BEZIER SURFACES

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INTRO TO BEZIER SURFACES

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Idea: use grid of control points

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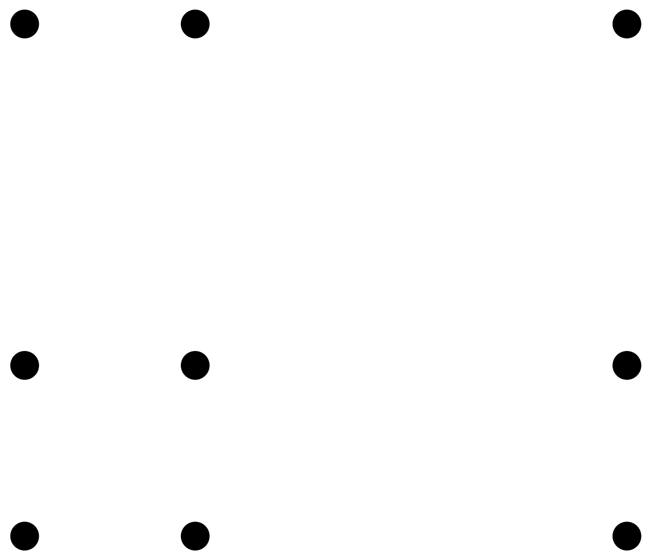

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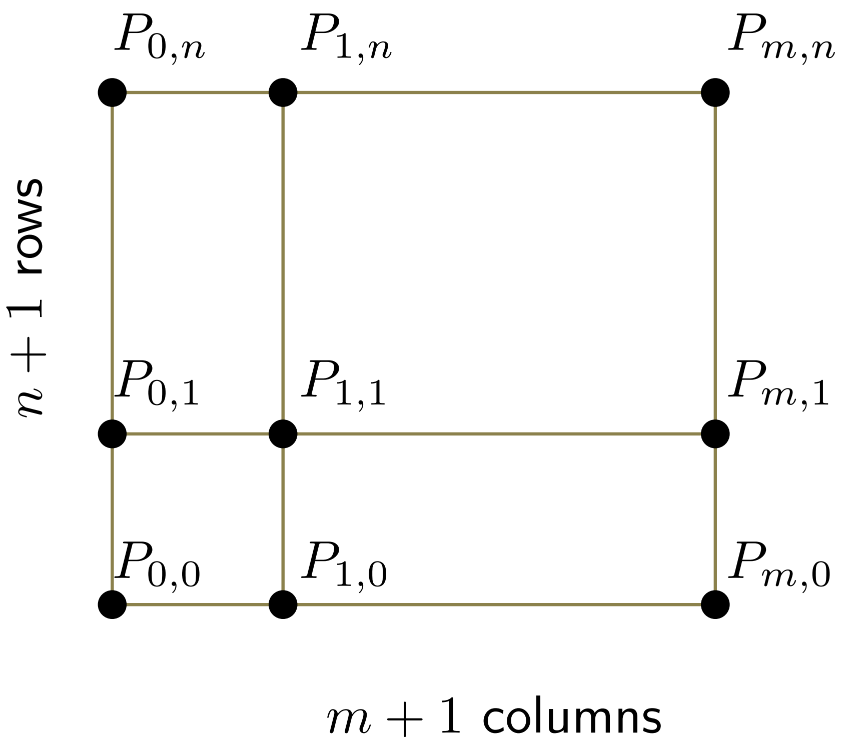

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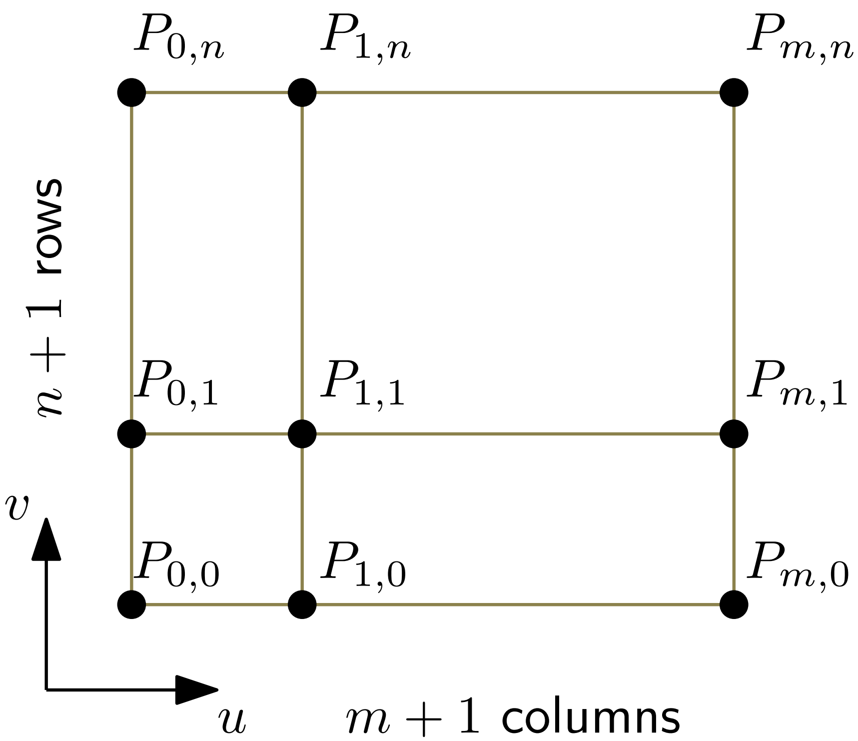

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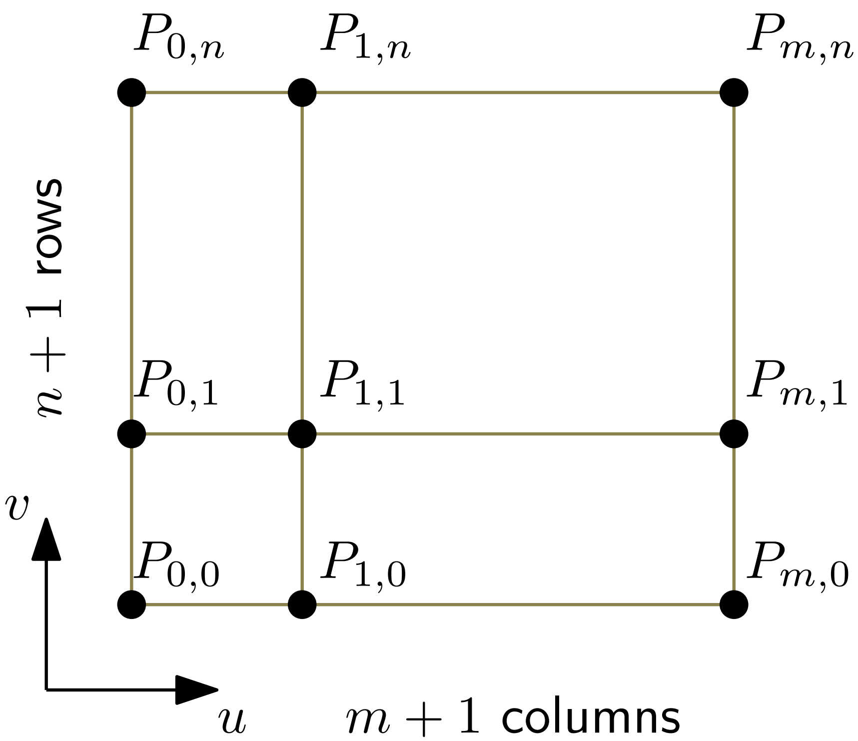


$$\boxed{S(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{n} B_{m,i}(u) B_{n,j}(v) P_{i,j}}$$

$$0 \leq u, v \leq 1$$

where the terms $B_{m,i}$ and $B_{n,j}$ are the Bernstein polynomials, same as in Bézier curves

Idea: use grid of control points

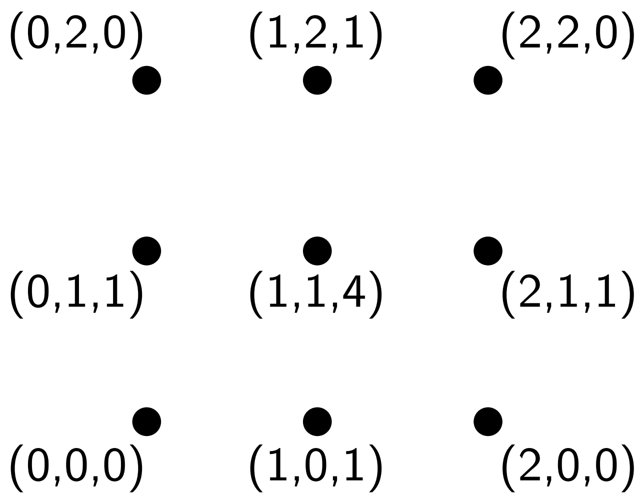
Consider a grid of $(m+1) \times (n+1)$ control points arranged in a rectangular grid

$$\boxed{S(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{n} B_{m,i}(u) B_{n,j}(v) P_{i,j}} \qquad 0 \leq u, v \leq 1$$

In matrix form:

$$S(u,v) = (B_{m,0}(u), B_{m,1}(u), \ldots, B_{m,m}(u)) \begin{pmatrix} P_{0,0} & P_{0,1} & \ldots & P_{0,n} \\ P_{1,0} & P_{1,1} & \ldots & P_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ P_{m,0} & P_{m,1} & \ldots & P_{m,n} \end{pmatrix} \begin{pmatrix} B_{n,0}(v) \\ B_{n,1}(v) \\ \vdots \\ B_{n,n}(v) \end{pmatrix}$$

Example



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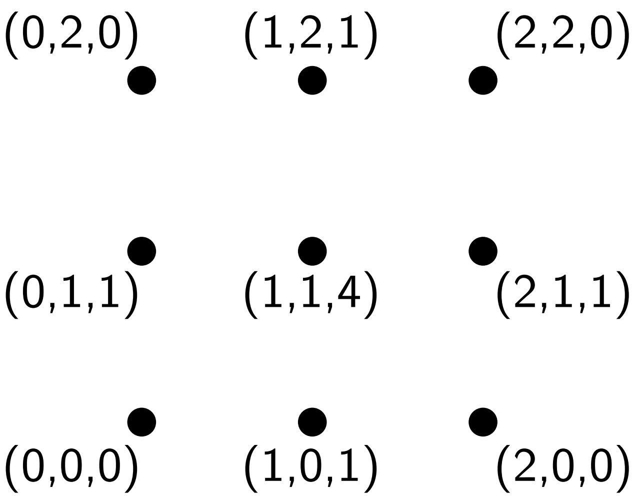

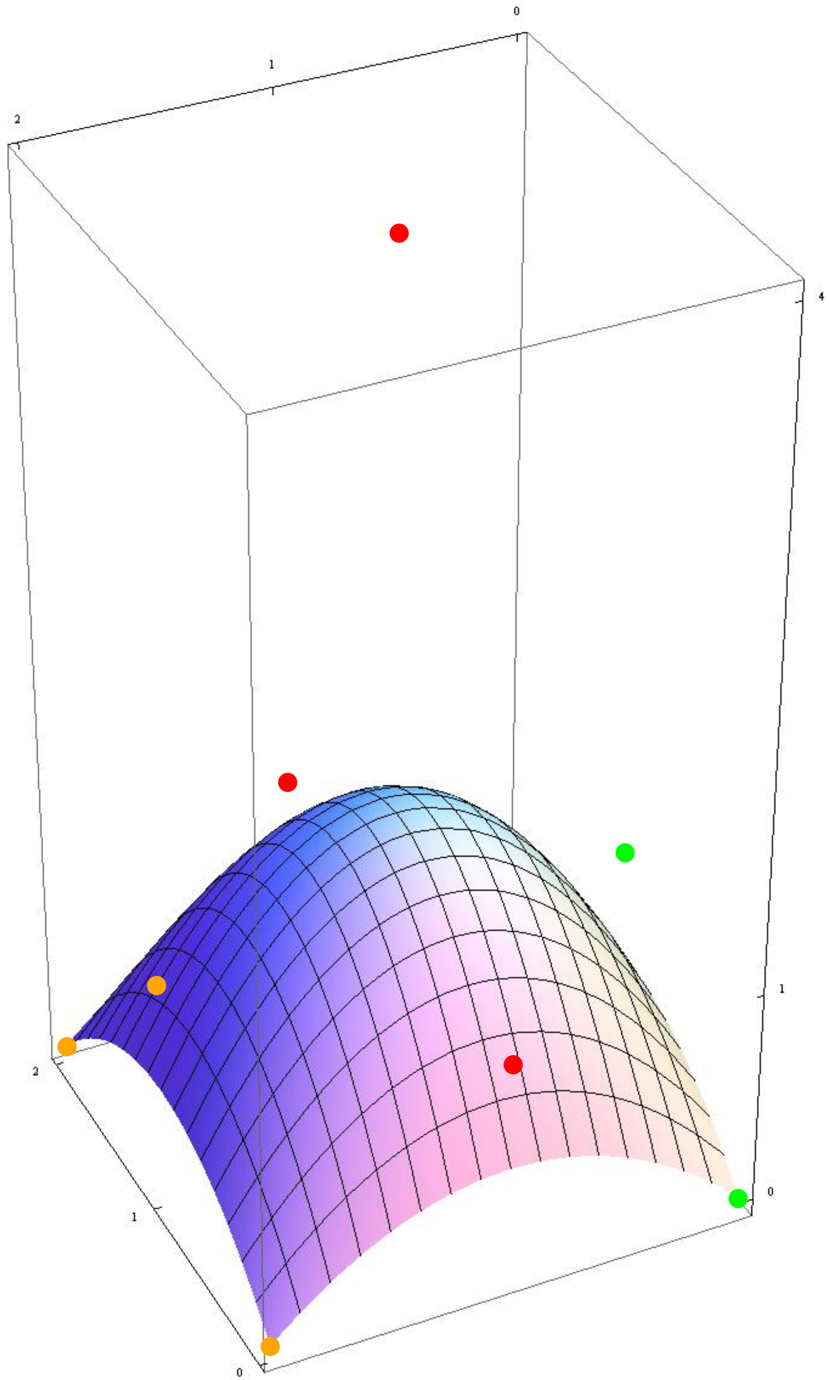

Biquadratic Bézier surface patch [Salomon, Fig 6.20]

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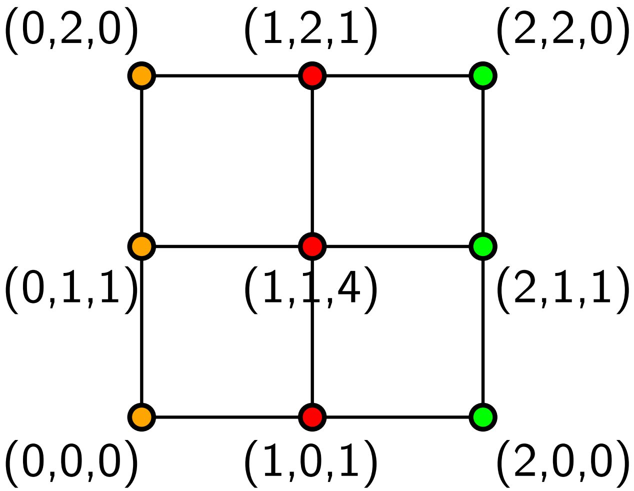

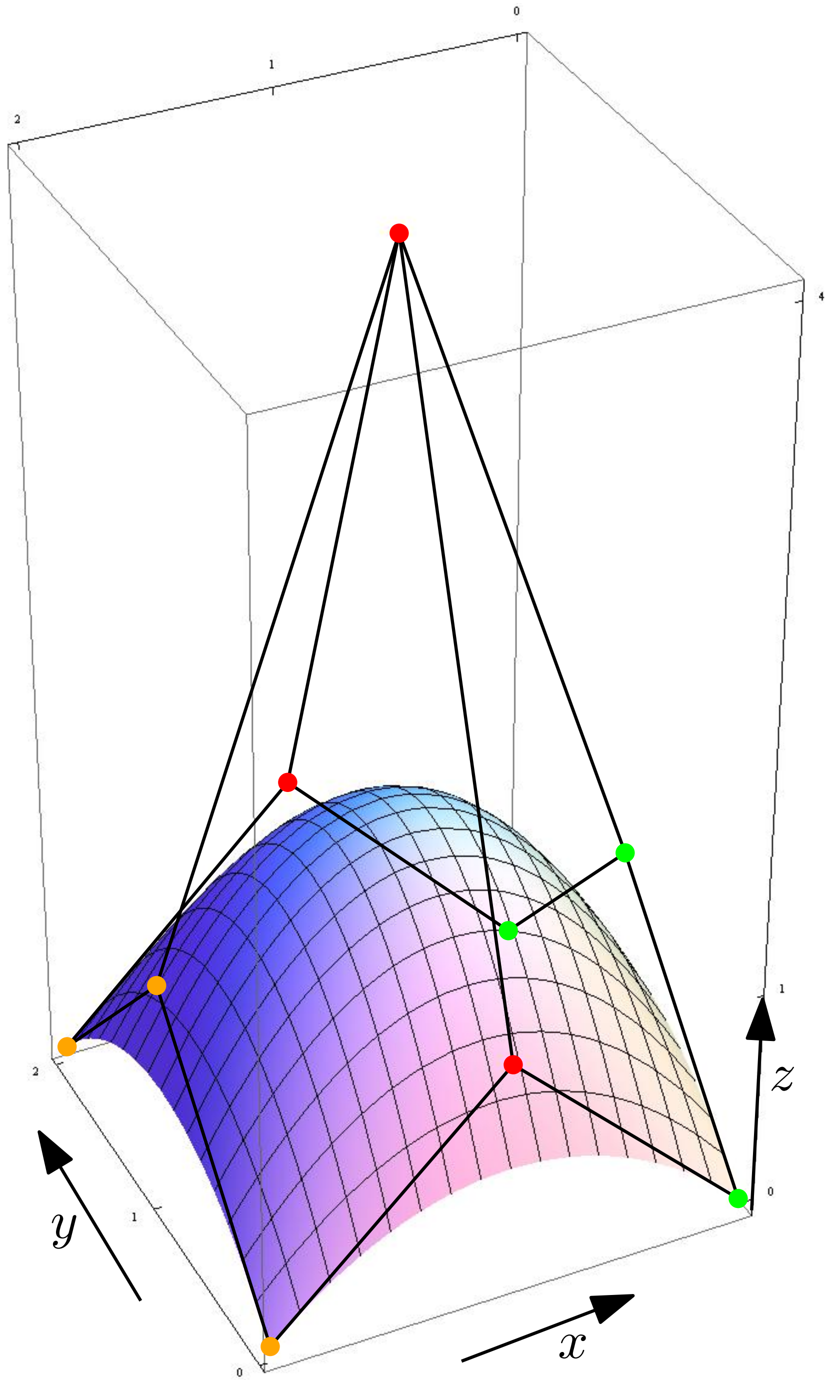

Biquadratic Bézier surface patch [Salomon, Fig 6.20]

Properties of Bézier surface (on rectangular grid)

$0 \leq u, v \leq 1$

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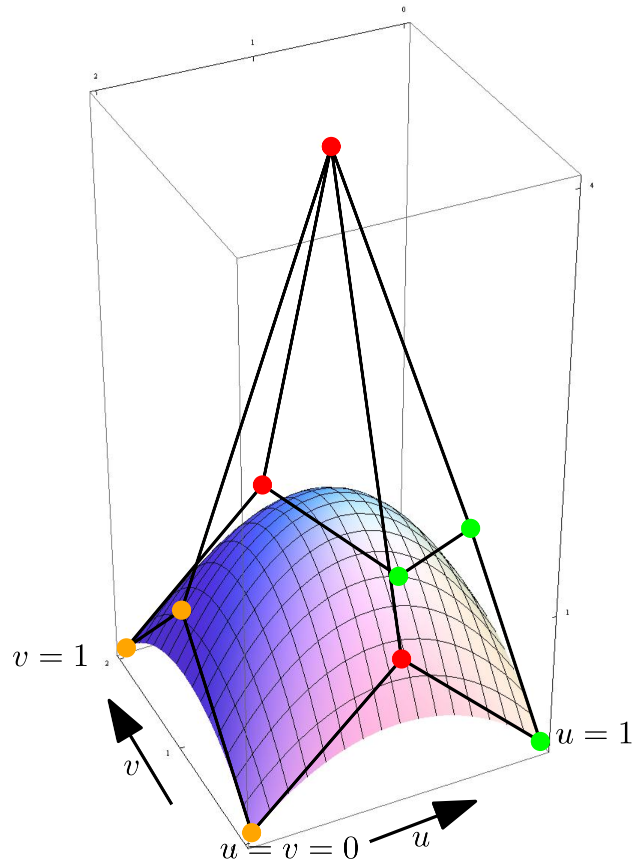
- Endpoints (patch corners)



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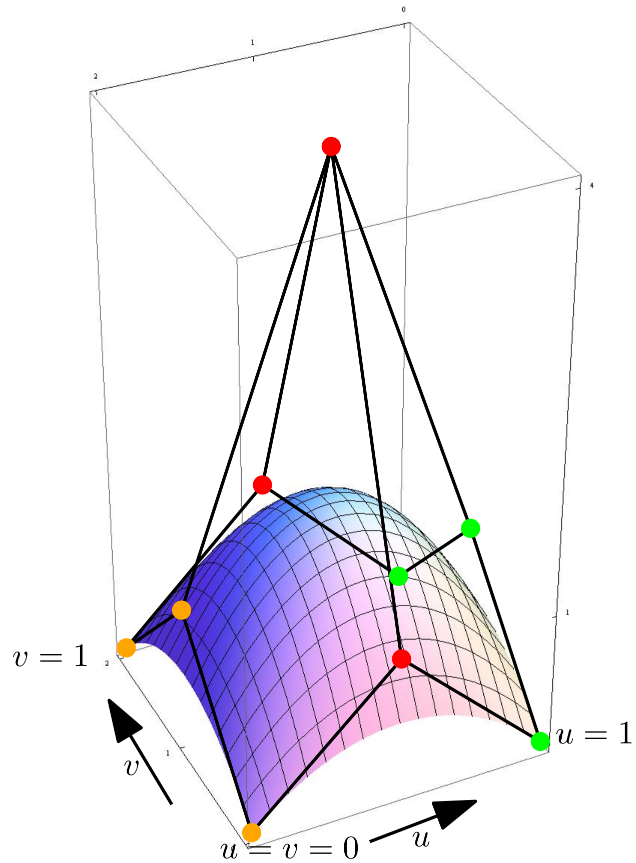
$S(0,0) = P_{0,0}$, $S(1,0) = P_{m,0}$, $S(0,1) = P_{0,n}$ and $S(1,1) = P_{m,n}$



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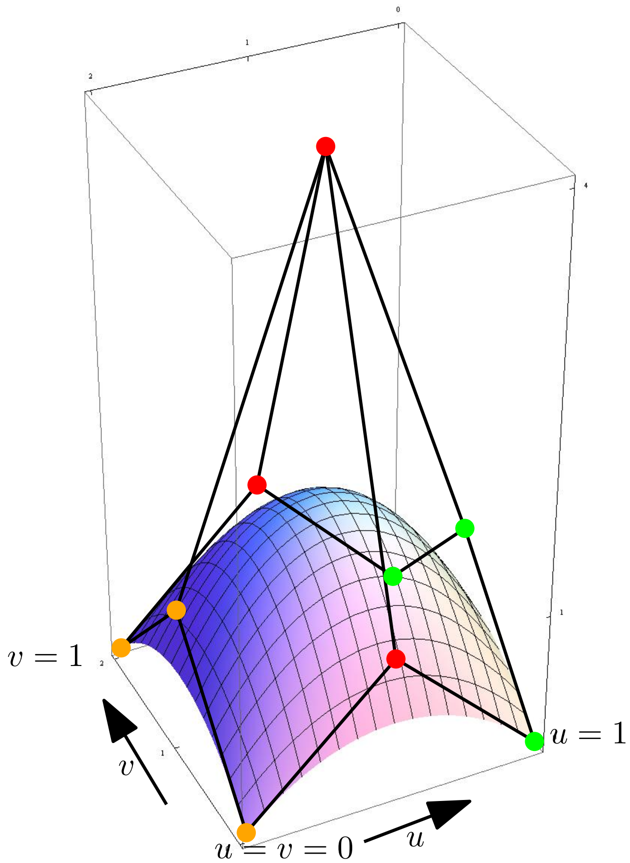

- Boundary curves

PROPERTIES OF BEZIER SURFACES

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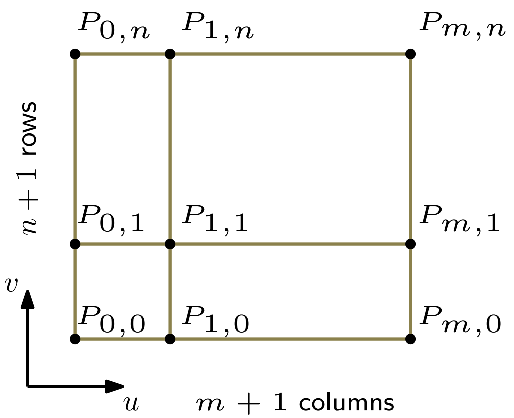
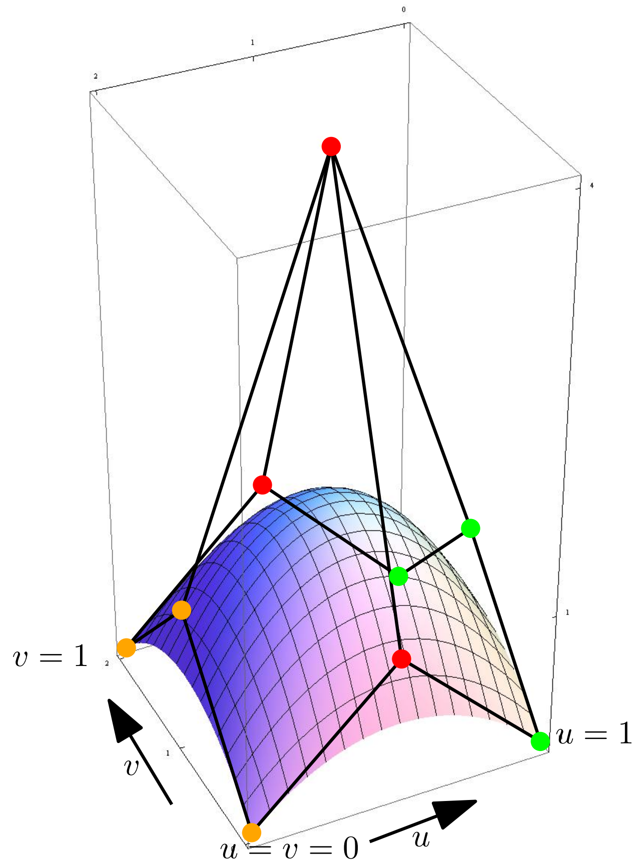
$S(u,0)$ is the Bézier curve defined by $P_{0,0}, P_{1,0}, \ldots, P_{n,0}$





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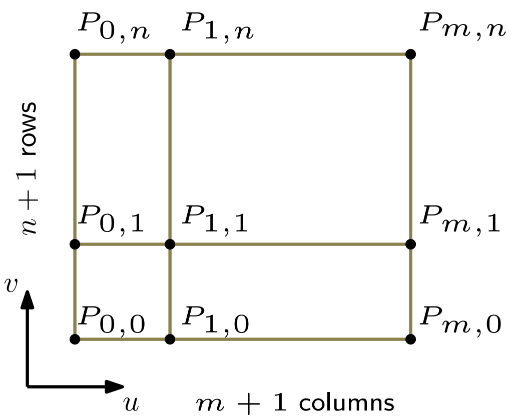

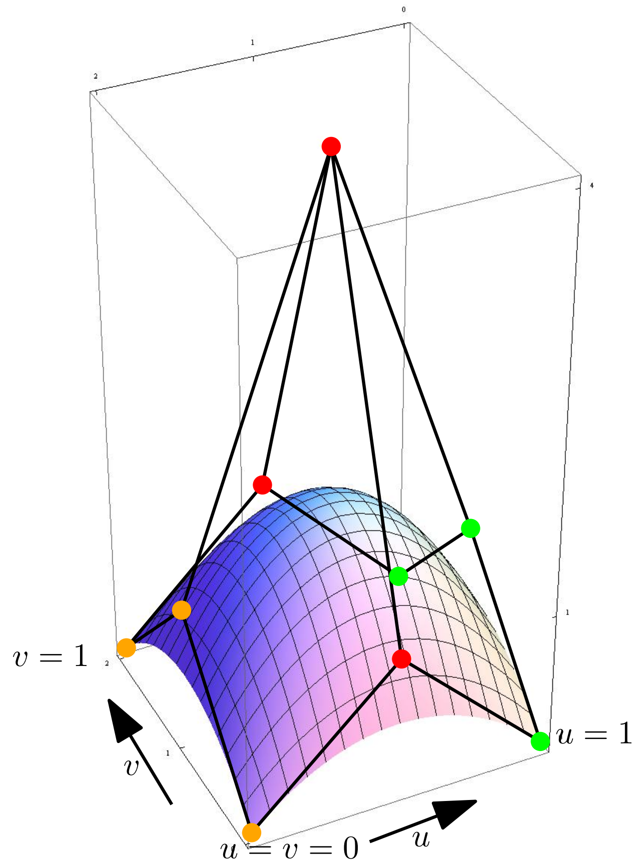

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- Uniparametric curves (i.e., fixed u or fixed v)

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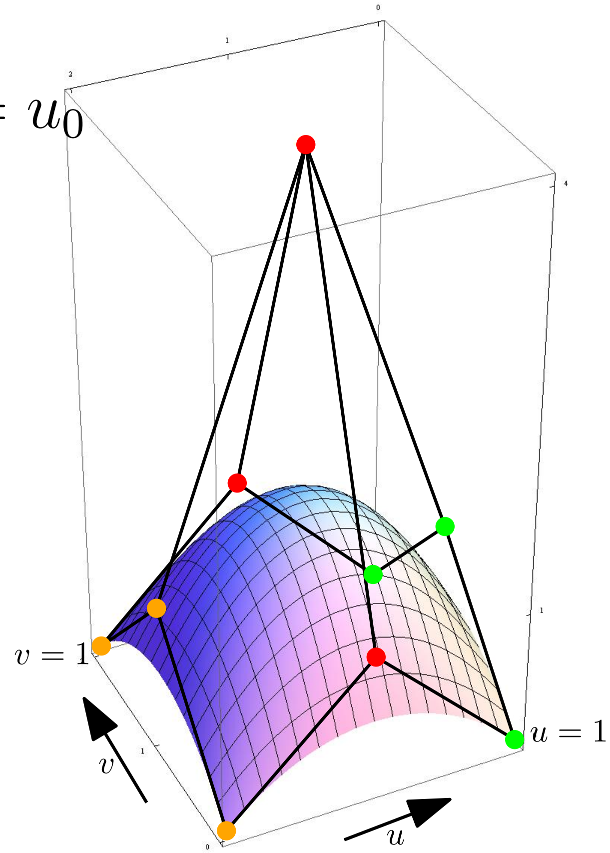

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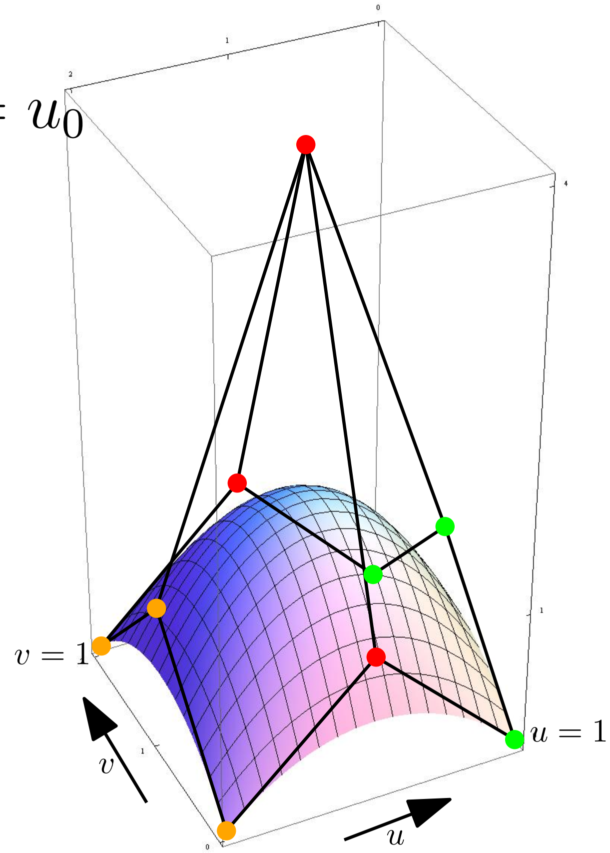

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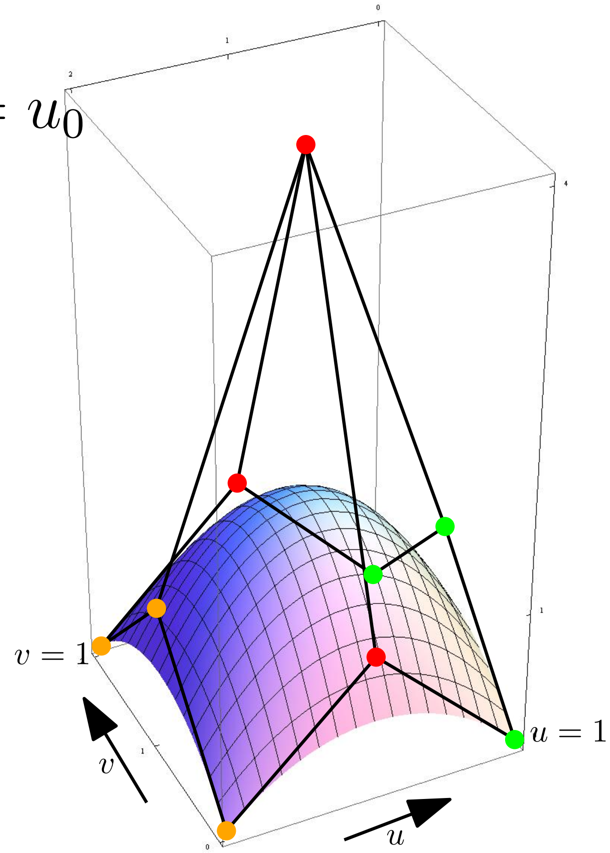

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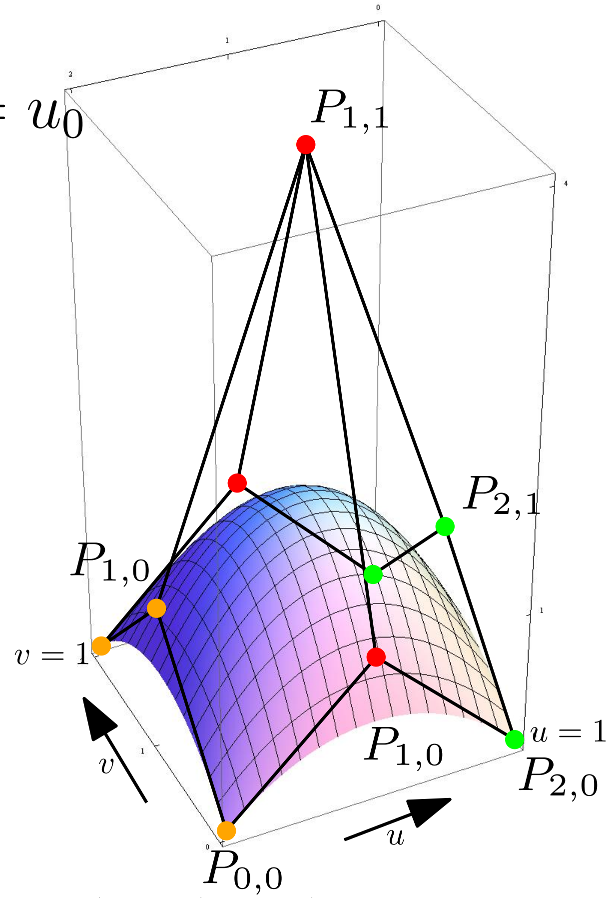

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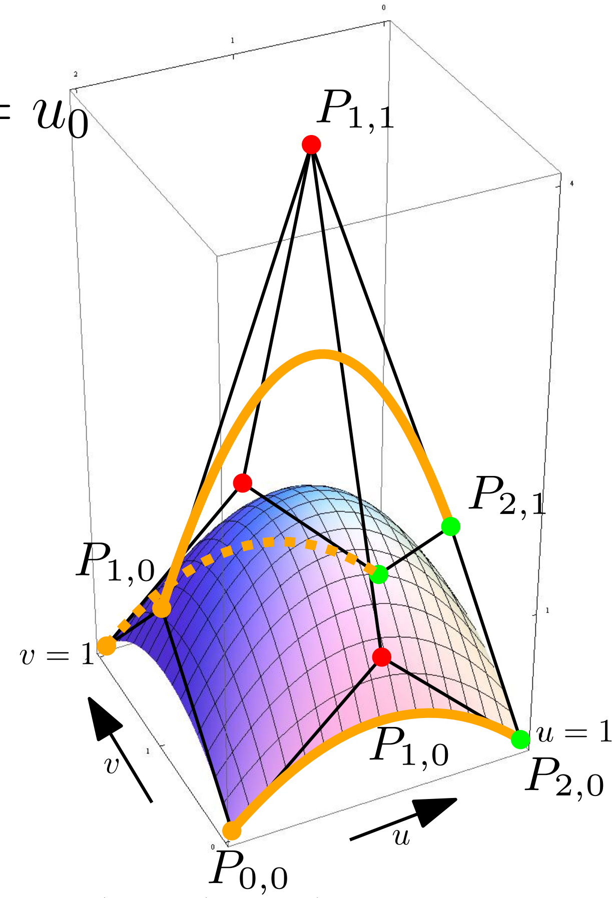

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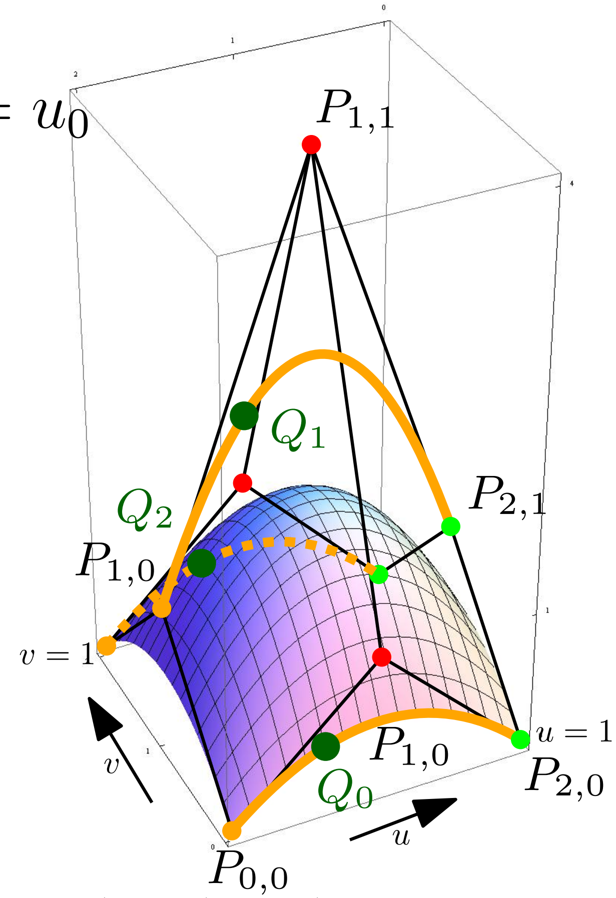

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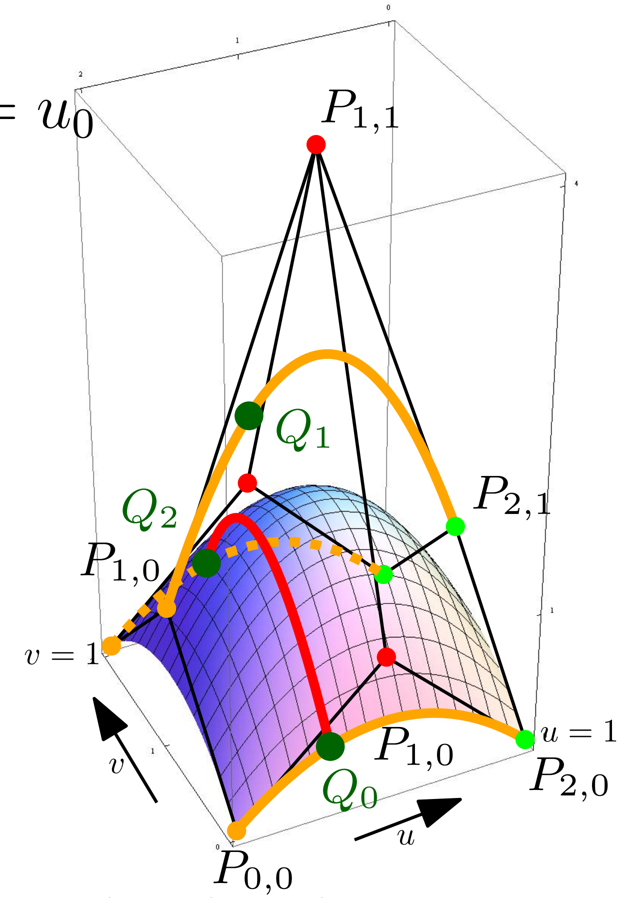

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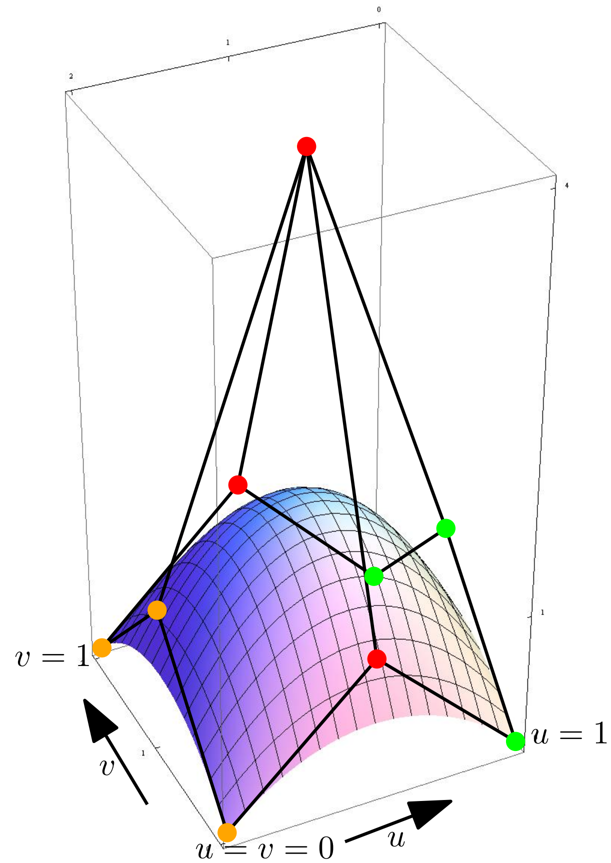
- Uniparametric curves (i.e., fixed u or fixed v)

  All uniparametric curves are Bézier curves

- Affine invariance

  $\sum_{i=0}^{m} \sum_{j=0}^{n} B_{m,i}(u) B_{n,j}(v) = 1$ for all (u, v)



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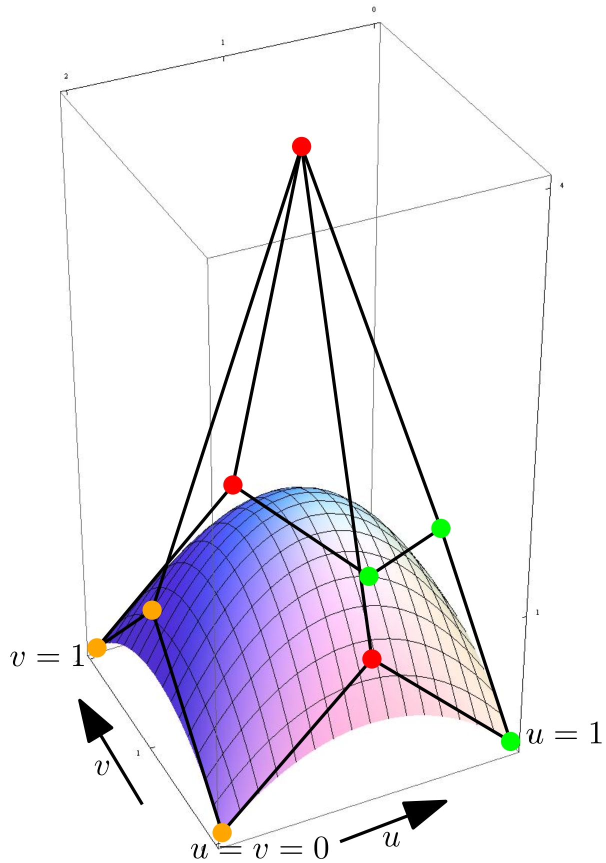

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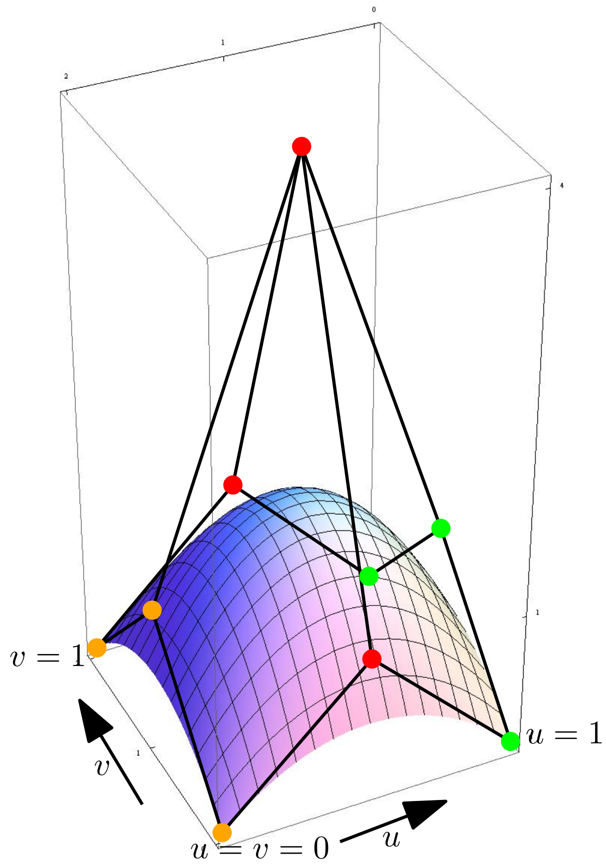
- No variation diminishing property



Smooth connection of rectangular Bézier patches

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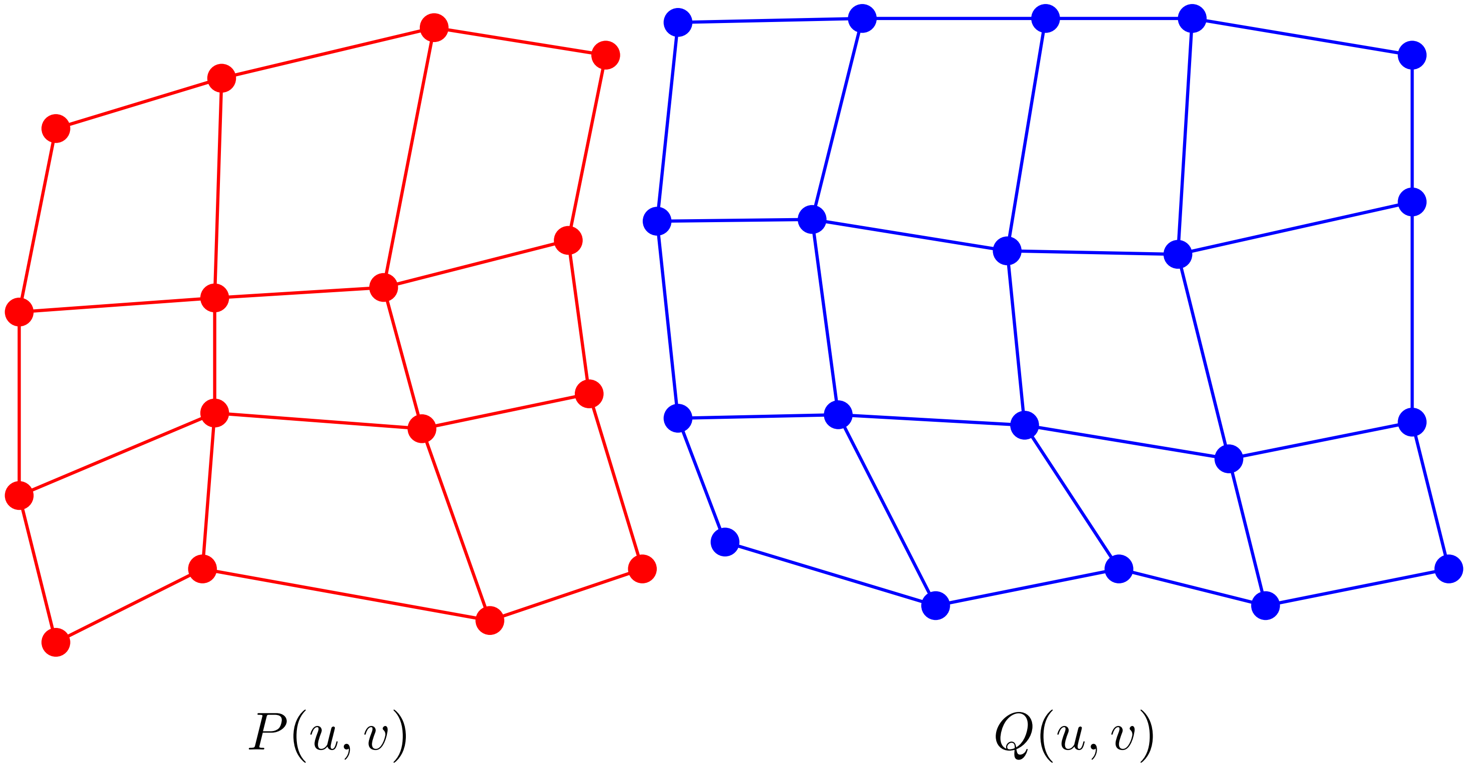

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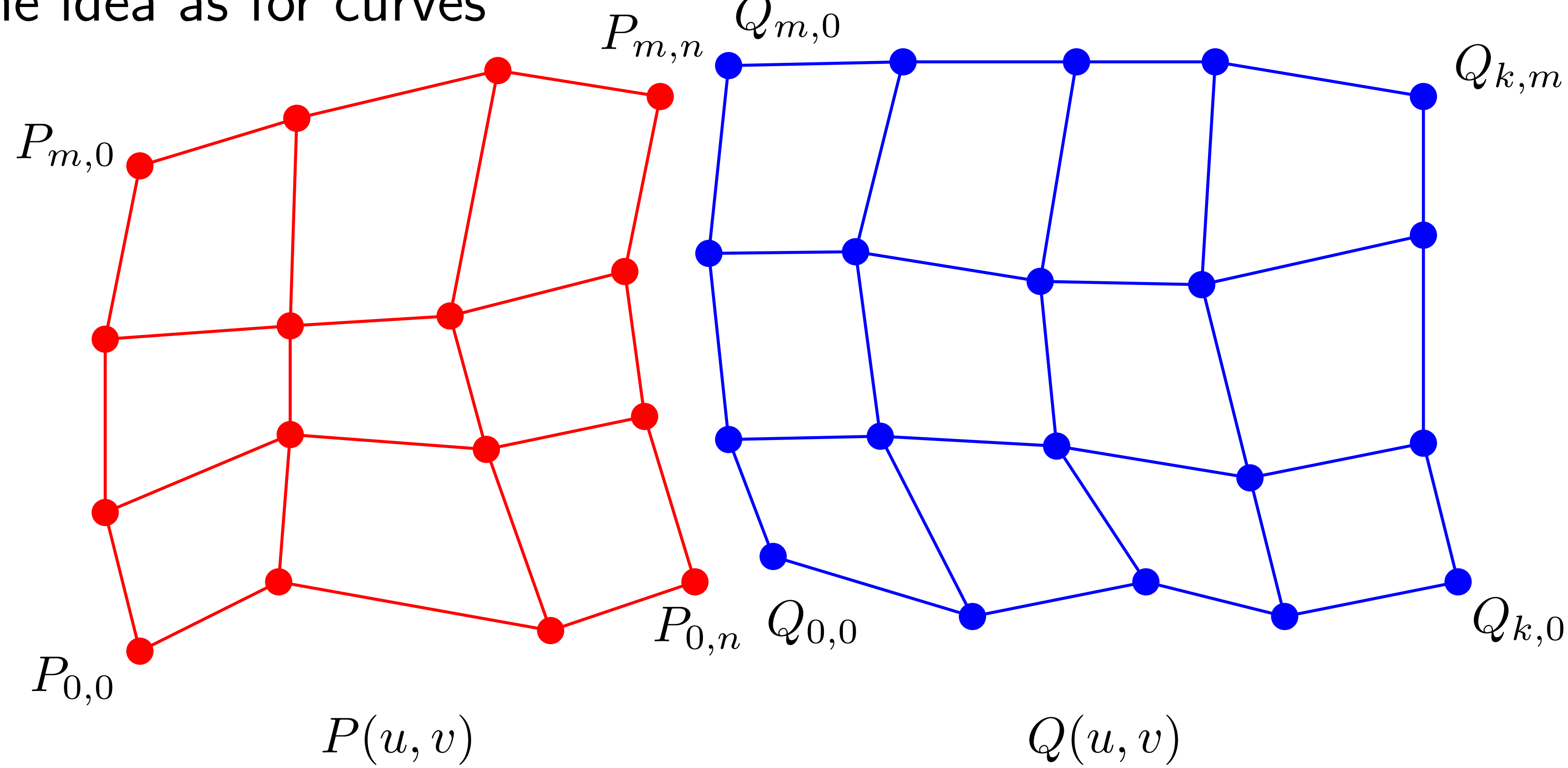

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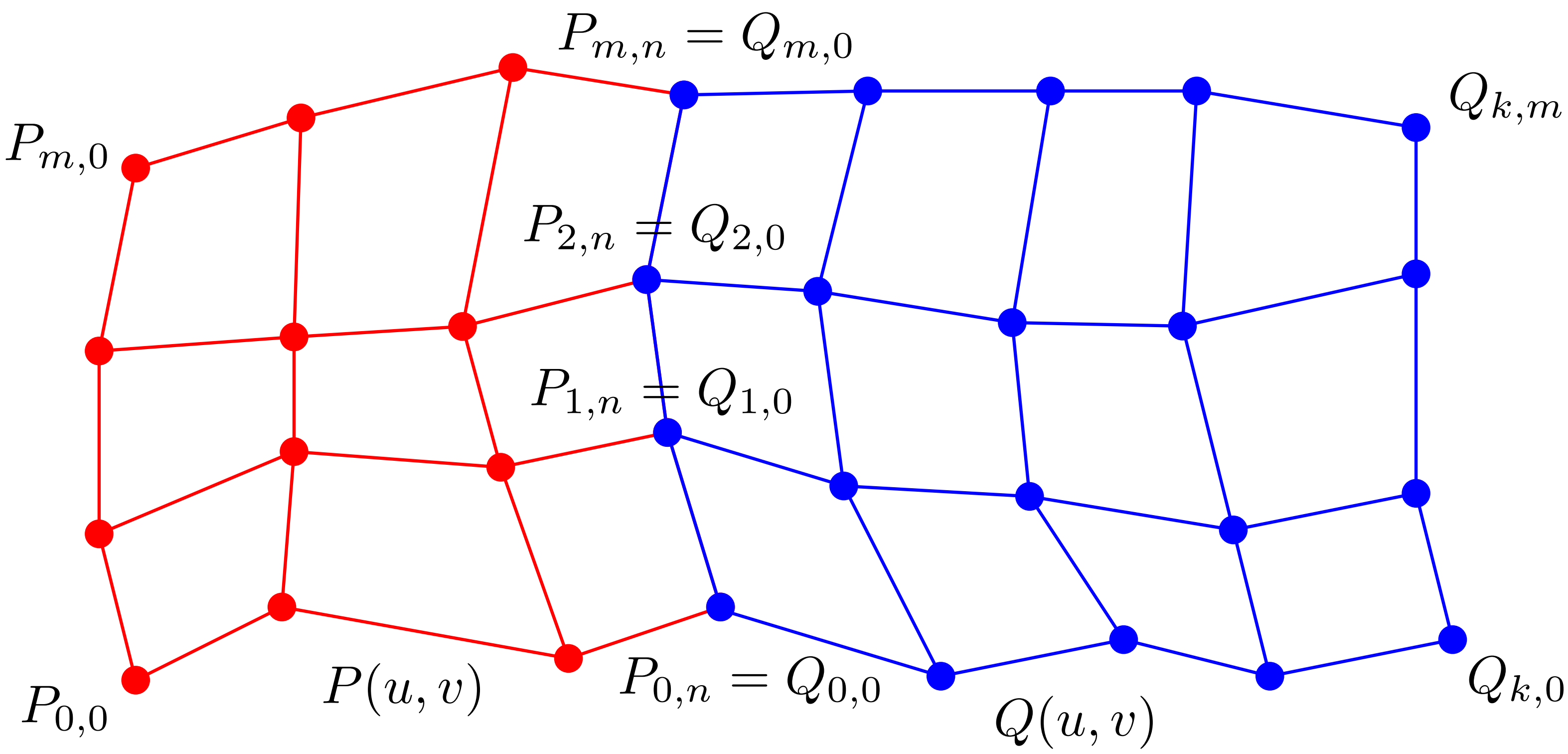

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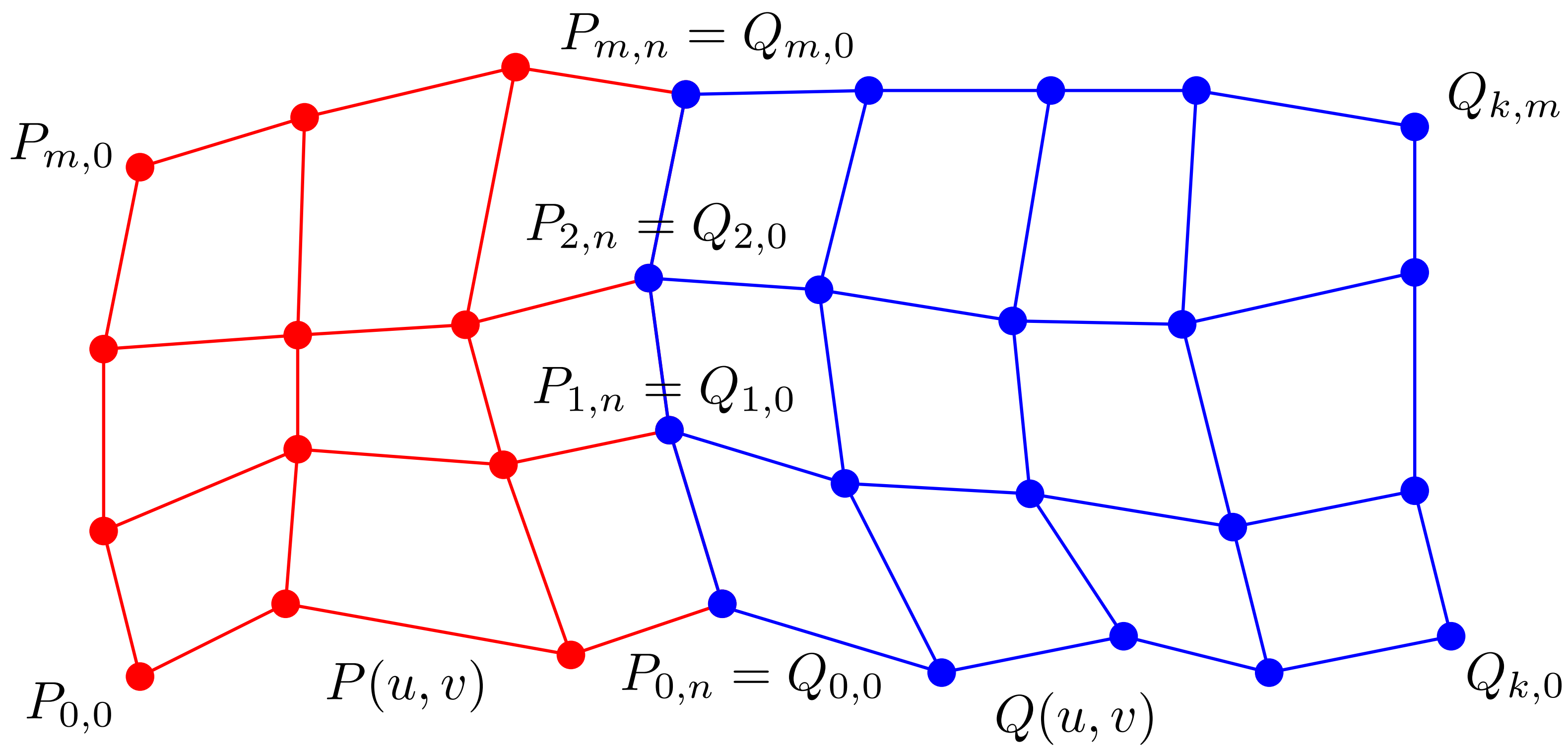
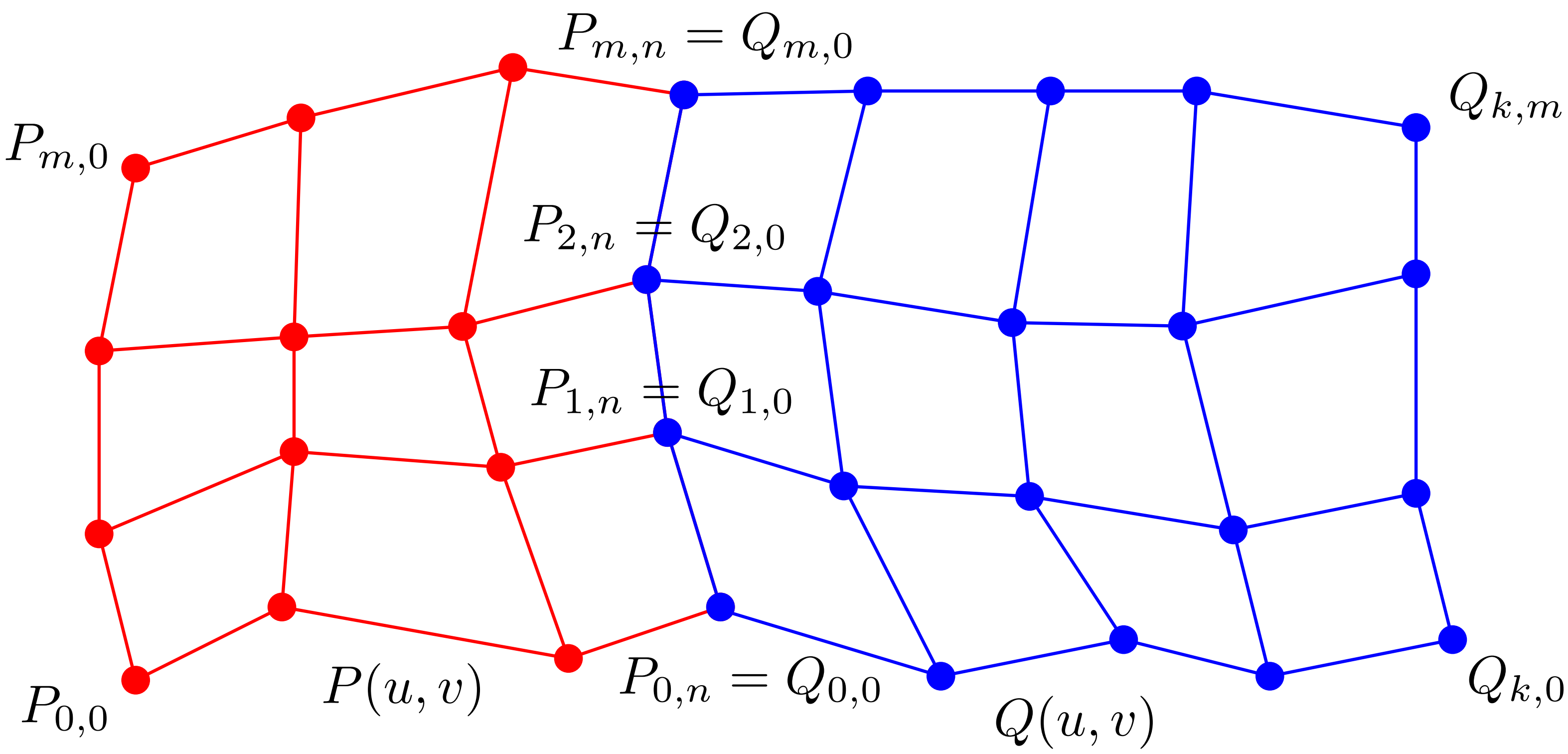


- Continuity (C^0-cont) $\qquad P_{i,n} = Q_{i,0} \quad \forall i = 0, \ldots, m$

Smooth connection of rectangular Bézier patches

- Same idea as for curves



- Continuity (C^0-cont) $\quad P_{i,n} = Q_{i,0} \quad \forall i = 0, \ldots, m$
- Smoothness (C^1-cont)

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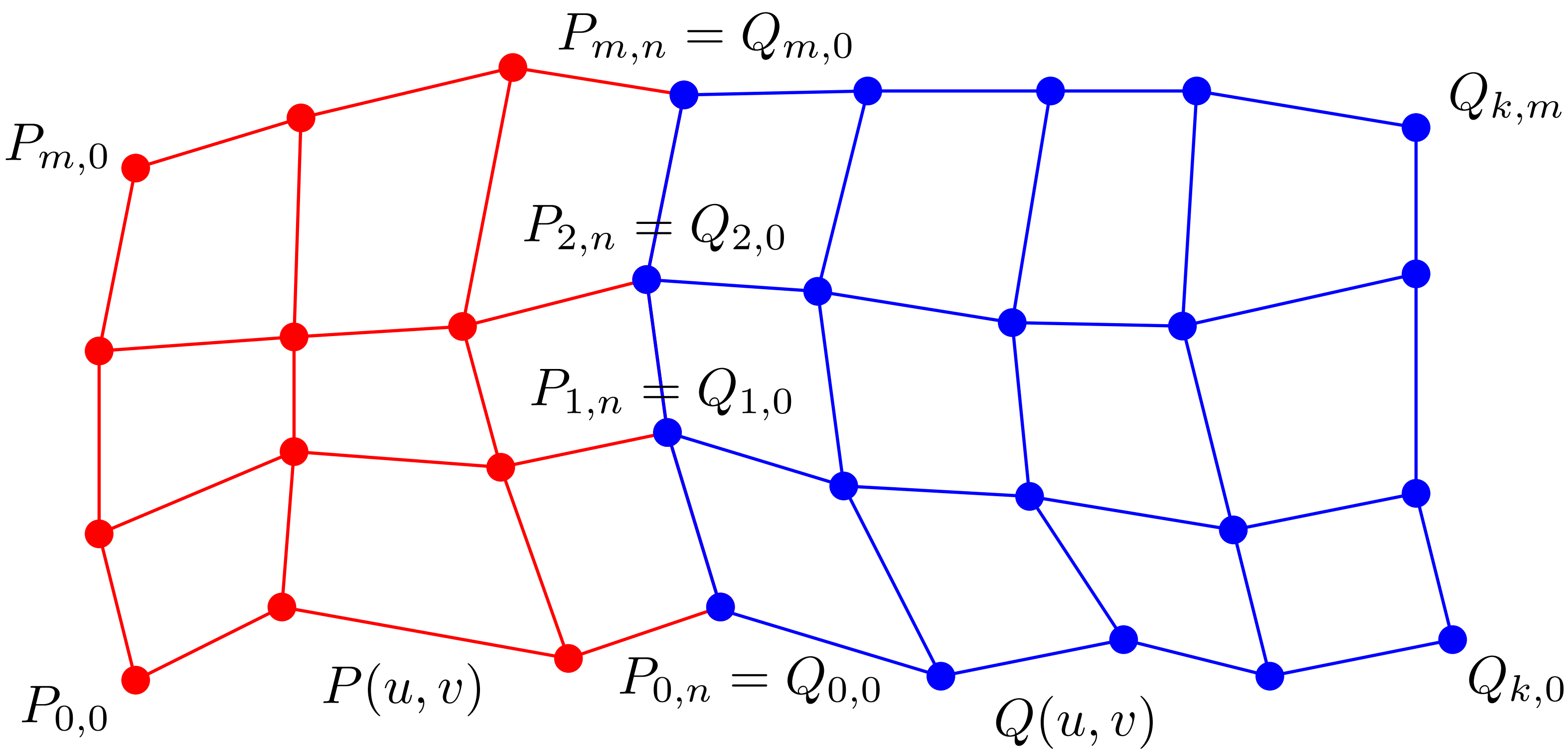

- Continuity (C^0-cont) $\quad P_{i,n} = Q_{i,0} \quad \forall i = 0, \ldots, m$
- Smoothness (C^1-cont) $\quad \left.\frac{\partial P(u,v)}{\partial u}\right|_{u=1} = \left.\frac{\partial Q(u,v)}{\partial u}\right|_{u=0}$

Smoothness condition (C^1-continuity)

$$\left.\frac{\partial P(u,v)}{\partial u}\right|_{u=1} = \left.\frac{\partial Q(u,v)}{\partial u}\right|_{u=0}$$

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$$\left.\frac{\partial P(u,v)}{\partial u}\right|_{u=1} = \left.\frac{\partial Q(u,v)}{\partial u}\right|_{u=0}$$

$$P(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{n} B_{m,i}(u) B_{n,j}(v) P_{i,j} = \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i} u^i (1-u)^{m-i} \binom{n}{j} v^j (1-v)^{n-j} P_{i,j}$$

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$$\frac{\partial P(u,v)}{\partial u}\bigg|_{u=1} = \sum_{i=0}^{m} \binom{m}{i}nv^i(1-v)^{m-i}P_{i,n} - \sum_{i=0}^{m} \binom{m}{i}nv^i(1-v)^{m-i}P_{i,n-1}$$

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$$\left.\frac{\partial P(u,v)}{\partial u}\right|_{u=1} = \sum_{i=0}^{m} \binom{m}{i} n v^i (1-v)^{m-i} P_{i,n} - \sum_{i=0}^{m} \binom{m}{i} n v^i (1-v)^{m-i} P_{i,n-1}$$

$$= n \sum_{i=0}^{m} \binom{m}{i} v^i (1-v)^{m-i} (P_{i,n} - P_{i,n-1})$$

Smoothness condition (C^1-continuity)

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$$= n\sum_{i=0}^{m}\binom{m}{i}v^i(1-v)^{m-i}(P_{i,n} - P_{i,n-1})$$

Analogously,

$$\frac{\partial Q(u,v)}{\partial u}\bigg|_{u=0} = k\sum_{i=0}^{m}\binom{m}{i}v^i(1-v)^{m-i}(Q_{i,1} - Q_{i,0})$$

Smoothness condition (C^1-continuity)

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Therefore, the condition for C^1-continuity is: $n(P_{i,n} - P_{i,n-1}) = k(Q_{i,1} - Q_{i,0}) \;\; \forall i = 0, \ldots, m$

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Analogously,

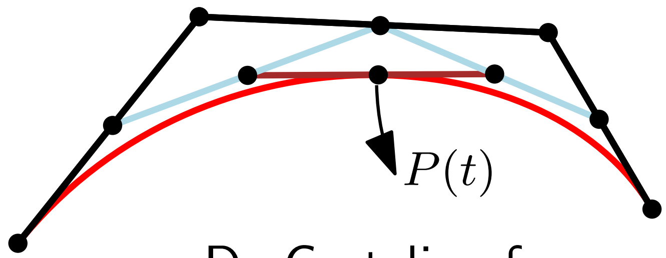
$$\left.\frac{\partial Q(u,v)}{\partial u}\right|_{u=0} = k\sum_{i=0}^{m}\binom{m}{i} v^i(1-v)^{m-i}(Q_{i,1} - Q_{i,0})$$

Therefore, the condition for C^1-continuity is: $\boxed{n(P_{i,n} - P_{i,n-1}) = k(Q_{i,1} - Q_{i,0}) \;\; \forall i = 0,\dots,m}$

If we just want G^1-cont, it is enough with $P_{i,n} - P_{i,n-1} = \alpha(Q_{i,1} - Q_{i,0})$, for some $\alpha \neq 0 \in \mathbb{R}$

Applying De Casteljau to each dimension

- For surfaces: apply it in two phases (along u, and along v)



De Casteljau for curves

Applying De Casteljau to each dimension

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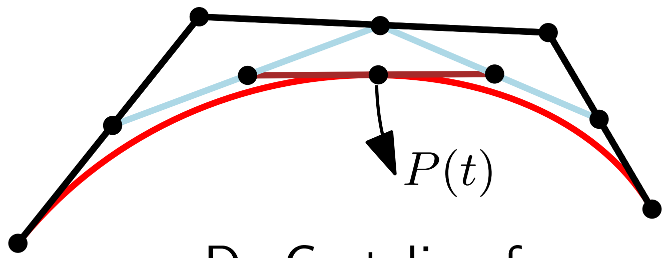

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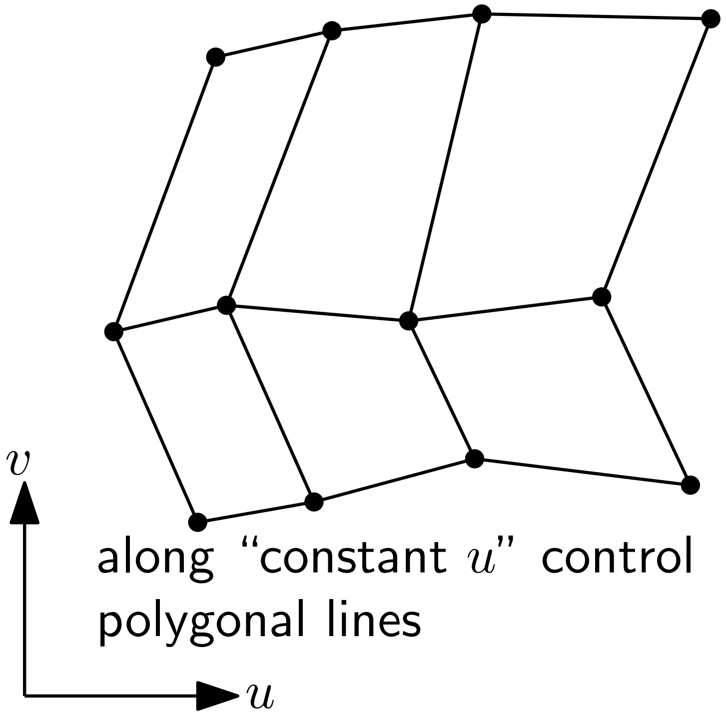
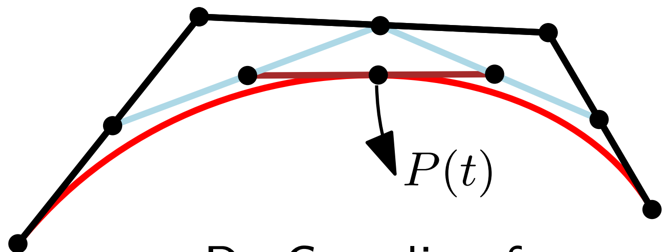


along "constant u" control polygonal lines

Applying De Casteljau to each dimension

- For surfaces: apply it in two phases (along u, and along v)



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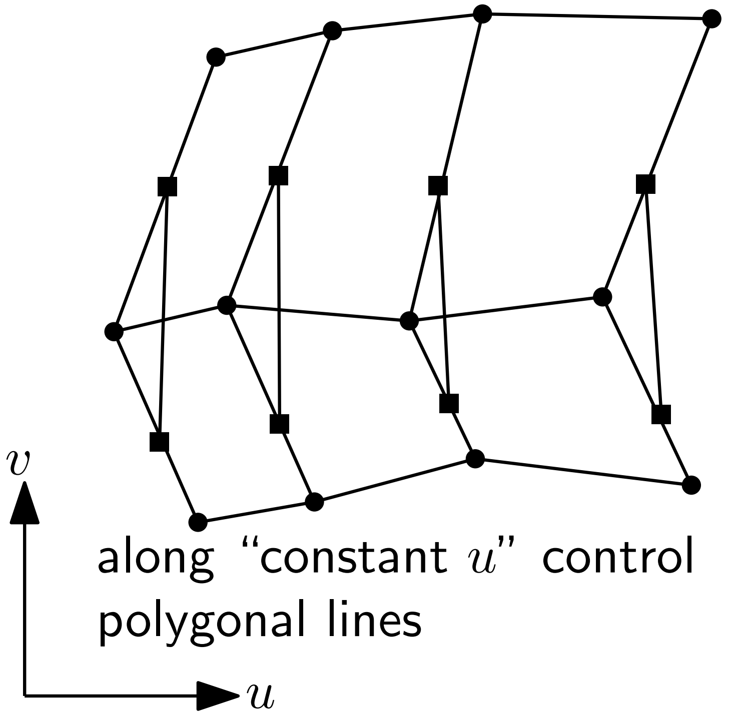

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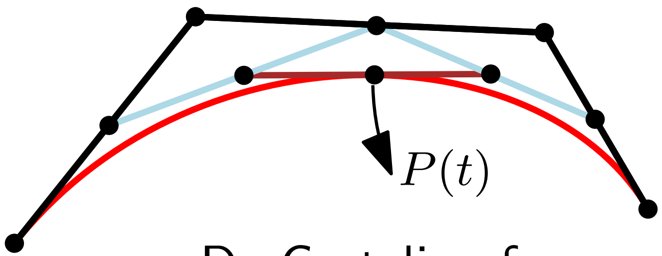
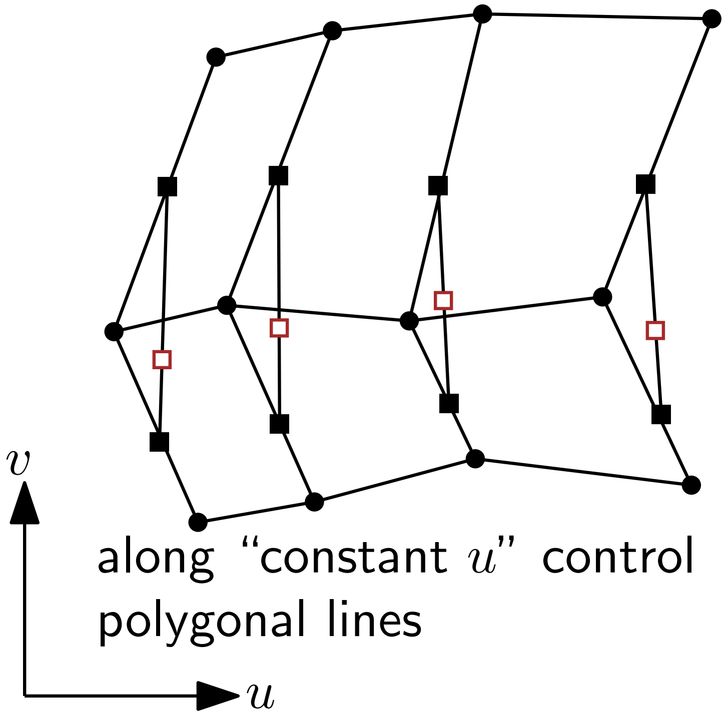


De Casteljau for curves



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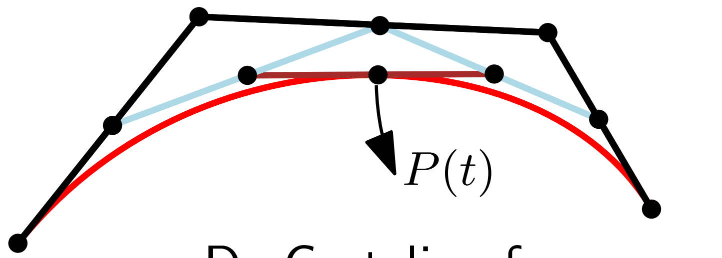

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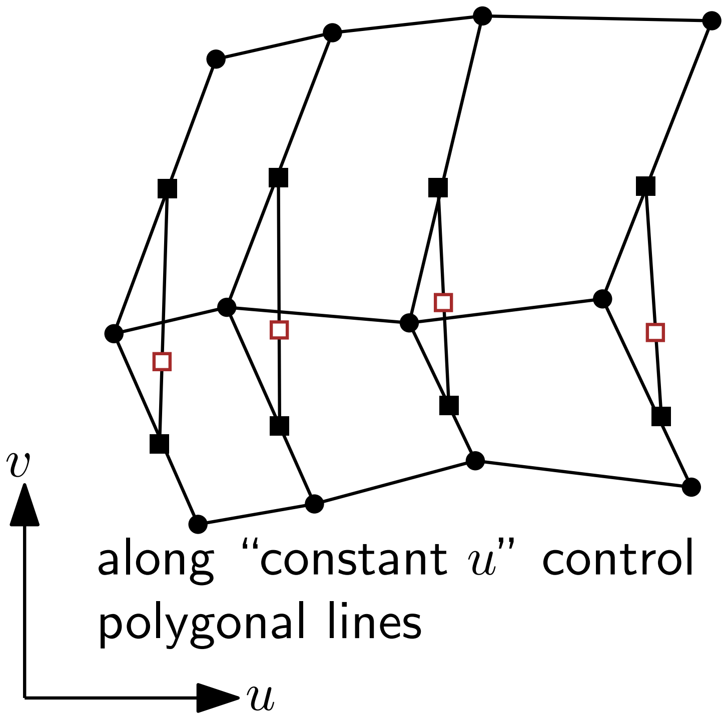

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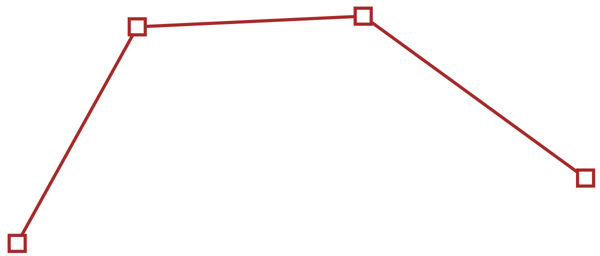
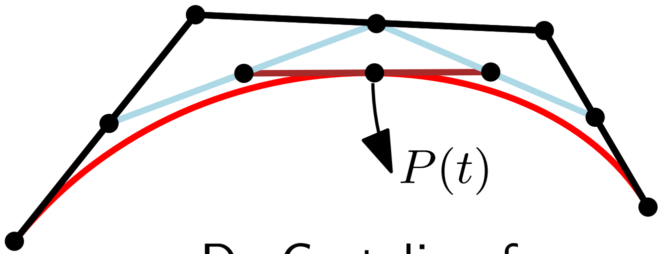


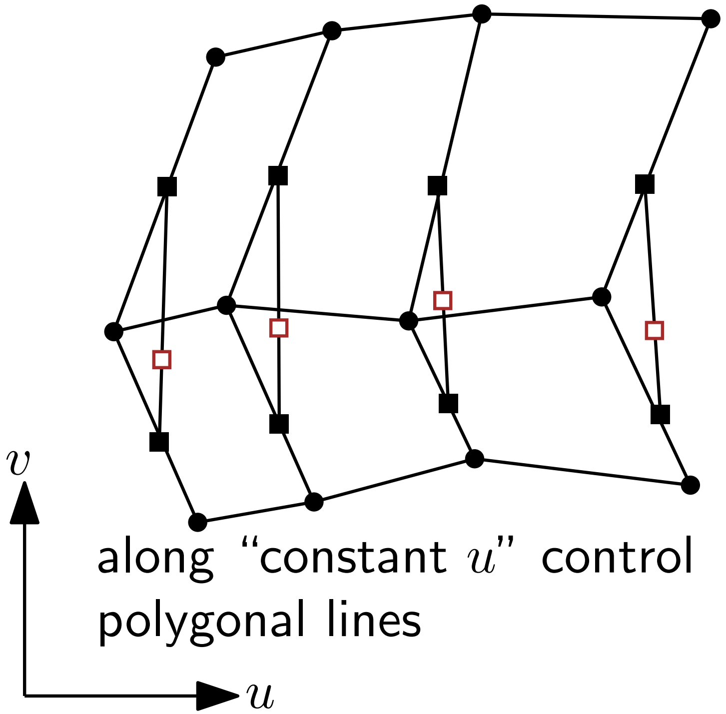
along one "constant v" polygon

Applying De Casteljau to each dimension

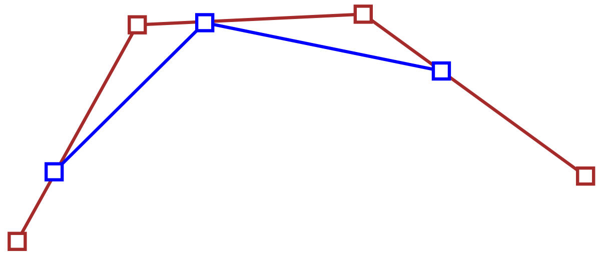
- For surfaces: apply it in two phases (along u, and along v)



De Casteljau for curves



along "constant u" control polygonal lines



along one "constant v" polygon

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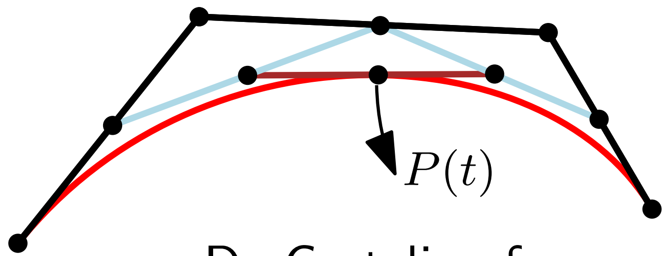
- For surfaces: apply it in two phases (along u, and along v)



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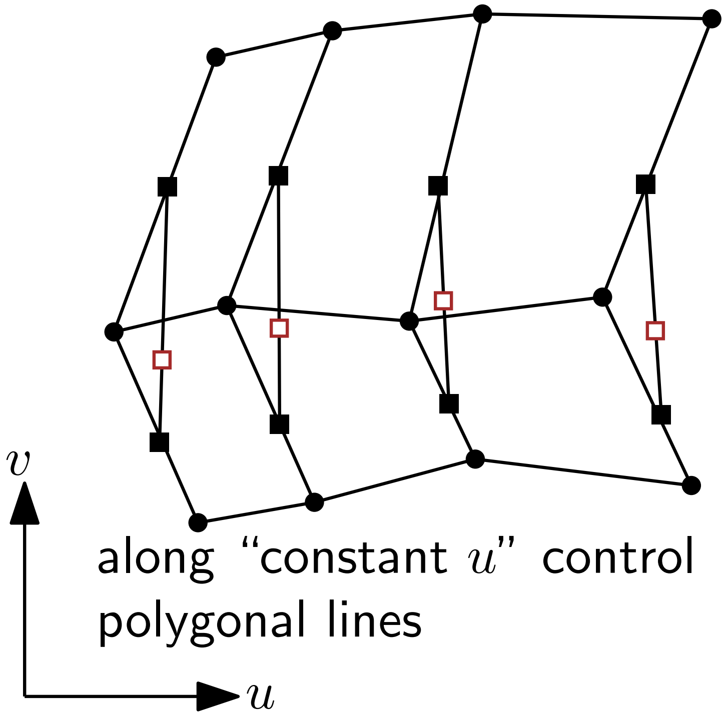
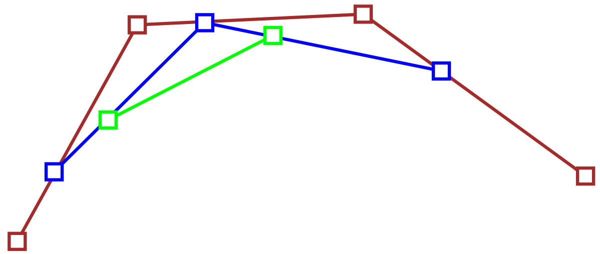


along "constant u" control polygonal lines



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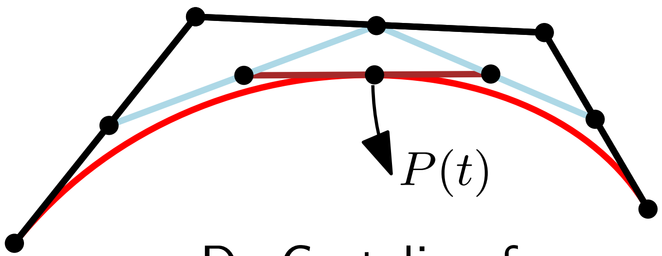

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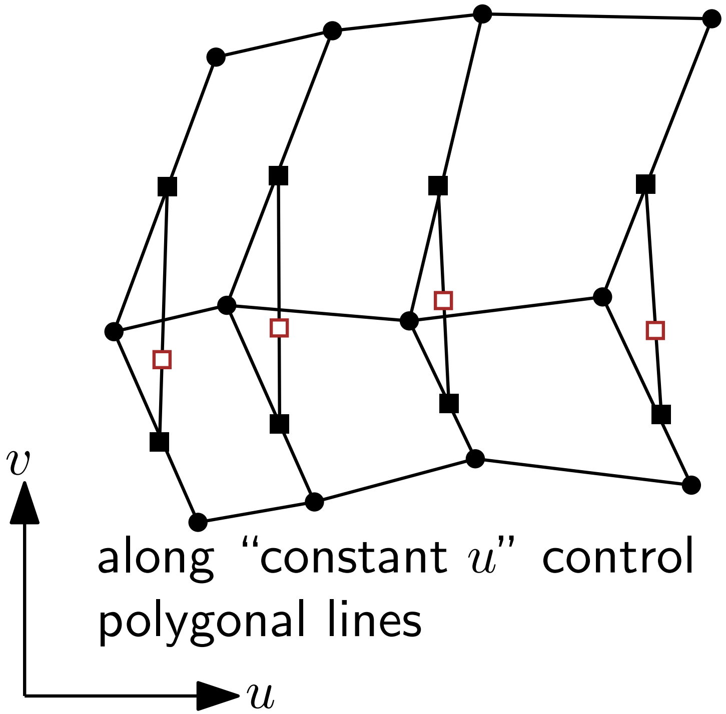

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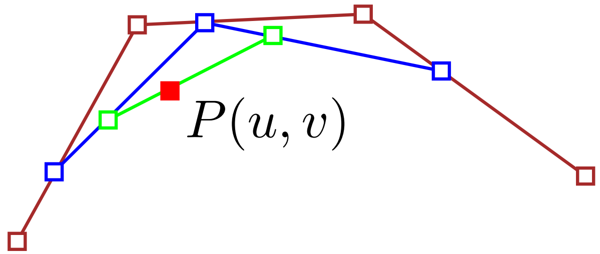
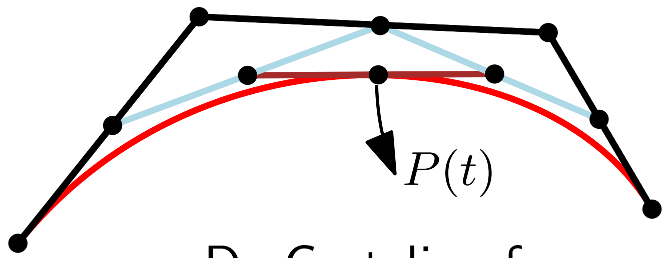


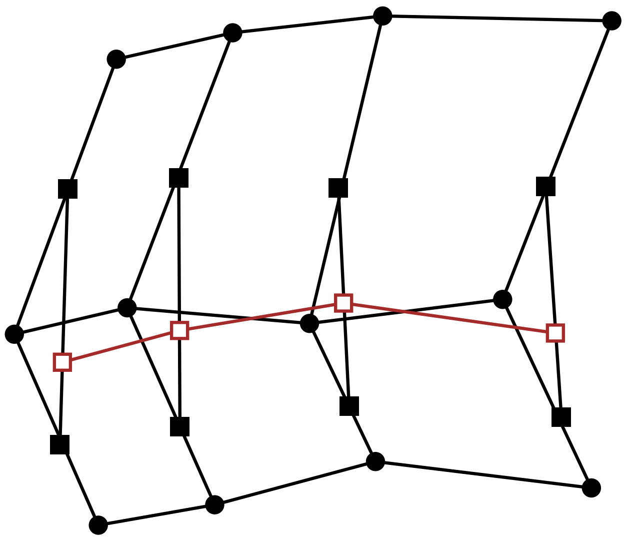
along one "constant v" polygon

Applying De Casteljau to each dimension

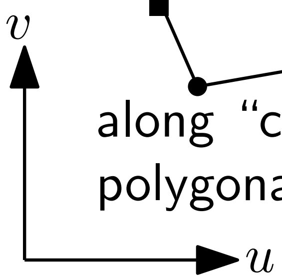
- For surfaces: apply it in two phases (along u, and along v)

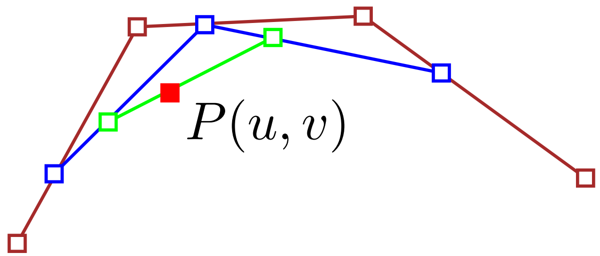


De Casteljau for curves



along "constant u" control polygonal lines

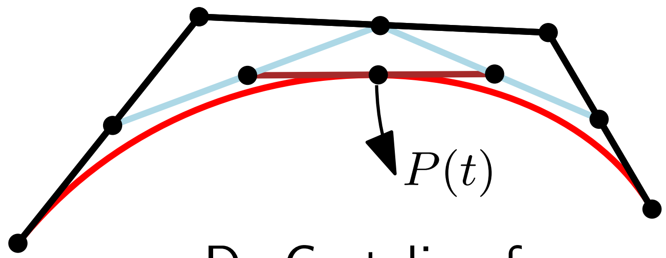




along one "constant v" polygon

Applying De Casteljau to each dimension

- For surfaces: apply it in two phases (along u, and along v)



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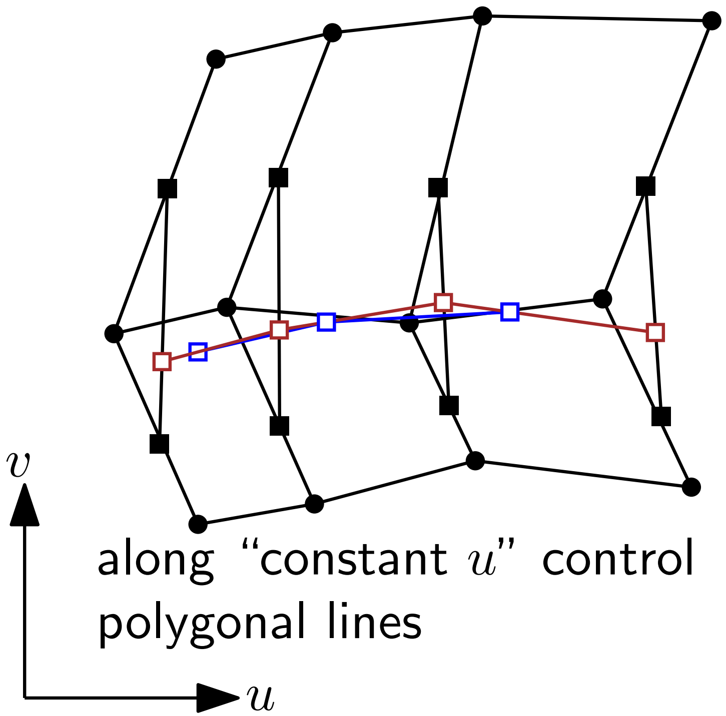

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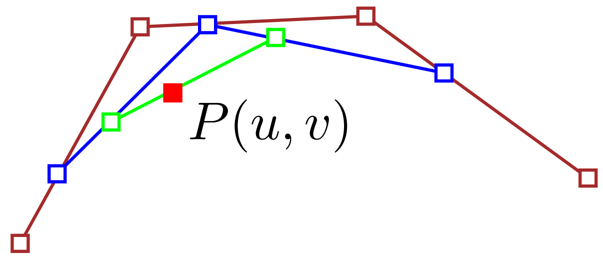

along one "constant v" polygon

Applying De Casteljau to each dimension

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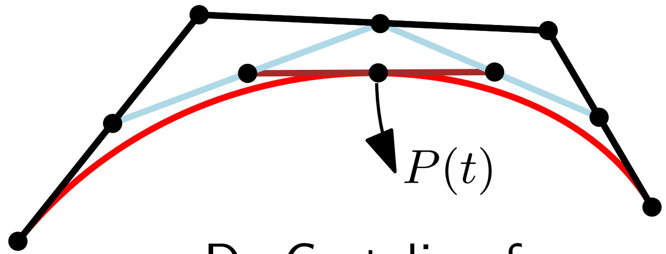

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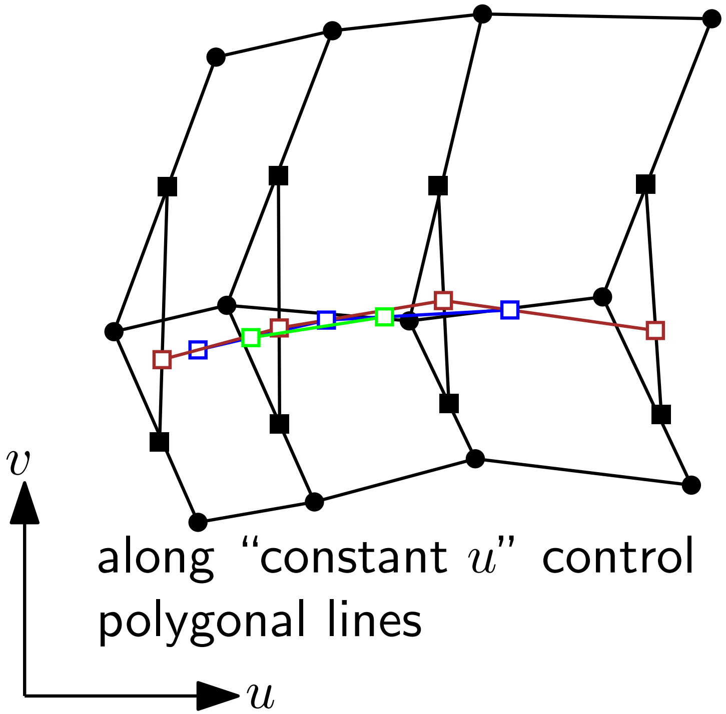

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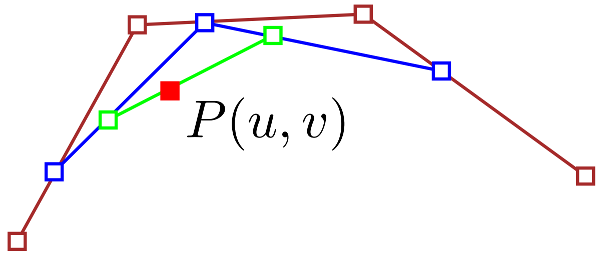
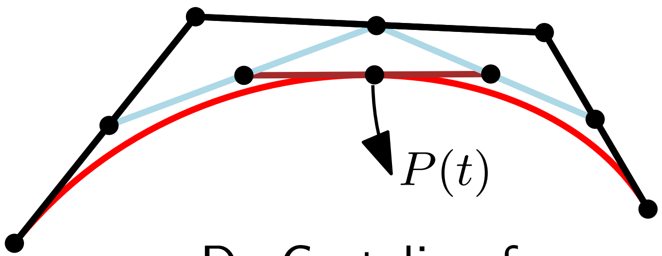


along one "constant v" polygon

Applying De Casteljau to each dimension

- For surfaces: apply it in two phases (along u, and along v)



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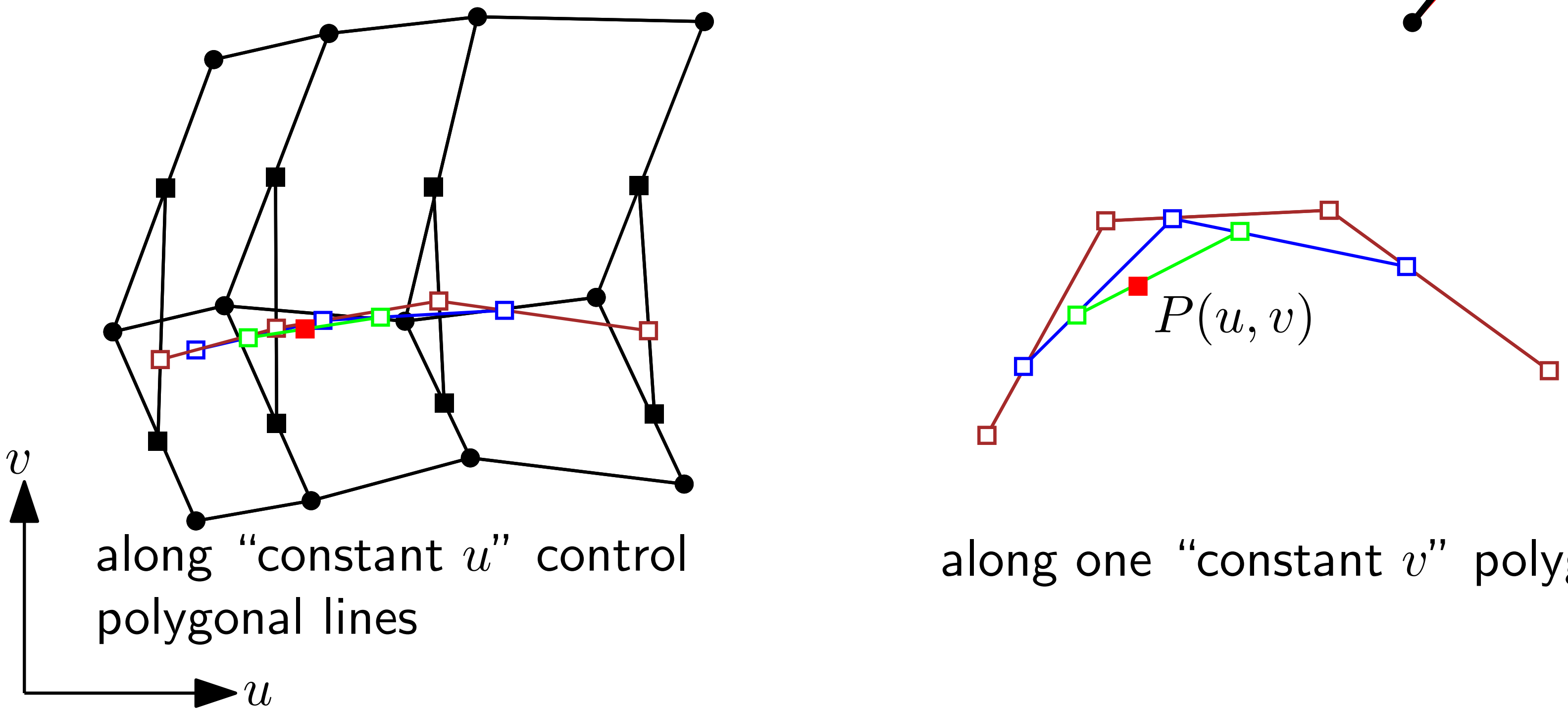

along "constant u" control polygonal lines

along one "constant v" polygon

- Applications of De Casteljau, e.g., to curve subdivision, also extend to surfaces

Interpolating Bézier surface patch

Problem: given $(m+1) \times (n+1)$ data points $Q_{k,l}$, compute a set of $(m+1) \times (n+1)$ control points $P_{i,j}$ such that the Bézier surface S defined by the points $P_{i,j}$ goes through the points $Q_{k,l}$

Interpolating Bézier surface patch

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Same approach as for curves:

Interpolating Bézier surface patch

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Same approach as for curves:

1) Select $m+1$ values u_k (for $k=0,\ldots,m$)

Interpolating Bézier surface patch

Problem: given $(m+1) \times (n+1)$ data points $Q_{k,l}$, compute a set of $(m+1) \times (n+1)$ control points $P_{i,j}$ such that the Bézier surface S defined by the points $P_{i,j}$ goes through the points $Q_{k,l}$

Same approach as for curves:

1) Select $m+1$ values u_k (for $k = 0, \ldots, m$)

2) Select $n+1$ values v_l (for $l = 0, \ldots, n$)

Interpolating Bézier surface patch

Problem: given $(m+1) \times (n+1)$ data points $Q_{k,l}$, compute a set of $(m+1) \times (n+1)$ control points $P_{i,j}$ such that the Bézier surface S defined by the points $P_{i,j}$ goes through the points $Q_{k,l}$

Same approach as for curves:

1) Select $m+1$ values u_k (for $k = 0, \ldots, m$)

2) Select $n+1$ values v_l (for $l = 0, \ldots, n$)

3) Require $S(u_k, v_l) = Q_{k,l}$ for all k, l

Interpolating Bézier surface patch

Problem: given $(m+1) \times (n+1)$ data points $Q_{k,l}$, compute a set of $(m+1) \times (n+1)$ control points $P_{i,j}$ such that the Bézier surface S defined by the points $P_{i,j}$ goes through the points $Q_{k,l}$

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Defines system of $(m+1) \times (n+1)$ equations that always has a solution

Interpolating Bézier surface patch

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Same approach as for curves:

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Defines system of $(m+1) \times (n+1)$ equations that always has a solution

$$S(u_k, v_l) = B_m^t(u_k) P B_n(v_l) = Q_{k,l}, \text{ for } k = 0, \ldots, m \text{ and } l = 0, \ldots, n$$

Interpolating Bézier surface patch

Problem: given $(m+1) \times (n+1)$ data points $Q_{k,l}$, compute a set of $(m+1) \times (n+1)$ control points $P_{i,j}$ such that the Bézier surface S defined by the points $P_{i,j}$ goes through the points $Q_{k,l}$

Same approach as for curves:

1) Select $m+1$ values u_k (for $k = 0, \ldots, m$)
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3) Require $S(u_k, v_l) = Q_{k,l}$ for all k, l

Defines system of $(m+1) \times (n+1)$ equations that always has a solution

$$S(u_k, v_l) = \underbrace{B_m^t(u_k) P B_n(v_l)}_{\text{matricial form of Bézier surface formula}} = Q_{k,l}, \text{ for } k = 0, \ldots, m \text{ and } l = 0, \ldots, n$$

Interpolating Bézier surface patch

Example from [Salomon, page 232]:

Example: We choose $m = 3$ and $n = 2$. The system of equations becomes

$$[(1-u_k)^3, 3u_k(1-u_k)^2, 3u_k^2(1-u_k), u_k^3] \begin{bmatrix} \mathbf{P}_{00} & \mathbf{P}_{01} & \mathbf{P}_{02} \\ \mathbf{P}_{10} & \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{20} & \mathbf{P}_{21} & \mathbf{P}_{22} \\ \mathbf{P}_{30} & \mathbf{P}_{31} & \mathbf{P}_{32} \end{bmatrix} \begin{bmatrix} (1-w_l)^2 \\ 2w_l(1-w_l) \\ w_l^2 \end{bmatrix} = \mathbf{Q}_{kl}$$

Interpolating Bézier surface patch

Example from [Salomon, page 232]:

Example: We choose $m = 3$ and $n = 2$. The system of equations becomes

$$[(1-u_k)^3, 3u_k(1-u_k)^2, 3u_k^2(1-u_k), u_k^3] \begin{bmatrix} \mathbf{P}_{00} & \mathbf{P}_{01} & \mathbf{P}_{02} \\ \mathbf{P}_{10} & \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{20} & \mathbf{P}_{21} & \mathbf{P}_{22} \\ \mathbf{P}_{30} & \mathbf{P}_{31} & \mathbf{P}_{32} \end{bmatrix} \begin{bmatrix} (1-w_l)^2 \\ 2w_l(1-w_l) \\ w_l^2 \end{bmatrix} = \mathbf{Q}_{kl}$$

12 equations, 12 unknowns

Rational rectangular Bézier surface patch

Rational rectangular Bézier surface patch

Definition anologous to the one for curves

Rational rectangular Bézier surface patch

Definition anologous to the one for curves

$$P(u,w) = \frac{\sum_{i=0}^{m} \sum_{j=0}^{n} w_{i,j} B_{m,i}(u) B_{n,j}(w) P_{i,j}}{\sum_{i=0}^{m} \sum_{j=0}^{n} w_{i,j} B_{m,i}(u) B_{n,j}(w)}$$

$0 \le u, w \le 1$
$w_{i,j} \in \mathbb{R}_{>0}$ for all i, j

If all weights are $w_{i,j} = 1$, it reduces to the ordinary Bézier surface

TRIANGULAR PATCHES

Surface patches don't need to be rectangular

$$\begin{array}{c} \mathbf{P}_{010} \\ \mathbf{P}_{001}\ \mathbf{P}_{100} \end{array} \qquad \begin{array}{c} \mathbf{P}_{020} \\ \mathbf{P}_{011}\ \mathbf{P}_{110} \\ \mathbf{P}_{002}\ \mathbf{P}_{101}\ \mathbf{P}_{200} \end{array} \qquad \begin{array}{c} \mathbf{P}_{030} \\ \mathbf{P}_{021}\ \mathbf{P}_{120} \\ \mathbf{P}_{012}\ \mathbf{P}_{111}\ \mathbf{P}_{210} \\ \mathbf{P}_{003}\ \mathbf{P}_{102}\ \mathbf{P}_{201}\ \mathbf{P}_{300} \end{array} \qquad \begin{array}{c} \mathbf{P}_{040} \\ \mathbf{P}_{031}\ \mathbf{P}_{130} \\ \mathbf{P}_{022}\ \mathbf{P}_{121}\ \mathbf{P}_{220} \\ \mathbf{P}_{013}\ \mathbf{P}_{112}\ \mathbf{P}_{211}\ \mathbf{P}_{310} \\ \mathbf{P}_{004}\ \mathbf{P}_{103}\ \mathbf{P}_{202}\ \mathbf{P}_{301}\ \mathbf{P}_{400} \end{array}$$

Surface patches don't need to be rectangular

Control points arranged as triangular array

$$\begin{array}{c} \mathbf{P}_{010} \\ \mathbf{P}_{001}\ \mathbf{P}_{100} \end{array} \quad \begin{array}{c} \mathbf{P}_{020} \\ \mathbf{P}_{011}\ \mathbf{P}_{110} \\ \mathbf{P}_{002}\ \mathbf{P}_{101}\ \mathbf{P}_{200} \end{array} \quad \begin{array}{c} \mathbf{P}_{030} \\ \mathbf{P}_{021}\ \mathbf{P}_{120} \\ \mathbf{P}_{012}\ \mathbf{P}_{111}\ \mathbf{P}_{210} \\ \mathbf{P}_{003}\ \mathbf{P}_{102}\ \mathbf{P}_{201}\ \mathbf{P}_{300} \end{array} \quad \begin{array}{c} \mathbf{P}_{040} \\ \mathbf{P}_{031}\ \mathbf{P}_{130} \\ \mathbf{P}_{022}\ \mathbf{P}_{121}\ \mathbf{P}_{220} \\ \mathbf{P}_{013}\ \mathbf{P}_{112}\ \mathbf{P}_{211}\ \mathbf{P}_{310} \\ \mathbf{P}_{004}\ \mathbf{P}_{103}\ \mathbf{P}_{202}\ \mathbf{P}_{301}\ \mathbf{P}_{400} \end{array}$$

Surface patches don't need to be rectangular

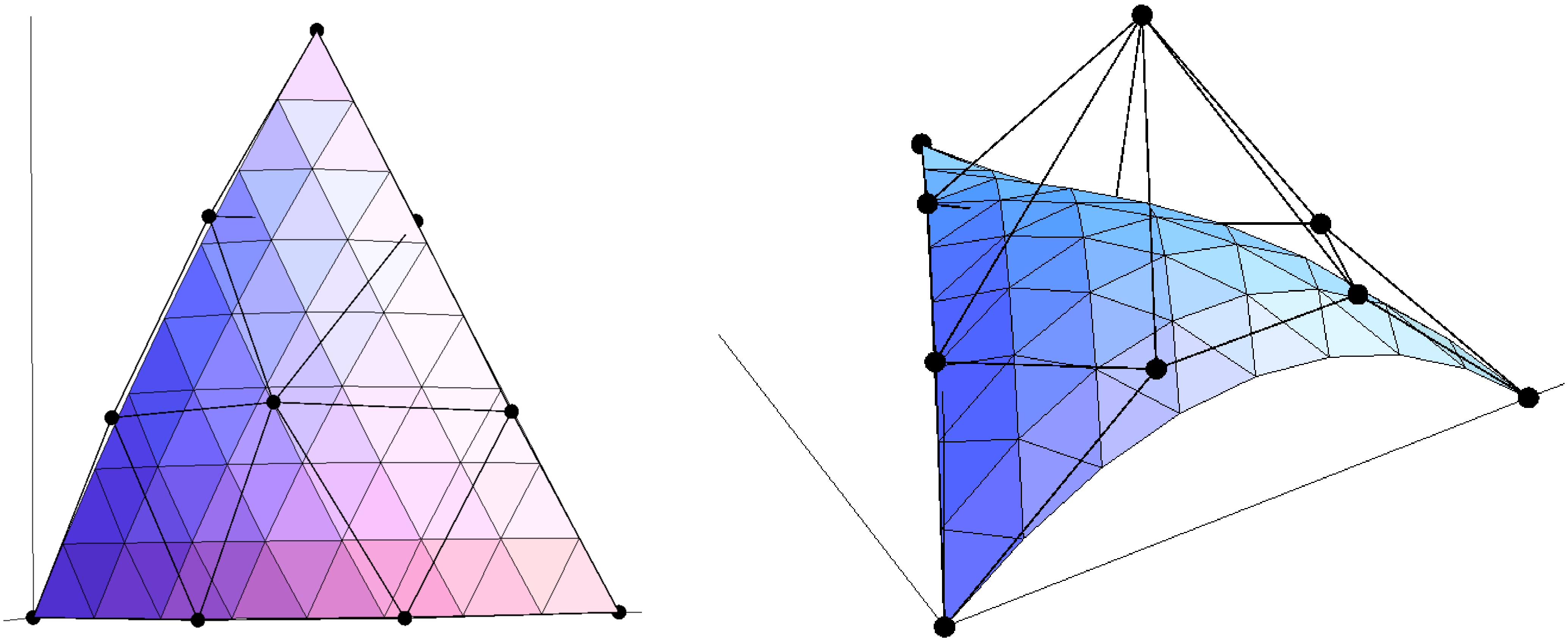
Control points arranged as triangular array

$$\begin{array}{c} \mathbf{P}_{010} \\ \mathbf{P}_{001}\ \mathbf{P}_{100} \end{array} \qquad \begin{array}{c} \mathbf{P}_{020} \\ \mathbf{P}_{011}\ \mathbf{P}_{110} \\ \mathbf{P}_{002}\ \mathbf{P}_{101}\ \mathbf{P}_{200} \end{array} \qquad \begin{array}{c} \mathbf{P}_{030} \\ \mathbf{P}_{021}\ \mathbf{P}_{120} \\ \mathbf{P}_{012}\ \mathbf{P}_{111}\ \mathbf{P}_{210} \\ \mathbf{P}_{003}\ \mathbf{P}_{102}\ \mathbf{P}_{201}\ \mathbf{P}_{300} \end{array} \qquad \begin{array}{c} \mathbf{P}_{040} \\ \mathbf{P}_{031}\ \mathbf{P}_{130} \\ \mathbf{P}_{022}\ \mathbf{P}_{121}\ \mathbf{P}_{220} \\ \mathbf{P}_{013}\ \mathbf{P}_{112}\ \mathbf{P}_{211}\ \mathbf{P}_{310} \\ \mathbf{P}_{004}\ \mathbf{P}_{103}\ \mathbf{P}_{202}\ \mathbf{P}_{301}\ \mathbf{P}_{400} \end{array}$$

Bézier formula needs version based on three variables

$$\mathbf{P}(u,v,w) = \sum_{i+j+k=n} \mathbf{P}_{ijk} \frac{n!}{i!\,j!\,k!} u^i v^j w^k = \sum_{i+j+k=n} \mathbf{P}_{ijk} B^n_{ijk}(u,v,w)$$

Example



$n = 3$