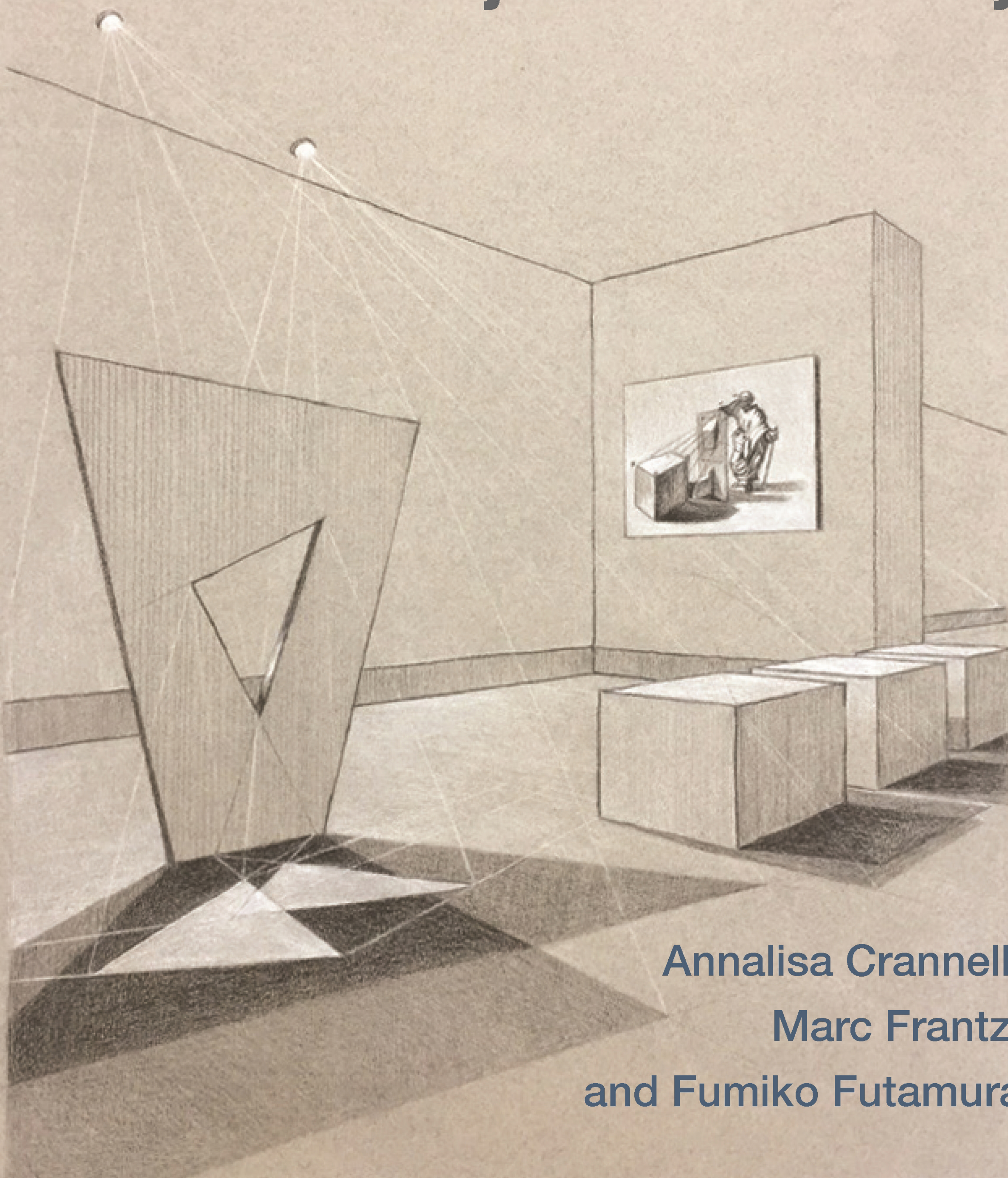
Perspective and Projective Geometry



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Contents

A comment on page numbering: almost every module begins with a one-page picture that is also a math/art puzzle meant to be removed for ease of drawing. The in-class worksheet follows immediately after.

PERSPECTIVE AND PROJECTIVE GEOMETRY

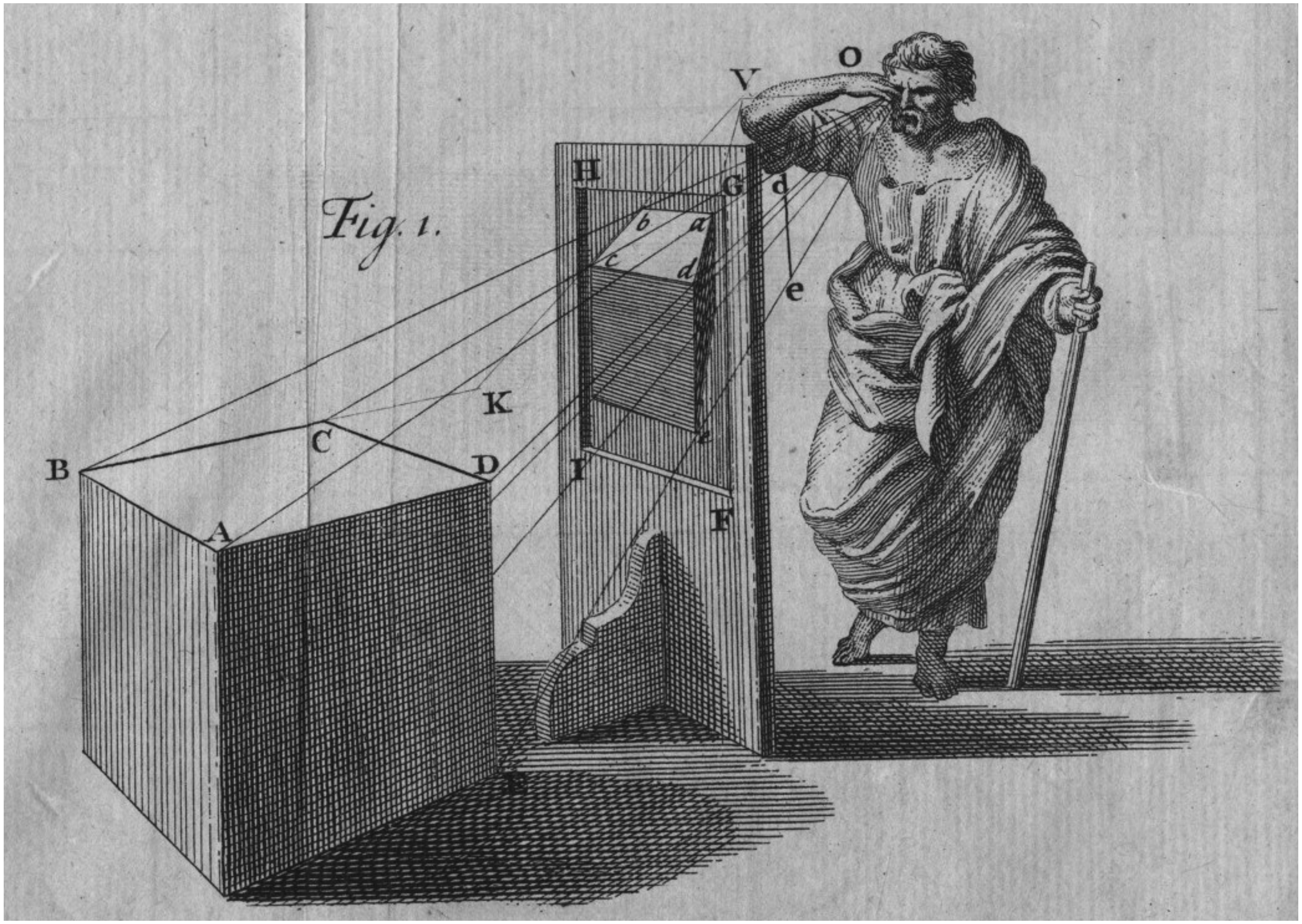

FIGURE 0: Looking at the world through a window. [For use with the INTRODUCTION AND FIRST ACTION module.]
Courtesy of the Max Planck Institute for the History of Science, Berlin

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Introduction and First Action

Looking at the World through the Window of Mathematics

Perspective and Projective Geometry is a course that will change the way that you look at the world, and we mean that literally.

In this course, you will take photographs, you will analyze perspective pictures and draw pictures of your own, and you will explore the geometry that explains how we fit a three-dimensional world onto a two-dimensional canvas. Along the way, you will get invaluable practice with making logical arguments (that is, writing mathematical proofs of statements that are true and refuting statements that are false). By combining art and geometry, we are following in a tradition that builds on centuries of exploration. Indeed, the very first treatise on perspective art—Leon Battista Alberti's *Della Pittura* (On Painting)—includes this exhortation from the author:

> It would please me if the painter were as learned as possible in all the liberal arts, but first of all I desire that he know geometry. . . . Our instruction in which all the perfect absolute art of painting is explained will be easily understood by the geometrician, but one who is ignorant in geometry will not understand these or any other rules of painting. Therefore, I assert that it is necessary for the painter to learn geometry. [3, p. 90]

Knowing how to look at the world is more than just a fancy, aesthetic luxury. The geometry that helped Renaissance artists create breathtaking, realistic images five or six centuries ago also helped those same societies construct maps that allowed them to navigate across the globe; it helped them understand the emerging warfare of ballistic cannons; and it helped them build fortresses that could withstand ballistic attacks.

Indeed, Girard Desargues—the author of one of the central theorems of this book—extolled the subject of geometry because of its usefulness not only in the world at large, but also to his own well-being:

> I freely confess that I never had taste for study or research either in physics or geometry except in so far as they could serve as a means of arriving at some sort of knowledge of the proximate causes for the good and convenience of life, in maintaining health, in the practice of some art . . . having observed that a good part of the arts is based on geometry . . . that of perspective in particular [33].

That very same geometry that defined an age named for "new birth" is making its own new strides in our modern world. During our lifetime, video games have moved from the two-dimensional mazes of *Pac-Man* into the immersive full-body experiences. Your parents and grandparents watched the flat worlds of the Flintstones and the Simpsons, but as the calendar ticked over into the most recent millennium, movies like *Shrek* and *Frozen* started bringing pixelated characters to life. And just as in the Renaissance, the uses of this projective geometry have spilled out over the edges of art into many areas of practical technology, paving the way for unprecedented progress in medical imaging, in geological exploration, and in 360-degree map views that have made commercial successes of applications like Zillow and Google. Knowing how to see the world is powerful, and this course will help you to harness that power.

We learn by doing, and so each lesson uses the following format (with occasional, minor variations). We begin each lesson with a picture puzzle. This puzzle comes with a module that has questions and occasional definitions that will help you and your fellow classmates construct an understanding of the geometry that allows us to solve that puzzle and others like it. At the end of the module, you will see three kinds of homework questions:

- short answer exercises (denoted by a Ⓔ symbol),
- "art" exercises (denoted by a △A symbol) that ask you to create drawings or photographs with certain properties, and
- proof or counterexample questions (denoted by a $\boxed{\text{P}}$ symbol) that build your reasoning and exposition skills.

At the end of this book, we include the key definitions and theorems in a Reference Manual to aid you in reviewing and studying.

Even before you begin the first module, we hope you will have the experience of looking at the world and projecting an image of it onto a two-dimensional canvas. In particular, you and your classmates will get to draw pictures on windows.

A Window into Perspective

The word *perspective* comes from the Medieval Latin roots *per* ("through") and *specere* ("look at"—the same root that gives us "spectacles"). So perspective art literally intends for us to look through a window to portray the objects that lie on the other side. As Alberti instructed aspiring painters, "When [artists] fill the circumbscribed places with colors, they should only seek to present the forms of things seen on this plane as if it were of transparent glass." Alberti's book had a huge influence on numerous scholars and artists of his time, including Leonardo da Vinci, Piero della Francesca, Albrecht Dürer, and Gerard Desargues. Three centuries after Alberti's treatise appeared, the mathematician Brook Taylor (of Taylor series fame) illustrated exactly such a *through-the-window* projection (his Figure 1, our Figure 0) in the preface of his book, *New Principles of Linear Perspective* published in 1719 [51].

The description of "looking through a window" wasn't merely a metaphor, and it wasn't meant as a mere illustration for descriptions that appeared in a book. Artists throughout the ages have practiced the actual physical act of drawing the world on a window they gazed through. When Leonardo da Vinci instructed painters on "how to draw a site correctly" [36, p. 65], he wrote,

> Have a sheet of glass as large as a half-sheet of royal folio paper, and place it firmly in front of your eyes; that is between the eye and the thing that you draw. Then place yourself at a distance of two-thirds of a braccio [arm's length] from your eye to the

glass, and hold your head with an instrument in such a way that you cannot move your head in the least. Then close or cover one eye, and with a brush or with a pencil of red chalk draw on the glass what appears beyond it.

Even before you start drawing images on paper, therefore, you ought to experience drawing through a window so that you can better understand some of the implications of projecting our three-dimensional world in this way.

The following set of instructions leads you through such an exercise.

Instructions for Window Taping

1. Get into a group of three or four people and choose one person to be the *Art Director*. The others will be *Artists* (and *Holders of Windows*, if using plexiglass).
   (a) *Art Director*: Stand or sit at a fixed location, close one eye, and look through the window with the other (see Figure 1). Direct the Artists to tape the outline and the most important and instructive features you see on the other side of the window.

   As Leonardo noted above, you will need to "hold your head … in such a way that you cannot move your head in the least"! In particular, keep your eye fixed in one location. When you start working on a new line, make sure the drafting tape that is already on the window correctly lines up with the features you've already worked on.

FIGURE 1: Art Director: Although you probably don't need to be taped to the wall, it's still very important to keep your eye fixed in one location!

FIGURE 2: Artists: Use the drafting tape and pay close attention to the Art Director's directions.

FIGURE 3: If you have a camera, take a photograph.

(b) *Artists*: Paying close attention to the Art Director's directions, place the drafting tape on the windows. It may help to have one person hold one end of the tape and the other person hold the other end; get the directions of lines correct first, and then break the tape to the correct length. At first this will be extremely difficult, because you can't see what the Art Director sees. The job might get easier as you place more tape on the window to use as guidelines.

(c) (*Holders of Windows*: hold the plexiglass window as still as possible!)

2. When the picture is "done" (or as done as possible for this session), you might want to take a photograph of the finished product from various places, including the

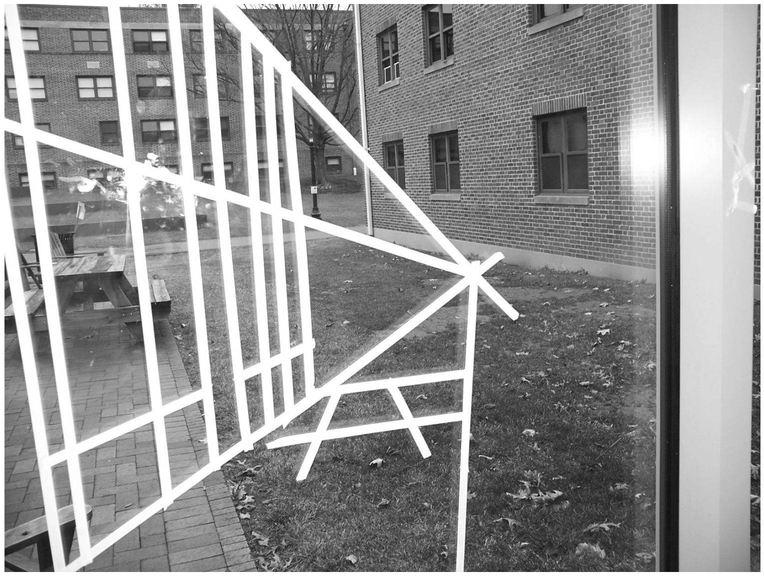


FIGURE 4: If we extend some of the tape lines by adding more tape, we might realize they seem to intersect in a common point.

point of view of the Art Director. If you like, print out your photograph on copier paper and bring it to class.

3. If there are other groups working on similar drawings, you might also wander around and look at the other groups' pictures. Try to see if you can stand or sit where the Art Director was, to see the drawing from the Art Director's point of view.
4. You will notice that some parallel lines in the real world (such as, probably, the vertical lines) have corresponding tape images that are likewise vertical. But some sets of parallel lines in the real world have tape images that seem to tilt. If you extend these lines by adding more tape, you might notice that all of these lines intersect in one spot on the window (see Figure 4). Understanding why this intersection happens, and what the geometric significance of this intersection point is, will be the topic of the first module.
5. Before you come back to the classroom, clean up the tape! Drafting tape usually leaves no residue on windows, especially if you remove it promptly.

Questions for Review

1. Which job was harder, the Art Director's, or the Artists'? Why?
2. Why did the Art Director need to cover one eye?
3. Why was it hard for the Artists to figure out where to put the tape?
4. What does it mean for a line in the real world to be parallel to the window?
5. If the lines in the real world are parallel to the window, what did you notice about their taped images?
6. One collection of lines met at a point. What did this point have to do with the Art Director?

7. What is the relationship between the window and the lines in the real world whose taped images are the lines that met at a point?
8. If you looked at other groups' pictures, could you figure out where the Art Director had been standing or sitting?

About the cover: The cover of this book depicts a room in an imaginary art museum that illustrates, literally, many important geometric concepts that will arise in this book. On the wall hangs a reproduction of Figure 0; the geometry of looking through a window will appear in Chapter 1, and the surprisingly tricky problem of drawing cubes is the subject of Chapters 9 and 10.

The regularly repeated cubes on the floor are inspired by the minimalist artwork of Donald Judd, hinting at the concepts of geometric division, cross ratio, and projective collineations found in chapters 6, 11, and 8, respectively.

The sculpture with the triangular opening was inspired by the work of Roger Jorgensen; the triangular pools of light cast by the sculpture depict a consequence of Desargues's theorem, which the reader will prove in Chapter 7. The space invites you to explore these and many other ideas found in the rooms beyond.

The work, by Fumiko Futamura, was planned in GeoGebra, then geometrically constructed and hand-drawn in black and white charcoal on gray paper.

FIGURE 1.0: Yuxun Sun (F&M Class of 2014) looks out of a window that he and his classmates have taped. See exercises below. [For use with the WINDOW TAPING: THE AFTER MATH module.]
With permission of Yuxun Sun

1

Window Taping

The After Math

Overview How do we draw a three-dimensional world on a two-dimensional picture plane? During a previous class, you and your classmates taped windows, making images of what the Art Director in each group saw outside the window. Your professor came around and made some comments, and then you took a photograph of the window from as close to where that Art Director stood as possible. This module asks you to build on that work, focusing particularly on the images of points and images of lines.

Here are some True/False questions about lines and planes in $\mathbb{R}^3$ (that is, three-dimensional Euclidean space). As you proceed, you do not need to come up with formal proofs, but you *should* work with your fellow students to justify your reasoning about these questions.

1. [T/F] If two distinct lines in $\mathbb{R}^3$ are parallel, then they do not intersect.

2. [T/F] If two distinct lines in $\mathbb{R}^3$ do not intersect, then they are parallel.

3. [T/F] If two distinct lines in $\mathbb{R}^3$ are parallel, then they lie in the same plane.

4. [T/F] If two distinct lines in $\mathbb{R}^3$ lie in the same plane, then they are parallel.

5. [T/F] If a plane in $\mathbb{R}^3$ and a line not lying in that plane are parallel, then they do not intersect.

6. [T/F] If a line and a plane, both in $\mathbb{R}^3$, do not intersect, then they are parallel.

7. [T/F] If the line ℓ is parallel to the plane ω, then there is some line $k \subset \omega$ that is parallel to ℓ.

8. [T/F] If there is some line $k \subset \omega$ that is parallel to a line ℓ, then ℓ is parallel to ω.

(Note: *Unless we say otherwise, we use the standard geometric convention of naming points with italicized capital letters, lines with italicized lowercase letters, and planes with lowercase Greek letters.*)

Real-World Lines and their Images

For the following questions, refer either to a photograph you took of your window-taping exercise, or to Figure 1.0. We will be looking at lines in the real world and at their images on the window (or even more specifically, the images of the tape in the photograph).

9. In the photograph, identify the images of a set of three lines that are parallel to each other in the real world and also parallel to the picture plane (the window). Use your straightedge to extend the line segments to the edges of the paper. Are these images in the photograph parallel?

10. Now identify the images of a set of three lines that are parallel to each other in the real world but **not** parallel to the picture plane. Use your straightedge to extend these line segments. Are these images in your photograph parallel?

11. In general, we expect that a pair of lines that are not parallel will intersect in a single point, but we might be surprised if a set of *three* lines that are not parallel happen to intersect in a single point. Consider the lines from question 10. Describe how the intersection point you located in that question is related to the plane of the window, to the original lines, and to the Art Director.

12. Formulate a conjecture:
    If a set of lines in $\mathbb{R}^3$ are parallel to each other and also parallel to the picture plane, then their images ______________________________.
    If a set of lines in $\mathbb{R}^3$ are parallel to each other but **not** parallel to the picture plane, then their images ______________________________.

Side Views and Top Views

In this class, we will often analyze an image by using a *top view* and a *side view* as in Figure 1.1. In this diagram, we see a *viewer*, the *picture plane* (which is drawn edge-on as a line in this diagram), and the *object*. In these two views, the object is a beach towel lying on the horizontal ground.

13. In the side view, the dot labeled A' on the picture plane is the *image* of either the closest corner or the far corner of the towel. Which corner is it? And how do you know? Label that corner of the towel A.
14. On the side-view, label the other corner (the far or the near corner, whichever you didn't choose for question 13) of the towel as B. Use a straightedge to determine B', and then shade in the image of the towel.

TOP VIEWS: LOOKING THROUGH THE NOSE

You discovered on the first day that looking through only one eye makes a huge difference. Rather than try to remember which eye our viewer looks through, when we draw top views, we'll assume that our viewers look through their **noses**. (The assumption is silly, but our diagrams will be *much* easier to draw!)

15. On the top view in Figure 1.1, label the corners A and B of the towel. Draw the four light rays connecting the four corners of the towel to the viewer's nose, and then draw the image of the towel. Label A' and B' on the picture plane.

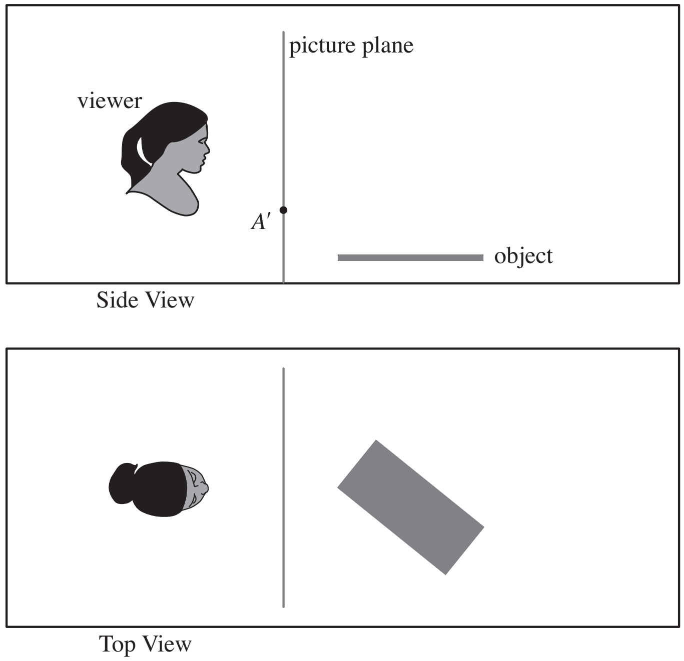


FIGURE 1.1: Two views of an artist looking at a towel through a picture plane.

Images of Lines

What we have done so far is to look at points and their images. What can we say about lines and their images?

In Figure 1.2, we show a side view of the towel lying on the pebble-strewn ground. We will take advantage of this seemingly obvious fact: *If you are looking at something, you can see it. If you are not looking at something, you can't see it.* In particular, if our viewer looks up at the sky (as we indicate in the side view), she does not see the ground. (When we talk about looking in one direction, it helps to think of a "line of sight," as though the viewer is looking through a straw and has no peripheral vision).

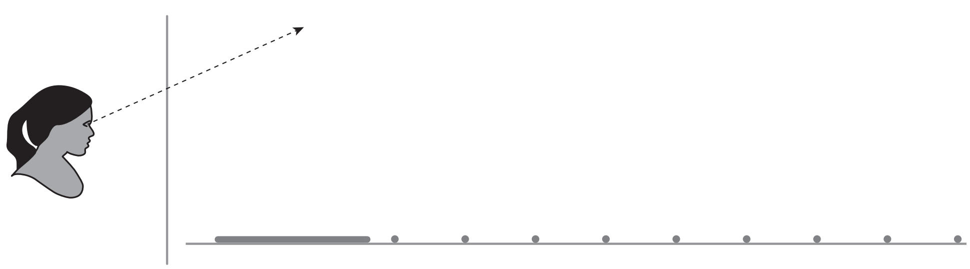


FIGURE 1.2: Side view showing a viewer facing a towel on the pebble-strewn ground. When the viewer looks up, she does not see the ground so she does not draw it.

16. In Figure 1.2, use a straightedge to draw the light rays connecting the viewer's eye to the pebbles.

(a) As the pebbles get further and further away from the viewer, what can you say about the light rays?

(b) Draw the images of the pebbles on the picture plane. As the pebbles get further from the viewer, what can you say about their images?

(c) Suppose the line of pebbles goes on forever. Locate the point on the picture plane where the viewer goes from *seeing* the pebbles to *not seeing* the pebbles. Because the ground appears to vanish at this point on the picture plane, we call this point the *vanishing point* of the line of pebbles.

(d) Draw the line of sight from the viewer to the vanishing point. How does this line relate to the line of pebbles?

(e) Fill in the blanks to complete the definition:

Given a line ℓ not parallel to the picture plane ω, the *vanishing point* V_ℓ of the line is the intersection of the plane ____ with the line through __________ that is parallel to ____.

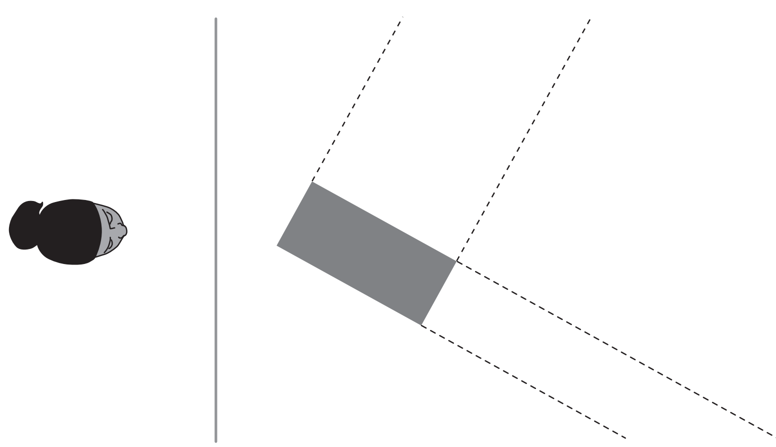


FIGURE 1.3: Top view showing a viewer looking at a towel, with the edges of the towel extended.

17. Let us repeat this procedure with the four lines in Figure 1.3. Add some pebbles to the lines that extend the edge of the towel, and then draw their images.

(a) Locate the vanishing points of these four lines. How many vanishing points are there?

(b) Draw the line(s) that connect(s) the viewer's nose to the vanishing point(s). What can you say about how the line(s) relate(s) to the edges of the towel?

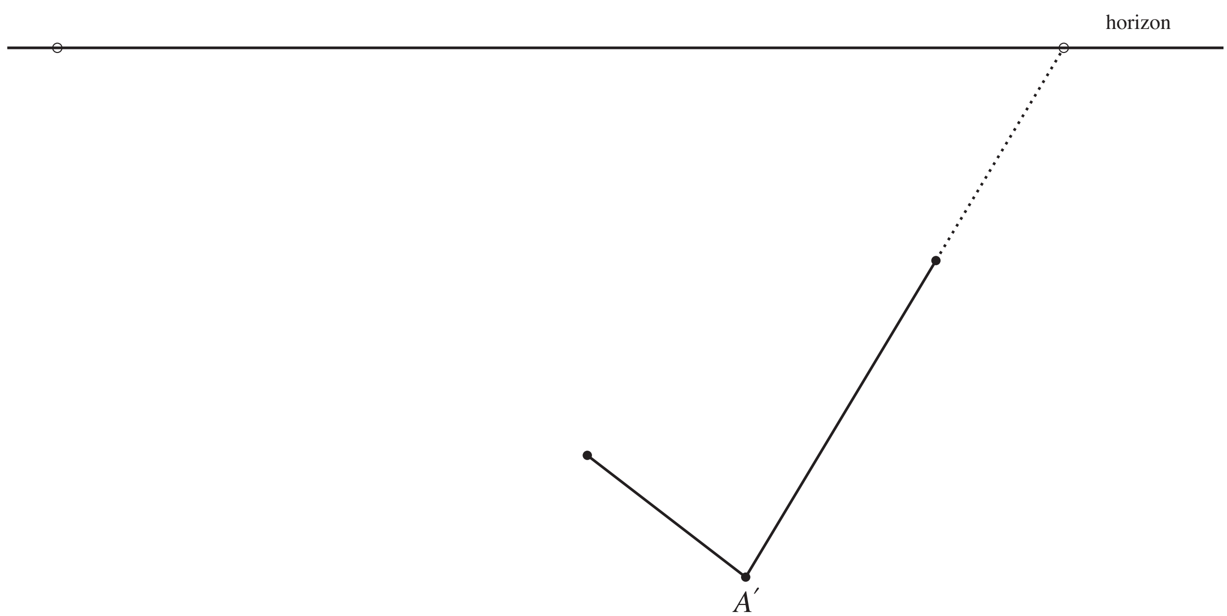
18. In the space below, draw your own side view and top view, showing a viewer, the vertical picture plane, and two horizontal lines that are parallel to the picture plane. Answer these questions about your side and top views:
(a) Must the two lines you draw be parallel to one another?
(b) Locate the image of each line in the side view.
(c) Locate the image of each line in the top view.
(d) How does the image of each line relate to the original line?
(e) How many vanishing points do each of these lines have?

CHECKING THE CONJECTURE

19. Compare your conjecture from question 12 to your answers to questions 16–18. Do your answers agree with one another?

20. In the drawing below, complete the sketch of the towel in perspective (that is, as it should appear in the picture plane), using your answers from above. Draw the edges of the towel as solid lines, and the extension of those edges as dashed lines. The point A' is already labeled; label point B' to correspond with your answers to questions 13–15.

21. Why don't points A and B appear in the picture you draw for question 20?

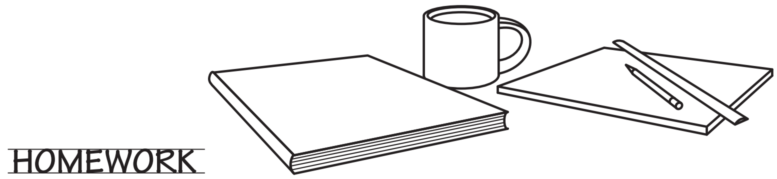


22. Complete the following set of statements.

If a line is not parallel to the picture plane, then it has a vanishing point. That vanishing point is located at the intersection of the ________ plane and the line through the artist's eye that is ______________________________.

A collection of lines that are parallel to each other and also parallel to the picture plane has exactly ________ vanishing point(s).

A collection of lines that are parallel to each other but not parallel to the picture plane has exactly ________ vanishing point(s).



EXERCISES

E 1.1. Consider Figure 1.4, which shows two sketches of hallways. For each sketch, determine whether the artist was standing, sitting, or lying down, and explain how you know.

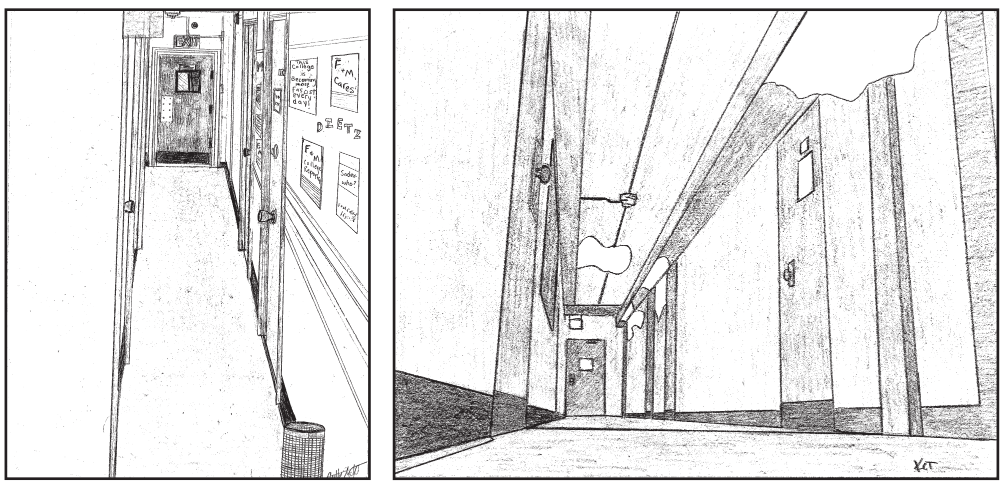
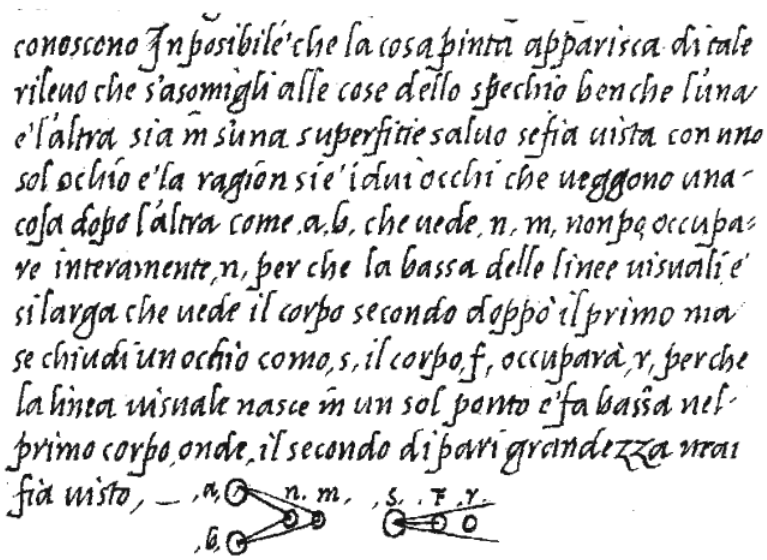


FIGURE 1.4: Two sketches of hallways for homework exercise E 1.1 (sketches reproduced with permission of Anthony Nocket and Kevin Toenboehm).
Courtesy of Anthony Nocket and Kevin Toenboehm

E 1.2. Suppose an artist is standing at a distance d from a picture plane, looking at a line segment that is parallel to the picture plane. If the length of the segment is L, can we determine the length l of its image? What other information might we need? Why do we need to know that the segment is parallel to the picture plane?

E 1.3. In all that we have done so far, we have assumed we look with only one eye. In fact, throughout this course, we will assume that we project images from a single point. But looking with only one eye is not the way that most people actually see the world.

We believe that objects that are further away from us appear to be smaller. But Leonardo da Vinci in his *Codex Urbinas* described why this is not always true when we observe the world with both eyes open. Recreate the argument that he presented in Figure 1.5. That is, explain why Leonardo writes that if we observe objects m and n, both smaller than the distance between our two eyes a and b, the near object "cannot conceal [the further object] entirely." However, if we observe same-sized objects f and r with just one eye s, then "the body f will cover r . . . hence the second body of equal size is never seen" [36].



conosceno Inposibile che la cosa pinta apparisca di tale
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cosa dopo l'altra come .a.b. che uede .n.m. non po occupa=
re interamente .n. per che la bassa delle linee uisuali e
si larga che uede il corpo secondo doppo il primo ma
se chiudi un ochio como .s. il corpo .f. occupara .r. perche
la linea uisuale nasce in un sol ponto e fa bassa nel
primo corpo onde il secondo di pari grandezza mai
fia uisto

FIGURE 1.5: From Parte Secuda (page 47) of Leonardo da Vinci's *Codex Urbinas Latinus 1270* [36].
Courtesy of Princeton University Press

Ⓔ 1.4. This question asks about the image of circles. Figure 1.6 shows a viewer looking at three same-sized circles. Perhaps this is a top view of an artist looking at a row of columns, for example.

(a) Which of these circles has the largest *apparent size* to the viewer? (We measure apparent size by the angle that the image subtends).

(b) Which of these circles has the largest image on the picture plane?

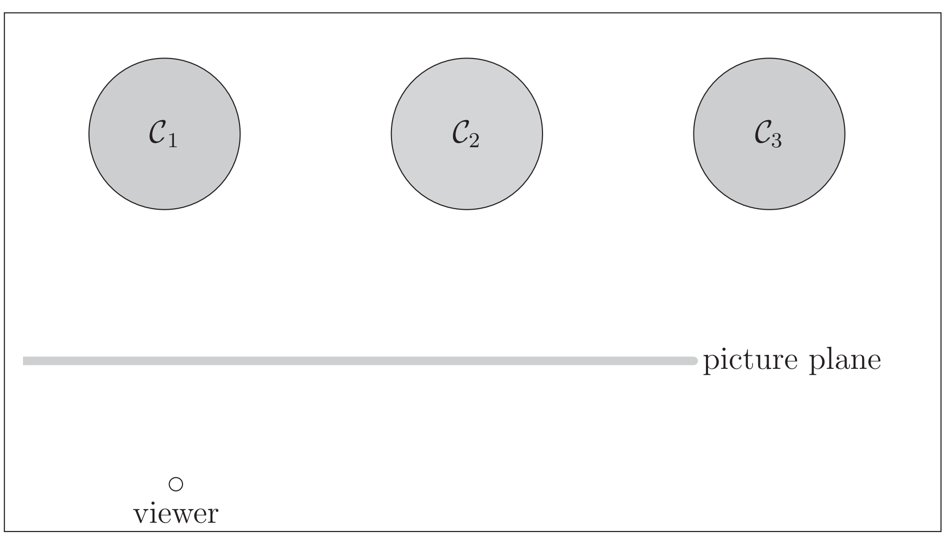

FIGURE 1.6: A viewer looks at three same-sized circles (see Exercise Ⓔ1.4).