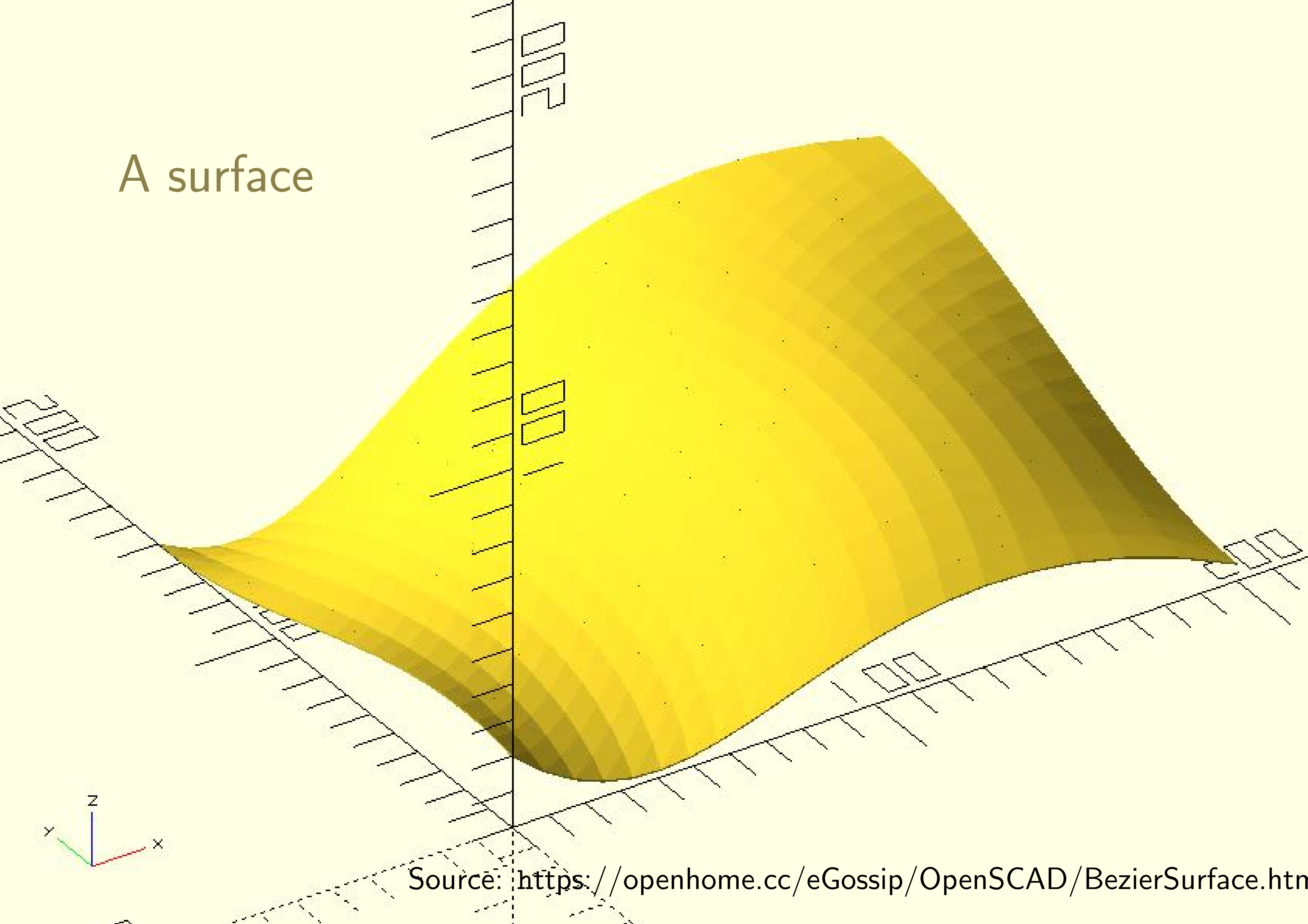
SURFACES

Rodrigo Silveira

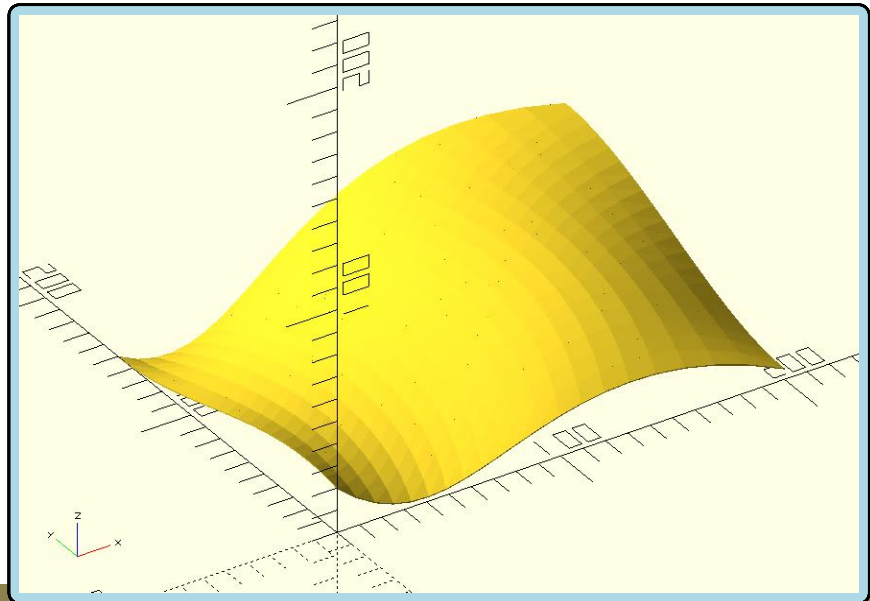
Curve and Surface Design
Facultat d'Informàtica de Barcelona
Universitat Politècnica de Catalunya

A surface

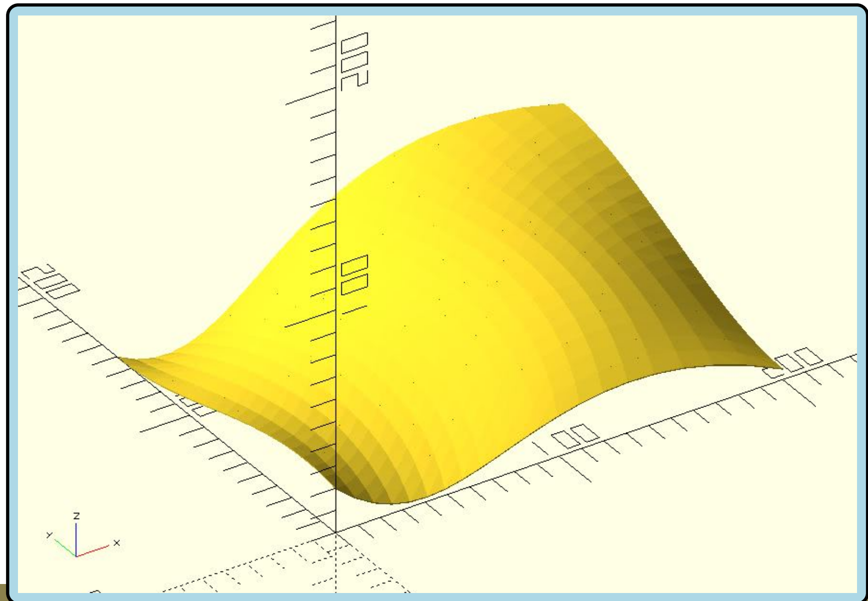


Source: https://openhome.cc/eGossip/OpenSCAD/BezierSurface.htm

PARAMETRIZING SURFACES

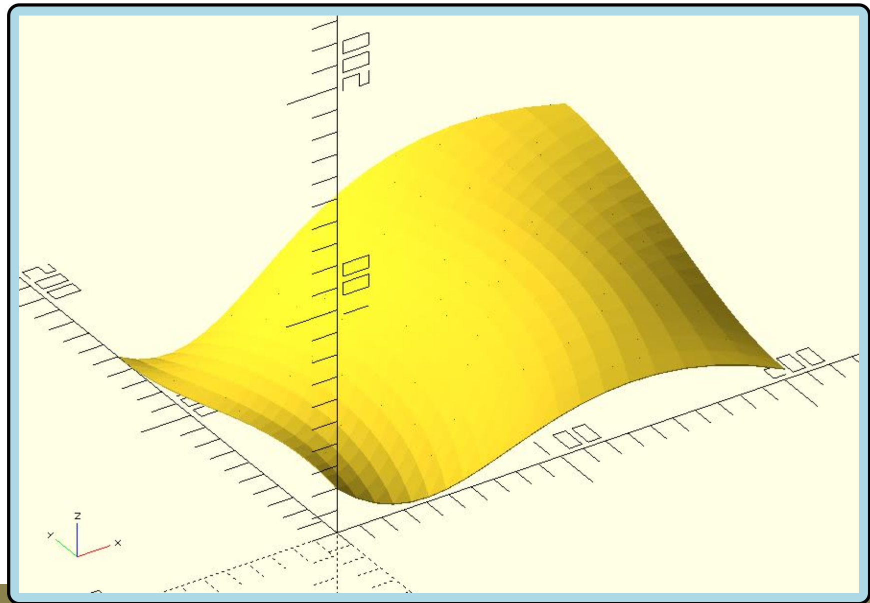


1) Explicit equation



1) Explicit equation

$z = f(x, y)$, for $x \in [x_0, x_1]$, $y \in [y_0, y_1]$, for f a continuous function



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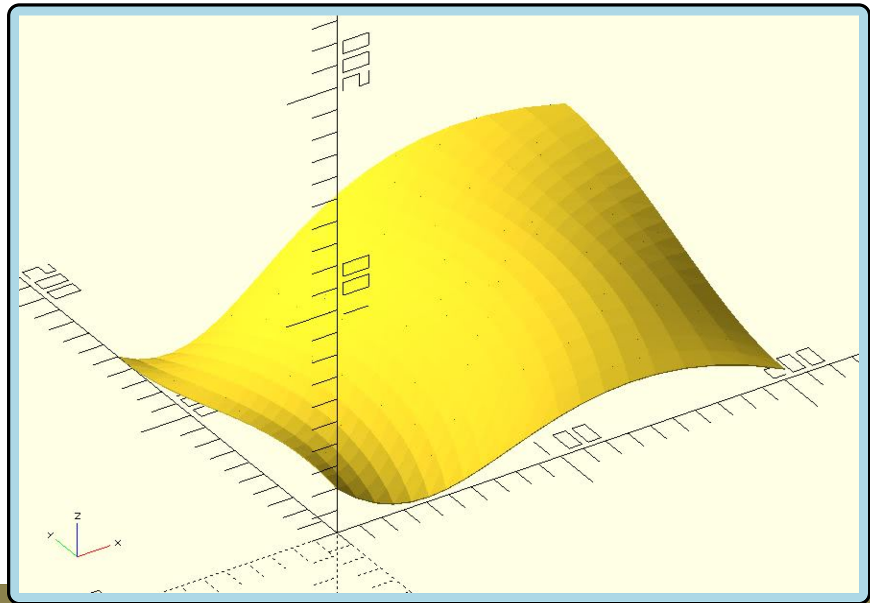

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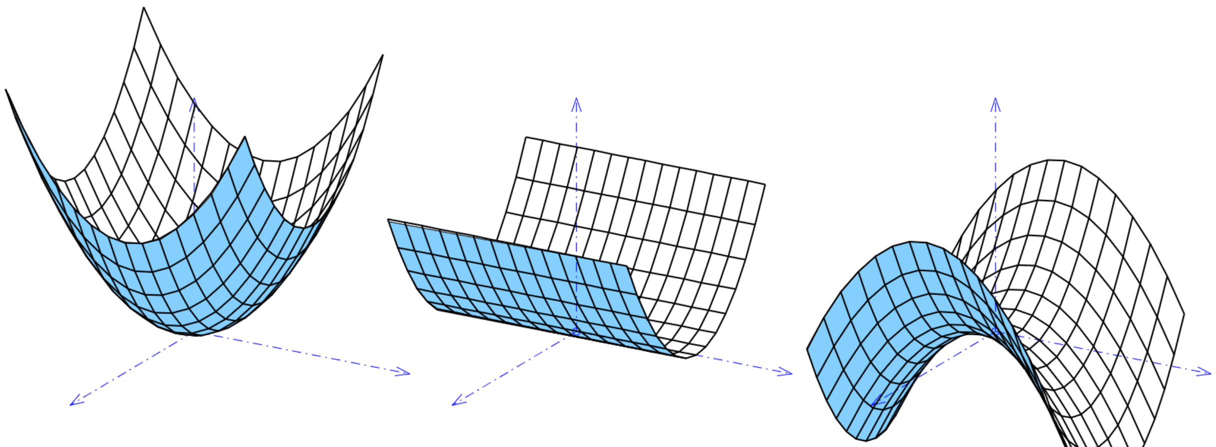
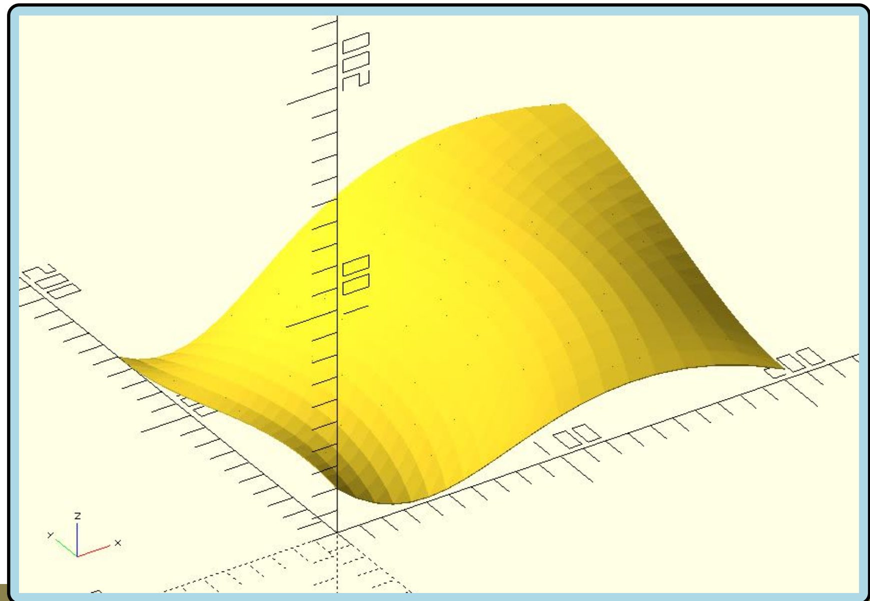


$z = x^2 + y^2$ $\quad$ $z = x^2$ $\quad$ $z = x^2 - y^2$

[From Wikimedia commons - by Ag2gaeh - Own work]



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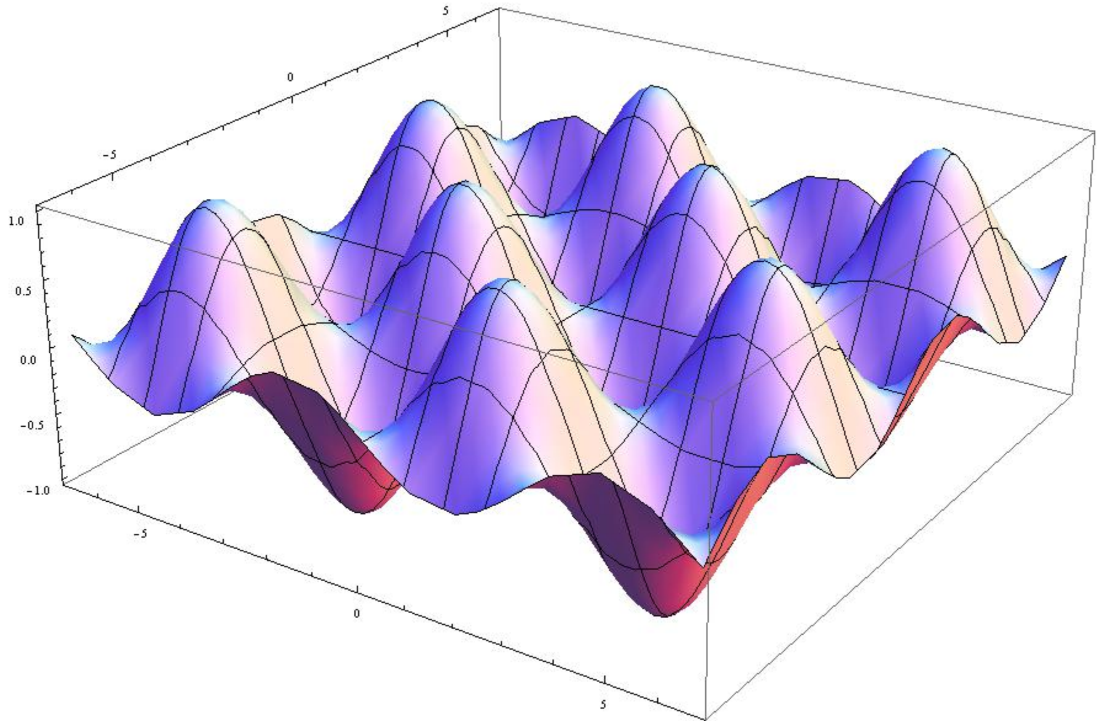
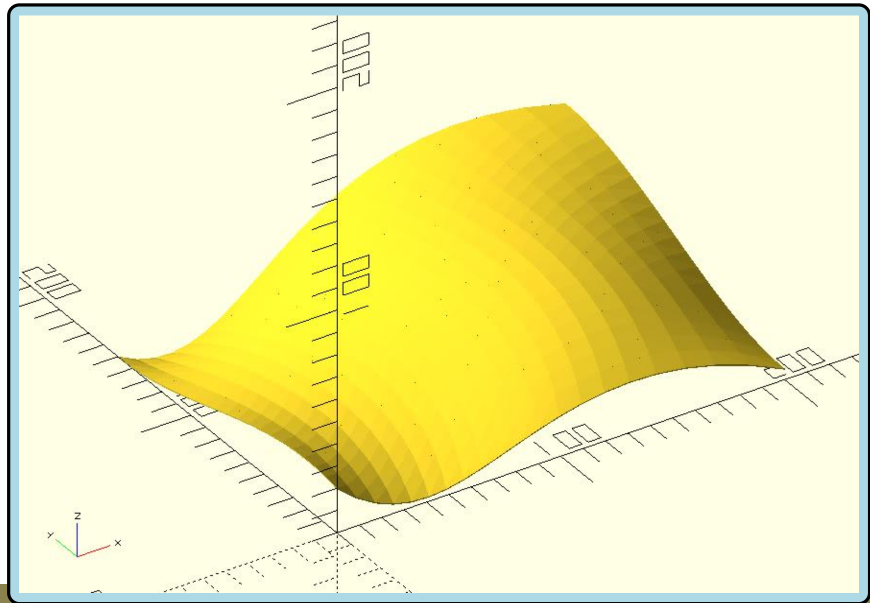
Example: $z = \sin x \sin y$





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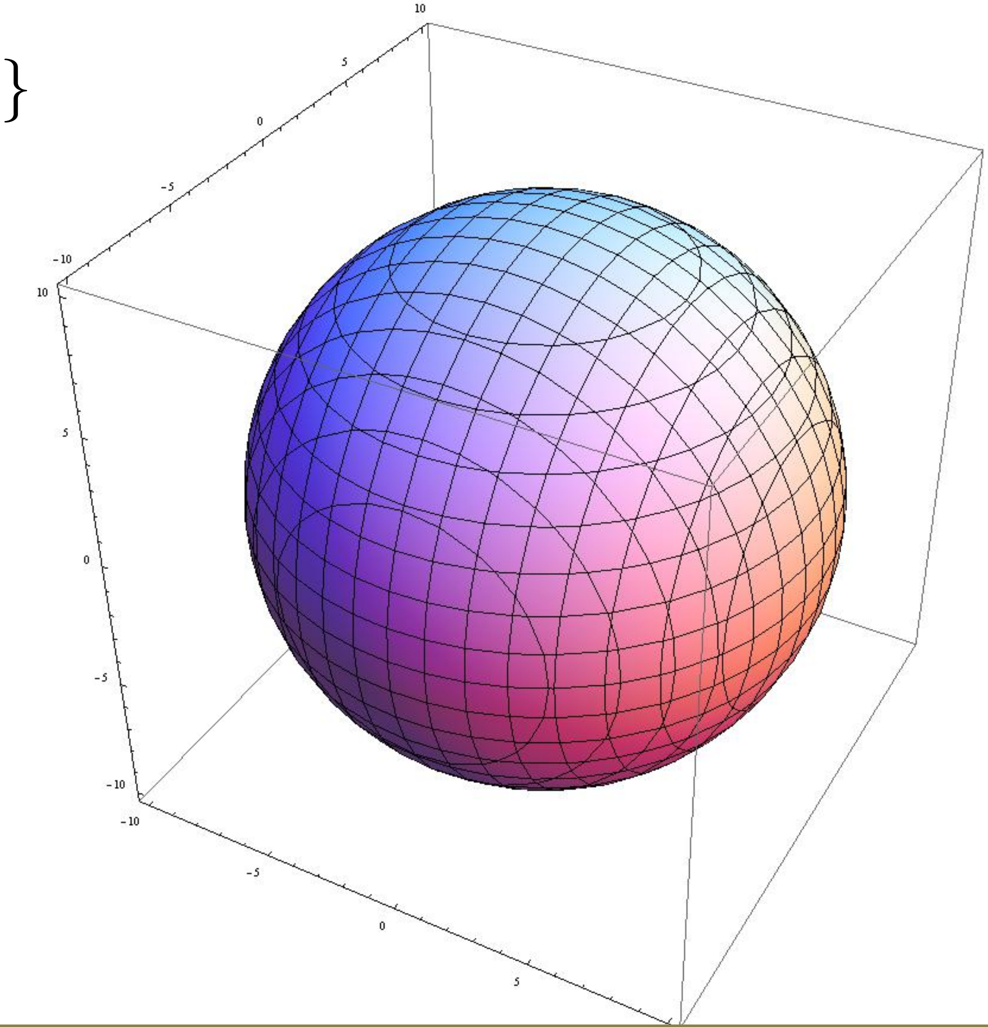

3) Parametric equation

$$\begin{aligned} x &= x(u,v) \\ y &= y(u,v) \\ z &= z(u,v) \end{aligned}$$

$\longrightarrow$ $u \in [u_0, u_1]$, $v \in [v_0, v_1]$, $x, y, z : \mathbb{R}^2 \to \mathbb{R}$ continuous

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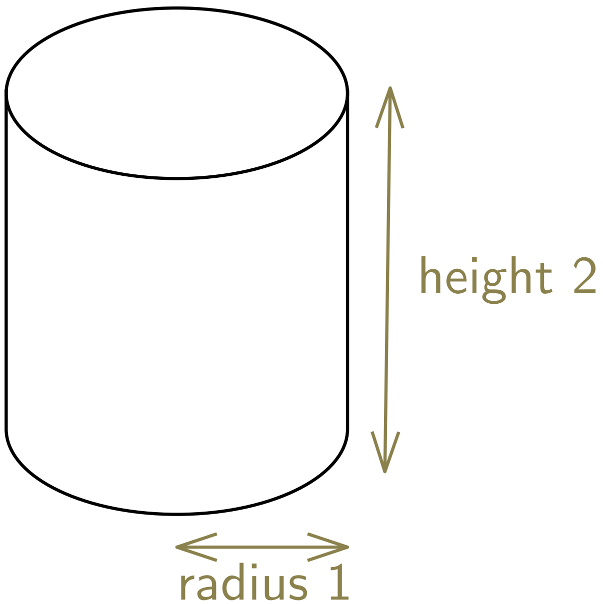

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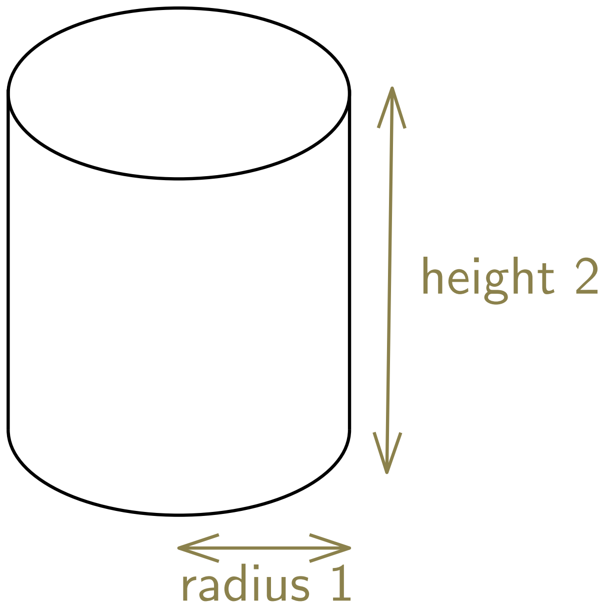

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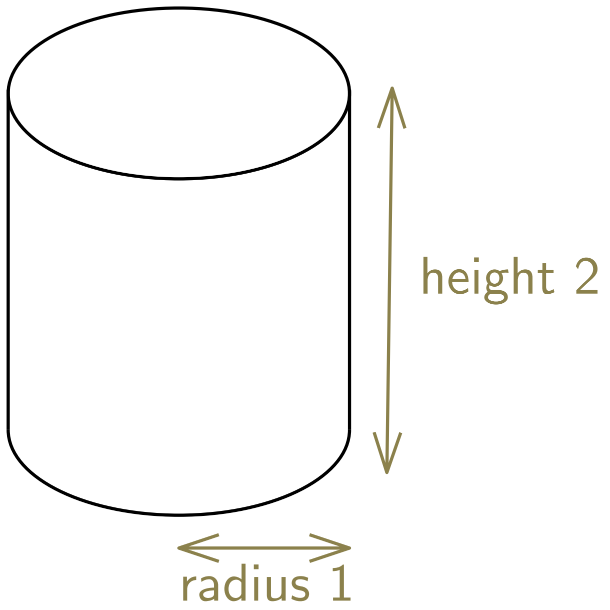

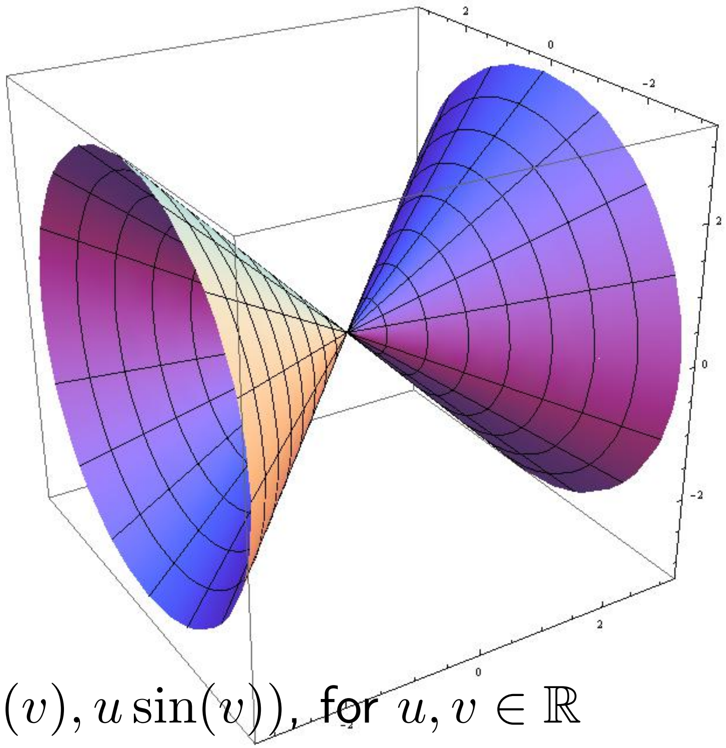

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→ the choice for surface design for CAD and graphics

EXAMPLES OF SURFACES

Some surfaces easier to parametrize: curve-based

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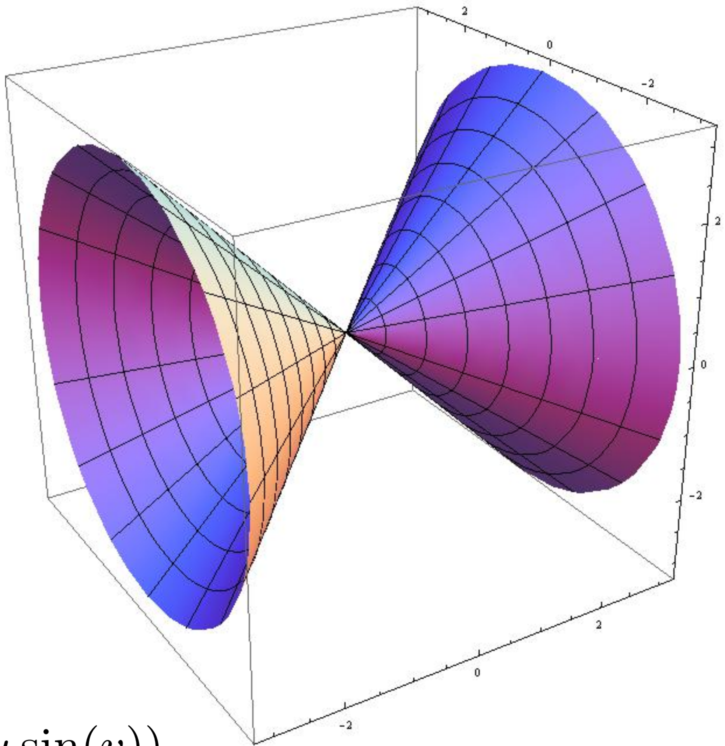

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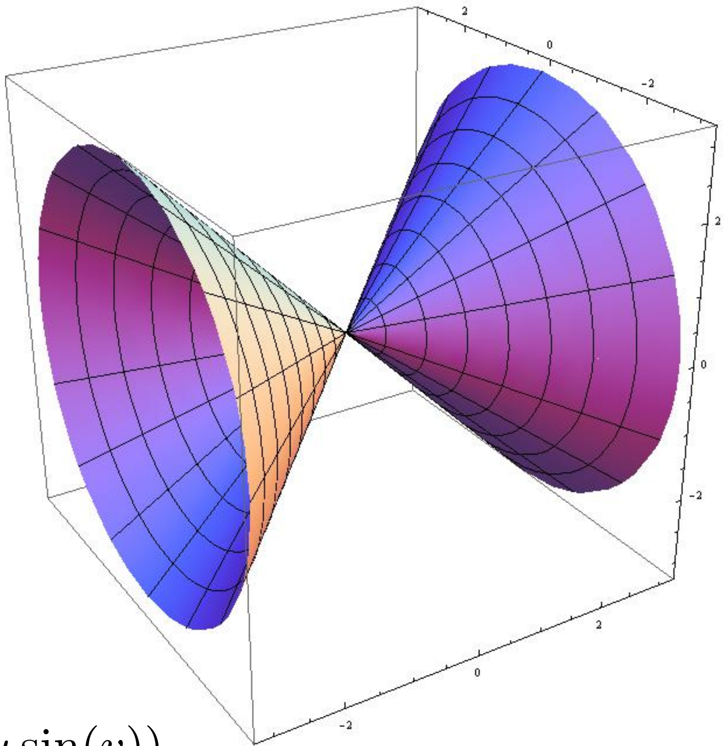

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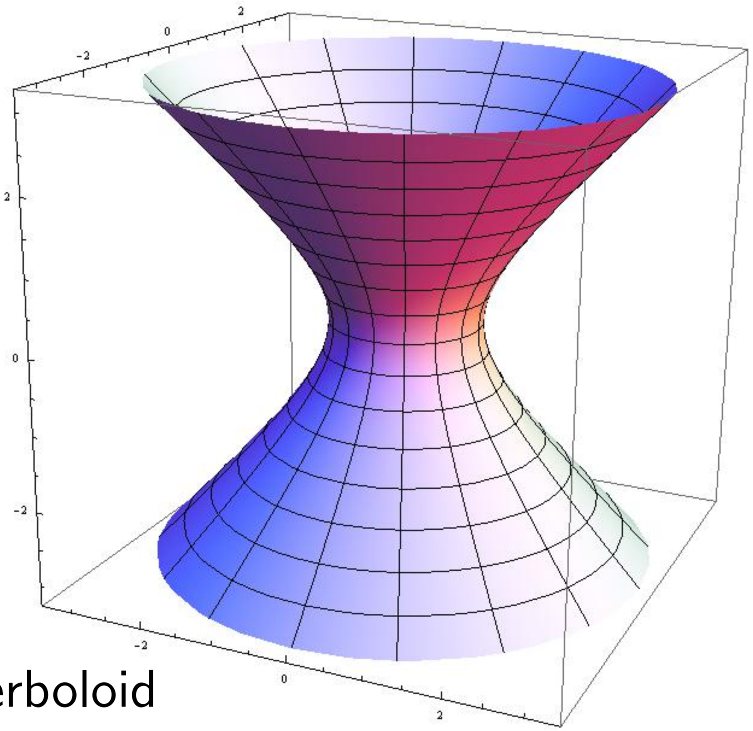

Hyperboloid

$(\sqrt{1+u^2}\cos v, \sqrt{1+u^2}\sin v, u)$

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Source: `http://math.arizona.edu/~models`

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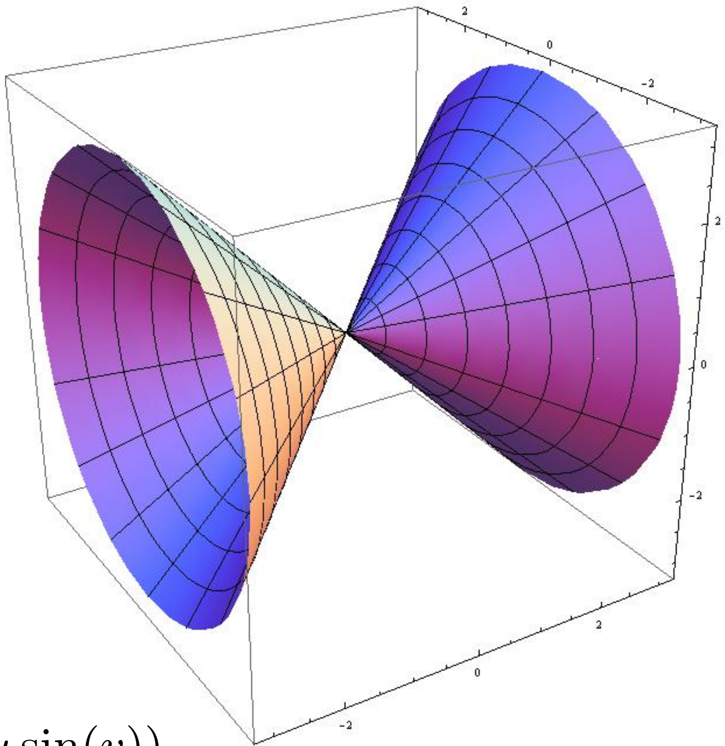

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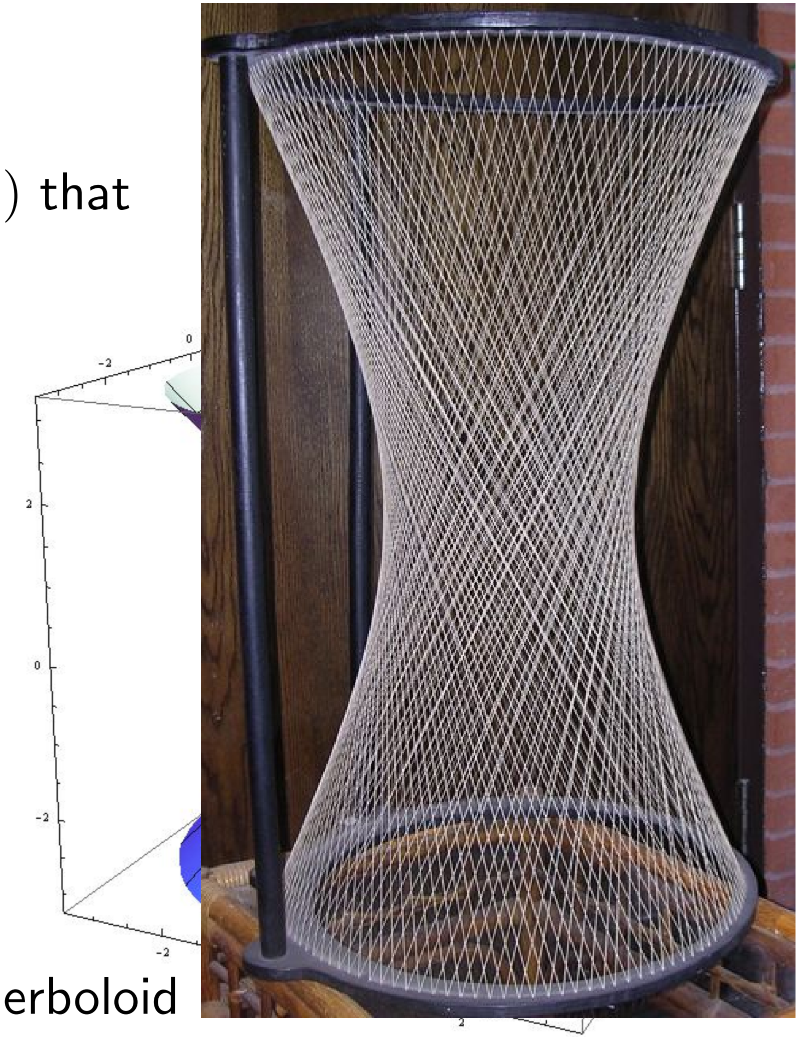

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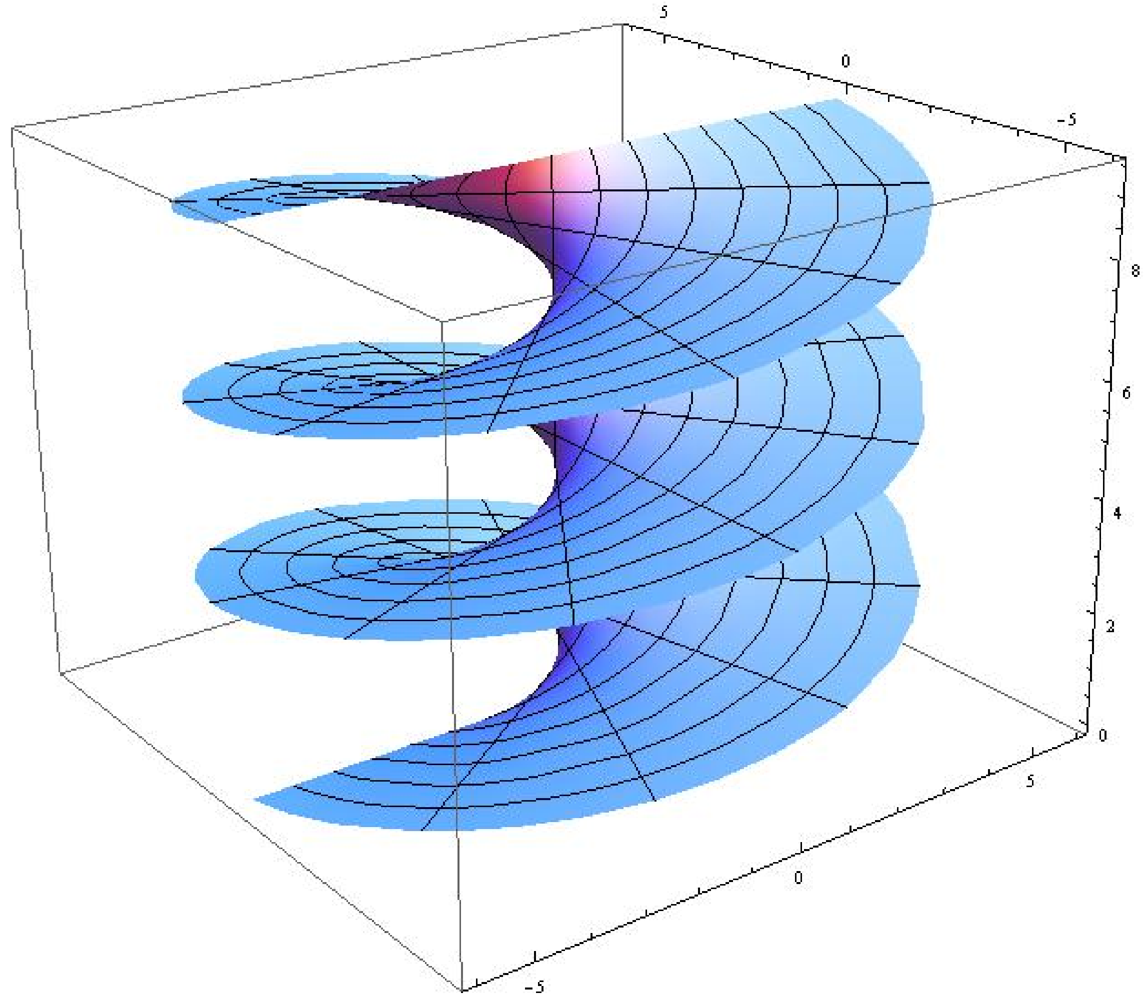
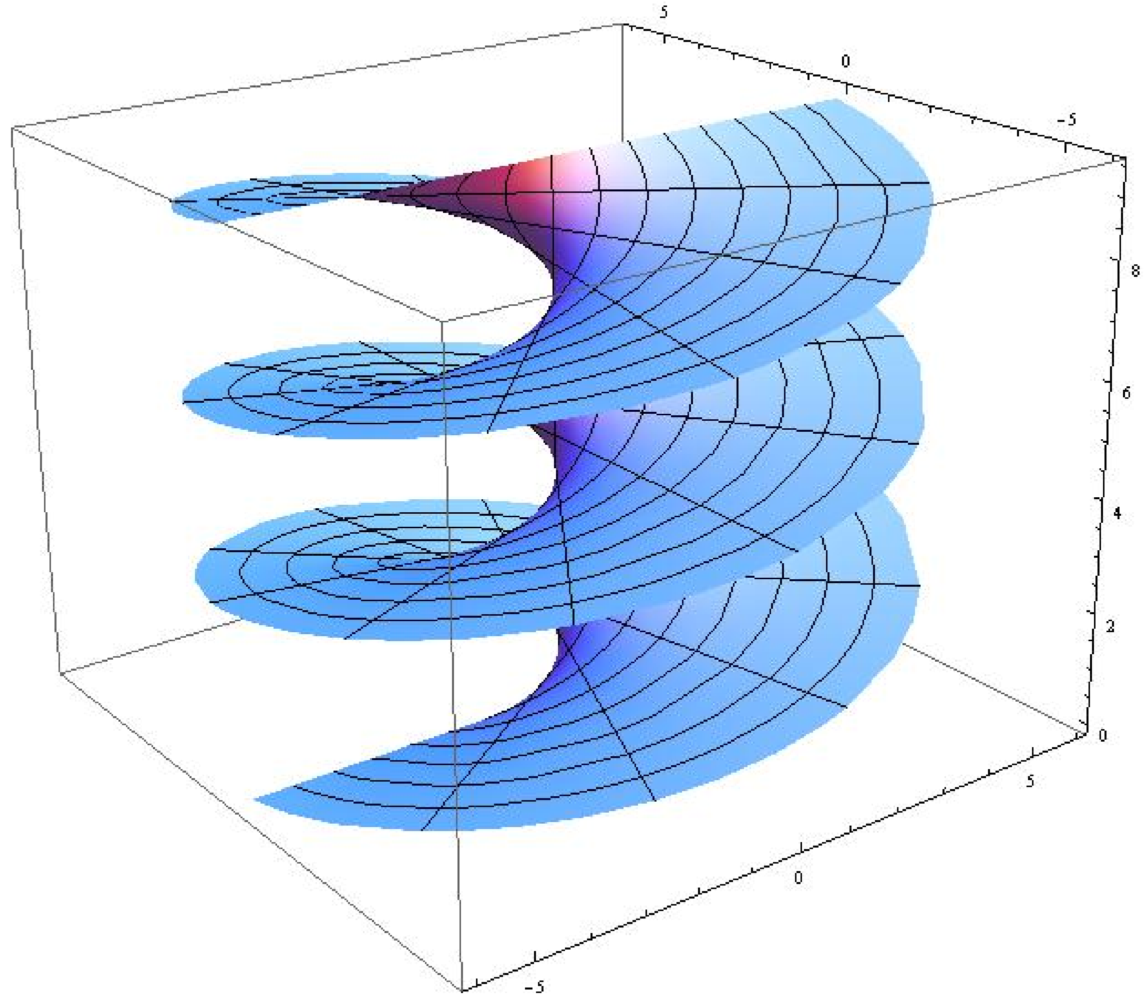


Ruled surface: helicoid

Consider a circular helix with axis $0z$

The *helicoid* associated is the set of all lines perpendicular to $0z$ that go through a point in $0z$ and one in the helix.



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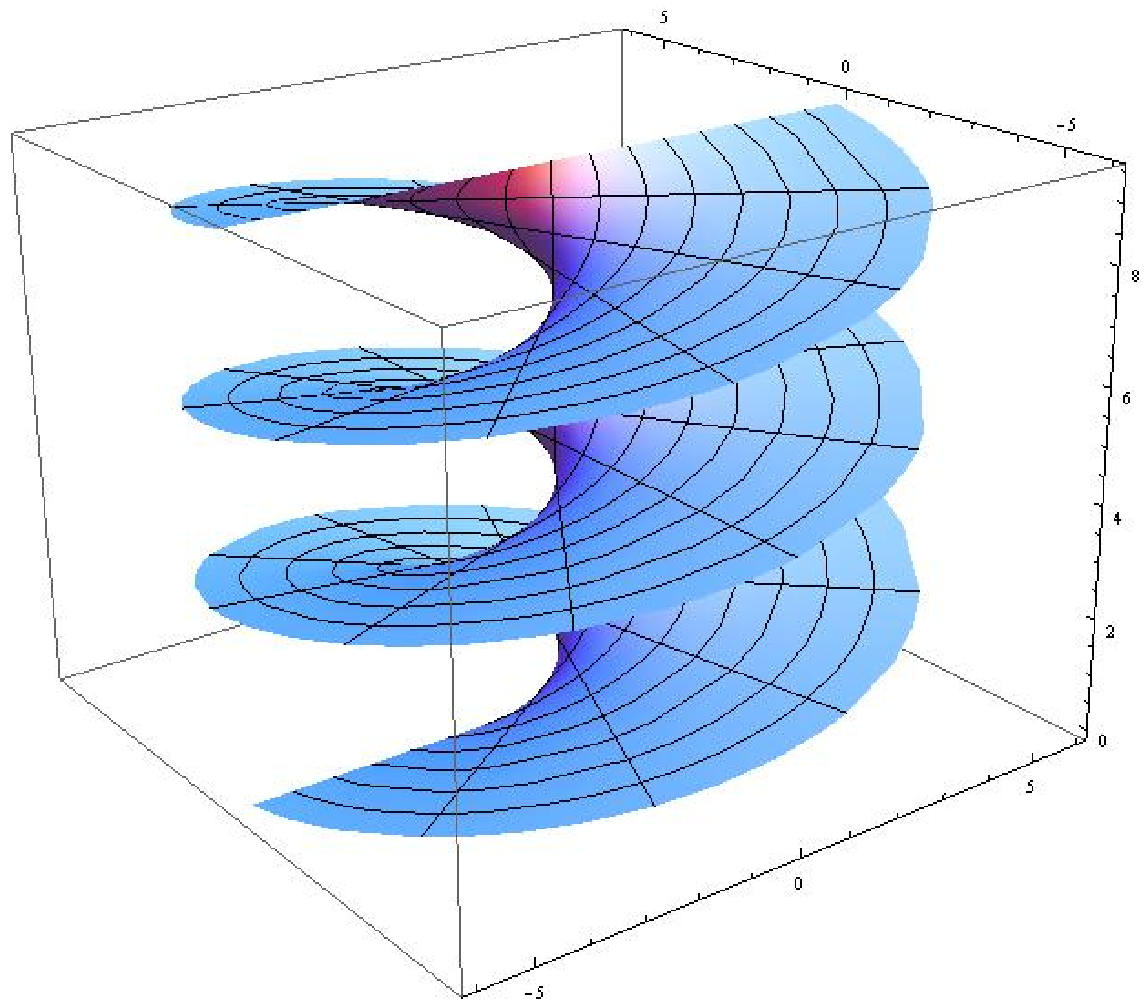

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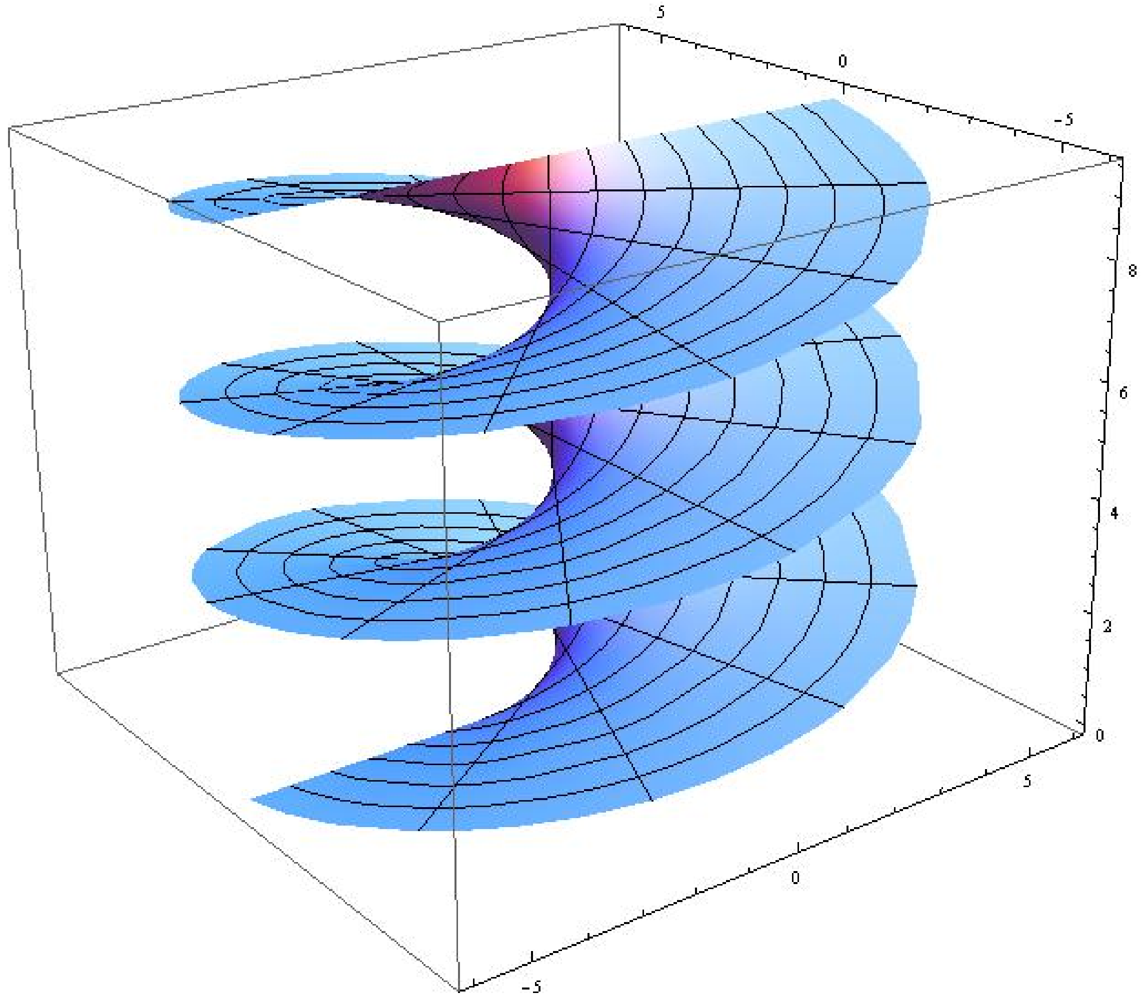

Parametrization

A point on circular helix:
$P(t) = (a\cos t, a\sin t, bt)$

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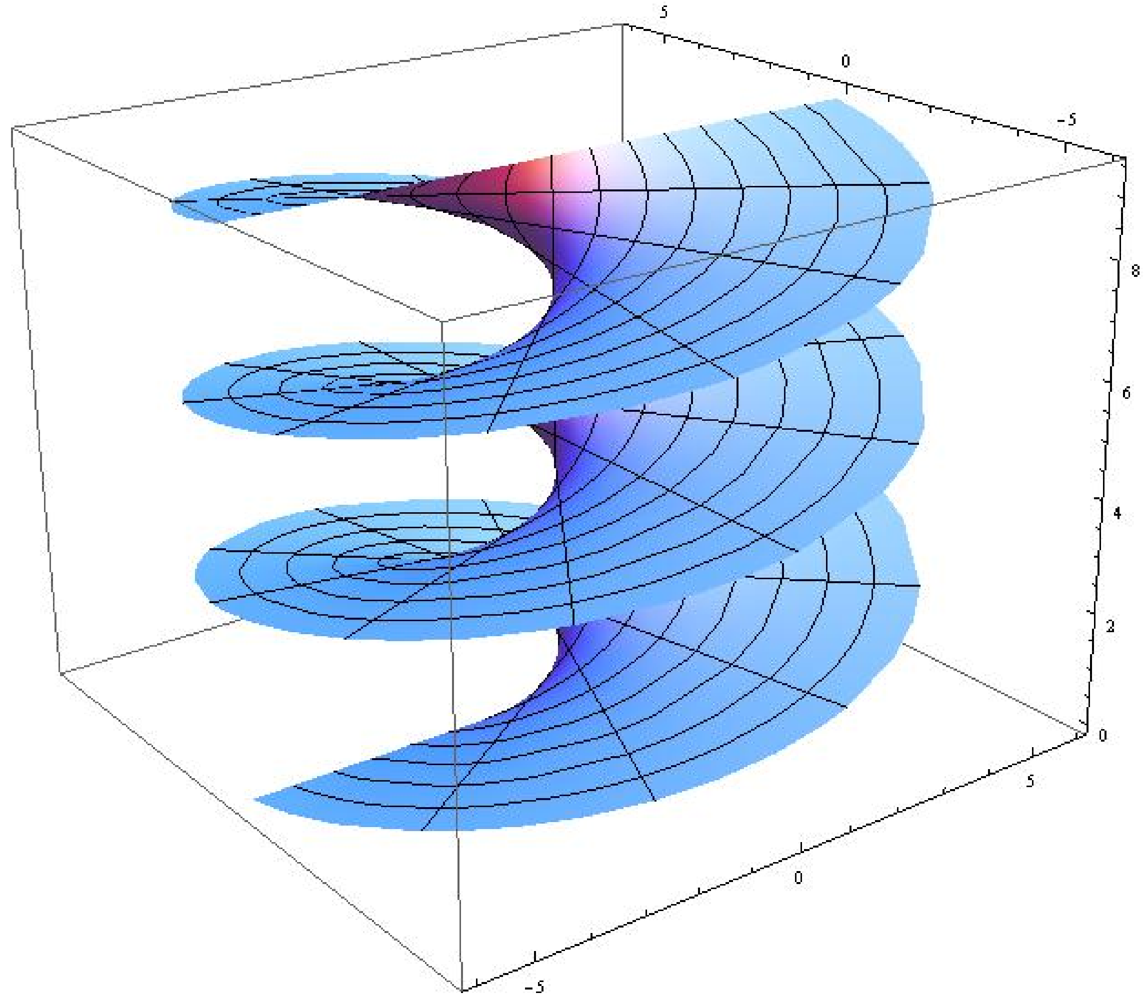

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To be horizontal, we set $s = bt$

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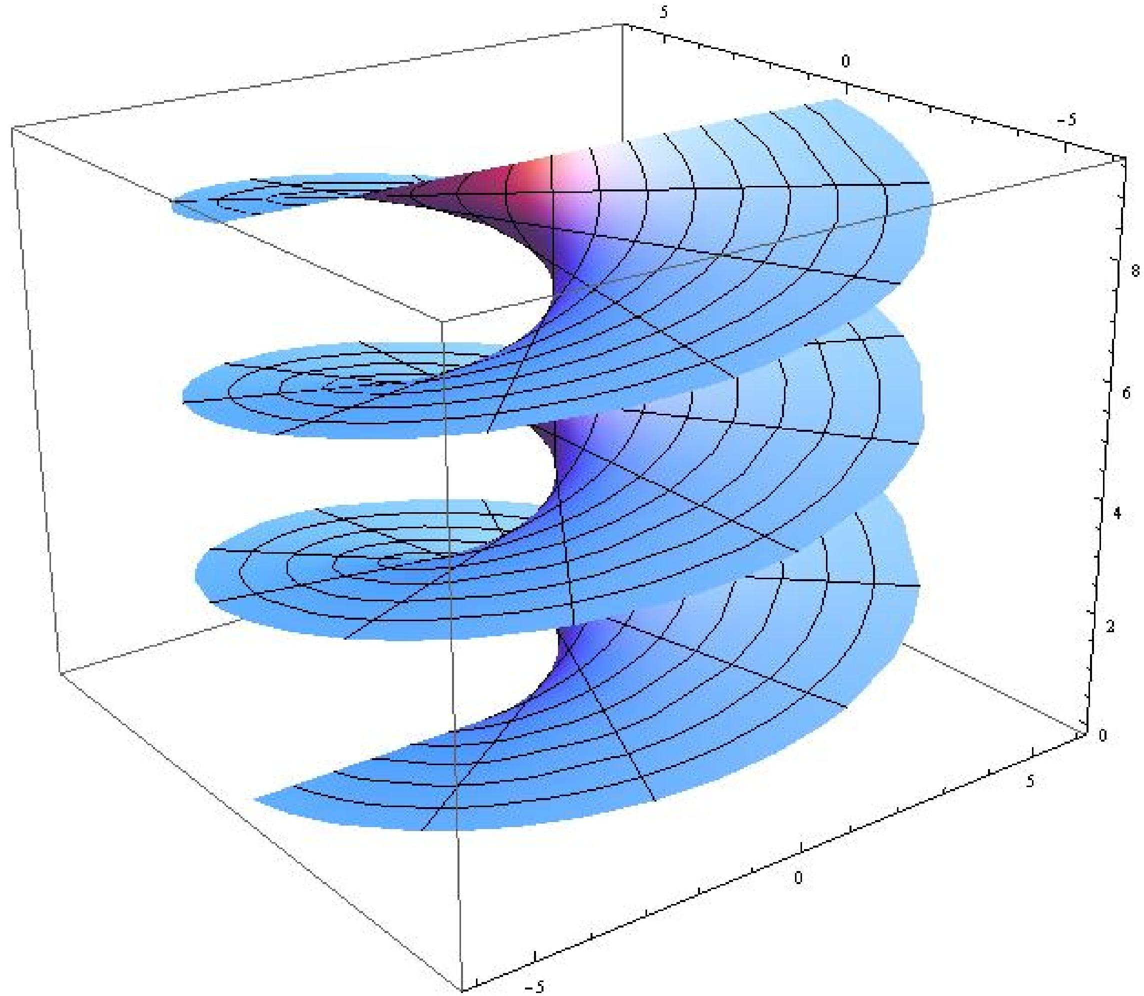

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To be horizontal, we set $s = bt$

Thus: $S(t,\lambda) = (1-\lambda)Q(t) + \lambda P(t)$
$= (1-\lambda)(0,0,bt) + \lambda(a\cos t, a\sin t, bt)$
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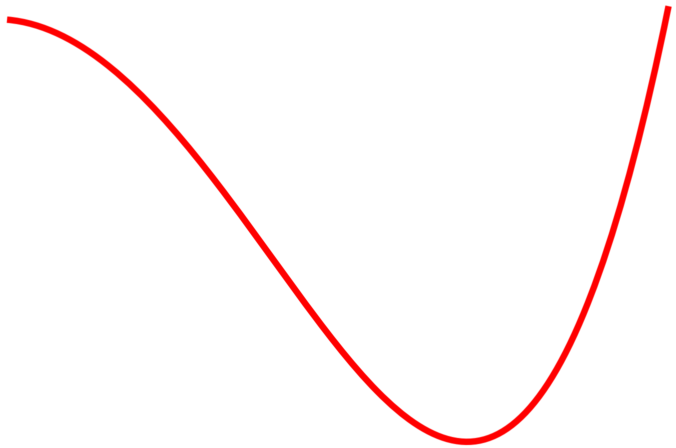

$$\gamma(t) = (x(t), y(t), z(t))$$

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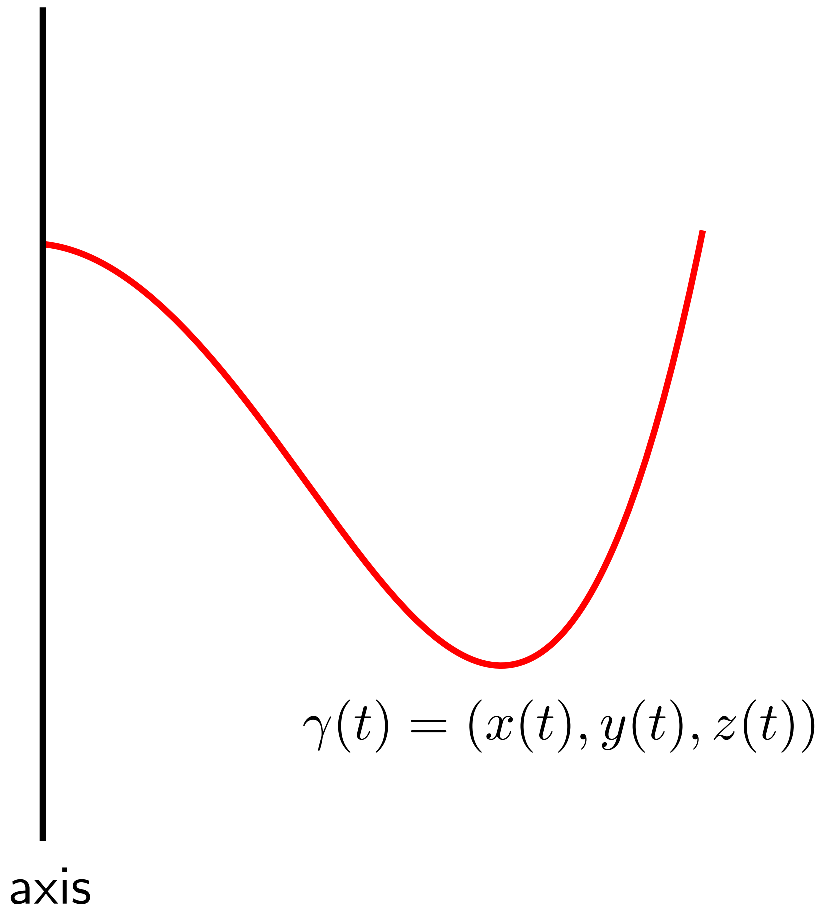

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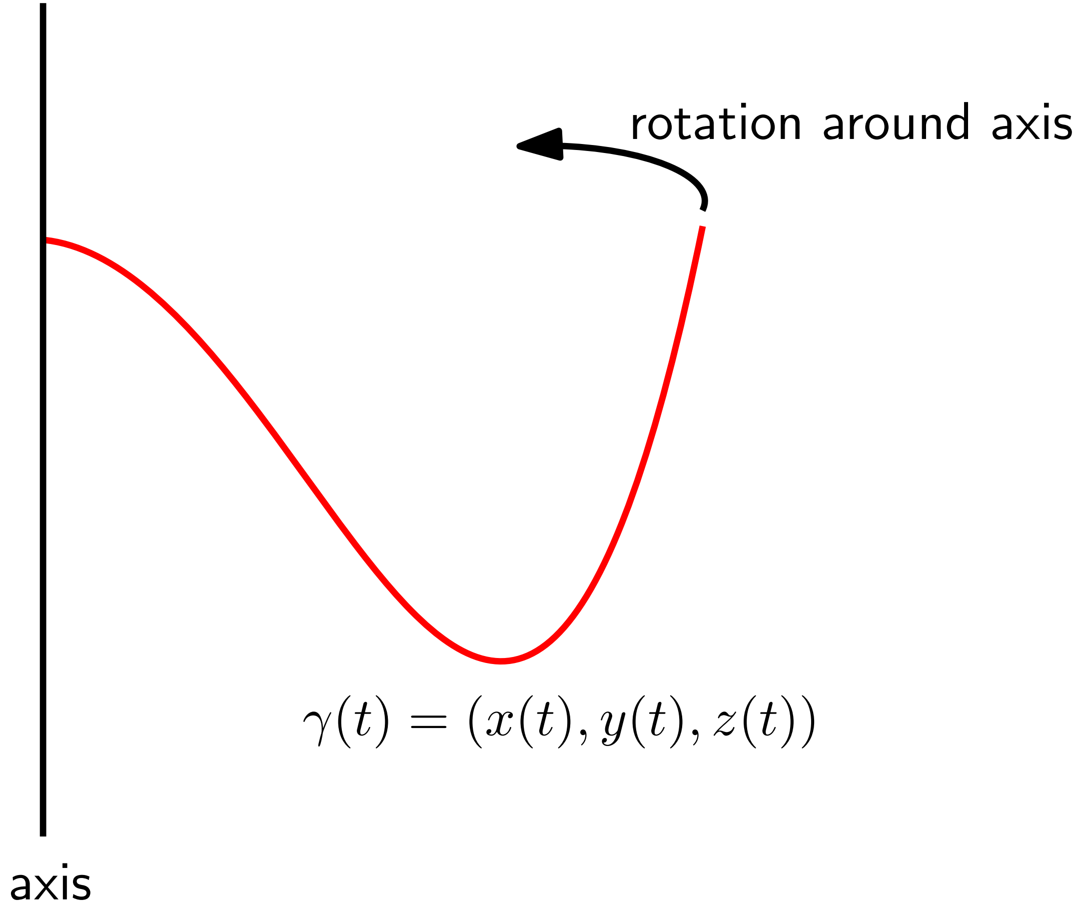

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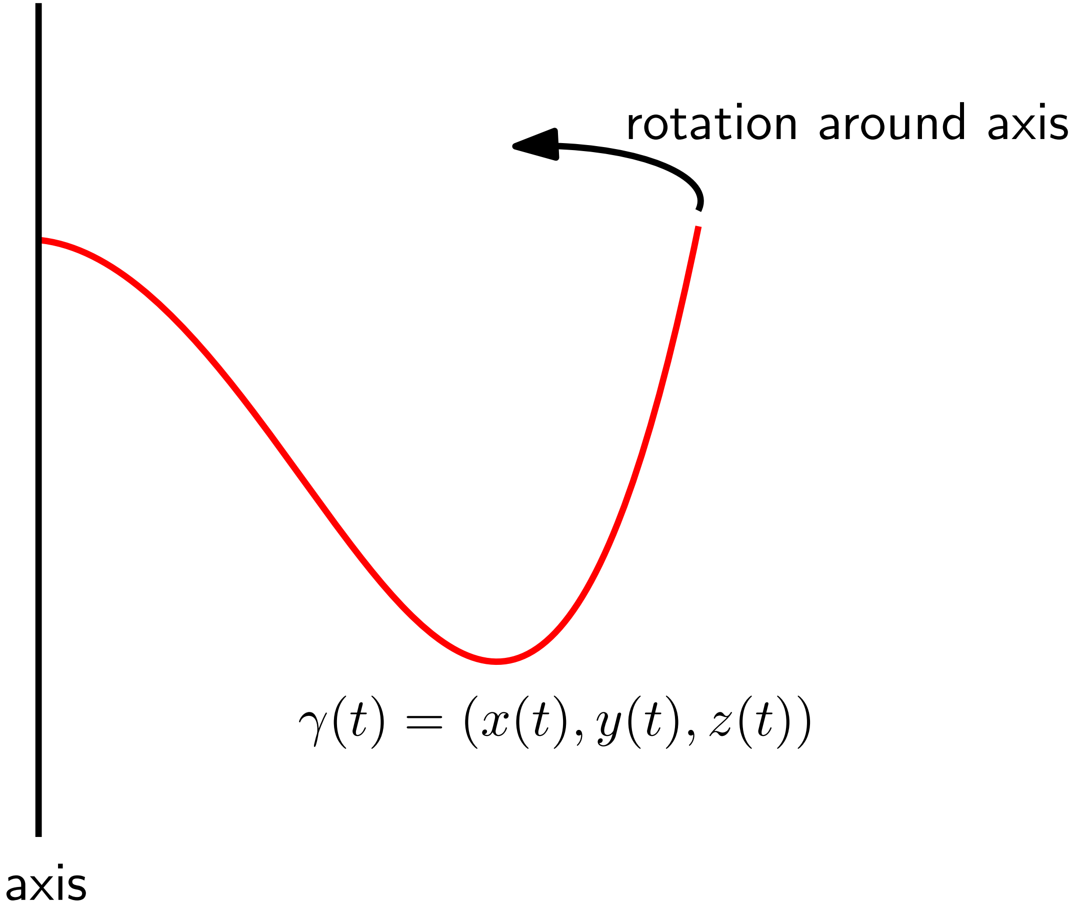

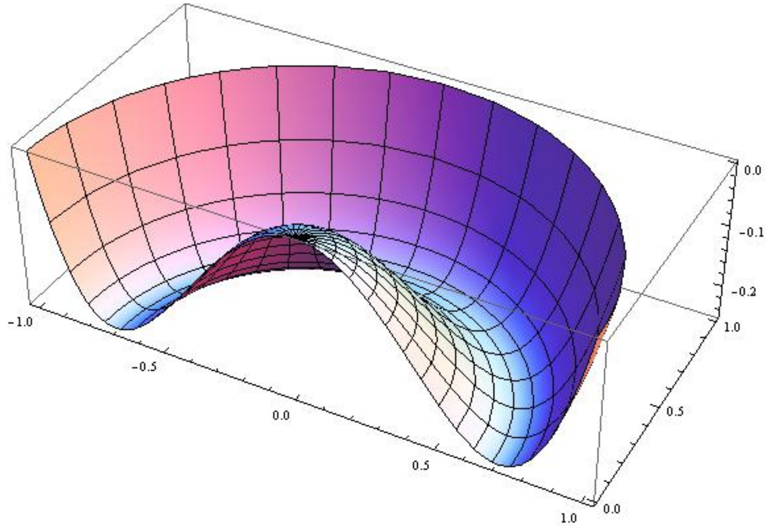

rotation from 0 to π

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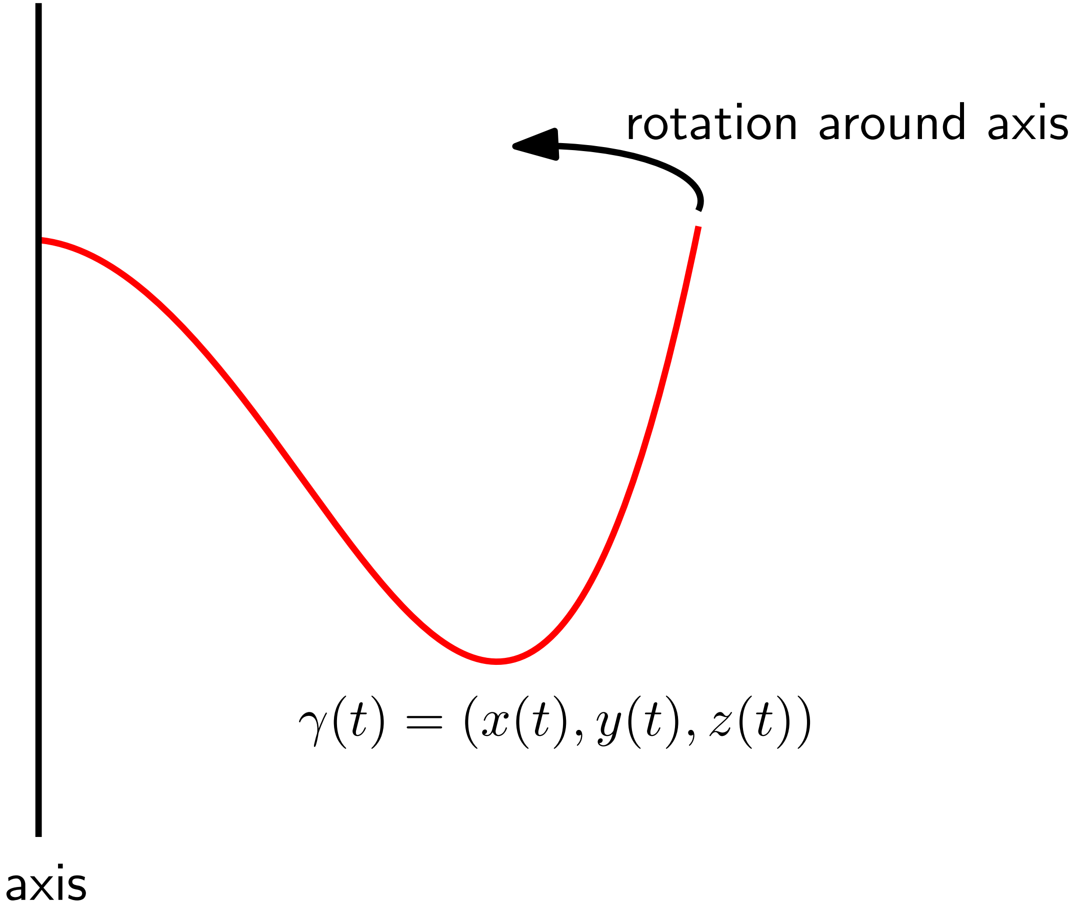

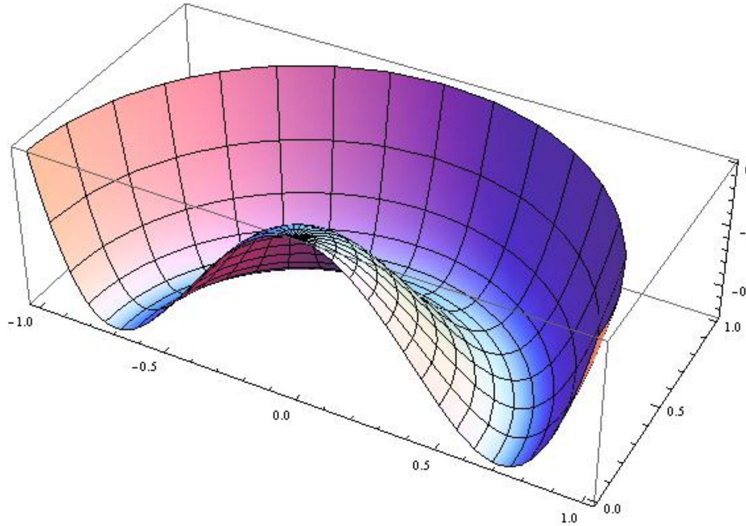

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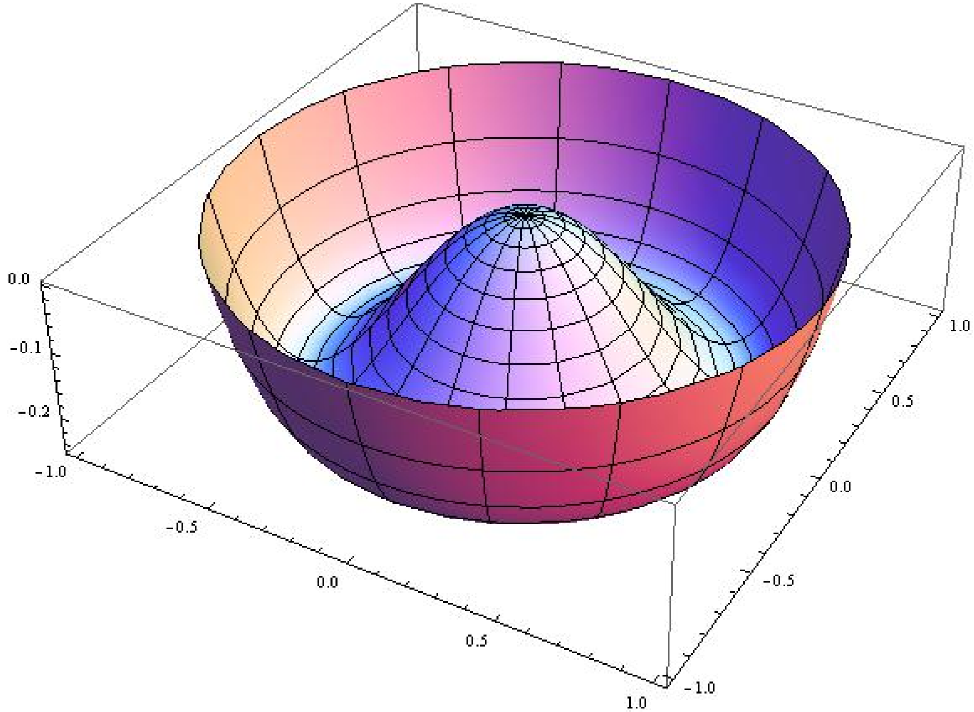

rotation from 0 to 2π

Surfaces of revolution

Surface created by rotating a curve (*generatrix*)
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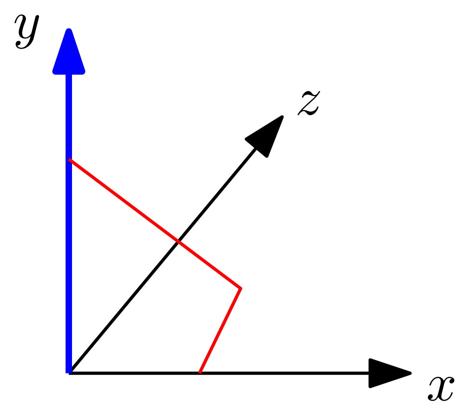
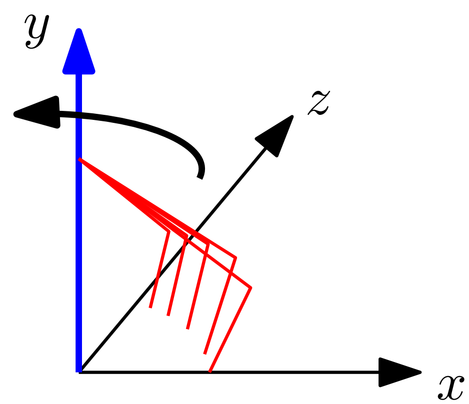
Surfaces of revolution



Surface created by rotating a curve (*generatrix*) around a line (*axis*)

Given a parametrization of the generatrix curve, say, in the xy-plane, so $P(t) = (x(t), y(t), 0)$, $t \in [0, 1]$, and an axis, say $0y$, we obtain the parametrization of the surface of revolution around tha axis as follows:

Surfaces of revolution



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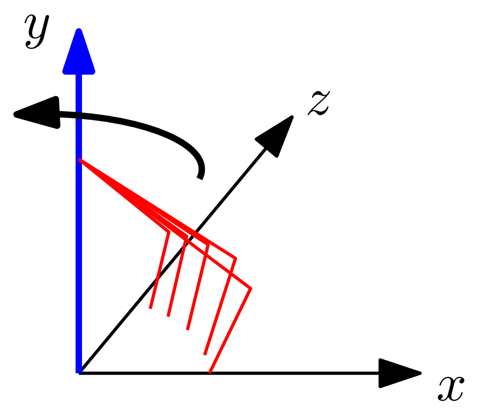

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rotation by angle w around y-axis

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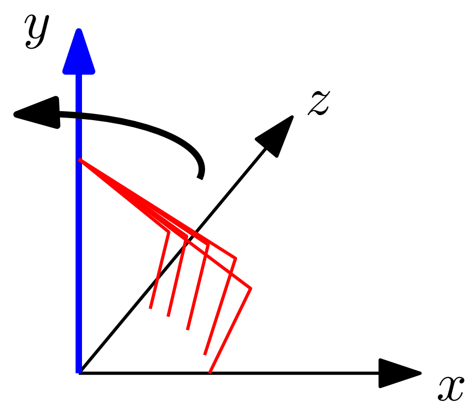

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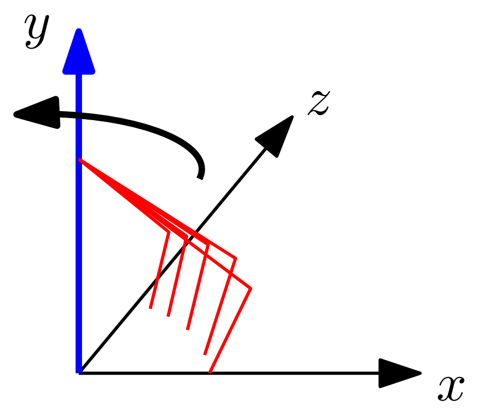
We apply the rotation as follows

$$P(u)T_y(w)$$

for $u \in [0, 1]$ and $w \in [0, 2\pi]$

Surfaces of revolution



Surface created by rotating a curve (*generatrix*) around a line (*axis*)

Given a parametrization of the generatrix curve, say, in the xy-plane, so $P(t) = (x(t), y(t), 0)$, $t \in [0,1]$, and an axis, say $0y$, we obtain the parametrization of the surface of revolution around tha axis as follows:

$$T_y(w) = \begin{pmatrix} \cos w & 0 & \sin w \\ 0 & 1 & 0 \\ -\sin w & 0 & \cos w \end{pmatrix}$$

rotation by angle w around y-axis

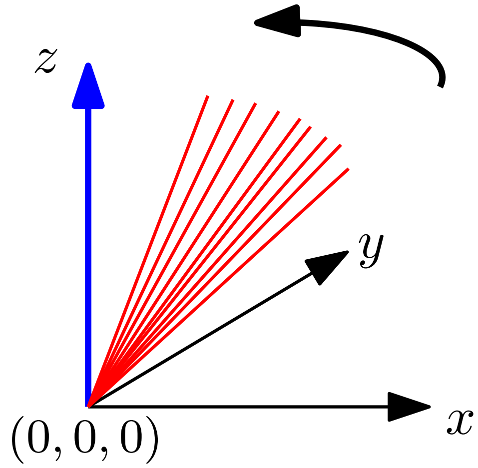
We apply the rotation as follows

$$P(u)T_y(w)$$

for $u \in [0,1]$ and $w \in [0, 2\pi]$

$$= (x(u)\cos w, y(u), x(u)\sin w)$$

Example of surface of revolution: cone



What is a cone? Let's start from the *implicit equation*

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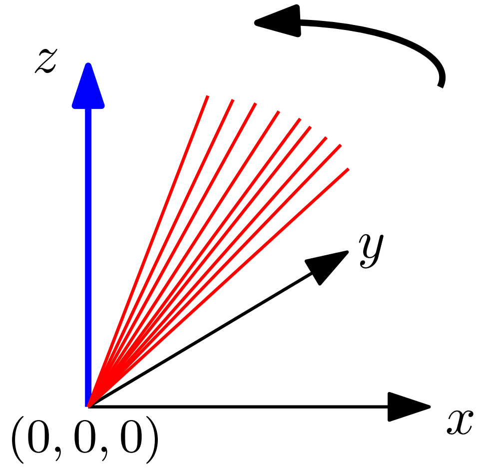

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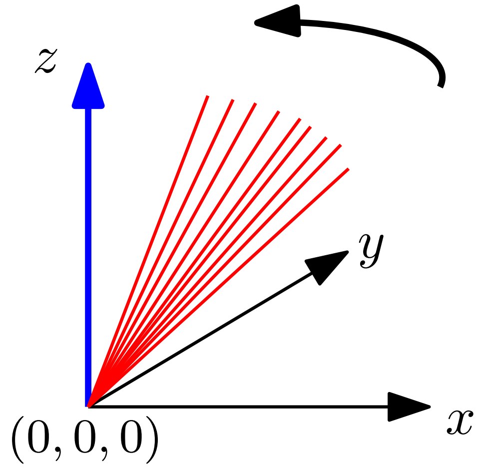

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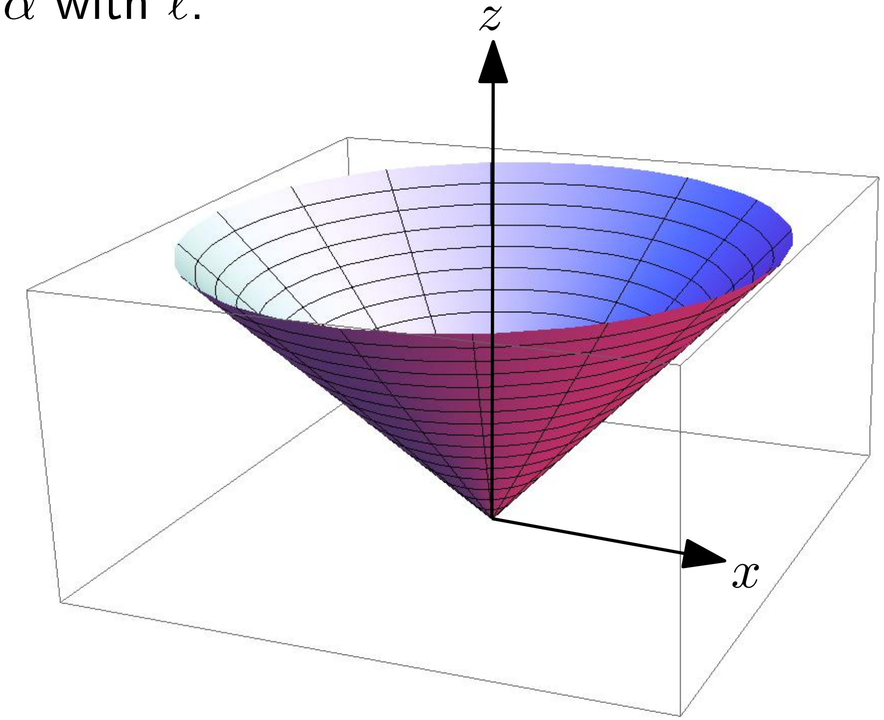

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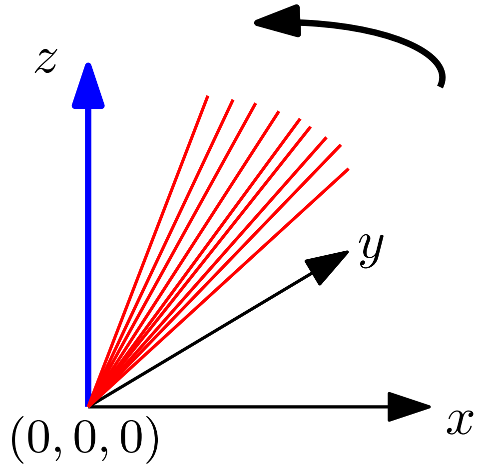

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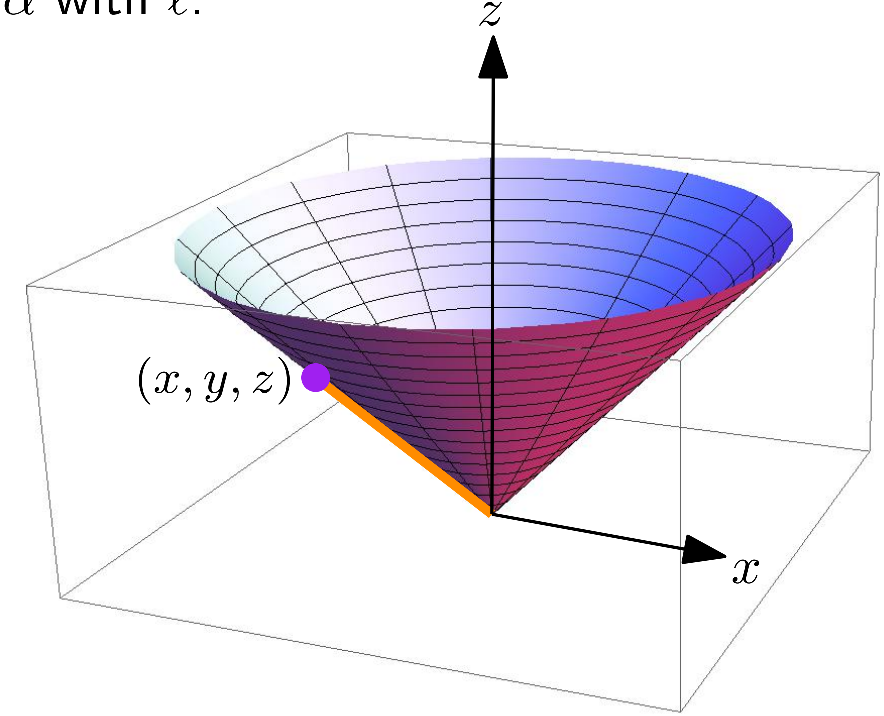

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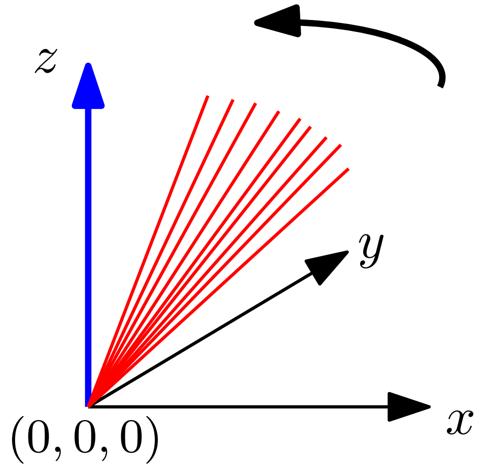

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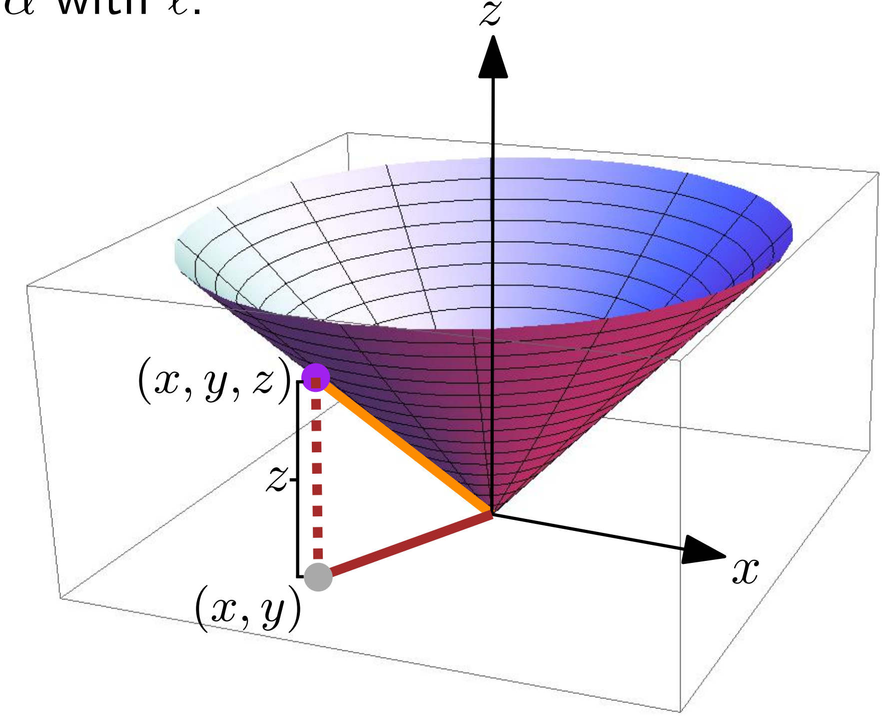

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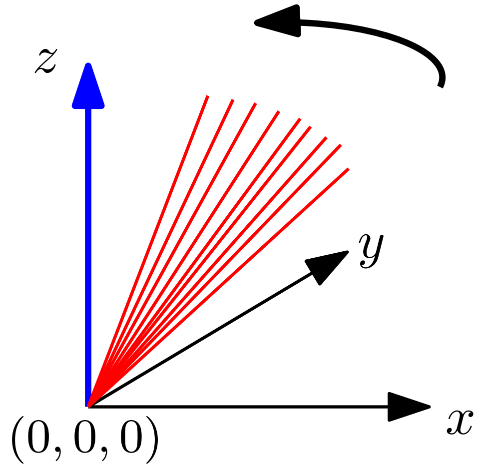

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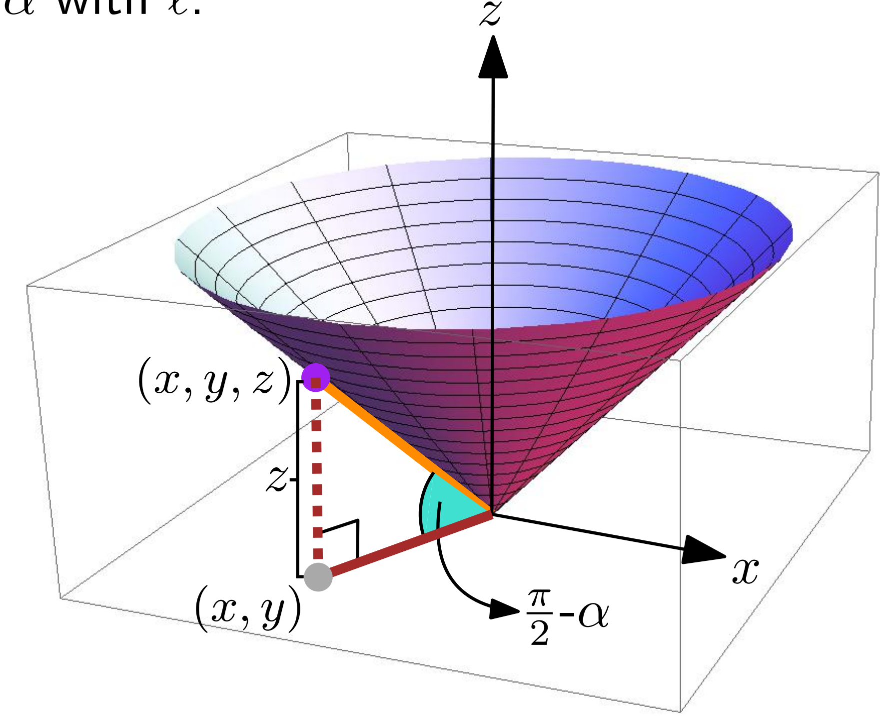

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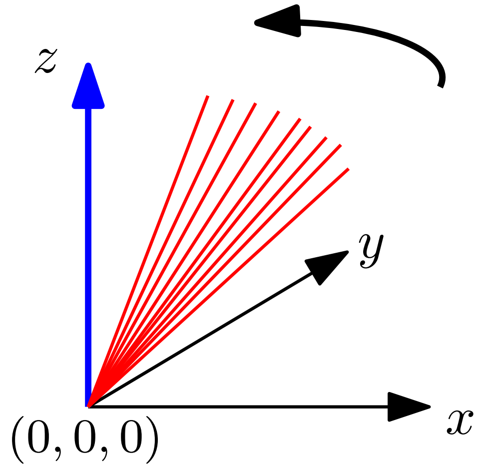

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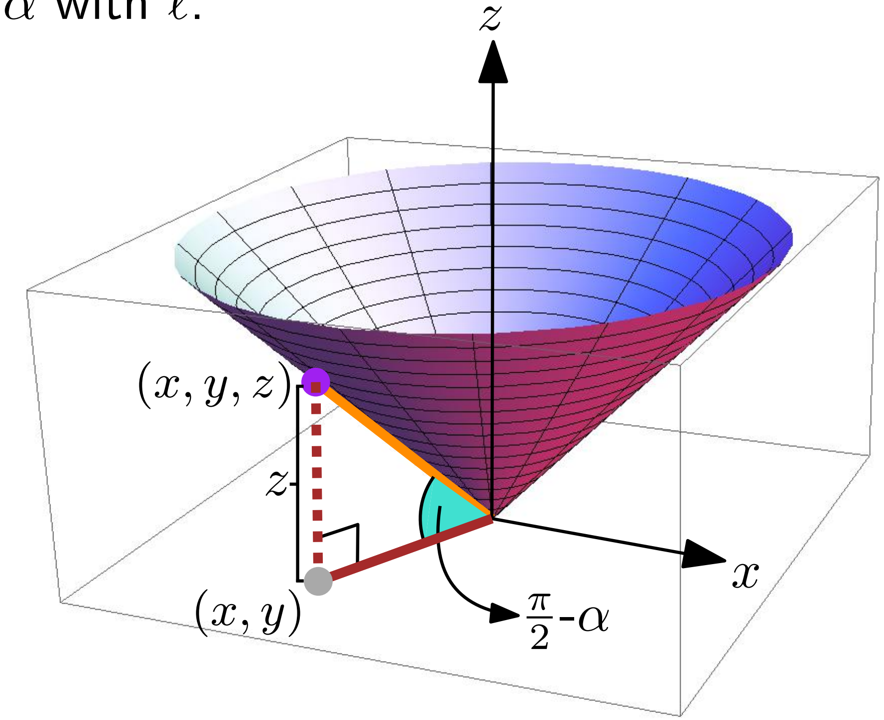

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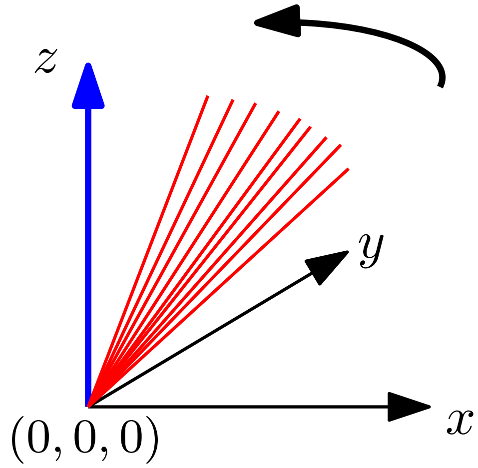

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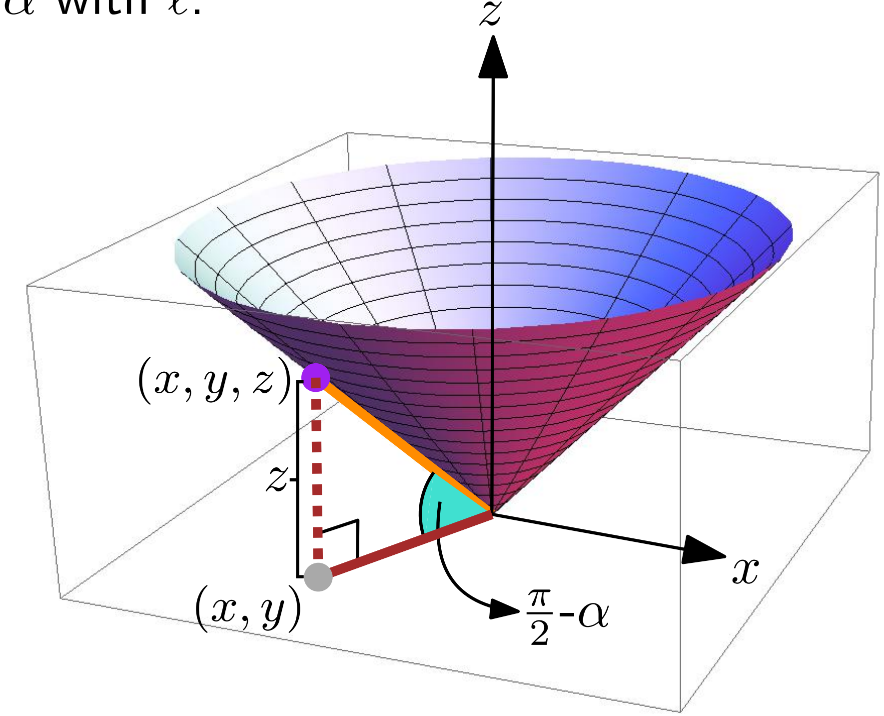

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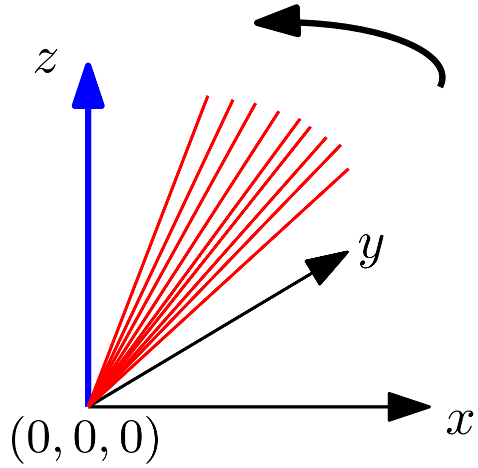

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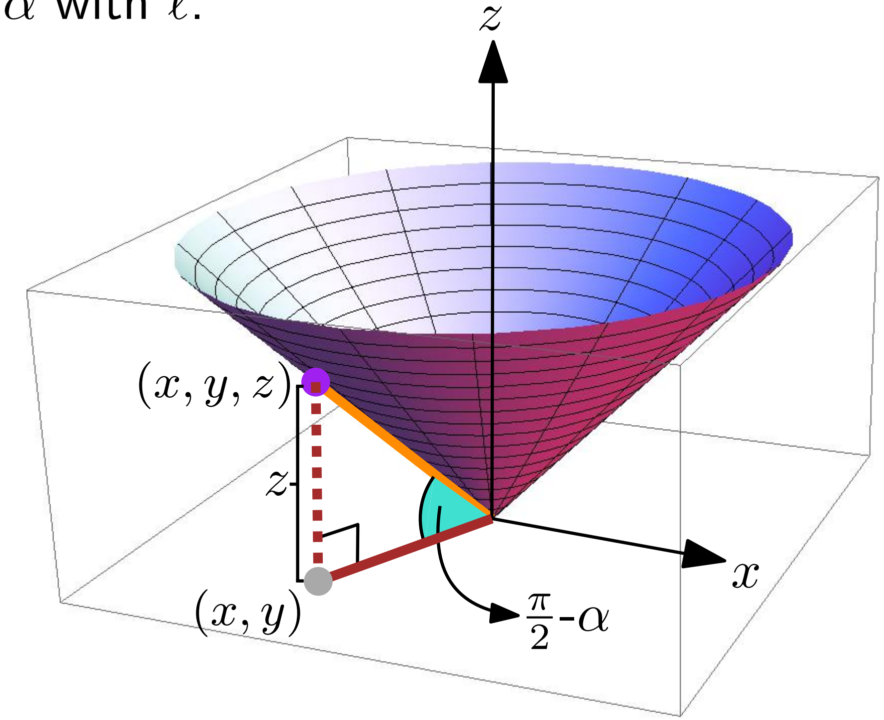

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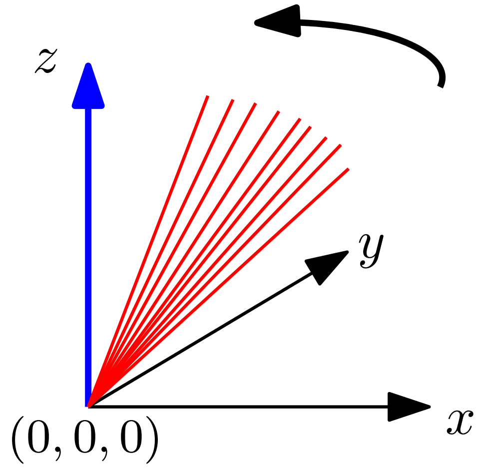

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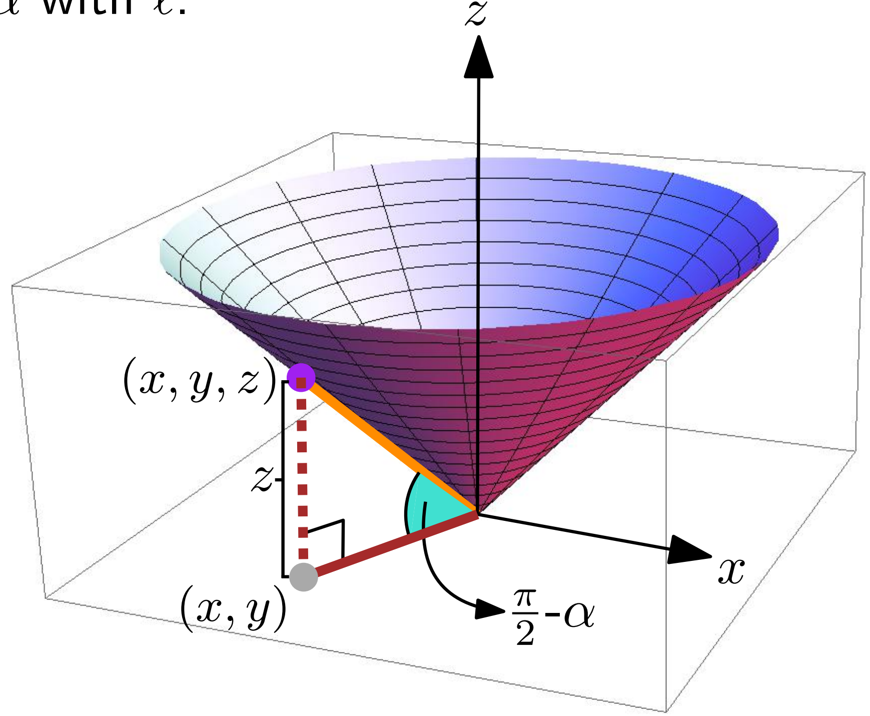
implicit equation of the cone



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Example of surface of revolution: cone

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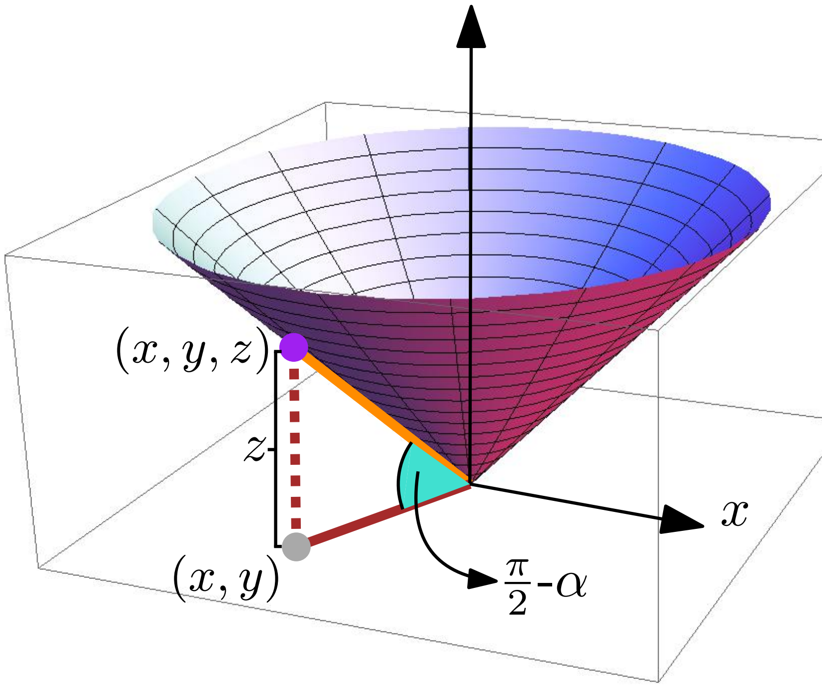

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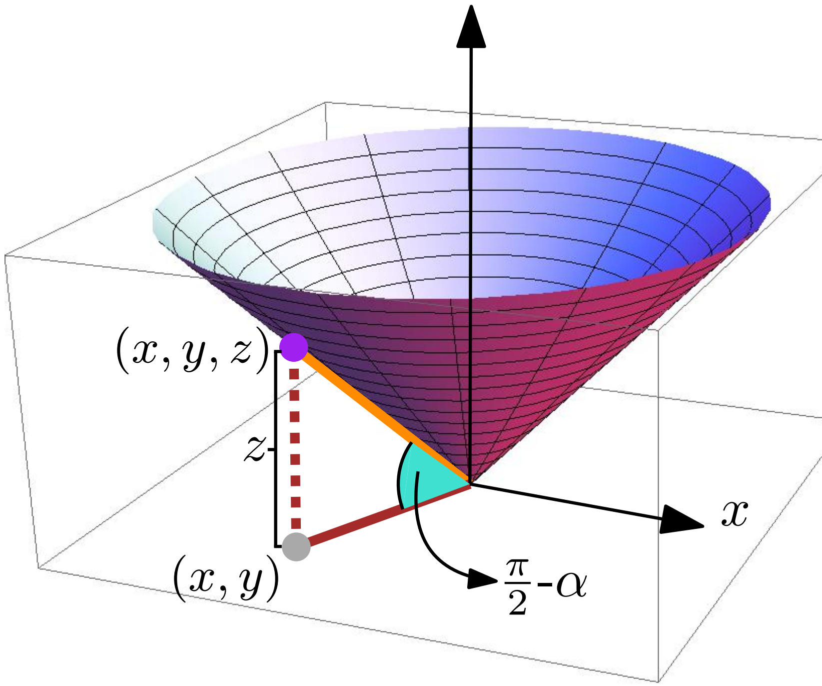

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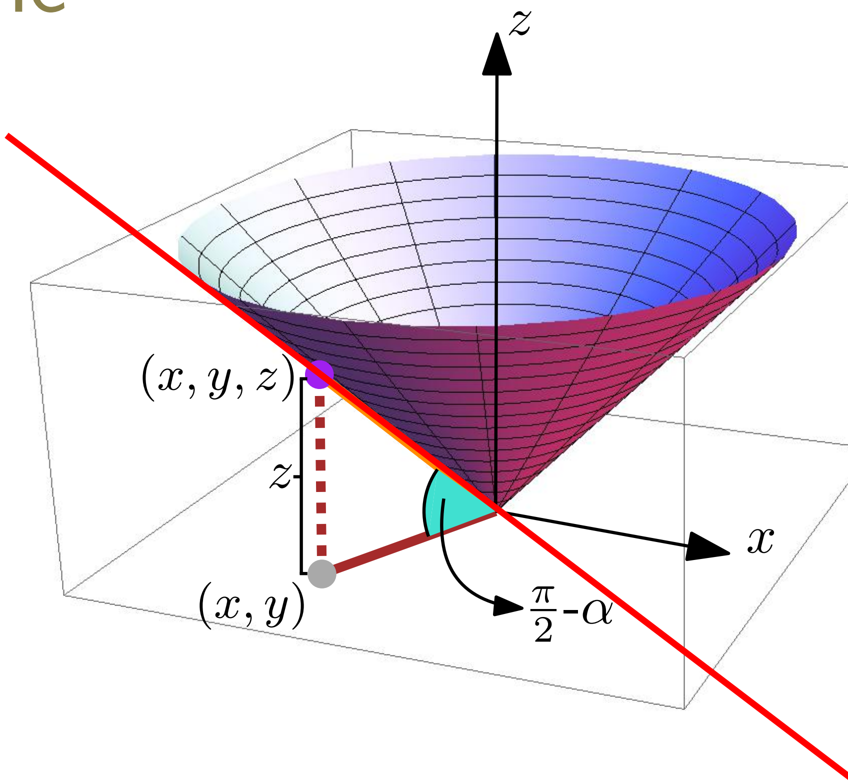

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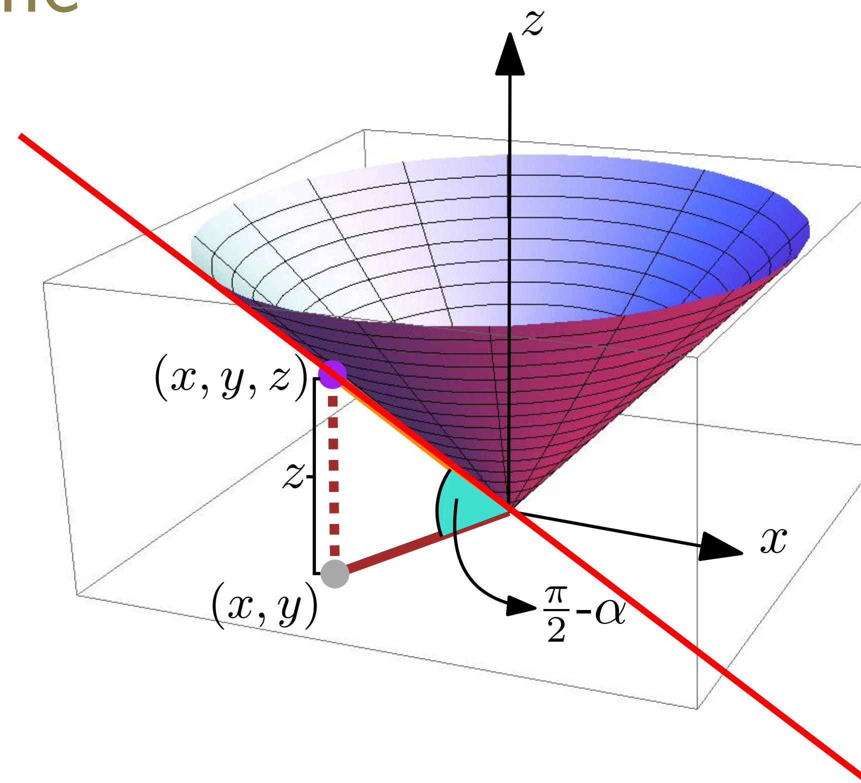

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$P(u) = u\vec{\mathbf{v}}$, for $u \in \mathbb{R}$, and some $\vec{\mathbf{v}}$ in $\mathbb{R}^3$

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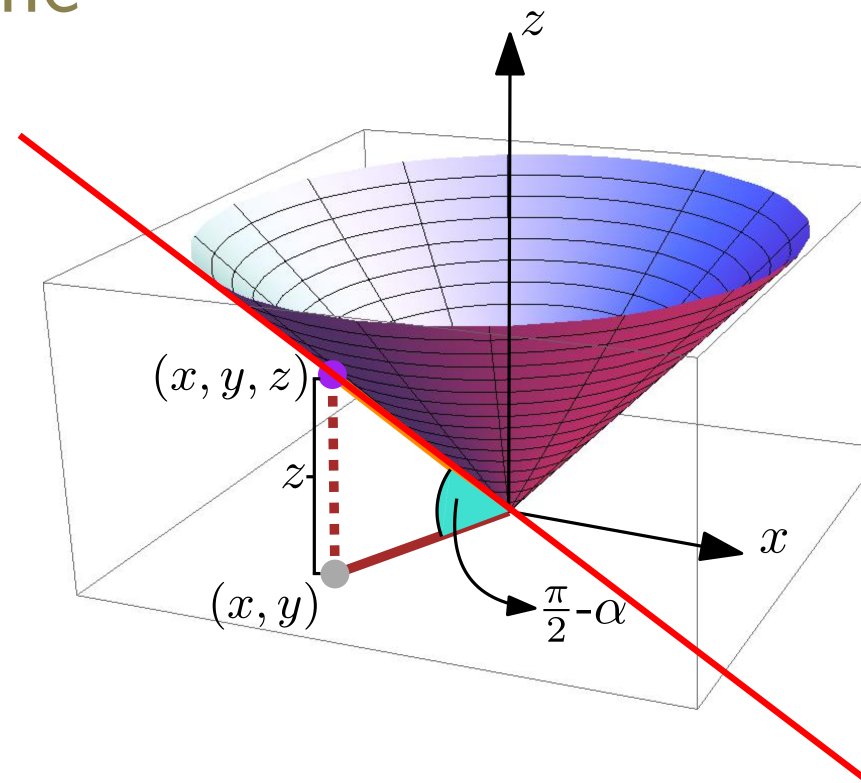

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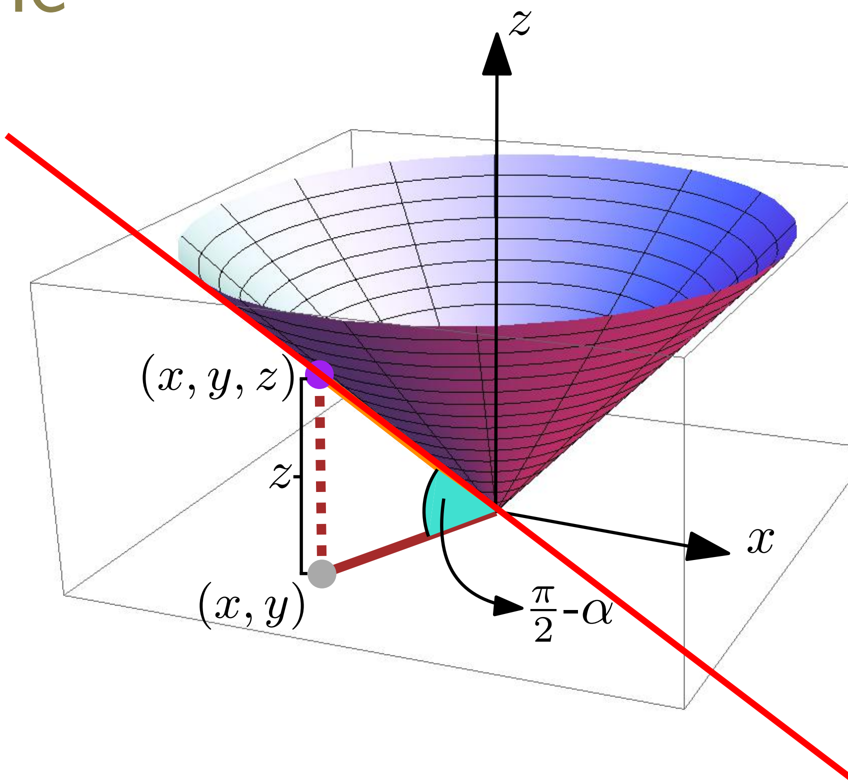

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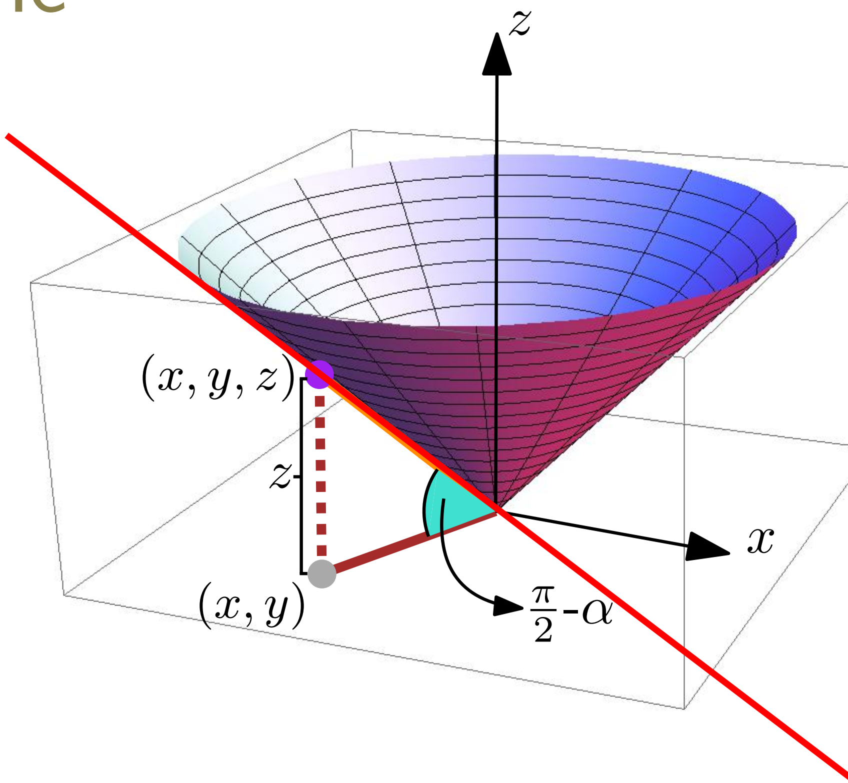

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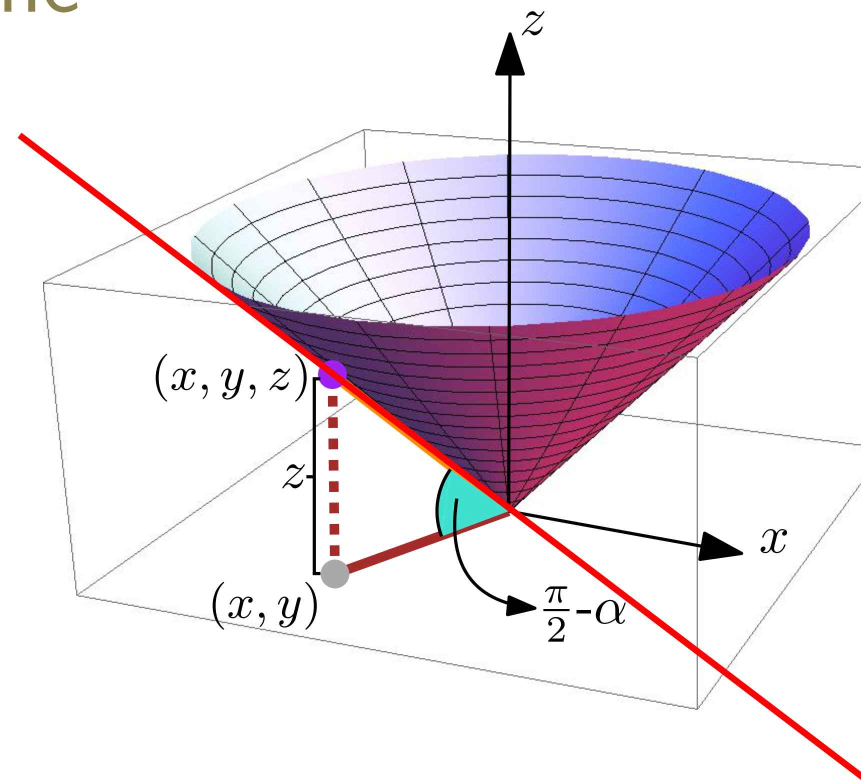

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$$C(u) = P(u)T_z(w) = (x(u)\cos w, x(u)\sin w, y(u))$$

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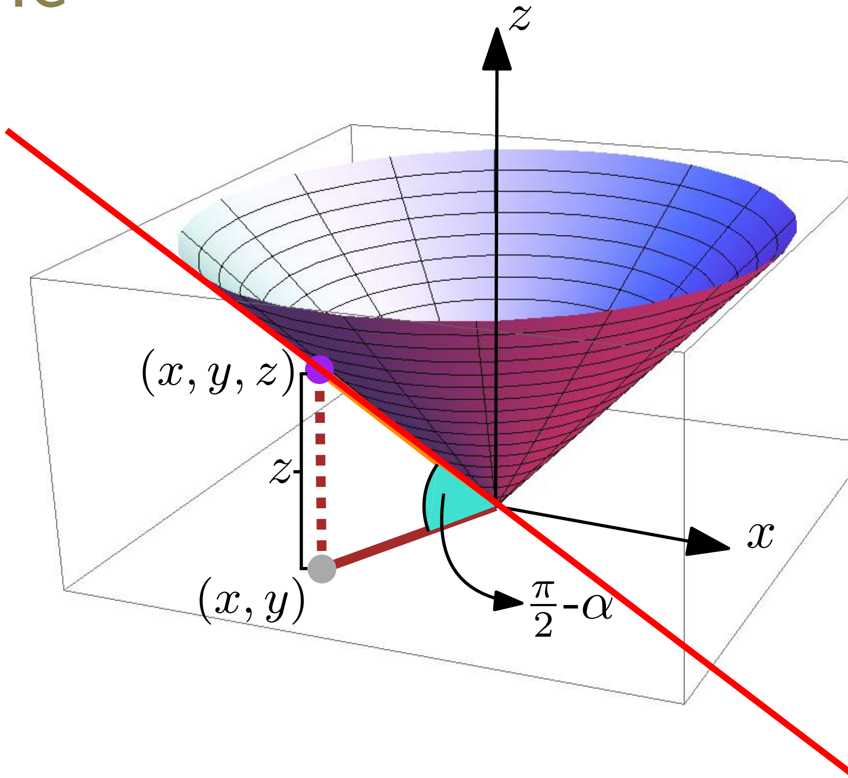

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$= \boxed{(u\cos w, u\sin w, \frac{u}{\tan\alpha})}$ for $u \in \mathbb{R}$, $w \in [0, 2\pi]$

parametric equation of the cone

Normal vector, tangent plane

Let S be a surface parametrized as
$S(u, v) = (x(u, v), y(u, v), z(u, v))$ for (u, v) in some domain

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If the functions x, y, z have partial derivatives, we consider $\begin{cases} \frac{\partial S}{\partial u} = (\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}) \\ \frac{\partial S}{\partial v} = (\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}) \end{cases}$

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A point P is called *regular* if $\frac{\partial S}{\partial u}(P)$ and $\frac{\partial S}{\partial v}(P)$:

- exist
- are continuous at P
- their cross product is not zero

If one of these conditions does not hold for P, it is called a *singular* point

A surface is called *regular* if all its points are regular

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Normal vector

If P is regular, then $\vec{N} = \frac{\partial S}{\partial u}(P) \times \frac{\partial S}{\partial v}(P)$ (normal vector to S at P)

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Tangent plane

The plane tangent to S at P is given by the plane defined by P and $\vec{N}$

Normal vector, tangent plane

$\frac{\partial S}{\partial u}(P)$ and $\frac{\partial S}{\partial v}(P)$ are tangent vectors in the u and v directions

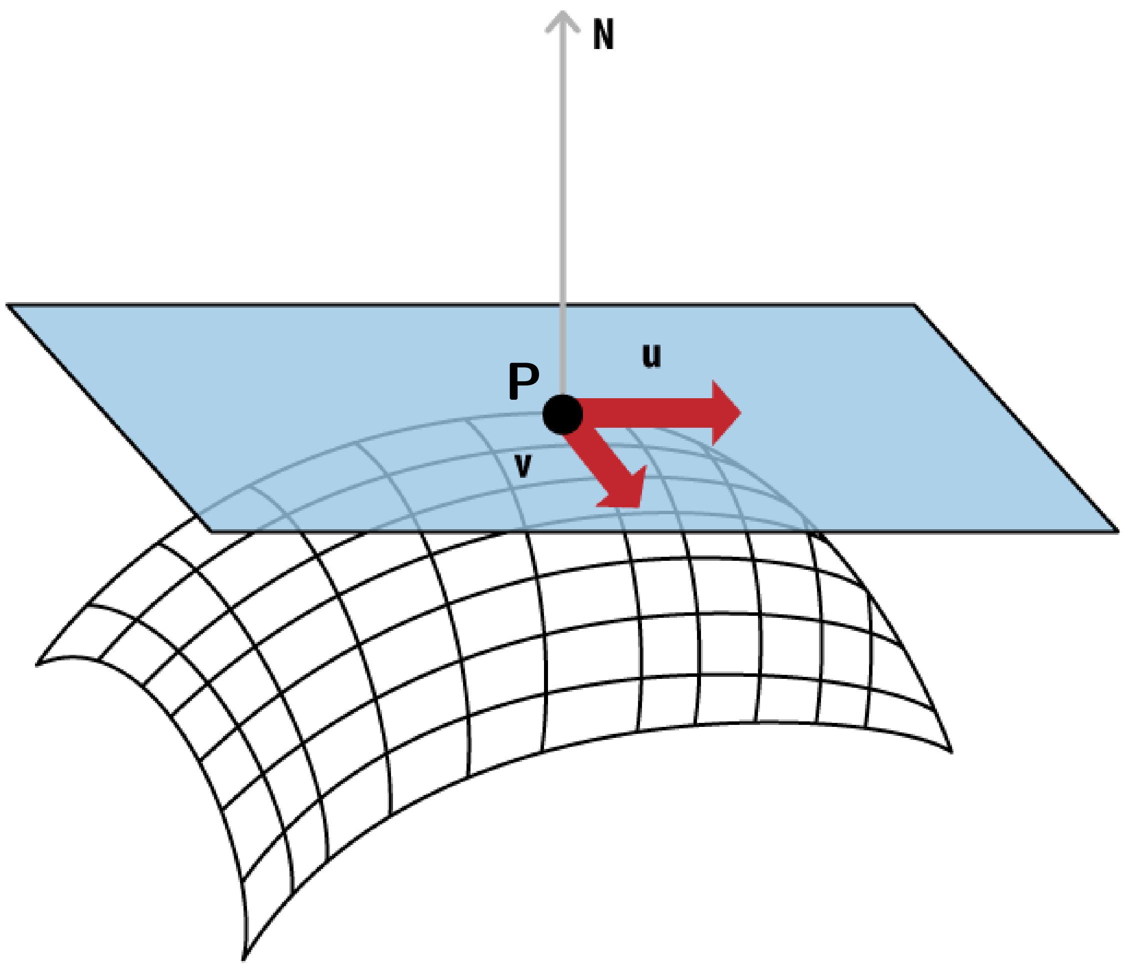

Figure from `oreilley.com`

Normal vector, tangent plane

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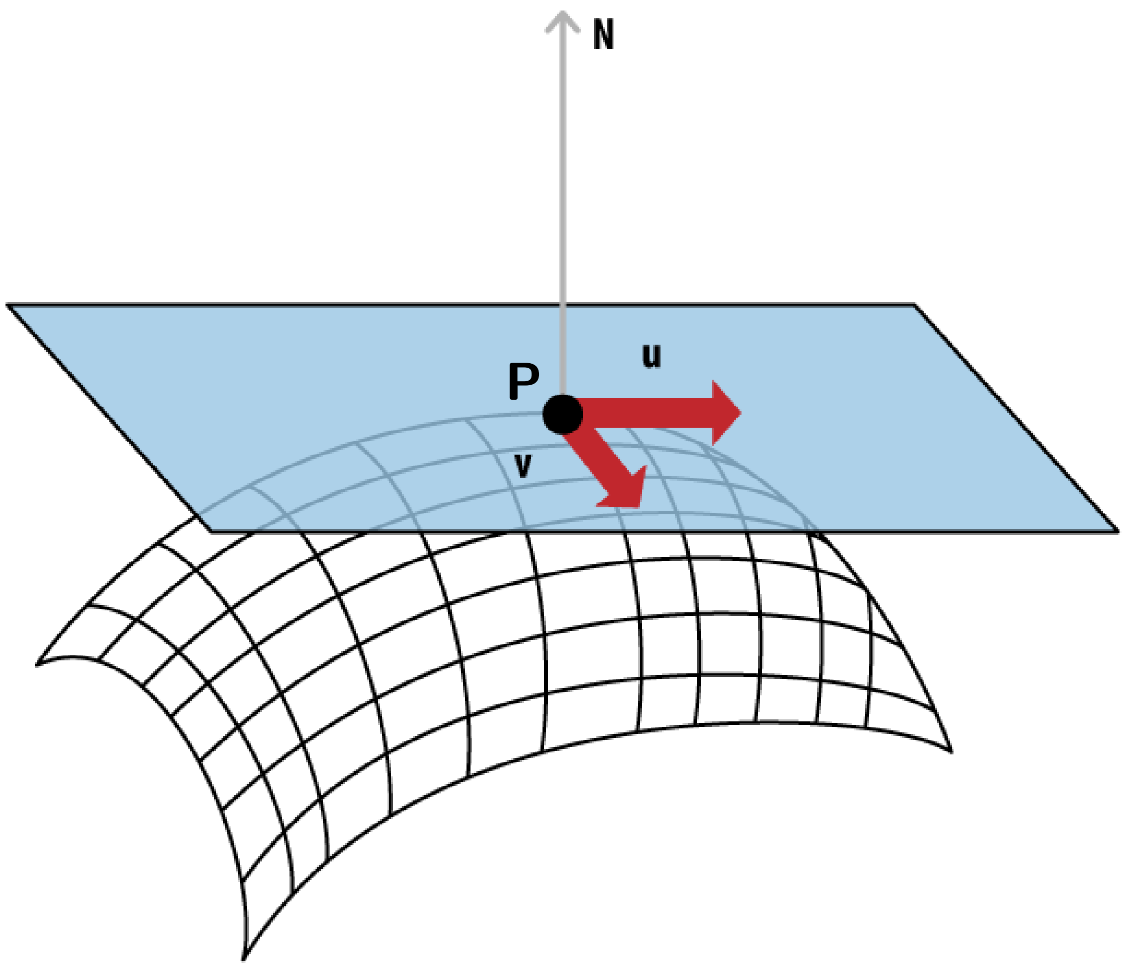

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Normal vector, tangent plane

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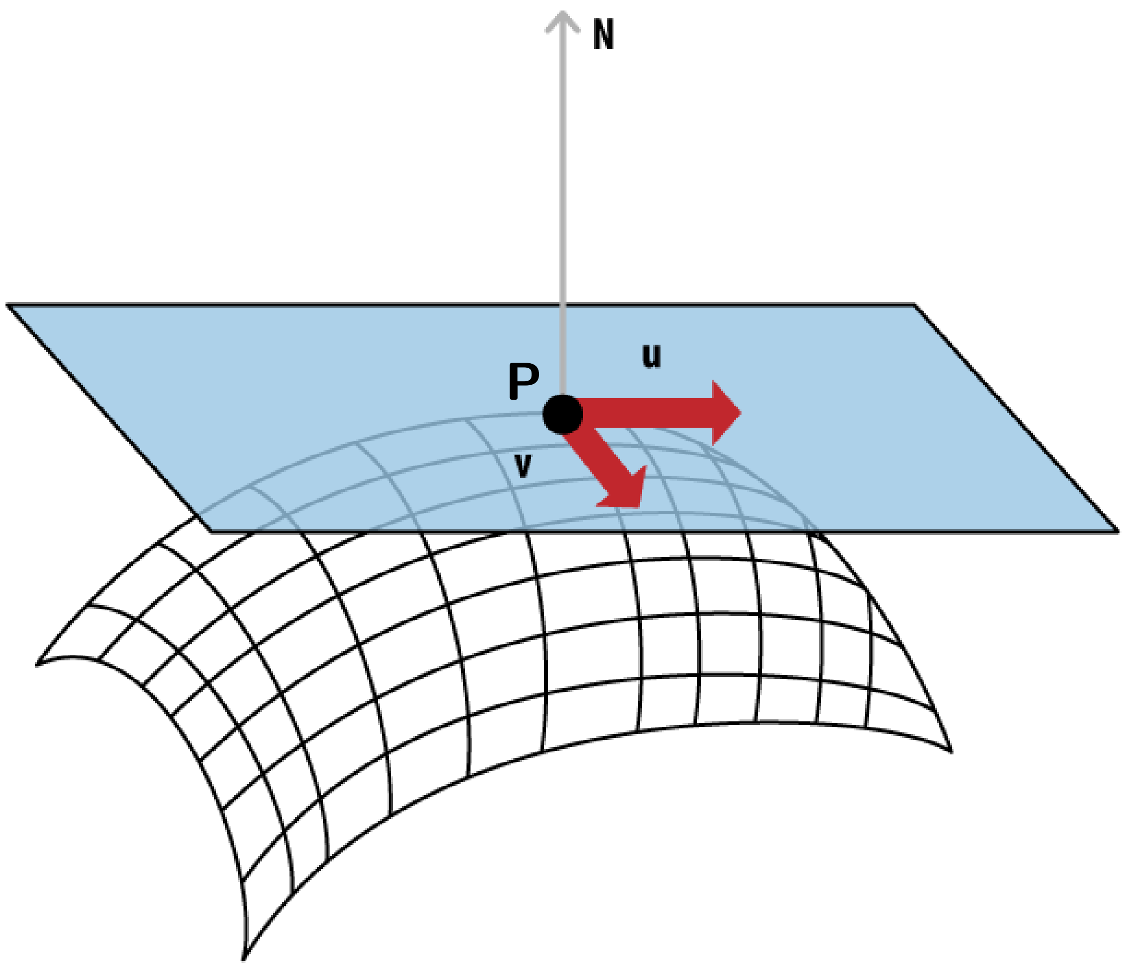
Exceptions:

- Surface does not have a tangent plane at P
- Parametrization is irregular at P



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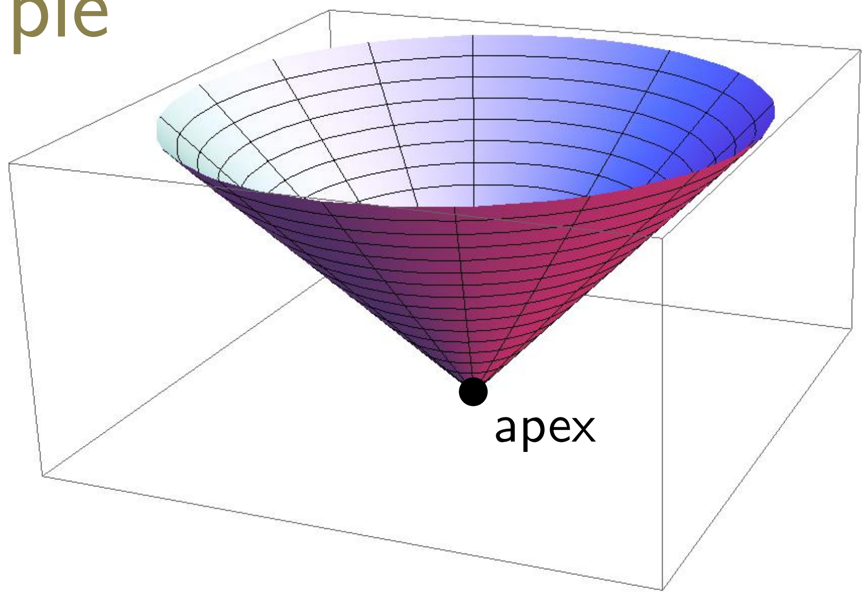
Normal vector, tangent plane: example

The apex of a cone is a **singular point**



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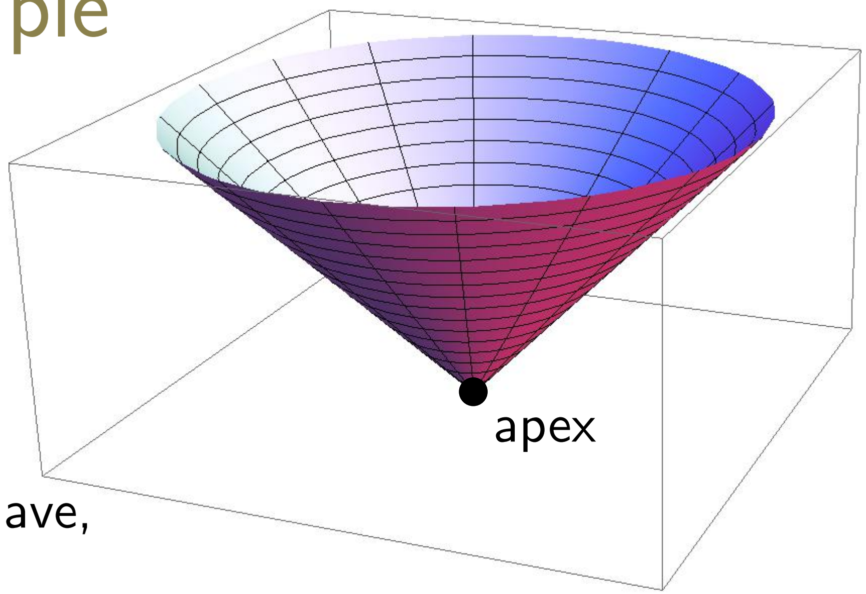

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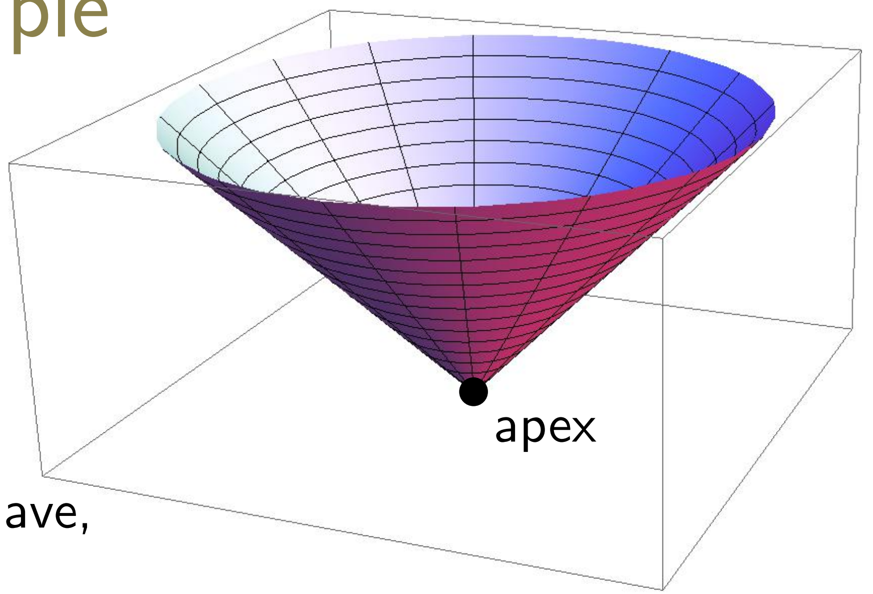

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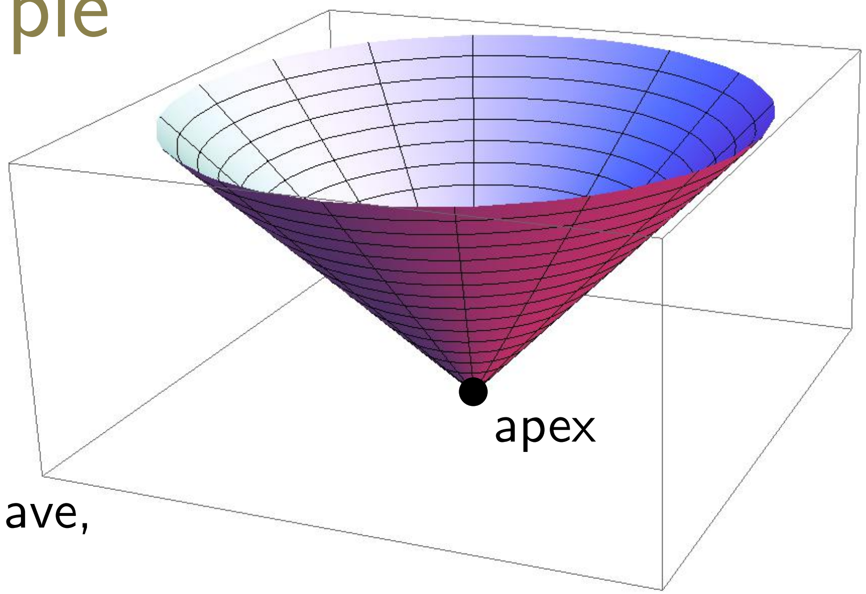
$S(u,v) = (u\cos v, u\sin v, u)$, for $u \in \mathbb{R}$ and $v \in [0, 2\pi)$

Normal vector, tangent plane: example



The apex of a cone is a **singular point**

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The tangent vectors are as follows:

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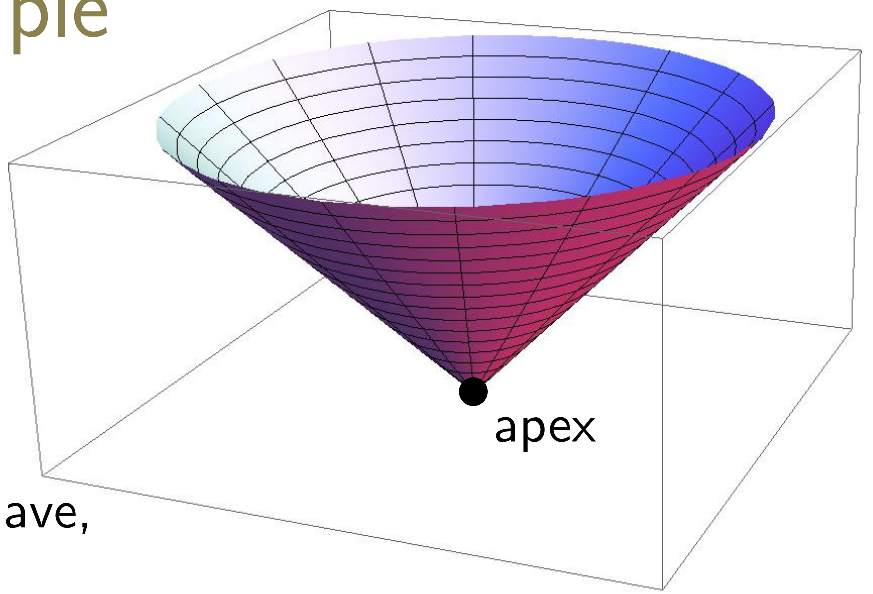
Normal vector, tangent plane: example



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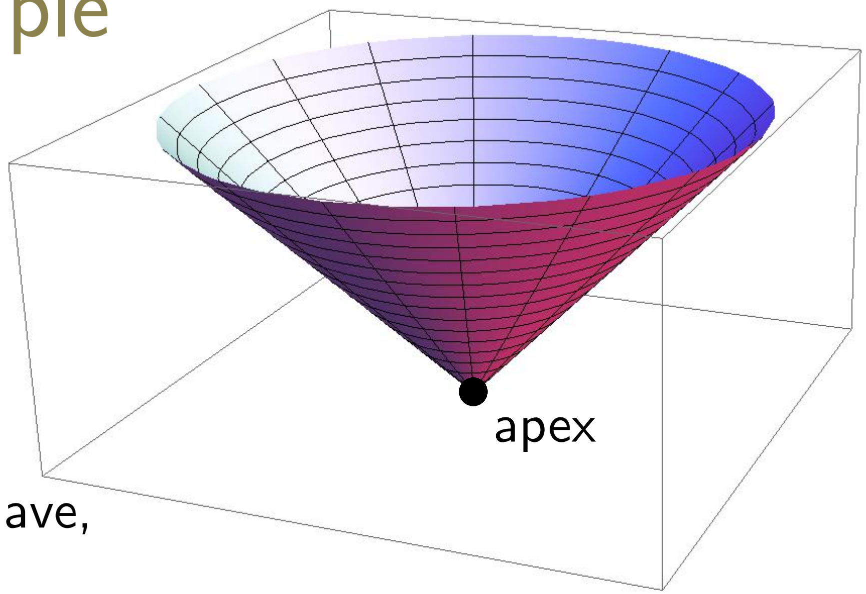

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For the apex, $P = (0, 0, 0)$, we get $\vec{N}(0, 0) = 0$

(thus, the apex is a singular point)

Normal vector, tangent plane: example

Consider the north hemisphere of the unit sphere

Normal vector, tangent plane: example

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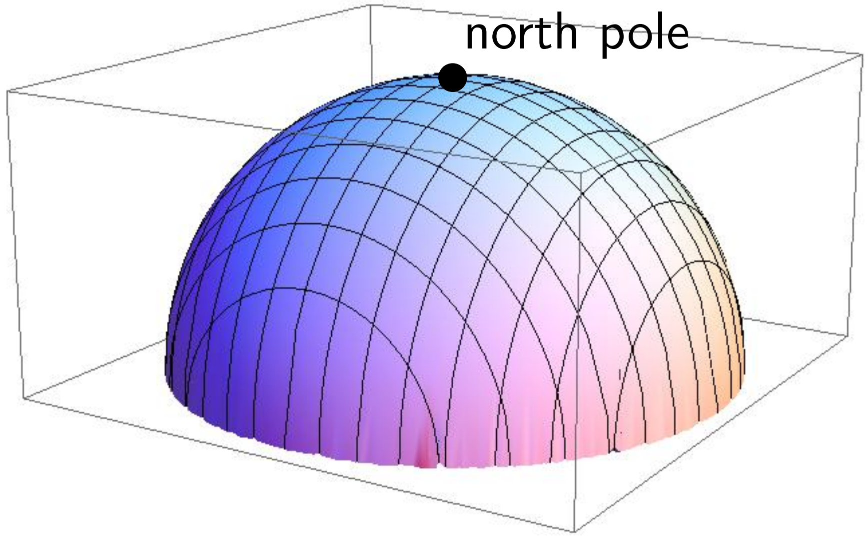
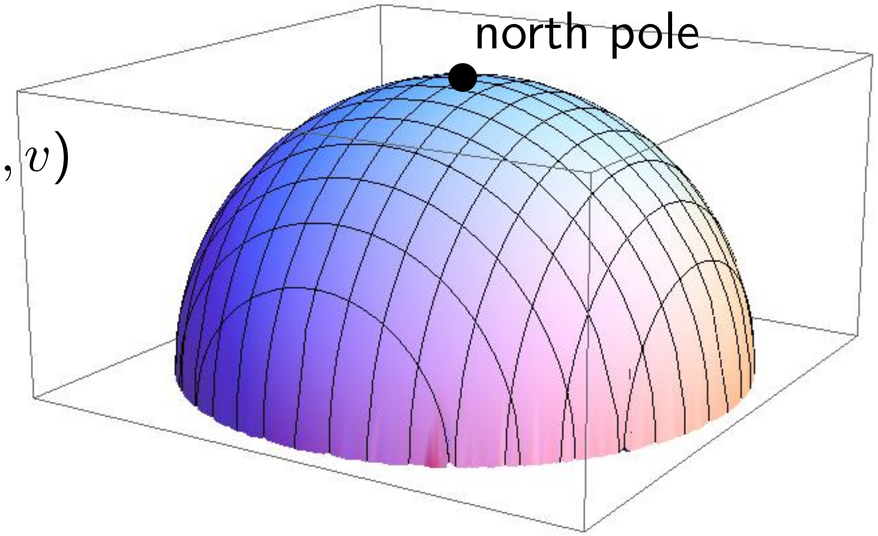


Normal vector, tangent plane: example

Consider the north hemisphere of the unit sphere

Parametrization 1

$S(u,v) = (u, v, \sqrt{1-u^2-v^2})$ (for the right values of u, v)



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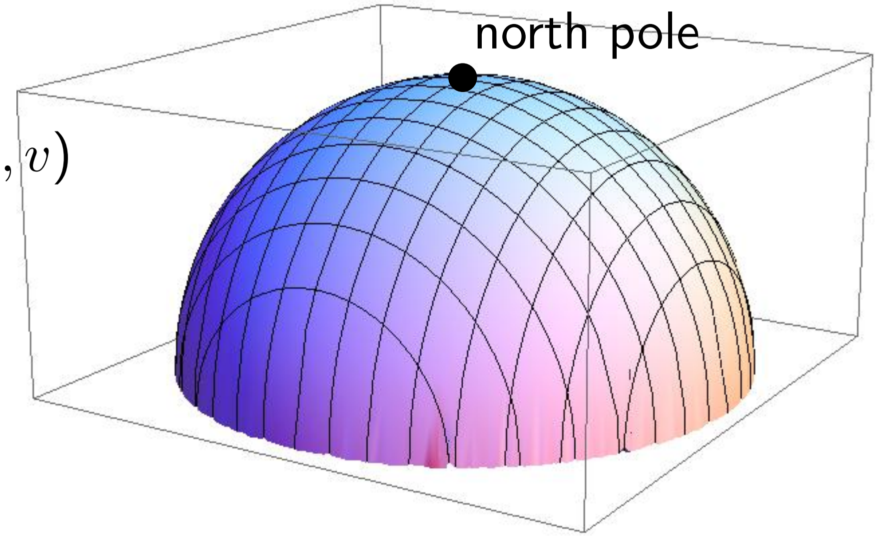

Parametrization 1

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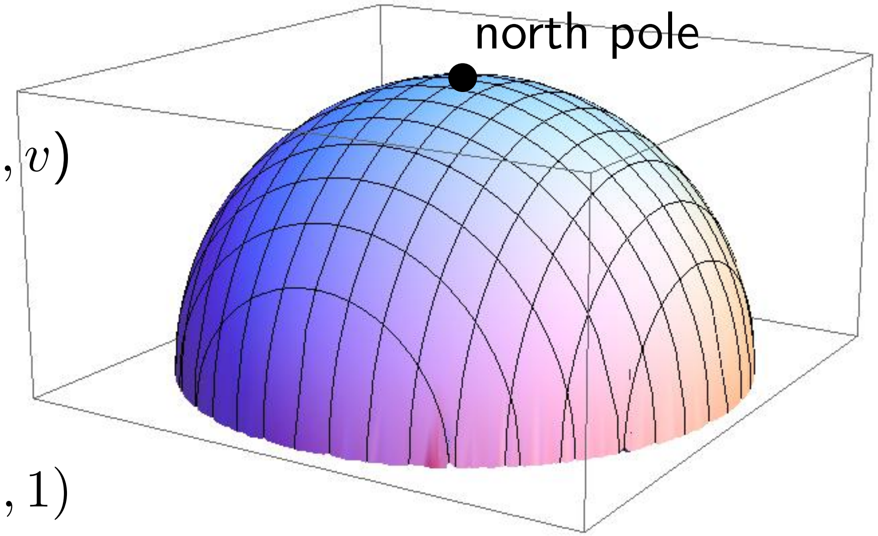

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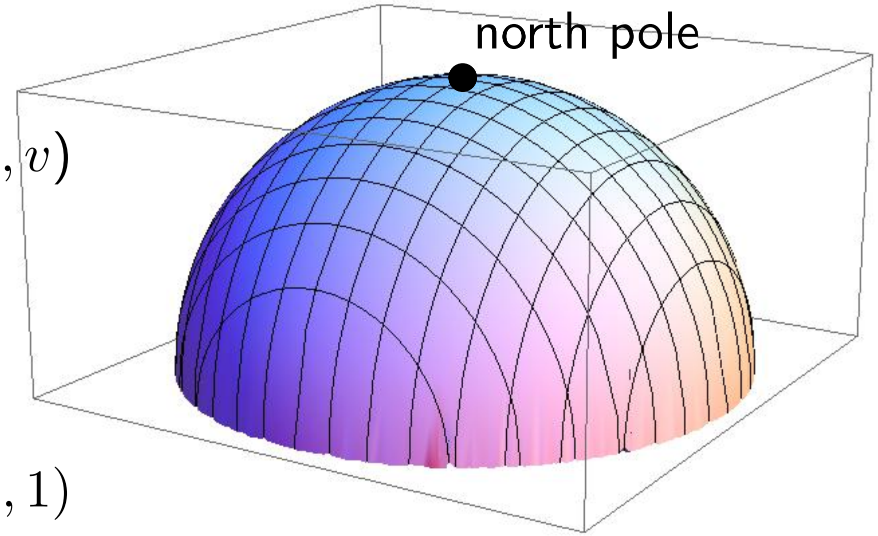

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Normal vector, tangent plane: example

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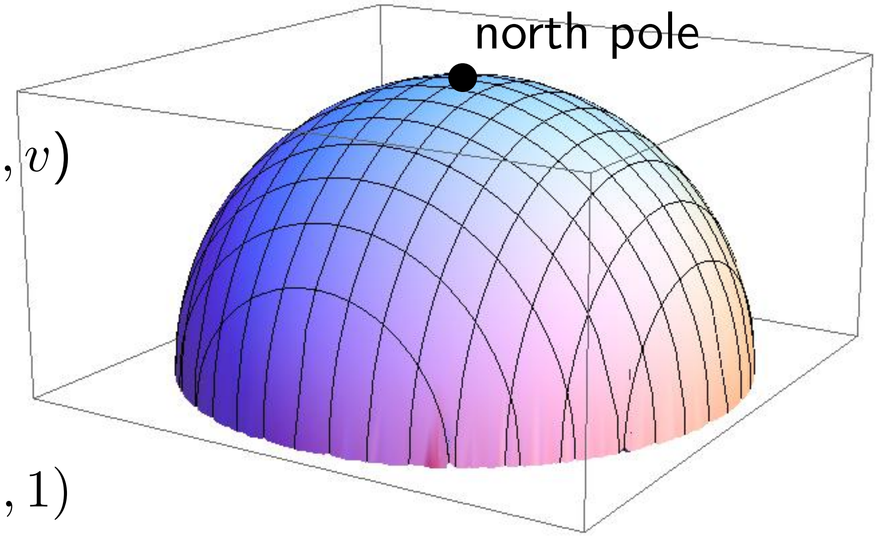

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Parametrization 2

Normal vector, tangent plane: example

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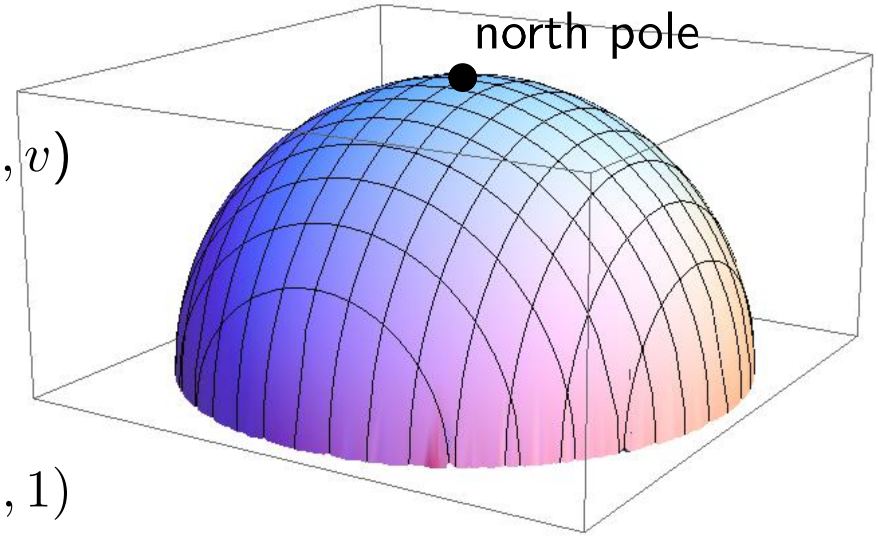

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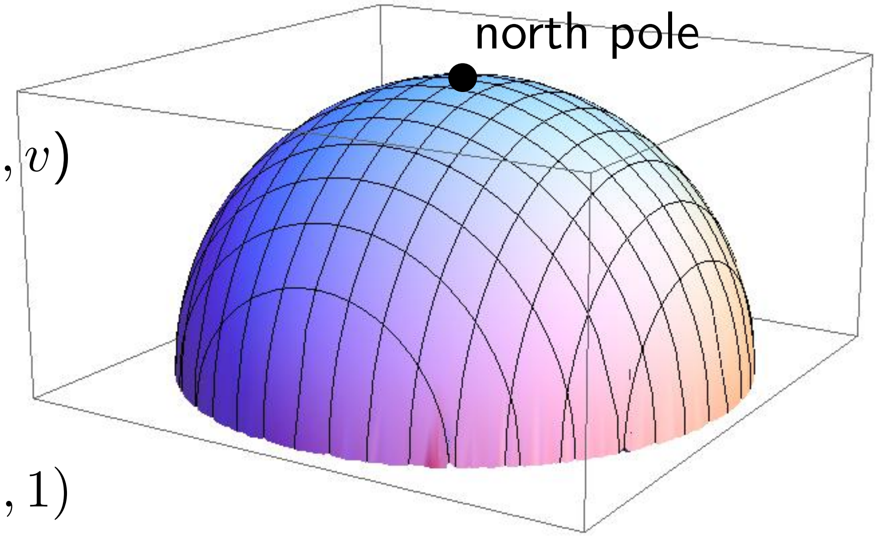
$S(\theta, \varphi) = (\cos\varphi\cos\theta, \cos\varphi\sin\theta, \sin\varphi)$ for $\theta \in [0, 2\pi)$ and $\varphi \in [0, \pi/2)$

Normal vector, tangent plane: example

Consider the north hemisphere of the unit sphere



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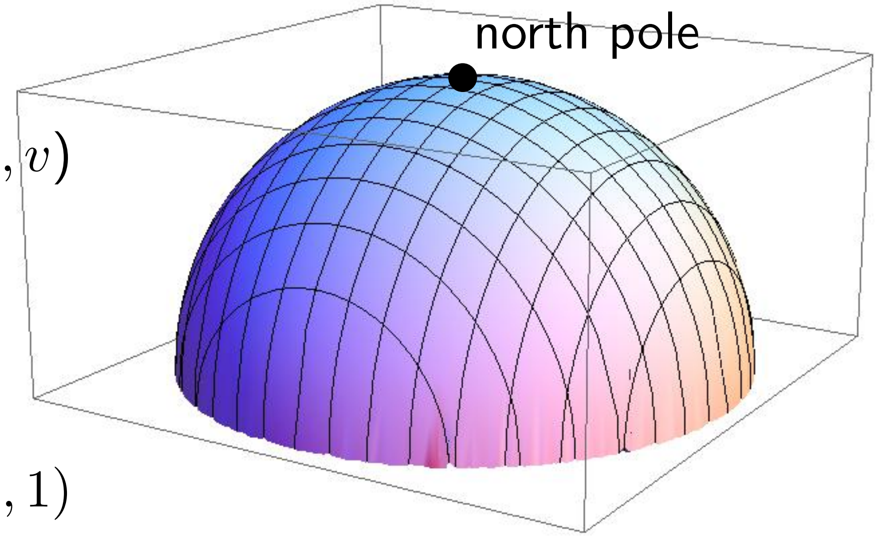

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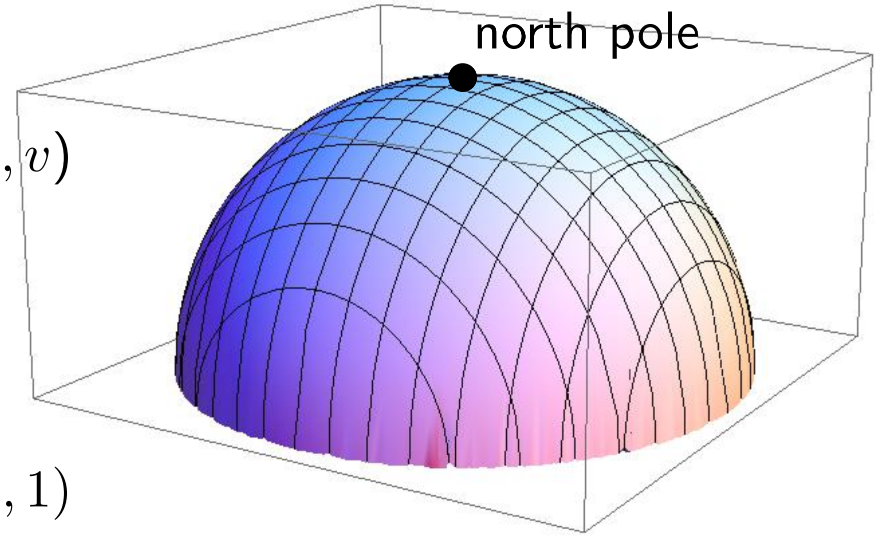

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$\rightarrow$ no issue with surface, but with parametrization