BÉZIER CURVES

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Interpolation or... approximation!

Previous curve design methods based on **interpolation**

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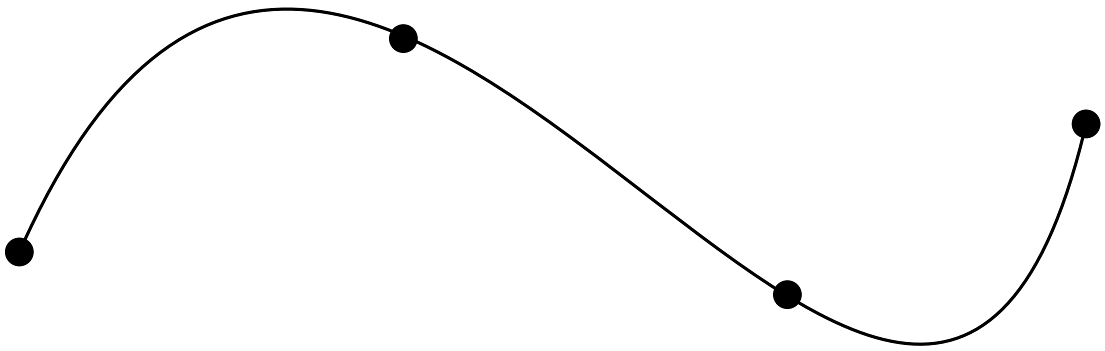

Interpolating curve

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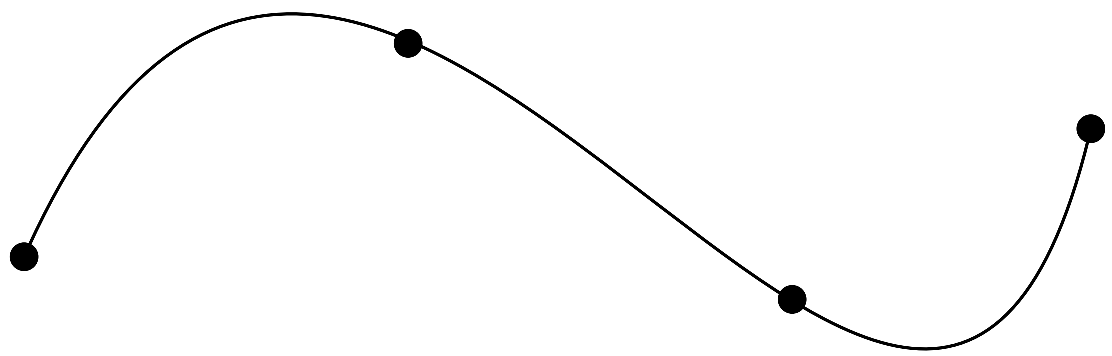
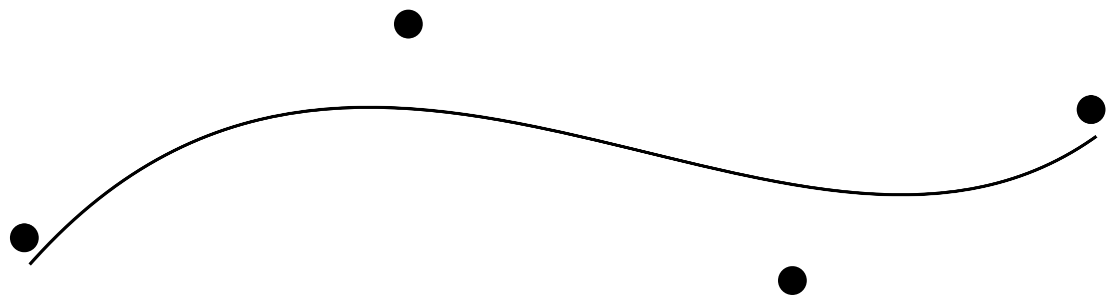


Interpolating curve

Curve passes exactly through given points



Approximating curve

Curve passes near the given points

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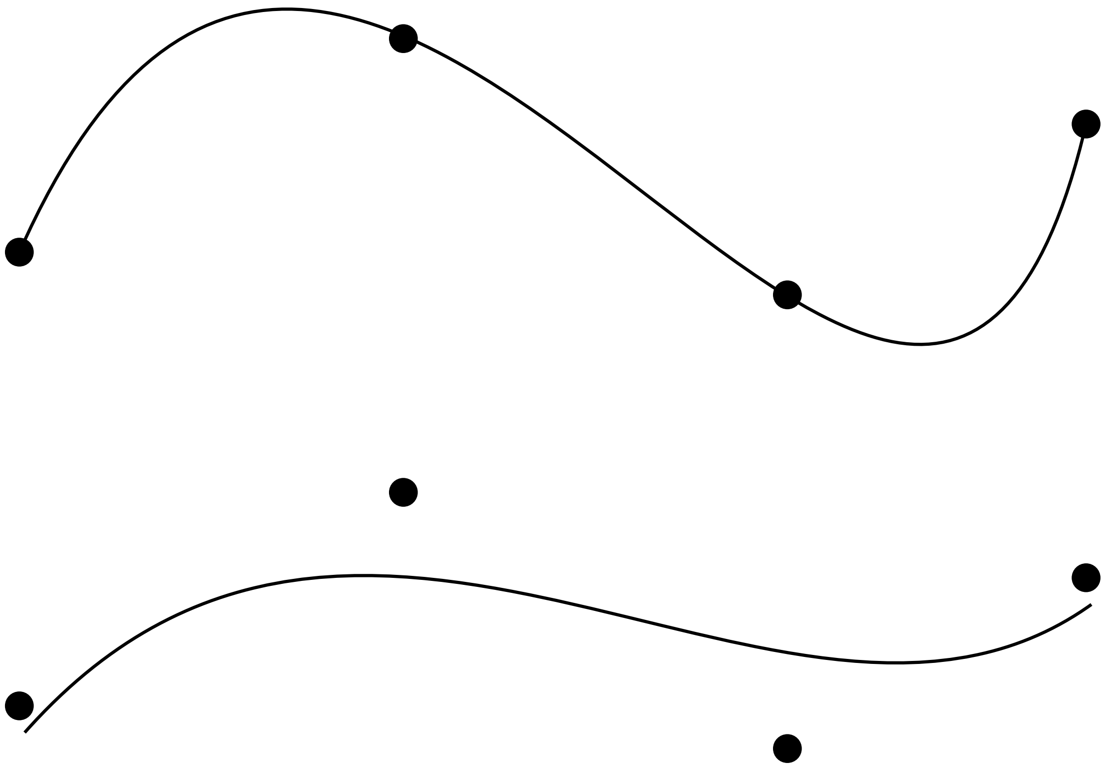

Interpolating curve

Curve passes exactly through given points

Approximating curve

Curve passes near the given points

What's wrong with interpolation? Curve change when moving points is unpredictable

Approximating curves can provide better "shape control"

INTRODUCTION TO BÉZIER CURVES

Bézier curves

Named after Pierre Bézier (1910-1999)

- Worked on automizing the process of designing cars
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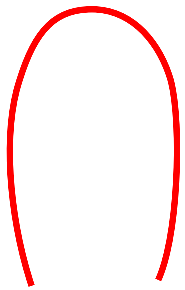
Bézier curve

- Parametric ($P(t)$)
- Polynomial
- Based on *control points*

Some examples of Bézier curves



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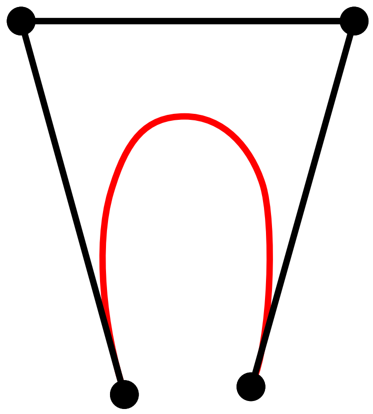

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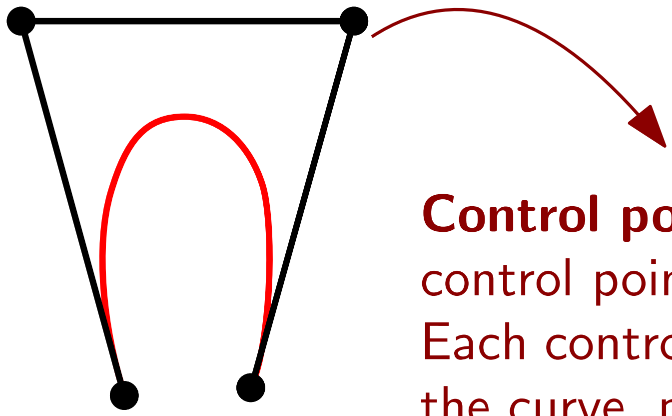
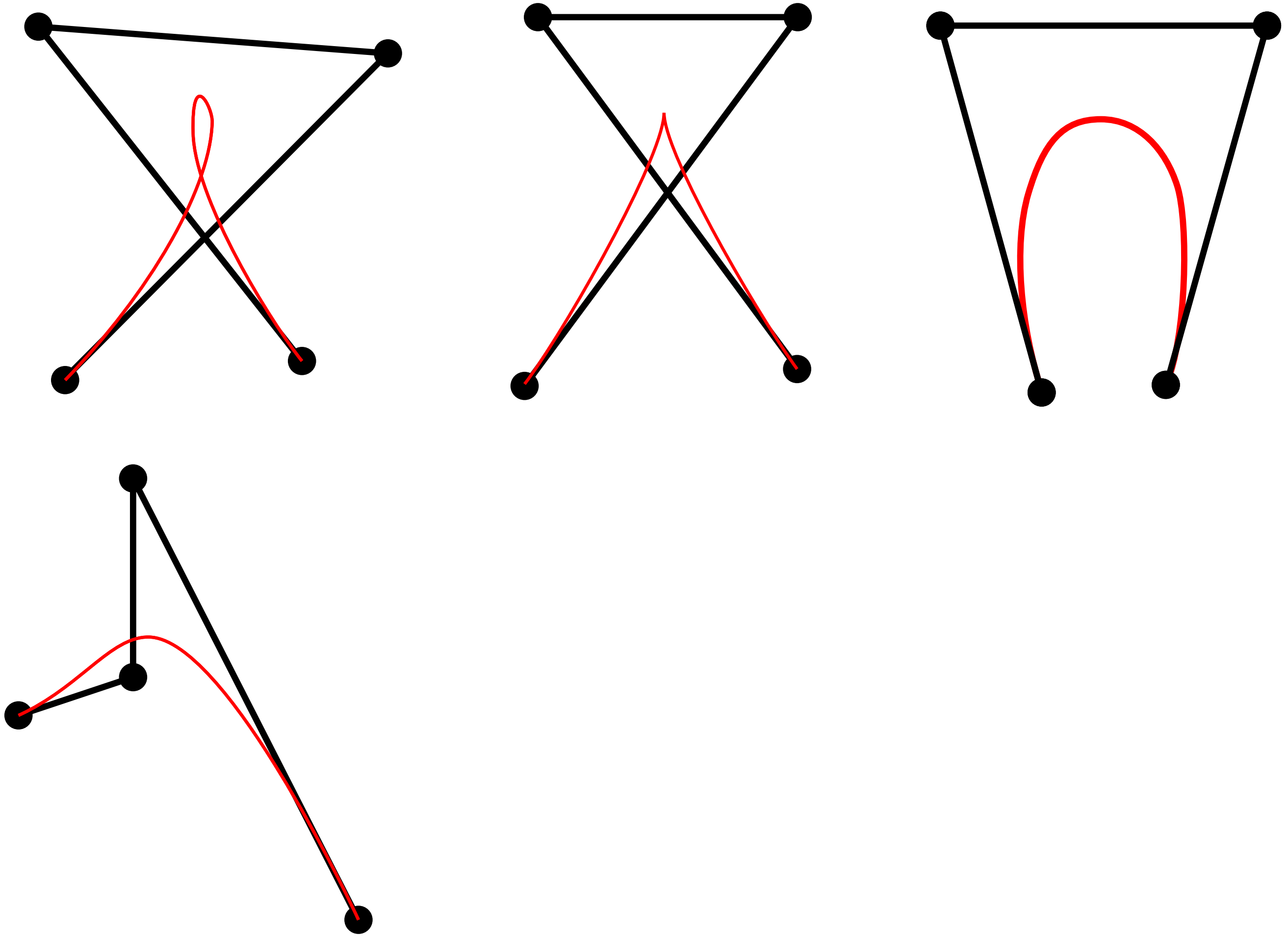


Control polygonal line, made of control points.
Each control point exerts a pull on the curve, pulling it towards itself

Some examples of Bézier curves



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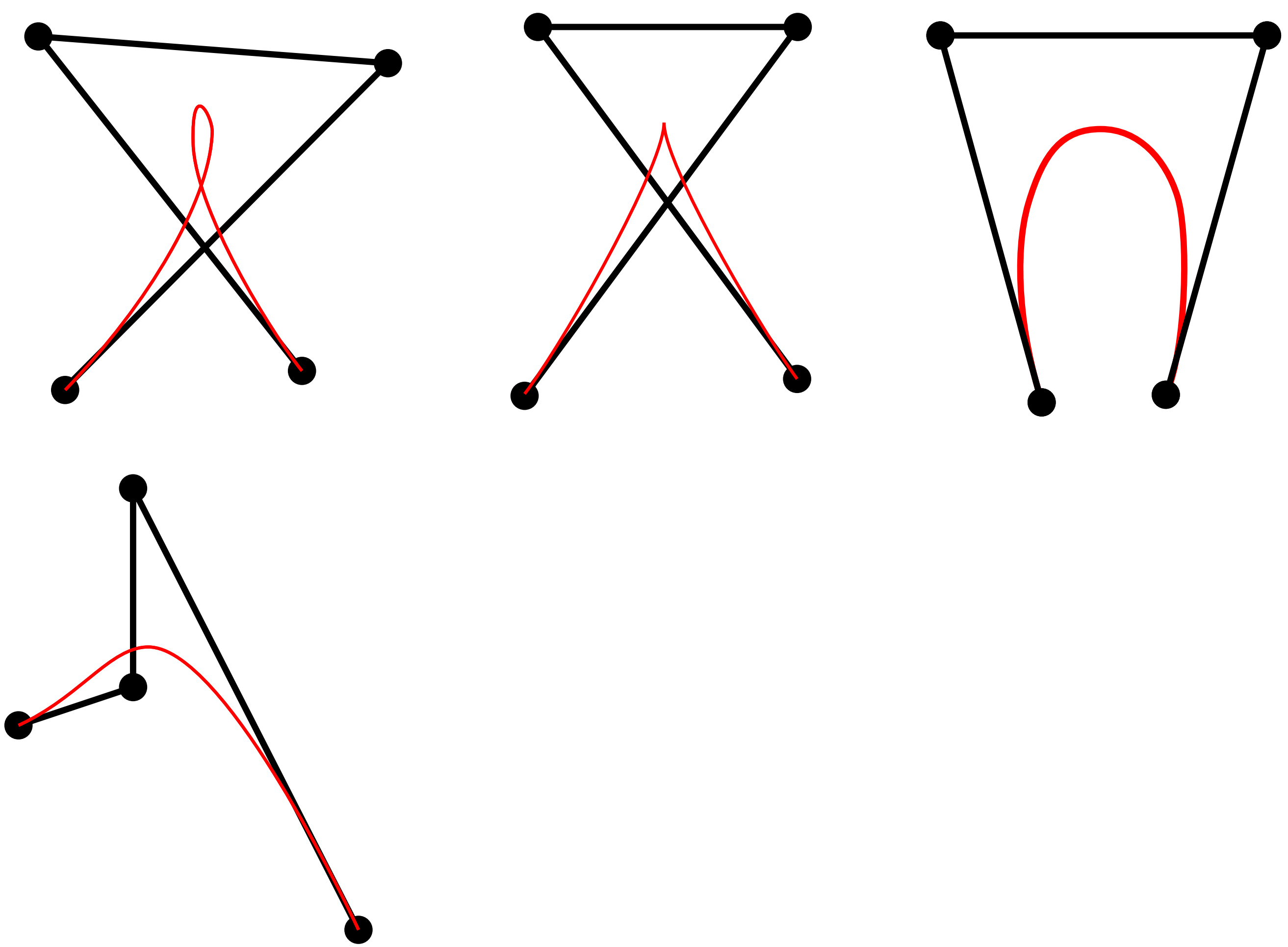

Each curve here is a polynomial, of degree....

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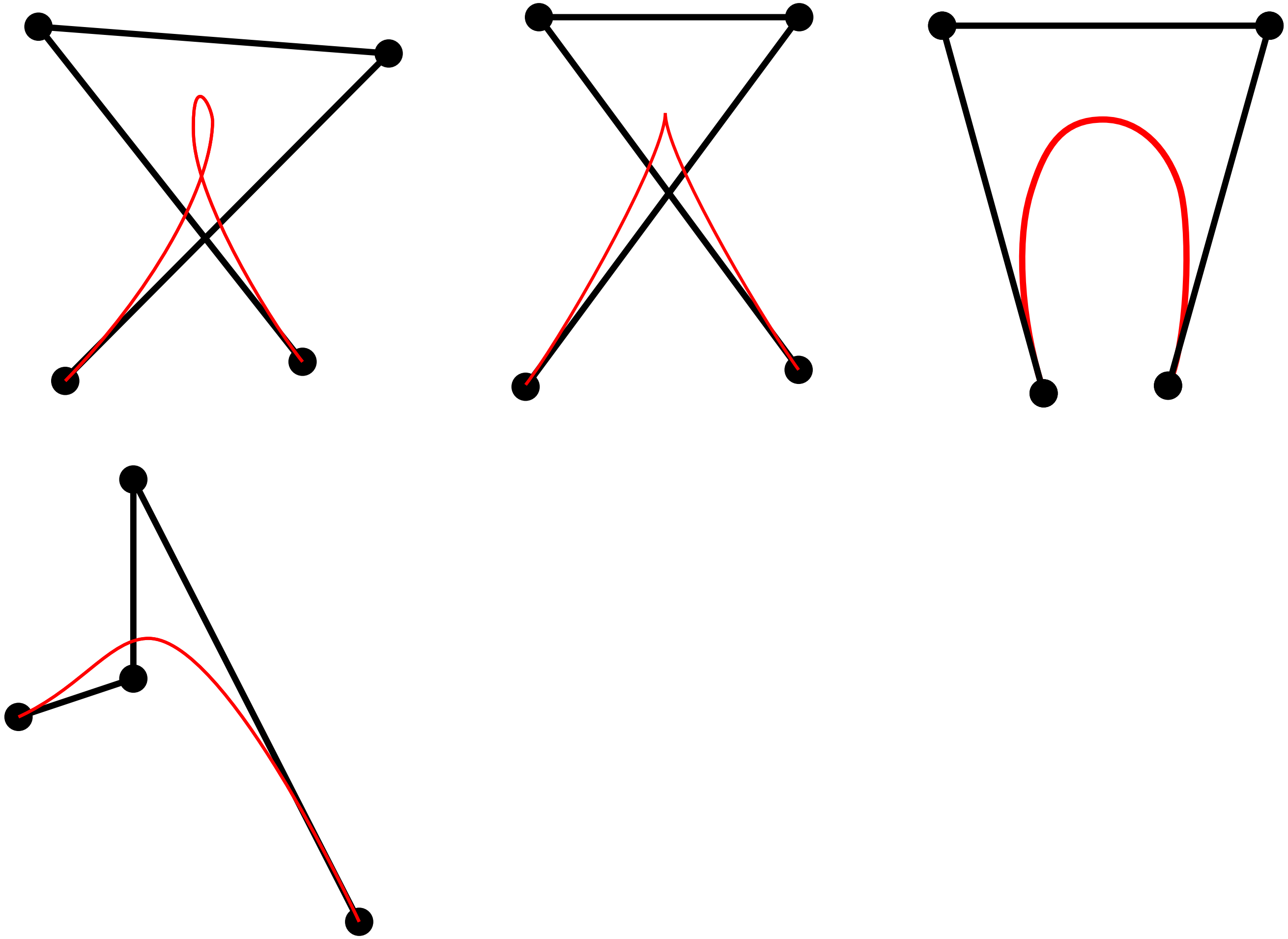

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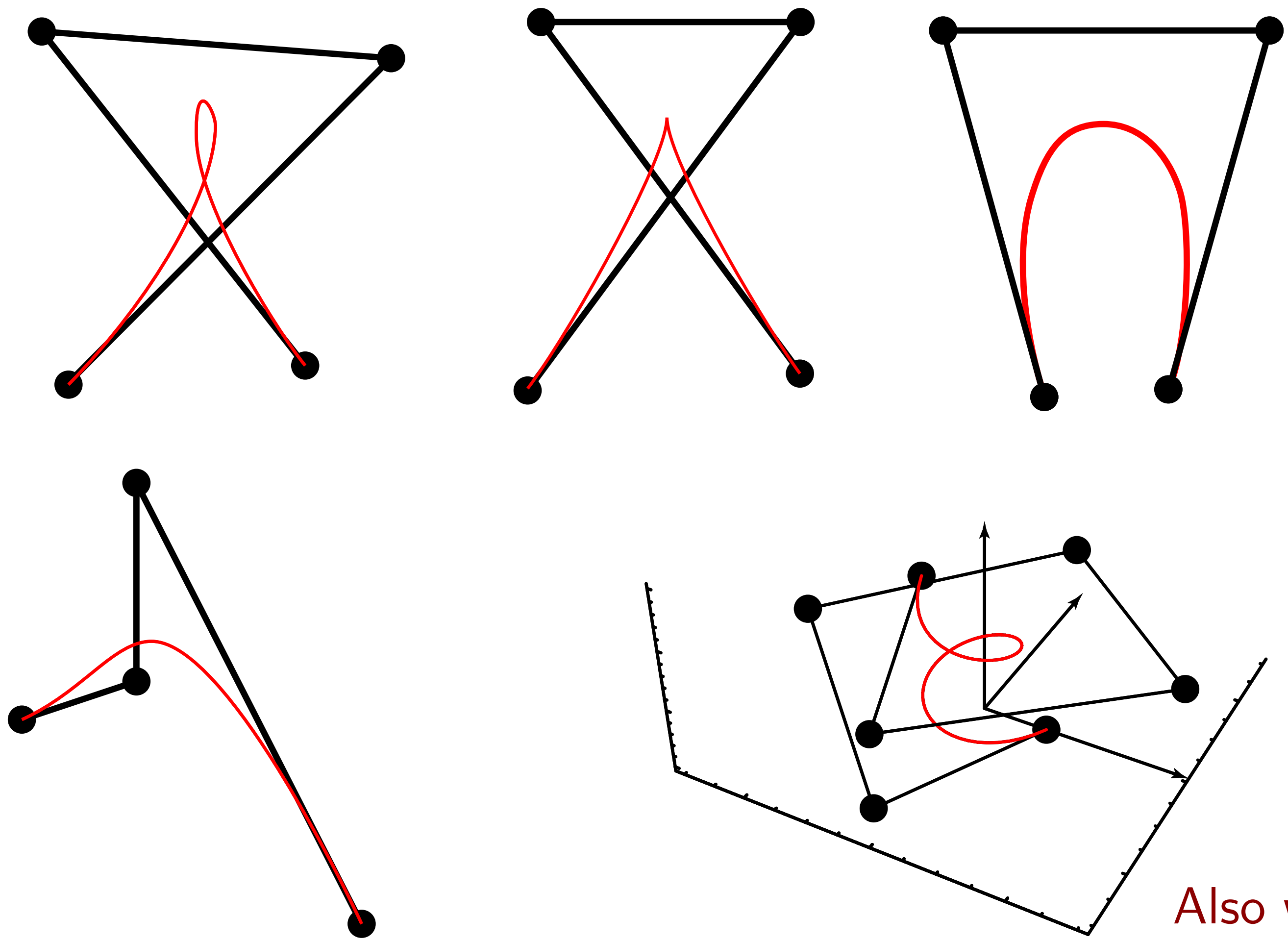
Some examples of Bézier curves



Also works in 3D!

What is a Bézier curve?

General form

$$P(t) = \sum_{i=0}^{n} P_i f_i(t) \qquad t \in [0, 1]$$

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basis function: gives weight of each point as function of t

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- The basis functions must be symmetric with respect to t and $(1-t)$ (so reversing the parameter and the order of control points gives the same curve)
- Control point weights are barycentric: shape independent from coordinate system. That is: $P(t)$ is an affine combination of control points, so curve is invariant under affinities

Basis functions

The family a functions used are **Bernstein polynomials**

$$f_i(t) = B_{n,i}(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

recall that
$0 \le i \le n$,
$\binom{n}{i} = \frac{n!}{i!(n-i)!}$, $0! = 1$
and assume $0^0 = 1$

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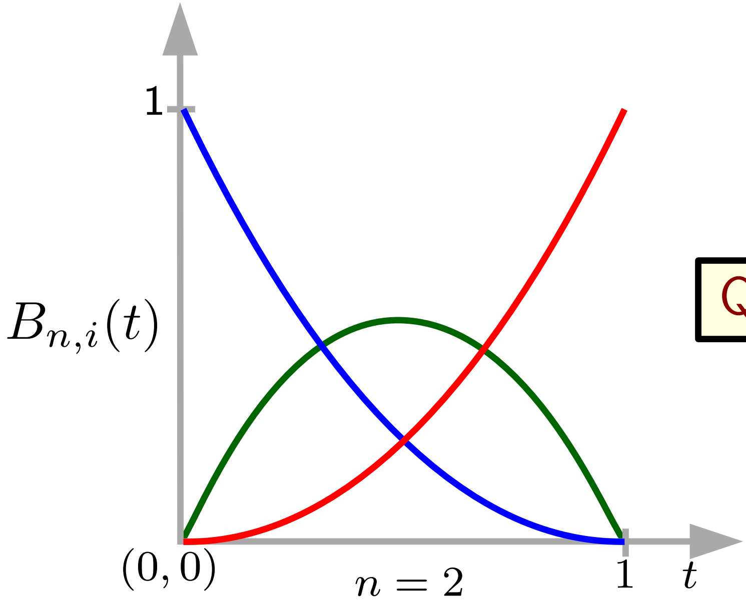

Question: Which basis function is which?

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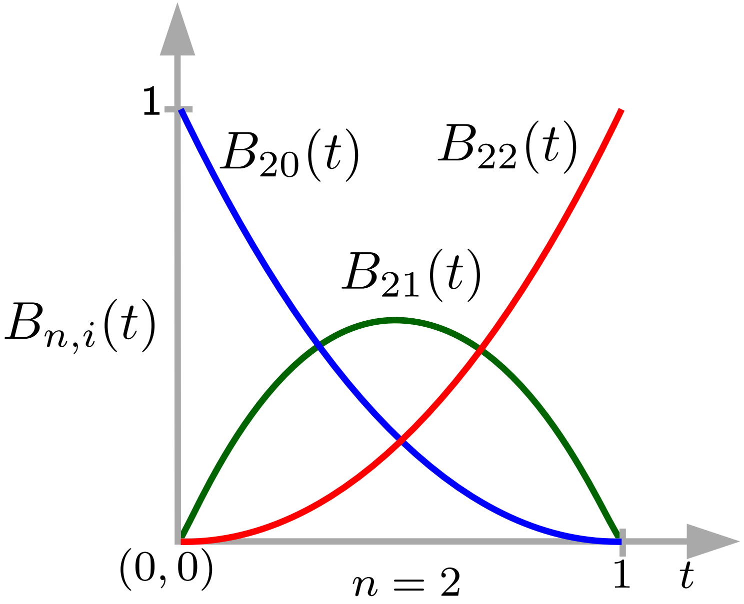

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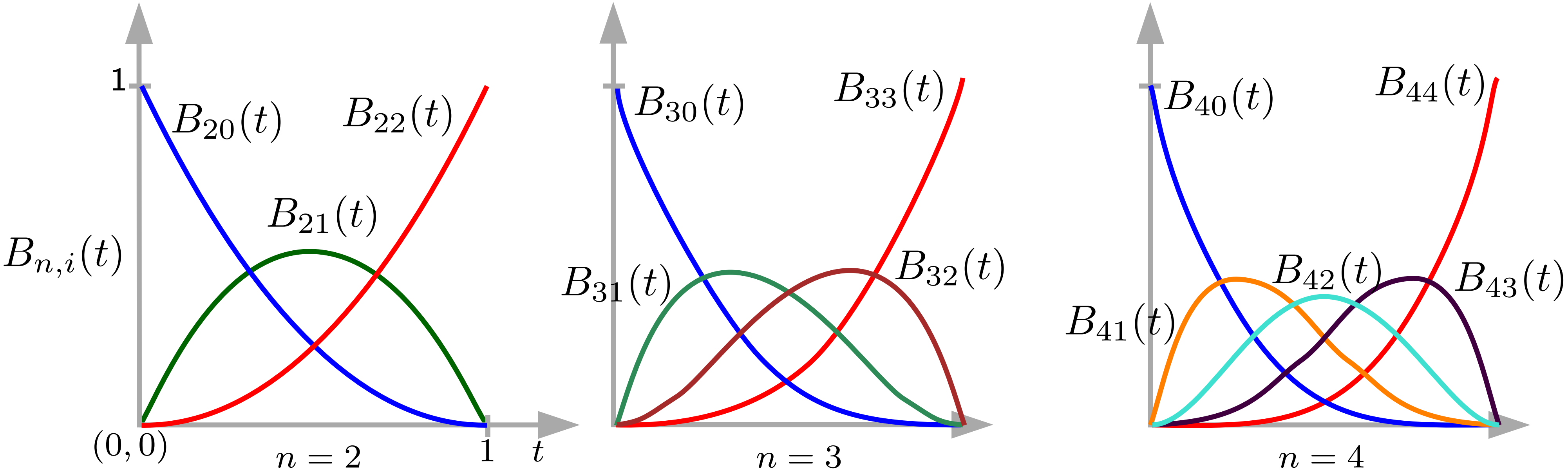

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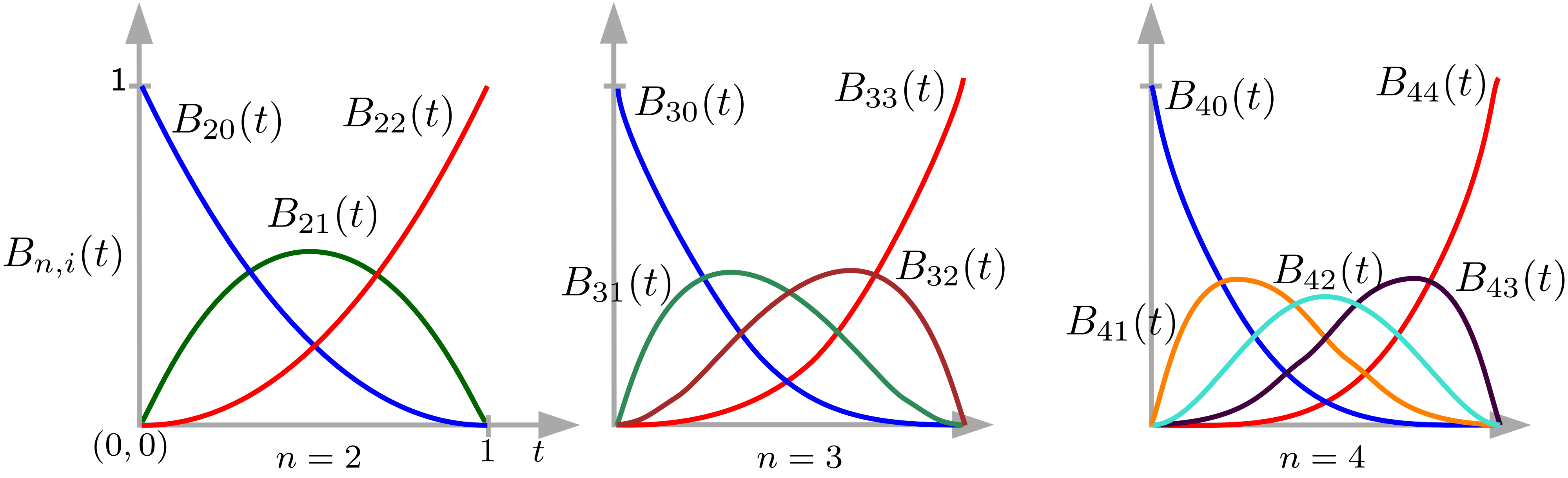

The Bézier curve becomes $P(t) = \sum_{i=0}^{n} P_i B_{n,i}(t) \qquad t \in [0,1]$

Example: degree-2 Bézier curve

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For $n = 2$, we have $B_{n,i}(t) = \binom{2}{i} t^i (1-t)^{2-i}$, for $0 \leq i \leq 2$

So, for $n = 2$, these are the three Bernstein polynomials:

- $B_{2,0}(t) = \binom{2}{0} t^0 (1-t)^{2-0} = (1-t)^2$
- $B_{2,1}(t) = \binom{2}{1} t^1 (1-t)^{2-1} = 2t(1-t)$
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$$P(t) = (1-t)^2 P_0 + 2t(1-t) P_1 + t^2 P_2$$

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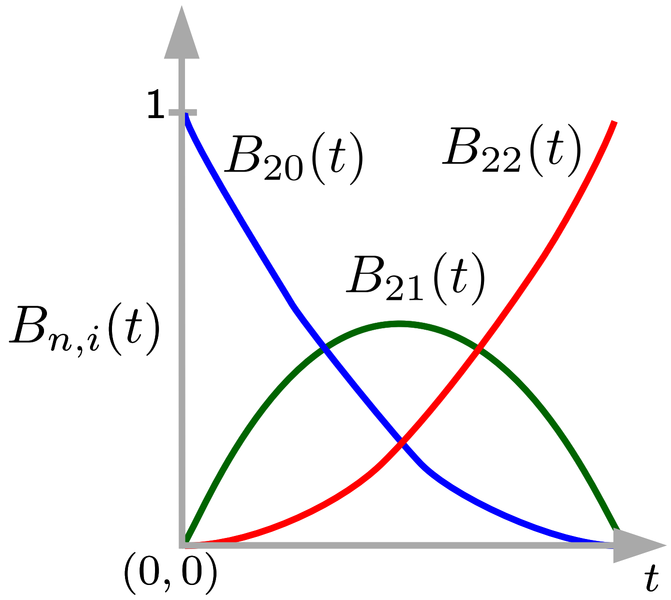
Example?

Question: Does this curve satisfy the properties in the previous slide?

Properties of Bézier curves

1. Endpoint interpolation
2. Symmetry
3. Affine invariance
4. Invariance under affine parameter transformations
5. Convex hull property
6. Pseudolocal control
7. Variation-diminishing property



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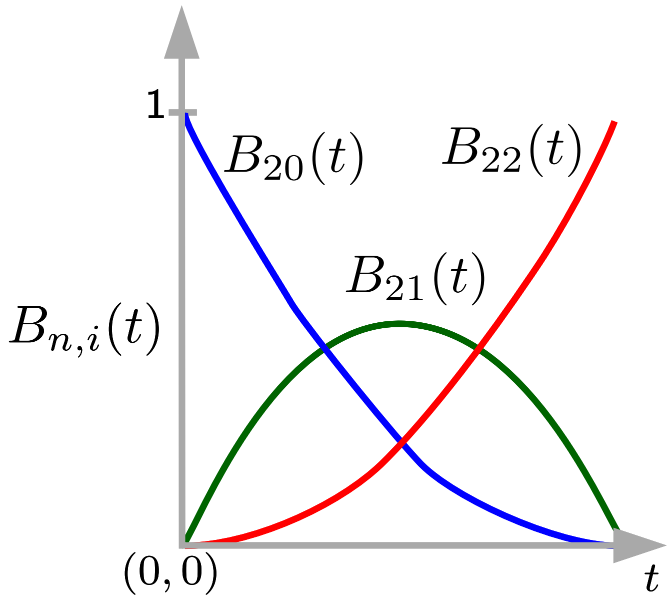

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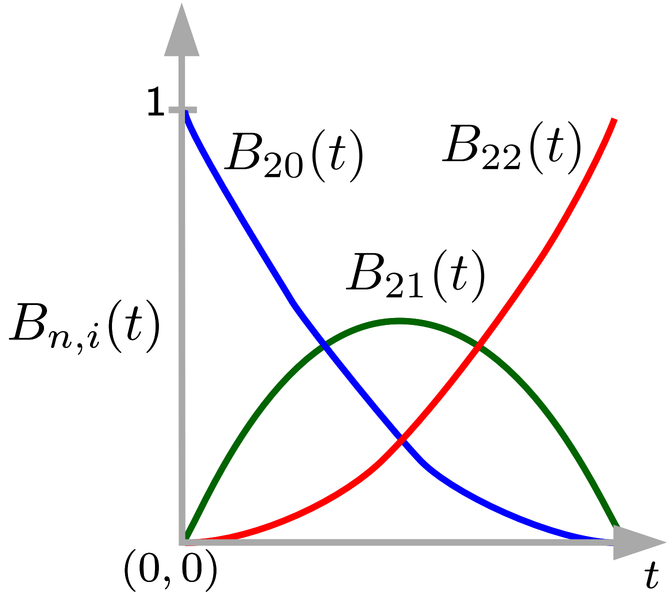
Properties of Bézier curves

3. Affine invariance

Applying an affine transformation to the curve is the same as applying the transformation to the control points

More precisely: $f(P(t)) = \sum_{i=0}^{n} f(P_i)B_{n,i}(t)$, for any affine map f, i.e., $f(v) = Av + W$



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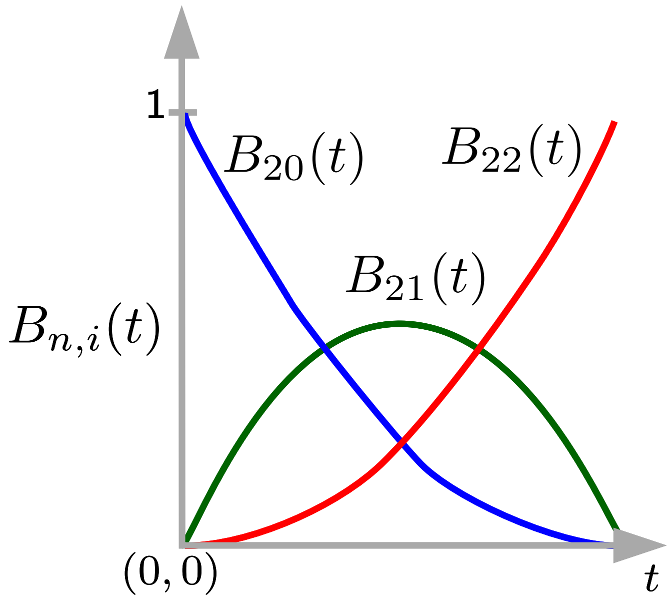
Why is that? Observe that $\sum_{i=0}^{n} B_{n,i}(t) = 1$

This follows from the binomial theorem:
$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$, with $a = t$ and $b = (1-t)$

Therefore Bézier curves are invariant under affine transformations!



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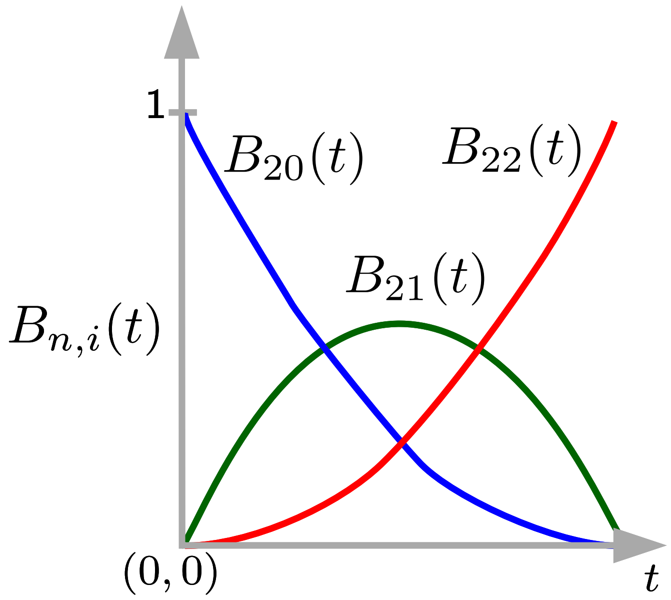

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4. Invariance under affine parameter transformations

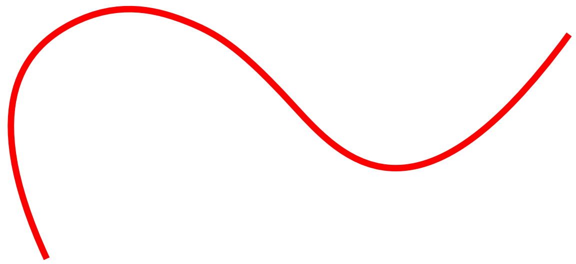
That is: $\sum_{i=0}^{n} P_i B_{n,i}(t) = \sum_{i=0}^{n} P_i B_{n,i}(\frac{u-a}{b-a})$

Practical consecuence: it is easy to have a curve defined over $[a, b]$ instead of $[0, 1]$

Properties of Bézier curves

5. Convex hull property

The curve lies inside the convex hull of the control points



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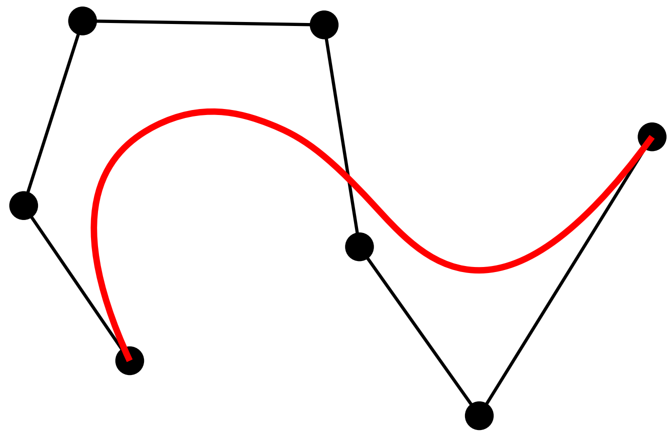

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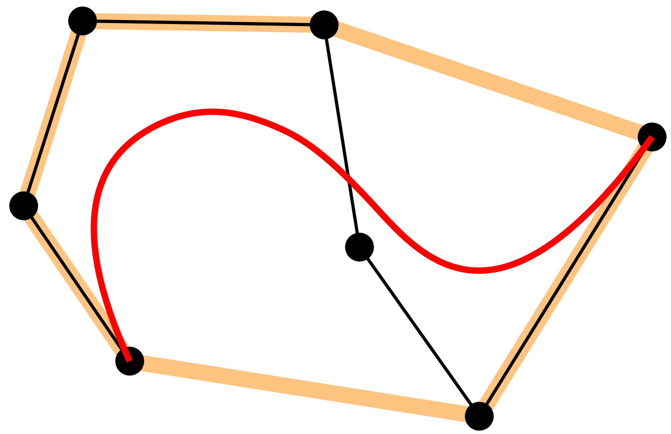

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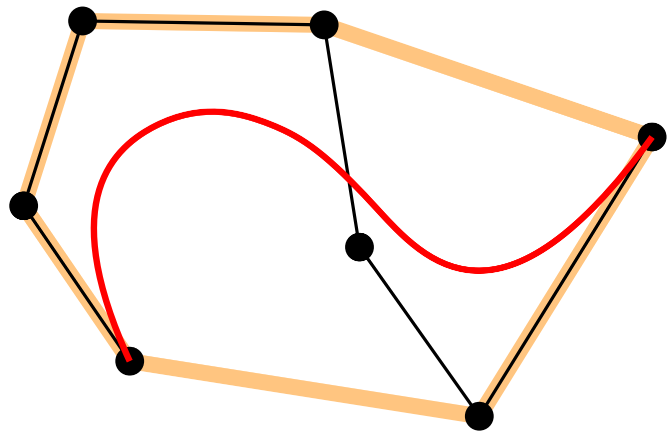
Why is this property true?

Properties of Bézier curves



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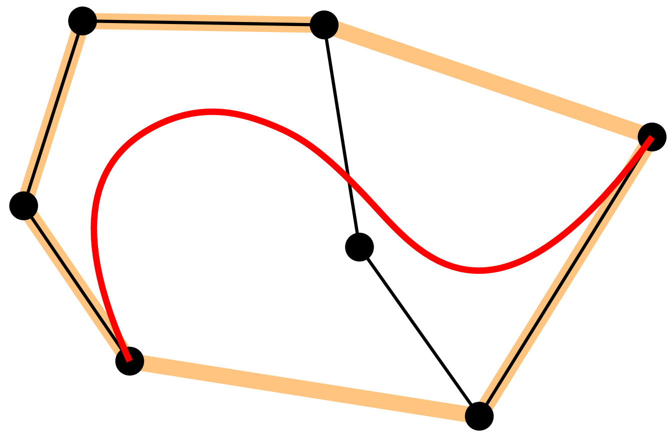
Point weights add up to one and are in $[0, 1]$, thus $P(t)$ is a **convex combination** of the control points.

The convex hull of the a set of points S is **exactly** the set of all convex combinations of points in S, thus all points in the curve belong to the convex hull.

Properties of Bézier curves



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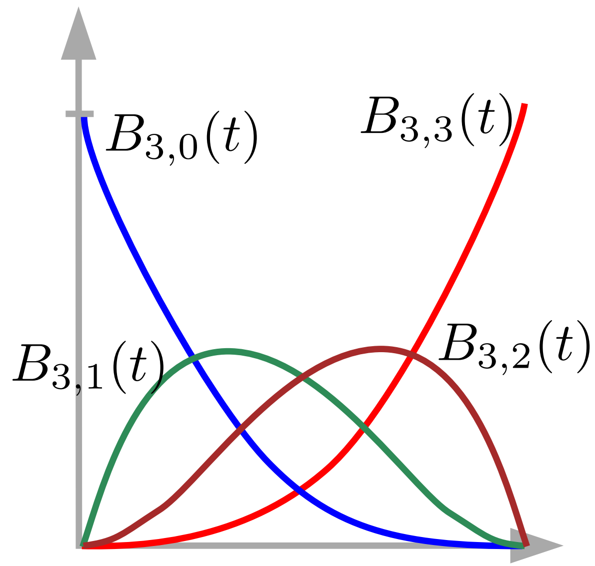
Question: What does this say about collinear control points?

Properties of Bézier curves

6. "Pseudolocal" control

Question: When does a control point influence the curve most?



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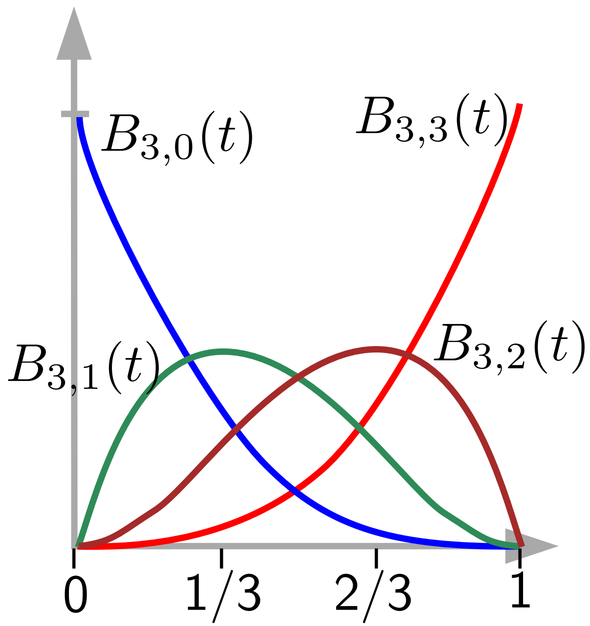

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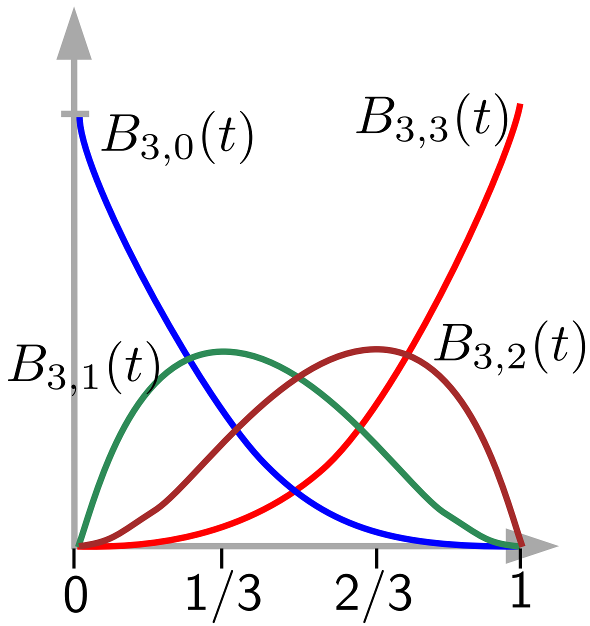
Consequence: if we move only one control point, P_i, the curve is mostly affected around $t = i/n$. This makes the effect of the change more or less predictable.

However, note that the change still affects the whole curve (so it is **global control**—not local).



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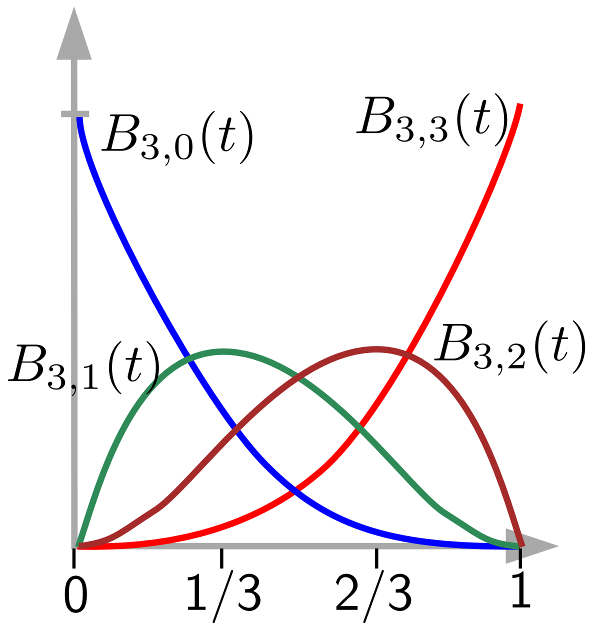


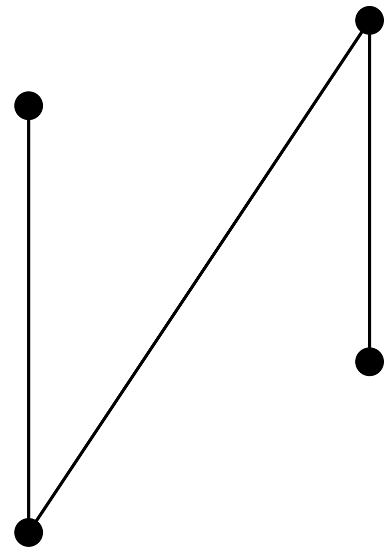
local maxima of $B_{n,i}$s

Question: What happens to $P(t)$ if P_k is moved by a vector (α, β)?

Properties of Bézier curves

7. Variation-diminishing property

The number of intersections of any line with a Bézier curve is at most the number of intersections of the line with the control polygonal line



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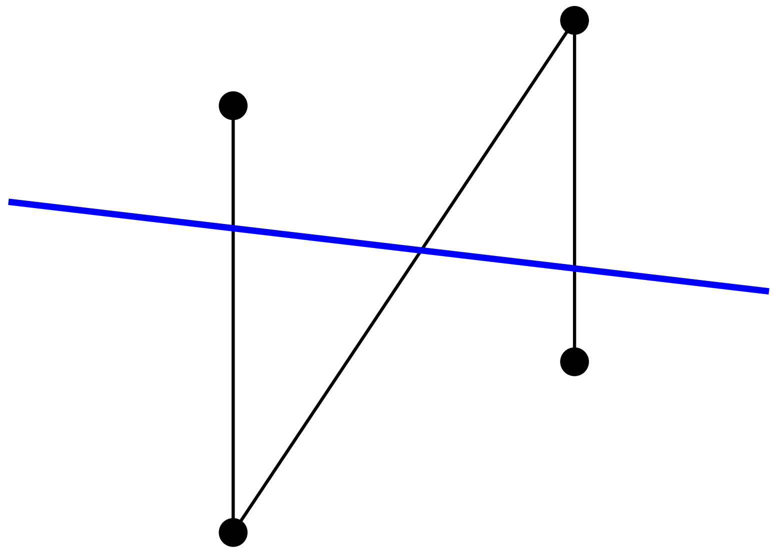

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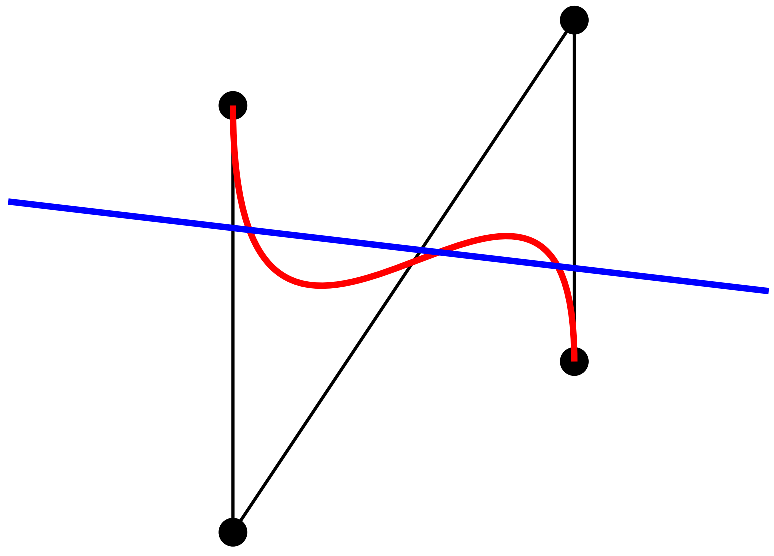

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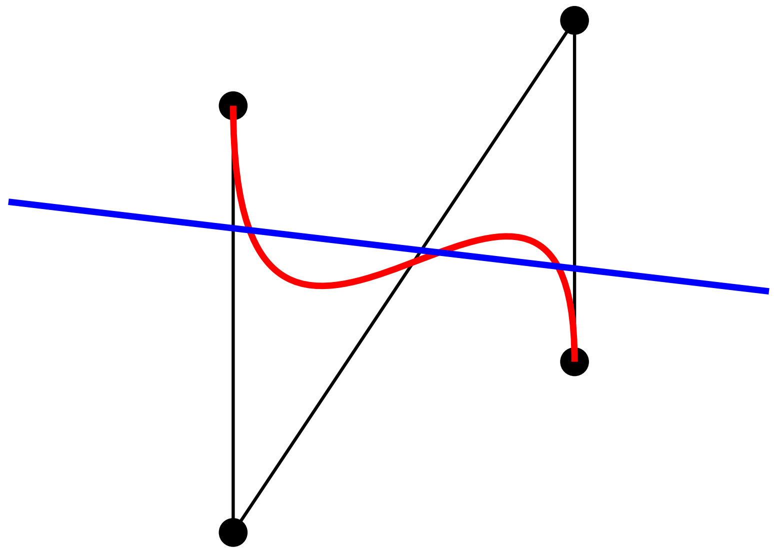

The number of intersections of any line with a Bézier curve is at most the number of intersections of the line with the control polygonal line

This means that, to some extent, the curve imitates the shape and is not "rougher" than the corresponding control polygon,

One consequence: if the control polygon is convex, then the Bézier curve is also convex

Proof? Later, after looking at **degree elevation**

Connecting two curves

- In practice, one should avoid high-degree Bézier curves
- Better use many low-degree curves (they give local control)
- Requires smooth connection between consecutive curves

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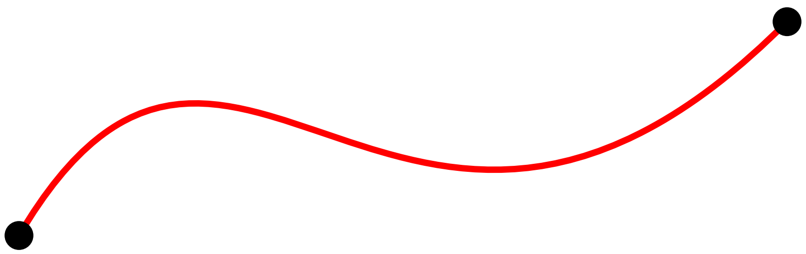

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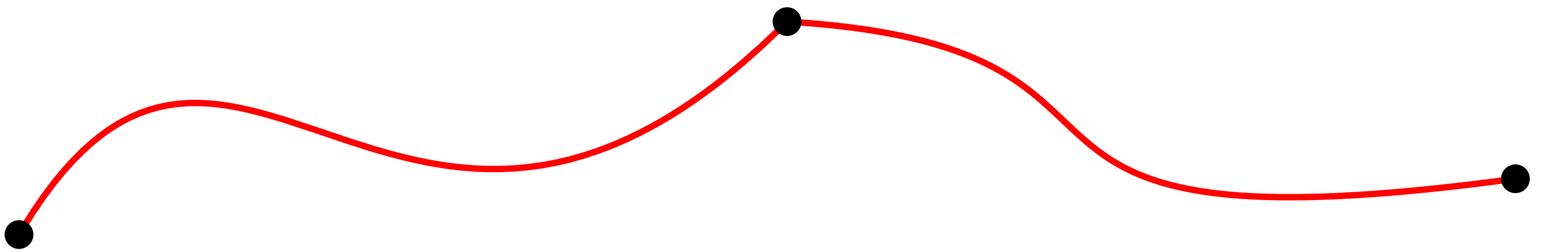

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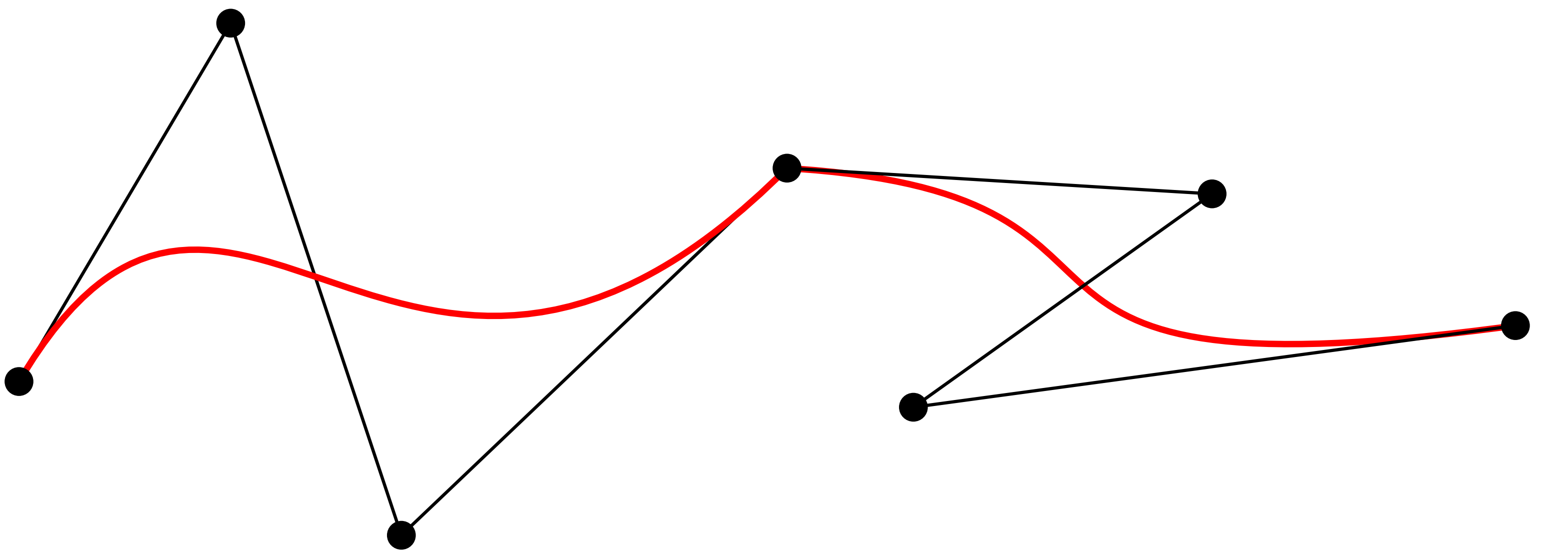

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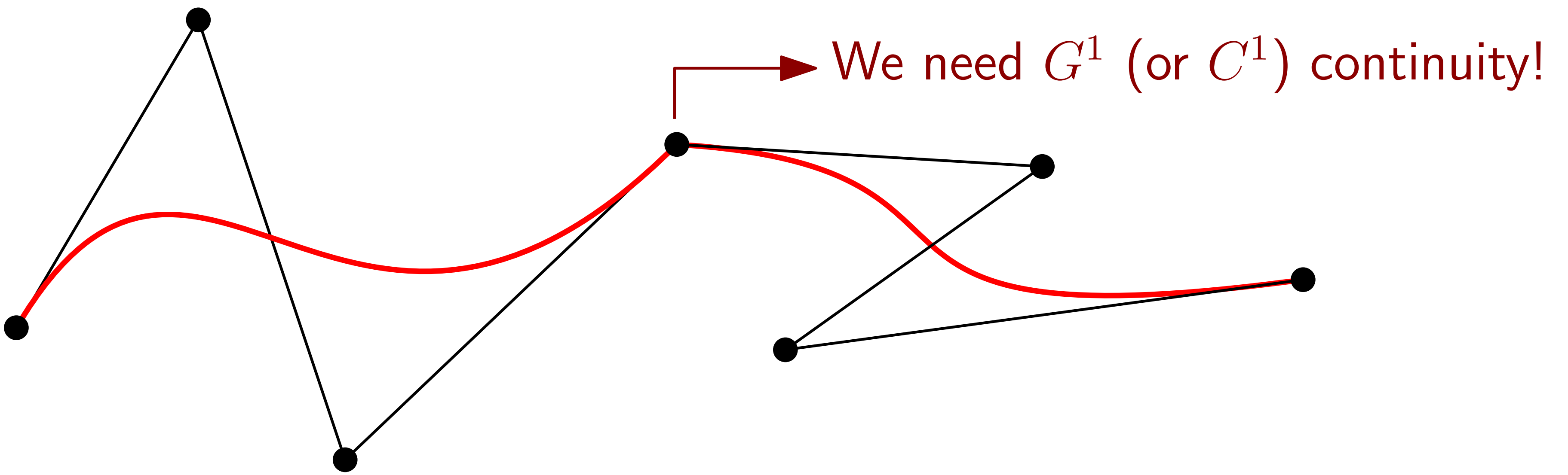

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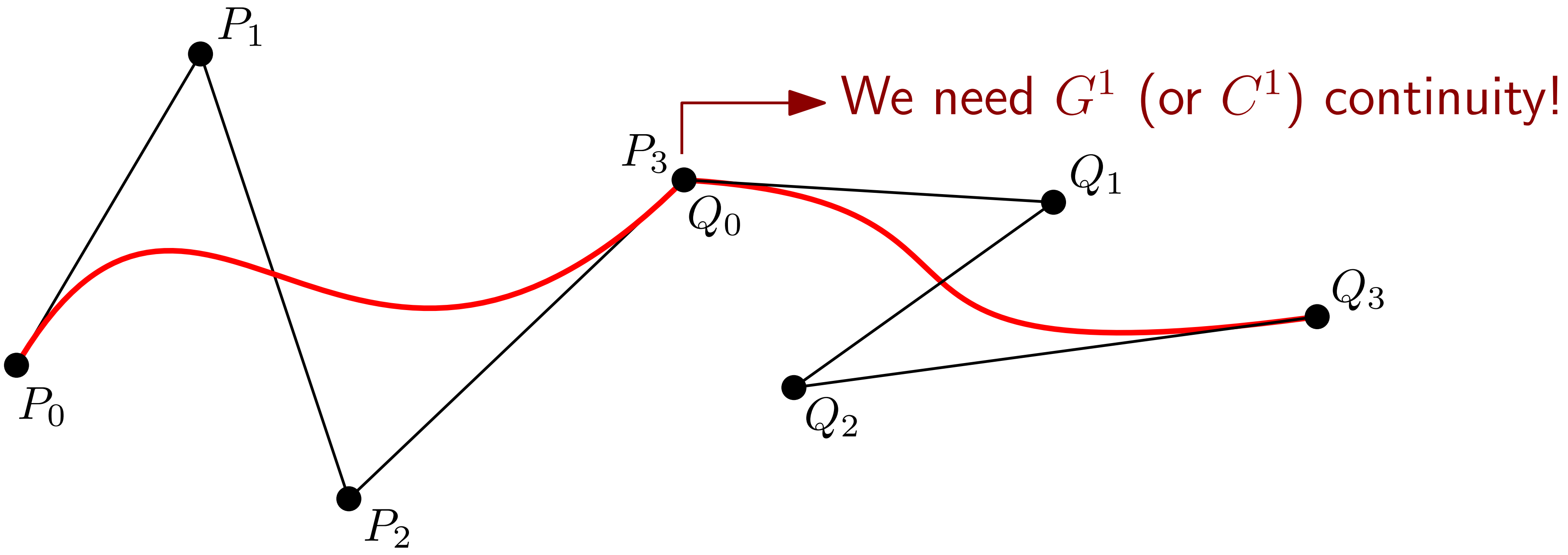

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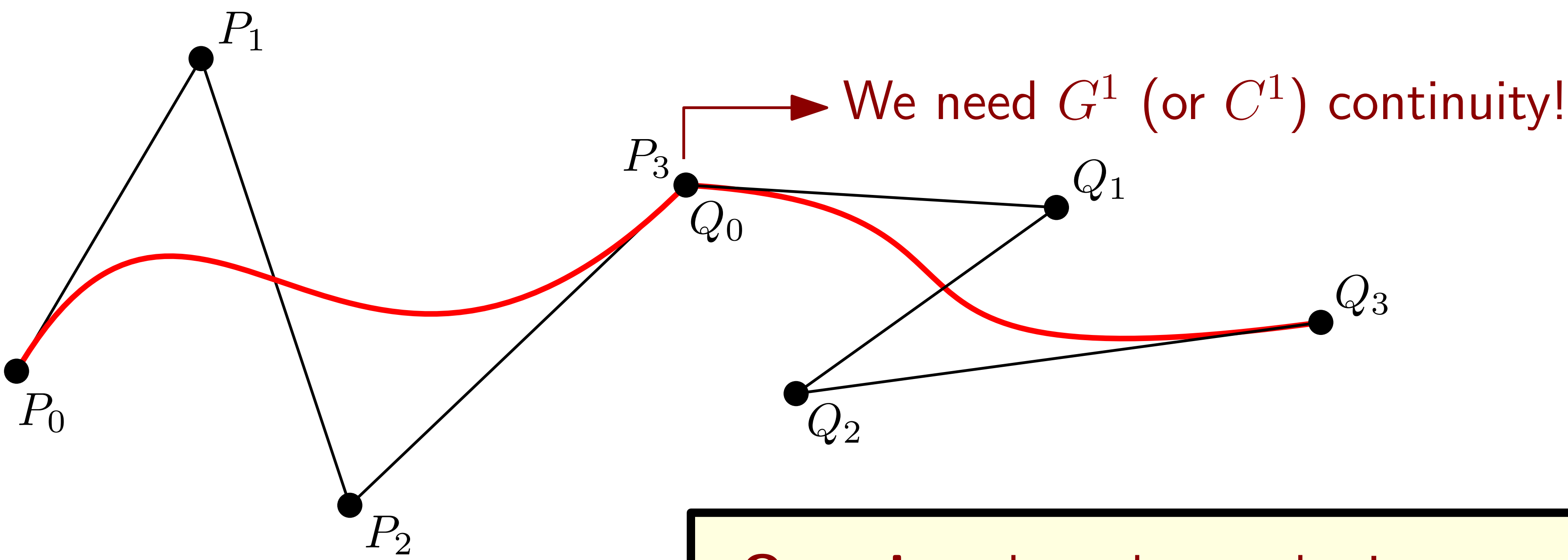
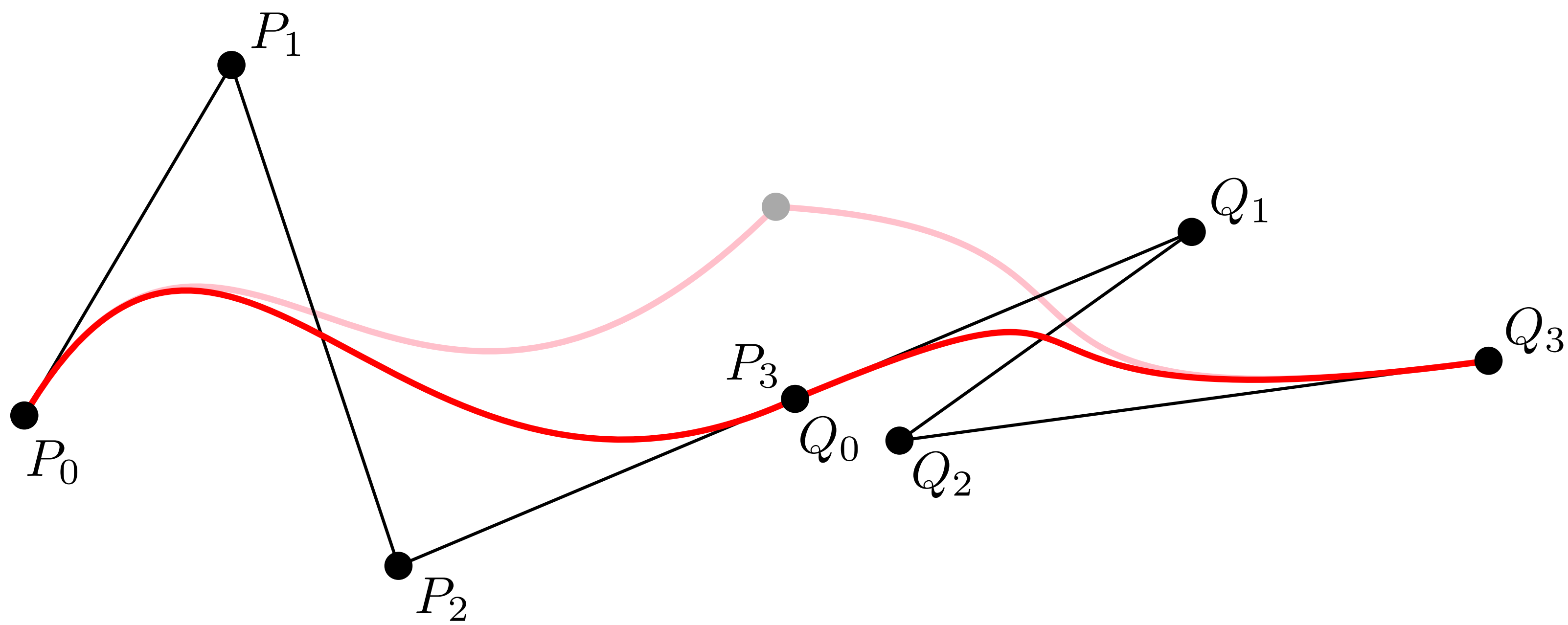


Question: how do we obtain a smooth joint?

Connecting two curves



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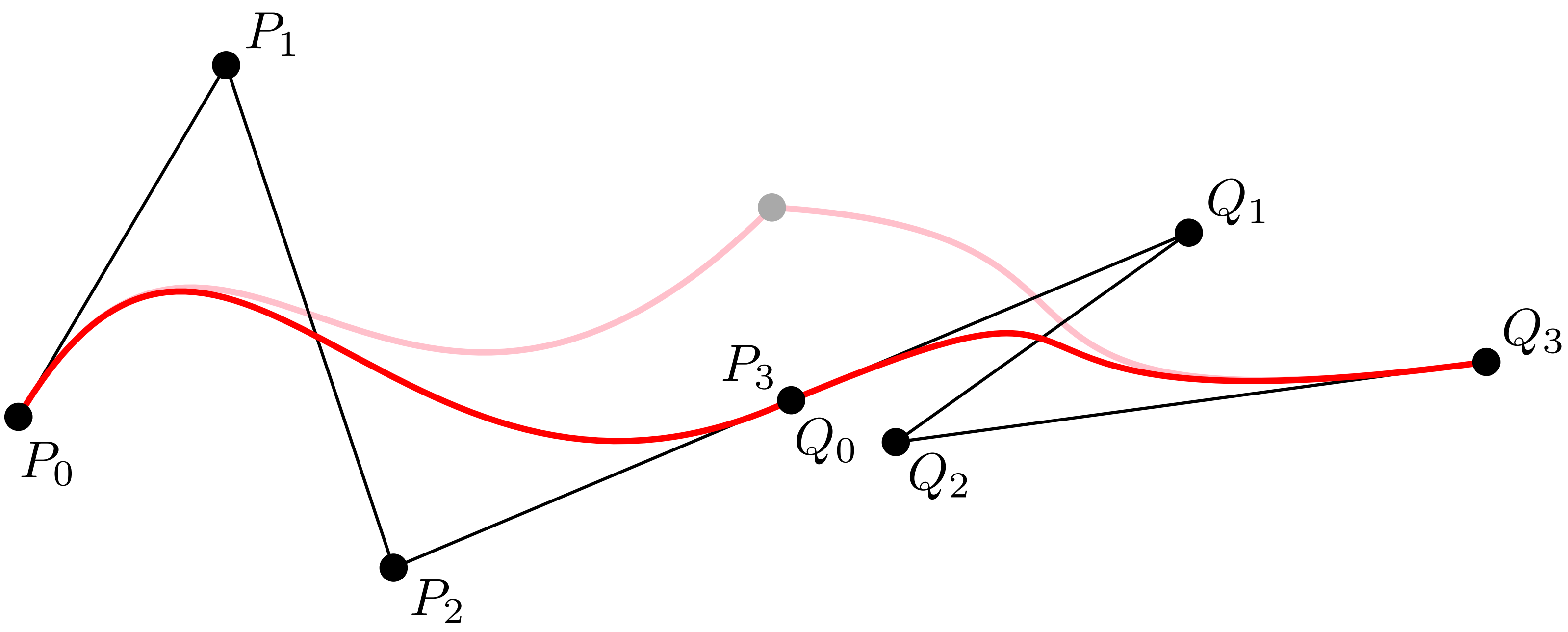
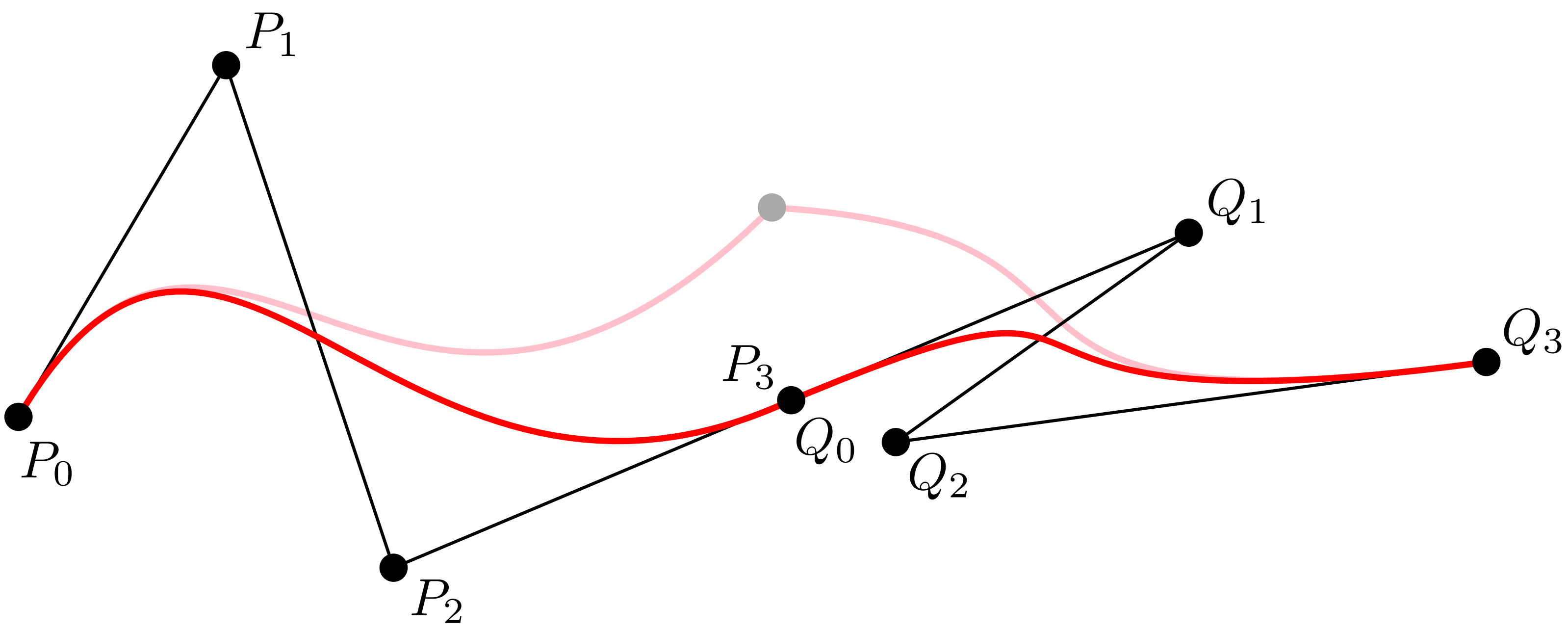


In general, for a curve P with $(n+1)$ control points and Q with $(m+1)$, the C^1-continuity condition is

$$Q_0 = P_n = \frac{m}{m+n} Q_1 + \frac{n}{m+n} P_{n-1}$$

Connecting two curves



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Question: how can you obtain higher-degree continuity?

EXAMPLE: FONT DESIGN

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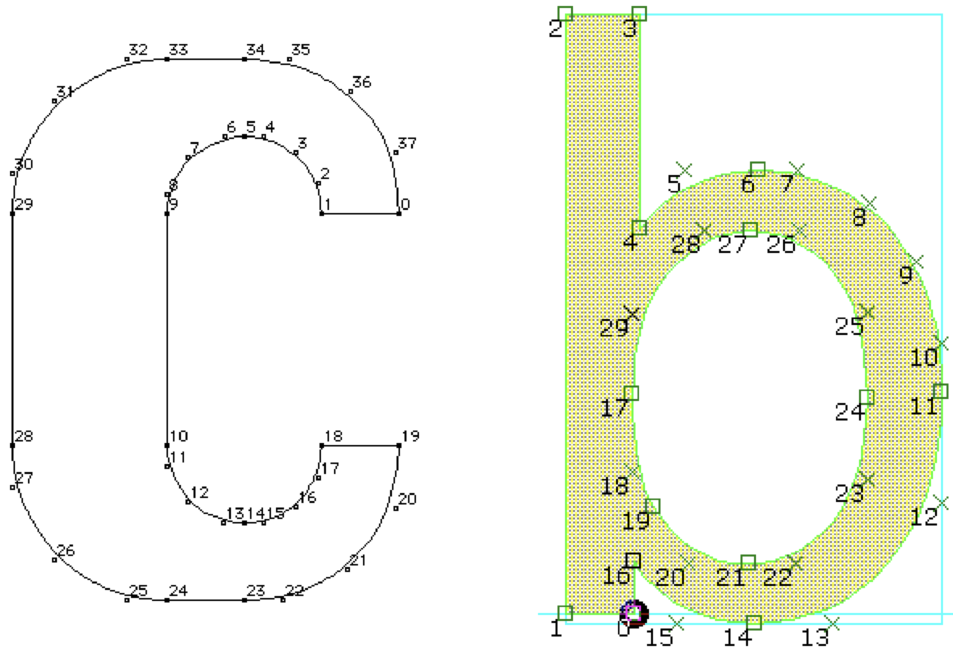

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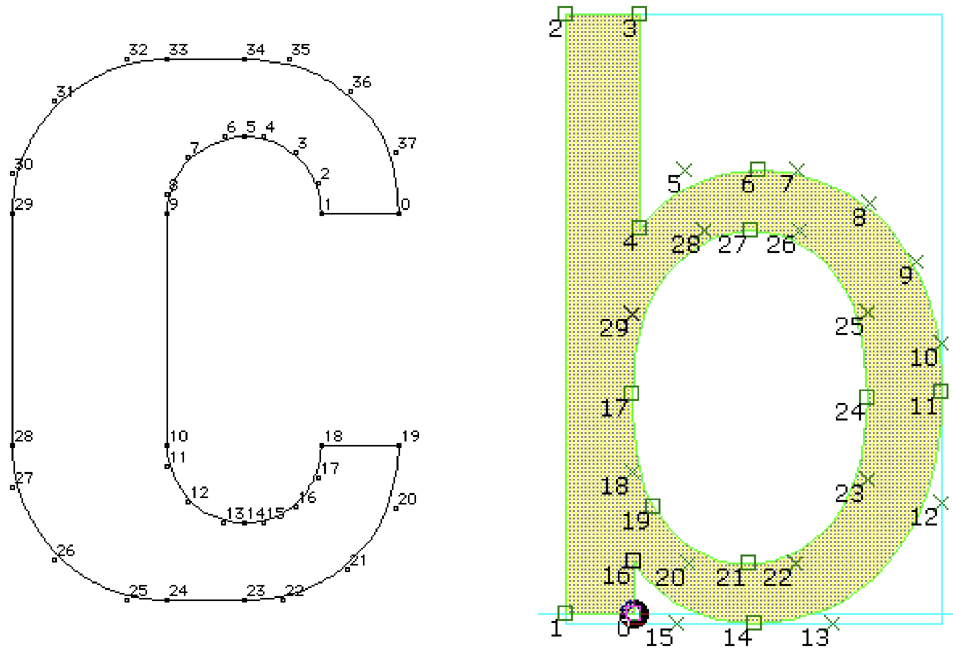

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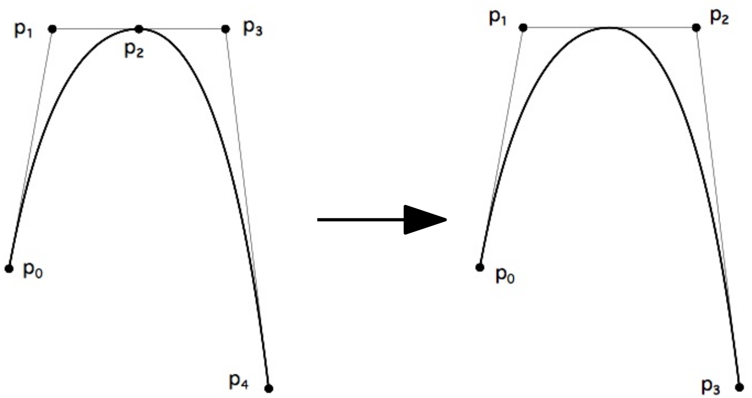
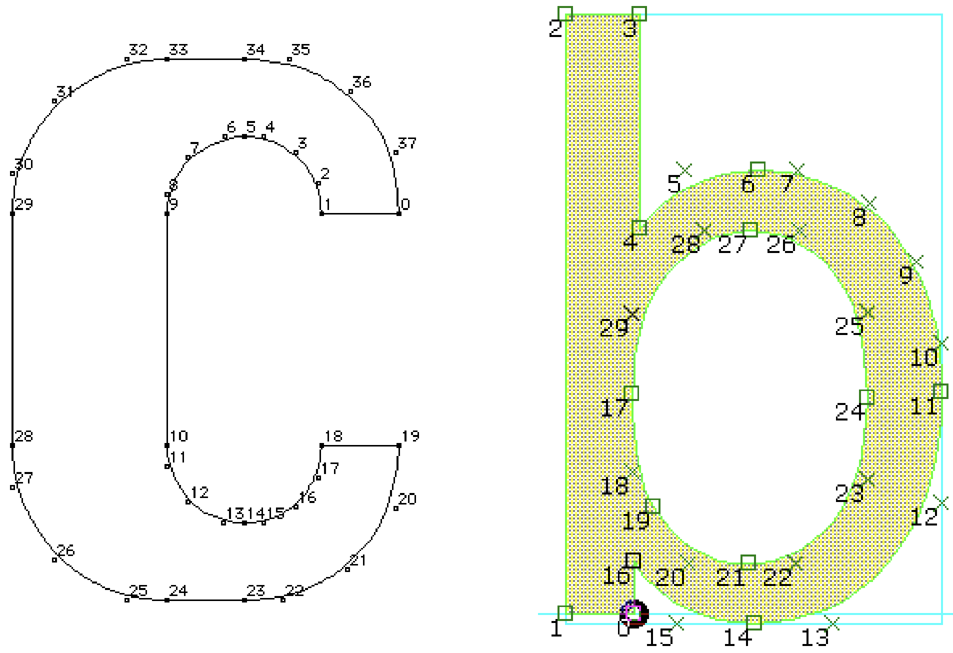
- Storage of glyphs in TTF:





Glyphs of two characters in a true-type font

Difference between points on-curve and off-curve (only off-curve points are stored)

An alternative approach to Bézier curves

De Casteljau (Citroën) followed a different approach based on **repeated linear interpolation**

BÉZIER CURVES AS LINEAR INTERPOLATION

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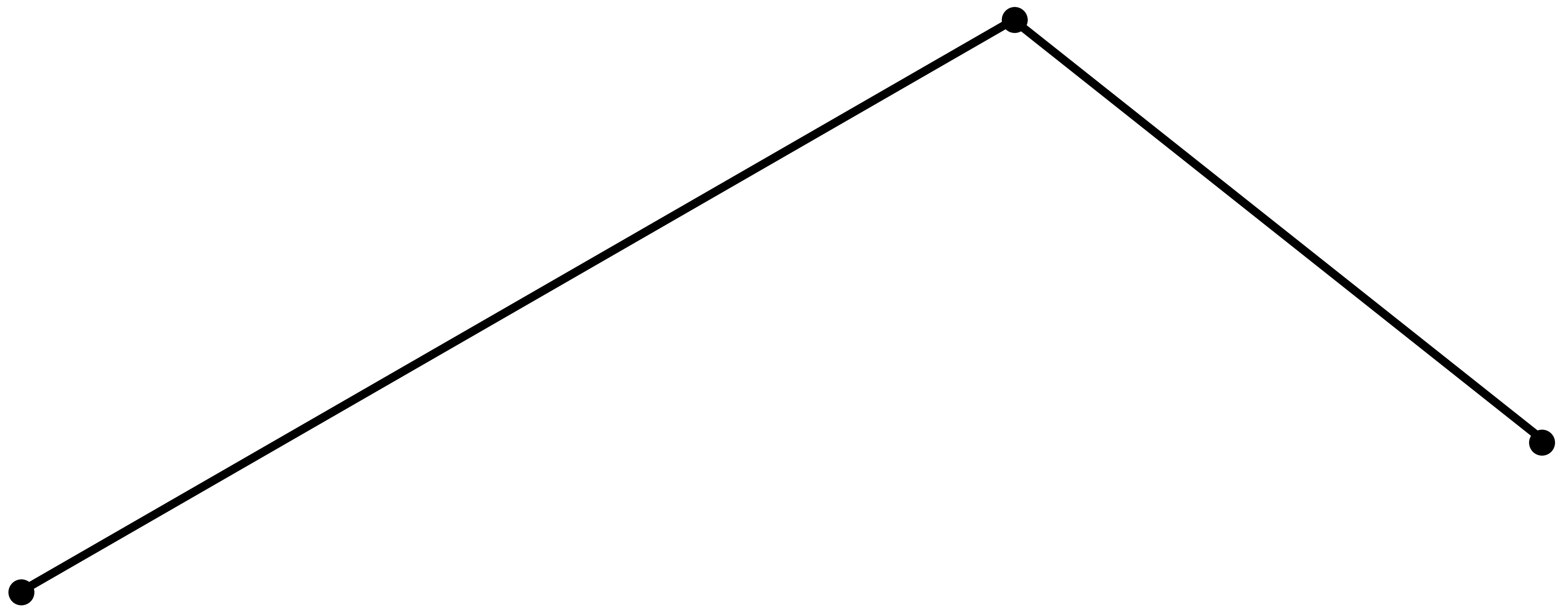

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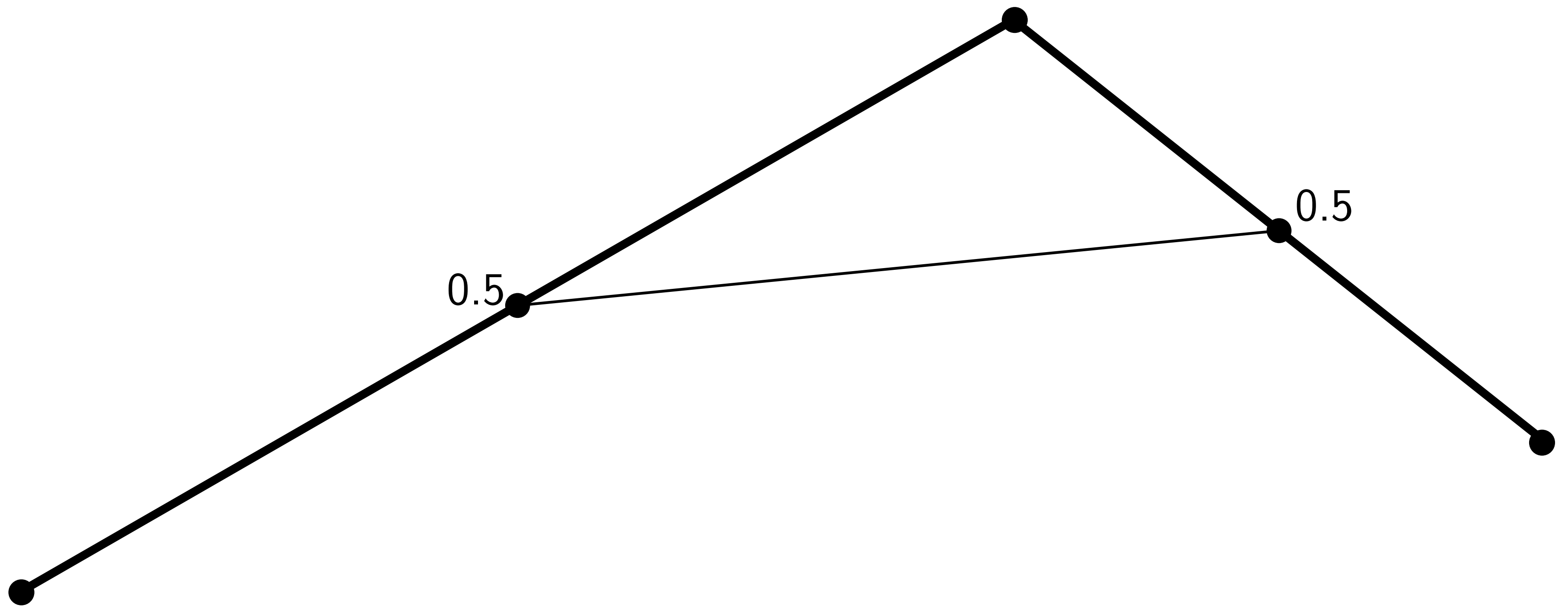

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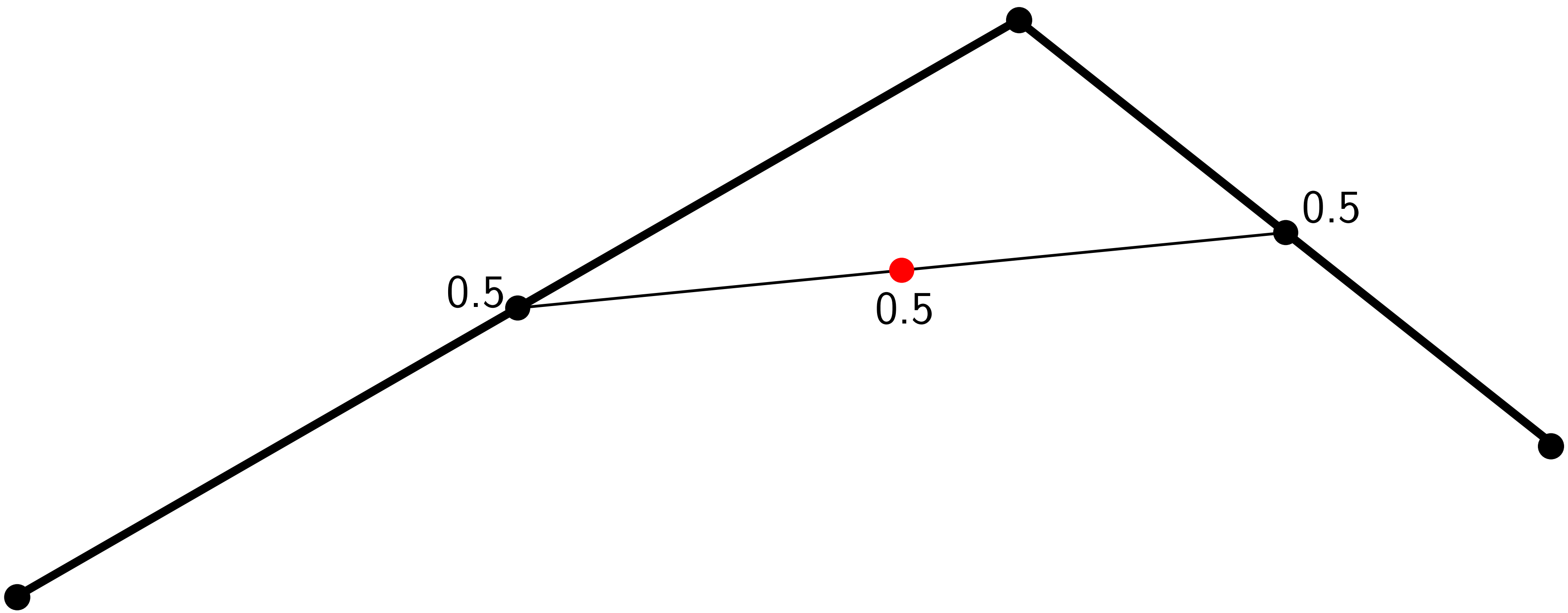

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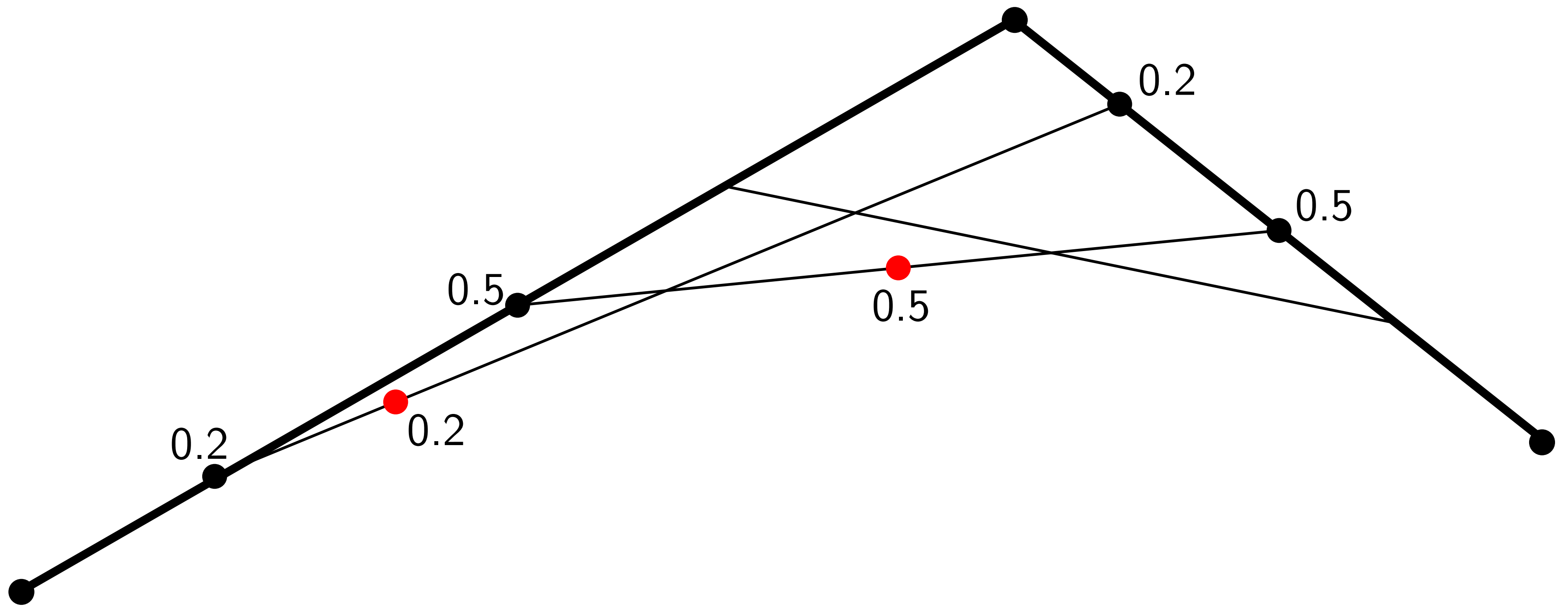

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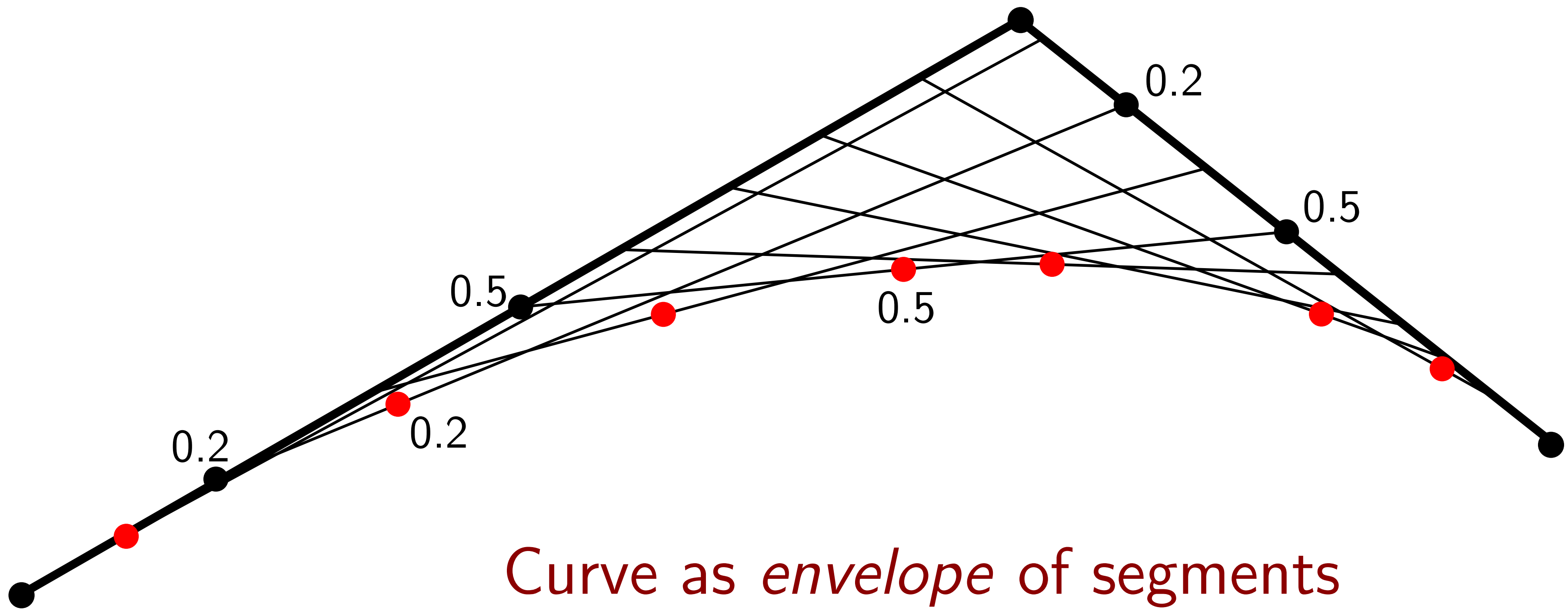

Curve as *envelope* of segments

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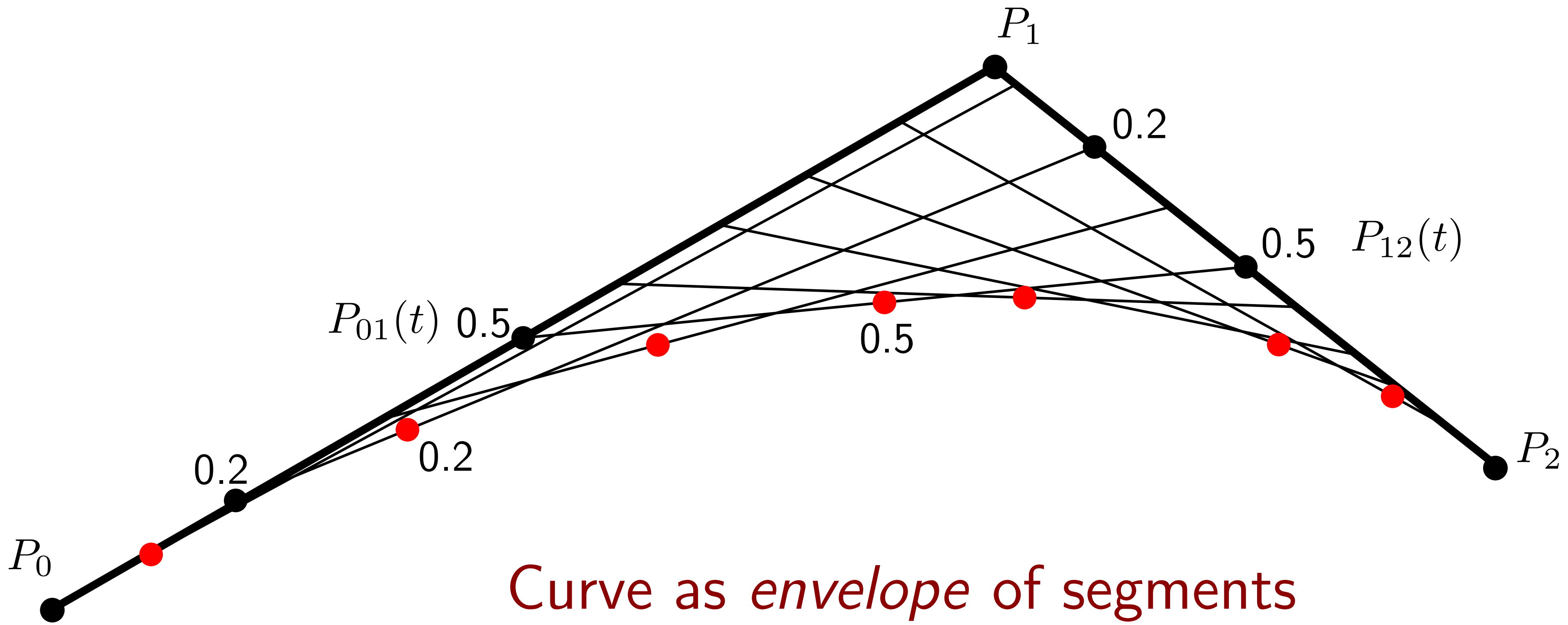

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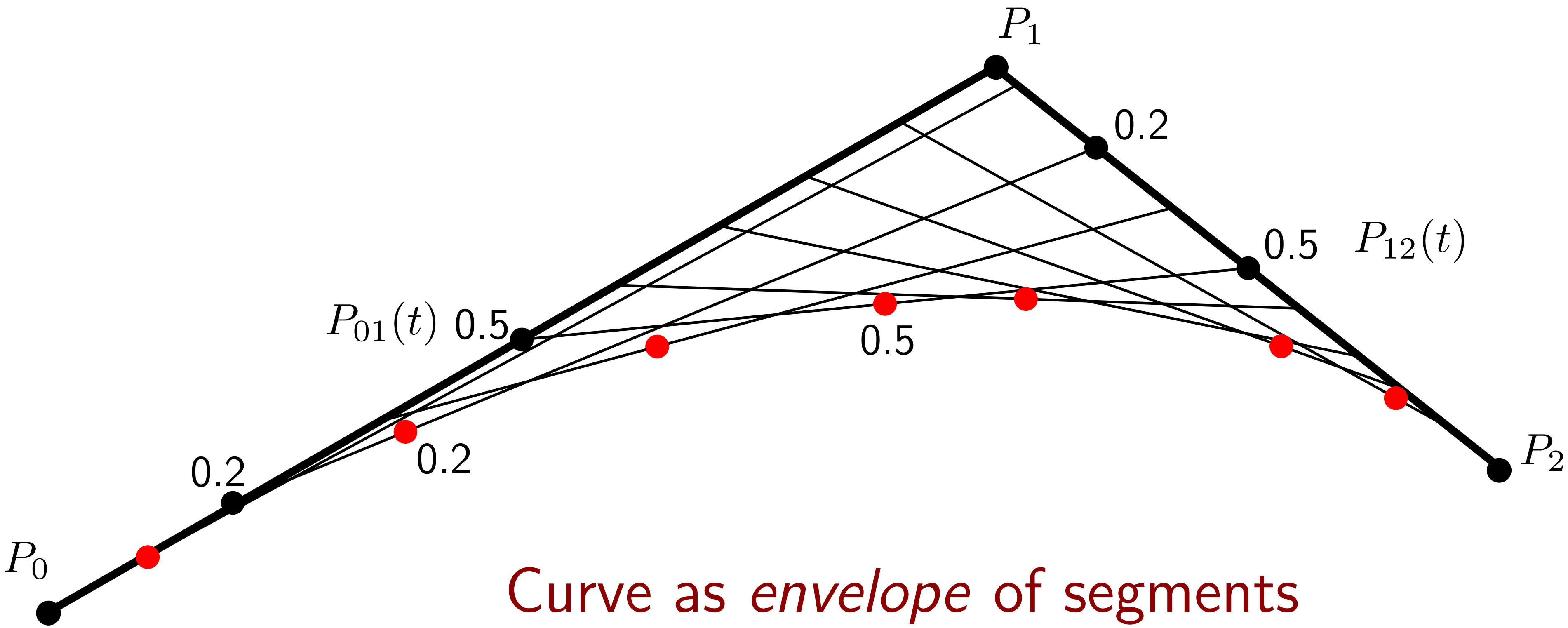

Curve as *envelope* of segments

Question: What is the expression of this envelope, as a function of t?

De Casteljau's algorithm

For 3 points P_0, P_1, P_2, and $0 \leq t \leq 1$, we have:

$$P_{01}(t) = (1-t)P_0 + tP_1$$
$$P_{12}(t) = (1-t)P_1 + tP_2$$
$$P(t) = P_{012}(t) = (1-t)P_{01}(t) + tP_{12}(t)$$

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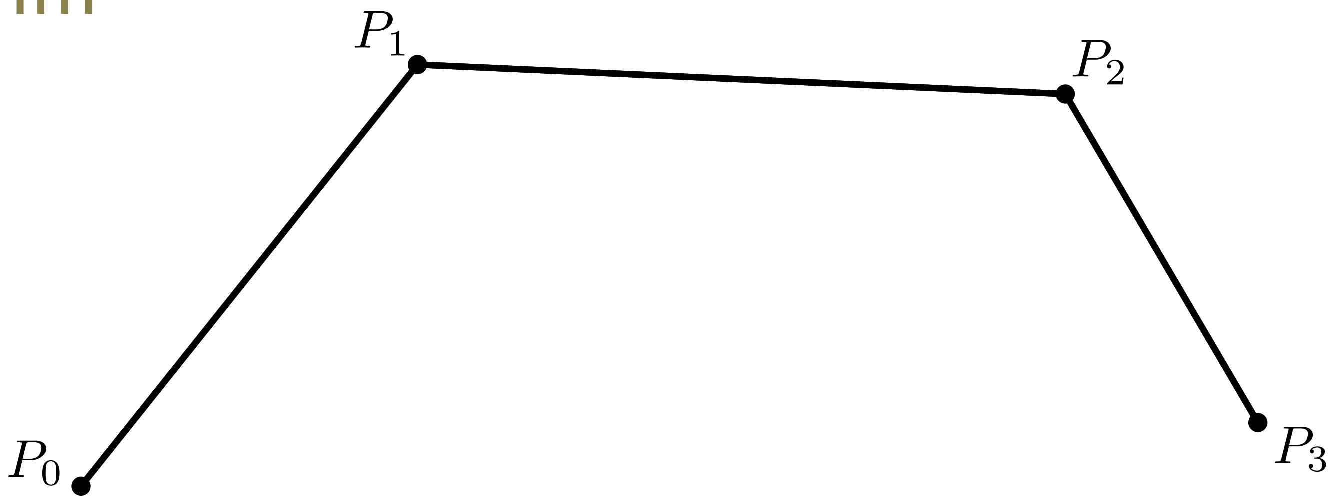
Recursive / geometric construction method

The final curve is given by $P(t) = P_{0\ldots n}(t)$

De Casteljau's algorithm

Example for $n = 3$ and $t = 1/2$

$$P(t) = P_{0123}(t)$$



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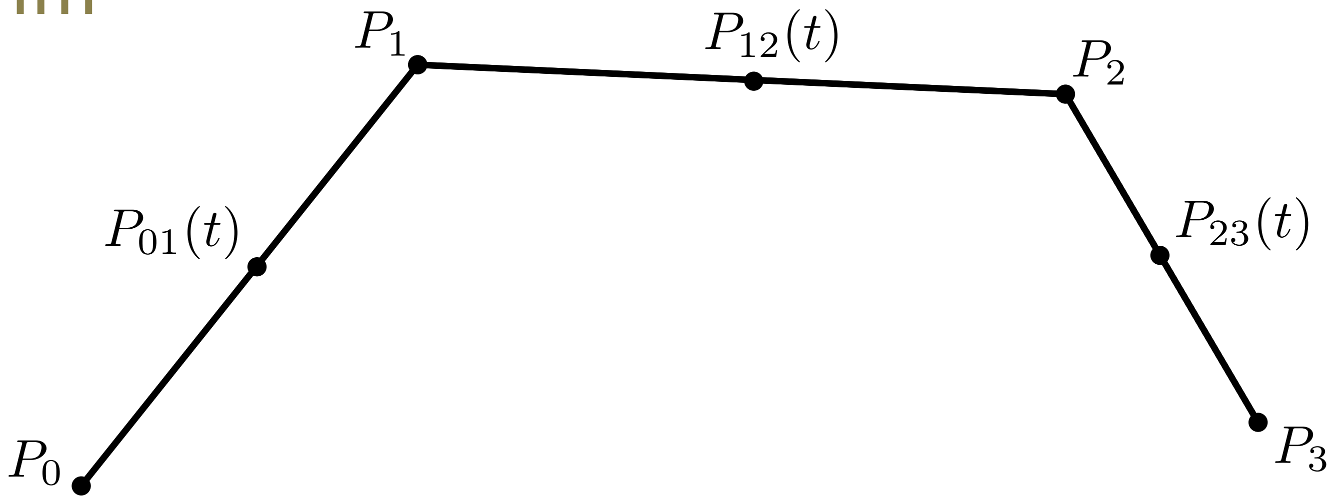

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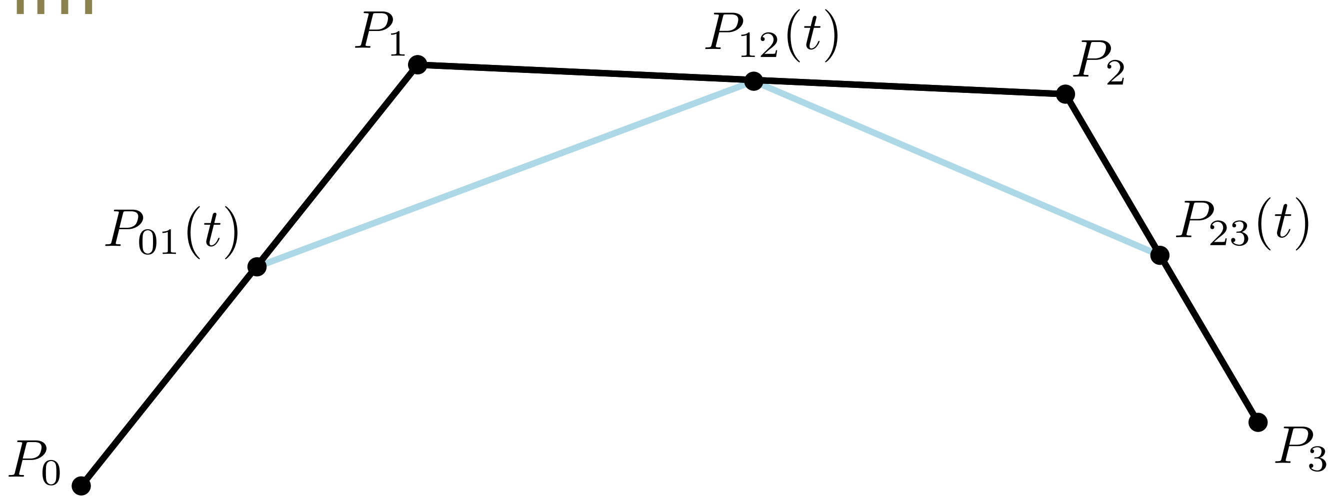

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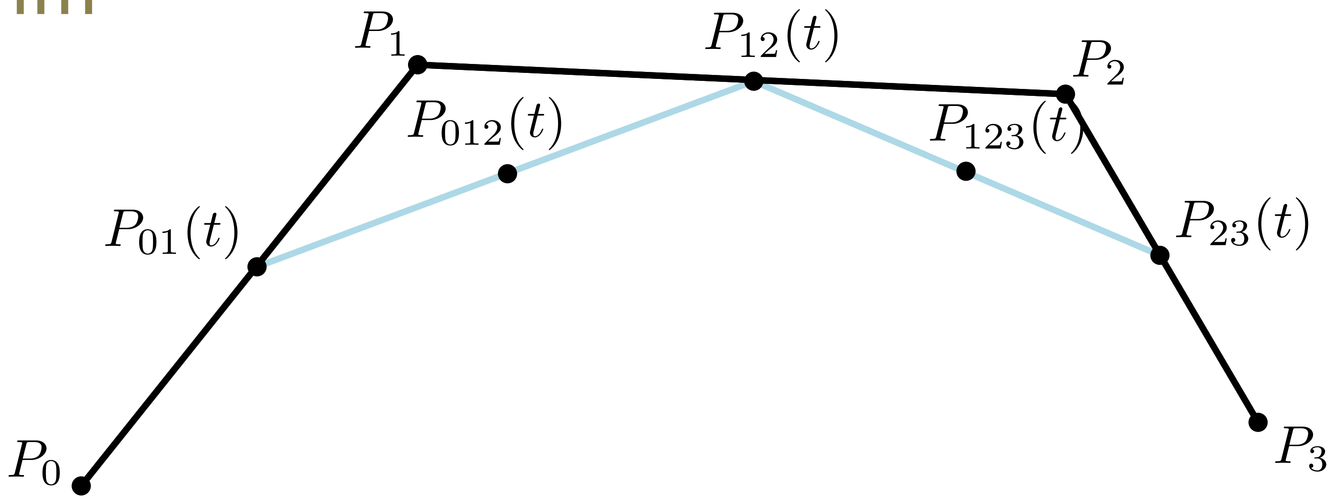

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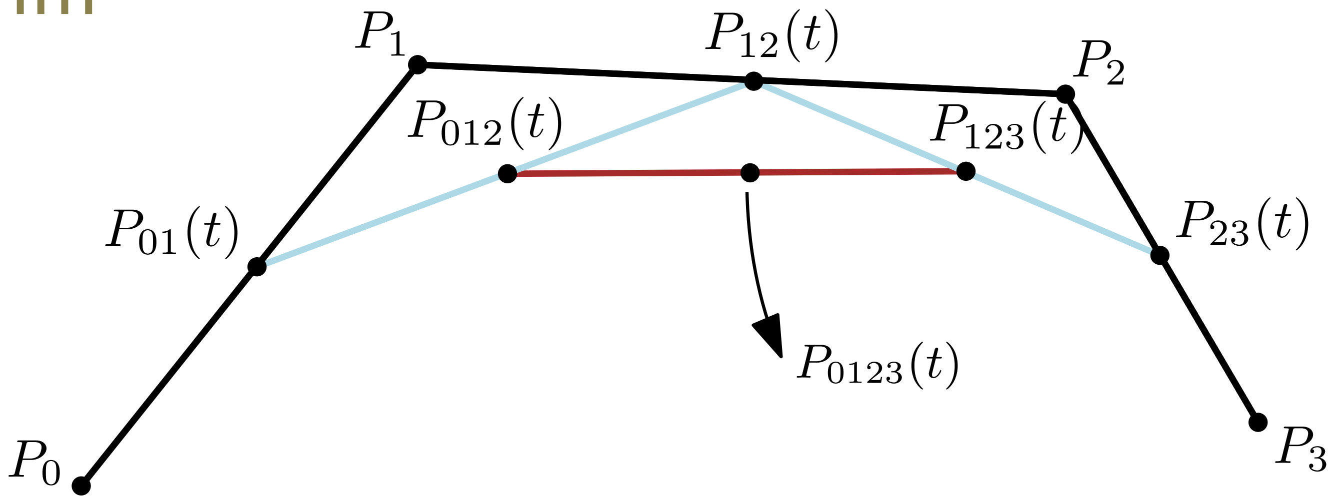

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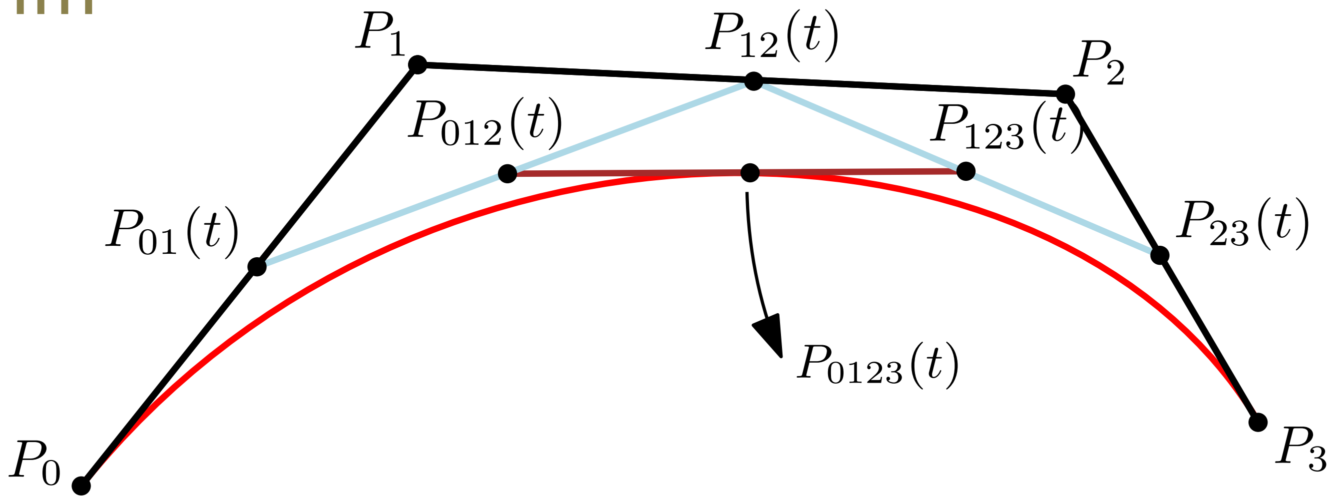

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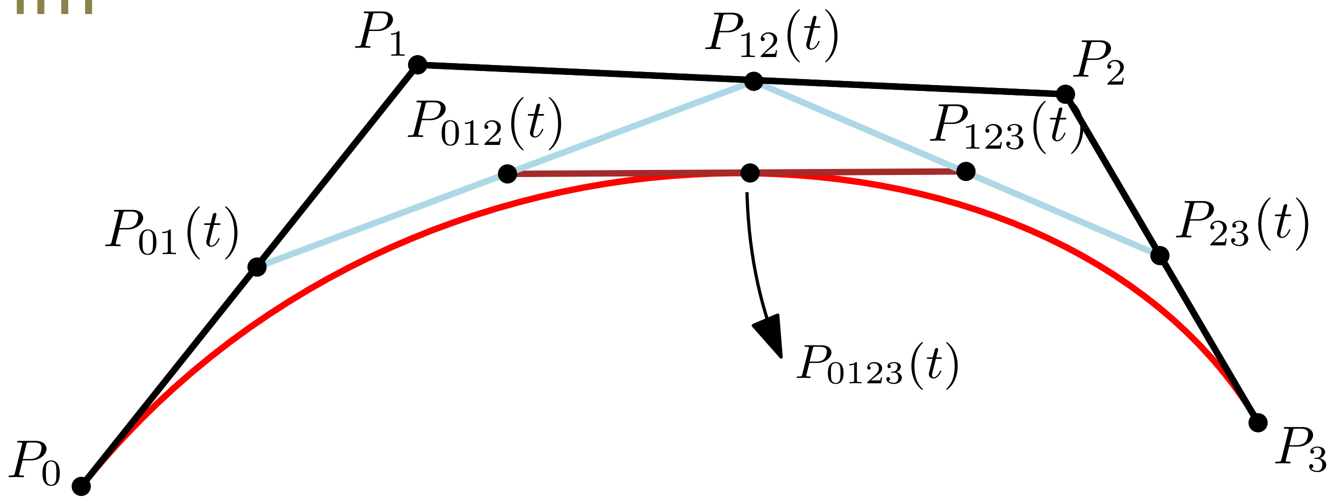

See `https://javascript.info/bezier-curve` for several animations

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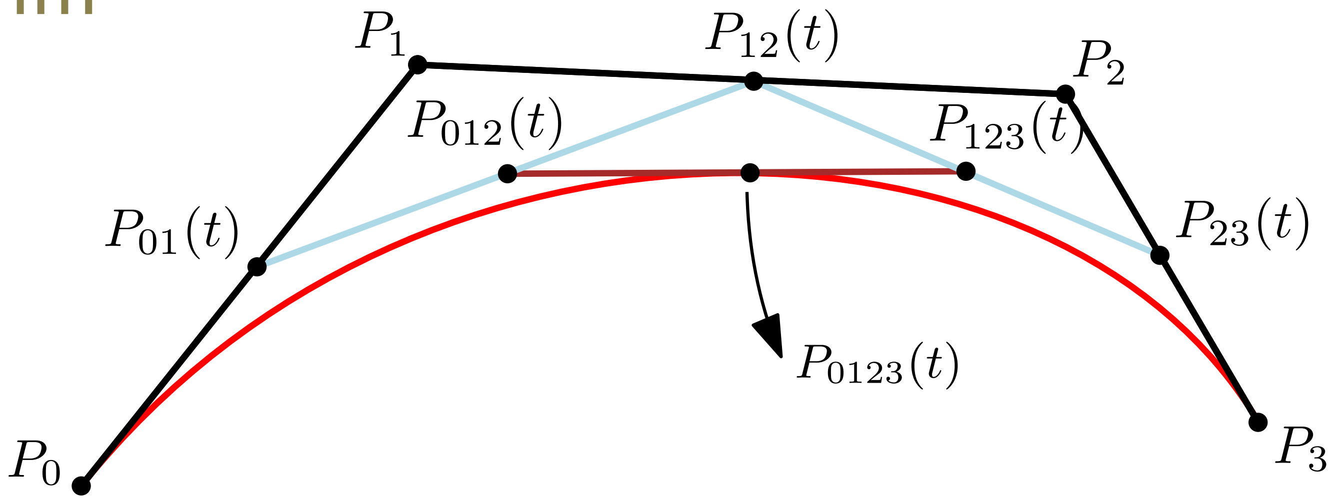

Implementation of the algorithm

How to evaluate $P(1/2)$?

BÉZIER CURVES AS LINEAR INTERPOLATION

De Casteljau's algorithm

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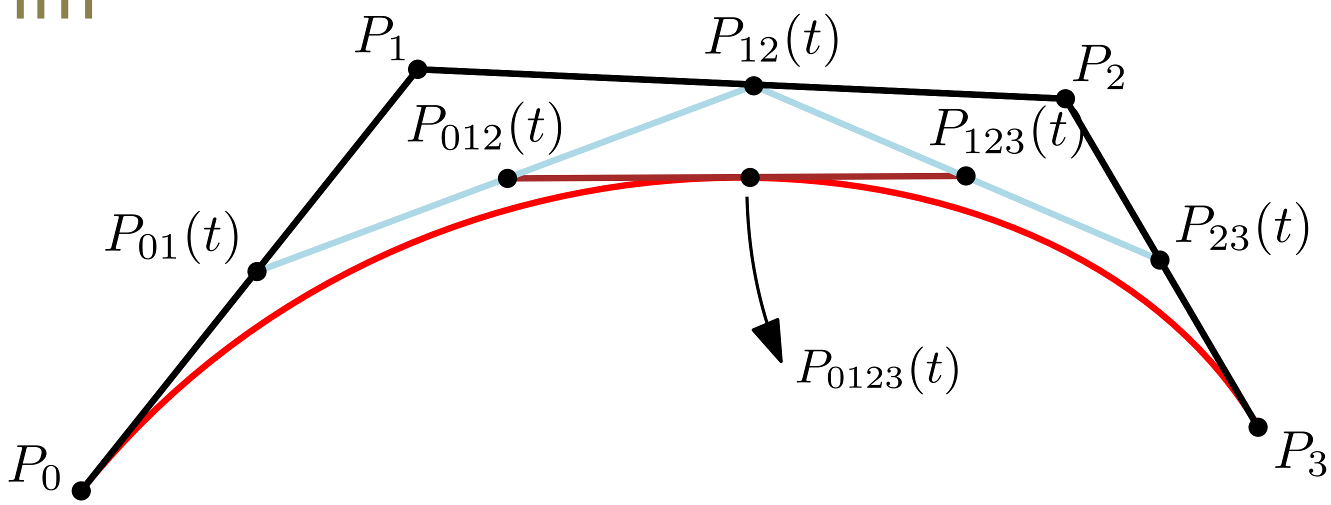

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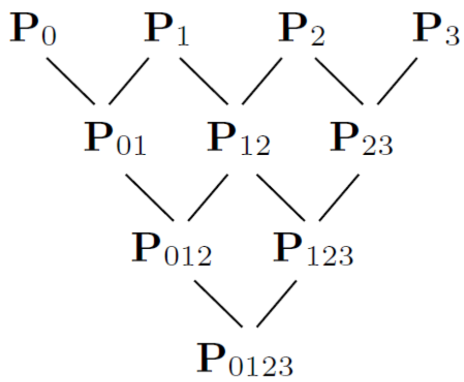
1	$\mathbf{P}_{01}\ \mathbf{P}_{12}\ \mathbf{P}_{23} \ldots \mathbf{P}_{n-1,n}$	n
2	$\mathbf{P}_{012}\ \mathbf{P}_{123}\ \mathbf{P}_{234} \ldots \mathbf{P}_{n-2,n-1,n}$	$n-1$
3	$\mathbf{P}_{0123}\ \mathbf{P}_{1234}\ \mathbf{P}_{2345} \ldots \mathbf{P}_{n-3,n-2,n-1,n}$	$n-2$
$\vdots$	$\vdots$	$\vdots$
n	$\mathbf{P}_{0123\ldots n}$	



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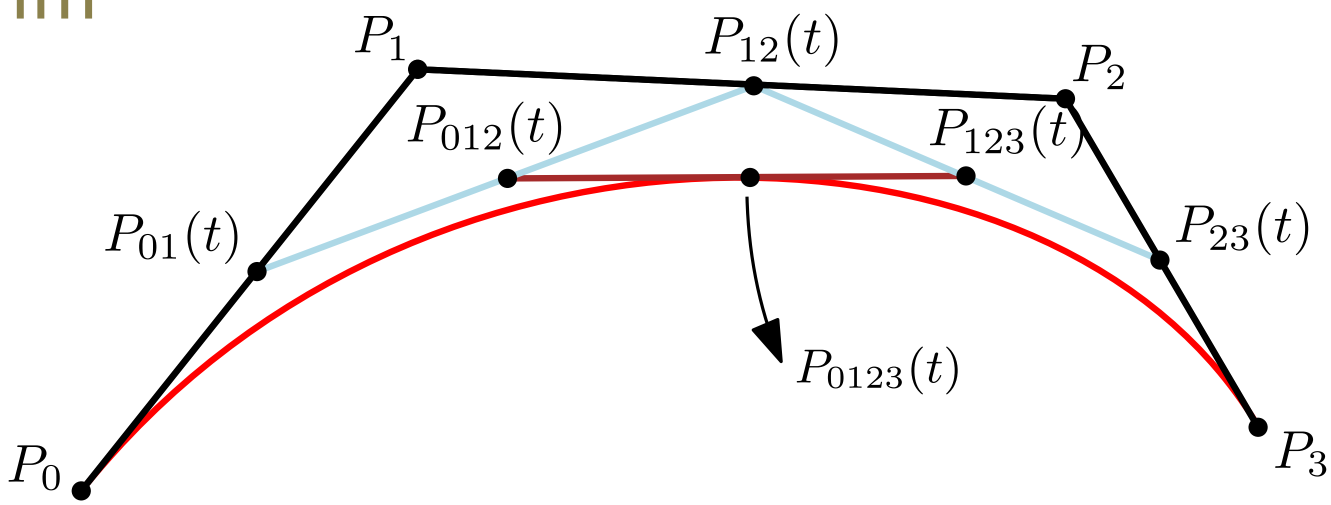

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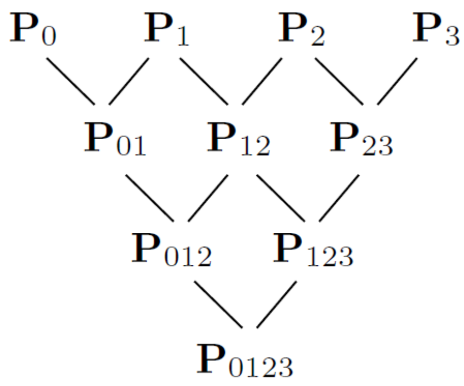

How many points computed in total?

BÉZIER CURVES AS LINEAR INTERPOLATION

De Casteljau's algorithm

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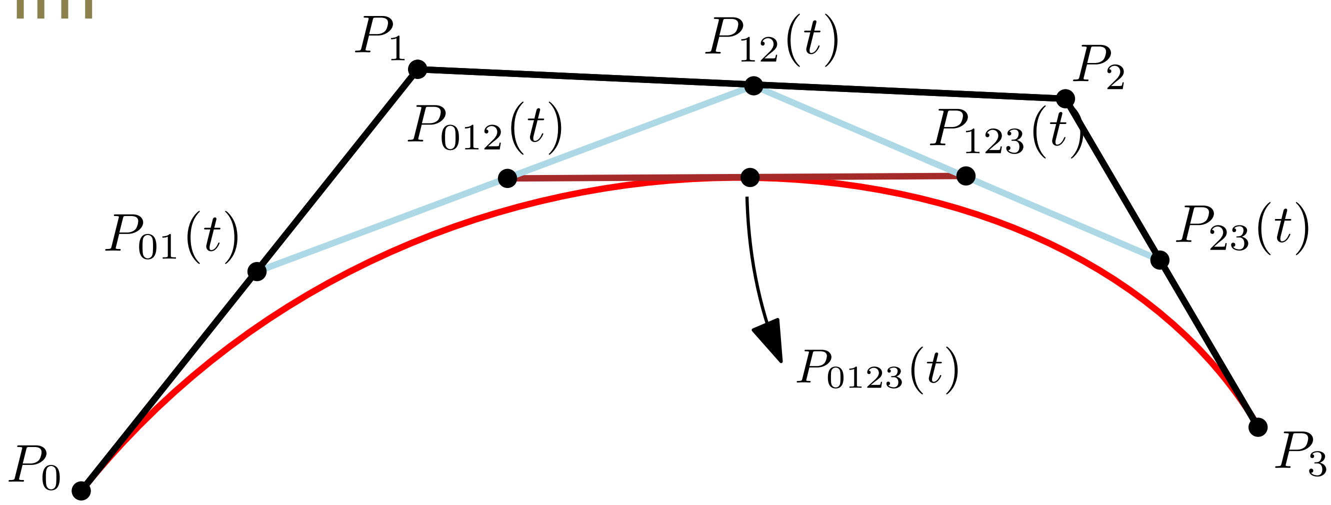

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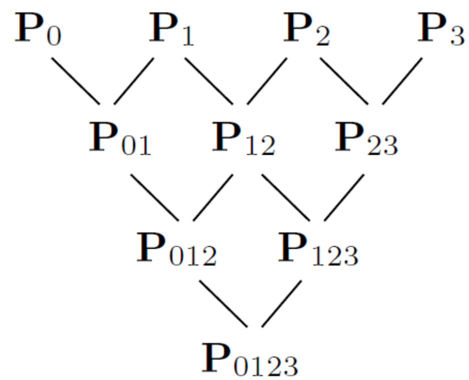

How many points computed in total?

$$n + (n-1) + (n-2) + \cdots + 2 + 1 = n(n+1)/2$$

De Casteljau's algorithm

Note: to generate one point on the curve, $\approx n^2/2$ computations is quite a lot...

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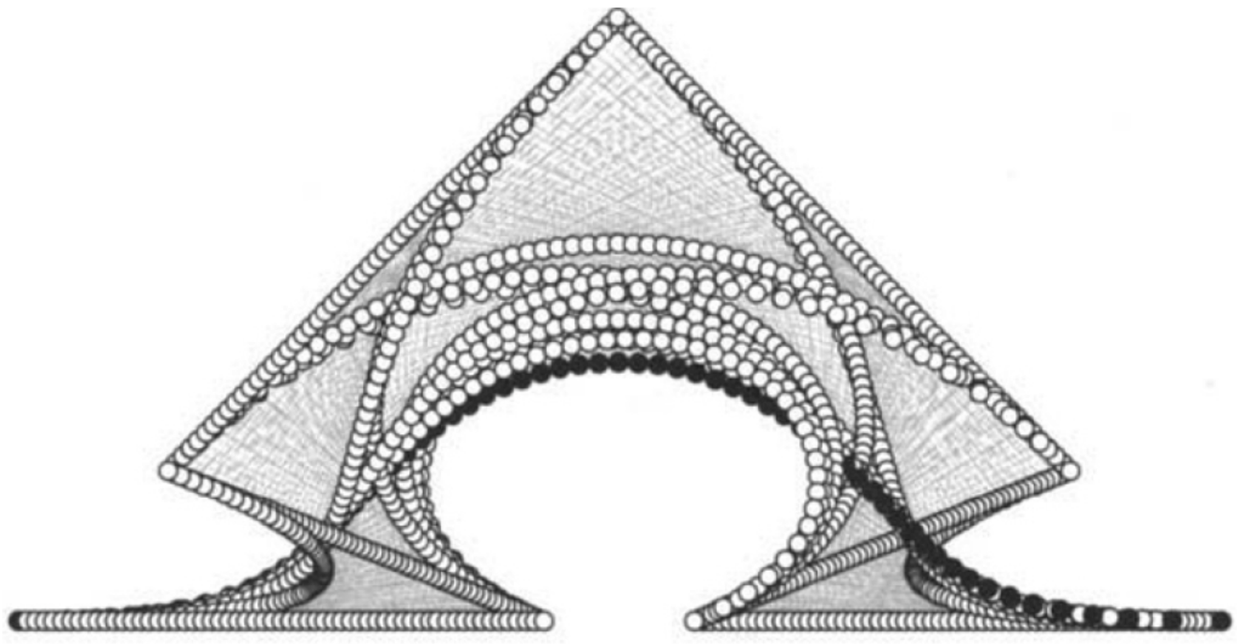

Figure 4.3 The de Casteljau algorithm: 60 points are computed on a degree six curve; all intermedediate points $\mathbf{b}_i^r$ are shown.

Figure from book by Farin (page 47)

De Casteljau's algorithm

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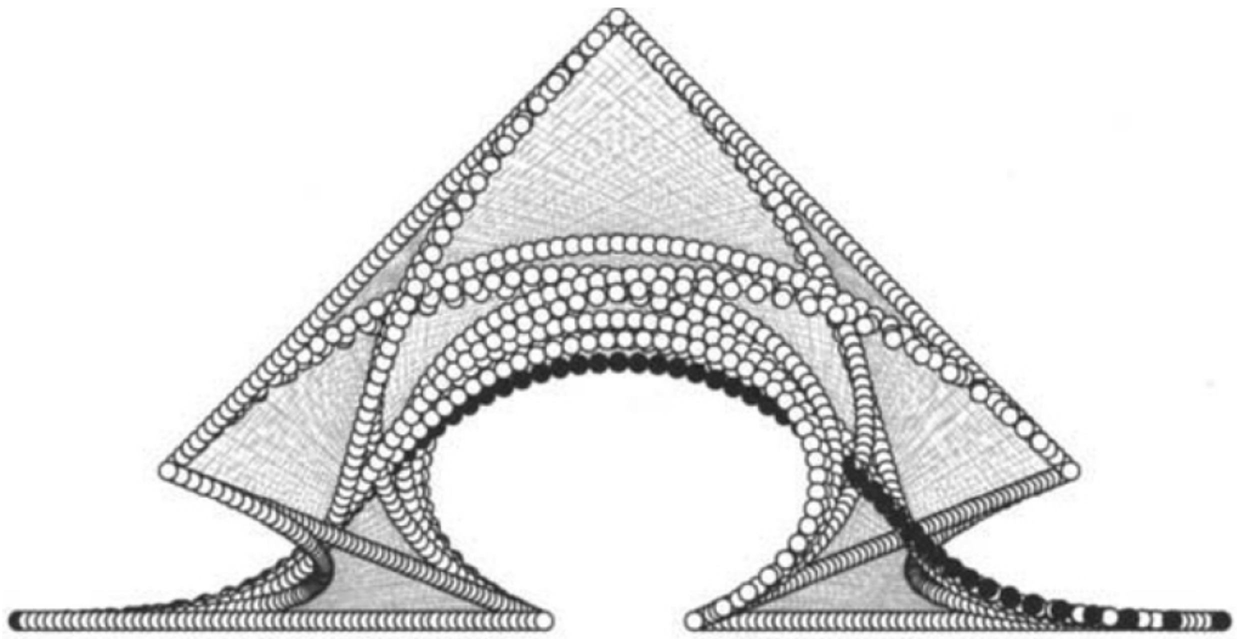

Figure 4.3 The de Casteljau algorithm: 60 points are computed on a degree six curve; all intermedediate points $\mathbf{b}_i^r$ are shown.

Figure from book by Farin (page 47)

Question for later: Is the computation based on Bernstein polynomials faster?

BÉZIER CURVES AS LINEAR INTERPOLATION

Using De Casteljau's to subdivide a curve

Using De Casteljau's to subdivide a curve

What if you want to add more points to a curve?
(We need this when we need more flexibility to design the curve)

Goal: increase number of points, but preserve shape of curve

Using De Casteljau's to subdivide a curve

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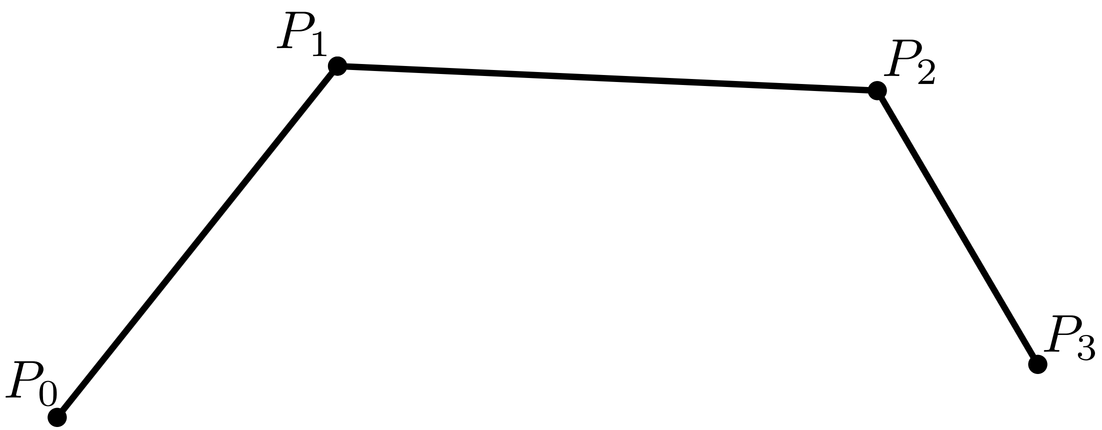

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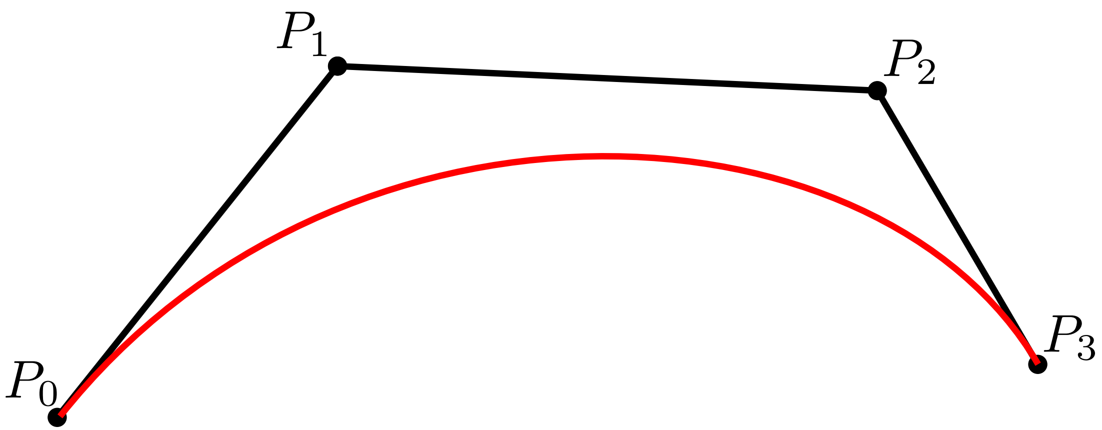

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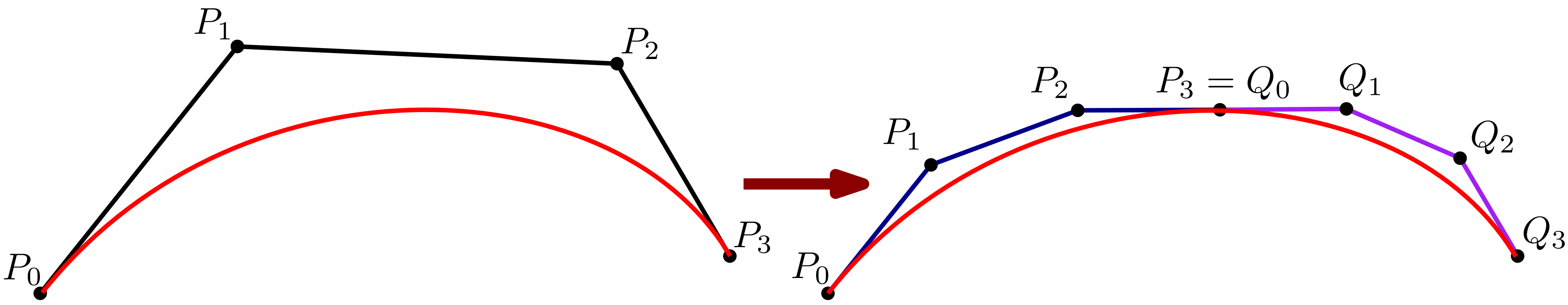

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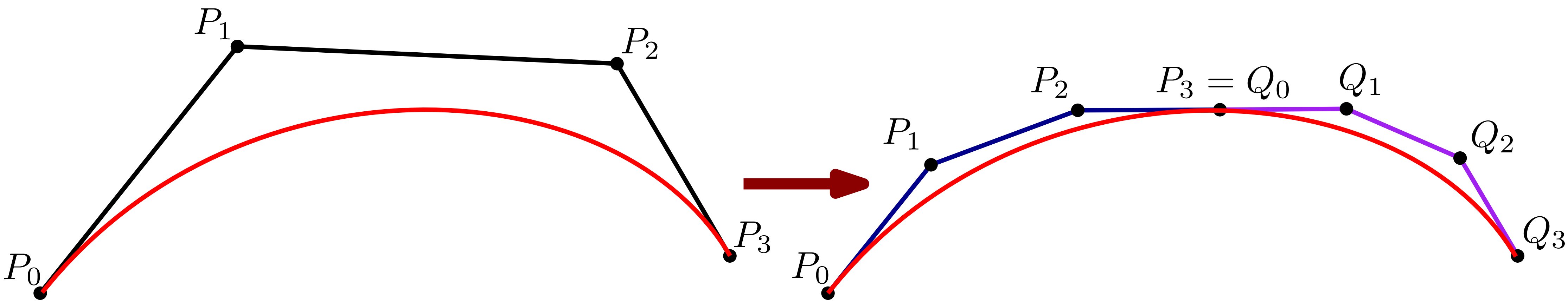

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BÉZIER CURVES AS LINEAR INTERPOLATION

Using De Casteljau's to subdivide a curve

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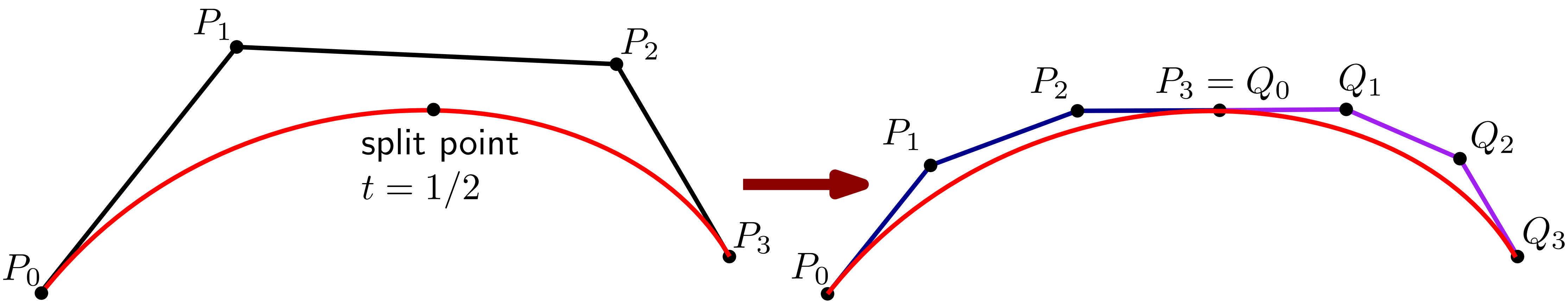

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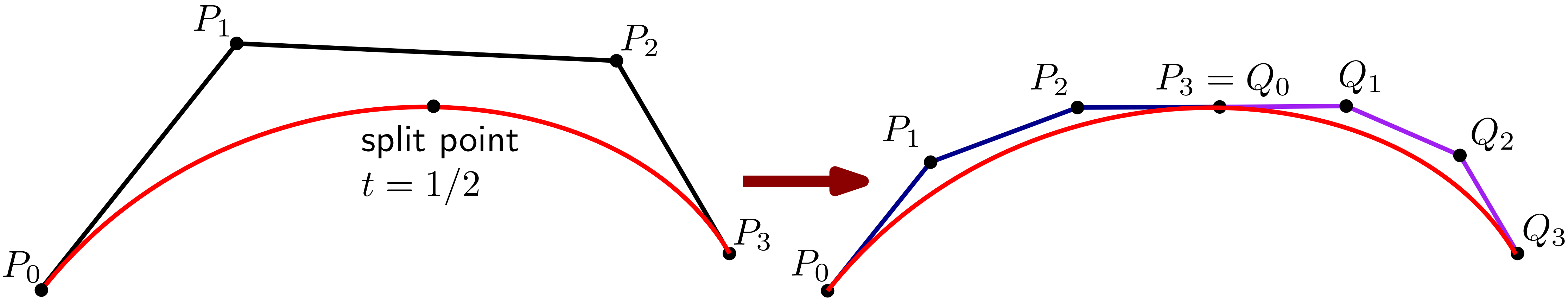

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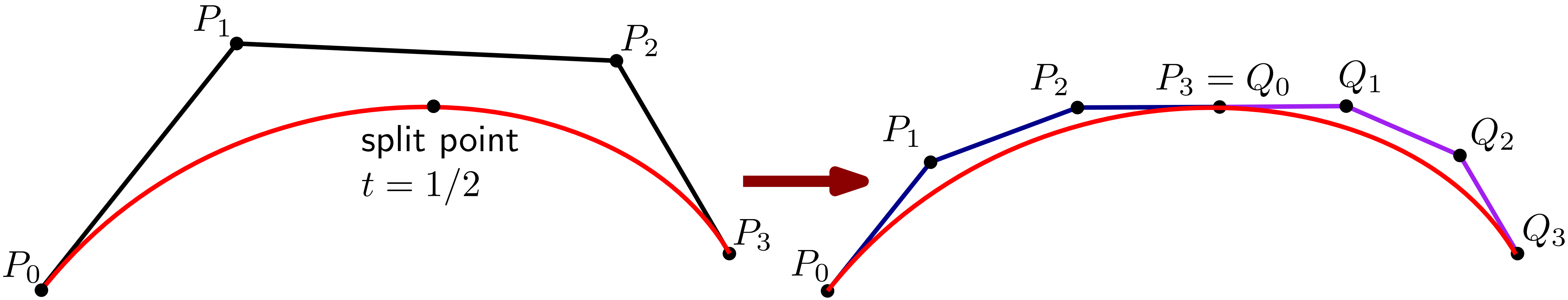

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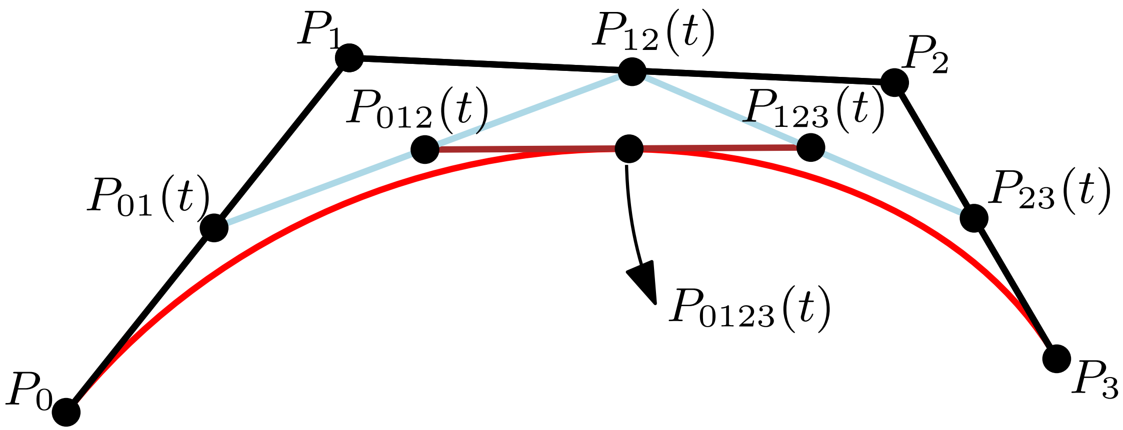

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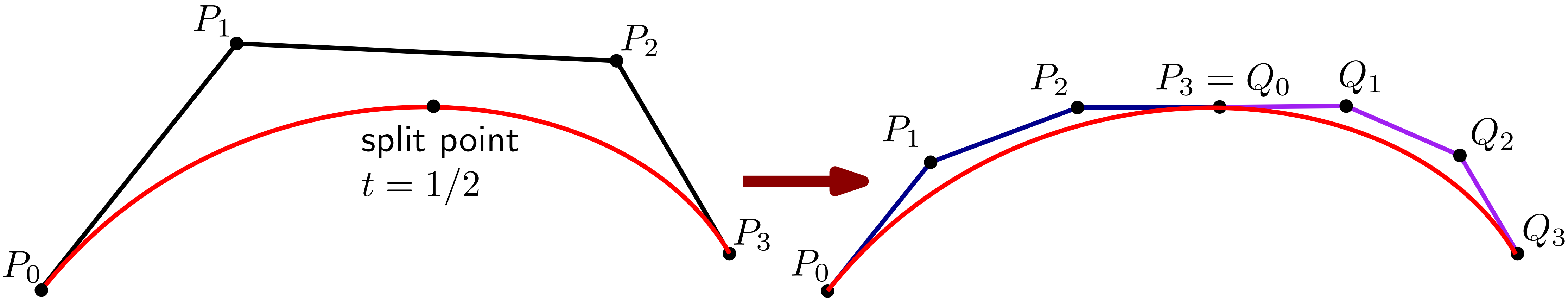

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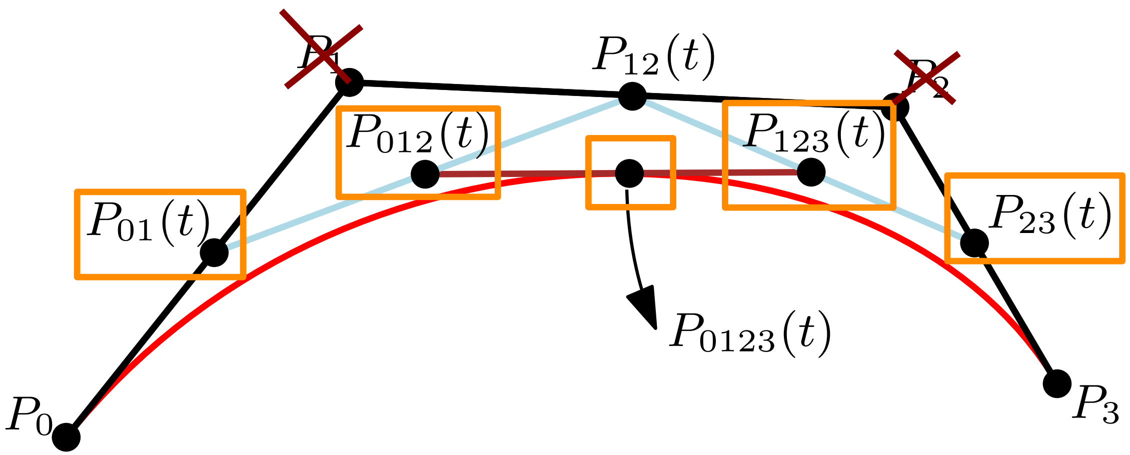

In general, the subdivision is done by:

- Discarding interior control points $P_1, \ldots, P_{n-1}$
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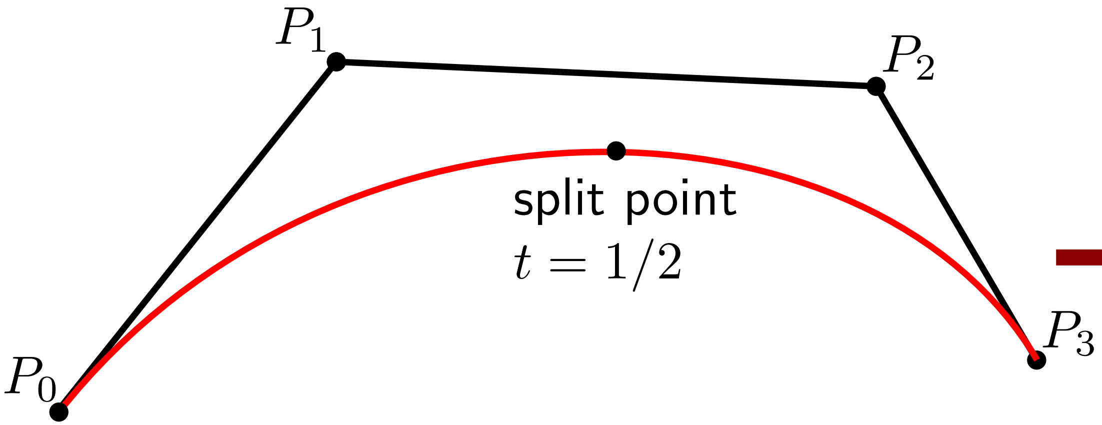

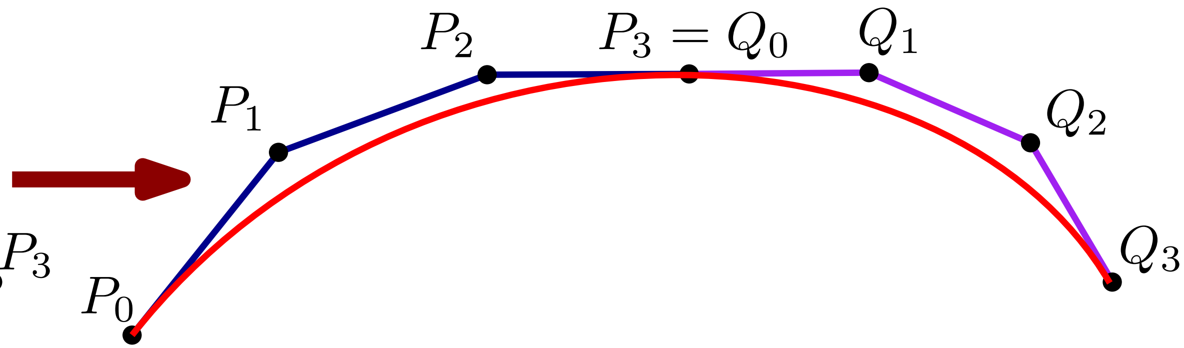

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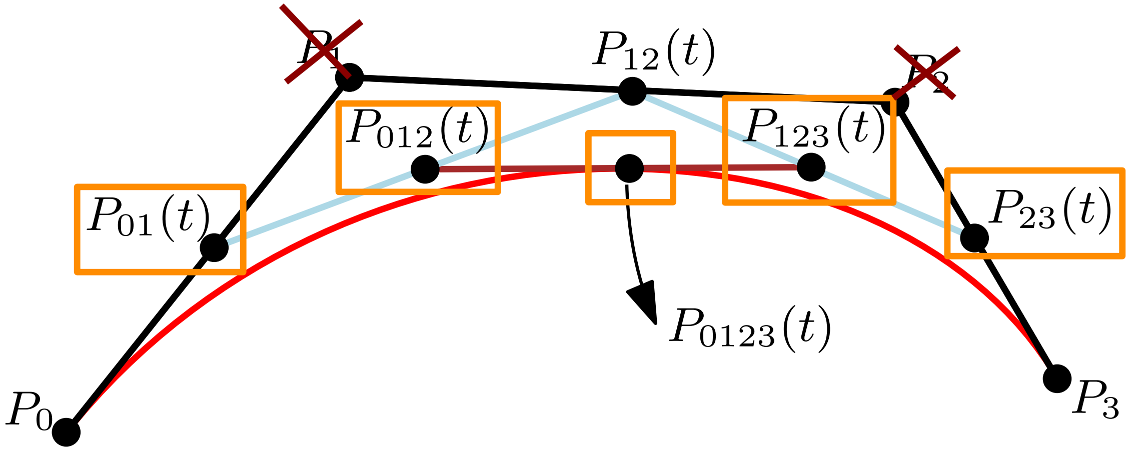

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This method is also useful for **clipping**

Computation of a Bézier curve

Recall definition

$B_{n,i}(t) = \binom{n}{i} t^i (1-t)^{n-i}$

recall that $0 \leq i \leq n$, $\binom{n}{i} = \frac{n!}{i!(n-i)!}$, and $0! = 1$

$P(t) = \sum_{i=0}^{n} P_i B_{n,i}(t)$

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This can also be stored in a table, and reused for other points

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If dP would exist, then we could do:

$$\begin{aligned}
P(0) &= P_0 \\
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$$P(t+\Delta) = P(t) + P'(t)\Delta + P''(t)\frac{\Delta^2}{2!} + P'''(t)\frac{\Delta^3}{3!} + P''''(t)\frac{\Delta^4}{4!} + \ldots$$

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Infinite series

But becomes finite if $P(t)$ has constant degree!

Forward differences for cubic Bézier curve

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or, equivalently,
$P(t) = at^3 + bt^2 + ct + d$
where:
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Problem: dP depends on t!

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We combine the three increments (dP, ddP and $dddP$) in an algorithms as follows:

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```
procedure FastCubicBézierSketch
    Compute dP, ddP and dddP for t=0
    P ← P_0
    for t = 0 to 1 step Δ do
        Draw point P
        P ← P + dP
        dP ← dP + ddP
        ddP ← ddP + dddP
```

Forward differences for cubic Bézier curve

Final code, trying to reuse computations as much as possible

procedure FASTCUBICBÉZIER

$Q_1 \leftarrow 3\Delta$

$Q_2 \leftarrow Q_1 \cdot \Delta$ $\qquad \triangleright\ 3\Delta^2$

$Q_3 \leftarrow \Delta^3$

$Q_4 \leftarrow 2Q_2$ $\qquad \triangleright\ 6\Delta^2$

$Q_5 \leftarrow 6Q_3$ $\qquad \triangleright\ 6\Delta^3$

$Q_6 \leftarrow P_0 - 2P_1 + P_2$

$Q_7 \leftarrow 3(P_1 - P_2) - P_0 + P_3$ $\qquad \triangleright\ a$

$P \leftarrow P_0$

$dP \leftarrow (P_1 - P_0)Q_1 + Q_6 \cdot Q_2 + Q_7 \cdot Q_3$

$ddP \leftarrow Q_6 \cdot Q_4 + Q_7 \cdot Q_5$

$dddP \leftarrow Q_7 \cdot Q_5$

for $t = 0$ to 1 step Δ **do**

  Draw point P

  $P \leftarrow P + dP$

  $dP \leftarrow dP + ddP$

  $ddP \leftarrow ddP + dddP$

Forward differences for cubic Bézier curve

Final code, trying to reuse computations as much as possible

procedure FASTCUBICBÉZIER

$Q_1 \leftarrow 3\Delta$

$Q_2 \leftarrow Q_1 \cdot \Delta$ $\qquad \triangleright\ 3\Delta^2$

$Q_3 \leftarrow \Delta^3$

$Q_4 \leftarrow 2Q_2$ $\qquad \triangleright\ 6\Delta^2$

$Q_5 \leftarrow 6Q_3$ $\qquad \triangleright\ 6\Delta^3$

$Q_6 \leftarrow P_0 - 2P_1 + P_2$

$Q_7 \leftarrow 3(P_1 - P_2) - P_0 + P_3$ $\qquad \triangleright\ a$

$P \leftarrow P_0$

$dP \leftarrow (P_1 - P_0)Q_1 + Q_6 \cdot Q_2 + Q_7 \cdot Q_3$

$ddP \leftarrow Q_6 \cdot Q_4 + Q_7 \cdot Q_5$

$dddP \leftarrow Q_7 \cdot Q_5$

for $t = 0$ to 1 step Δ **do**

  Draw point P

  $P \leftarrow P + dP$

  $dP \leftarrow dP + ddP$

  $ddP \leftarrow ddP + dddP$

The reduction in $\#$ of operations is huge: Ignoring the initialization, 3 sums for each evaluation of t

Matrix formulation

Bézier curves are often expressed in matrix form

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For $n = 2$, we had $P(t) = (1-t)^2 P_0 + 2t(1-t)P_1 + t^2 P_2$

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For $n = 2$, we had
$$\begin{aligned} P(t) &= (1-t)^2 P_0 + 2t(1-t)P_1 + t^2 P_2 \\ &= ((1-t)^2, 2t(1-t), t^2)(P_0, P_1, P_2)^t \end{aligned}$$

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Note: matrix notation is very practical, but not necessarily very efficient, or numerically stable

Question: how many sums/products to evaluate $P(t)$?

Another way to increase number of points

- Recall: curve subdivision took a degree-n curve and produced two curves of degree-n ($2n + 1$ control points in total)
- Alternative: add points (increase degree) while preserving curve

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original curve		new curve
$P_n(t)$	$\longrightarrow$	$P_{n+1}(t)$
P: $n+1$ control points		Q: $n+2$ new control points

DEGREE ELEVATION

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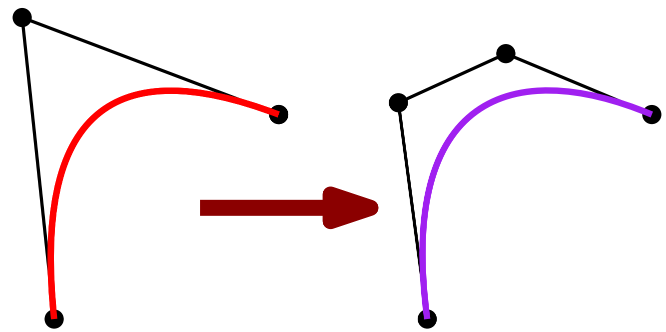

Adding one more point

original curve

$P_n(t)$

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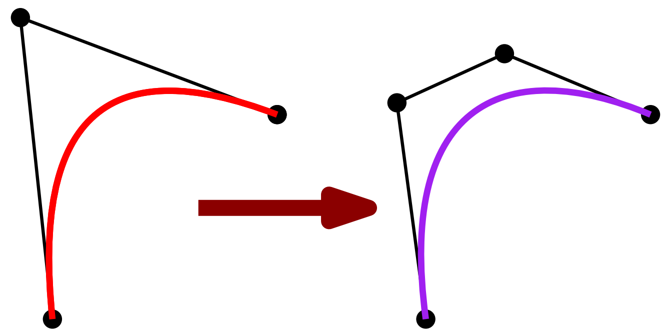

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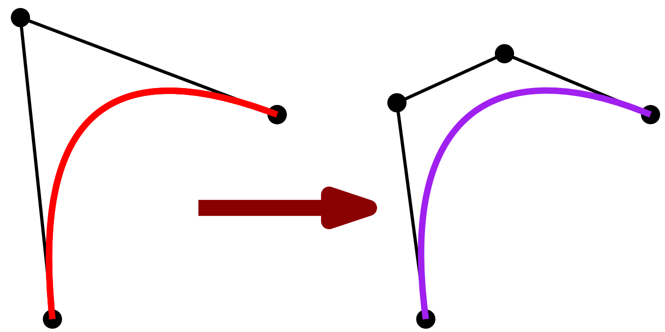

Producing control points for $P_{n+1}(t)$

Adding one more point

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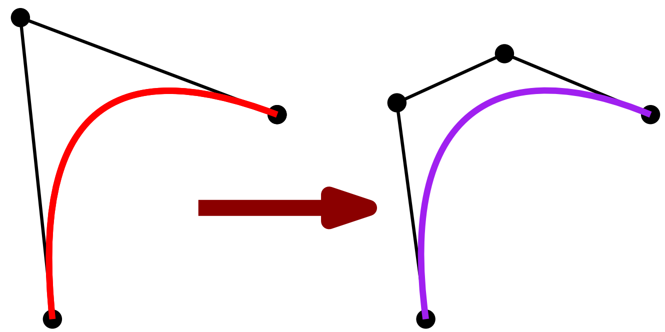

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With some basic algebraic tricks one can write $P(t)$ as an $(n+1)$-degree Bézier curve

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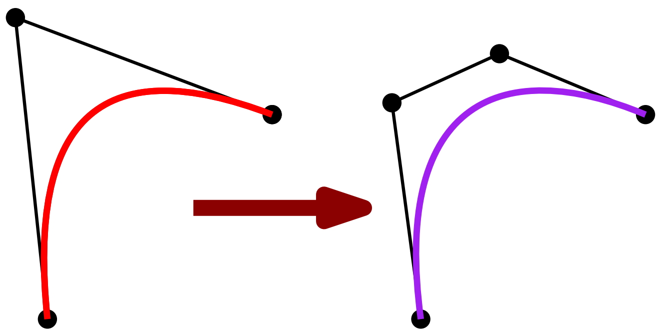

Producing control points for $P_{n+1}(t)$

> With some basic algebraic tricks one can write $P(t)$ as an $(n+1)$-degree Bézier curve

Start from trivial identity $P(t) = (t + (1-t))P(t) = tP(t) + (1-t)P(t)$

Use that $P(t) = \sum_{i=0}^{n} P_i B_{n,i}(t) = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} P_i$, and extract coefficientes of new $(n+2)$ control points

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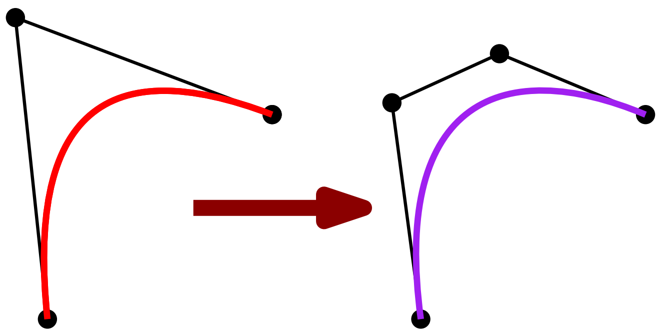

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Result:

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DEGREE ELEVATION

Adding one more point

original curve

$P_n(t)$

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new curve

$P_{n+1}(t)$

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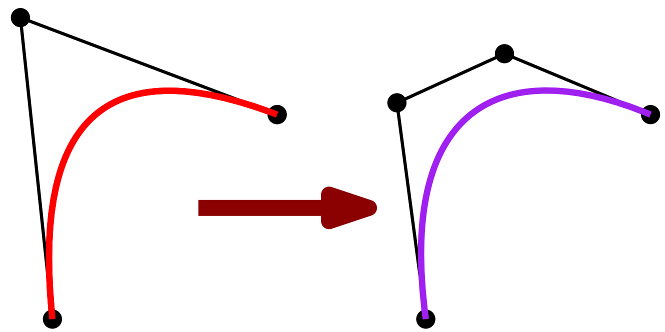

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new control points

Bézier curve of degree $(n+1)$!

$Q_i = \alpha_i P_{i-1} + (1-\alpha_i) P_i$

Note: here we assume $P_{-1} = 0$ and $P_{n+1} = 0$

Summary

The expression obtained for $P_{n+1}(t)$ is:

$$P_{n+1}(t) = \sum_{i=0}^{n+1} \binom{n+1}{i} t^i (1-t)^{n+1-i} \left(\frac{i}{n+1} P_{i-1} + \left(1 - \frac{i}{n+1}\right) P_i \right)$$

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degree-4 curve (5 control points)

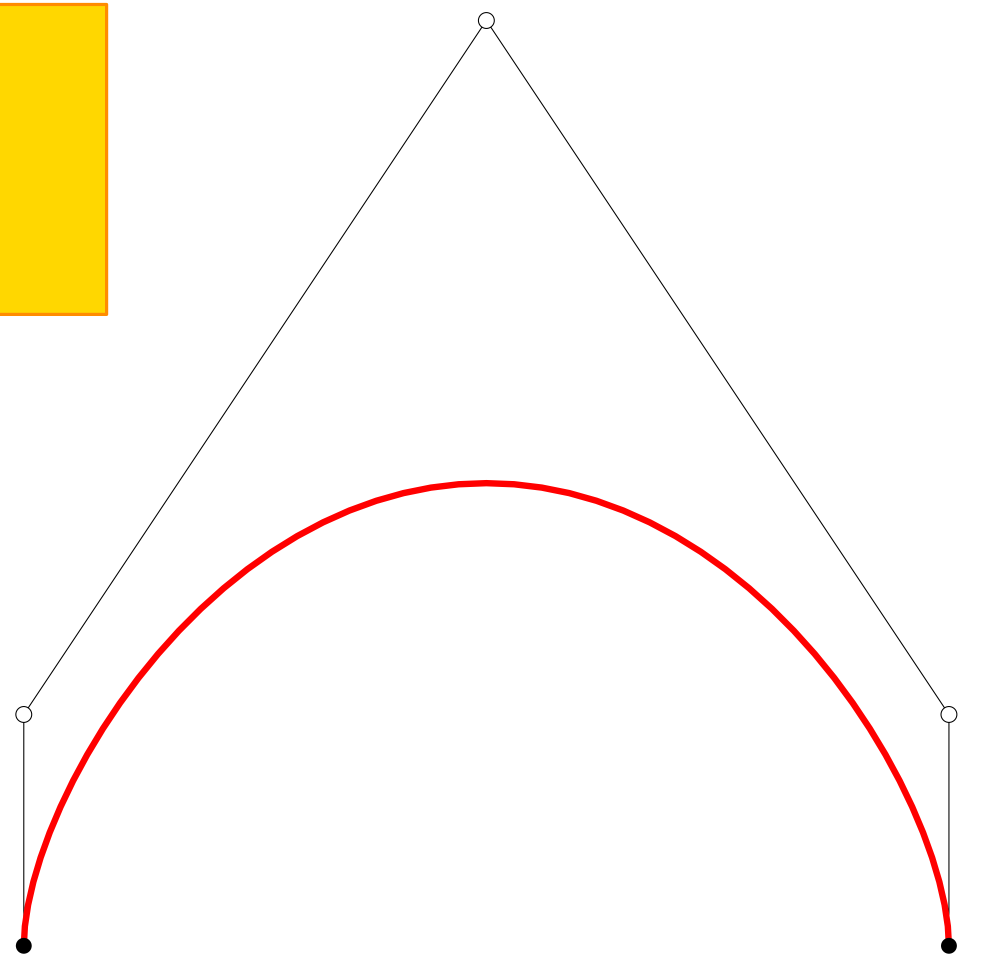

Figure from [G. Farin and D. Hansford, *The Essentials of CAGD*, AK Peters, 2000].

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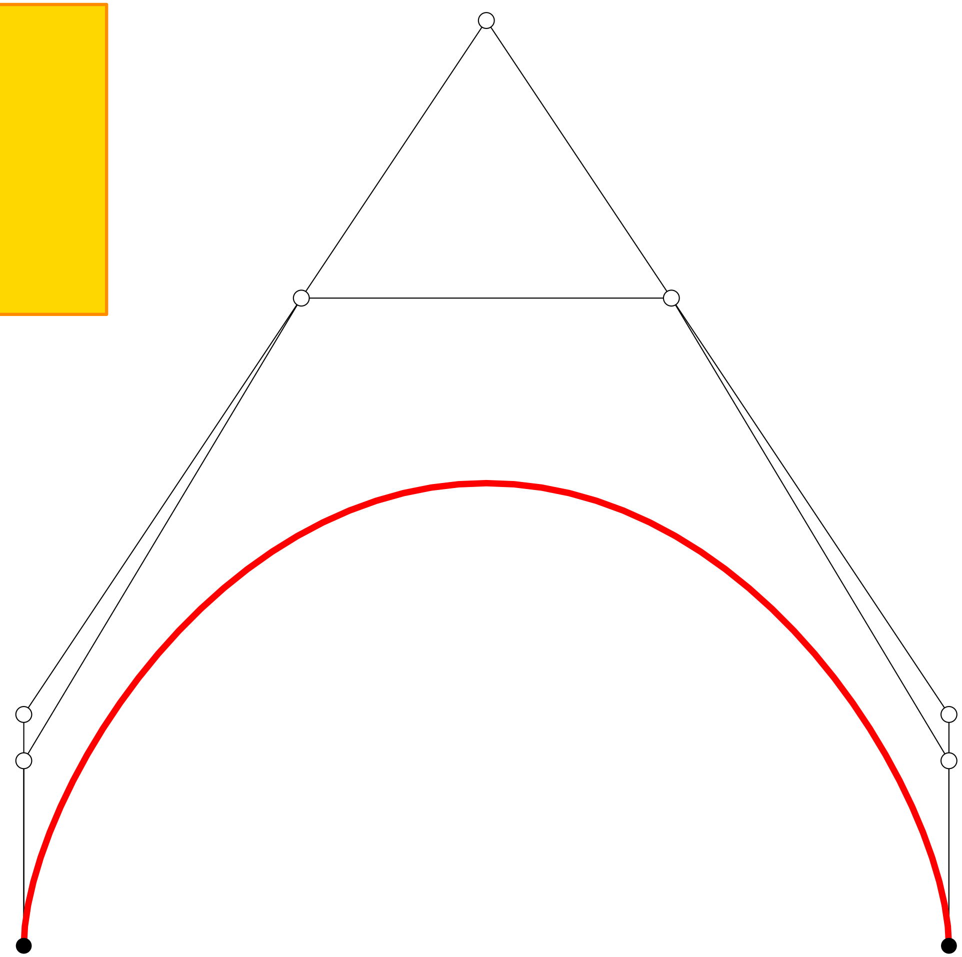
Example

degree-4 curve (5 control points)

degree-5 curve (6 control points)



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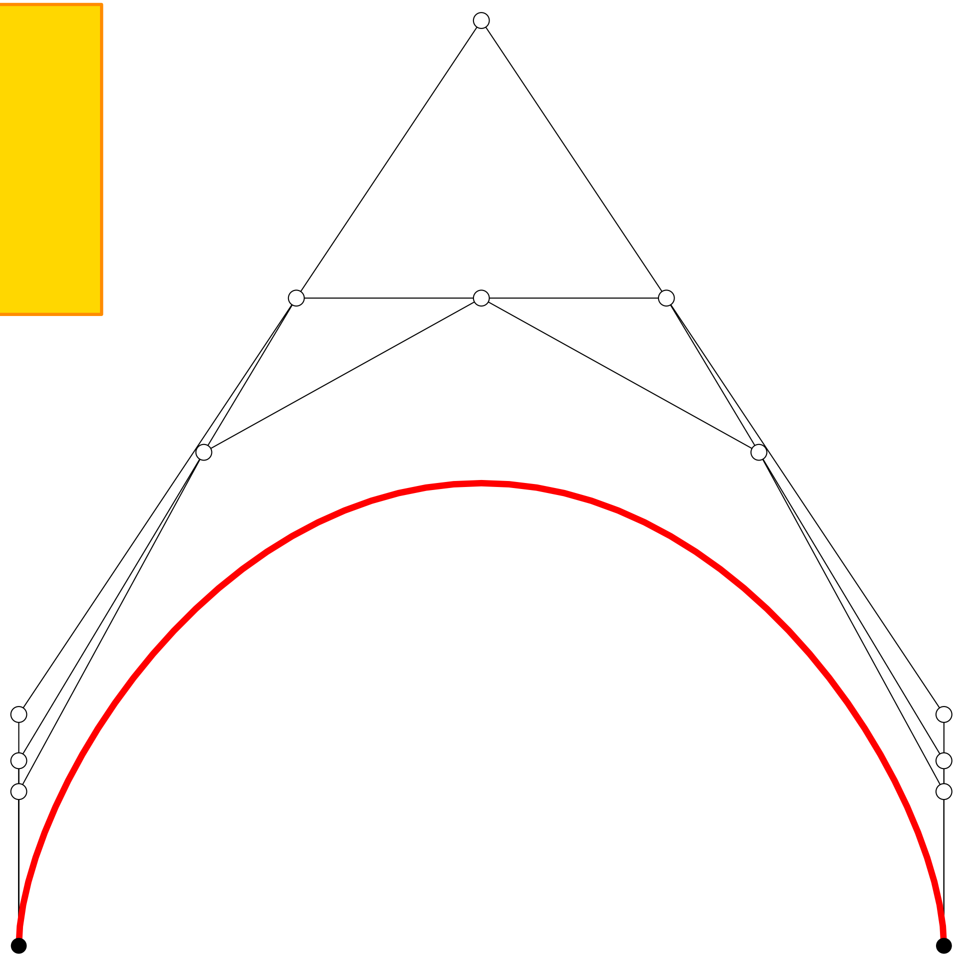
degree-4 curve (5 control points)

degree-5 curve (6 control points)

degree-6 curve (7 control points)



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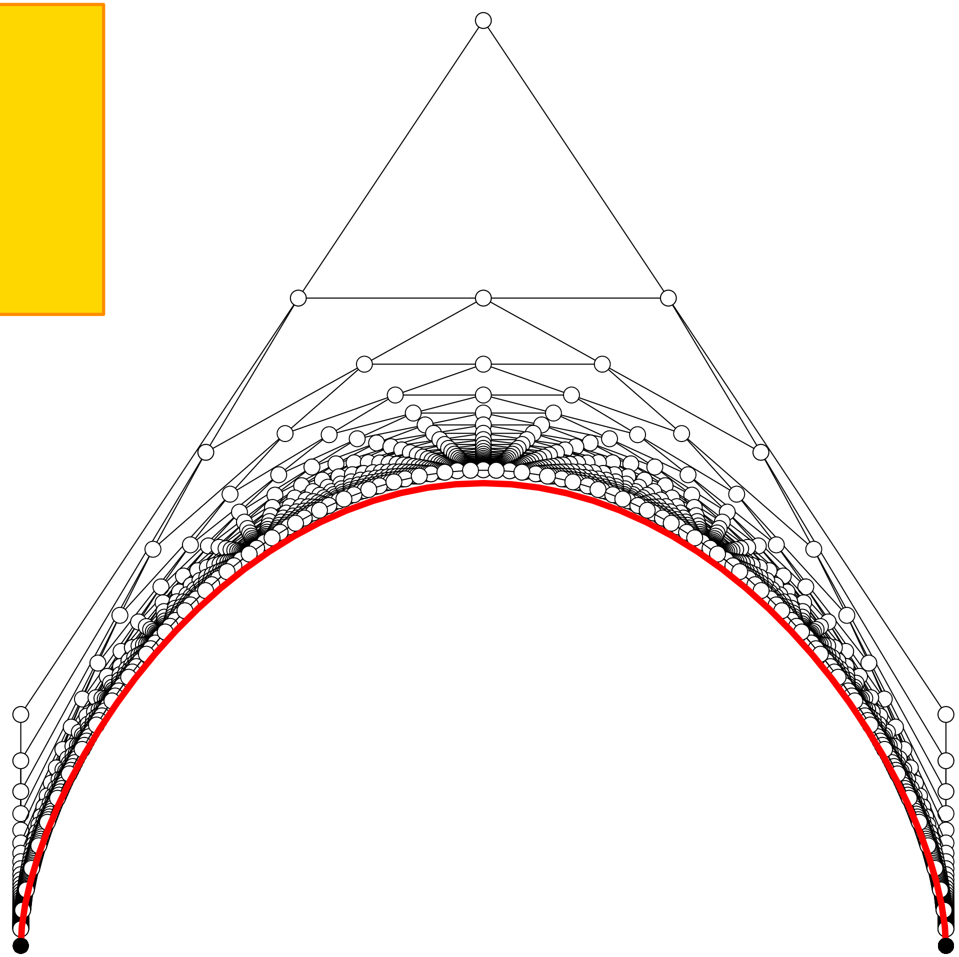
degree-n curve ($n+1$ control points)



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DEGREE ELEVATION

Summary

Question: Can you do degree reduction?

The expression obtained for $P_{n+1}(t)$ is:

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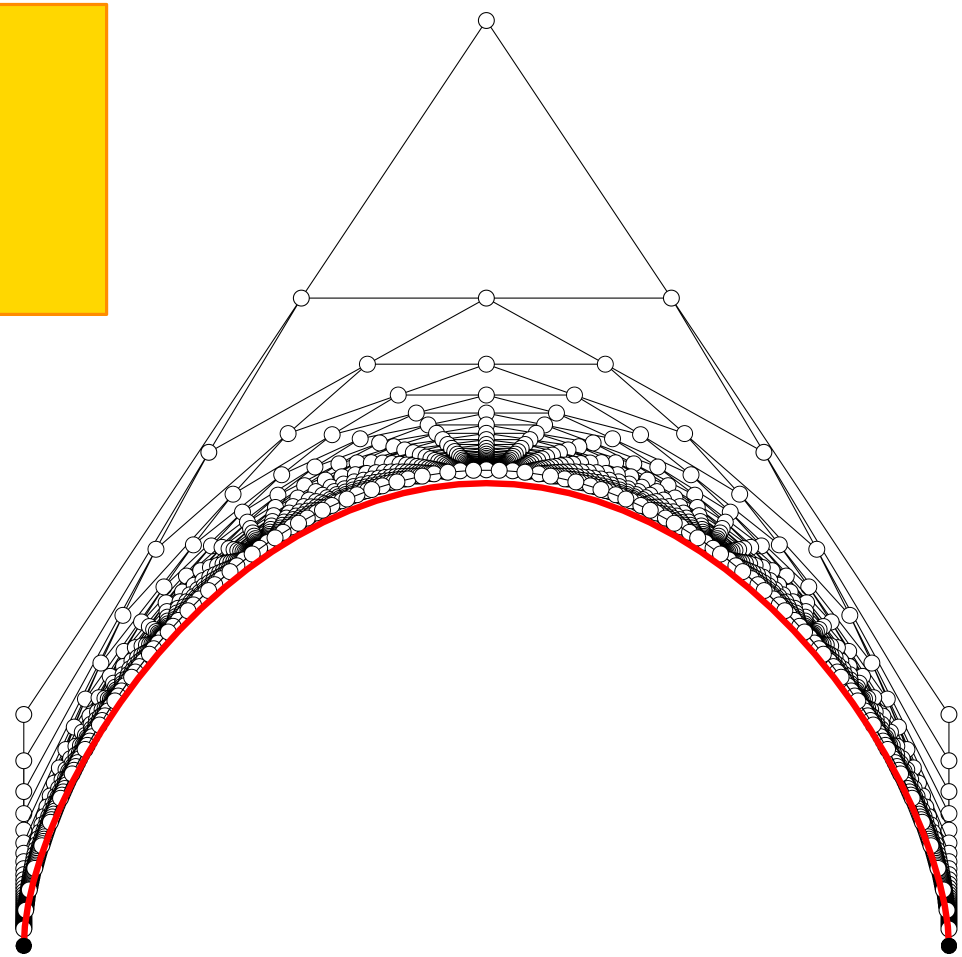

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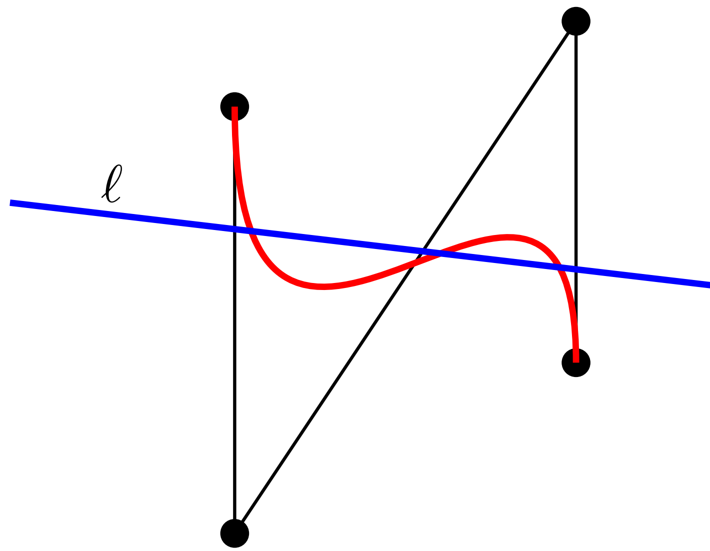
Back to the variation diminishing property

Recall the property: The number of intersections of any line with a Bézier curve is at most the number of intersections of the line with the control polygon



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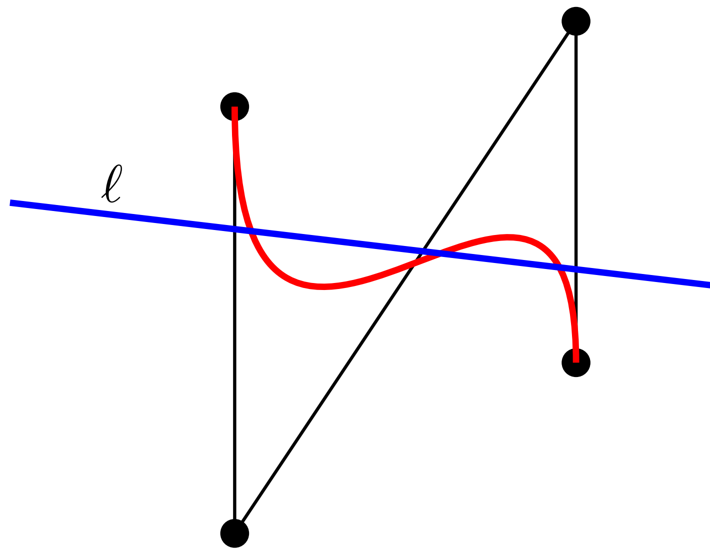

Proof sketch using degree elevation

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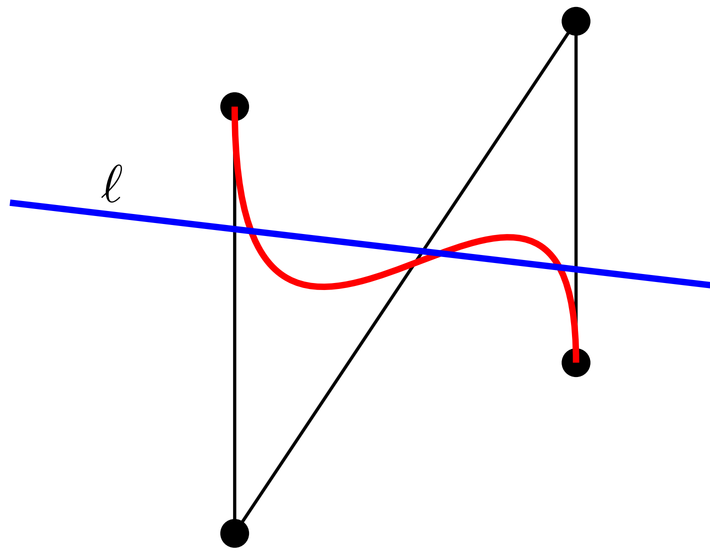

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Recall: linear interpolation is variation diminishing

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Proof sketch using degree elevation

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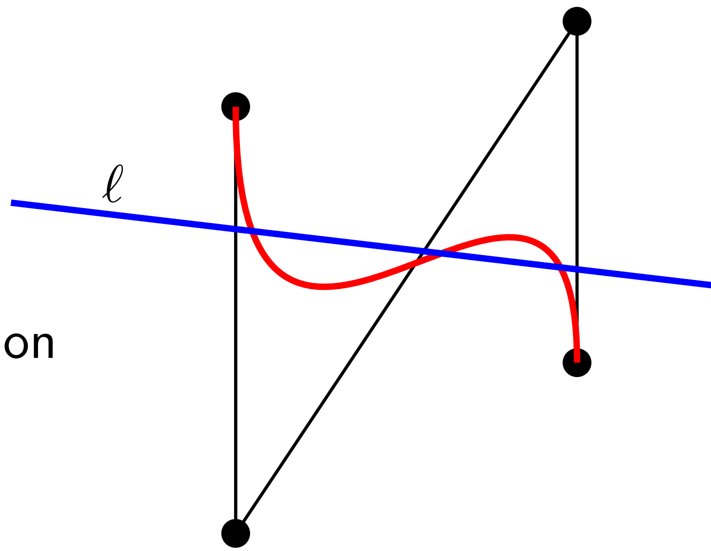
Observation 2: degree elevation is an instance of linear interpolation



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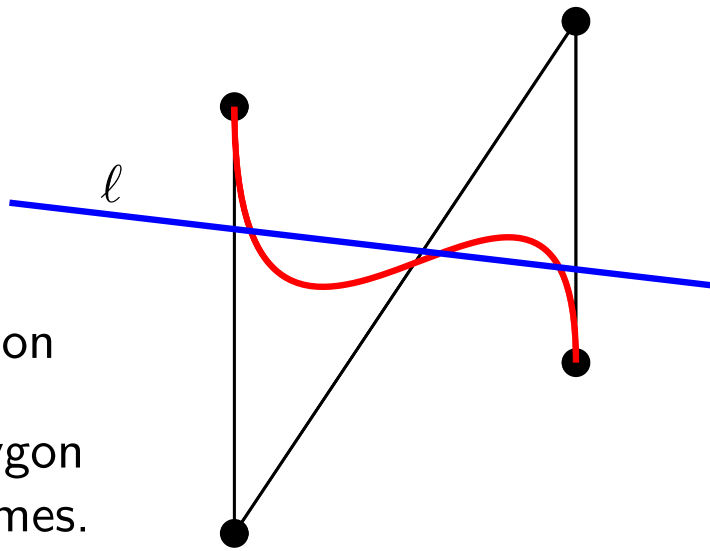

Proof sketch using degree elevation

Recall: linear interpolation is variation diminishing

Observation 2: degree elevation is an instance of linear interpolation

Let R_0 be the control polygon of $P(t)$, let R_1 be the control polygon after increasing the degree by one, and R_k after increasing it k times.

Let ℓ be a given line. Then R_k has no more intersections with ℓ than R_0.

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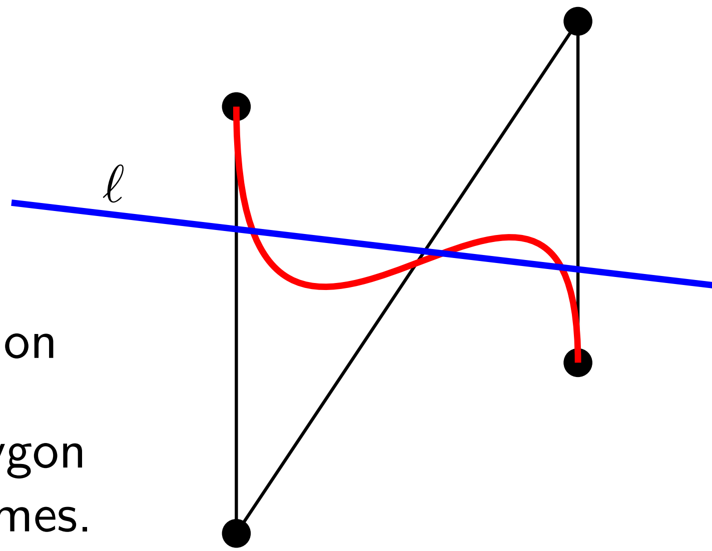

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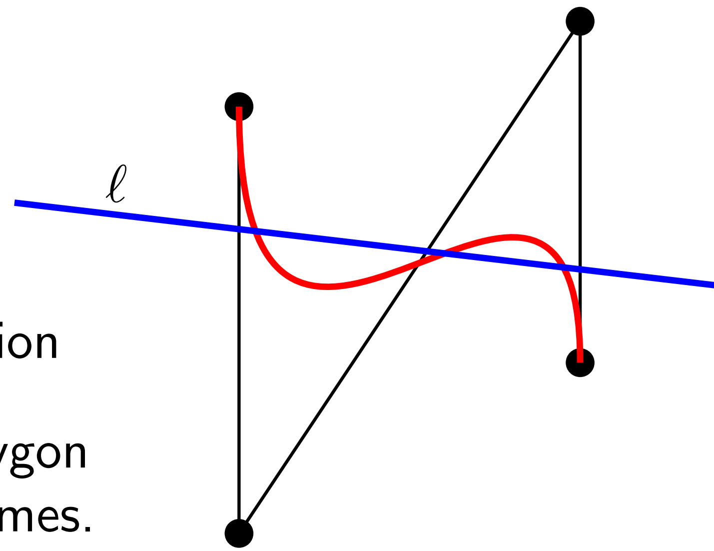

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Corollary: the Bézier curve $P(t)$ has no more intersections with ℓ than its control polygon

Goal: find Bézier curve that interpolates given points

- In general, the Bézier curve does not interpolate its control points
- There are situations in which the user may want to force the curve through some points

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INTERPOLATION WITH BÉZIER CURVES

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We are free to choose the values of the t_is!

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=
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$$

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
\frac{8}{27} & \frac{4}{9} & \frac{2}{9} & \frac{1}{27} \\
\frac{1}{27} & \frac{2}{9} & \frac{4}{9} & \frac{8}{27} \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix}
=
\begin{pmatrix} Q_0 \\ Q_1 \\ Q_2 \\ Q_3 \end{pmatrix}
\Longrightarrow
\begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix}
=
\begin{pmatrix}
1 & 0 & 0 & 0 \\
-\frac{5}{6} & 3 & -\frac{3}{2} & \frac{1}{3} \\
\frac{1}{3} & -\frac{3}{2} & 3 & -\frac{5}{6} \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix} Q_0 \\ Q_1 \\ Q_2 \\ Q_3 \end{pmatrix}
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Concrete example, take $Q_0 = (0,0)$, $Q_1 = (1,1)$, $Q_2 = (2,1)$, $Q_3 = (3,0)$

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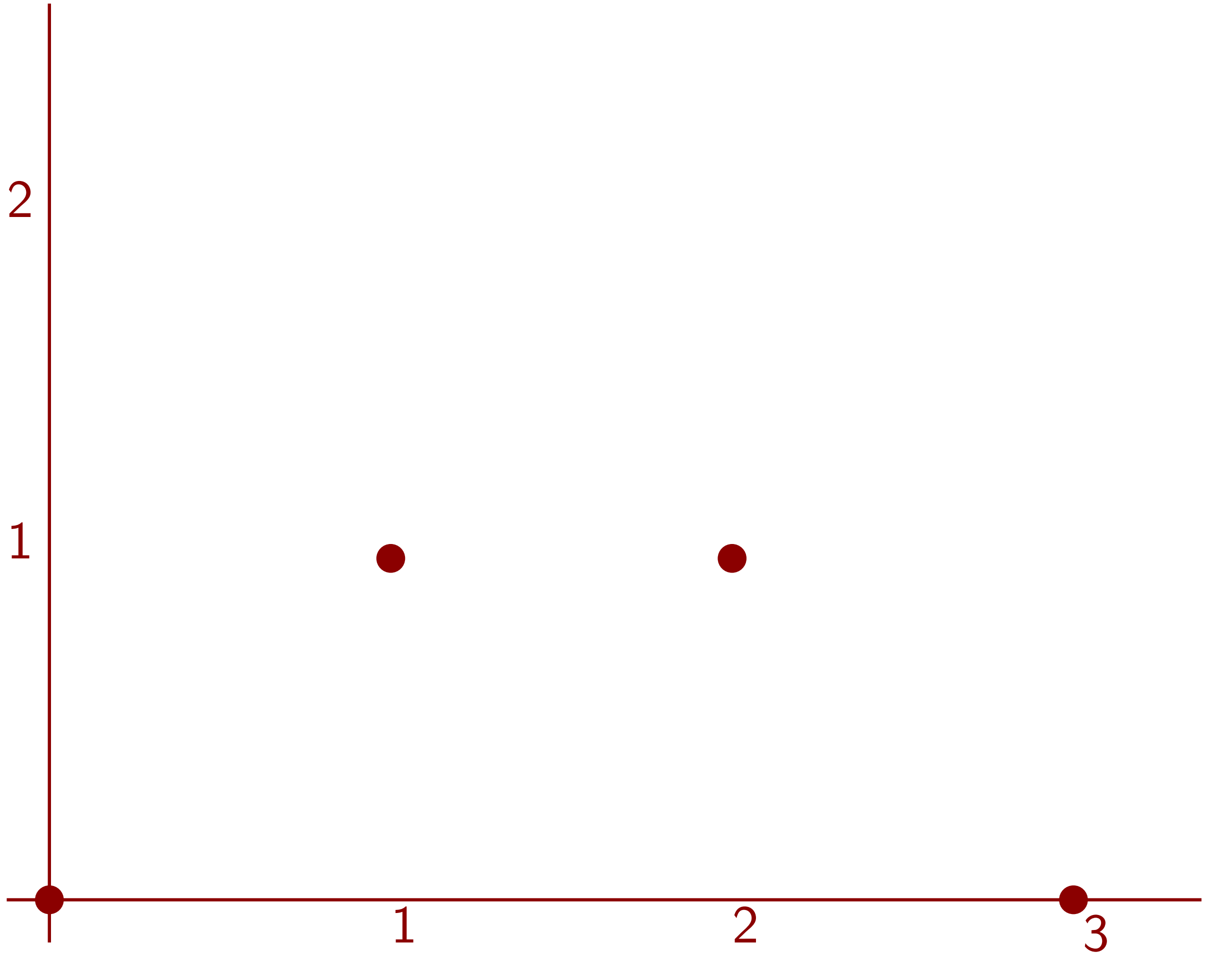

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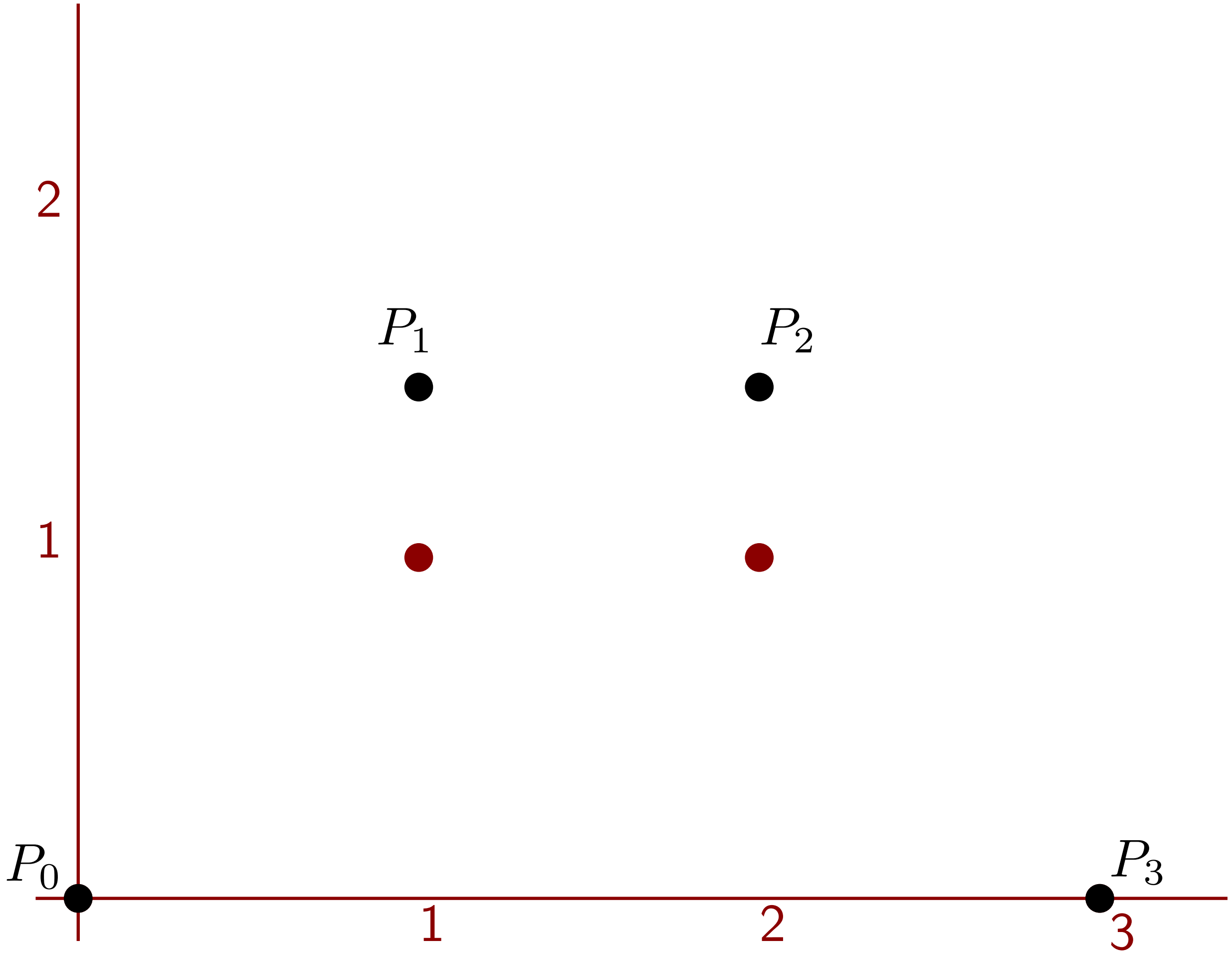

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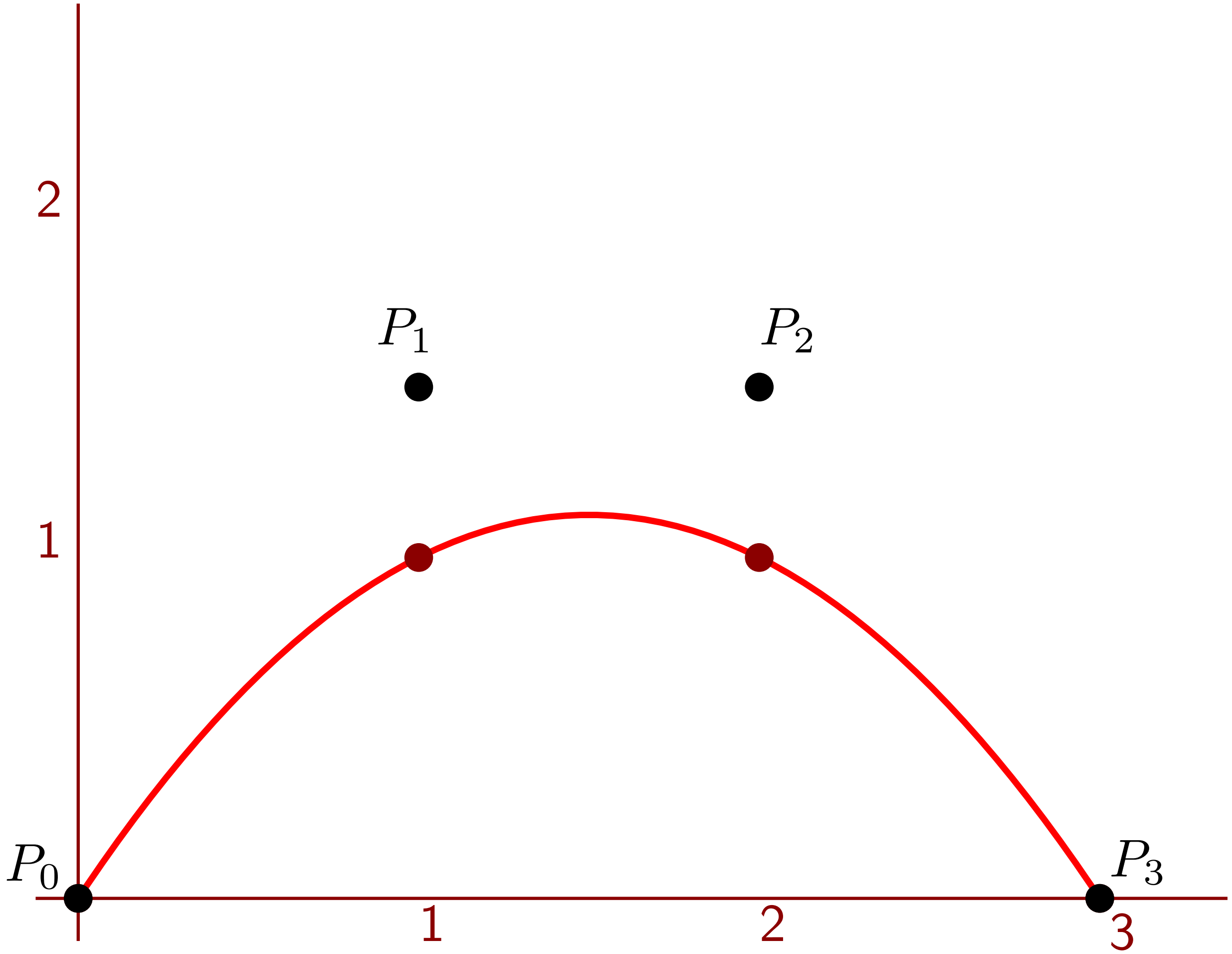

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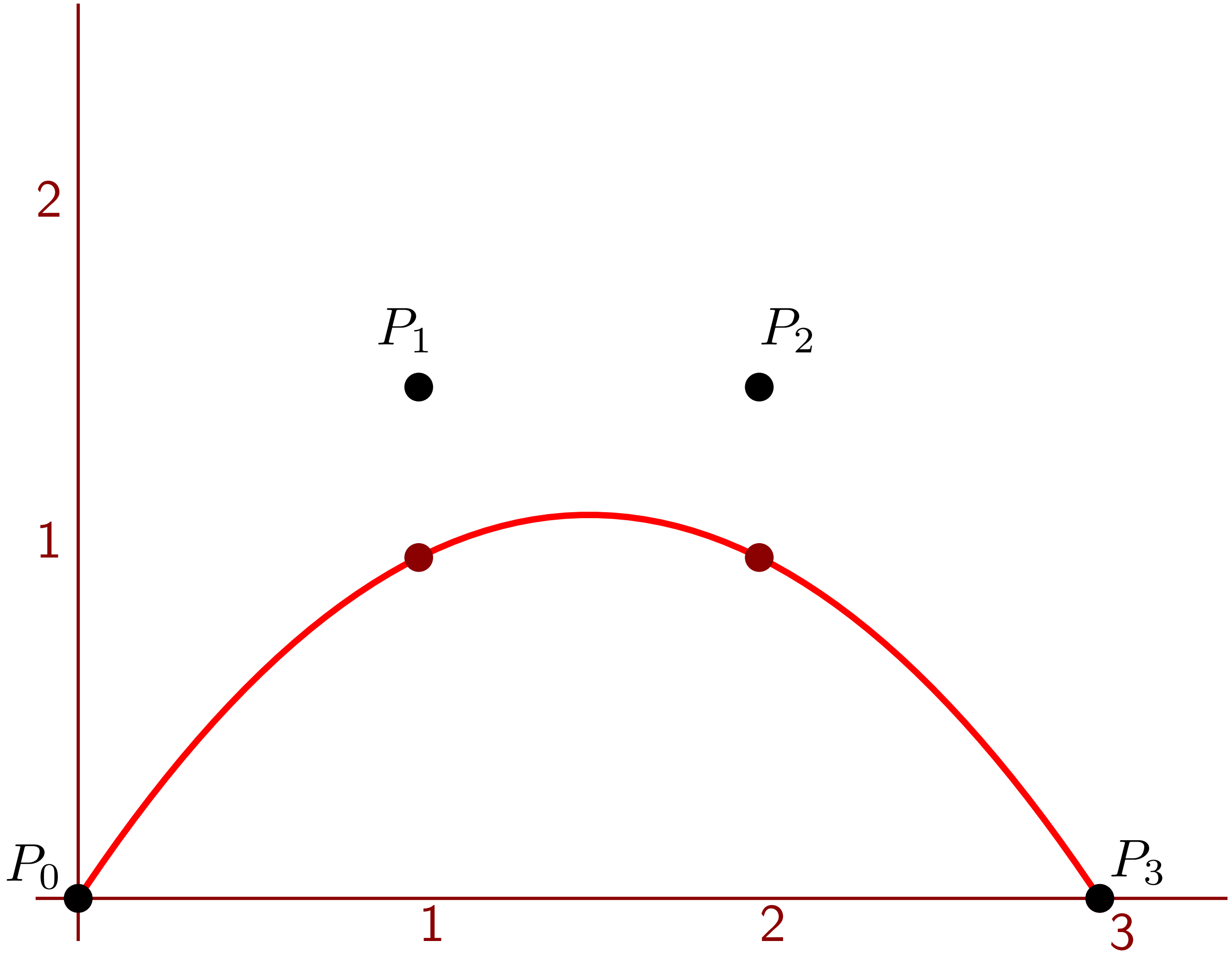
This is only one way to interpolate with Bézier curves, others are possible



Rational Bézier curves

Rational Bézier curves

Each control point has a weight, giving more flexibility to shape the curve

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$$P(t) = \sum_{i=0}^{n} P_i B_{n,i}(t)$$

Bézier curve

$$P(t) = \frac{\sum_{i=0}^{n} w_i P_i B_{n,i}(t)}{\sum_{j=0}^{n} w_j B_{n,j}(t)} = \sum_{i=0}^{n} P_i \left(\frac{w_i B_{n,i}(t)}{\sum_{j=0}^{n} w_j B_{n,j}(t)} \right)$$

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Bézier curve — Rational Bézier curve

- Weights are usually non-negative (otherwise denominator could be zero)

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Bézier curve

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Advantages: why complicate things so much?

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Bézier curve   Rational Bézier curve

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Rational Bézier curve

- Weights are usually non-negative (otherwise denominator could be zero)

Advantages: why complicate things so much?

- Invariant under projections
- It can represent conic curves (impossible with Hermite or Bézier curves) (e.g., segments of circles, ellipses, hyperbolas and parabolas)

Understanding rational Bézier curves

Effect of the weights

$$P(t) = \frac{\sum_{i=0}^{n} w_i P_i B_{n,i}(t)}{\sum_{j=0}^{n} w_j B_{n,j}(t)} = \sum_{i=0}^{n} P_i \left(\frac{w_i B_{n,i}(t)}{\sum_{j=0}^{n} w_j B_{n,j}(t)} \right)$$

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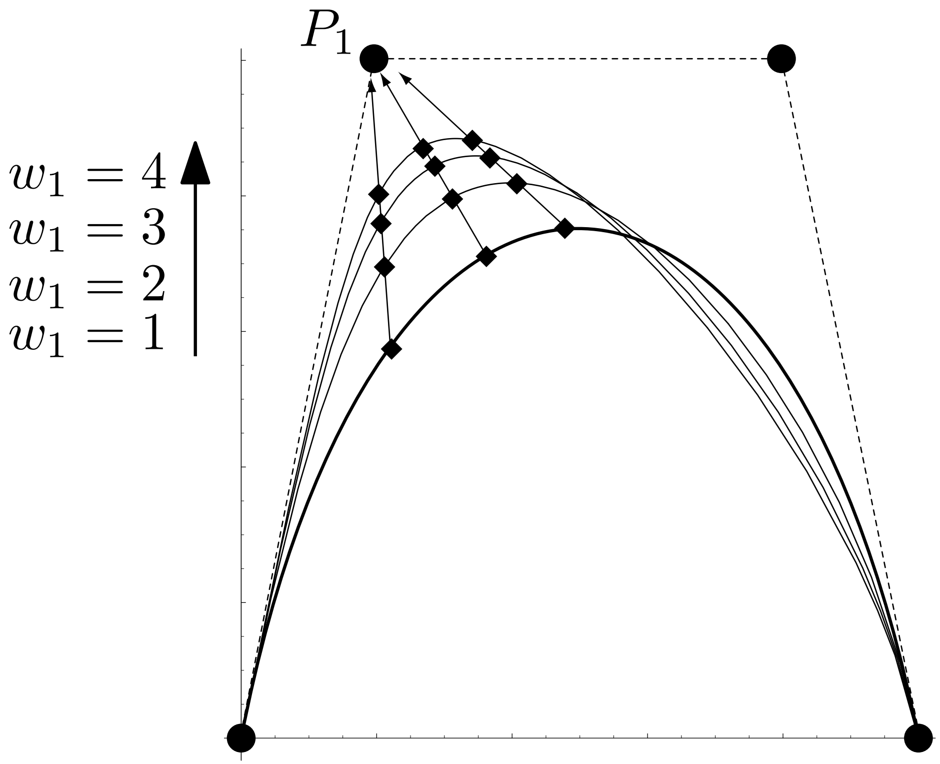
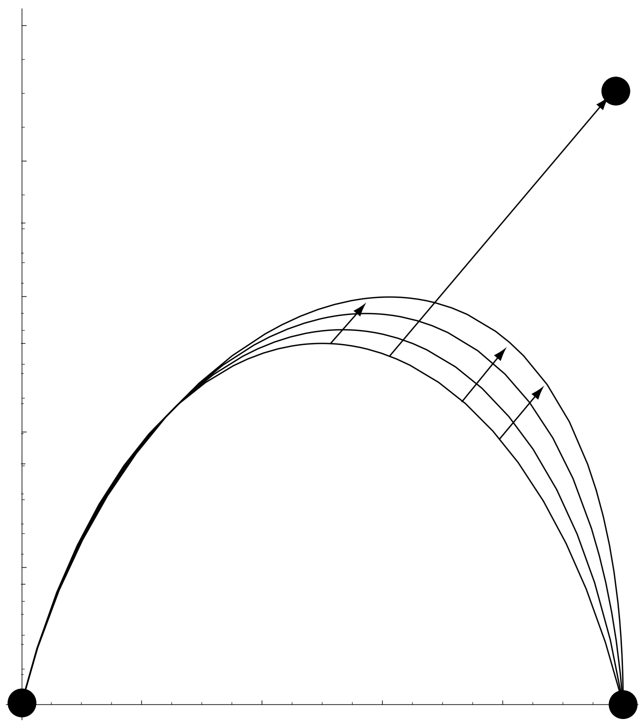
- If $w_i > 1$, the curve gets closer to P_i
- If $w_i < 1$, the curve moves away from P_i



changing weights



moving control point (weight fixed at 1)

Figure from book by Salomon (page 219)

Rational Bézier curves as curves in projective space

2D rational Bézier curve =projection of 3D nonrational Bézier curve onto 2D space!

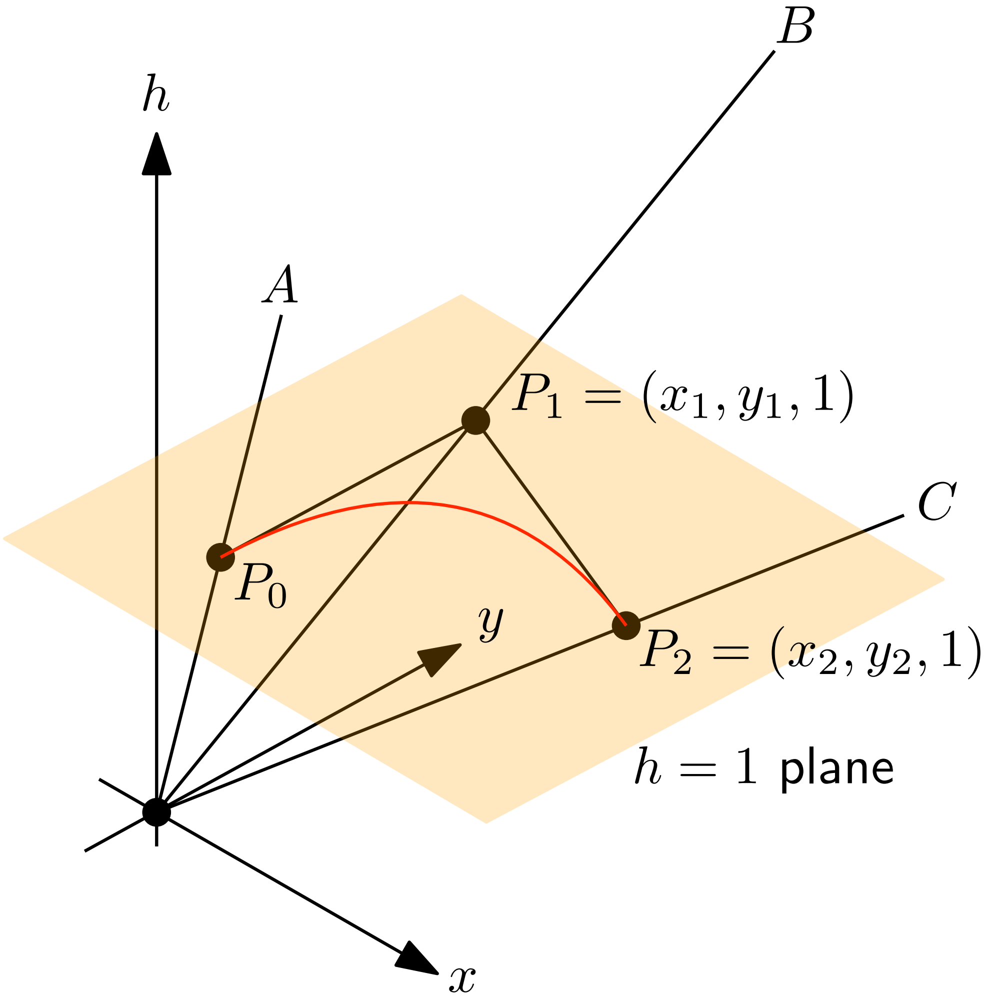


Figure adapted from book by Mortenson

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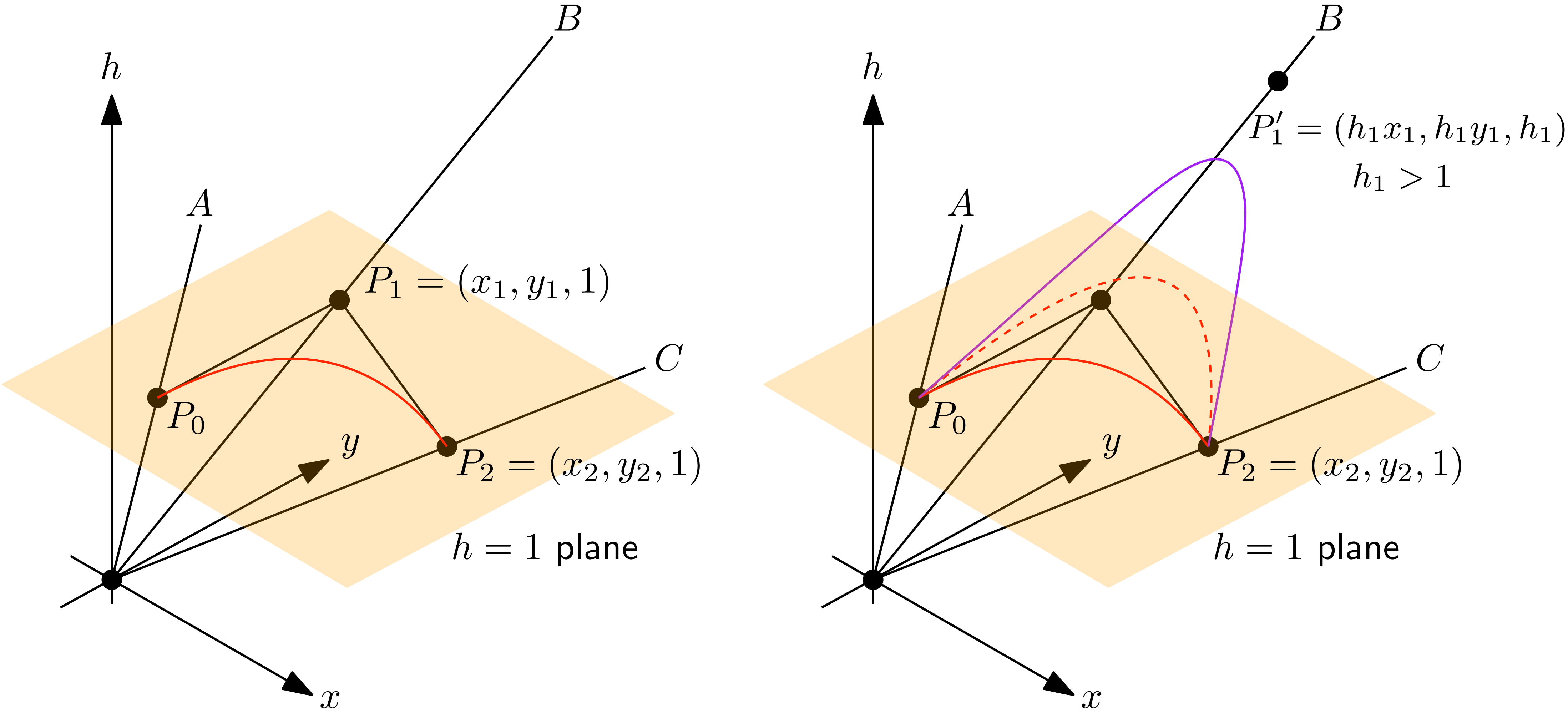

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Representing conics with rational Bézier curves

We can represent a conic curve *exactly* with a quadratic rational Bézier curve:

Representing conics with rational Bézier curves

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Theorem Consider a conic curve $C(t)$. Then there exist weights w_0, w_1, w_2 and control points P_0, P_1, P_2 such that

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Representing conics with rational Bézier curves

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Example

Take $w_0 = w_2 = 1$ and let $s = \frac{w_1}{1+w_1}$

- $s = 1/2$ produces a **parabolic** arc
- $s < 1/2$ produces an **elliptic** arc
- $s > 1/2$ produces a **hyperbolic** arc

for any three non-colinear control points