# Lecture 04: Discrete Frequency Domain Analysis ($z$-transform)

# Outline

# Lecture Contents

# Outline

# Introduction to the $z$-transform

The $z$-transform

- Transforms a digital signal to a frequency respresentation
- Used for stability analysis using the *poles* and *zeros*
- Also used for frequency analysis

# Outline

# $z$-Transform

Two $z$-transforms are common in digital signal processing. The one-sided or unilateral $z$-transform:

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

and the bilateral $z$-transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}.$$

The one-sided transform is considered here only.

## $z$-Transform *Example*

**Q.** Find the $z$-Transform of a step function:

$$x[n] = \begin{cases} 1 & \text{for} \quad n \geq 0 \\ 0 & \quad \text{elsewhere} \end{cases}$$

*A.*

$$\begin{aligned} X(z) = \sum_{n=0}^{\infty} x[n] z^{-n} &= \sum_{n=0}^{\infty} z^{-n} \\ &= (z^0 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + \ldots) \\ &= \left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \ldots\right) \end{aligned}$$

This is a geometric series of the form:

$$s = \sum_{k=0}^{\infty} a r^k = \frac{a}{1-r}$$

where $a = 1$ and $r = z^{-1}$ so that

$$X(z) = \frac{1}{1 - z^{-1}} = \frac{z}{z-1}.$$

# $z$-Transform *Example*

**Q.** Find the $z$-Transform of a square pulse:

$$x[n] = \begin{cases} 0.2 & \text{for} \quad 0 \leq n < 5 \\ 0 & \text{elsewhere} \end{cases}$$

**A.**

$$\begin{aligned} X(z) = \sum_{n=0}^{\infty} x[n] z^{-n} &= 0.2 \sum_{n=0}^{4} z^{-n} \\ &= 0.2(z^0 + z^{-1} + z^{-2} + z^{-3} + z^{-4}) \\ &= 0.2 \left( 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} \right) \end{aligned}$$

This is a geometric series of the form:

$$s = \sum_{k=0}^{n-1} a r^k = a \frac{1 - r^n}{1 - r}$$

where $a = 0.2$, $n = 5$ and $r = z^{-1}$ so that

$$X(z) = 0.2 \frac{1 - z^{-5}}{1 - z^{-1}} = 0.2 \frac{z^5 - 1}{z^5 - z^4} = 0.2 \frac{z^5 - 1}{z^4 (z - 1)}.$$

# Time delays in $z$-Transform Representations

Each $z^{-1}$ in a $z$-Transform can be considered as a single time delay. Consider the time-shifted or delayed unit impulse:

$$x[n] = \delta[n - \tau]$$

then the $z$-Transform is given by:

$$X(z) = \sum_{n=0}^{\infty} \delta[n - \tau] z^n = z^{-\tau}$$

indicating a delay of $\tau$ samples.

# The $z$-Transform and Time Delay *Example*

**Q.** Find the signal corresponding to the $z$-Transform:

$$X(z) = \frac{z}{z - 0.5}.$$

**A.** Remember the geometric series formula: $s = \sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$. Need to find the form of $X(z)$ to easily find $r$ and $a$... Dividing the numerator and denominator by $z$ gives

$$X(z) = \frac{1}{1 - 0.5z^{-1}}.$$

So that $r = 0.5z^{-1}$ and $a = 1$ then

$$\begin{aligned} X(z) = &\sum_{k=0}^{\infty} \left(0.5z^{-1}\right)^k = 1 + 0.5z^{-1} + (0.5z^{-1})^2 + (0.5z^{-1})^3 + (0.5z^{-1})^4 + ... \\ =&1 + 0.5z^{-1} + 0.25z^{-2} + 0.125z^{-1} + 0.0625z^{-4} + ... \end{aligned}$$

Remembering that each $z^{-1}$ is a delay of 1 time instance, the signal $x[n]$ is then given by the coefficients for each time instance, *i.e.* $x[0] = 1$, $x[1] = 0.5$, $x[2] = 0.25$, $x[3] = 0.125$, $x[4] = 0.0625$ *etc.*

# The $z$-Transform and Time Delay *Example 2*

**Q.** Find the signal corresponding to the $z$-Transform:

$$X(z) = \frac{z^2 - 0.2}{z(z-0.2)}$$

**A.** Remember the geometric series formula: $s = \sum_{k=0}^{n-1} ar^k = a\frac{1-r^n}{1-r}$. Need to find the form of $X(z)$ to easily find $r$, $a$ and $n$... Dividing through by $z^2$ gives

$$X(z) = \frac{1 - 0.2z^{-2}}{1 - 0.2z^{-1}}.$$

So that $r = 0.2z^{-1}$, $a = 1$ and $n = 2$ resulting in:

$$X(z) = \sum_{k=0}^{n-1} ar^k = \sum_{k=0}^{1} (0.2z^{-1})^k = 1 + 0.2z^{-1}.$$

Therefore the original signal, $x[n]$ is given by $x[0] = 1$ and $x[1] = 0.2$.

# Outline

# The Inverse $z$-Transform

The inverse $z$-Transform $\mathcal{Z}^{-1}(X(z))$ is given by

$$x[n] = \mathcal{Z}^{-1}(X(z)) = \frac{1}{2\pi j}\int X(z)z^{n-1}\mathrm{d}z$$

- The inverse $z$-Transform is **not usually computed directly**.
- Instead the $z$-Transform is split into parts using partial fractions
- And then the inverse $z$-Transform of the parts are found using a table of $z$-Transform pairs.

# Table of (unilateral) $z$-Transform Pairs

*Lynn* and *Fuerst* give the following table of $z$-Transform pairs:

| **Signal** $x[n]$ | $z$-**Transform** $X(z)$ |
|---|---|
| $\delta[n]$ | $1$ |
| $u[n] = \begin{cases} 1 & \text{for} \quad n \geq 0 \\ 0 & \quad \text{elsewhere} \end{cases}$ | $\frac{z}{z-1}$ |
| $r[n] = \begin{cases} n & \text{for} \quad n \geq 0 \\ 0 & \quad \text{elsewhere} \end{cases}$ | $\frac{z}{(z-1)^2}$ |
| $a^n u[n]$ | $\frac{z}{z-a}$ |
| $(1-a^n)u[n]$ | $\frac{z(1-a)}{(z-a)(z-1)}$ |
| $\cos(n\Omega_0)u[n]$ | $\frac{z(z-\cos(\Omega_0))}{z^2-2z\cos(\Omega_0)+1}$ |
| $\sin(n\Omega_0)u[n]$ | $\frac{z\sin(\Omega_0)}{z^2-2z\cos(\Omega_0)+1}$ |
| $a^n \sin(n\Omega_0)u[n]$ | $\frac{az\sin(\Omega_0)}{z^2-2az\cos(\Omega_0)+a^2}$ |

The $z$-Transform representation has therefore to be separated using partial fractions into parts, each of the form of the expressions on the right hand side of the above table.

# Method of Partial Fractions *Example*

**Q.** Decompose the following function into partial fractions:

$$\frac{1}{(z+3)(z-2)}.$$

**A.** Let

$$\frac{1}{(z+3)(z-2)} = \frac{A}{z+3} + \frac{B}{z-2}.$$

Then

$$A(z-2) + B(z+3) = 1.$$

So that

$$Az - 2A + Bz + 3B = 1$$

$$z(A+B) - 2A + 3B = 1$$

Therefore $z(A+B) = 0 \Rightarrow A = -B$ and $-2A + 3B = 1$ so that $-2A - 3A = 1$ giving $A = -\frac{1}{5}$ and $B = \frac{1}{5}$.
**Check:**

$$\frac{A}{z+3} + \frac{B}{z-2} = \frac{-\frac{1}{5}}{z+3} + \frac{\frac{1}{5}}{z-2} = \frac{\frac{1}{5}(5)}{(z+3)(z+2)} = \frac{1}{(z+3)(z+2)}$$

# Method of Partial Fractions: Cover up method *Example*

**Q.** Decompose the following function into partial fractions:

$$\frac{z}{(z+3)(z-2)}.$$

**A.** Let $\frac{z}{(z+3)(z-2)} = \frac{A}{z+3} + \frac{B}{z-2}$. To find $A \Rightarrow z+3=0 \Rightarrow z=-3$.

$$A = \left.\frac{z}{(z+3)(z-2)}\right|_{z=-3} = \frac{-3}{-3-2} = \frac{3}{5}.$$

To find $B \Rightarrow z-2=0 \Rightarrow z=2$.

$$B = \left.\frac{z}{(z+3)(z-2)}\right|_{z=2} = \frac{2}{2+3} = \frac{2}{5}.$$

Hence

$$\frac{z}{(z+3)(z-2)} = \frac{\frac{3}{5}}{z+3} + \frac{\frac{2}{5}}{z-2}.$$

**Check:**

$$\frac{\frac{3}{5}}{z+3} + \frac{\frac{2}{5}}{z-2} = \frac{\frac{3}{5}(z-2) + \frac{2}{5}(z+3)}{(z+3)(z-2)} = \frac{\frac{3}{5}z - \frac{6}{5} + \frac{2}{5}z + \frac{6}{5}}{(z+3)(z-2)} = \frac{z}{(z+3)(z-2)}.$$

# Inverse $z$-Transform *Example*

**Q.** Find the inverse $z$-Transform of:

$$X(z) = \frac{1}{(z+3)(z-2)} = \frac{1}{5}\left(\frac{1}{z-2} - \frac{1}{z+3}\right). \tag{1}$$

**A.** Re-writing (1) to

$$X(z) = \frac{z^{-1}}{5}\left(\frac{z}{z-2} - \frac{z}{z+3}\right). \tag{2}$$

Enables us to find inverse $z$-Transforms for the two terms inside the brackets:

$$\mathcal{Z}^{-1}\left(\frac{z}{z-2}\right) = 2^n u[n]$$

and

$$\mathcal{Z}^{-1}\left(-\frac{z}{z+3}\right) = -((-3)^n)u[n].$$

The two terms are multiplied by $z^{-1}$ which is equivalent to a time delay hence the final signal is given by:

$$x[n] = \mathcal{Z}^{-1}(X(z)) = \frac{1}{5}\left(2^{(n-1)}u[n-1] - ((-3)^{(n-1)})u[n-1]\right).$$

# Inverse $z$-Transform *Example*

**Q.** Find the inverse $z$-Transform of:

$$X(z) = \frac{z}{(z+3)(z-2)}.$$

**A.** *From earlier the partial fraction expansion is given by:* $\frac{z}{(z+3)(z-2)} = \frac{\frac{3}{5}}{z+3} + \frac{\frac{2}{5}}{z-2}$.

**(i)** However it is more convenient if we divide both sides by $z$ first. Hence

$$\frac{X(z)}{z} = \frac{1}{(z+3)(z-2)}.$$

The Right Hand Side (RHS) has partial fractions (see earlier slide):

$$\frac{X(z)}{z} = \frac{-\frac{1}{5}}{z+3} + \frac{\frac{1}{5}}{z-2}.$$

Multiplying both sides by $z$ then gives:

$$X(z) = \frac{1}{5}\left(\frac{-z}{z+3} + \frac{z}{z-2}\right).$$

**(ii)** We saw earlier:

$$\mathcal{Z}^{-1}\left(\frac{z}{z-2}\right) = 2^n u[n]$$

and

$$\mathcal{Z}^{-1}\left(-\frac{z}{z+3}\right) = -((-3)^n)u[n]$$

so that

$$\begin{aligned} x[n] =& \mathcal{Z}^{-1}(X(z)) \\ =& \frac{1}{5}\left(2^{(n)}u[n] - ((-3)^{(n)})u[n]\right). \end{aligned}$$

# Inverse $z$-Transform: Algebraic Long Division

The numerator and the denominator of the $z$-Transform can be divided using algebraic long division to find coefficients that correspond to the original signal.

*Example*

**Q.** Given $H(z) = \frac{z}{(z-1)(z+2)} = \frac{z}{z^2+z-2}$, determine the coefficients.

**A.** Via algebraic or polynomial long division:

$$
\begin{array}{r l}
 & z^{-1} - z^{-2} + 3z^{-3} - 5z^{-4} \ldots \\
z^2 + z - 2 \, \lceil & \overline{z \qquad\qquad\qquad\qquad} \\
 & \underline{z \;\; +1 \;\; -2z^{-1}} \\
 & \quad -1 \;\; +2z^{-1} \\
 & \quad \underline{-1 \;\; -z^{-1} + 2z^{-2}} \\
 & \qquad\quad 3z^{-1} - 2z^{-2} \\
 & \qquad\quad \underline{3z^{-1} + 3z^{-2} - 5z^{-3}} \\
 & \qquad\qquad\quad -5z^{-2} + 5z^{-3} \\
 & \qquad\qquad\quad \ldots
\end{array}
$$

So the coefficients of the original signal are given by:
$x[0] = 0$, $x[1] = 1$, $x[2] = -1$, $x[3] = 3$, $x[4] = -5$, *etc.*

This can be checked by performing the inverse $z$-Transform on $H(z)$.

Expansion with partial fractions gives:
$H(z) = \frac{1}{3}\left(\frac{z}{z-1} - \frac{z}{z+2}\right)$

Inverse $z$-Transform: $x[n] = \mathcal{Z}^{-1}(H(z)) = \frac{1}{3}\left(u[n] - (-2)^n u[n]\right)$

Then $x[0] = \frac{1}{3}(1-1) = 0$,
$x[1] = \frac{1}{3}(1+2) = 1$,
$x[2] = \frac{1}{3}(1-4) = -1$,
$x[3] = \frac{1}{3}(1+8) = 3$,
$x[4] = \frac{1}{3}(1-15) = -5$, etc.

This confirms the long division result.

# Inverse $z$-Transform *Example*

**Q.** Find the inverse $z$-Transform of:

$$X(z) = \frac{0.5z}{z^2 - z + 0.5} \tag{3}$$

**A.** The table of $z$-Transform pairs has the following definition:

$$\mathcal{Z}^{-1}\left(\frac{az\sin(\Omega_0)}{(z^2 - 2az\cos(\Omega_0) + a^2)}\right) = a^n \sin(n\Omega_0)u[n]. \tag{4}$$

Therefore we can try to equate the terms inside (4) and (3).

$$\text{In the numerator: } a\sin(\Omega_0) = 0.5, \text{ and in the denominator } a^2 = 0.5$$

$$\Rightarrow a = \sqrt{0.5}, \text{ then } \sin(\Omega_0) = 0.5/\sqrt{0.5}, \Rightarrow \Omega_0 = \sin^{-1}(0.5/\sqrt{0.5}) = \frac{\pi}{4}.$$

We can therefore *plug* these values into the result of (4) to find the inverse $z$-Transform of (3):

$$x[n] = \mathcal{Z}^{-1}\left(\frac{0.5z}{z^2 - z + 0.5}\right) = (\sqrt{0.5})^n \sin(n\pi/4)u[n].$$

# Outline

# BIBO Stability

- A linear system is said to be stable if it has:
  - **A Bounded Output for A Bounded Input**
- Bounded means the signal does not exceed a particular value.
- System stability is expressed using a function of the impulse response $h[n]$ of the system:

$$\sum_{-\infty}^{\infty} |h[n]| < \infty \tag{5}$$

  where $|h[n]|$ is the absolute value of $h[n]$. *i.e.* $|-h[n]| = |h[n]|$.
- Equation (5) states that the sum across all values of the impulse function should be smaller than infinity ($\infty$).
- This ensures that the system is bounded and will not be larger than infinity for some input
- If equation (5) is true then the system can be described as being BIBO **stable**.
- A system is not usually very useful if it goes to $\pm$infinity.

# $z$-Transform and Stability

- The $z$-Transform can be used to determine if a system is stable.
- The $z$-Transform results in a rational function consisting of a numerator $N(z)$ and a denominator $D(z)$

$$X(z) = \frac{N(z)}{D(z)}$$

- $X(z)$ can be
  - A system input
  - A system output
  - A system transfer function
- The stability of $X(z)$ can be found by the roots of $N(z)$ and $D(z)$:

$$X(z) = \frac{N(z)}{D(z)} = \frac{K(z - z_1)(z - z_2)(z - z_3)...}{(z - p_1)(z - p_2)(z - p_3)...}$$

# $z$-Transform and Stability

$$X(z) = \frac{N(z)}{D(z)} = \frac{K(z-z_1)(z-z_2)(z-z_3)...}{(z-p_1)(z-p_2)(z-p_3)...}$$

- The roots of the numerator are $z_1$, $z_2$, $z_3$... known as **zeros**
- The roots of the denominator are $p_1$, $p_2$, $p_3$... known as **poles**
- The **zeros** are values of $z$ that make $X(z) \to 0$
  - *e.g.* if $z = z_1$ then

$$\begin{aligned} X(z=z_1) = &\frac{N(z=z_1)}{D(z=z_1)} = \frac{K(z_1-z_1)(z_1-z_2)(z_1-z_3)...}{(z_1-p_1)(z_1-p_2)(z_1-p_3)...} \\ = &\frac{K \times 0 \times (z_1-z_2)(z_1-z_3)...}{(z_1-p_1)(z_1-p_2)(z_1-p_3)...} = 0 \end{aligned}$$

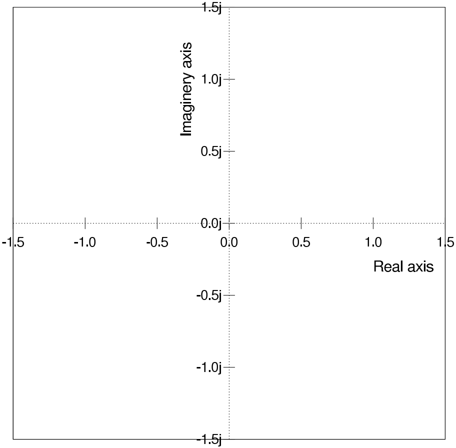
- The **poles** are values of $z$ that make $X(z) \to \infty$
  - *e.g.* if $z = p_1$ then

$$\begin{aligned} X(z=p_1) = &\frac{N_1}{D_1} = \frac{K(p_1-z_1)(p_1-z_2)(p_1-z_3)...}{(p_1-p_1)(p_1-p_2)(p_1-p_3)...} \\ = &\frac{K(p_1-z_1)(p_1-z_2)(p_1-z_3)...}{0 \times (p_1-p_2)(p_1-p_3)...} = \frac{K...}{0} = \infty \end{aligned}$$

# $z$-Transform and Stability: $z$-Plane

A $z$-Transform can be represented *graphically* with the $z$-Plane.

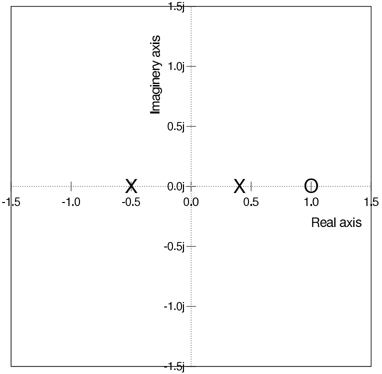
- The $z$-Plane is complex $(a + jb)$ where $j = \sqrt{-1}$
- The vertical axis ($\uparrow$) is imaginary ($b$)
- The horizontal axis ($\rightarrow$) is real ($a$)
- The $z$-Plane is also known as an Argand diagram



# $z$-Transform and Stability: $z$-Plane

$$X(z) = \frac{N(z)}{D(z)} = \frac{K(z-z_1)(z-z_2)(z-z_3)\dots}{(z-p_1)(z-p_2)(z-p_3)\dots}$$

- Each **zero**: $z_1$, $z_2$, $z_3$, ... is represented by a **circle**: **O**
- Each **pole**: $p_1$, $p_2$, $p_3$, ... is represented by a **cross**: **X**
- *e.g.* $X(z) = \frac{z-1}{(z+0.5)(z-0.4)}$ then
  - $z_1 = 1 + 0j$
  - $p_1 = -0.5 + 0j$
  - $p_2 = 0.4 + 0j$



# $z$-Transform and Stability: $z$-Plane

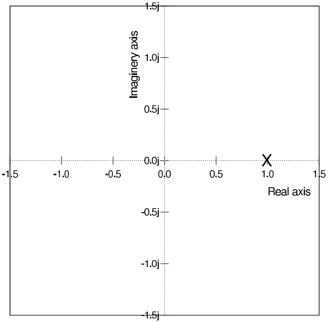
Stability is determined by the location of the **poles** in the $z$-plane.
*Example*

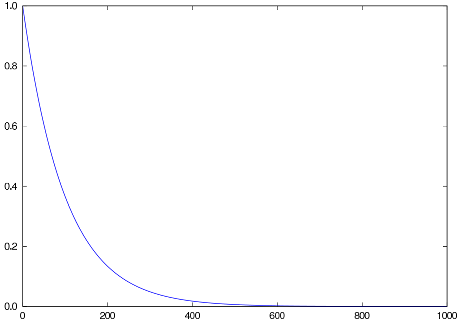
$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{(z-a)}$$

- From the table of $z$-Transform pairs:
- $\mathcal{Z}^{-1}\left(\frac{z}{z-a}\right) = a^n u[n]$
- $\therefore$ let $H(z) = z^{-1}\left(\frac{z}{z-a}\right)$
- $z^{-1}$ is a unit delay hence:
- $x[n] = a^{n-1}u[n-1]$
- So that $x[0] = 0$, $x[1] = 1$, $x[2] = a$, $x[3] = a^2$, $x[4] = a^3$ *etc.*

# $z$-Transform and Stability: $z$-Plane

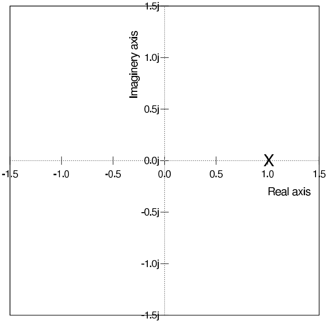
- $x[n] = a^{n-1}u[n-1]$
- $x[0] = 0,\ x[1] = 1,\ x[2] = a,\ x[3] = a^2,\ x[4] = a^3$ *etc.*
- If $a = 0.99$
- Decreasing and tending to zero ($x[n] \to 0$) when $a < 1$

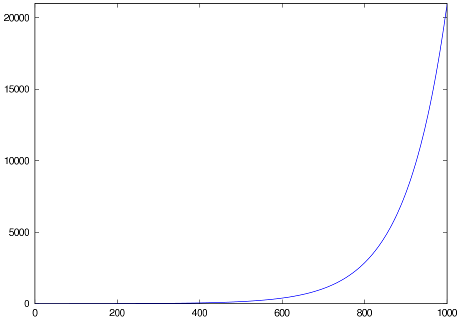


# $z$-Transform and Stability: $z$-Plane

- $x[n] = a^{n-1}u[n-1]$
- $x[0] = 0$, $x[1] = 1$, $x[2] = a$, $x[3] = a^2$, $x[4] = a^3$ *etc.*
- If $a = 1.01$
- Increasing and tending to infinity $(x[n] \to \infty)$ when $a > 1$



# $z$-Transform and Stability: $z$-Plane

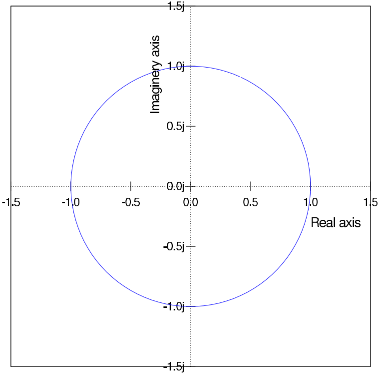
$x[n] = a^{n-1}u[n-1]$

- Decreasing and tending to zero ($x[n] \to 0$) when $a < 1$
- Increasing and tending to infinity ($x[n] \to \infty$) when $a > 1$

These observations are true more generally:

$$X(z) = \frac{N(z)}{D(z)} = \frac{K(z-z_1)(z-z_2)(z-z_3)...}{(z-p_1)(z-p_2)(z-p_3)...}$$

*If the **magnitude** of any pole ($p_i$) is greater than $1$ then the system will tend to infinity.*




- A unit circle is drawn on the $z$-plane.
- If any pole is outside of the unit circle then the system is **not stable**.

# $z$-Plane and Stability: *Magnitude Example*

*If the* **magnitude** *of any pole* $(p_i)$ *is greater than* $1$ *then the system will tend to infinity.*
**Q.** Determine whether the following system is stable:

$$H(z) = \frac{1}{(z - 0.7 + 0.8j)(z - 0.7 - 0.8j)}$$

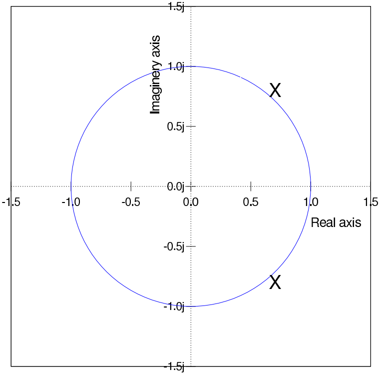
**A.** The system has two poles at:

$$p_1 = 0.7 - 0.8j \text{ and } p_2 = 0.7 + 0.8j.$$

The distance from the origin of these poles is given by the magnitude:

$$r = \sqrt{0.7^2 + 0.8^2} = 1.063 > 1.$$

These poles are beyond the unit circle, therefore this system is **not stable**.



# $z$-Plane and Stability: *Magnitude Example*

**Q.** Determine whether the following system is stable:

$$H(z) = \frac{1}{(z - 0.5 - 0.5j)(z - 0.5 + 0.5j)}$$

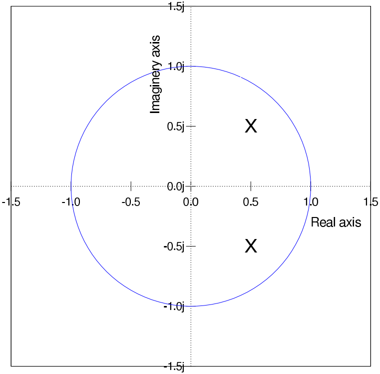
**A.** The system has two poles at:

$p_1 = 0.5 + 0.5j$ and $p_2 = 0.5 - 0.5j$

The distance from the origin of these poles is given by the magnitude:

$$r = \sqrt{0.5^2 + 0.5^2} = 0.707 < 1.$$

These poles are inside the unit circle, therefore this system is **stable**.



# $z$-Plane and The Zeros

The $z$-Plane zeros:

- Do **not** determine stability:
  - They can be located anywhere in the $z$-Plane without directly affecting stability
- If a **zero** is located at the origin then there is a time advance of a signal
- If there are more zeros than poles then the system starts before $n = 0$ and is therefore **not causal**
- It is usually desirable to have the same number of poles and zeros in a system to:
  - Ensure minimum delay or time lag
  - Ensure the system is causal

# $z$-Plane and The Zeros *Example*

The inverse $z$-Transform of:

$$H(z) = \frac{1}{z - 0.4} = z^{-1}\left(\frac{z}{z - 0.4}\right)$$

is given by (using the table of $z$-Transform pairs):

$$x[n] = 0.4^{n-1}u[n-1],$$

which has a delay of 1 time interval. If we provide $H(z)$ with a zero at the origin (*i.e.* $z_1 = 0$) so that:

$$H(z) = \frac{z - z_1}{z - 0.4} = \frac{z}{z - 0.4}$$

then the inverse $z$-Transform is given by:

$$x[n] = 0.4^{n}u[n],$$

which has **no time delay**.

# Lecture Summary

Today's lecture has covered:

- The $z$-Transform
- The inverse $z$-Transform
- Stability analysis using the $z$-plane
- Partial fractions
- The $z$-Transform and time delays