Three Species Model of Zika Virus

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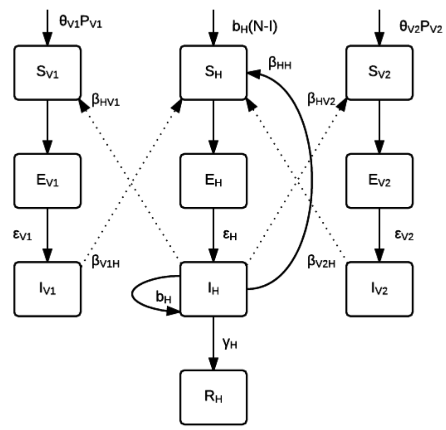

Host ODEs

$\frac{dS_H}{dt} = b_H(N_H - I_H) - \beta_{V1H}\frac{S_H}{N_H}I_{V1} - \beta_{V2H}I_{V2} - \beta_{HH}\frac{S_H I_H}{N_H^2} - d_H S_H$

$\frac{dE_H}{dt} = \left(\beta_{V1H}I_{V1} + \beta_{V2H}I_{V2} + \beta_{HH}\frac{I_H}{N_H}\right)\frac{S_H}{N_H} - \epsilon_H E_H - d_H E_H$

$\frac{dI_H}{dt} = b_H I_H + \epsilon_H E_H - \gamma_H I_H - d_H I_H$

$\frac{dR_H}{dt} = \gamma_H I_H - d_H R_H$

$$\implies \frac{dN_H}{dt} = b_H N_H - d_H N_H = (b_H - d_H)N_H$$

Vector 1 ODEs

$\frac{dS_{V1}}{dt} = \theta_{V1}P_{V1} - \beta_{HV1}S_{V1}\frac{I_H}{N_H} - d_{V1}S_{V1}\frac{N_{V1}}{K_{V1}}$

$\frac{dE_{V1}}{dt} = \beta_{HV1}S_{V1}\frac{I_H}{N_H} - \epsilon_{V1}E_{V1} - d_{V1}E_{V1}\frac{N_{V1}}{K_{V1}}$

$\frac{dI_{V1}}{dt} = \epsilon_{V1}E_{V1} - d_{V1}I_{V1}\frac{N_{V1}}{K_{V1}}$

$$\implies \frac{dN_{V1}}{dt} = \theta_{V1}P_{V1} - d_{V1}N_{V1}\frac{N_{V1}}{K_{V1}}$$

Vector 2 ODEs

$\frac{dS_{V2}}{dt} = \theta_{V2}P_{V2} - \beta_{HV2}S_{V2}\frac{I_H}{N_H} - d_{V2}S_{V2}\frac{N_{V2}}{K_{V2}}$

$\frac{dE_{V2}}{dt} = \beta_{HV2}S_{V2}\frac{I_H}{N_H} - \epsilon_{V2}E_{V2} - d_{V2}E_{V2}\frac{N_{V2}}{K_{V2}}$

$$\implies \frac{dN_{V2}}{dt} = \theta_{V2}P_{V2} - d_{V2}N_{V2}\frac{N_{V2}}{K_{V2}}$$

Per van den Driessche (2001), we focus our attention only on pools with incoming new infections. For vertical transmission, this is only found in the host species:

$$\frac{dE_H}{dt} = \left(\beta_{V1H} I_{V1} + \beta_{V2H} I_{V2} + \beta_{HH}\frac{I_H}{N_H}\right)\frac{S_H}{N_H} - \epsilon_H E_H - d_H E_H$$

$$= \mathscr{F}_V - \mathscr{V}_V = \left(\beta_{HH}\frac{I_H S_H}{N_H^2}\right) - \left(\epsilon_H E_H + d_H E_H - \left(\beta_{V1H} I_{V1} + \beta_{V2H} I_{V2} + \beta_{HH}\frac{I_H}{N_H}\right)\right)$$

$$\frac{dI_H}{dt} = b_H I_H + \epsilon_H E_H - \gamma_H I_H - d_H I_H$$

$$= \mathscr{F}_V - \mathscr{V}_V = (b_H I_H) - (\gamma_H I_H + d_H I_H - \epsilon_H E_H)$$

Converting to matrix notation,

$$\mathscr{F}_V = \begin{bmatrix} \beta_{HH}\frac{I_H S_H}{N_H^2} \\ b_H I_H \end{bmatrix}, \mathscr{V}_V = \begin{bmatrix} \epsilon_H E_H + d_H E_H - \left(\beta_{V1H} I_{V1} + \beta_{V2H} I_{V2} + \beta_{HH}\frac{I_H}{N_H}\right) \\ \gamma_H I_H + d_H I_H - \epsilon_H E_H. \end{bmatrix}$$

It follows that F_V and V_V are found as the Jacobian of $\mathscr{F}_V$ and $\mathscr{V}_V$:

$$F_V = \begin{bmatrix} 0 & \beta_{HH}\frac{S_H}{N_H^2} \\ 0 & b_H \end{bmatrix}, \text{ and } V_V = \begin{bmatrix} \epsilon_H + dH & \frac{\beta_{HH}}{N_H} \\ -\epsilon_H & \gamma_H + d_H \end{bmatrix}$$

For the horizontal transmission component, focus is on pools with incoming new horizontally-acquired infection,

$$\frac{d}{dt}\begin{bmatrix} E_H \\ I_H \\ E_{V1} \\ I_{V1} \\ E_{V2} \\ I_{V2} \end{bmatrix} = \mathscr{F}_H - \mathscr{V}_H = \begin{bmatrix} \left(\beta_{V1H} I_{V1} + \beta_{V2H} I_{V2} + \beta_{HH}\frac{I_H}{N_H}\right)\frac{S_H}{N_H} \\ 0 \\ \beta_{HV1} S_{V1}\frac{I_H}{N_H} \\ 0 \\ \beta_{HV2} S_{V2}\frac{I_H}{N_H} \\ 0 \end{bmatrix} \text{ - } \begin{bmatrix} \epsilon_H E_H + d_H E_H \\ \gamma_H I_H + d_H I_H - \epsilon_H E_H - b_H I_H \\ \epsilon_{V1} E_{V1} + d_{V1} E_{V1}\frac{N_{V1}}{K_{V1}} \\ d_{V1} I_{V1}\frac{N_{V1}}{K_{V1}} - \epsilon_{V1} E_{V1} \\ \epsilon_{V2} E_{V2} + d_{V2} E_{V2}\frac{N_{V2}}{K_{V2}} \\ d_{V2} I_{V2}\frac{N_{V2}}{K_{V2}} - \epsilon_{V2} E_{V2} \end{bmatrix}.$$

F_H and V_H are again the Jacobian of $\mathscr{F}_H$ and $\mathscr{V}_H$. Thus,

$$F_H = \begin{bmatrix} 0 & \beta_{HH}\frac{S_H}{N_H^2} & 0 & \beta_{V1H}\frac{S_H}{N_H} & 0 & \beta_{V2H}\frac{S_H}{N_H} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta_{HV1}\frac{S_{V1}}{N_H} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta_{HV2}\frac{S_{V2}}{N_H} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$V_H = \begin{bmatrix} \epsilon_H + d_H & 0 & 0 & 0 & 0 & 0 \\ -\epsilon_H & \gamma_H + d_H - b_H & 0 & 0 & 0 & 0 \\ 0 & 0 & \epsilon_{V1} + d_{V1}\kappa 1 & 0 & 0 & 0 \\ 0 & 0 & -\epsilon_{V1} & d_{V1}\kappa 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \epsilon_{V2} + d_{V2}\kappa 2 & 0 \\ 0 & 0 & 0 & 0 & -\epsilon_{V2} & d_{V2}\kappa 2 \end{bmatrix}$$

where $\kappa 1 = \frac{N_{V1}}{K_{V1}}, \kappa 2 = \frac{N_{V1}}{K_{V2}}$.

The reproductive number, R_0, equals the sum of the spectral radii of the next generation matrices, $F_V V_V^{-1} and F_H V_H^{-1}$. That is, $R_0 = \rho(F_V V_V^{-1}) + \rho(F_H V_H^{-1})$.

For the vertical portion,

$$V_V^{-1} = \frac{-N_H}{\epsilon_H \beta_{HH} + N_H(\epsilon_H + d_H)(\gamma_H + d_H)} \begin{bmatrix} -(\gamma_H + d_H) & \frac{\beta_{HH}}{N_H} \\ -\epsilon_H & -(\epsilon_H + d_H) \end{bmatrix}$$

$$F_V V_V^{-1} = \frac{1}{2} * \frac{N_H}{\epsilon_H \beta_{HH} + N_H(\epsilon_H + d_H)(\gamma_H + d_H)} \begin{bmatrix} \epsilon_H \beta_{HH} \frac{S_H}{N_H^2} & \beta_{HH} \frac{S_H}{N_H^2}(\epsilon_H + d_H) \\ \epsilon_H b_H & b_H(\epsilon_H + d_H) \end{bmatrix}$$

Letting A $= \frac{N_H}{\epsilon_H \beta_{HH} + N_H(\epsilon_H + d_H)(\gamma_H + d_H)}$,

$$\rho(F_V V_V^{-1}) = \frac{1}{2A} \left[\left(\frac{\epsilon_H \beta_{HH} S_H}{N_H^2} \right)^2 - 2 \left(\frac{\epsilon_H \beta_{HH} S_H b_H}{N_H^2} (\epsilon_H + d_H) \right) + 4 \left(\frac{\beta_{HH} S_H \epsilon_H b_H}{N_H^2} (\epsilon_H + d_H) \right) + b_H^2 (\epsilon_H + d_H)^2 \right]^{-1/2}$$

$$+ \frac{1}{2A} \left[\frac{\epsilon_H \beta_{HH} S_H}{N_H^2} + b_H(\epsilon_H + d_H) \right]$$

For the horizontal portion,

$$V_H^{-1} = \begin{bmatrix} \frac{1}{\epsilon_H + d_H} & 0 & 0 & 0 & 0 & 0 \\ \frac{\epsilon_H}{(\gamma_H + d_H - b_H)(\epsilon_H + d_H)} & \frac{1}{\gamma_H + d_H - b_H} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\epsilon_{V1} + d_{V1}\kappa_1} & 0 & 0 & 0 \\ 0 & 0 & \frac{\epsilon_{V1}}{(\epsilon_{V1} + d_{V1}\kappa_1)(d_{V1}\kappa 1)} & \frac{1}{d_{V1}\kappa_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\epsilon_{V2} + d_{V2}\kappa_2} & 0 \\ 0 & 0 & 0 & 0 & \frac{\epsilon_{V2}}{(\epsilon_{V2} + d_{V2}\kappa_2)(d_{V2}\kappa_2)} & \frac{1}{d_{V2}\kappa_2} \end{bmatrix}$$

$$F_H V_H^{-1} = \begin{bmatrix} \frac{\beta_{HH} S_H \epsilon_H}{N_H^2 (\gamma_H + d_H - b_H)(\epsilon_H + d_H)} & \frac{\beta_{HH} S_H}{(\gamma_H + d_H - b_H) N_H^2} & \frac{\beta_{V1H} S_H \epsilon_{V1}}{N_H d_{V1}\kappa_1 (\epsilon_{V1} + d_{V1}\kappa_1)} & \frac{\beta_{V1H} S_H}{N_H d_{V1}\kappa_1} & \frac{\epsilon_{V2} \beta_{V2H} S_H}{N_H d_{V2}\kappa_2 (\epsilon_{V2} + d_{V2}\kappa_2)} & \frac{\beta_{V2H} S_H}{N_H d_{V2}\kappa_2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\beta_{HV1} S_{V1} \epsilon_H}{N_H (\gamma_H + d_H - b_H)(\epsilon_H + d_H)} & \frac{\beta_{HV1} S_{V1}}{N_H (\gamma_H + d_H - b_H)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\beta_{HV2} S_{V2} \epsilon_H}{N_H (\gamma_H + d_H - b_H)(\epsilon_H + d_H)} & \frac{\beta_{HV2} S_{V2}}{N_H (\gamma_H + d_H - b_H)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Letting $A = \left(\frac{\beta_{HHS_H}}{N_H^2} \right) \left(\frac{\epsilon_H}{\gamma_H + d_H - b_H} \right) \left(\frac{1}{\epsilon_H + d_H} \right)$, C $= \left(\frac{\beta_{V1H} S_H}{N_H} \right) \left[\frac{\epsilon_{V1}}{(\epsilon_{V1} + d_{V1}\kappa_1) d_{V1}\kappa_1} \right]$, G $= \left(\frac{\beta_{HV1} S_{V1}}{N_H} \right) \left[\frac{\epsilon_H}{(\gamma_H + d_H - b_H)(\epsilon_H + d_H)} \right]$, E $= \frac{\epsilon_{V2} \beta_{V2H} S_H}{(\epsilon_{V2} + d_{V2}\kappa_2) d_{V2}\kappa_2 N_H}$, and I $= \left(\frac{\beta_{HV2} S_{V2}}{N_H} \right) \left[\frac{\epsilon_H}{(\gamma_H + d_H - b_H)(\epsilon_H + d_H)} \right]$,

$$\rho(F_H V_H^{-1}) = \frac{1}{2}(A + \sqrt{(}A^2 + 4CG + 4EI)).$$