Outline

Lecture Contents

Outline

Frequency-Domain Analysis

- Point 1:
  - Sinusoidal and exponential signals occur everywhere. Most signals can be decomposed into component frequencies;
  - Response of a LTI processor to each frequency component can only alter the amplitude and phase, not the frequency;
  - The output can then be found with the *Principle of Superposition*.
- Point 2:
  - If an input signal has a frequency spectrum and the LTI processor has a frequency response then the output signal spectrum is found by multiplication;
  - Simpler than time-domain convolution.
- Point 3:
  - Frequency response is a typical requirement for DSP algorithms
  - Such as low-pass, bandstop or bandpass filters.

Continuous-time Fourier Analysis

- A signal can be decomposed into sinusoidal components;
- Even functions are composed of only cosine functions;
- Odd functions are composed of only sine functions;
- A finite number of frequency components can be used to approximate a signal;
- Frequency components of a periodic signal are harmonically related with discrete spectral lines, *line spectrum*, described by Fourier series;
- Fourier series can be expressed in exponential form;
- Aperiodic (non-repetitive) signals are decomposed into non-harmonically related sinusoids. Resulting spectrum is continuous, described by Fourier Transform;
- The inverse Fourier Transform reverses the process;
- The frequency response of an LTI system specifies how each sinusoidal or exponential component is modified in amplitude and phase;
- Multiplication of the frequency response along with the input signal spectrum gives the output signal spectrum.

Discrete-time Fourier Analysis

- Parallel set of Fourier techniques applicable to digital signals
- Two representations:
  - A discrete-time Fourier Series, applicable to periodic digital signals
  - A discrete-time Fourier Transform, applicable to aperiodic digital signals and LTI processors

Outline

Spectra of Periodic Digital Signals

- Periodic digital signal $x[n]$ can be represented by Fourier Series
- Line spectrum coefficients can be found using the *analysis equation*:

$$a[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] \exp\left(\frac{-j2\pi kn}{N}\right)$$

  where $a[k]$ is the k spectral component or harmonic and N is the number of sample values in each period of the signal.
- Regeneration of the original signal $x[n]$ can be found using the *synthesis equation*:

$$x[n] = \sum_{k=0}^{N-1} a[k] \exp\left(\frac{j2\pi kn}{N}\right).$$

Finding Line Spectra *with a computer*

- Remember Euler's identity:

$$\exp\left(\frac{-j2\pi kn}{N}\right) = \underbrace{\overbrace{\cos\left(\frac{2\pi kn}{N}\right)}^{\text{real part}}}_{\mathrm{Re}(\cdot)} \underbrace{\overbrace{-j\sin\left(\frac{2\pi kn}{N}\right)}^{\text{imaginary part}}}_{\mathrm{Im}(\cdot)}$$

$$\mathrm{Re}\left(\exp\left(\frac{-j2\pi kn}{N}\right)\right) = \cos\left(\frac{2\pi kn}{N}\right)$$

$$\mathrm{Im}\left(\exp\left(\frac{-j2\pi kn}{N}\right)\right) = -j\sin\left(\frac{2\pi kn}{N}\right)$$

- The real $\mathrm{Re}(a[k])$ and imaginary $\mathrm{Im}(a[k])$ components can be calculated individually

Finding Line Spectra *with a computer*

$$a[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\exp\left(\frac{-j2\pi kn}{N}\right) = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\left(\cos\left(\frac{2\pi kn}{N}\right) - j\sin\left(\frac{2\pi kn}{N}\right)\right)$$

Two loops:

- A loop over n
- A loop over k

Pseudo-code:

1. Given $x[n]$ and N:
2. Create $a[k]$ size N and set all elements to zero $a[k] = 0$
3. For k=0 to N-1 % *outer loop*
   - 3.1 For n=0 to N-1 % *inner loop*
     - 3.1.1 $\mathrm{Re}(a[k]) = a[k] + x[n] \times \cos(2\pi kn/N)$ % *real part*
     - 3.1.2 $\mathrm{Im}(a[k]) = a[k] - x[n] \times \sin(2\pi kn/N)$ % *imaginary part*
   - 3.2 End For
   - 3.3 Let $\mathrm{Re}(a[k]) = \mathrm{Re}(a[k])/N$ and Let $\mathrm{Im}(a[k]) = \mathrm{Im}(a[k])/N$
4. End For

Finding Line Spectra *Simple Example - manually*

Let $N = 2$ and $x[0] = 7$, $x[1] = 1$. Find $a[k]$. To start find the real part of $a[k]$...

- *To start* Let $n = 0$ and $k = 0$, then

$$\mathrm{Re}(a[0]) = \mathrm{Re}(a[0]) + x[0] \times \cos(2\pi kn/2) = 0 + 7 \times \cos(2\pi 0 \times 0/2) = 7$$

- Increment n, so $n = 1$, then

$$\mathrm{Re}(a[0]) = \mathrm{Re}(a[0]) + x[1] \times \cos(2\pi kn/2) = 7 + 1 \times \cos(2\pi 0 \times 1/2) = 8$$

- Divide by N: $\mathrm{Re}(a[0]) = \mathrm{Re}(a[0])/N = 4$
- *If we increment n any more then it will be larger than $N - 1$ so*
  Let $n = 0$ and increment k so that $k = 1$, then

$$\mathrm{Re}(a[1]) = \mathrm{Re}(a[1]) + x[0] \times \cos(2\pi kn/2) = 0 + 7 \times \cos(2\pi 1 \times 0/2) = 7$$

- Increment n, so $n = 1$, then

$$\mathrm{Re}(a[1]) = \mathrm{Re}(a[1]) + x[1] \times \cos(2\pi kn/2) = 7 + 1 \times \cos(2\pi 1 \times 1/2) = 6$$

- Divide by N: $\mathrm{Re}(a[0]) = \mathrm{Re}(a[0])/N = 3$

So the real part of $a[k]$ is given by $\mathrm{Re}(a[0]) = 4$ and $\mathrm{Re}(a[1]) = 3$.

Finding Line Spectra *Simple Example - manually*

Finding the imaginary part of $a[k]$...

- *To start* Let $n = 0$ and $k = 0$, then

$$\mathrm{Im}(a[0]) = \mathrm{Im}(a[0]) - x[0] \times \sin(2\pi kn/2) = 0 + 7 \times \sin(2\pi 0 \times 0/2) = 0$$

- Increment n, so $n = 1$, then

$$\mathrm{Im}(a[0]) = \mathrm{Im}(a[0]) - x[1] \times \sin(2\pi kn/2) = 0 + 1 \times \sin(2\pi 0 \times 1/2) = 0$$

- Divide by N: $\mathrm{Im}(a[0]) = \mathrm{Im}(a[0])/N = 0$
- *If we increment n any more then it will be larger than $N-1$ so* Let $n = 0$ and increment k so that $k = 1$, then

$$\mathrm{Im}(a[1]) = \mathrm{Im}(a[1]) - x[0] \times \sin(2\pi kn/2) = 0 + 7 \times \sin(2\pi 1 \times 0/2) = 0$$

- Increment n, so $n = 1$, then

$$\mathrm{Im}(a[1]) = \mathrm{Im}(a[1]) - x[1] \times \sin(2\pi kn/2) = 0 + 1 \times \sin(2\pi 1 \times 1/2) = 0$$

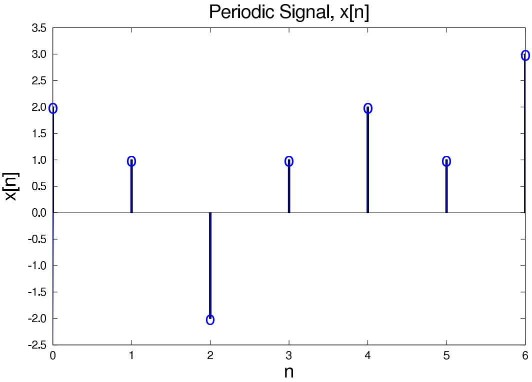
- Divide by N: $\mathrm{Im}(a[0]) = \mathrm{Im}(a[0])/N = 0$

So the imaginary part of $a[k]$ is given by $\mathrm{Im}(a[0]) = 0$ and $\mathrm{Im}(a[1]) = 0$.

Finding Line Spectra *Another Example - but by computer*

This time let $N = 7$ so that $x = \begin{pmatrix} +2 \\ +1 \\ -2 \\ +3 \\ -1 \\ -1 \\ +1 \end{pmatrix} = (+2 \ +1 \ -2 \ +3 \ -1 \ -1 \ +1)^{\mathrm{T}}$ or

$x[0] = +2$, $x[1] = +1$, ..., $x[6] = +1$. A plot of this *periodic* signal is given by



Finding Line Spectra *Another Example - but by computer*

This time let $N = 7$ so that $x = \begin{pmatrix} +2 \\ +1 \\ -2 \\ +3 \\ -1 \\ -1 \\ +1 \end{pmatrix} = (+2\ \ +1\ \ -2\ \ +3\ \ -1\ \ -1\ \ +1)^{\mathrm{T}}$ or

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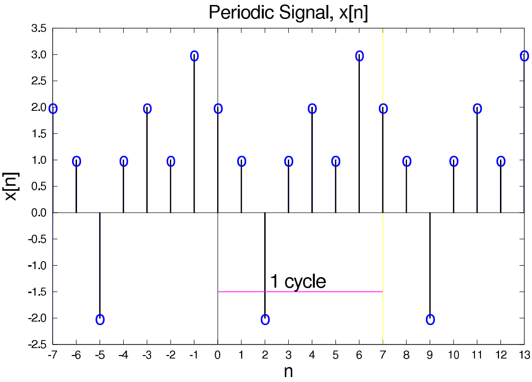
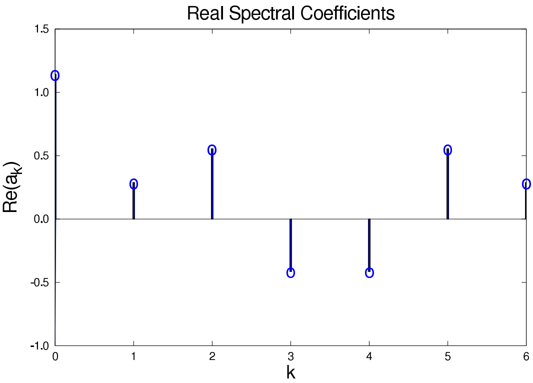


Finding Line Spectra *Another Example - but by computer*

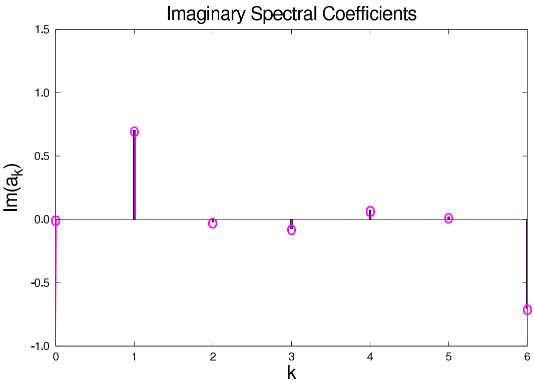
The line spectra in this case are found by a computer program using the algorithm described earlier.



Notice (for the real coefficients) the mirror image, where $a[1] = a[6]$ and $a[2] = a[5]$ *etc.*

Finding Line Spectra *Another Example - but by computer*

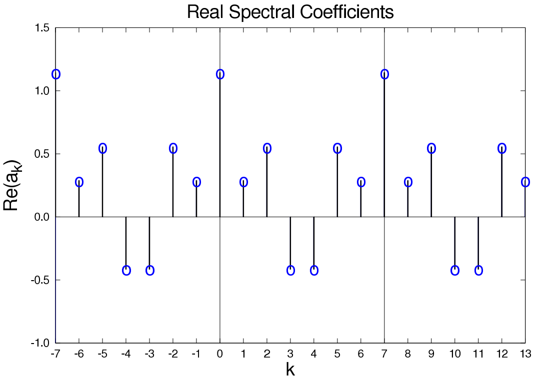
The line spectra in this case are found by a computer program using the algorithm described earlier.



Notice (for the imaginary coefficients) a similar effect, except for a change of sign, *e.g.* $a[1] = -a[6]$.

Finding Line Spectra *Another Example - but by computer*

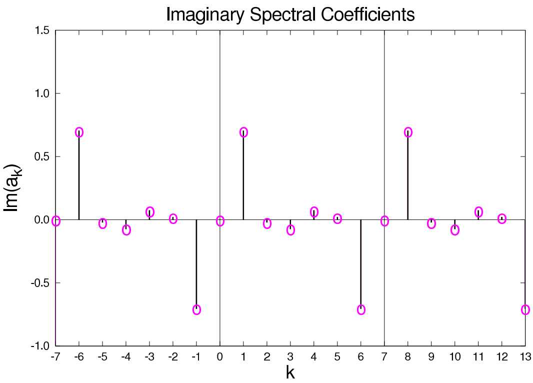
The line spectra in this case are found by a computer program using the algorithm described earlier.



The line spectra are also periodic. The line spectra have the same periodicity as the original signal.

Finding Line Spectra *Another Example - but by computer*

The line spectra in this case are found by a computer program using the algorithm described earlier.



The line spectra are also periodic. The line spectra have the same periodicity as the original signal.

Magnitude and Phase of Line Spectra

- The real and imaginary components are often quoted in terms of magnitude and phase, or more commonly magnitude only.
- The magnitude can be calculated with:

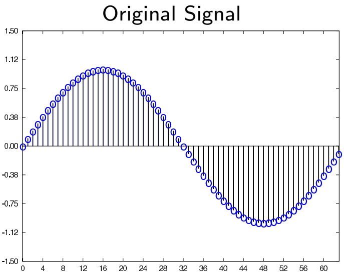
$$\mathrm{Mag}(a[k]) = \sqrt{\mathrm{Re}(a[k])^2 + \mathrm{Im}(a[k])^2}$$

- and the phase with:

$$\phi(a[k]) = \tan^{-1}\left(\frac{\mathrm{Im}(a[k])}{\mathrm{Re}(a[k])}\right).$$

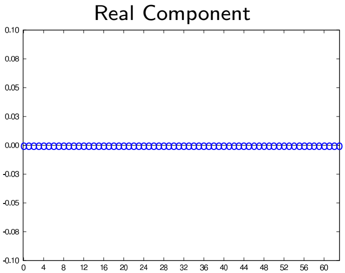
- The magnitude indicates the relative strength of the signal at different frequencies;
- The phase indicates the phase angle of the signal at different frequencies;
- The strength of the signal at different frequencies is often the more important information for a description of a system or a signal.

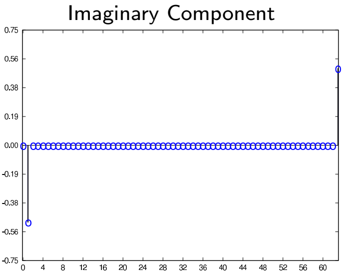
Magnitude and Phase of Line Spectra *Example*



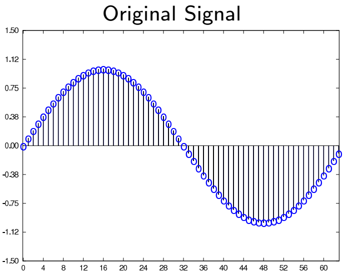
$$x[n] = \sin(2\pi n/64)$$

The real and imaginary components contain separate information about the frequency content of the signal. The sine function is not present in the real part.



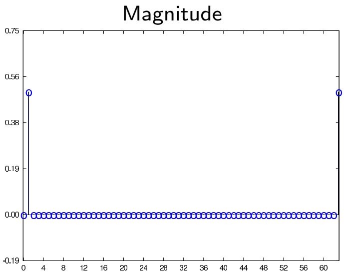


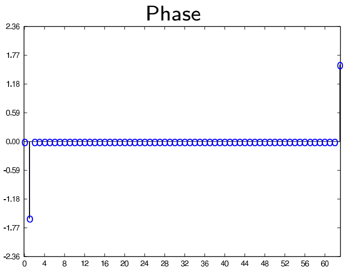
Magnitude and Phase of Line Spectra *Example*



$$x[n] = \sin(2\pi n/64)$$

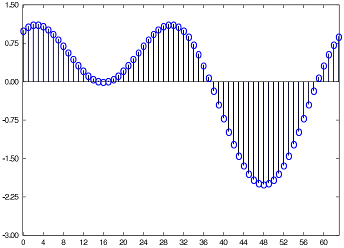
The magnitude provides a convenient overview of the frequency content of the signal.





Magnitude and Phase of Line Spectra *Example*

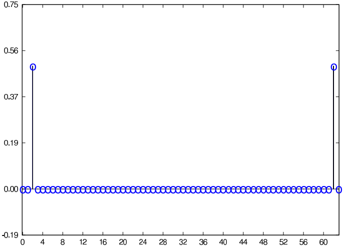
Original Signal



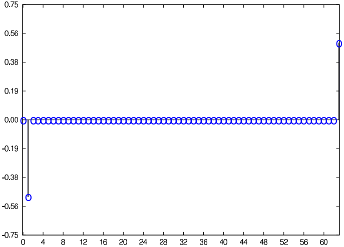
$$x[n] = \sin(2\pi n/64) + \cos(2\pi n/32)$$

The cosine part of the signal is present in the real part.

Real Component

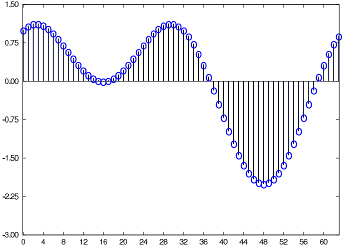


Imaginary Component



Magnitude and Phase of Line Spectra *Example*

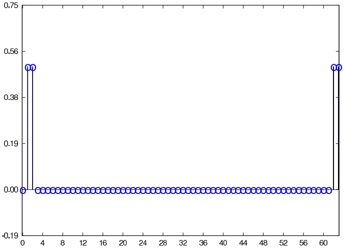
Original Signal



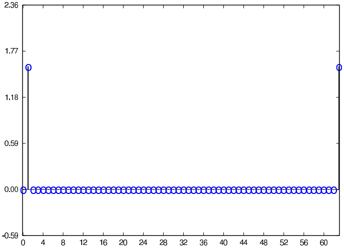
$$x[n] = \sin(2\pi n/64) + \cos(2\pi n/32)$$

The cosine frequency is not present in the phase as it has zero phase. The sine part is present as it has $\pm\pi/2$ phase.

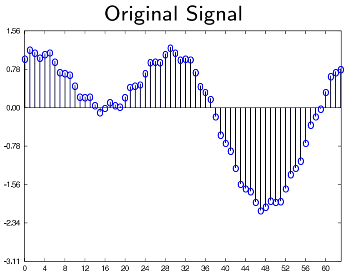
Magnitude



Phase

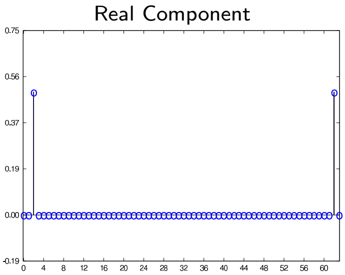


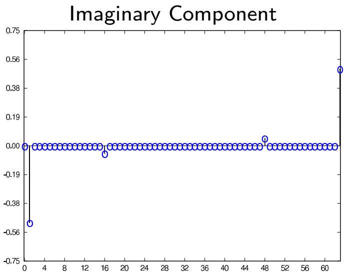
Magnitude and Phase of Line Spectra *Example*



$$x[n] = \sin(2\pi n/64) + \cos(2\pi n/32)$$

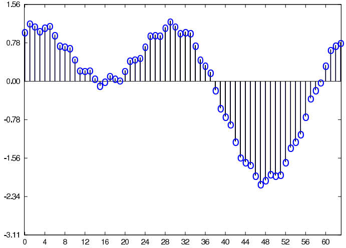
$$+0.1\sin(2\pi n/4)$$





Magnitude and Phase of Line Spectra *Example*

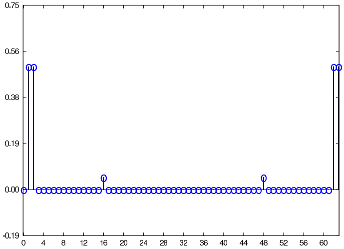
Original Signal



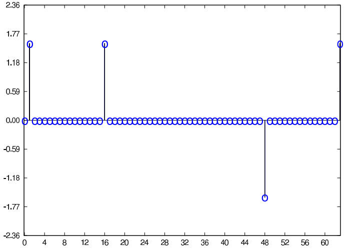
$$x[n] = \sin(2\pi n/64) + \cos(2\pi n/32)$$

$$+0.1\sin(2\pi n/4)$$

Magnitude

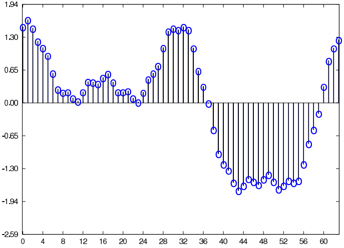


Phase



Magnitude and Phase of Line Spectra *Example*

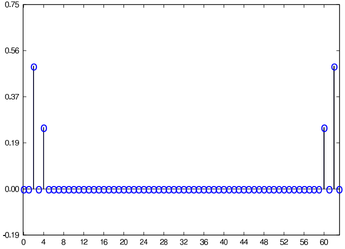
Original Signal



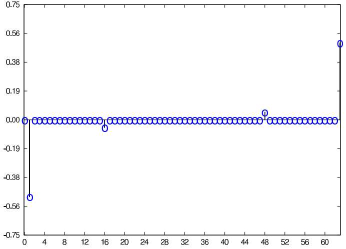
$$x[n] = \sin(2\pi n/64) + \cos(2\pi n/32)$$

$$+0.1\sin(2\pi n/4) + 0.5\cos(2\pi n/16)$$

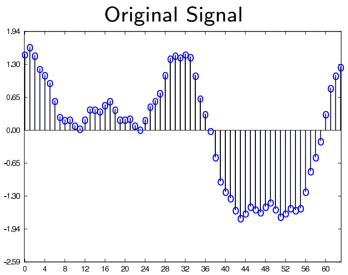
Real Component



Imaginary Component

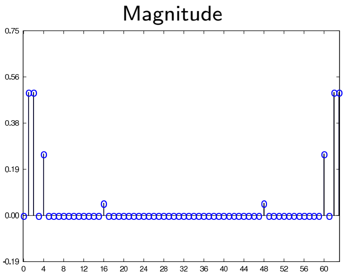


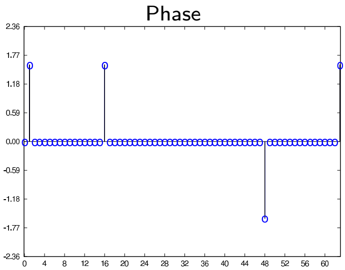
Magnitude and Phase of Line Spectra *Example*



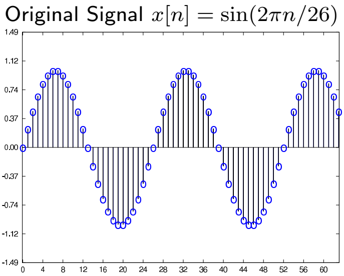
$$x[n] = \sin(2\pi n/64) + \cos(2\pi n/32)$$

$$+0.1\sin(2\pi n/4) + 0.5\cos(2\pi n/16)$$

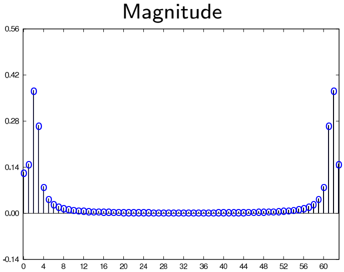


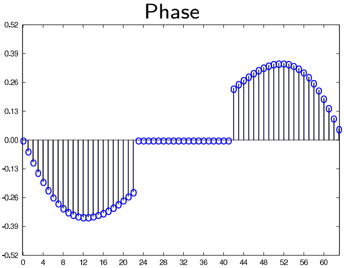


Magnitude and Phase of Signals with Discontinuities



- This signal is not periodic as the sine function is analyzed over $\sim 2\frac{2}{5}$ periods. The end of the signal does not join up with the beginning, resulting in a discontinuity.
- The discontinuity has many frequency components with different phases.

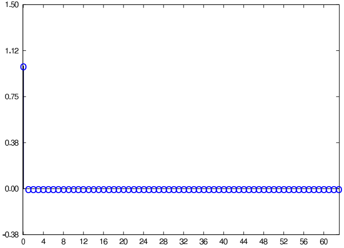




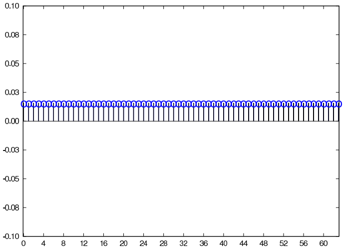
Magnitude and Phase of Impulse Function

- The line spectra of a (periodic) impulse function is composed of all frequencies
- This illustrates the usefulness of an impulse function in characterizing a system's frequency response
- Zero phase because the function is even, *i.e.* $x[n] = x[-n]$, frequency response composed cosine functions only.

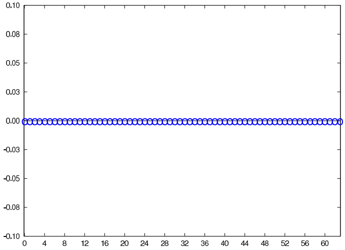
Original Signal $x[n] = \delta[0]$



Magnitude

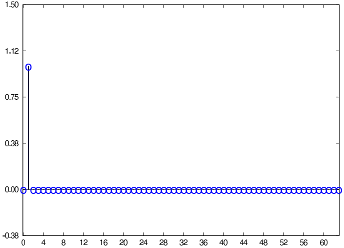


Phase



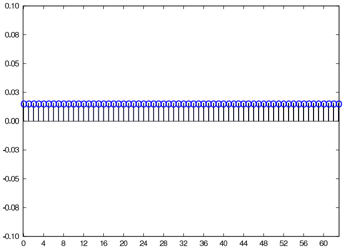
Magnitude and Phase of Impulse Function

Original Signal $x[n] = \delta[1]$

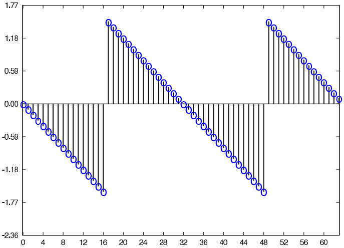


- The line spectra of a (periodic) impulse function is composed of all frequencies
- This illustrates the usefulness of an impulse function in characterizing a system's frequency response
- Phase components present when odd function, *i.e.* $x[n] = -x[-n]$, composed of sine functions only.

Magnitude



Phase



Useful Properties of Discrete Fourier Series

Parseval's theorem
Equates the total power of a signal in the time and frequency domains:

$$\frac{1}{N}\sum_{n=0}^{N-1}(x[n])^2 = \sum_{k=0}^{N-1}(\mathrm{Mag}(a[k]))^2$$

Example
Impulse function, $\delta[0] = 1$

$$\frac{1}{N}\sum_{n=0}^{N-1}(x[n])^2 = \frac{1}{N}$$

and

$$\sum_{k=0}^{N-1}(\mathrm{Mag}(a[k]))^2 = N \times \left(\frac{1}{N}\right)^2 = \frac{1}{N}$$

which are equal.

Other Example Useful Properties of Discrete Fourier Series

$x[n] \leftrightarrow a[k]$ symbolizes $a[k]$ is the discrete Fourier Series of $x[n]$.

- **Linearity:**
  If $x_1[n] \leftrightarrow a_1[k]$ and $x_2[n] \leftrightarrow a_2[k]$ then

$$w_1 x_1[n] + w_2 x_2[n] \leftrightarrow w_1 a_1[k] + w_2 a_2[k]$$

- **Time-shifting (invariance):**
  If $x[n] \leftrightarrow a[k]$ then

$$x[n - n_0] \leftrightarrow a[k] \exp(-j2\pi k n_0 / N),$$

i.e. The shift is just a phase shift and does not affect the magnitude.

Outline

Aperiodic Digital Sequences

- Most signals do not endlessly repeat (*i.e.* not periodic);
- Most signals are therefore known as *aperiodic*.
- Different analysis and synthesis equations are necessary for aperiodic sequences, known as the **Fourier Transform** for aperiodic digital sequences

$$X(\Omega) = \mathcal{F}(x[n]) = \sum_{n=-\infty}^{\infty} x[n] \exp(-j\Omega n)$$

- and the **inverse Fourier Transform** for aperiodic digital sequences

$$x[n] = \mathcal{F}^{-1}(X(\Omega)) = \frac{1}{2\pi} \int_{2\pi} X(\Omega) \exp(j\Omega n) \mathrm{d}\Omega.$$

- *Note: $X(\Omega)$ is a continuous function. It is also periodic which is a result of the ambiguities in discretely sampled signals.*

Fourier Transform for Aperiodic Digital Sequences

Comparing the Fourier Transform:

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] \exp(-j\Omega n), \tag{1}$$

with the Fourier Series analysis equations:

$$a[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] \exp\left(\frac{-j2\pi kn}{N}\right). \tag{2}$$

We can see that $\Omega = \frac{2\pi k}{N}$. n has also been taken to $\pm\infty$ and because of this the Fourier Transform is no longer divided by N (otherwise $X(\Omega)$ would be zero) so that $X(\Omega)$ can in some way be equated with $Na[k]$.

Fourier Transform *Boxcar Example*

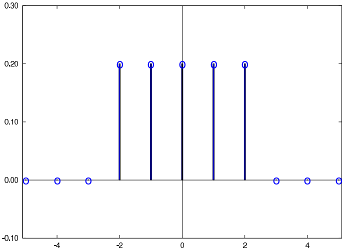
The Fourier Transform of the impulse function $\delta[0]$:

$$\begin{aligned} x[n] &= \delta[0] \\ \therefore \;\; X(\Omega) &= \sum_{n=-\infty}^{\infty} \delta[0] \exp(-j\Omega n) \\ &= \exp(-j\Omega \times 0) \\ &= 1. \end{aligned}$$

In other words, the Fourier Transform of an impulse function consists of all frequencies. Similar to the Fourier Series representation of a periodic impulse function, calculated earlier.

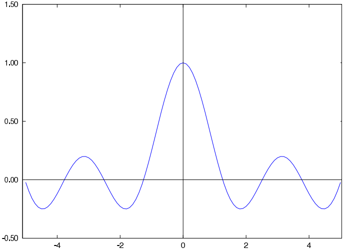
Fourier Transform *Example*

If $x[n] = \begin{cases} 0.2 & \text{if} \quad -2 \leq n \leq 2, \\ 0 & \text{otherwise.} \end{cases}$



$\overset{\mathcal{F}}{\Rightarrow}$

$X(\Omega) =$
$0.2 \times (2\cos(\Omega 2) + 2\cos(\Omega 1) + 1).$

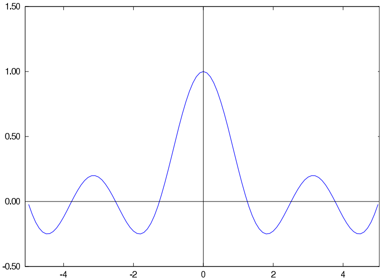


Then

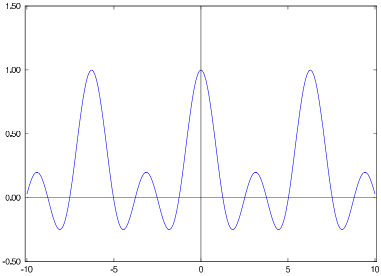
$$
\begin{aligned}
X(\Omega) = & \sum_{n=-\infty}^{\infty} x[n] \exp(-j\Omega n) = \sum_{n=-2}^{2} 0.2 \exp(-j\Omega n) \\
= & 0.2 \times (\exp(j\Omega 2) + \exp(j\Omega 1) + \exp(-j\Omega 0) + \exp(-j\Omega 1) + \exp(-j\Omega 2)) \\
= & 0.2 \times (\cos(\Omega 2) + j\sin(\Omega 2) + \cos(\Omega 1) + j\sin(\Omega 1) + 1 \\
& + \cos(\Omega 1) - j\sin(\Omega 1) + \cos(\Omega 2) - j\sin(\Omega 2)) \\
= & 0.2 \times (2\cos(\Omega 2) + 2\cos(\Omega 1) + 1).
\end{aligned}
$$

Periodicity of Fourier Transform

Also note that the Fourier Transform of an aperiodic signal is periodic.



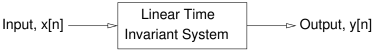
$\Rightarrow$



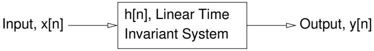
The periodicity is every 2π periods, a result of the sampling in the digitisation process.

Frequency Response of LTI Systems

An LTI system has an input $x[n]$ and an output $y[n]$:



Recall (see lecture 2) that an LTI system has an impulse response, $h[n]$:



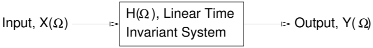
which describes the response of the system when an impulse function is given as the input. The impulse response is useful as it can be used to calculate the output signal for a given input signal:

$$y[n] = x[n] * h[n]$$

where $*$ is convolution *NOT* multiplication.

Frequency Response of LTI Systems

An LTI system can also be described in the frequency domain:



where

- The input frequency domain signal is $X(\Omega) = \mathcal{F}(x[n])$,
- The output frequency domain signal is $Y(\Omega) = \mathcal{F}(y[n])$
- The LTI system is described by $H(\Omega) = \mathcal{F}(h[n])$ which is known as the *frequency response* of the system and is the Fourier Transform of the impulse response.

In the frequency domain, the output can be calculated more easily:

$$Y(\Omega) = X(\Omega) \times H(\Omega),$$

where multiplication *IS* used here. In other words, *convolution* is performed by multiplication in the frequency domain.

Frequency Response of LTI Systems

The frequency domain convolution (multiplication) equation:

$$Y(\Omega) = X(\Omega) \times H(\Omega),$$

can be re-arranged so that:

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)}$$

so if we want to find the frequency response of a system then we can find it via this equation or via the Fourier transform of the time domain representation $h[n]$.

Frequency Response of LTI Systems

Recall the general form of LTI difference equations (see lecture 2):

$$\sum_{m=0}^{N} a[m]y[n-m] = \sum_{m=0}^{M} b[m]x[n-m].$$

Using the linearity and time-shifting properties of Fourier transforms we can convert it to an expression using frequency domain terms:

$$\sum_{m=0}^{N} a[m]\exp(-jk\Omega)Y(\Omega) = \sum_{m=0}^{M} b[m]\exp(-jk\Omega)X(\Omega).$$

Therefore the frequency response of a system can also be described by

$$H(\Omega) = \frac{\sum\limits_{m=0}^{M} b[m]\exp(-jm\Omega)}{\sum\limits_{m=0}^{N} a[m]\exp(-jm\Omega)}.$$

This equation can be used to directly find the frequency response of a system even if only the coefficients $a[m]$ and $b[m]$ are known.

Frequency Response *Example*

Q. A moving average filter has $y[n] = \frac{1}{3}\left(x[n] + x[n-1] + x[n-2]\right)$. Find the frequency response of this filter.

A. We can find the frequency response by using the coefficients:

- There is only 1 output coefficient, $a[0] = 1$.
- There are 3 input coefficients, $b[0] = b[1] = b[2] = \frac{1}{3}$
- Therefore

$$\begin{aligned}
H(\Omega) =& \frac{\frac{1}{3}\sum_{m=0}^{M} \exp(-jm\Omega)}{\exp(-j0\Omega)} = \frac{1}{3}(1 + \exp(-j\Omega) + \exp(-j2\Omega)) \\
=& \frac{1}{3}(1 + \cos(\Omega) - j\sin(\Omega) + \cos(2\Omega) - j\sin(2\Omega)) \\
=& \frac{1}{3}(1 + \cos(\Omega) + \cos(2\Omega) - j(\sin(\Omega) + \sin(2\Omega)))
\end{aligned}$$

Frequency Response *Example cont'd.*

- Magnitude:

$$\text{Mag}(H(\Omega)) = \sqrt{\frac{1}{3}\left((1+\cos(\Omega)+\cos(2\Omega))^2 + (\sin(\Omega)+\sin(2\Omega))^2\right)}$$

- Phase:

$$\phi(H(\Omega)) = \tan^{-1}\left(-\frac{(\sin(\Omega)+\sin(2\Omega))}{(1+\cos(\Omega)+\cos(2\Omega))}\right)$$

The magnitude can be simplified using:

- $2\sin(\Omega)\sin(2\Omega) = \cos(\Omega) - \cos(3\Omega)$
- $2\cos(\Omega)\cos(2\Omega) = \cos(\Omega) + \cos(3\Omega)$
- $\sin^2(\Omega) = \frac{1-\cos(2\Omega)}{2}$
- $\sin^2(2\Omega) = \frac{1-\cos(4\Omega)}{2}$
- $\cos^2(\Omega) = \frac{1+\cos(2\Omega)}{2}$
- $\cos^2(2\Omega) = \frac{1+\cos(4\Omega)}{2}$

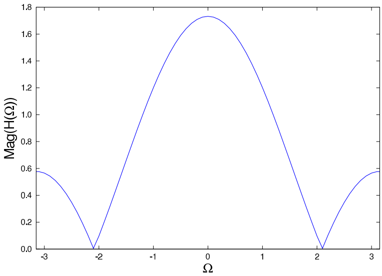
Resulting in:

- $\text{Mag}(H(\Omega)) = \sqrt{\frac{1}{3}(3 + 2(2\cos(\Omega) + \cos(2\Omega)))}$

Frequency Response *Example cont'd.*

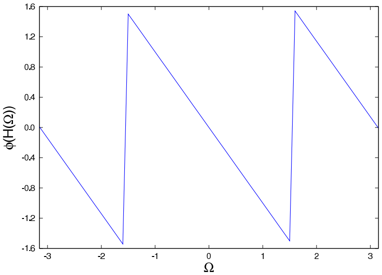
Moving Average Filter (k=3) Frequency Response

Magnitude



$$\mathrm{Mag}(H(\Omega)) = \sqrt{\frac{1}{3}(3 + 2(2\cos(\Omega) + \cos(2\Omega)))}$$

Phase



$$\phi(H(\Omega)) = \tan^{-1}\left(-\frac{(\sin(\Omega)+\sin(2\Omega))}{(1+\cos(\Omega)+\cos(2\Omega))}\right)$$

Lecture Summary

This lecture has covered...

- Introduction to frequency domain analysis
- Discrete Fourier Series
- Spectra of Periodic Digital Signals
- Magnitude and Phase of Line Spectra
- The Fourier Transform for aperiodic digital sequences