Two-Host Two-Vector Species Model of Zika Virus

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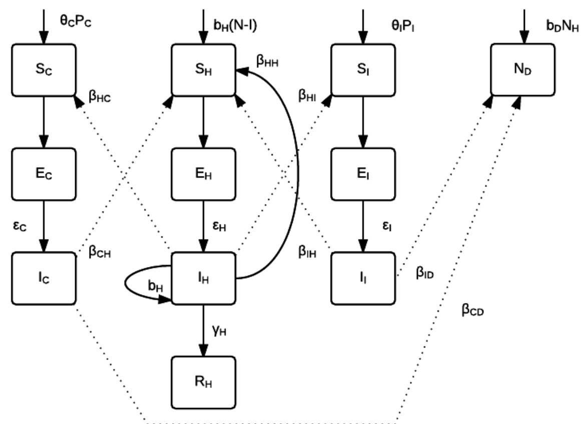

Competent Host ODEs

$$\frac{dS_H}{dt} = b_H(N_H - I_H) - \beta_{V_C H}\frac{S_H}{N_H}I_{V_C} - \beta_{V_I H}I_{V_I} - \beta_{HH}\frac{S_H I_H}{N_H^2} - d_H S_H$$

$$\frac{dE_H}{dt} = \left(\beta_{V_C H}I_{V_I} + \beta_{V_I H}I_{V_I} + \beta_{HH}\frac{I_H}{N_H}\right)\frac{S_H}{N_H} - \epsilon_H E_H - d_H E_H$$

$$\frac{dI_H}{dt} = b_H I_H + \epsilon_H E_H - \gamma_H I_H - d_H I_H$$

$$\frac{dR_H}{dt} = \gamma_H I_H - d_H R_H$$

$$\implies \frac{dN_H}{dt} = b_H N_H - d_H N_H = (b_H - d_H)N_H$$

Dead-End Host ODEs

$$\frac{dN_D}{dt} = b_D N_D - d_D N_D \frac{N_D}{K_D}$$

Competent Vector ODEs

$$\frac{dS_C}{dt} = \theta_C P_C - \beta_{HC} S_C \frac{I_H}{N_H} - d_C S_C \frac{N_C}{K_C}$$

$$\frac{dE_C}{dt} = \beta_{HC} S_C \frac{I_H}{N_H} - \epsilon_C E_C - d_C E_C \frac{N_C}{K_C}$$

$$\frac{dI_C}{dt} = \epsilon_C E_C - d_C I_C \frac{N_C}{K_C}$$

$$\implies \frac{dN_C}{dt} = \theta_C P_C - d_C N_C \frac{N_C}{K_C}$$

Incompetent Vector ODEs

$\frac{dS_I}{dt} = \theta_I P_I - \beta_{HI} S_I \frac{I_H}{N_H} - d_I S_I \frac{N_I}{K_I}$

$\frac{dE_I}{dt} = \beta_{HI} S_I \frac{I_H}{N_H} - \epsilon_I E_I - d_I E_I \frac{N_I}{K_I}$

$\frac{dI_I}{dt} = \epsilon_I E_I - d_I I_I \frac{N_I}{K_I}$

$$\implies \frac{dN_I}{dt} = \theta_I P_I - d_I N_I \frac{N_I}{K_I}$$

Per van den Driessche (2001), we focus our attention only on pools with incoming new infections. For vertical transmission, this is only found in the host species:

$$\frac{dE_H}{dt} = \left(\beta_{CH} I_C + \beta_{IH} I_I + \beta_{HH} \frac{I_H}{N_H}\right) \frac{S_H}{N_H} - \epsilon_H E_H - d_H E_H$$

$$= \mathscr{F}_V - \mathscr{V}_V = \left(\beta_{HH} \frac{I_H S_H}{N_H^2}\right) - \left(\epsilon_H E_H + d_H E_H - \left(\beta_{CH} I_C + \beta_{IH} I_I + \beta_{HH} \frac{I_H}{N_H}\right)\right)$$

$$\frac{dI_H}{dt} = b_H I_H + \epsilon_H E_H - \gamma_H I_H - d_H I_H$$

$$= \mathscr{F}_V - \mathscr{V}_V = (b_H I_H) - (\gamma_H I_H + d_H I_H - \epsilon_H E_H)$$

Converting to matrix notation,

$$\mathscr{F}_V = \begin{bmatrix} \beta_{HH} \frac{I_H S_H}{N_H^2} \\ b_H I_H \end{bmatrix}, \mathscr{V}_V = \begin{bmatrix} \epsilon_H E_H + d_H E_H - \left(\beta_{CH} I_C + \beta_{IH} I_I + \beta_{HH} \frac{I_H}{N_H}\right) \\ \gamma_H I_H + d_H I_H - \epsilon_H E_H. \end{bmatrix}$$

It follows that F_V and V_V are found as the Jacobian of $\mathscr{F}_V$ and $\mathscr{V}_V$:

$$F_V = \begin{bmatrix} 0 & \beta_{HH} \frac{S_H}{N_H^2} \\ 0 & b_H \end{bmatrix}, \text{ and } V_V = \begin{bmatrix} \epsilon_H + dH & \frac{\beta_{HH}}{N_H} \\ -\epsilon_H & \gamma_H + d_H \end{bmatrix}$$

For the horizontal transmission component, focus is on pools with incoming new horizontally-acquired infection,

$$\frac{d}{dt} \begin{bmatrix} E_H \\ I_H \\ E_C \\ I_C \\ E_I \\ I_I \end{bmatrix} = \mathscr{F}_H - \mathscr{V}_H = \begin{bmatrix} \left(\beta_{CH} I_C + \beta_{IH} I_I + \beta_{HH} \frac{I_H}{N_H}\right) \frac{S_H}{N_H} \\ 0 \\ \beta_{HC} S_C \frac{I_H}{N_H} \\ 0 \\ \beta_{HI} S_I \frac{I_H}{N_H} \\ 0 \end{bmatrix} - \begin{bmatrix} \epsilon_H E_H + d_H E_H \\ \gamma_H I_H + d_H I_H - \epsilon_H E_H - b_H I_H \\ \epsilon_C E_C + d_C E_C \frac{N_C}{K_C} \\ d_C I_C \frac{N_C}{K_C} - \epsilon_C E_C \\ \epsilon_I E_I + d_I E_I \frac{N_I}{K_I} \\ d_I I_I \frac{N_I}{K_I} - \epsilon_I E_I \end{bmatrix}.$$

F_H and V_H are again the Jacobian of $\mathscr{F}_H$ and $\mathscr{V}_H$. Thus,

$$F_H = \begin{bmatrix} 0 & \beta_{HH} \frac{S_H}{N_H^2} & 0 & \beta_{CH} \frac{S_H}{N_H} & 0 & \beta_{IH} \frac{S_H}{N_H} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta_{HC} \frac{S_C}{N_H} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta_{HI} \frac{S_I}{N_H} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$V_H = \begin{bmatrix} \epsilon_H + d_H & 0 & 0 & 0 & 0 & 0 \\ -\epsilon_H & \gamma_H + d_H - b_H & 0 & 0 & 0 & 0 \\ 0 & 0 & \epsilon_C + d_C\kappa_1 & 0 & 0 & 0 \\ 0 & 0 & -\epsilon_C & d_C\kappa_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \epsilon_I + d_I\kappa_2 & 0 \\ 0 & 0 & 0 & 0 & -\epsilon_I & d_I\kappa_2 \end{bmatrix}$$

where $\kappa_1 = \frac{N_C}{K_C}, \kappa_2 = \frac{N_C}{K_I}$.

The reproductive number, R_0, equals the sum of the spectral radii of the next generation matrices, $F_V V_V^{-1} and F_H V_H^{-1}$. That is, $R_0 = \rho(F_V V_V^{-1}) + \rho(F_H V_H^{-1})$.

For the vertical portion,

$$V_V^{-1} = \frac{-N_H}{\epsilon_H\beta_{HH} + N_H(\epsilon_H + d_H)(\gamma_H + d_H)} \begin{bmatrix} -(\gamma_H + d_H) & \frac{\beta_{HH}}{N_H} \\ -\epsilon_H & -(\epsilon_H + d_H) \end{bmatrix}$$

$$F_V V_V^{-1} = \frac{1}{2} * \frac{N_H}{\epsilon_H\beta_{HH} + N_H(\epsilon_H + d_H)(\gamma_H + d_H)} \begin{bmatrix} \epsilon_H\beta_{HH}\frac{S_H}{N_H^2} & \beta_{HH}\frac{S_H}{N_H^2}(\epsilon_H + d_H) \\ \epsilon_H b_H & b_H(\epsilon_H + d_H) \end{bmatrix}$$

Letting A $= \frac{N_H}{\epsilon_H\beta_{HH} + N_H(\epsilon_H + d_H)(\gamma_H + d_H)}$,

$$\rho(F_V V_V^{-1}) = \tfrac{1}{2A}\left[\left(\tfrac{\epsilon_H\beta_{HH}S_H}{N_H^2}\right)^2 - 2\left(\tfrac{\epsilon_H\beta_{HH}S_H b_H}{N_H^2}(\epsilon_H + d_H)\right) + 4\left(\tfrac{\beta_{HH}S_H\epsilon_H b_H}{N_H^2}(\epsilon_H + d_H)\right) + b_H^2(\epsilon_H + d_H)^2\right]^{-1/2}$$

$$+\tfrac{1}{2A}\left[\tfrac{\epsilon_H\beta_{HH}S_H}{N_H^2} + b_H(\epsilon_H + d_H)\right]$$

For the horizontal portion,

$$V_H^{-1} = \begin{bmatrix} \frac{1}{\epsilon_H + d_H} & 0 & 0 & 0 & 0 & 0 \\ \frac{\epsilon_H}{(\gamma_H + d_H - b_H)(\epsilon_H + d_H)} & \frac{1}{\gamma_H + d_H - b_H} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\epsilon_C + d_C\kappa_1} & 0 & 0 & 0 \\ 0 & 0 & \frac{\epsilon_C}{(\epsilon_C + d_C\kappa_1)(d_C\kappa_1)} & \frac{1}{d_C\kappa_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\epsilon_I + d_I\kappa_2} & 0 \\ 0 & 0 & 0 & 0 & \frac{\epsilon_I}{(\epsilon_I + d_I\kappa_2)(d_I\kappa_2)} & \frac{1}{d_I\kappa_2} \end{bmatrix}$$

$$F_H V_H^{-1} = \begin{bmatrix} \frac{\beta_{HH}S_H\epsilon_H}{N_H^2(\gamma_H + d_H - b_H)(\epsilon_H + d_H)} & \frac{\beta_{HH}S_H}{(\gamma_H + d_H - b_H)N_H^2} & \frac{\beta_{CH}S_H\epsilon_C}{N_H d_C\kappa_1(\epsilon_C + d_C\kappa_1)} & \frac{\beta_{CH}S_H}{N_H d_C\kappa_1} & \frac{\epsilon_I\beta_{IH}S_H}{N_H d_I\kappa_2(\epsilon_I + d_I\kappa_2)} & \frac{\beta_{IH}S_H}{N_H d_I\kappa_2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\beta_{HC}S_C\epsilon_H}{N_H(\gamma_H + d_H - b_H)(\epsilon_H + d_H)} & \frac{\beta_{HC}S_C}{N_H(\gamma_H + d_H - b_H)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\beta_{HI}S_I\epsilon_H}{N_H(\gamma_H + d_H - b_H)(\epsilon_H + d_H)} & \frac{\beta_{HI}S_I}{N_H(\gamma_H + d_H - b_H)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Letting $A = \left(\frac{\beta_{HH}s_H}{N_H^2}\right)\left(\frac{\epsilon_H}{\gamma_H + d_H - b_H}\right)\left(\frac{1}{\epsilon_H + d_H}\right)$, C $= \left(\frac{\beta_{CH}S_H}{N_H}\right)\left[\frac{\epsilon_C}{(\epsilon_C + d_C\kappa_1)d_C\kappa_1}\right]$, G $= \left(\frac{\beta_{HC}S_C}{N_H}\right)\left[\frac{\epsilon_H}{(\gamma_H + d_H - b_H)(\epsilon_H + d_H)}\right]$, E $= \frac{\epsilon_I\beta_{IH}S_H}{(\epsilon_I + d_I\kappa_2)d_I\kappa_2 N_H}$, and I $= \left(\frac{\beta_{HI}S_I}{N_H}\right)\left[\frac{\epsilon_H}{(\gamma_H + d_H - b_H)(\epsilon_H + d_H)}\right]$,we conclude that the basic reproductive number,

$$R_0 = \rho(F_H V_H^{-1}) = \tfrac{1}{2}(A + \sqrt(A^2 + 4CG + 4EI)).$$