate?

B-spline curve: given $n+1$ given control points, produces curve through $n-1$ joints K_j

We want: given $n+1$ data points $K_0, \ldots, K_n$, produce $\boxed{n\text{-segment curve}}$ **through them**

To produce n segments, a <u>cubic</u> B-spline requires $\boxed{n+3 \text{ control points, } P_{-1}, \ldots, P_{n+1}}$

Recall formula of cubic B-spline:

$$P_i(t) = \frac{1}{6}(-t^3+3t^2-3t+1)P_{i-1}+\frac{1}{6}(3t^3-6t^2+4)P_i+\frac{1}{6}(-3t^3+3t^2+3t+1)P_{i+1}+\frac{t^3}{6}P_{i+2}$$

$$P_i(1) = \frac{1}{6}(P_i + 4P_{i+1} + P_{i+2}) = K_{i+1}$$

we require this $\longrightarrow$

this gives one equation for each K_j ($n+1$ in total)

We need $n+3$ equations, so we need two more: we fix extreme tangent vectors T_1 and T_n

$$P_0'(0) = \frac{1}{2}(P_1 - P_{-1})$$

$$P_n'(1) = \frac{1}{2}(P_{n+1} - P_{n-1})$$

How to build an interpolating cubic B-spline curve Unknowns

B-spline curve approximates control points. How can we make it interpolate?

B-spline curve: given $n+1$ given control points, produces curve through $n-1$ joints K_j

We want: given $n+1$ data points $K_0, \ldots, K_n$, produce $\boxed{n\text{-segment curve}}$ **through them**

To produce n segments, a <u>cubic</u> B-spline requires $\boxed{n+3 \text{ control points, } P_{-1}, \ldots, P_{n+1}}$

Recall formula of cubic B-spline:

$$P_i(t) = \frac{1}{6}(-t^3+3t^2-3t+1)P_{i-1}+\frac{1}{6}(3t^3-6t^2+4)P_i+\frac{1}{6}(-3t^3+3t^2+3t+1)P_{i+1}+\frac{t^3}{6}P_{i+2}$$

$$P_i(1) = \frac{1}{6}(P_i + 4P_{i+1} + P_{i+2}) = K_{i+1}$$

we require this — this gives one equation for each K_j ($n+1$ in total)

We need $n+3$ equations, so we need two more: we fix extreme tangent vectors T_1 and T_n

$$P_0'(0) = \frac{1}{2}(P_1 - P_{-1}) = T_1$$

$$P_n'(1) = \frac{1}{2}(P_{n+1} - P_{n-1}) = T_n$$

two more equations (T_1 and T_2 given by user)

How to build an interpolating cubic B-spline curve Unknowns

B-spline curve approximates control points. How can we make it interpolate?

B-spline curve: given $n+1$ given control points, produces curve through $n-1$ joints K_j

We want: given $n+1$ data points $K_0, \ldots, K_n$, produce $\boxed{n\text{-segment curve}}$ **through them**

To produce n segments, a <u>cubic</u> B-spline requires $\boxed{n+3 \text{ control points}, P_{-1}, \ldots, P_{n+1}}$

Recall formula of cubic B-spline:

$$P_i(t) = \frac{1}{6}(-t^3+3t^2-3t+1)P_{i-1} + \frac{1}{6}(3t^3-6t^2+4)P_i + \frac{1}{6}(-3t^3+3t^2+3t+1)P_{i+1} + \frac{t^3}{6}P_{i+2}$$

$$P_i(1) = \frac{1}{6}(P_i + 4P_{i+1} + P_{i+2}) = K_{i+1}$$

we require this $\longrightarrow$ this gives one equation for each K_j ($n+1$ in total)

We need $n+3$ equations, so we need two more: we fix extreme tangent vectors T_1 and T_n

$$P'_0(0) = \frac{1}{2}(P_1 - P_{-1}) = T_1$$

$$P'_n(1) = \frac{1}{2}(P_{n+1} - P_{n-1}) = T_n$$

two more equations (T_1 and T_2 given by user)

$\boxed{\text{System with } n+3 \text{ unknowns and } n+3 \text{ equations that (one can check) is nonsingular}}$

An different way to look at B-splines

An different way to look at B-splines

Recall Bézier curves: $\quad P(t) = \sum_{i=0}^{n} P_i B_{n,i}(t) \quad t \in [0,1]$

An different way to look at B-splines

Recall Bézier curves: $\quad P(t) = \sum_{i=0}^{n} P_i B_{n,i}(t) \quad t \in [0,1]$

We can do the same with B-splines!   Approach based on using a *knot vector*

An different way to look at B-splines

Recall Bézier curves: $P(t) = \sum_{i=0}^{n} P_i B_{n,i}(t) \quad t \in [0, 1]$

We can do the same with B-splines!   Approach based on using a *knot vector*

For now, we consider **cubic uniform B-splines** (order 4)
                                            Example: 5 control points $\rightarrow$ two cubic segments

An different way to look at B-splines

Recall Bézier curves: $P(t) = \sum_{i=0}^{n} P_i B_{n,i}(t) \quad t \in [0, 1]$

We can do the same with B-splines! Approach based on using a *knot vector*

For now, we consider **cubic uniform B-splines** (order 4)

Example: 5 control points $\rightarrow$ two cubic segments

cubic = order 4

We need to find five weight (basis) functions $B_{4,0}(t), \ldots, B_{4,4}(t)$

- In this approach we assume each cubic segment is defined for one interval of one unit $[u, u + 1]$

- Each integer value u is called a *knot* (the sequence of knot values is called *knot vector*)

- In *uniform* B-splines, knots are equally spaced, e.g., $0, 1, 2, \ldots$

An different way to look at B-splines

Recall Bézier curves: $P(t) = \sum_{i=0}^{n} P_i B_{n,i}(t) \quad t \in [0,1]$

We can do the same with B-splines! Approach based on using a *knot vector*

For now, we consider **cubic uniform B-splines** (order 4)

Example: 5 control points $\rightarrow$ two cubic segments

cubic = order 4

We need to find five weight (basis) functions $B_{4,0}(t), \ldots, B_{4,4}(t)$

- In this approach we assume each cubic segment is defined for one interval of one unit $[u, u+1]$

- Each integer value u is called a *knot* (the sequence of knot values is called *knot vector*)

- In *uniform* B-splines, knots are equally spaced, e.g., $0, 1, 2, \ldots$

in our example, since there are two segments, the parameter for the first one lives in $[0,1]$ and for the second one in $[1,2]$

The cubic B-spline basis functions

The cubic B-spline basis functions

Each function should:
- Be a cubic polynomial

- Have it maximum near "its" control point

- Drop to zero away from "its" control point

The cubic B-spline basis functions

Each function should:

- Be a cubic polynomial

- Have it maximum near "its" control point

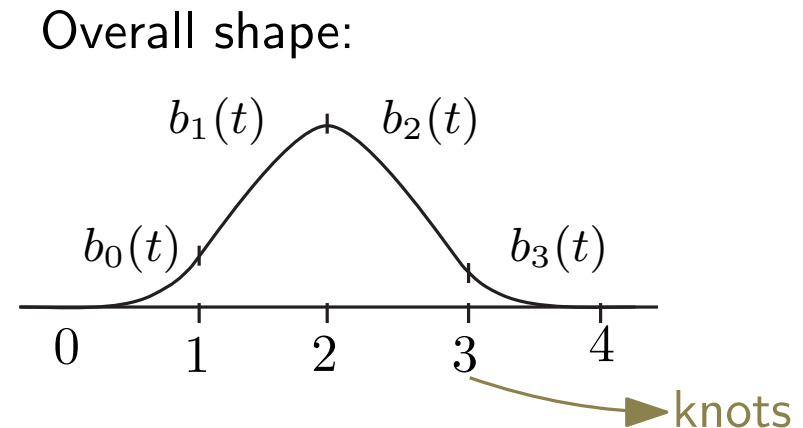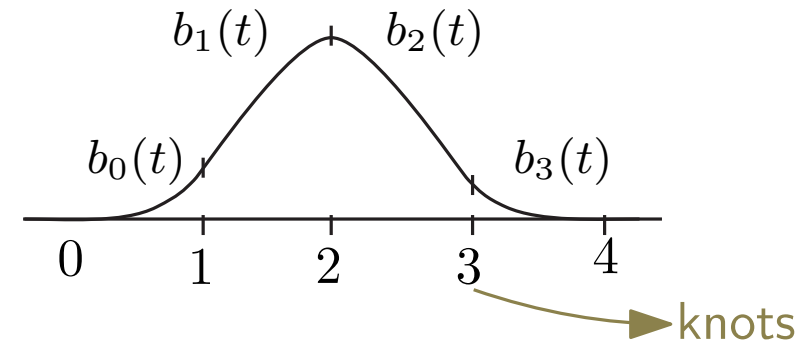- Drop to zero away from "its" control point

Overall shape:

knots

The cubic B-spline basis functions
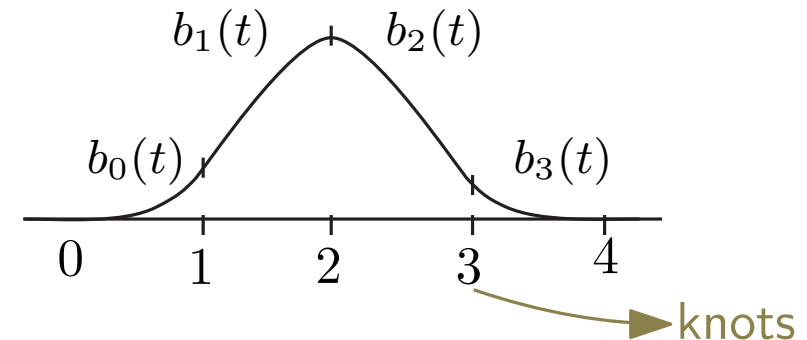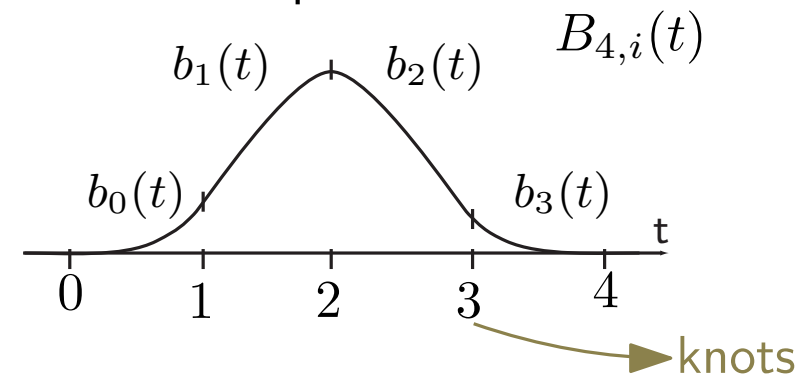
Each function should:

- Be a cubic polynomial

- Have it maximum near "its" control point

- Drop to zero away from "its" control point

We write the basis function as the union of **four parts**

Overall shape:

The cubic B-spline basis functions

Overall shape:

Each function should:

- Be a cubic polynomial

- Have it maximum near "its" control point

- Drop to zero away from "its" control point

We write the basis function as the union of **four parts** $\longrightarrow$ each $b_i(t)$ is a cubic polynomial defined over $[0, 1]$

The cubic B-spline basis functions

Each function should:
- Be a cubic polynomial

- Have it maximum near "its" control point

- Drop to zero away from "its" control point

Overall shape:

We write the basis function as the union of **four parts** $\longrightarrow$ each $b_i(t)$ is a cubic polynomial defined over $[0, 1]$

Conditions sought for the $b_i(t)$ functions:
- Affine invariant

- C^2-continuous at three joints

- $b_0(t)$, $b_0'(t)$, $b_0''(t)$ should be zero at the start point $b_0(0)$

- $b_3(t)$, $b_3'(t)$, $b_3''(t)$ should be zero at the end point $b_3(1)$

The cubic B-spline basis functions

Overall shape:

Each function should:
- Be a cubic polynomial

- Have it maximum near "its" control point

- Drop to zero away from "its" control point

We write the basis function as the union of **four parts** $\longrightarrow$ each $b_i(t)$ is a cubic polynomial defined over $[0, 1]$

Conditions sought for the $b_i(t)$ functions:
- Affine invariant

- C^2-continuous at three joints

- $b_0(t)$, $b_0'(t)$, $b_0''(t)$ should be zero at the start point $b_0(0)$

- $b_3(t)$, $b_3'(t)$, $b_3''(t)$ should be zero at the end point $b_3(1)$

How many equations on the coefficients of the $b_i(t)$ functions do we obtain from these conditions?

The cubic B-spline basis functions

Solution to the equations:

$$b_0(t) = \frac{1}{6}t^3$$

$$b_1(t) = \frac{1}{6}(1 + 3t + 3t^2 - 3t^3)$$

$$b_2(t) = \frac{1}{6}(4 - 6t^2 + 3t^3)$$

$$b_3(t) = \frac{1}{6}(1 - 3t + 3t^2 - t^3)$$

The cubic B-spline basis functions

Solution to the equations:

$$\mathbf{N_{i,4}(t)} = B_{4,i}(t) = \begin{cases} b_0(t) = \frac{1}{6}t^3 & u_{i-3} \leq t \leq u_{i-2} \\ b_1(t) = \frac{1}{6}(1 + 3t + 3t^2 - 3t^3) & u_{i-2} \leq t \leq u_{i-1} \\ b_2(t) = \frac{1}{6}(4 - 6t^2 + 3t^3) & u_{i-1} \leq t \leq u_i \\ b_3(t) = \frac{1}{6}(1 - 3t + 3t^2 - t^3) & u_i \leq t \leq u_{i+1} \end{cases}$$
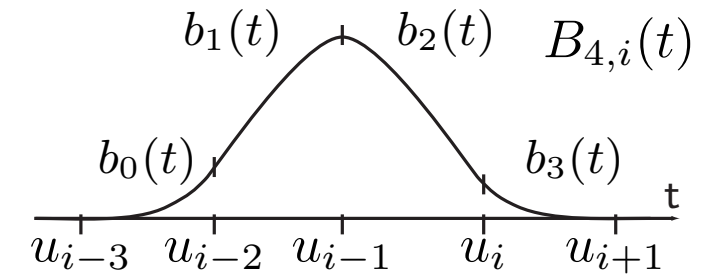

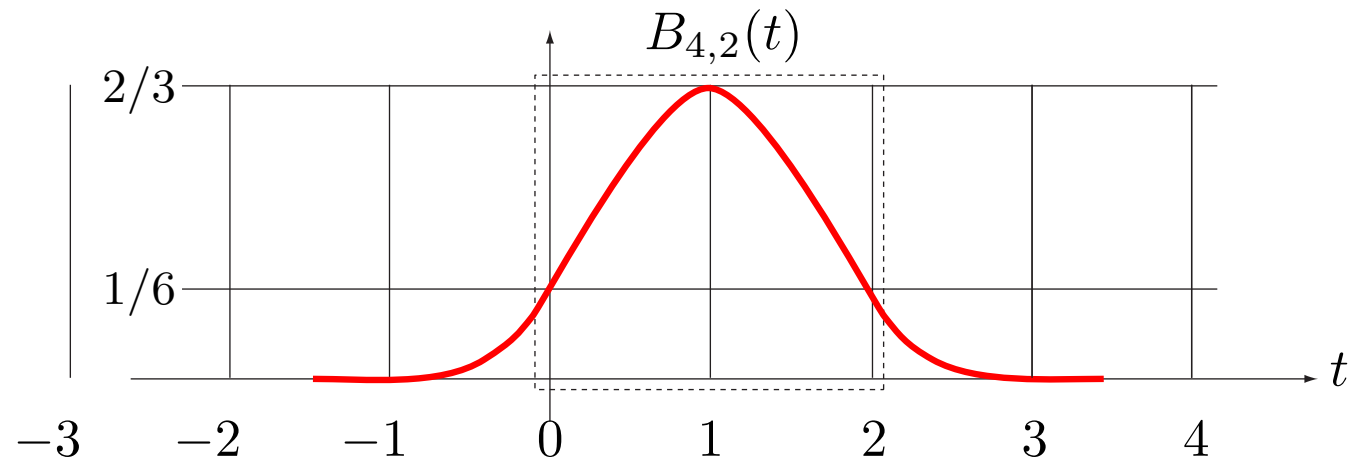Each control point P_i gets multiplied by a shifted copy of the basis function:

The cubic B-spline basis functions

Solution to the equations:

$$\mathbf{N_{i,4}(t)} = \\ B_{4,i}(t) = \begin{cases} b_0(t) = \frac{1}{6}t^3 & u_{i-3} \leq t \leq u_{i-2} \\ b_1(t) = \frac{1}{6}(1 + 3t + 3t^2 - 3t^3) & u_{i-2} \leq t \leq u_{i-1} \\ b_2(t) = \frac{1}{6}(4 - 6t^2 + 3t^3) & u_{i-1} \leq t \leq u_i \\ b_3(t) = \frac{1}{6}(1 - 3t + 3t^2 - t^3) & u_i \leq t \leq u_{i+1} \end{cases}$$

Each control point P_i gets multiplied by a shifted copy of the basis function: $P(t) = \sum_{i=0}^{4} P_i B_{4,i}(t)$
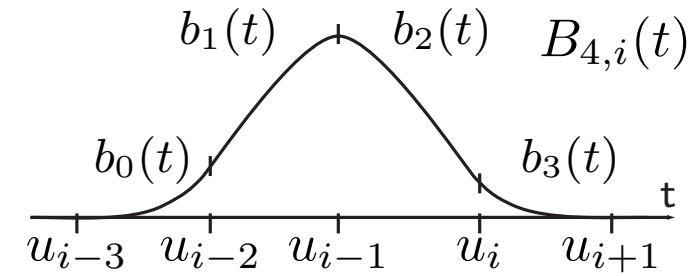
The cubic B-spline basis functions

Solution to the equations:

$$\mathbf{N_{i,4}(t)} = \\ B_{4,i}(t) = \begin{cases} b_0(t) = \frac{1}{6}t^3 & u_{i-3} \le t \le u_{i-2} \\ b_1(t) = \frac{1}{6}(1 + 3t + 3t^2 - 3t^3) & u_{i-2} \le t \le u_{i-1} \\ b_2(t) = \frac{1}{6}(4 - 6t^2 + 3t^3) & u_{i-1} \le t \le u_i \\ b_3(t) = \frac{1}{6}(1 - 3t + 3t^2 - t^3) & u_i \le t \le u_{i+1} \end{cases}$$

Each control point P_i gets multiplied by a shifted copy of the basis function: $P(t) = \sum_{i=0}^{4} P_i B_{4,i}(t)$
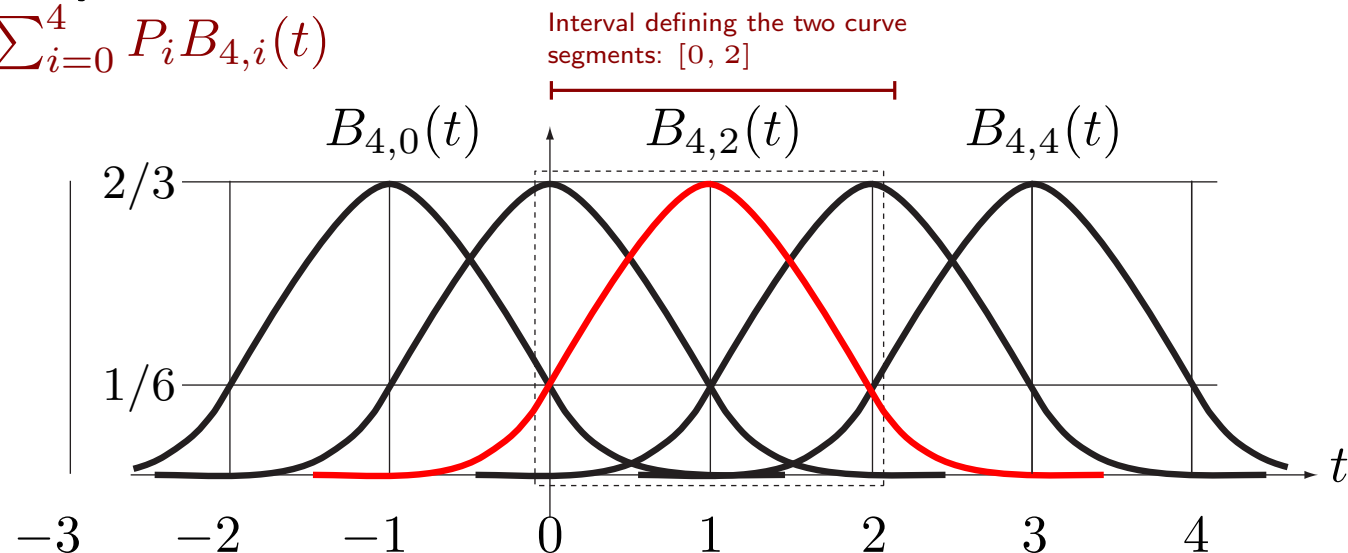
The cubic B-spline basis functions

Solution to the equations:

$$\mathbf{N_{i,4}(t)} = B_{4,i}(t) = \begin{cases} b_0(t) = \frac{1}{6}t^3 & u_{i-3} \leq t \leq u_{i-2} \\[2mm] b_1(t) = \frac{1}{6}(1 + 3t + 3t^2 - 3t^3) & u_{i-2} \leq t \leq u_{i-1} \\[2mm] b_2(t) = \frac{1}{6}(4 - 6t^2 + 3t^3) & u_{i-1} \leq t \leq u_i \\[2mm] b_3(t) = \frac{1}{6}(1 - 3t + 3t^2 - t^3) & u_i \leq t \leq u_{i+1} \end{cases}$$

Each control point P_i gets multiplied by a shifted copy of the basis function: $P(t) = \sum_{i=0}^{4} P_i B_{4,i}(t)$

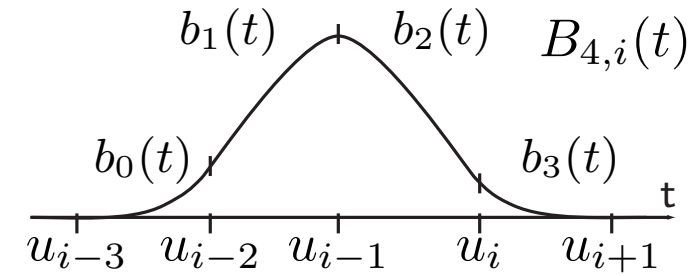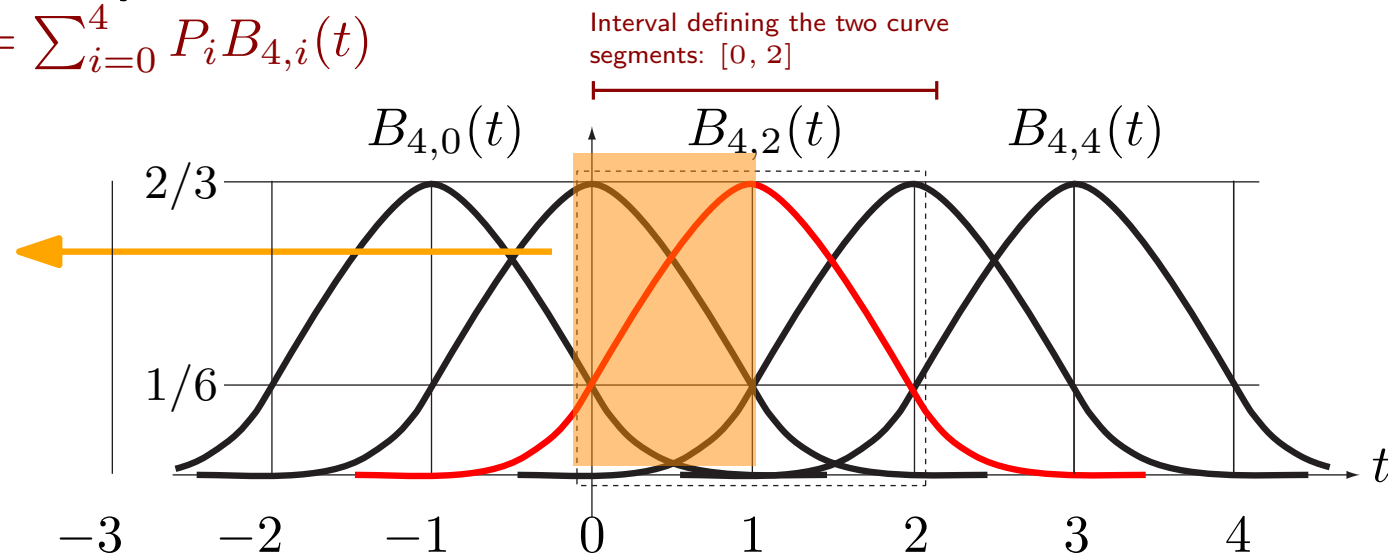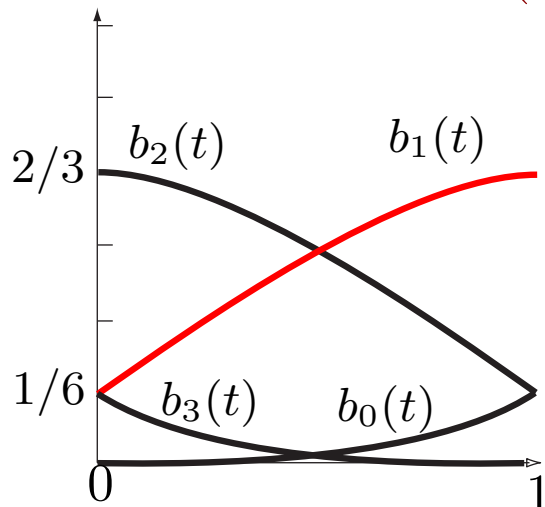Interval defining the two curve segments: $[0, 2]$

The cubic B-spline basis functions

Solution to the equations:

$$\mathbf{N_{i,4}(t)} = \\ B_{4,i}(t) = \begin{cases} b_0(t) = \frac{1}{6}t^3 & u_{i-3} \leq t \leq u_{i-2} \\[2mm] b_1(t) = \frac{1}{6}(1 + 3t + 3t^2 - 3t^3) & u_{i-2} \leq t \leq u_{i-1} \\[2mm] b_2(t) = \frac{1}{6}(4 - 6t^2 + 3t^3) & u_{i-1} \leq t \leq u_i \\[2mm] b_3(t) = \frac{1}{6}(1 - 3t + 3t^2 - t^3) & u_i \leq t \leq u_{i+1} \end{cases}$$

Each control point P_i gets multiplied by a shifted copy of the basis function: $P(t) = \sum_{i=0}^{4} P_i B_{4,i}(t)$

The cubic B-spline basis functions

Solution to the equations:

$$\mathbf{N_{i,4}(t)} = B_{4,i}(t) = \begin{cases} b_0(t) = \frac{1}{6}t^3 & u_{i-3} \le t \le u_{i-2} \\ b_1(t) = \frac{1}{6}(1 + 3t + 3t^2 - 3t^3) & u_{i-2} \le t \le u_{i-1} \\ b_2(t) = \frac{1}{6}(4 - 6t^2 + 3t^3) & u_{i-1} \le t \le u_i \\ b_3(t) = \frac{1}{6}(1 - 3t + 3t^2 - t^3) & u_i \le t \le u_{i+1} \end{cases}$$

Each control point P_i gets multiplied by a shifted copy of the basis function: $P(t) = \sum_{i=0}^{4} P_i B_{4,i}(t)$
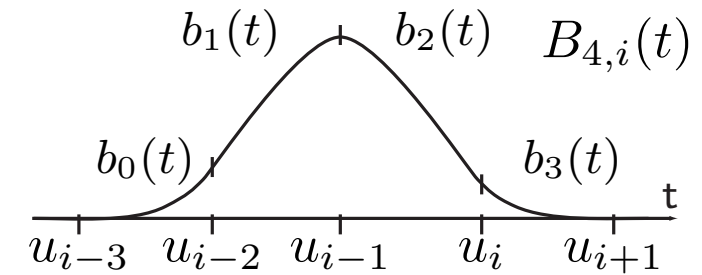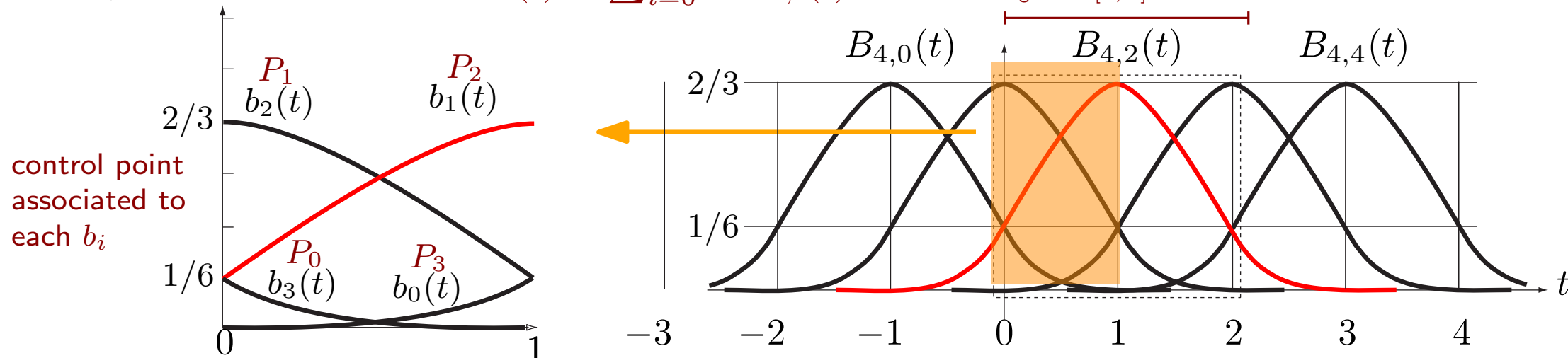
control point associated to each b_i

Interval defining the two curve segments: [0, 2]

General B-spline basis functions

This can be generalized to $(n+1)$ control points and order k as follows:

General B-spline basis functions

This can be generalized to $(n+1)$ control points and order k as follows:

$$P(t) = \sum_{i=0}^{n} P_i N_{i,k}(t)$$

General B-spline basis functions

This can be generalized to $(n+1)$ control points and order k as follows:

$$P(t) = \sum_{i=0}^{n} P_i \boxed{N_{i,k}(t)} \longrightarrow \text{B-spline basis function}$$

General B-spline basis functions

This can be generalized to $(n+1)$ control points and order k as follows:

$$P(t) = \sum_{i=0}^{n} P_i \boxed{N_{i,k}(t)} \longrightarrow \text{B-spline basis function}$$

Given: control points $P_0, \ldots, P_n$, knots: $t_0 \leq t_1 \leq \cdots \leq t_{n+k}$, order: k

General B-spline basis functions

This can be generalized to $(n+1)$ control points and order k as follows:

$$P(t) = \sum_{i=0}^{n} P_i \boxed{N_{i,k}(t)} \longrightarrow \text{B-spline basis function}$$

Given: control points $P_0, \ldots, P_n$, knots: $t_0 \leq t_1 \leq \cdots \leq t_{n+k}$, order: k

Note that #knots is # control points + order: $n + k + 1$ knots

General B-spline basis functions

This can be generalized to $(n + 1)$ control points and order k as follows:

$$P(t) = \sum_{i=0}^{n} P_i \boxed{N_{i,k}(t)} \longrightarrow \text{B-spline basis function}$$

Given: control points $P_0, \ldots, P_n$, knots: $t_0 \leq t_1 \leq \cdots \leq t_{n+k}$, order: k

Order 1 ($k = 1$, degree 0)

Note that #knots is # control points + order: $n + k + 1$ knots

$$N_{i,1}(t) = \begin{cases} 1, & \text{if } t \in [t_i, t_{i+1}) \\ 0, & \text{otherwise} \end{cases}$$

General B-spline basis functions

This can be generalized to $(n+1)$ control points and order k as follows:
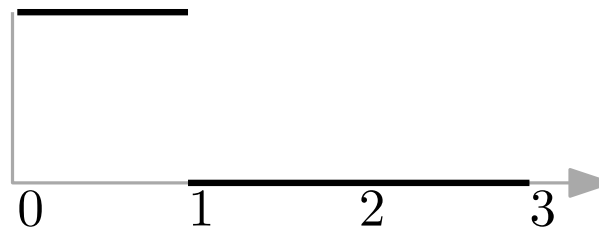
$$P(t) = \sum_{i=0}^{n} P_i \boxed{N_{i,k}(t)} \longrightarrow \text{B-spline basis function}$$

Given: control points $P_0, \ldots, P_n$, knots: $t_0 \le t_1 \le \cdots \le t_{n+k}$, order: k

Order 1 ($k = 1$, degree 0)

Note that #knots is # control points + order: $n + k + 1$ knots

$$N_{i,1}(t) = \begin{cases} 1, & \text{if } t \in [t_i, t_{i+1}) \\ 0, & \text{otherwise} \end{cases}$$

$N_{0,1}(t)$

$t_i\text{s} = [0, 1, 2, 3, \ldots]$

0   1   2   3

General B-spline basis functions

This can be generalized to $(n + 1)$ control points and order k as follows:
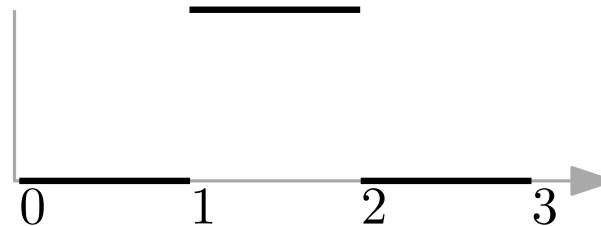
$$P(t) = \sum_{i=0}^{n} P_i \boxed{N_{i,k}(t)} \longrightarrow \text{B-spline basis function}$$

Given: control points $P_0, \ldots, P_n$, knots: $t_0 \leq t_1 \leq \cdots \leq t_{n+k}$, order: k

Note that #knots is # control points + order: $n + k + 1$ knots

Order 1 ($k = 1$, degree 0)

$$N_{i,1}(t) = \begin{cases} 1, & \text{if } t \in [t_i, t_{i+1}) \\ 0, & \text{otherwise} \end{cases}$$

$N_{1,1}(t)$

$t_i\text{s} = [0, 1, 2, 3, \ldots]$

General B-spline basis functions

This can be generalized to $(n+1)$ control points and order k as follows:
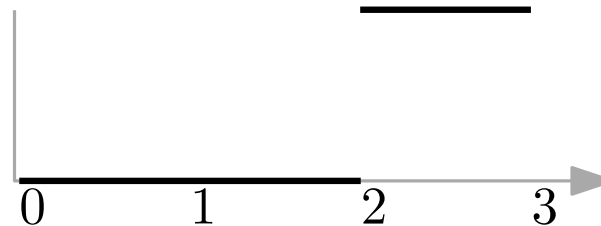
$$P(t) = \sum_{i=0}^{n} P_i \boxed{N_{i,k}(t)} \longrightarrow \text{B-spline basis function}$$

Given: control points $P_0, \ldots, P_n$, knots: $t_0 \leq t_1 \leq \cdots \leq t_{n+k}$, order: k

Note that #knots is # control points + order: $n + k + 1$ knots

Order 1 ($k = 1$, degree 0)

$$N_{i,1}(t) = \begin{cases} 1, & \text{if } t \in [t_i, t_{i+1}) \\ 0, & \text{otherwise} \end{cases}$$

$N_{2,1}(t)$

$t_i\text{s} = [0, 1, 2, 3, \ldots]$

General B-spline basis functions

This can be generalized to $(n + 1)$ control points and order k as follows:
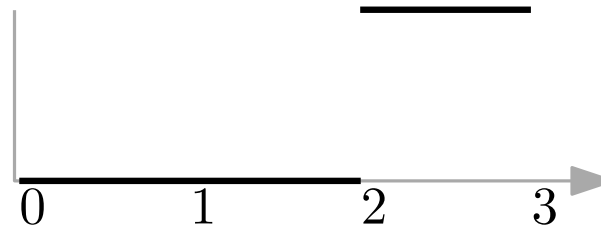
$$P(t) = \sum_{i=0}^{n} P_i \boxed{N_{i,k}(t)} \longrightarrow \text{B-spline basis function}$$

Given: control points $P_0, \ldots, P_n$, knots: $t_0 \leq t_1 \leq \cdots \leq t_{n+k}$, order: k

Order 1 ($k = 1$, degree 0)

Note that #knots is # control points + order: $n + k + 1$ knots

$$N_{i,1}(t) = \begin{cases} 1, & \text{if } t \in [t_i, t_{i+1}) \\ 0, & \text{otherwise} \end{cases}$$

$N_{2,1}(t)$

$t_i\mathsf{s} = [0, 1, 2, 3, \ldots]$

$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t)$$

(taking $0/0$ as 0)

Examples of B-spline basis functions

$$N_{i,1} = \begin{cases} 1, & \text{if } t \in [t_i, t_{i+1}) \\ 0, & \text{otherwise} \end{cases} \qquad N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t)$$

Examples of B-spline basis functions

$$N_{i,1} = \begin{cases} 1, & \text{if } t \in [t_i, t_{i+1}) \\ 0, & \text{otherwise} \end{cases} \qquad N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t)$$

Order 2 ($k = 2$, degree 1)

Question: how do the basis functions look for order 2 ($k = 2$) when knots are uniform (i.e., $t_i = i$)?

Examples of B-spline basis functions

$$N_{i,1} = \begin{cases} 1, & \text{if } t \in [t_i, t_{i+1}) \\ 0, & \text{otherwise} \end{cases} \qquad N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t)$$

Order 2 ($k = 2$, degree 1)

Question: how do the basis functions look for order 2 ($k = 2$) when knots are uniform (i.e., $t_i = i$)?

Order 3 ($k = 3$, degree 2)
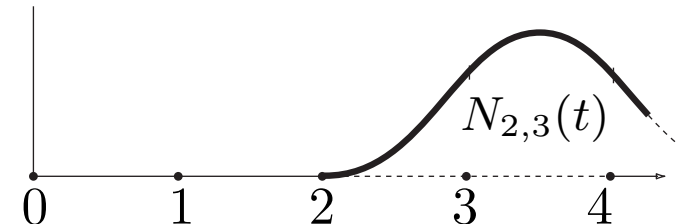
Examples of B-spline basis functions
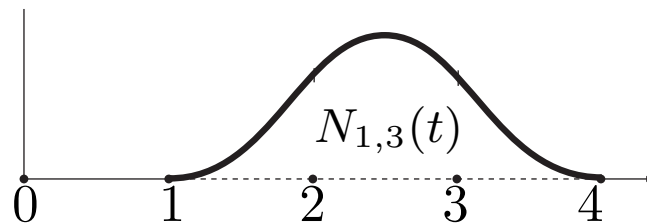
$$N_{i,1} = \begin{cases} 1, & \text{if } t \in [t_i, t_{i+1}) \\ 0, & \text{otherwise} \end{cases} \qquad N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t)$$

Order 2 ($k = 2$, degree 1)

Question: how do the basis functions look for order 2 ($k = 2$) when knots are uniform (i.e., $t_i = i$)?

Order 3 ($k = 3$, degree 2)

Some important properties

$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t)$$

Some important properties

$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t)$$

1) Shifted basis $N_{i,k}(t) = N_{0,k}(t - i)$

Some important properties

$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t)$$

1) Shifted basis $\quad N_{i,k}(t) = N_{0,k}(t - i)$

2) Local support $\quad N_{i,k}(t) \neq 0$ only for $t \in [t_i, t_{i+k})$

Some important properties

$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t)$$

1) Shifted basis $\quad N_{i,k}(t) = N_{0,k}(t - i)$

2) Local support $\quad N_{i,k}(t) \neq 0$ only for $t \in [t_i, t_{i+k})$

3) Non-zero bases for fixed t

In any interval $[t_j, t_{j+1})$, at most k of the $N_{i,k}$ are non-zero: $N_{(j-k+1),k}, \ldots, N_{j,k}$

Some important properties

$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t)$$

1) Shifted basis $\quad N_{i,k}(t) = N_{0,k}(t - i)$

2) Local support $\quad N_{i,k}(t) \neq 0$ only for $t \in [t_i, t_{i+k})$

3) Non-zero bases for fixed t

In any interval $[t_j, t_{j+1})$, at most k of the $N_{i,k}$ are non-zero: $N_{(j-k+1),k}, \dots, N_{j,k}$

4) Non-negativity $\quad N_{i,k}(t) \geq 0$ for all i, k, t

Some important properties

$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t)$$

1) Shifted basis $N_{i,k}(t) = N_{0,k}(t - i)$

2) Local support $N_{i,k}(t) \neq 0$ only for $t \in [t_i, t_{i+k})$

3) Non-zero bases for fixed t

In any interval $[t_j, t_{j+1})$, at most k of the $N_{i,k}$ are non-zero: $N_{(j-k+1),k}, \ldots, N_{j,k}$

4) Non-negativity $N_{i,k}(t) \geq 0$ for all i, k, t

5) Affine invariance

For any $t \in [t_j, t_{j+1})$, $\sum_{i=0}^{n} N_{i,k}(t) = 1$

Some important properties

$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t)$$

1) Shifted basis $N_{i,k}(t) = N_{0,k}(t - i)$

2) Local support $N_{i,k}(t) \neq 0$ only for $t \in [t_i, t_{i+k})$

3) Non-zero bases for fixed t

In any interval $[t_j, t_{j+1})$, at most k of the $N_{i,k}$ are non-zero: $N_{(j-k+1),k}, \ldots, N_{j,k}$

4) Non-negativity $N_{i,k}(t) \geq 0$ for all i, k, t

5) Affine invariance

For any $t \in [t_j, t_{j+1})$, $\sum_{i=0}^{n} N_{i,k}(t) = 1$

6) Continuity

For uniform knots, the curve and its $k - 1$ derivatives are continuous
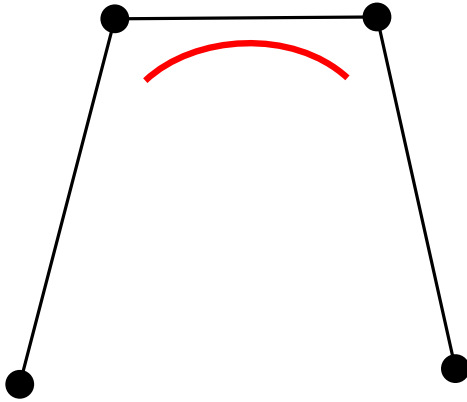(Non-uniform B-Splines can have discontinuities at knot values!)

Open (or clampled) uniform B-Splines

Open (or clampled) uniform B-Splines
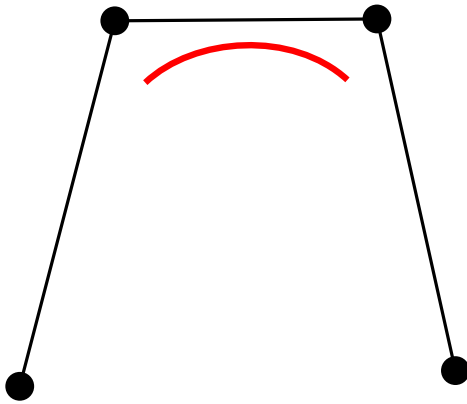
Uniform knot vector except at ends: at the beginning and end knot values are repeated k times

Open (or clampled) uniform B-Splines

Uniform knot vector except at ends: at the beginning and end knot values are repeated k times

$n = 3$ (4 control points)

$k = 4$ (cubic B-spline)

uniform knot vector: $(0, 1/7, 2/7, 3/7, 4/7, 5/7, 6/7, 1)$

Open (or clampled) uniform B-Splines

Uniform knot vector except at ends: at the beginning and end knot values are repeated k times

$n = 3$ (4 control points)

$k = 4$ (cubic B-spline)

uniform knot vector: $(0, 1/7, 2/7, 3/7, 4/7, 5/7, 6/7, 1)$

"open" knot vector: $(0, 0, 0, 0, 1, 1, 1, 1)$

Open (or clampled) uniform B-Splines

Uniform knot vector except at ends: at the beginning and end knot values are repeated k times

$n = 3$ (4 control points)

$k = 4$ (cubic B-spline)

uniform knot vector: $(0, 1/7, 2/7, 3/7, 4/7, 5/7, 6/7, 1)$

"open" knot vector: $(0, 0, 0, 0, 1, 1, 1, 1)$

Open (or clampled) uniform B-Splines

Uniform knot vector except at ends: at the beginning and end knot values are repeated k times
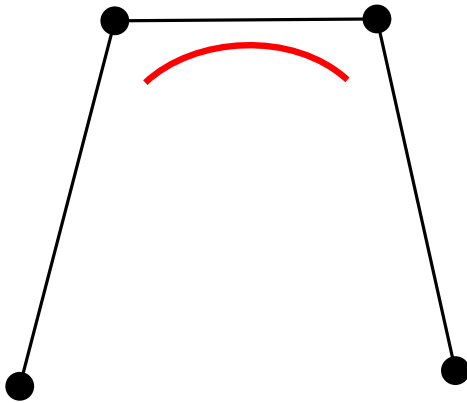

$n = 3$ (4 control points)

$k = 4$ (cubic B-spline)

uniform knot vector: $(0, 1/7, 2/7, 3/7, 4/7, 5/7, 6/7, 1)$

"open" knot vector: $(0, 0, 0, 0, 1, 1, 1, 1)$

Cubic Bézier curve?

Open (or clampled) uniform B-Splines

Uniform knot vector except at ends: at the beginning and end knot values are repeated k times

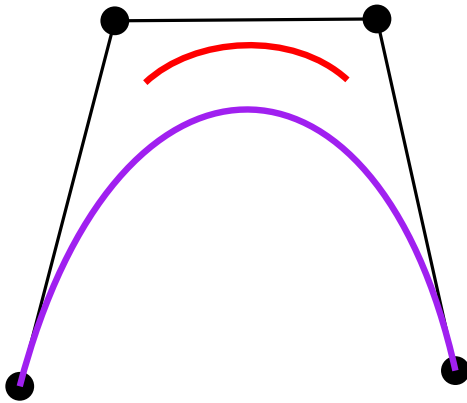$n = 3$ (4 control points)

$k = 4$ (cubic B-spline)

uniform knot vector: $(0, 1/7, 2/7, 3/7, 4/7, 5/7, 6/7, 1)$

"open" knot vector: $(0, 0, 0, 0, 1, 1, 1, 1)$

Cubic Bézier curve?

Open (or clampled) uniform B-Splines

Uniform knot vector except at ends: at the beginning and end knot values are repeated k times

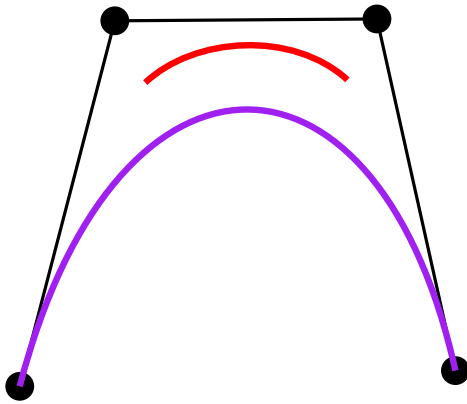$n = 3$ (4 control points)

$k = 4$ (cubic B-spline)

uniform knot vector: $(0, 1/7, 2/7, 3/7, 4/7, 5/7, 6/7, 1)$

"open" knot vector: $(0, 0, 0, 0, 1, 1, 1, 1)$

Cubic Bézier curve? $\longrightarrow$ **Always the case when** $k = n + 1$

Open (or clampled) uniform B-Splines

Uniform knot vector except at ends: at the beginning and end knot values are repeated k times
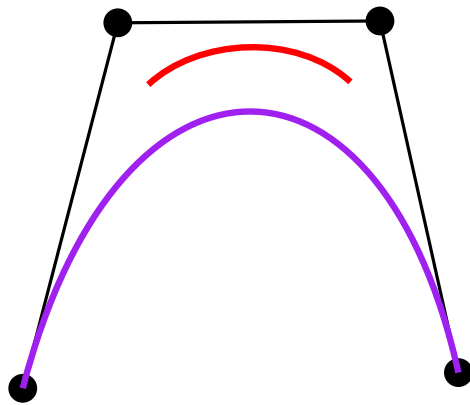
$n = 3$ (4 control points)

$k = 4$ (cubic B-spline)

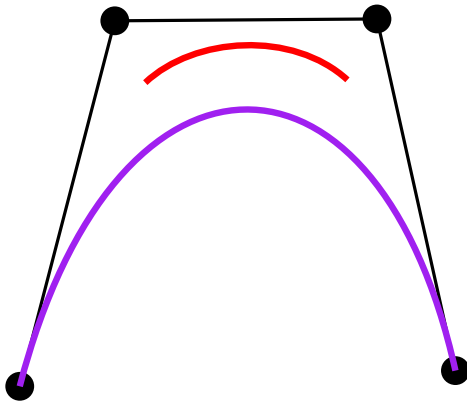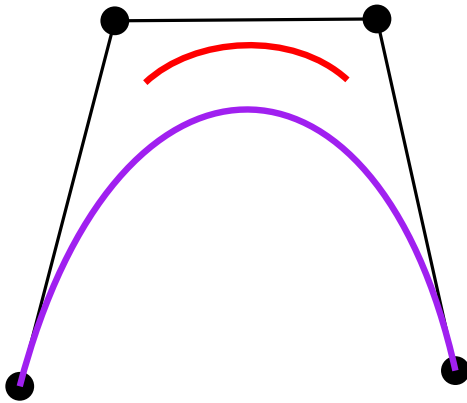uniform knot vector: $(0, 1/7, 2/7, 3/7, 4/7, 5/7, 6/7, 1)$

"open" knot vector: $(0, 0, 0, 0, 1, 1, 1, 1)$
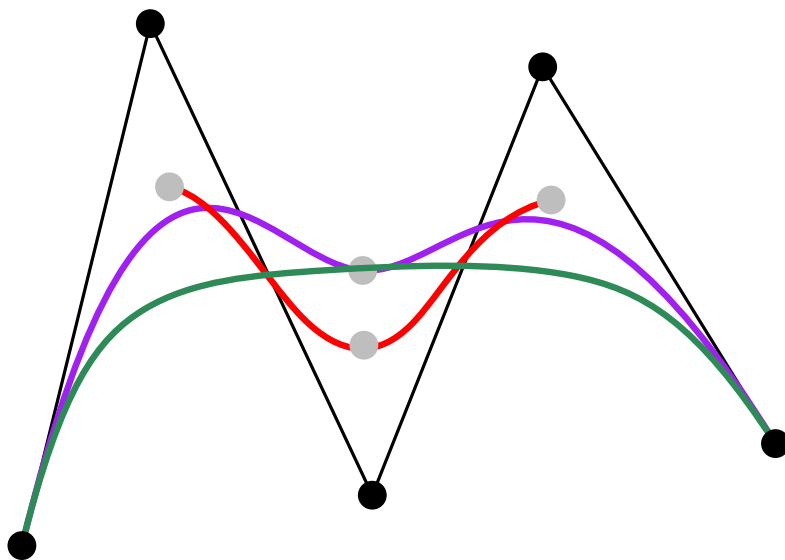
Cubic Bézier curve?

$n = 4$ (5 control points)

$k = 4$ (cubic B-spline)

uniform knot vector: $(0, 1/8, 2/8, 3/8, 4/8, 5/8, 6/8, 7/8, 1)$

"open" knot vector: $(0, 0, 0, 0, 0.5, 1, 1, 1, 1)$

degree-4 Bézier—knot vector: $(0, 0, 0, 0, 0, 1, 1, 1, 1, 1)$

Open uniform B-spline curves always start at P_0 and end at P_n. Tangents are also like in Bézier curves

Example for quadratic open B-Splines

Example: compute basis functions for 5 control points $(n = 4)$ and $k = 3$ (i.e., quadratic open B-splines)

Example for quadratic open B-Splines

Example: compute basis functions for 5 control points $(n = 4)$ and $k = 3$ (i.e., quadratic open B-splines)

Recall:

- knot vector: $(0, 0, 0, 1, 2, 3, 3, 3)$
- t goes from $t_{k-1} = t_2 = 0$ to $t_{n+1} = t_5 = 3$
- need to compute 5 bases: $N_{0,3}(t)$ to $N_{4,3}(t)$

$$N_{i,1} = \begin{cases} 1, & \text{if } t \in [t_i, t_{i+1}) \\ 0, & \text{otherwise} \end{cases} \qquad N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t)$$

More examples

Bézier vs open B-Spline of order 3
where $n = 9$ and $k = 3$
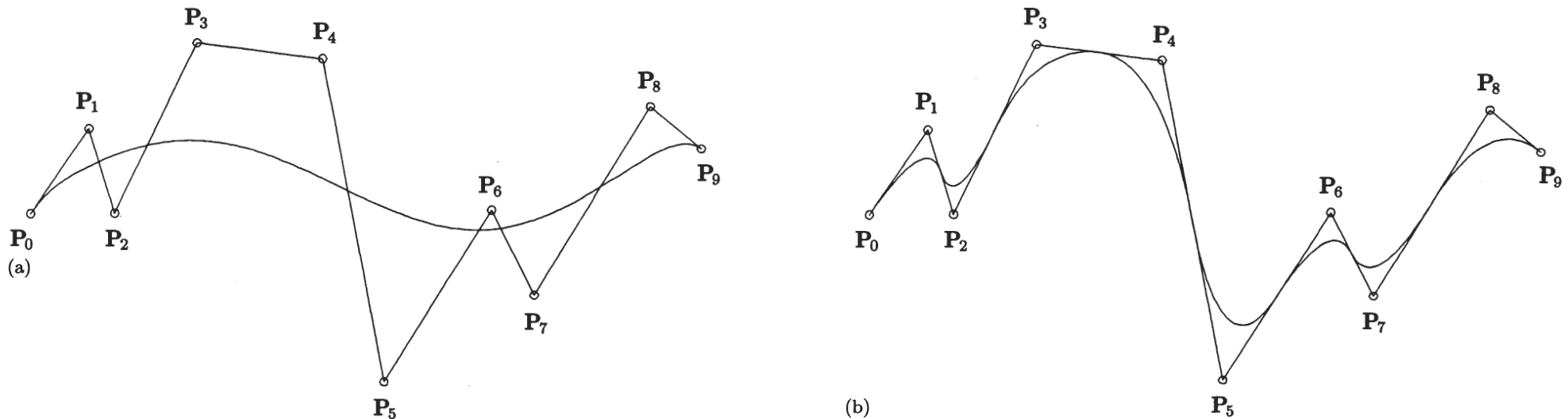
Figure 3.8. B-spline curves. (a) A ninth-degree Bézier curve on the knot vector $U = \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1\}$; (b) a quadratic curve using the same control polygon defined on $U = \{0, 0, 0, 1/8, 2/8, 3/8, 4/8, 5/8, 6/8, 7/8, 1, 1, 1\}$.

Figure from [Piegl and Tiller]

When knots are *not* equally spaced

When knots are *not* equally spaced

ignoring the first and last k knots, in case of open b-splines

When knots are *not* equally spaced

ignoring the first and last k knots, in case of open b-splines

- Only restriction: non-decreasing knots

When knots are *not* equally spaced

ignoring the first and last k knots, in case of open b-splines

- Only restriction: non-decreasing knots
- Knots can have multiplicity larger than one

When knots are *not* equally spaced

↳ ⟶ ignoring the first and last k knots, in case of open b-splines

- Only restriction: non-decreasing knots
- Knots can have multiplicity larger than one

Effect of knot multiplicity for $k = 4$ (cubic)

When knots are *not* equally spaced

$\longrightarrow$ ignoring the first and last k knots, in case of open b-splines

- Only restriction: non-decreasing knots
- Knots can have multiplicity larger than one

Effect of knot multiplicity for $k = 4$ (cubic)

When knots are *not* equally spaced

↳ ignoring the first and last k knots, in case of open b-splines

- Only restriction: non-decreasing knots
- Knots can have multiplicity larger than one

Effect of knot multiplicity for $k = 4$ (cubic)

Knot vector:
$(-3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8)$

When knots are *not* equally spaced

→ ignoring the first and last k knots, in case of open b-splines

- Only restriction: non-decreasing knots
- Knots can have multiplicity larger than one

Effect of knot multiplicity for $k = 4$ (cubic)

Knot vector:
$(-3, -2, -1, 0, 1, 1, 2, 3, 4, 5, 6, 7)$

When knots are *not* equally spaced

ignoring the first and last k knots, in case of open b-splines

- Only restriction: non-decreasing knots
- Knots can have multiplicity larger than one

Effect of knot multiplicity for $k = 4$ (cubic)

Knot vector:
$(-3, -2, -1, 0, 1, 1, 1, 2, 3, 4, 5, 6)$

When knots are *not* equally spaced

ignoring the first and last k knots, in case of open b-splines

- Only restriction: non-decreasing knots
- Knots can have multiplicity larger than one

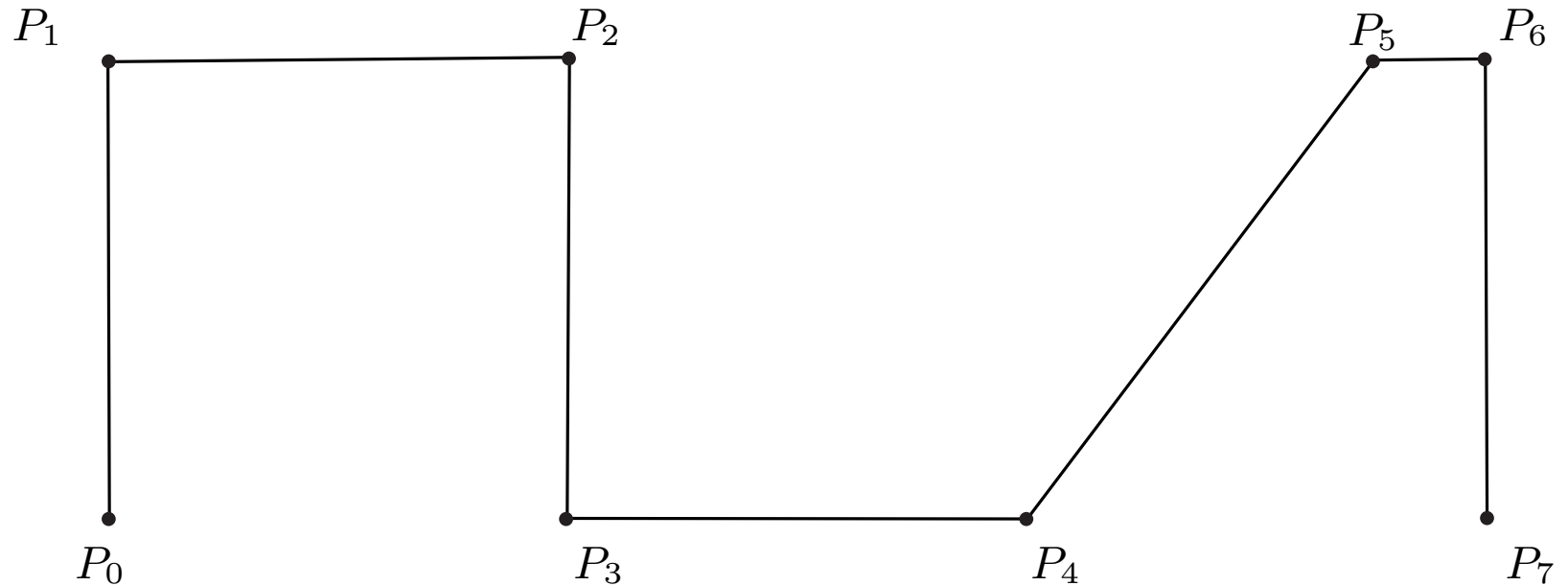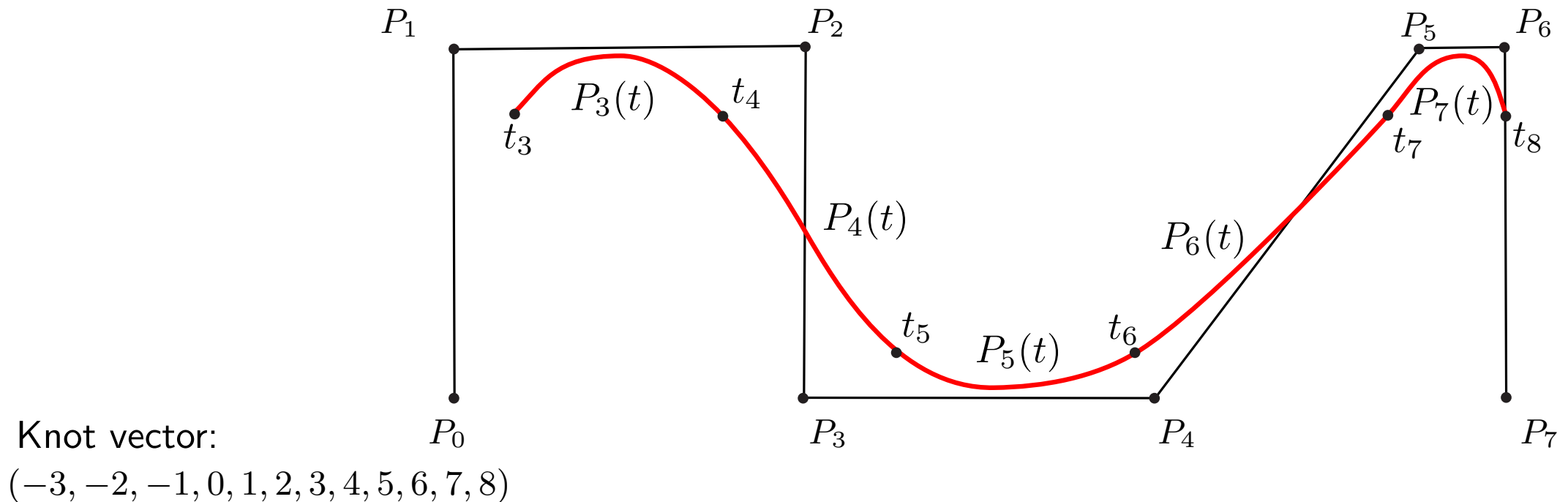## Effect of knot multiplicity for $k = 4$ (cubic)

Knot vector:
$(-3, -2, -1, 0, 1, 1, 1, 1, 2, 3, 4, 5)$

Understanding knot vectors

Understanding knot vectors

Open uniform B-splines interpolate the first and last control points due to the knot multiplicity

In general: continuity at the knots depends on multiplicity

$N_{i,k}(t)$ is $(k - m - 1)$ times continuously differentiable, where m is the multiplicity of the knot

Understanding knot vectors

Open uniform B-splines interpolate the first and last control points due to the knot multiplicity

In general: continuity at the knots depends on multiplicity

$N_{i,k}(t)$ is $(k - m - 1)$ times continuously differentiable, where m is the multiplicity of the knot

(m=number of repetitions of knot value)

Examples:

- If all knots are different, a cubic $(k = 4)$ B-spline is C^2-continuous at every knot

Understanding knot vectors

Open uniform B-splines interpolate the first and last control points due to the knot multiplicity

In general: continuity at the knots depends on multiplicity

$N_{i,k}(t)$ is $(k - m - 1)$ times continuously differentiable, where m is the multiplicity of the knot

(m=number of repetitions of knot value)

Examples:

- If all knots are different, a cubic ($k = 4$) B-spline is C^2-continuous at every knot
- If a knot appears twice, the cubic B-spline will be only C^1-continuous there

Understanding knot vectors

Open uniform B-splines interpolate the first and last control points due to the knot multiplicity

In general: continuity at the knots depends on multiplicity

$N_{i,k}(t)$ is $(k - m - 1)$ times continuously differentiable, where m is the multiplicity of the knot

(m=number of repetitions of knot value)

Examples:

- If all knots are different, a cubic ($k = 4$) B-spline is C^2-continuous at every knot
- If a knot appears twice, the cubic B-spline will be only C^1-continuous there
- If a knot appears three times, the cubic B-spline will be only C^0-continuous there

See example is
http://geometrie.foretnik.net/files/NURBS-en.swf
or play in http://nurbscalculator.in/

Matrix form

Matrix-based expressions to compute non-uniform B-splines also exist

Matrix form

Matrix-based expressions to compute non-uniform B-splines also exist

Linear case $(k = 2)$

$$N_{i2} = \frac{u - u_i}{u_{i+1} - u_i} N_{i1}(u) + \frac{u_{i+2} - u}{u_{i+2} - u_{i+2}} N_{i+1,1}(u)$$

$$= \begin{cases} \dfrac{u - u_i}{u_{i+1} - u_i} & \text{for } u \in [u_i, u_{i+1}), \\ \dfrac{u_{i+2} - u}{u_{i+2} - u_{i+1}} & \text{for } u \in [u_{i+1}, u_{i+2}), \\ 0 & \text{otherwise.} \end{cases}$$

$$\Delta = u_2 - u_1$$

$$t = \frac{u - u_1}{\Delta} = \frac{u - u_1}{u_2 - u_1}.$$

For $i = 0$, this becomes

$$N_{02} = \begin{cases} \dfrac{u - u_0}{u_1 - u_0} & \text{for } u \in [u_0, u_1), \\ \dfrac{u_2 - u}{u_2 - u_1} & \text{for } u \in [u_1, u_2), \\ 0 & \text{otherwise.} \end{cases}$$

$$\boxed{\mathbf{P}(t) = (t, 1) \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \end{pmatrix}}$$

$$t \in [0, 1]$$

$N_{12}(u)$ is obtained by incrementing all the indices

Matrix form

Matrix-based expressions to compute non-uniform B-splines also exist

Quadratic case $(k = 3)$ $\longrightarrow$ $N_{13}(u)$ and $N_{23}(u)$ are obtained by incrementing all the indices

$$N_{03}(u) = \begin{cases} \dfrac{u - u_0}{u_2 - u_0} \cdot \dfrac{u - u_0}{u_1 - u_0} & \text{for } u \in [u_0, u_1), \\[2ex] \dfrac{u - u_0}{u_2 - u_0} \cdot \dfrac{u_2 - u}{u_2 - u_1} + \dfrac{u_3 - u}{u_3 - u_1} \cdot \dfrac{u - u_1}{u_2 - u_1} & \text{for } u \in [u_1, u_2), \\[2ex] \dfrac{u_3 - u}{u_3 - u_1} \cdot \dfrac{u_3 - u}{u_3 - u_2} & \text{for } u \in [u_2, u_3), \\[2ex] 0 & \text{otherwise.} \end{cases}$$

over subinterval $[u_2, u_3)$

$$N_{03}(u) = \frac{u_3 - u}{u_3 - u_1} \cdot \frac{u_3 - u}{u_3 - u_2},$$

$$N_{13}(u) = \frac{u - u_1}{u_3 - u_1} \cdot \frac{u_3 - u}{u_3 - u_2} + \frac{u_4 - u}{u_4 - u_2} \cdot \frac{u - u_2}{u_3 - u_2},$$

$$N_{23}(u) = \frac{u - u_2}{u_4 - u_2} \cdot \frac{u - u_2}{u_3 - u_2}.$$

Need notation for difference between consecutive knots:

$$\Delta_1 = u_2 - u_1, \quad \Delta_2 = u_3 - u_2, \quad \Delta_3 = u_4 - u_3.$$

We also define $t = (u - u_2)/\Delta_2$, which implies

$$u - u_1 = t\Delta_2 + \Delta_1,$$
$$u - u_2 = t\Delta_2,$$
$$u - u_3 = (t - 1)\Delta_2,$$
$$u - u_4 = t\Delta_2 - (\Delta_2 + \Delta_3).$$

$$\mathbf{P}(t) = (t^2, t, 1) \begin{pmatrix} a & -a - b & b \\ -2a & 2a & 0 \\ a & 1 - a & 0 \end{pmatrix} \begin{pmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \end{pmatrix}$$

$$a = \frac{\Delta_2}{\Delta_1 + \Delta_2}, \qquad b = \frac{\Delta_2}{\Delta_2 + \Delta_3},$$

$$t \in [0, 1]$$

Matrix form

Matrix-based expressions to compute non-uniform B-splines also exist

Cubic case ($k = 4$)

$$N_{04}(u) = \begin{cases} \dfrac{u - u_0}{u_3 - u_0} \cdot \dfrac{u - u_0}{u_2 - u_0} \cdot \dfrac{u - u_0}{u_1 - u_0} & \text{for } u \in [u_0, u_1), \\[2ex] \dfrac{u - u_0}{u_3 - u_0} \cdot \dfrac{u - u_0}{u_2 - u_0} \cdot \dfrac{u_2 - u}{u_2 - u_1} \\[1ex] + \dfrac{u - u_0}{u_3 - u_0} \cdot \dfrac{u_3 - u}{u_3 - u_1} \cdot \dfrac{u - u_1}{u_2 - u_1} \\[1ex] + \dfrac{u_4 - u}{u_4 - u_1} \cdot \dfrac{u - u_1}{u_3 - u_1} \cdot \dfrac{u - u_1}{u_2 - u_1} & \text{for } u \in [u_1, u_2), \\[2ex] \dfrac{u - u_0}{u_3 - u_0} \cdot \dfrac{u_3 - u}{u_3 - u_1} \cdot \dfrac{u_3 - u}{u_3 - u_2} \\[1ex] + \dfrac{u_4 - u}{u_4 - u_1} \cdot \dfrac{u - u_1}{u_3 - u_1} \cdot \dfrac{u_3 - u}{u_3 - u_2} \\[1ex] + \dfrac{u_4 - u}{u_4 - u_1} \cdot \dfrac{u_4 - u}{u_4 - u_2} \cdot \dfrac{u - u_2}{u_3 - u_2} & \text{for } u \in [u_2, u_3), \\[2ex] \dfrac{u_4 - u}{u_4 - u_1} \cdot \dfrac{u_4 - u}{u_4 - u_2} \cdot \dfrac{u_4 - u}{u_4 - u_3} & \text{for } u \in [u_3, u_4), \\[2ex] 0 & \text{otherwise.} \end{cases}$$

$N_{14}(u), N_{24}(u)$ and $N_{34}(u)$ are obtained by incrementing all the indices

Only in $[u_3, u_4)$ all four are nonzero, with values:

$$N_{04}(u) = \frac{u_4 - u}{u_4 - u_1} \cdot \frac{u_4 - u}{u_4 - u_2} \cdot \frac{u_4 - u}{u_4 - u_3},$$

$$N_{14}(u) = \frac{u - u_1}{u_4 - u_1} \cdot \frac{u_4 - u}{u_4 - u_2} \cdot \frac{u_4 - u}{u_4 - u_3} + \frac{u_5 - u}{u_5 - u_2} \cdot \frac{u - u_2}{u_4 - u_2} \cdot \frac{u_4 - u}{u_4 - u_3} + \frac{u_5 - u}{u_5 - u_2} \cdot \frac{u_5 - u}{u_5 - u_3} \cdot \frac{u - u_3}{u_4 - u_3},$$

$$N_{24}(u) = \frac{u - u_2}{u_5 - u_2} \cdot \frac{u - u_2}{u_4 - u_2} \cdot \frac{u_4 - u}{u_4 - u_3} + \frac{u - u_2}{u_5 - u_2} \cdot \frac{u_5 - u}{u_5 - u_3} \cdot \frac{u - u_3}{u_4 - u_3} + \frac{u_6 - u}{u_6 - u_3} \cdot \frac{u - u_3}{u_5 - u_3} \cdot \frac{u - u_3}{u_4 - u_3},$$

$$N_{34}(u) = \frac{u - u_3}{u_6 - u_3} \cdot \frac{u - u_3}{u_5 - u_3} \cdot \frac{u - u_3}{u_4 - u_3}.$$

take

$$\Delta_1 = u_2 - u_1, \quad \Delta_2 = u_3 - u_2, \quad \Delta_3 = u_4 - u_3,$$
$$\Delta_4 = u_5 - u_4, \quad \Delta_5 = u_6 - u_5, \quad t = (u - u_3)/\Delta_3.$$

Matrix form

Matrix-based expressions to compute non-uniform B-splines also exist

Cubic case $(k = 4)$

$$N_{04}(u) = \begin{cases} \dfrac{u - u_0}{u_3 - u_0} \cdot \dfrac{u - u_0}{u_2 - u_0} \cdot \dfrac{u - u_0}{u_1 - u_0} \\[2mm] \dfrac{u - u_0}{u_3 - u_0} \cdot \dfrac{u - u_0}{u_2 - u_0} \cdot \dfrac{u_2 - u}{u_2 - u_1} \\[2mm] \quad + \dfrac{u - u_0}{u_3 - u_0} \cdot \dfrac{u_3 - u}{u_3 - u_1} \cdot \dfrac{u - u_1}{u_2 - u_1} \\[2mm] \quad + \dfrac{u_4 - u}{u_4 - u_1} \cdot \dfrac{u - u_1}{u_3 - u_1} \cdot \dfrac{u - u_1}{u_2 - u_1} \\[2mm] \dfrac{u - u_0}{u_3 - u_0} \cdot \dfrac{u_3 - u}{u_3 - u_1} \cdot \dfrac{u_3 - u}{u_3 - u_2} \\[2mm] \quad + \dfrac{u_4 - u}{u_4 - u_1} \cdot \dfrac{u - u_1}{u_3 - u_1} \cdot \dfrac{u_3 - u}{u_3 - u_2} \\[2mm] \quad + \dfrac{u_4 - u}{u_4 - u_1} \cdot \dfrac{u_4 - u}{u_4 - u_2} \cdot \dfrac{u - u_2}{u_3 - u_2} \\[2mm] \dfrac{u_4 - u}{u_4 - u_1} \cdot \dfrac{u_4 - u}{u_4 - u_2} \cdot \dfrac{u_4 - u}{u_4 - u_3} \\[2mm] 0 \end{cases}$$

We obtain:

$$\mathbf{P}(t) = (t^3, t^2, t, 1) \begin{pmatrix} -a & a+b+c & -b-c-d & d \\ 3a & -3a-3b & 3b & 0 \\ -3a & 3a-3e & 3e & 0 \\ a & 1-a-f & f & 0 \end{pmatrix} \begin{pmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{pmatrix},$$

where

$$a = \frac{\Delta_3^2}{(\Delta_1 + \Delta_2 + \Delta_3)(\Delta_2 + \Delta_3)}, \qquad d = \frac{\Delta_3^2}{(\Delta_3 + \Delta_4 + \Delta_5)(\Delta_3 + \Delta_4)},$$

$$b = \frac{\Delta_3^2}{(\Delta_2 + \Delta_3 + \Delta_4)(\Delta_2 + \Delta_3)}, \qquad e = \frac{\Delta_2 \Delta_3}{(\Delta_2 + \Delta_3 + \Delta_4)(\Delta_2 + \Delta_3)},$$

$$c = \frac{\Delta_3^2}{(\Delta_2 + \Delta_3 + \Delta_4)(\Delta_3 + \Delta_4)}, \qquad f = \frac{\Delta_2^2}{(\Delta_2 + \Delta_3 + \Delta_4)(\Delta_2 + \Delta_3)}.$$

for $u \in [u_3, u_4)$,

otherwise.

$$N_{34}(u) = \frac{u}{u_6 - u_3} \cdot \frac{u}{u_5 - u_3} \cdot \frac{u}{u_4 - u_3}.$$

$N_{14}(u), N_{24}(u)$ and $N_{34}(u)$ are obtained by incrementing all the indices

take

$$\Delta_1 = u_2 - u_1, \quad \Delta_2 = u_3 - u_2, \quad \Delta_3 = u_4 - u_3,$$
$$\Delta_4 = u_5 - u_4, \quad \Delta_5 = u_6 - u_5, \quad t = (u - u_3)/\Delta_3.$$

The most general parametric curve

Same idea as for rational Bézier: each control point P_i has a weight, $w_i \geq 0$. This gives even more flexibility to shape the curve

The most general parametric curve

Same idea as for rational Bézier: each control point P_i has a weight, $w_i \geq 0$. This gives even more flexibility to shape the curve

Advantages

- Invariant under projections

The most general parametric curve

Same idea as for rational Bézier: each control point P_i has a weight, $w_i \geq 0$. This gives even more flexibility to shape the curve

Advantages
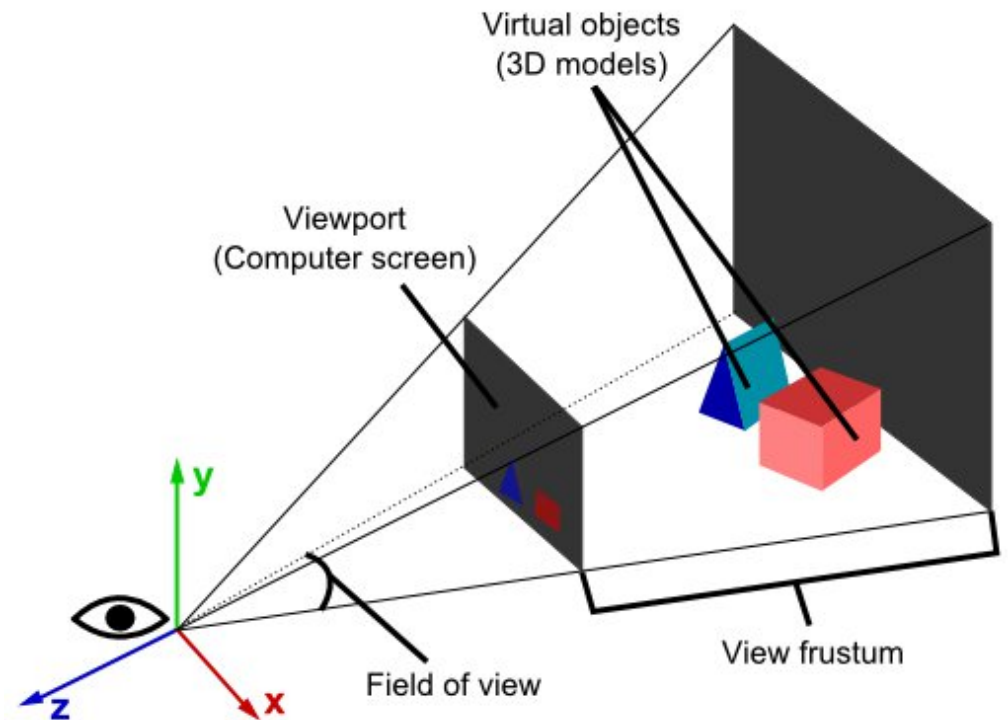
- Invariant under projections

Figure from real3dtutorials.com

NON-UNIFORM RATIONAL B-SPLINES (NURBS)

The most general parametric curve

Same idea as for rational Bézier: each control point P_i has a weight, $w_i \geq 0$. This gives even more flexibility to shape the curve

Advantages

- Invariant under projections

- It can represent conic curves exactly (e.g., segments of circles, ellipses, hyperbolas and parabolas)

The most general parametric curve

Same idea as for rational Bézier: each control point P_i has a weight, $w_i \geq 0$. This gives even more flexibility to shape the curve

Advantages

- Invariant under projections

- It can represent conic curves exactly (e.g., segments of circles, ellipses, hyperbolas and parabolas)

- It is more general, so it includes as particular cases all other B-splines and Bézier curves

Related to homogeneous coordinates

Any 2D point (x, y) is equivalent to a 3D point: (wx, wy, w)

Related to homogeneous coordinates

Any 2D point (x, y) is equivalent to a 3D point: (wx, wy, w)

└──▶ the projection of 3D point (a, b, w) to 2D is $(a/w, b/w)$

Related to homogeneous coordinates

Any 2D point (x, y) is equivalent to a 3D point: (wx, wy, w)

$\quad\longrightarrow$ the projection of 3D point (a, b, w) to 2D is $(a/w, b/w)$

Suppose your control points Q_i have one extra dimension $P_i \in \mathbb{R}^2 \to Q_i \in \mathbb{R}^3$

$\quad\longrightarrow$ e.g., each point $P_i = (x_i, y_i)$, becomes $Q_i = (w_i x_i, w_i y_i, w_i)$, for some $w_i \geq 0$

Related to homogeneous coordinates

Any 2D point (x, y) is equivalent to a 3D point: (wx, wy, w)

$\quad\quad \longrightarrow$ the projection of 3D point (a, b, w) to 2D is $(a/w, b/w)$

Suppose your control points Q_i have one extra dimension $P_i \in \mathbb{R}^2 \rightarrow Q_i \in \mathbb{R}^3$

$\quad\quad \longrightarrow$ e.g., each point $P_i = (x_i, y_i)$, becomes $Q_i = (w_i x_i, w_i y_i, w_i)$, for some $w_i \geq 0$

Compute a 3D B-spline curve in the usual (non-rational) way: $\quad P_{\mathsf{nr}}(t) = \sum_{i=0}^{n} Q_i N_{i,k}(t)$

Related to homogeneous coordinates

Any 2D point (x, y) is equivalent to a 3D point: (wx, wy, w)

$\qquad\longmapsto$ the projection of 3D point (a, b, w) to 2D is $(a/w, b/w)$

Suppose your control points Q_i have one extra dimension $P_i \in \mathbb{R}^2 \rightarrow Q_i \in \mathbb{R}^3$

$\qquad\longmapsto$ e.g., each point $P_i = (x_i, y_i)$, becomes $Q_i = (w_i x_i, w_i y_i, w_i)$, for some $w_i \geq 0$

Compute a 3D B-spline curve in the usual (non-rational) way: $\quad P_{\mathsf{nr}}(t) = \sum_{i=0}^{n} Q_i N_{i,k}(t)$

Now, we can project any point on the 3D curve $P_{\mathsf{nr}}(t)$ to 2D:

NON-UNIFORM RATIONAL B-SPLINES (NURBS)

Related to homogeneous coordinates

Any 2D point (x, y) is equivalent to a 3D point: (wx, wy, w)

$\quad\quad\longrightarrow$ the projection of 3D point (a, b, w) to 2D is $(a/w, b/w)$

Suppose your control points Q_i have one extra dimension $P_i \in \mathbb{R}^2 \to Q_i \in \mathbb{R}^3$

$\quad\quad\longrightarrow$ e.g., each point $P_i = (x_i, y_i)$, becomes $Q_i = (w_i x_i, w_i y_i, w_i)$, for some $w_i \geq 0$

Compute a 3D B-spline curve in the usual (non-rational) way: $\quad P_{\mathsf{nr}}(t) = \sum_{i=0}^{n} Q_i N_{i,k}(t)$

Now, we can project any point on the 3D curve $P_{\mathsf{nr}}(t)$ to 2D:

$\quad\quad\longrightarrow$ we need to isolate the coefficients w_i multiplying each point and divide by them

Related to homogeneous coordinates

Any 2D point (x, y) is equivalent to a 3D point: (wx, wy, w)

$\hookrightarrow$ the projection of 3D point (a, b, w) to 2D is $(a/w, b/w)$

Suppose your control points Q_i have one extra dimension $P_i \in \mathbb{R}^2 \to Q_i \in \mathbb{R}^3$

$\hookrightarrow$ e.g., each point $P_i = (x_i, y_i)$, becomes $Q_i = (w_i x_i, w_i y_i, w_i)$, for some $w_i \geq 0$

Compute a 3D B-spline curve in the usual (non-rational) way: $P_{\mathsf{nr}}(t) = \sum_{i=0}^{n} Q_i N_{i,k}(t)$

Now, we can project any point on the 3D curve $P_{\mathsf{nr}}(t)$ to 2D:

$\hookrightarrow$ we need to isolate the coefficients w_i multiplying each point and divide by them

$$P_{\mathsf{r}}(t) = \frac{\sum_{i=0}^{n} w_i P_i N_{i,k}(t)}{\sum_{j=0}^{n} w_j N_{j,k}(t)} = \sum_{i=0}^{n} P_i \left(\frac{w_i N_{i,k}(t)}{\sum_{j=0}^{n} w_j N_{j,k}(t)} \right) = \sum_{i=0}^{n} P_i R_{i,k}(t)$$

NON-UNIFORM RATIONAL B-SPLINES (NURBS)

Related to homogeneous coordinates

Any 2D point (x, y) is equivalent to a 3D point: (wx, wy, w)

└──▶ the projection of 3D point (a, b, w) to 2D is $(a/w, b/w)$

Suppose your control points Q_i have one extra dimension $P_i \in \mathbb{R}^2 \to Q_i \in \mathbb{R}^3$

└──▶ e.g., each point $P_i = (x_i, y_i)$, becomes $Q_i = (w_i x_i, w_i y_i, w_i)$, for some $w_i \geq 0$

Compute a 3D B-spline curve in the usual (non-rational) way: $P_{\mathsf{nr}}(t) = \sum_{i=0}^{n} Q_i N_{i,k}(t)$

Now, we can project any point on the 3D curve $P_{\mathsf{nr}}(t)$ to 2D:

└──▶ we need to isolate the coefficients w_i multiplying each point and divide by them

$$P_{\mathsf{r}}(t) = \frac{\sum_{i=0}^{n} w_i P_i N_{i,k}(t)}{\sum_{j=0}^{n} w_j N_{j,k}(t)} = \sum_{i=0}^{n} P_i \boxed{\left(\frac{w_i N_{i,k}(t)}{\sum_{j=0}^{n} w_j N_{j,k}(t)} \right)} = \sum_{i=0}^{n} P_i \boxed{R_{i,k}(t)}$$

rational blending functions

Rational curves as curves in projective space
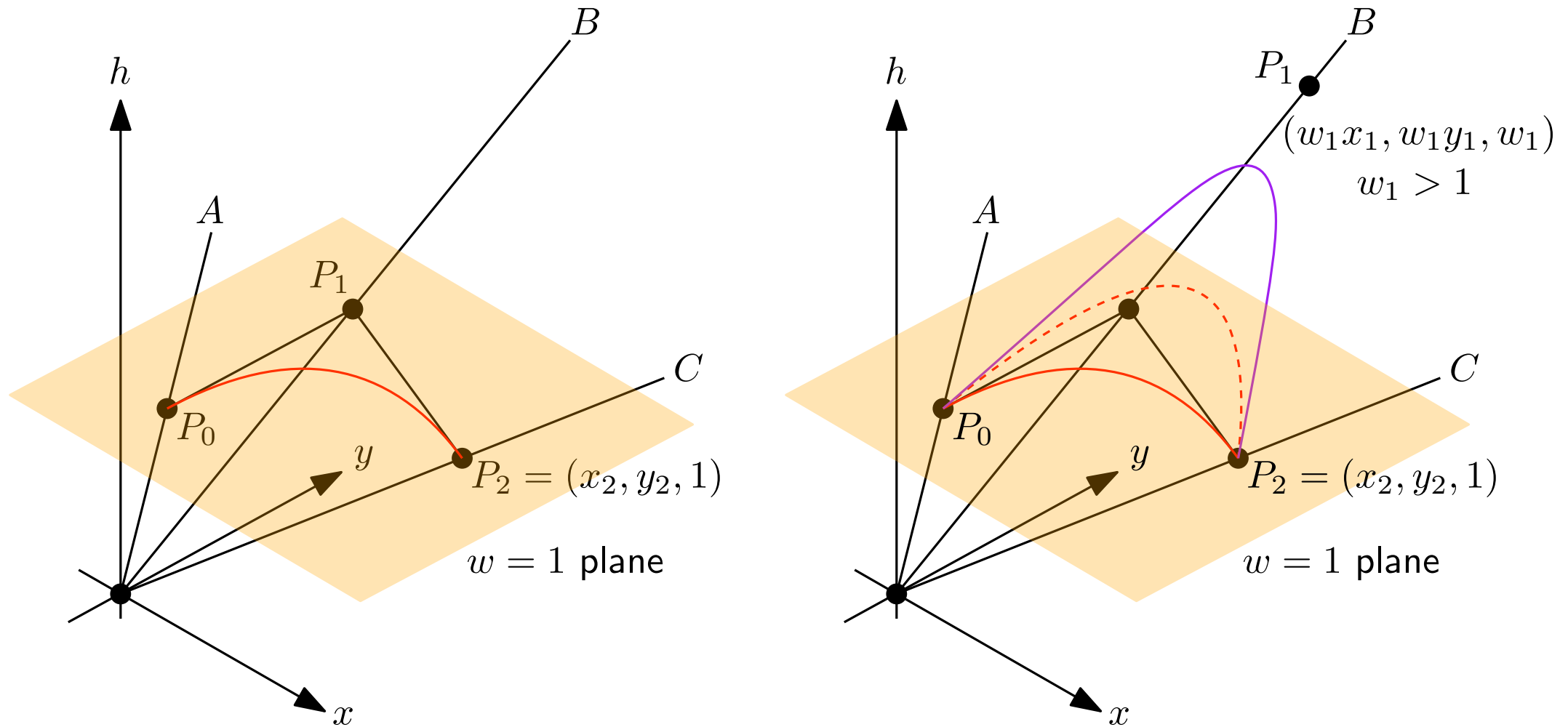
Figure adapted from book by Mortenson

Rational curves as curves in projective space

Another picture
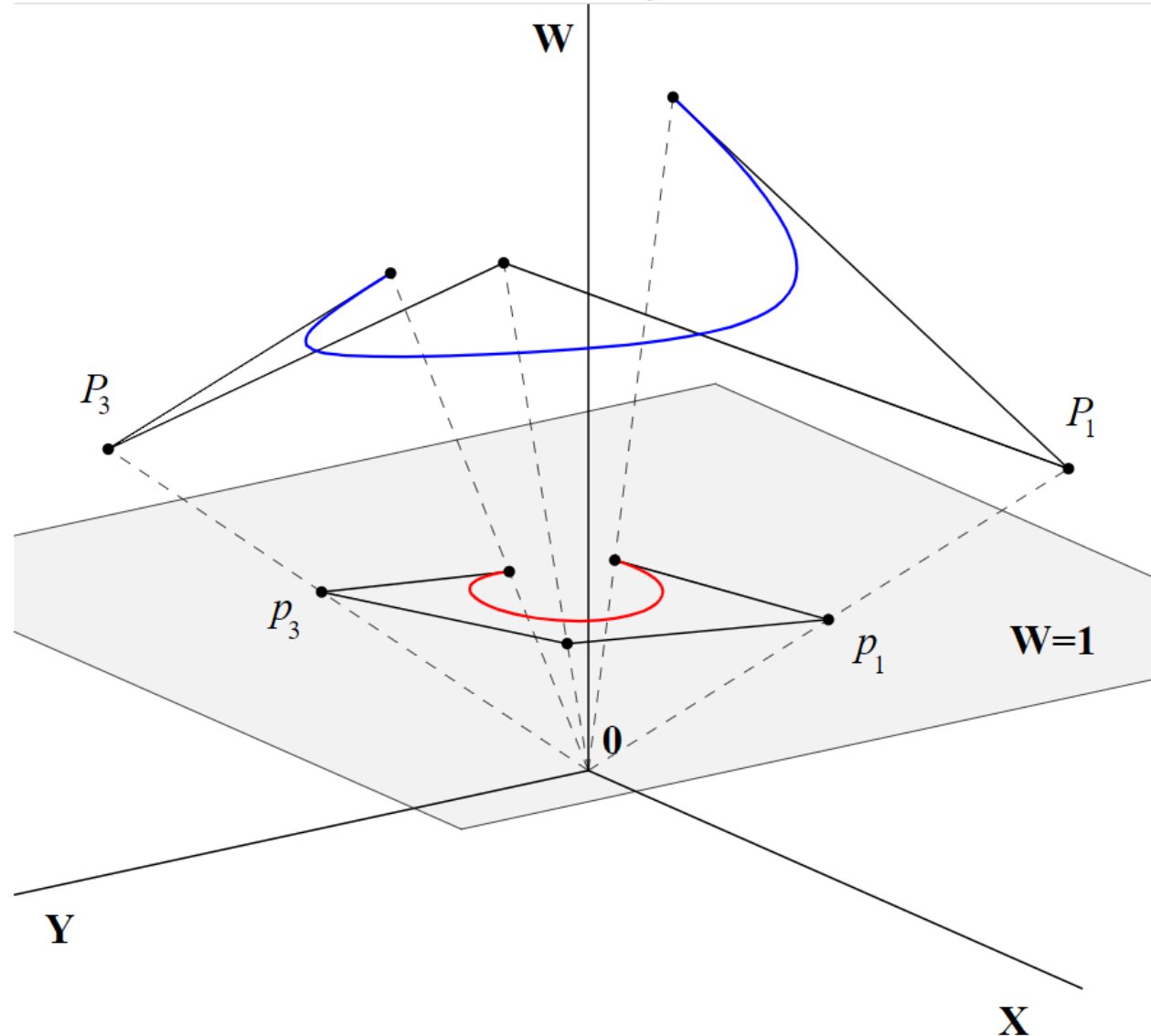
Figure by Wojciech Muła - Own work (Python script, final touches Inkscape), CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=1196334

Properties of rational basis functions and NURBS

Properties of rational basis functions and NURBS

Has most properties of the non-rational basis functions, plus a few more:

Properties of rational basis functions and NURBS

Has most properties of the non-rational basis functions, plus a few more:

- Non-negativity, partition of unity, unimodality, local support (for control point position and weight), convex hull property, etc..

Properties of rational basis functions and NURBS

Has most properties of the non-rational basis functions, plus a few more:

- Non-negativity, partition of unity, unimodality, local support (for control point position and weight), convex hull property, etc..

- Effect of changing weight: increasing weight moves curve closer to P_i

Properties of rational basis functions and NURBS

Has most properties of the non-rational basis functions, plus a few more:

- Non-negativity, partition of unity, unimodality, local support (for control point position and weight), convex hull property, etc..

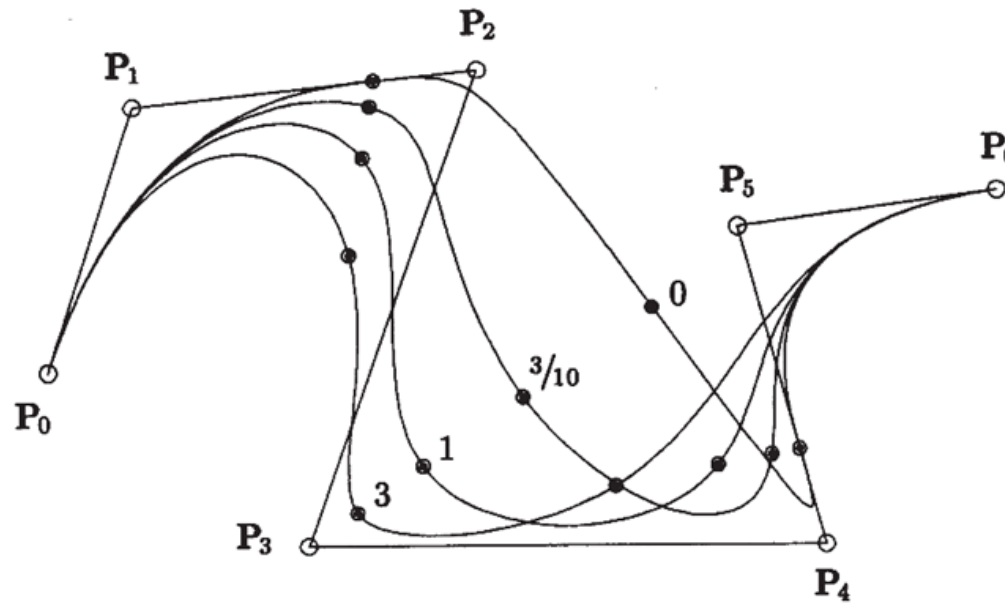- Effect of changing weight: increasing weight moves curve closer to P_i

Figure 4.2. Rational cubic B-spline curves, with w_3 varying.

Figure from [Piegl and Tiller]

Properties of rational basis functions and NURBS

Properties of rational basis functions and NURBS

- When all weights are the same, the curve becomes non-rational

Properties of rational basis functions and NURBS

- When all weights are the same, the curve becomes non-rational

- Curve is invariant under projective transformations
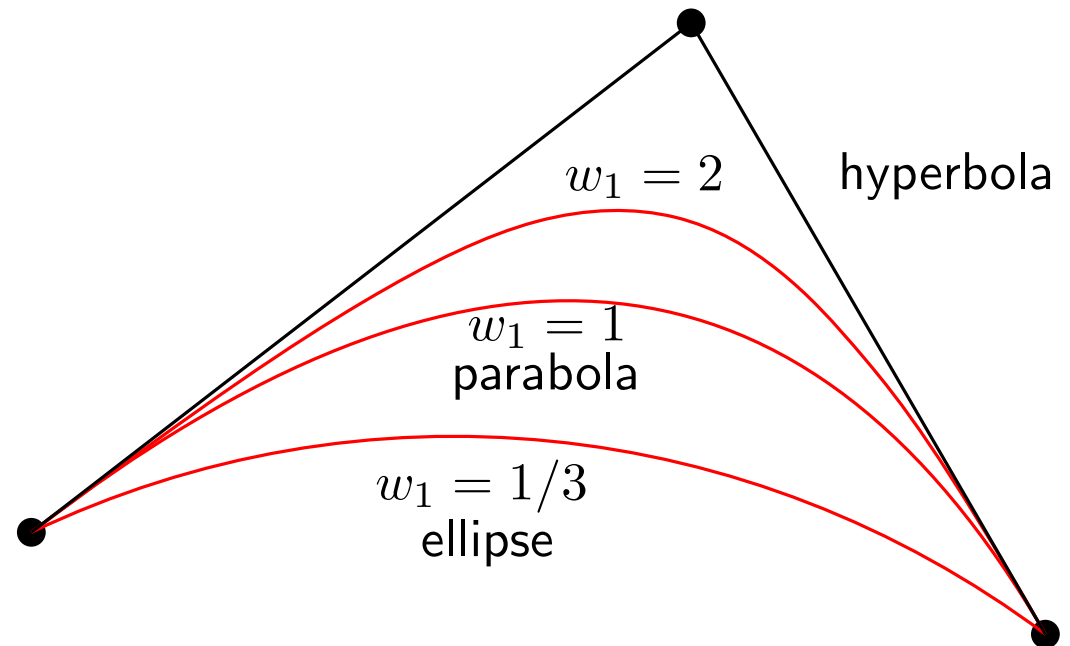
Properties of rational basis functions and NURBS

- When all weights are the same, the curve becomes non-rational

- Curve is invariant under projective transformations

- Conic sections can be represented *exactly* (same as with rational Bézier curves)

$w_1 = 2$

hyperbola

$w_1 = 1$
parabola

$w_1 = 1/3$
ellipse

Properties of rational basis functions and NURBS

- When all weights are the same, the curve becomes non-rational

- Curve is invariant under projective transformations

- Conic sections can be represented *exactly* (same as with rational Bézier curves)

Play with NURBS at
http://geometrie.foretnik.net/files/NURBS-en.swf
or play in http://nurbscalculator.in/

$w_1 = 2$ hyperbola

$w_1 = 1$
parabola

$w_1 = 1/3$
ellipse