EE 325 Probability and Random Processes

Homework 5: Computational Assignment

- The following computational assignments will have to be done in batches of at most two people per batch. You can code in either of Matlab, Python, or C.
- The code will be assessed during the weekend of 03-04 November. You will have to demo the codes in an interactive session in a slot. The slot options will be announced shortly.
- Both team members should know the code, and the theory, well enough to make minor changes to the code and answer the questions that may be asked based on this theory.
- Bring your computer to the evaluation session.
- The coding effort is nominal but the learning by experimenting with the parameters.

1. **Analysing uniform pseudo random sequences:** Generate a sequence of $N = m \times k$ iid 0/1 Bernouilli random variables with mean μ. For different values of k, the N samples can be treated as m samples of $\mathsf{X}_1, \ldots, \mathsf{X}_k$. From the samples, obtain the joint pmf of $\mathsf{X}_1, \ldots, \mathsf{X}_k$ and verify the independence.

2. **Generating pseudo random sequences:**

   (a) Using iid uniformly distributed random numbers, generate a sequence of iid exponential random variables with mean $1/\lambda$. Plot the sample mean and sample variance as a function of the number of samples.

   (b) Use the Box-Mueller method to generate samples of iid Gaussian random variables of a given μ and σ. Plot the sample mean and sample variance as a function of the number of samples.

   (c) We want to generate a sequence of iid non negative random variables with pdf $\sqrt{2/\pi}e^{-x^2/2}$. Generate these RVs using the Accept/Reject method using iid exponential andom variables with unit mean. Generate a sufficiently large number of samples and plot the sample pdf and verify against the expected pdf. Also determine the probability of acceptance.

   (d) A machine has two components and the lifetime of the two components, denoted by X_1 and X_2 respectively, is correlated as follows. Generate three independent exponential random variables, Y_1, Y_2 and Y_{12} with means $1/\lambda_1$, $1/\lambda_2$, and $1/\lambda_{12}$ respectively. We will have $\mathsf{X}_1 = \min(\mathsf{Y}_1, \mathsf{Y}_{12})$ and $\mathsf{X}_2 = \min(\mathsf{Y}_2, \mathsf{Y}_{12})$. Via simulation, verify that

   $$\mathsf{Prob}(\mathsf{X}_1 > x_1, \mathsf{X}_2 > x_2) = e^{-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_{12} \max x_1, x_2}$$

   Also use the simulation code to determine the mean and variances of X_1 and X_2, and the covariance.

3. **Estimation:** Consider the following autoregressive AR sequence

   $$\mathsf{X}(t+1) = a\mathsf{X}(t) + b\mathsf{X}(t-1) + c\mathsf{X}(t-2)\mathsf{W}(t).$$

   The observation $\mathsf{Y}(t) = \mathsf{X}(t) + \mathsf{V}(t)$ where $\mathsf{V}(t)$ $\mathsf{W}(t)$ are zero mean IID Gaussian with variances, respectively, 1 and 0.1. Use $a = 2.5$, $b = -2.0$, and $c = 0.7$. The objective is to predict $\mathsf{X}(t+1)$

using the observations upto t using linear estimator. Implement a vector Kalman filter to perform the estimation. Note that in a simulation you actually know the true value and also the estimate.

- Plot $\mathsf{X}(t)$, $\hat{E}(\mathsf{X}(t+1)|\mathsf{Y}^t)$. and $\mathsf{Y}(t)$ as a function of t.
- Plot the Kalman gain, the actual error, and the predicted error as function of t. Be prepared to discuss any 'discrepancies.'

4. **A discrete time queue:** Consider the following discrete time queue. In slot, a new packet joins the the queue with probability λ. If there are packets in the queue, one packet departs with probability μ. Two random sequences are of interest—$\mathsf{Q}(t)$, the number of in packets in the queue at time t, and W_n the time spent in the queue by the n-th packet to arrive into the system. Quantities of interest will be the 'time' averages of the mean and variance of these two quantities. Write a program to simulate this queue and also capture these statistics for these two random sequences. Write a program that plots the sample mean and sample variance of $\mathsf{Q}(t)$ and W_n as a function of t and n. Also for a given T and N, plot the ensemble average of $\mathsf{Q}(T)$ and $\mathsf{W}(N)$ over K sample paths, as a function of K. Experiment with different combination of values for λ and μ.

5. **NRI wedding problem:** Just before finishing PhD abroad, Sri was visiting the family. The family had a surprise for Pat: they had decided that Sri would select the spouse during the visit and that n potential spousal candidates were readied for interview. Sri gives a score to X_k to the k-th interviewee. The catch is that if Sri decides to choose candidate k, the interviews stops there. On the other hand, if Sri does not choose k right after the interview, Sri cannot go back and the candidate is lost forever. Sri decides to use the following strategy: Decline all the candidates upto some k and then choose the first candidate that is better than all the preceding k. Sri has to decide the k before the interviews begin. Of course k is a function of n; if k is too lareg the best candidate may be interviewed (and discarded) but not chosen. If k is too small, then the best candidate may not be interviewed at all. Either of these cases will be an error. Quantities of interest will be

   - $p(n,k)$: For a given k,n the probability of an error in the choice.
   - $q(n,k)$: For a given k,n the probability of not choosing a spouse.
   - k_n^*, the optimal k as a function of n that minimises the error probability and (p_n*, q_n^*) for this choice of k_n^*.

   Assume that X_k is uniformly distributed between in $(0,1)$. Write a simulation program to plot the preceding quantities as a function of n.