Problem 1. [**2 points**] Suppose you need to design a *spur gear* like the one shown below, as accurately as possible.

Review the different types of curves and surfaces seen in the course, and argue which type(s) would be the most suitable one(s) to represent the shape of this piece accurately. You can combine more than one type of curve/surface.

Important: provide a detailed justification of your choice, comparing against other types of curves, and stating clearly which properties of the chosen type of curve make it the best choice.

Solution sketch. The most important part in this exercise is to analyze the different parts that form the piece, and to find a suitable type of design for each. At least two parts need to be differentiated: the central cylindrical part (maybe composed of two nested cylinders), and the part holding the dents.

The central cylindrical part(s) requires a perfect cylinder. So for this there are basically two options: (i) explicitly use a cylinder equation, (ii) use rational Bézier or B-Splines curves, extruded to create the surface.

The dents can be tackled in many ways. Some people tried to use a sinusoidal curve along a circle. This will product something reasonable, although much smoother than the original piece. A better design requires drawing the profile of the dents, and then extruding it to produce a surface. The profile can be done in many ways, although the most general way is using NURBS, which will allow to have discontinuities at the dents, and to follow the shape of the circle.

All solutions that presented a clear decomposition of the object into different parts, justified the tool chosen for each, and did not mention things that were wrong, obtained the full score in this problem.

Problem 2. [2 points] Let $C(t)$ be the quadratic Bézier curve defined by the three control points $P_0, P_1, P_2 = [(0,0); (3,3); (6,0)]$.

1. Give five control points such that the degree-4 Bézier curve $P(t)$ defined by them is exactly the same as $C(t)$.

2. Prove that the curves $C(t)$ and $P(t)$ are indeed the same.

Solution sketch. The idea in this problem is to use *degree elevation* to compute the new control points, and then to argue that the corresponding curve is the same as the original. A maximum of one point was given to each of these two parts.

Since we have to increase the number of control points by two, degree elevation needs to be applied twice: first to go from 3 to 4 control points, and then to go from 4 to 5 control points. Some of you tried to deduce conditions on the curve with 5 control points directly, but that's a much more complicated way.

The control points resulting from applying degree elevation twice are: $Q_0 = (0,0), Q_1 = (3/2, 3/2), Q_2 = (3,2), Q_3 = (9/2, 3/2), Q_4 = (6,0)$.

To prove that the two curves are the same, the easiest way is to just write the full expression of both curves: $C(t)$ and $P(t)$. To this end, just apply the formula based on Bernstein polynomials: $C(t)$ will use three control points, and $P(t)$ will use five. The important thing here is that one has to replace the values of the points in each equation, and verify that the expression one gets for each curve are exactly the same polynomials. We have:

$C(t) = (1-t)^2 P_0 + 2t(1-t)P_1 + t^2 P_2$.

$P(t) = (1-t)^4 Q_0 + 4t(1-t)^3 Q_1 + 6t^2(1-t)^2 Q_2 + 4t^3(1-t)Q_3 + t^4 Q_4$.

Now we need to insert the values of the points P_0, P_1, P_2 and $Q_0, \ldots, Q_4$ in the formulas above. Since the points are two-dimensional, we will have to analyze separately the x and the y coordinates.

For example, for the x-coordinate, after plugging in the x-coordinates of the eight points involved, one gets:

$x(C(t)) = 2t(1-t)3 + t^2 6 = 6t$.

Now we do the same for $P(t)$, and see if we get the same:

$x(P(t)) = 4t(1-t)^3(3/2) + 6t^2(1-t)^2 3 + 4t^3(1-t)(9/2) + 6t^4$.

Simplifying the expression for $x(P(t))$, we get:

$x(P(t)) = 6t(1-t)^3 + 18t^2(1-t)^2 + 18t^3(1-t) + 6t^4$

$x(P(t)) = 6(t - 3t^2 + 3t^3 - t^4) + 18(t^2 - 2t^3 + t^4) + 18(t^3 - t^4) + 6t^4$

Now, reagrupping by each of the powers of t, we obtain:

$x(P(t)) = 6t + (-18 + 18)t^2 + (18 - 36 + 18)t^3 + (-6 + 18 - 18 + 6)t^4$

which is equal to $6t$! Thus we verify that $x(C(t)) = x(P(t))$. Next it remains to do the same with the y-coordinate.

Problem 3. [**2 points**] Consider the cubic uniform Bézier curve defined by four control points P_0, P_1, P_2, P_3, where all four control points are colinear (i.e., there exists a line that contains all four points). Prove that the resulting Bézier curve has degree 1, and give its expression.

Solution sketch. There are two main ways to solve this problem. First, for simplicity, we can assume without loss of generality that P_0 is the leftmost control point and P_3 is the rightmost, and that the points are not vertically aligned.

The simplest option is to use the convex hull property: we know that the Bézier curve always stays inside the convex hull of its control points. Now, for points that are collinear, the convex hull is just the line segment between the two most extreme control points, assumed to be P_0 and P_1. Therefore the resulting curve will be identical to the line segment from P_0 to P_1, that is, $C(t) = (1-t)P_0 + tP_1$, which is clearly a degree-1 curve.

An alternative option, but more complicated, is to use the formula of the cubic Bézier curve based on Bernstein polynomials. Then one can write P_1 and P_2 as a combinations of P_0 and P_3, which makes the whole formula depend only on P_0 and P_3. To complete the proof, one has to show that the coefficients that multiply P_0 and P_3 are positive and add up to one, concluding that the whole curve is a convex combination of those two points.

When grading this exercise, 1.5 point was given to solutions with a good justification of why the result is a line segment, with an extra 0.5 point for those that also gave an expression for the resulting segment.

Problem 4. [**4 points**] Write a program to draw a glass bottle similar to the one below, which you can buy in a well-known Swedish-founded furniture retailer.

You may want to split your design in several pieces, to simplify your task. You can use a different method to design each piece. Any of the methods seen in the course can be used.

Very important: In the HTML file that you will deliver, include a description of the strategy that you have followed and the design decisions made. All explanations should be visible by simply opening the HTML file with a web browser (**code comments do not count as explanations**).

Solution sketch.

Three main aspects of the solution were evaluated: the design of the glass bottle (2 points max), the lid (without the handle, 1 point max), and the lid handle (1 point max).

The best way to model the glass bottle was using a surface of revolution. To design the profile of the bottle, a Bézier curve was the most common options. This is enough to capture the overall shape of the bottle, including the different widths. Note that a cylinder is not useful to model the bottle, since it has a fixed radius, while the radius of the bottle is variable. Approaches based on Bézier surfaces cannot easily obtain a real cylinder, and even obtaining C^1-continuity is not so straightforward.

The lid is easily modeled as one cylinder, although something extra is needed to make it closed from the top (otherwise, the water will easily fall!). Some people directly addressed both aspects by modeling the lid also as a surface of revolution, with an Γ-shaped profile.

Finally, the handle could be done in several ways. While some of you used Bézier surface patches (with good results), most parametrized a half-circle, and then made it wider. However, this produces a thin handle, not thick as the one in the real bottle. Something extra is needed to make it thick: for instance, to extrude a little rectangle along the half-circle, or to have several copies of it.