INTERPOLATING CURVES

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Curve and Surface Design

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INTERPOLATING A SET OF POINTS

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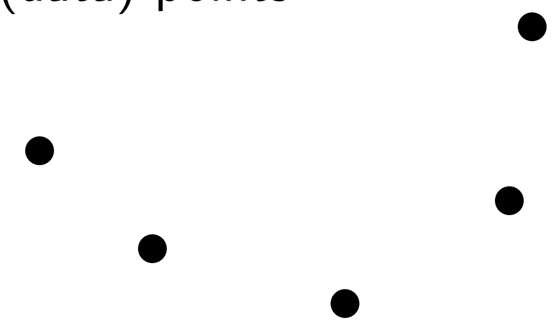

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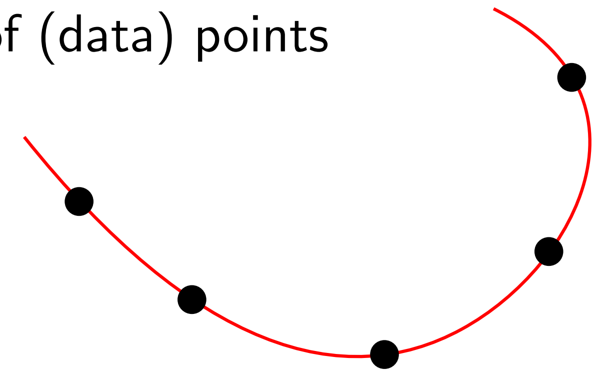

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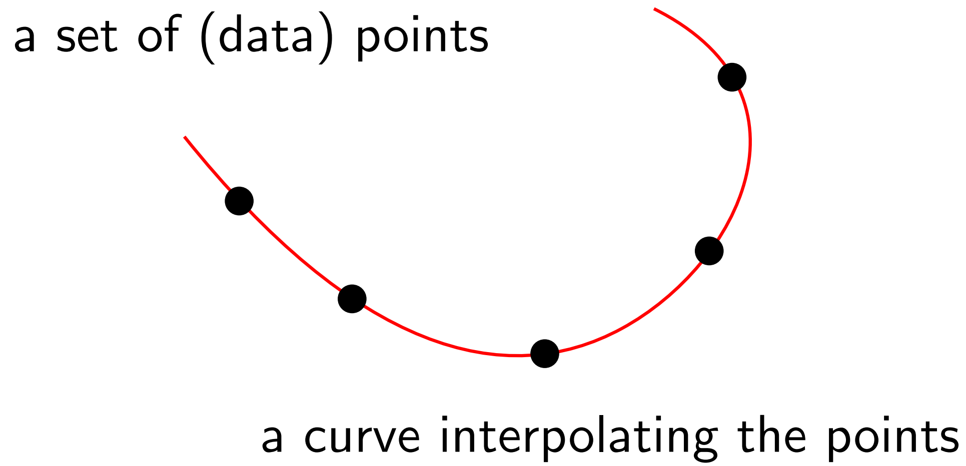

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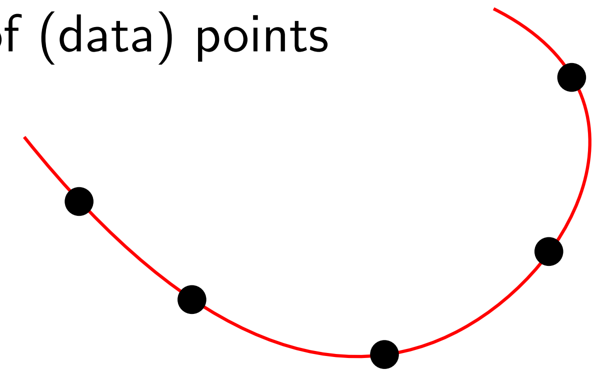

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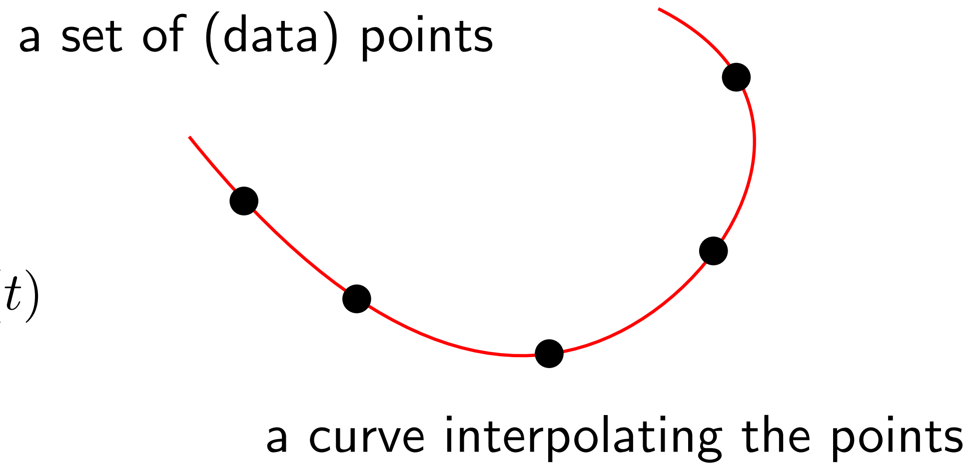

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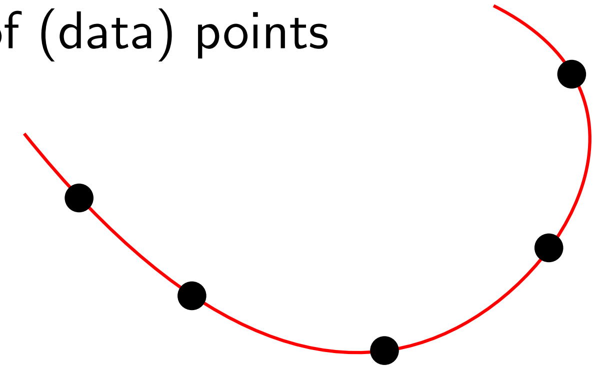

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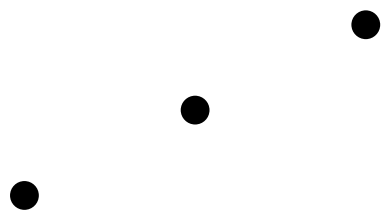

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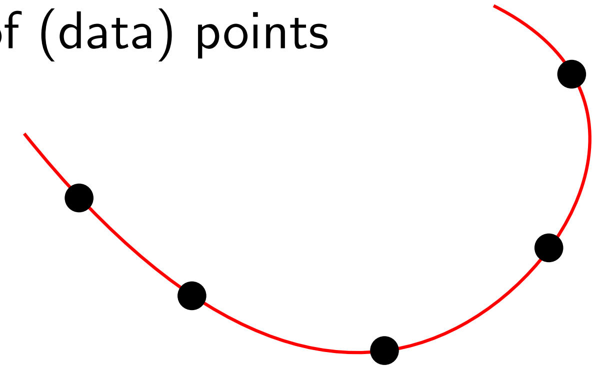

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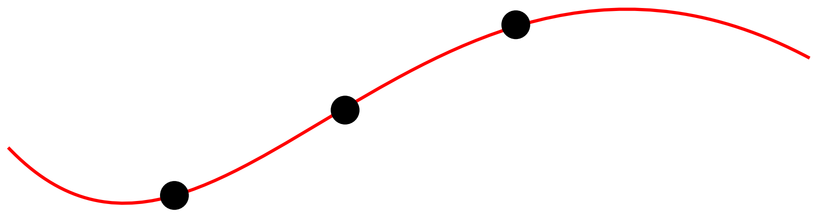

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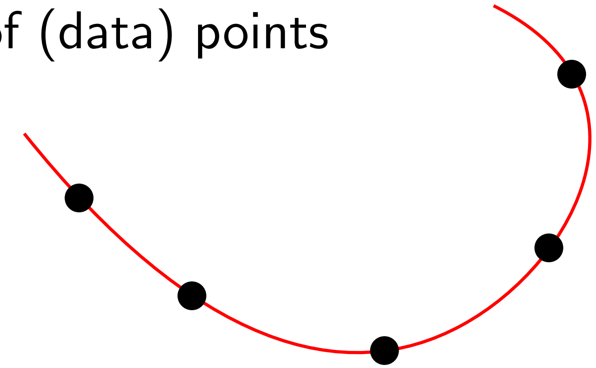

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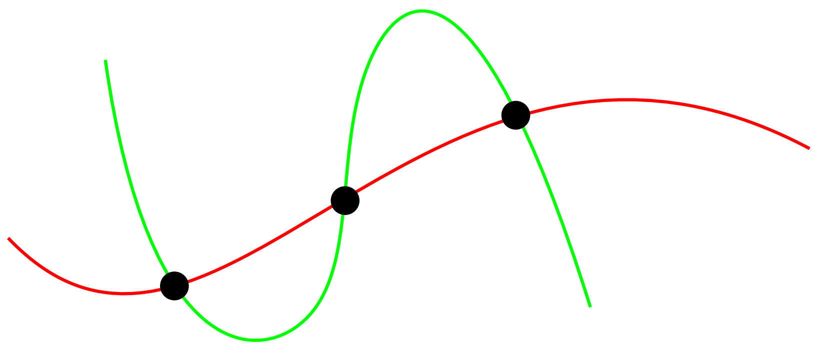

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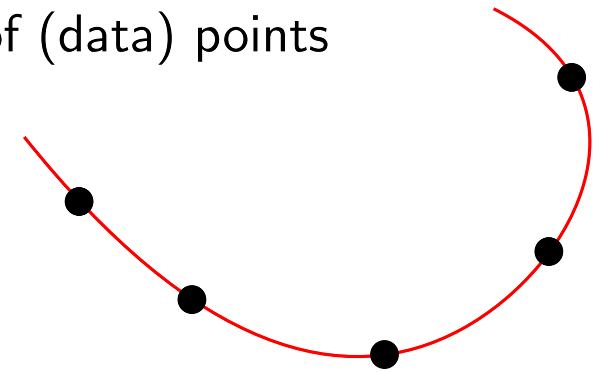

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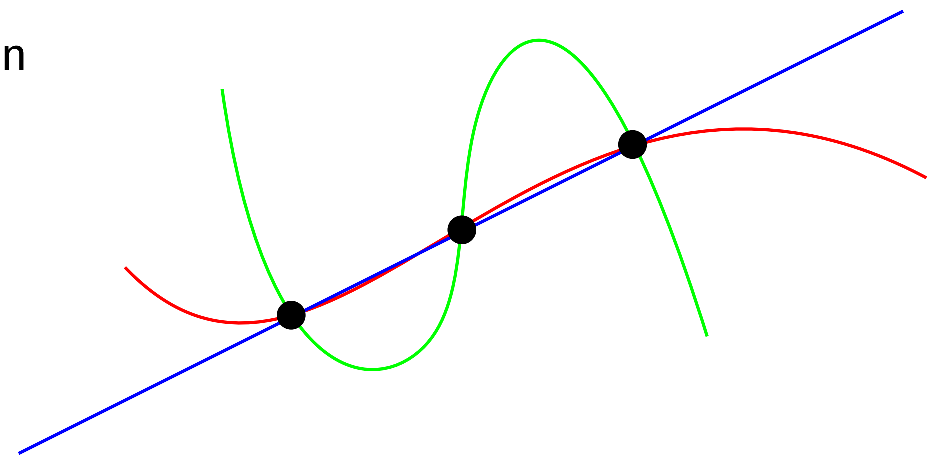

Piecewise linear interpolation

The simplest solution is piecewise linear interpolation: use a *polygonal line* that has the points $P_1, \ldots, P_n$ as vertices, and line segments $P_{i-1}P_i$ as edges.

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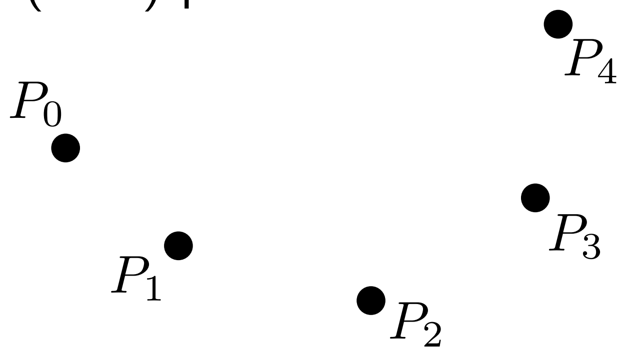

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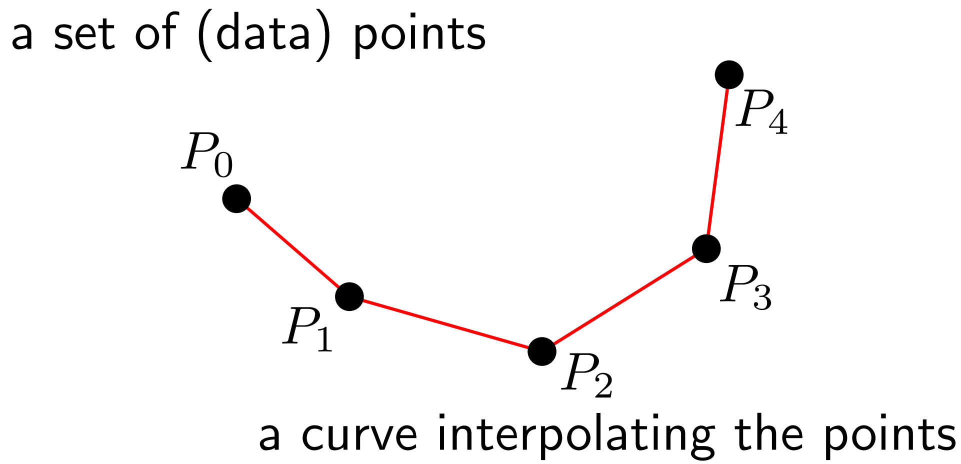

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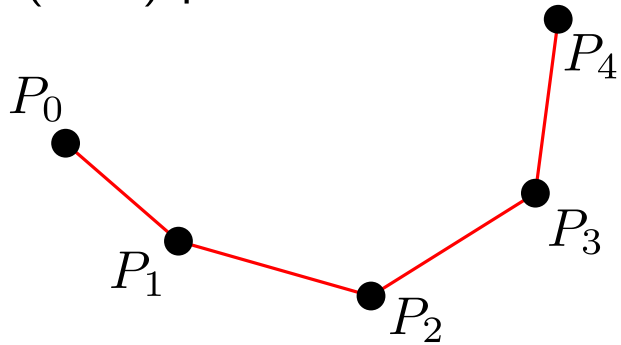

a curve interpolating the points

How can we parametrize this curve?

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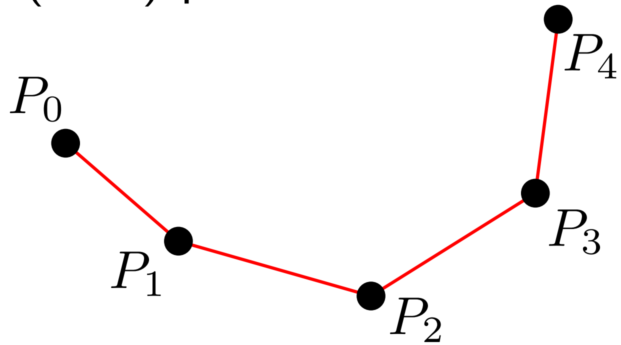
Parametrization

a set of (data) points



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Given a set P of $n+1$ points $P_0, P_1, \ldots, P_n$ in $\mathbb{R}^d$, and an increasing sequence of $n+1$ real values $t_0 < t_1 < \cdots < t_n$, the following curve interpolates the points in P:

$\gamma : [t_0, t_n] \to \mathbb{R}^d$

$$\gamma(t) = \frac{t_i - t}{t_i - t_{i-1}} P_{i-1} + \frac{t - t_{i-1}}{t_i - t_{i-1}} P_i \quad \text{if } t \in [t_{i-1}, t_i] \text{ for all } i = 1, \ldots, n$$

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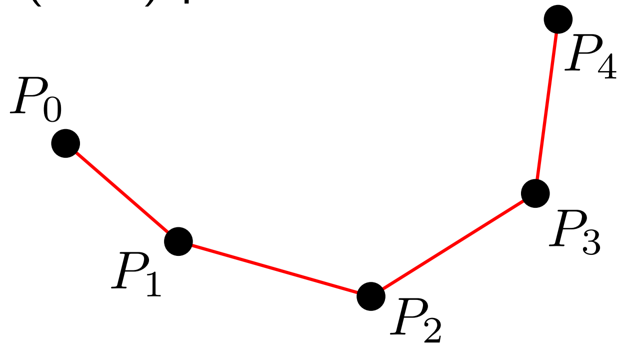

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Observations

- γ_{t-1} is continuous: trivially in (t_{i-1}, t_i) for all i, and also at each t_i for all i because $\gamma(t_i)$ is well defined (i.e., consecutive line segments coincide at data points)
- γ_{t-1} is not differentiable at the points t_i (unless three consecutive points are aligned)

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  Parametrization $[0, n] \to \mathbb{R}^d$. *Speed* possibly different on each edge.

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  Parametrization $[0, \sum_{k=1}^{n} d_k] \to \mathbb{R}^d$, **unit-speed parametrization**

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An important property for any interpolating curve

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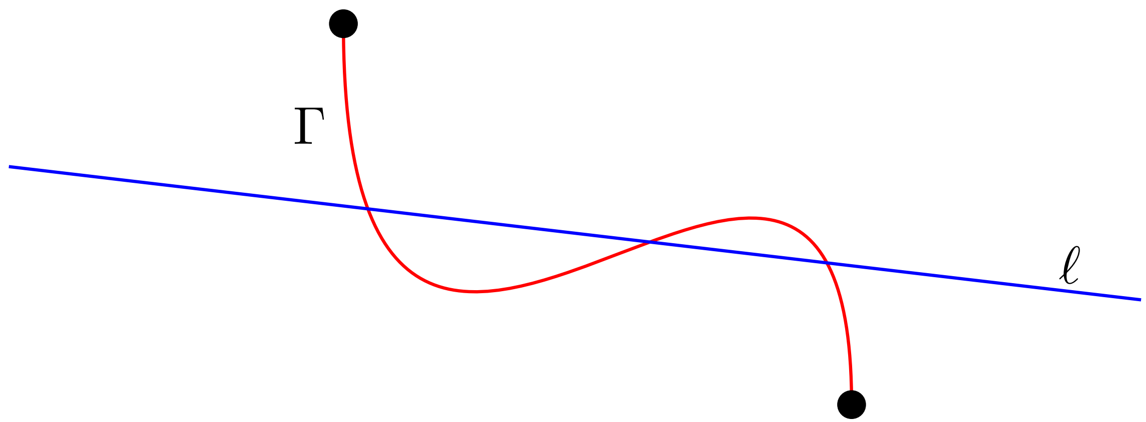
Given a curve Γ in $\mathbb{R}^2$ or $\mathbb{R}^3$, and **any** line ℓ (in $\mathbb{R}^2$) or plane π (in $\mathbb{R}^3$), let us denote $cross(\Gamma, \ell)$ or $cross(\Gamma, \pi)$ the number of crossings (i.e., intersection points) of Γ and ℓ or π.



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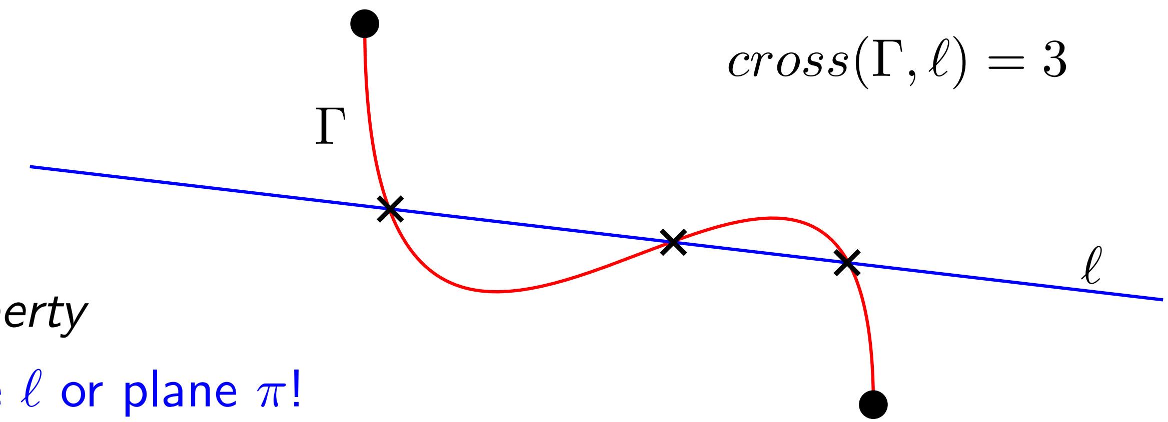

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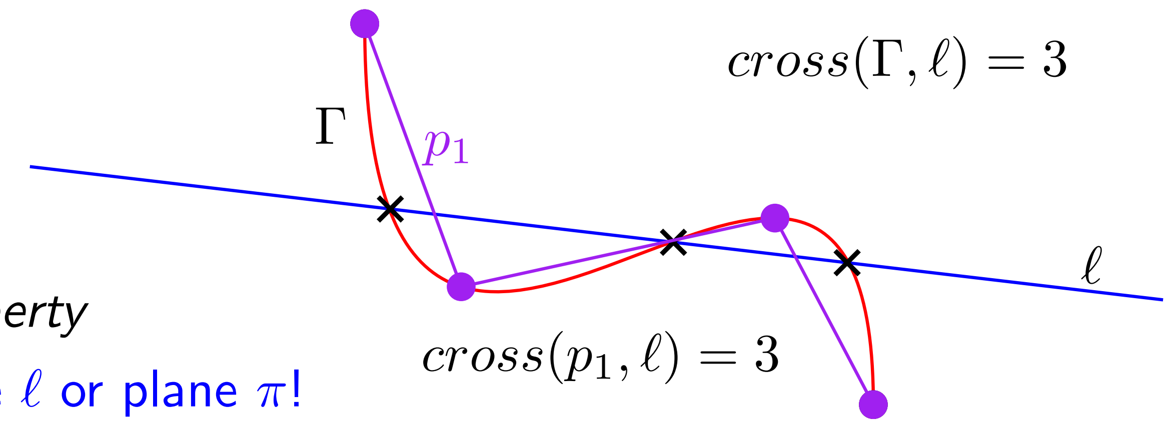

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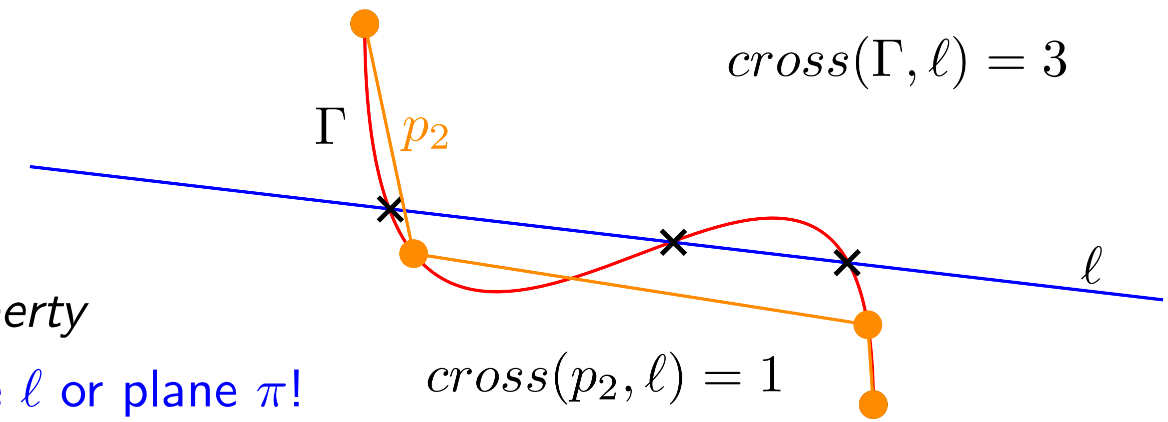

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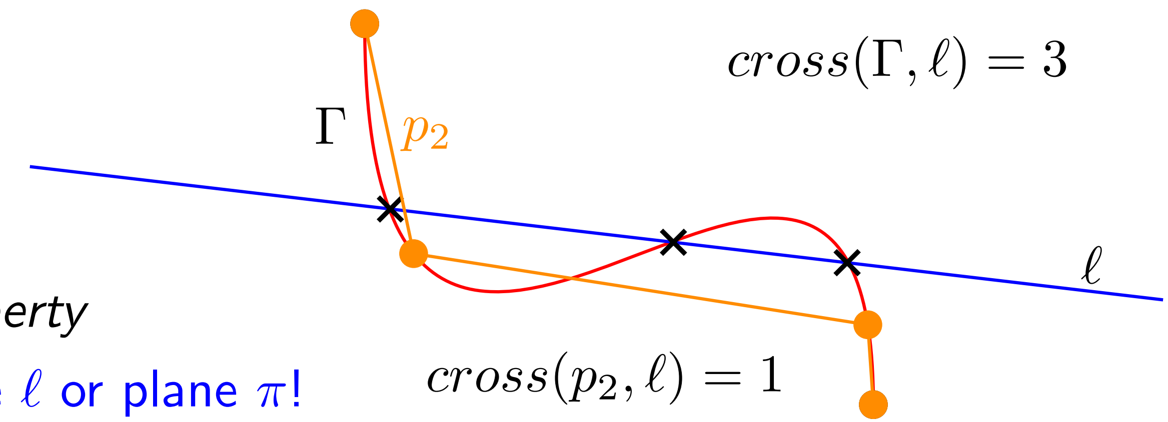

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> Implies that the interpolating curve (p) does not wiggle much more than the original one (Γ)

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Let $\gamma(t)$ be a curve in $\mathbb{R}^d$, and let $p(t)$ be a piecewise linear interpolation of $\gamma(t)$. Let f be any affine transformation $f : \mathbb{R}^d \to \mathbb{R}^d$.

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- for all i, $f(P_i)$ is a point in $f(\gamma(t))$, since P_i is a point in $\gamma(t)$

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Thus it is the same to (i) first linearly interpolate, then apply affine transformation, than (ii) first apply affine transformation, then linearly interpolate

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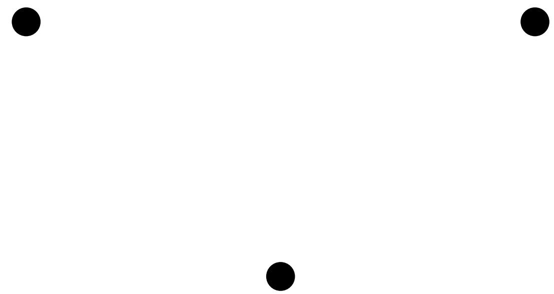

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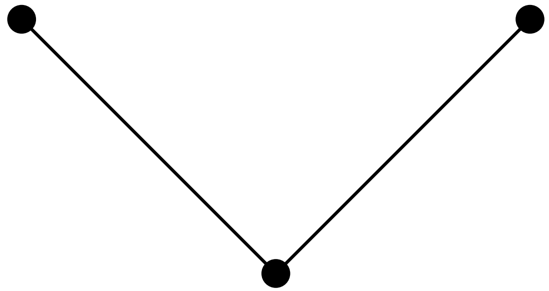
degree 1 (linear)



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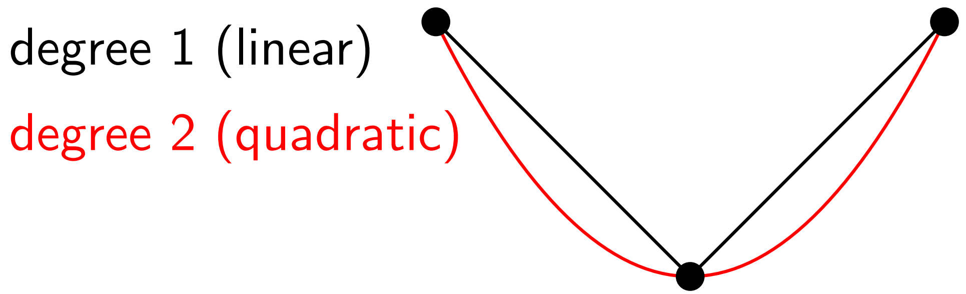

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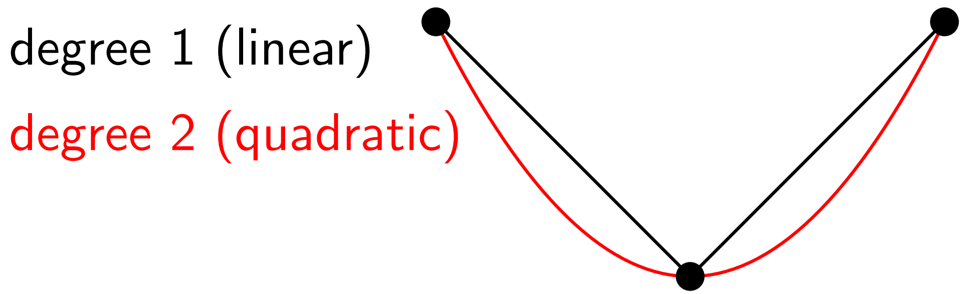

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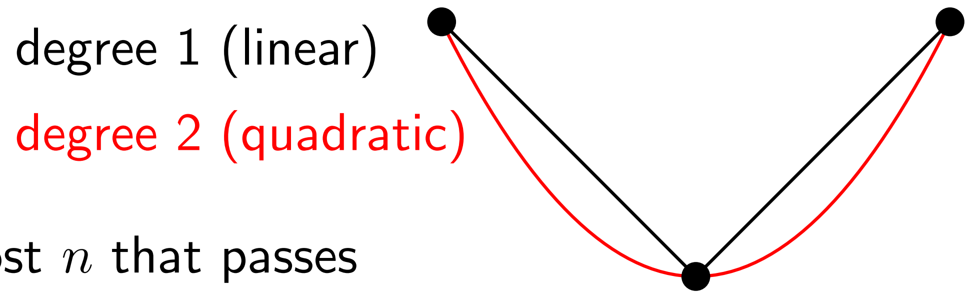

Interpolation theorem

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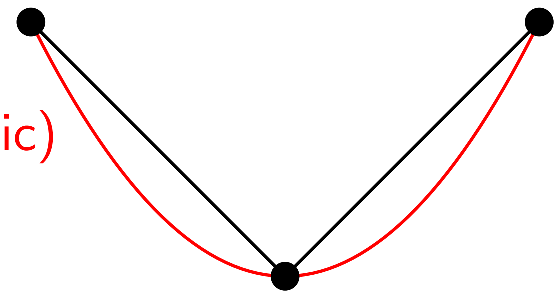

Interpolation theorem

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Proof

1) Uniqueness (if it exists, it is unique)

- Suppose that there are two polynomials $p(x)$ and $q(x)$ that interpolate the points. Consider then $r(x) = p(x) - q(x)$
- $r(x)$ is also a polynomial of degree at most n, but it has $n+1$ different roots: one at each x_i (since $r(x_i) = p(x_i) - q(x_i) = y_i - y_i = 0$)
- But a degree-n polynomial different from zero can have at most n roots! Then $r(x)$ must be the zero polynomial, i.e., $r(x) = 0$, implying that $p(x) = q(x)$!

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Proof (cont'd)

2) Existence (it exists!)

- We define the following auxiliary polynomials (known as *Lagrange weights*)

$$L_i^n(x) = \frac{\prod_{j \neq i}(x - x_j)}{\prod_{j \neq i}(x_i - x_j)}$$

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Observe:

- $L_i^n(x)$ is a polynomial of degree n for all $i = 0, \ldots, n$
- $L_i^n(x_j) = 1$ if $j = i$, and $L_i^n(x_j) = 0$ if $j \neq i$

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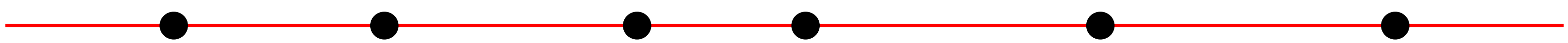
That is: the only degree that *always* allows to interpolate $n+1$ points is degree n

The exception is when two or more data points lie on a low-degree polynomial.

Example: n points on a line can be interpolated with a polynomial of degree just 1



Lagrange polynomial

Lemma: n is the minimum degree that guarantees the existence of an interpolating polynomial for *any* set of $n+1$ distinct points.

Why? **Proof sketch:** (by induction on n)

- Base case: $n = 1$. Then we have only two points P_0, P_1, and we know that two points are required to determine a line (i.e., a polynomial of degree 1)

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- Induction step: assume that there exists a set of n points $P_1, \ldots, P_n$ whose interpolating polynomial has degree exactly $n - 1$ (i.e., with lower degree it is not possible)

  Let P_0 be a point that does not lie on the polynomial curve that interpolates $P_1, \ldots, P_n$. Since that polynomial of degree $n - 1$ is unique, and it does not go through P_0, then the polynomial through $P_0, P_1, \ldots, P_n$ must be different, and thus must have higher degree.

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$$p(x) = \sum_{i=0}^{n} L_i^n(x) y_i$$

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Is the Lagrange polynomial affine invariant?

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Clever idea: let's choose a really simple set of points to interpolate (we can choose any, because the $L_i^n(x)$ weights are independent of the y-values)

Consider $n+1$ distinct points on the function $f(x) = 1$, that is: $P_i = (x_i, 1)$ for $i = 0, \ldots, n$. Then the unique polynomial of degree at most n that interpolates them is easy: $p(x) = 1$

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$$1 = p(x) = \sum_{i=0}^{n} L_i^n(x) y_i = \sum_{i=0}^{n} L_i^n(x) \cdot 1 = \sum_{i=0}^{n} L_i^n(x) \;\rightarrow \sum_{i=0}^{n} L_i^n(x) = 1$$

Therefore, the Lagrange polynomial is affine invariant

Parametric version of Lagrange polynomial

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$L_i^n(t) = \frac{\prod_{j \neq i}(t - t_j)}{\prod_{j \neq i}(t_i - t_j)}$, for $i = 0, \ldots, n$ $\quad \gamma(t) = \sum_{i=0}^{n} L_i^n(t) P_i$

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- $L_i^n(t)$ is a polynomial of degree n in $t \rightarrow$ so is $\gamma(t)$
- $L_i^n(t_j) = 1$ if $j = i$, and $L_i^n(t_j) = 0$ if $j \neq i$
- $\gamma(t_i) = P_i$
- For each value of t, $\gamma(t)$ is an affine combination of $P_0, \ldots, P_n$, with weights $L_i^n(t)$ that (you can prove) add up to 1 $\rightarrow$ this interpolation is also affine invariant

Parametric version of Langrange polynomial

Remarks

- As before, the parametric Lagrange interpolation is not unique: we are free to choose the parameter values $t_1, \ldots, t_{n-1}$.

  - Uniform version: $t_i - t_{i-1} = 1$ or $= \frac{1}{n}$
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$$p(t) = L_0^1(t)P_0 + L_1^1(t)P_1 =$$

$$= (t,1)\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} P_0 \\ P_1 \end{pmatrix}$$

See bibliography for the matrix version for larger n

Issues with polynomial interpolation

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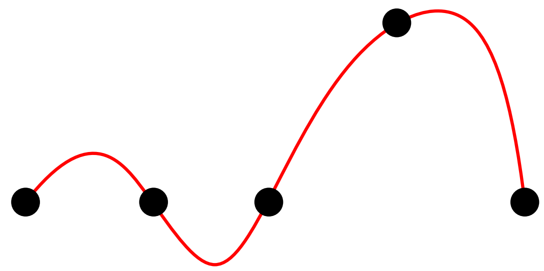

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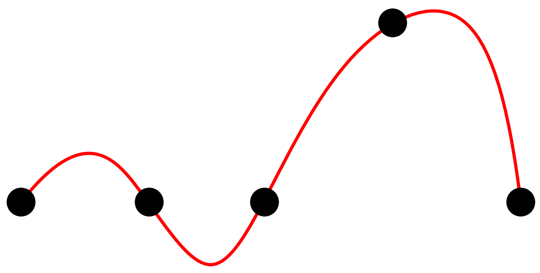

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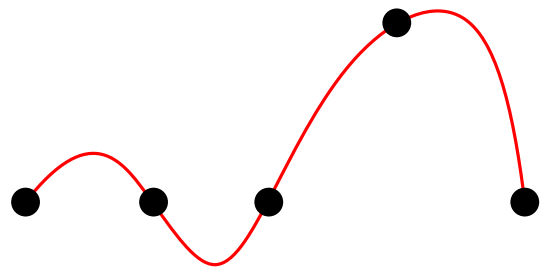

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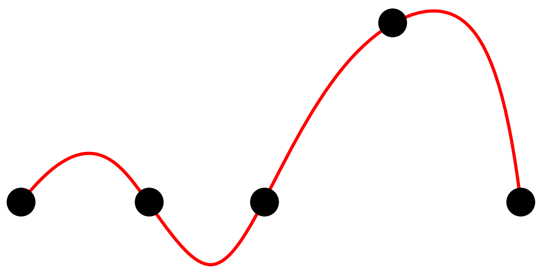

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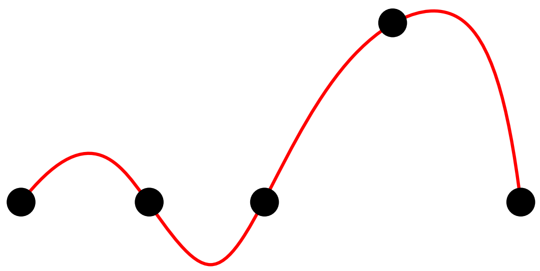

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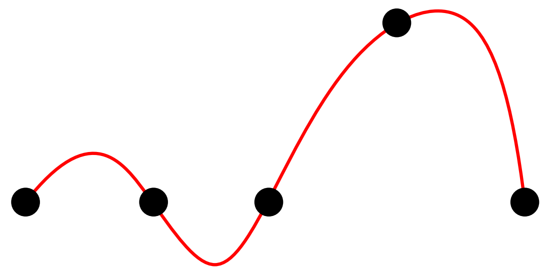

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- Lagrange's formula is not numerically stable: small variations in the input points can produce large variations in the final curve

- The method is not easy to make interactive: if the curve is not what one wants, (and you cannot modify the data points) all you can do is to add more points