Course: Discrete Mathematics

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Module 1: Proof Methods
Lesson 1: Basics of Logic

Topic: Predicates and Quantifiers

Reading Objective

This reading supplements the concepts you have learned in the video. You will learn about the basic definitions of predicates and then define quantifiers. You will work through multiple examples to gain mastery of these topics.

1 Predicate or Propositional Function

Definition 1.1

A statement of the form P(x_1,x_2,...x_n) is the value of the propositional function P at the n-tuple (x_1,x_2,...x_n), and P is called an **n-place predicate** or an **n-ary predicate**.

Examples

- Consider the predicate P(x): [5x = 15]

  Here, P(1) is F, P(3) is T
- Consider the 3-ary or a 3-place predicate P(x,y,z): [x+y > z]

  Here, P(1,1,1) is T, P(1,1,2) is F

Definition 1.2

A **domain** or the **universe of discourse** is the set from which the n-tuple in the predicates takes their values.

Examples

- In the P(x): [$x - 4 > 0$], the domain of x can be the set of integers $\mathbb{Z}$. Alternatively, the domain of x could also be the set of real numbers $\mathbb{R}$.
- Consider the predicate P(x,y): "The student x has taken the course y." Here, if the domain of x is the set of students studying in BITS Pilani and the domain of y is the courses offered by BITS Pilani. Let "Seema" be a student of BITS Pilani who has taken the courses "Discrete Mathematics", "Introduction to Programming". "Data Structures and Algorithms" is another course being offered. Here, P("Seema", "Discrete Mathematics") is T, and P("Seema", "Data Structures and Algorithms") is F.

2 Universal Quantifier

Definition 2.1

The **universal quantification** of P(x) is the statement: "P(x) is true for all values of x in the domain." denoted by $\forall x P(x)$. Here, $\forall$ is called the universal quantifier.

Therefore, "$\forall x P(x)$" is T if P(x) is true for every element in the domain and F if P(x) is false for at least one element in the domain.

Definition 2.2

An element for which P(x) is false is called a **counterexample** of $\forall x P(x)$.

Note that the truth value of the quantification of a statement depends on the domain or the universe of discourse defined.

Examples

- $\forall x[x > 0]$, where the domain is the set of natural numbers $\mathbb{N}$ is T.
- $\forall x[x > 0]$, where the domain is the set of integers $\mathbb{Z}$ is F. Here, the counterexample can be x = -2.

3 Existential Quantifier

Definition 3.1

The **existential quantification** of P(x) is the proposition: "There exists an element x in the domain such that P(x) is true." denoted by $\exists x P(x)$. Here, $\exists$ is called the existential quantifier.

Therefore, "$\exists x P(x)$" is T if P(x) is true for at least one element in the domain, and F if P(x) is false for all the elements in the domain.

Examples

- $\exists x[x < 0]$, where the domain is the set of natural numbers $\mathbb{N}$ is F.
- $\exists x[x < 0]$, where the domain is the set of integers $\mathbb{Z}$ is T.

4 Logical Equivalences using Quantifiers

Definition 4.1

Statements involving predicates and quantifiers are **logically equivalent** if and only if they have the same truth value, no matter which predicates are substituted into these statements and which domain of discourse is used for the variables in these propositional functions.

We use the notation $S \equiv T$ to indicate that two statements, S and T, involving predicates and quantifiers, are logically equivalent.

Examples

Example 4.1

$\forall x[P(x) \wedge Q(x)] \equiv \forall x P(x) \wedge \forall x Q(x)$
Explanation
Suppose $\forall x[P(x) \wedge Q(x)]$ is T, that means for all values of x in domain P(x), and Q(x) are T. Therefore, $\forall x P(x)$ and $\forall x Q(x)$ are true, that is, $\forall x P(x) \wedge \forall x Q(x)$ is T. If $\forall x[P(x) \wedge Q(x)]$ is F, means there is at least one x in the domain such that at least one of P(x) or Q(x) is F. Then at least one of $\forall x P(x)$ or $\forall x Q(x)$ is F. Hence,$\forall x P(x) \wedge \forall x Q(x)$ is F.

Conversely, if $\forall x P(x) \wedge \forall x Q(x)$ is T, then both $\forall x P(x)$ and $\forall x Q(x)$ are T, that is, P(x) and Q(x) are both true for all elements in the domain. Hence, $\forall x[P(x) \wedge Q(x)]$ is T. If $\forall x P(x) \wedge \forall x Q(x)$ is F, then at least one of $\forall x P(x)$ or $\forall x Q(x)$ is F, that is, at least one of P(x) or Q(x) is F for at least one x in the domain. Thus, at least for one value of x in domain of $P(x) \wedge Q(x)$ will be F, so $\forall x[P(x) \wedge Q(x)]$ is F. Therefore, $\forall x[P(x) \wedge Q(x)] \equiv \forall x P(x) \wedge \forall x Q(x)$.

Example 4.2

$\forall x[P(x) \vee Q(x)] \not\equiv \forall x P(x) \vee \forall x Q(x)$
Explanation: Consider the following counterexample

- P(x): x > 0.
- Q(x): x <= 0.

where the domain of x is the set of integers $\mathbb{Z}$.
$\forall x[P(x) \vee Q(x)]$ will be true since every integer is either greater than zero or less than or equal to zero. However, for any positive integer, Q(x) will be F, so $\forall x Q(x)$ is F, and for any negative integer, P(x) will be F, so $\forall x P(x)$ is F. Therefore, $\forall x P(x) \vee \forall x Q(x)$ is F.

Reading Summary

In this reading, you have learned the following:

- Basic definitions of predicates and quantifiers
- When two predicates are equivalent
- Some additional examples to master the concepts